L1.1

Analysis I by Jake Rasmussen

Fundamental Theorem of Calculus:

If $f: [a,b] \longrightarrow \mathbb{R}$ is continuous,

$F: [a,b] \longrightarrow \mathbb{R}$ is diffble and

$F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Obvious Problem

What do f is continuous / F is diff ble / $\int_a^b f(x)\,dx$ mean?

Subtle Problem This is a theorem about $\mathbb{R}$

Spose $F' = f = G'$. Then $F(x) - F(a) = G(x) - G(a)$, i.e.

$F - G \equiv C$.

Ex: $F(x) = \begin{cases} 1 & \text{if } x^2 < 2, \\ 0 & \text{if } x^2 \geqslant 2. \end{cases}$

Then F is differentiable at all $x \in \mathbb{Q}$ and $F'(x) = 0 \ \forall x \in \mathbb{Q}$

But $F \not\equiv C$.

§0 The Real Numbers

Axiom: There is a set $\mathbb{R}$ (the real numbers) s.t.

1) $\mathbb{Q} \subset \mathbb{R}$
2) $\mathbb{R}$ is an ordered field
3) $\mathbb{R}$ satisfies the least upper bound property

Explanation: $\mathbb{R}$ is a field means that there are binary operations $+, \cdot$ on $\mathbb{R}$ and elements $0, 1 \in \mathbb{R}$ such that

1) $(\mathbb{R}, +, 0)$ and $(\mathbb{R}\setminus\{0\}, \times, 1)$ are abelian groups
2) $a\cdot(b+c) = a\cdot b + a\cdot c \quad \forall a,b,c \in \mathbb{R}$

Ex: $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$ for p prime are fields
$\mathbb{Z}/6\mathbb{Z}$ not a field

$\mathbb{R}$ is an ordered field means there is a total order $<$ on $\mathbb{R}$ such that 1) $a < b \Rightarrow a + c < b + c \quad \forall a, b, c \in \mathbb{R}$
2) $a < b$ and $c > 0 \Rightarrow ac < bc$ "

Def^n: Spose X is an ordered set, and $A \subset X$. We say $x \in X$ is an upper bound of A if $a \leq x$ for all $a \in A$. We say x is the least upper bound or supremum of A if x is an upper bound for A and if x' is an upper bound for A, then $x \leq x'$.

Remarks: 1) If A has an upper bound, we say A is bounded above
2) There are similar notions of lower bound, greatest lower bound (infimum)
3) These subsets of $\mathbb{Q}^{=X}$ are bounded above, but have no supremum
i) $A = \emptyset$ ii) $A = \{a \in \mathbb{Q} : a^2 < 2\}$
4) If x and x' are both least upper bounds for A, then
$x \leq x'$ and $x' \leq x$, so $x' = x$.
$\Rightarrow$ least upper bound is unique, if it exists

Notation $a = \sup A$ is the least upper bound for A.

Least Upper Bound Property:
If $A \subset \mathbb{R}$ is non-empty and bounded above, then it has a supremum.
(Thm) If $\mathbb{F}$ is an ordered field satisfying the least upper bound property, then $\mathbb{F} \cong \mathbb{R}$, in the sense that there's a bijection $\varphi: \mathbb{F} \to \mathbb{R}$ that respects $+, \cdot, <$.

<u>Prop:</u> (Further Properties of $\mathbb{R}$)

1) If $x \in \mathbb{R}$, then there is some $N \in \mathbb{N}$ with $x < N$. (Archimedean Axiom)
2) If $x \in \mathbb{R}$ and $x > 0$, then there is $N \in \mathbb{N}$ such that $0 < 1/N < x$.
3) If $x < z$, $x, z \in \mathbb{R}$, then $\exists\, y \in \mathbb{Q}$ with $x < y < z$ (density of $\mathbb{Q}$ in $\mathbb{R}$)
4) If $x \in \mathbb{R}$, $x \geq 0$, then there is a unique $y \geq 0$ in $\mathbb{R}$ with $y^2 = x$.

<u>Proof of 1)</u> By contradiction. Spose $x \in \mathbb{R}$, but there is no $N \in \mathbb{N}$ with $x < N$. Then x is an upper bound for $\mathbb{N}$. So $\mathbb{N}$ is non-empty and bounded above, so $y = \sup \mathbb{N}$ exists.

Consider $y - 1 < y$, so $y-1$ is not an upper bound for $\mathbb{N}$, i.e. there exists some $N \in \mathbb{N}$ with $y - 1 < N$. But then $y < N+1$ but $N+1 \in \mathbb{N}$, which contradicts y being an upper bound for $\mathbb{N}$. $\square$

§0.1 The Complex Numbers

<u>Def:</u> $\mathbb{C} = \left\{ A \in M_{2\times 2}(\mathbb{R}) : A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\}$

$$= \{ a \cdot 1 + b \cdot i : a, b \in \mathbb{R} \}$$

where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$\mathbb{C}$ is a field, $+, \cdot$ are given by matrix addition and multiplication

N.B. $i^2 = -1 < 0 \Rightarrow \mathbb{C}$ cannot be ordered

$\mathbb{R} = \{a \cdot 1 + 0 \cdot i\} \subset \mathbb{C}$

<u>Def:</u>" If $z = a + bi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{C}$, then $|z| = (a^2 + b^2)^{1/2}$.
$|z| = \left(\det \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right)^{1/2}$ ergo $|zw| = |z||w|$.

L1.4

<u>Prop:</u> (Triangle Inequality)
If $z, w \in \mathbb{C}$, then $|z+w| \leq |z| + |w|$.

<u>Idea of Proof:</u> Identify $\mathbb{C}$ with $\mathbb{R}^2$, $a+bi \longrightarrow (a,b)$

$|\vec{v}| + |\vec{w}| \geq |\vec{v}+\vec{w}|$ in $\mathbb{R}^2$

follows from Cauchy-Schwarz.

I) Convergence

1.1) Sequences

- Let X be a set. A sequence in X is a function $\mathbb{N} \longrightarrow X$.

The sequence $n \longrightarrow x_n$ is written (x_n).

Ex: If $x \in X$, then (x) is a constant sequence

$(\sin(n^3 + 17))$ is a sequence in $\mathbb{R}$

Def: Let (z_n) be a sequence in $\mathbb{C}$. We say (z_n) converges to $z \in \mathbb{C}$ (written $z_n \to z$ or $\lim_{n\to\infty} z_n = z$) if for every $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$) there is some $N \in \mathbb{N}$ such that

- $|z_n - z| < \varepsilon$ whenever $n > N$.

NB: N depends on ε

Ex: i) If $c \in \mathbb{C}$, then the constant sequence (c) converges to c.

ii) $\frac{1}{n} \to 0$

Proof: Given $\varepsilon > 0$, I can find some $N \in \mathbb{N}$ with $\frac{1}{\varepsilon} < N$.

Then for $n > N$, $\frac{1}{\varepsilon} < N < n$, so

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$ □

- If (z_n) is a sequence in $\mathbb{C}$ which does not converge to any $z \in \mathbb{C}$, we say (z_n) diverges.

Ex: (n) diverges since for $z \in \mathbb{C}$, $|z - n| < 1$ for at most two values of n.

$((-1)^n)$ diverges since for any $z \in \mathbb{C}$, there are ∞ many n with $|(-1)^n - z| \geq 1$.

Equivalent Formulations: $z_n \to z$ if and only if

- 1) $z_n - z \longrightarrow 0$
- 2) $|z_n - z| \longrightarrow 0$
- 3) For every $m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ s.t. $|z_n - z| < \frac{1}{m}$ for all $n > N$

Prop (Laws of Limits I)

Suppose (z_n) and (w_n) are sequences in $\mathbb{C}$, and that $z_n \to z \in \mathbb{C}$, $w_n \to w \in \mathbb{C}$. Then

1) $z_n + w_n \longrightarrow z + w$

2) $z_n w_n \longrightarrow zw$

3) If $z \neq 0$, then $\exists N \in \mathbb{N}$ s.t. $z_n \neq 0$ for $n > N$ and $(1/z_n)_{n>N} \longrightarrow 1/z$.

4) If $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbb{R}$ and $z = x + iy$, then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

Notation: $(x_n)_{n \geq k} = (x_{n+k-1})$

$(x_n)_{n \geq k} \longrightarrow x$ iff $(x_n) \longrightarrow x$

Ex: 1) If $c \in \mathbb{C}$, and $z_n \to z$, then $cz_n \to cz$ by Rule 2

If $w_n \to w$, then $w_n + cz_n \to w + cz$ by Rule 1

2) Know $\frac{1}{n} \to 0$, so $\frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} \to 0 \cdot 0 = 0$ by Rule 2

$1 - 1/n^2 \to 1 - 0 = 1$ by Rule 1

Similarly $1 + 1/n^2 \to 1 + 0 = 1$

Then $\frac{n^2-1}{n^2+1} = \frac{1 - 1/n^2}{1 + 1/n^2} \to \frac{1}{1} = 1$ by Rules 2 and 3

Proof: 1) Given $\varepsilon > 0$, must find N s.t.

(*) $|(z_n + w_n) - (z + w)| < \varepsilon$ for $n > N$.

i) Estimate (*) in terms of $|z - z_n|$ and $|w - w_n|$

$$|(z_n + w_n) - (z + w)| = |z_n - z + w_n - w| \leq |z_n - z| + |w_n - w|$$

ii) Make (*) $< \varepsilon$.

Since $z_n \to z$, I can find N_1 s.t. $|z_n - z| < \varepsilon/2$ for $n > N_1$

Since $w_n \to w$, $\exists N_2$ s.t. $|w_n - w| < \varepsilon/2$ for $n > N_2$.

L2.3 Then for $n > \max(N_1, N_2)$,

$$|(z_n+w_n)-(z+w)| \leq |z_n-z|+|w_n-w| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Which is what I wanted. $\square$

2) Estimate
$$\begin{aligned}|z_n w_n - zw| &= |z_n w_n - z_n w + z_n w - zw| \\ &\leq |z_n||w_n-w| + |w||z_n-z| \\ &\leq (|z|+1)|w_n-w| + |w||z_n-z| \quad \text{for } n > N_1\end{aligned}$$

Estimate $|z_n|$:

Choose N_1 s.t. $|z_n - z| < 1$ for $n > N_1$.

Then $|z_n| < |z|+1$ for $n > N_1$

Choose N_2 s.t. $|w_n - w| < \frac{\varepsilon}{2(|z|+1)}$ for $n > N_2$

N_3 s.t. $|z_n - z| < \frac{\varepsilon}{2(|w|+1)}$ for $n > N_3$.

Then if $n > \max\{N_1, N_2, N_3\}$

$$|z_n w_n - zw| \leq (|z|+1)\frac{\varepsilon}{2(|z|+1)} + |w|\frac{\varepsilon}{2(|w|+1)}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

3) Show that $z_n \neq 0$ for $n > N$:

Choose N so that $|z_n - z| < |z|/2$ for $n > N$.

Then for $n > N$, have $|z_n| \geq |z| - |z_n - z|$

$> |z|/2 > 0.$ so for $n > N$, $z_n \neq 0$

Estimate $\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z_n - z|}{|z_n||z|} \leq \frac{2|z_n - z|}{|z|^2}$ for $n > N$.

Choose N_1 so that $|z_n - z| < \frac{|z|^2\varepsilon}{2}$ for $n > N_1$. Then for $n > \max(N, N_1)$ we have

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| < \frac{2}{|z|^2} \cdot \frac{|z|^2\varepsilon}{2} = \varepsilon. \quad \square$$

4) Suppose $z_n = x_n + iy_n$, $z = x+iy$. If $x_n \to x$ and $y_n \to y$ then $x_n + iy_n \to x+iy$ by Rules 1 and 2.

Suppose that $z_n \to z$. Estimate $|x_n - x| \leq |x_n - x + i(y_n - y)| = |z_n - z|$.

So given $\varepsilon > 0$, choose N with $|z_n - z| < \varepsilon$ for $n > N$.

Then $|x_n - x| < \varepsilon$ for $n > N \Rightarrow x_n \to x$.

Similarly $y_n \to y$. □

L3.1

1.1) Cont'd

Real Sequences:

Recall that $x_n \to x$ means that for every $\varepsilon > 0$, there is a $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n > N$.

i.e. given $\varepsilon > 0$, all but finitely many x_n are within distance ε of x.

Prop: (Laws of Limits II)

Suppose (x_n) and (y_n) are sequences in $\mathbb{R}$, and that $x_n \to x$, $y_n \to y$.

1) If $x_n \geq 0$ for all n, then $x \geq 0$.

2) If $x_n \geq y_n$ for all n, then $x \geq y$.

3) (Squeeze Rule)

If $x_n \geq c_n \geq y_n$ for all n and $x = y = c$, then $c_n \to c$.

Proof: 1) By contradiction. Spose $x < 0$. Since $x_n \geq 0$,

$|x_n - x| = x_n + |x| \geq |x| > 0$.

Now $x_n \to x$ means $|x_n - x| < \varepsilon$ for all but finitely many n.

Take $\varepsilon = |x| > 0$. Then $|x_n - x| < \varepsilon$ for no n. ※ □

2) $x_n \to x$, $y_n \to y$, so by LOL I $x_n - y_n \to x - y$.

$x_n \geq y_n$ for all n, so $x_n - y_n \geq 0$ for all n.

By 1), $x - y \geq 0$ i.e. $x \geq y$. □

3) Given $\varepsilon > 0$, choose N_1 s.t. $|x_n - c| < \varepsilon$ for $n > N_1$ and choose N_2 s.t. $|y_n - c| < \varepsilon$ for $n > N_2$.

Then $x_n < c + \varepsilon$ for $n > N_1$, $c - \varepsilon < y_n$ for $n > N_2$

so for $n > \max\{N_1, N_2\}$,

$c - \varepsilon < y_n \leq c_n \leq x_n < c + \varepsilon$

$\Rightarrow |c_n - c| < \varepsilon$ for $n > \max\{N_1, N_2\}$

$\Rightarrow c_n \to c$. □

L3.2

Ex: $\frac{n^2+\sin n}{n^2} \leq \frac{n^2+1}{n^2}$

$\geq$

$\frac{n^2-1}{n^2}$ So $\frac{n^2+\sin n}{n^2} \longrightarrow 1$

Monotone Sequences:

Def.^{n} If (x_n) is a sequence in $\mathbb{R}$, we say (x_n) is monotone increasing (MI) if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

We say (x_n) is bounded above if $\{x_n : n \in \mathbb{N}\}$ is bounded ~~a~~ above.

Thm: (Monotone Sequence Thm)

If (x_n) is monotone increasing and bounded above, then (x_n) converges.

Remarks: 1) If we replace $\mathbb{R}$ with $\mathbb{Q}$, statement is false

E.g. 3, 3.1, 3.14, 3.141, ...

does not converge in $\mathbb{Q}$

To prove it, must use LUBP.

2) If (x_n) is monotone increasing, then $x_n \geq x_m$ whenever $n \geq m$ (Induction)

Proof: Let $X = \{x_n : n \in \mathbb{N}\}$. Then $X \neq \emptyset$ and by hypothesis, X is bounded above. By LUBP of $\mathbb{R}$, X has a supremum $x \in \mathbb{R}$.

We claim that $x_n \longrightarrow x$. Given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $x_N > x - \varepsilon$. Otherwise $x - \varepsilon$ would be an upper bound for X, contradicting x being $\sup X$. Since (x_n) is monotone increasing, $x_n \geq x_N > x - \varepsilon$ for $n \geq N$.

Since x is an upper bound for X, $x_n \leq x$ for all n.

So $x - \varepsilon < x_n \leq x$ for all $n > N$

$\Rightarrow |x_n - x| < \varepsilon$ for all $n > N$

i.e. $x_n \to x$. $\square$

Corollary: If (x_n) is MI and $x_n \to x$ then $x_n \leq x$ for all n.

Def: If (x_n) is a sequence in $\mathbb{R}$, we say $x_n \to \infty$ if for all $M \in \mathbb{N}$ there is some $N \in \mathbb{N}$ s.t. $x_n > M$ for all $n > N$.

NB: $x_n \to \infty \Rightarrow (x_n)$ diverges

Ex: 1) If (x_n) is MI, either (x_n) converges or $x_n \to \infty$

2) If $|z_n| \to \infty$, then (z_n) diverges

1.2) Series

Q: (Zeno) What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

A: Suppose (a_n) is a sequence in $\mathbb{C}$

To study $\sum_{n=1}^{\infty} a_n$, consider the

partial sums $\quad s_n = \sum_{i=1}^{n} a_i$.

Def: We say $\sum_{n=1}^{\infty} a_n$ converges or diverges if the sequence (s_n) converges or diverges.

If $s_n \to s \in \mathbb{C}$, write $\sum_{n=1}^{\infty} a_n = s$.

Ex: (Geometric Series)

$a_n = z^n$, $z \in \mathbb{C}$

Then $s_n = \sum_{i=0}^{n} z^i = \frac{1-z^{n+1}}{1-z}$.

If $x \in [0,1)$, $x^n \to 0$ (Ex Sheet)

$\Rightarrow$ if $|z| < 1$, $|z^n - 0| = |z|^n \to 0$ so $z^n \to 0$.

Thus if $|z| < 1$, $\frac{1-z^{n+1}}{1-z} \to \frac{1}{1-z}$

i.e. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$.

Lemma: (n^{th} term test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Proof: $\sum_{n=1}^{\infty} a_n$ converges means $s_n \to s \in \mathbb{C}$
$\Rightarrow s_{n-1} \to s \in \mathbb{C}$

By LOL, $s_n - s_{n-1} \to s - s = 0$
i.e. $a_n \to 0$ $\square$

Ex: If $|z| \geq 1$, $|z^n| \geq 1$ for all n
$\Rightarrow z^n \not\to 0$, so
$\sum_{n=0}^{\infty} z^n$ diverges for $|z| > 1$.

Convergence Tests for Series

Thm (Comparison Test):
Suppose (a_n) and (b_n) are real sequences and that $a_n \geq b_n \geq 0$ $\forall n$
If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ converges as well.

Proof: Let $s_n = \sum_{i=1}^{n} a_n$, $t_n = \sum_{i=1}^{n} b_n$.
~~Observe~~

Then $a_i \geq b_i$ for all i $\Rightarrow$ $s_n \geq t_n$ for all n.

$a_n \geq 0$ for all n $\Rightarrow (s_n)$ is monotone increasing.

Similarly, $b_n \geq 0$ for all n $\Rightarrow$ (t_n) is monotone increasing.

Know that (s_n) converges, so by Cor to MST, have $s_n \leq s$ where $s = \sum_{i=1}^{\infty} a_i$.

Hence $t_n \leq s_n \leq s$ for all n, so (t_n) is bounded above.

By MST, (t_n) converges.

Hence $\sum_{n=1}^{\infty} b_n$ converges. $\square$

Remark: Argument also shows that $\sum_n b_n \leq \sum_n a_n$ since s is an upper bound for the set $T = \{t_n : n \in \mathbb{N}\}$ and $t_n \to \sup T$

1.2) Cont'd

Recall $\sum_{n=1}^{\infty} a_n = S$ means that $\sum_{i=1}^{n} a_i \longrightarrow S$.

<u>Thm (Comparison Test)</u>: If $a_n \geq b_n \geq 0 \ \forall n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges too.

<u>Contrapositive</u>: If $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} a_n$.

<u>Lemma</u>: 1) For any $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=k}^{\infty} a_n \text{ converges}$$

2) If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then $\sum_{n=1}^{\infty} a_n + cb_n = A + cB$

for any $c \in \mathbb{C}$.

<u>Proof</u>: 1) Let $s_n = \sum_{i=1}^{n} a_i \quad s_n' = \sum_{i=k}^{n} a_i$.

note a sequence converging does not depend on the first few terms

Then $s_n = s_n' + t$ where $t = s_{k-1}$.

By LOL, if (s_n') converges, so does (s_n).

Also $s_n' = s_n - t$, so if (s_n) converges so does s_n.

2) If $\sum_{i=1}^{n} a_i \longrightarrow A$ and $\sum_{i=1}^{n} b_i \longrightarrow B$, then by LOL

$$\sum_{i=1}^{n} a_i + c\sum_{i=1}^{n} b_i \longrightarrow A + cB$$

i.e. $\sum_{i=1}^{n} a_i + cb_i \longrightarrow A + cB$

<u>Notation</u>: $\sum_n a_n$ converges if $\sum_{n=k}^{\infty} a_n$ converges for some k

$\iff \sum_{n=k}^{\infty} a_n$ converges for all k

<u>Def</u>: $\sum_n a_n$ <u>converges absolutely</u> if $\sum_n |a_n|$ converges.

<u>Prop</u>: If $\sum_n a_n$ converges absolutely, then it converges.

<u>Proof</u>: 1) Suppose (a_n) is ~~the~~ a sequence in $\mathbb{R}$ and $\sum_n |a_n|$ converges.

Let $b_n = \max\{0, a_n\}$ so $0 \leq b_n \leq |a_n|$

$c_n = \max\{0, -a_n\}$ so $0 \leq c_n \leq |a_n|$ and $a_n = b_n - c_n$.

L4.2 By Comparison, $\sum_n b_n$ and $\sum_n c_n$ converge. By part 2) of Lemma $\sum_n b_n - c_n$ converges, i.e. $\sum_n a_n$ converges. VERY NICE

2) Now spose (a_n) is a sequence of complex numbers $a_n = \alpha_n + i\beta_n$ for $\alpha_n, \beta_n \in \mathbb{R}$

Then $0 \leq |\alpha_n| \leq |a_n|$ and $0 \leq |\beta_n| \leq |a_n|$, so if $\sum_n |a_n|$ converges, then $\sum_n |\alpha_n|$ and $\sum_n |\beta_n|$ converge by Comparison. By 1), $\sum_n \alpha_n$ and $\sum_n \beta_n$ converge

$\Rightarrow \sum_n \alpha_n + i\beta_n$ converges. $\square$

<u>Cor</u>: (Strong Comparison) Suppose (c_n) is a sequence in $\mathbb{C}$, that $0 \leq |c_n| \leq a_n$ for all n and $\sum_n a_n$ converges.
Then $\sum_n c_n$ converges.

<u>Prop</u>: (Ratio Test) Suppose (a_n) is a sequence in $\mathbb{C} \setminus \{0\}$ and that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r.$$

Then if $r < 1$, then $\sum_n a_n$ converges

if $r > 1$, $\sum_n a_n$ diverges.

<u>Proof</u>: Since $r \neq 1$, $\frac{|1-r|}{2} > 0$.

Choose N such that $\left| \frac{|a_{n+1}|}{|a_n|} - r \right| < \frac{|1-r|}{2}$ for all $n \geq N$.

Then if $r < 1$, $\frac{|a_{n+1}|}{|a_n|} < \frac{1+r}{2}$

if $r > 1$, $\frac{|a_{n+1}|}{|a_n|} > \frac{1+r}{2}$.

Let $c = \frac{1+r}{2}$. Then if $r < 1$, $|a_{n+1}| < c|a_n|$ for $n \geq N$.

By induction $|a_{N+k}| < c^k |a_N|$.

Now compare with geometric ~~sequence~~ series $\sum_n c^n$. If $r < 1$, $c < 1$,

so $\sum_n c^n$ converges $\Rightarrow \sum_n |a_N| c^n$ converges.

By Strong Comparison, $\sum_n a_n$ converges

If $r>1$, $c>1$ as well and $|a_{n+1}| > c|a_n|$ for $n \geq N$.

By induction, we see $|a_{N+k}| \geq |a_N| > 0$.

This implies that $a_n \not\to 0$ so if $r>1$, $\Sigma_n a_n$ diverges by the n^{th} term test. $\square$

<u>Example</u> Fix $z \in \mathbb{C}$, consider $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z}{n+1}\right|$.

But $\frac{|z|}{n+1} \to 0$. By Ratio Test, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges $\forall z \in \mathbb{C}$.

<u>Def^n:</u> $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

<u>N.B.</u> For this to make sense, we need

<u>Lemma:</u> If $a_n \to a$ and $a_n \to a'$ then $a = a'$.

<u>Proof:</u> 1) Consider the sequence (0). Suppose $0 \to x$. If $x \neq 0$, then I can find N s.t. $|0_n - x| < \frac{|x|}{2}$ for all $n > N$, i.e. $|x| < |x|/2$ which is false. Hence if $0 \to x$, $x \neq 0$.

2) If $a_n \to a$ and $a_n \to a'$ then by LOL, $a_n - a_n \to a - a'$ i.e. $0 \to a - a' \Rightarrow a - a' = 0$ by 1). $\square$

<u>Example:</u> Harmonic Series

$\sum_{n \geq 1} \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n/_{n+1} = 1$

Ratio Test tells us nothing, but

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

In general $s_n = \sum_{i=1}^{n} 1/i$, then $s_{2^k} \geq 1 + \frac{k}{2} \to \infty$

so $\sum_n \frac{1}{n}$ diverges by comparison.

<u>Prop:</u> (Cauchy Condensation Test) Suppose that $a_n \geq a_{n+1} \geq 0$ for all n. Then $\sum_n a_n$ converges $\Leftrightarrow$ $\sum_k 2^k a_{2^k}$ converges

<u>Ex:</u> ~~Proof~~ p-series $\sum_{n \geq 1} 1/n^p$ converges iff $\sum_k 2^k \frac{1}{(2^k)^p}$ converges

which is $\sum_k 2^{k(1-p)}$.

This is a geometric series with ratio $c = 2^{1-p}$ so it converges iff $2^{1-p} < 1$, i.e. $p > 1$.

<u>Summary:</u> $\sum_{n \geq 1} \frac{1}{n^p}$ converges iff $p > 1$

<u>Remark:</u> Could define a function $\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k}$.

Makes sense for $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 1$

<u>Ex:</u> $\zeta(2) = \pi^2/6$ $\quad$ $\zeta(3)$ is irrational

$\zeta(4) = \pi^4/90$

1.2) Cont'd

Abbreviation: I'll write $\Leftrightarrow$ or iff for if and only if.

Proposition: (Cauchy Condensation)

Spose $a_n \geq a_{n+1} \geq 0$ for all n.

Then $\sum_m a_m$ converges iff $\sum_k 2^k a_{2^k}$ converges.

Substitute for

(Integral Test): If $f(x) \geq f(y) \geq 0$ for all $x \leq y$, then

$\sum_n f(n)$ converges iff $\int_1^\infty f(x)\,dx$ converges.

Idea of Proof:

Compare
$$\begin{array}{ccc} a_1+a_2+a_3+a_4+a_5+\cdots & \geq & a_1+a_2+a_4+a_4+a_8+\cdots \\ \leq & & \\ a_1+a_2+a_2+a_4+a_4+\cdots & & \end{array}$$

Proof: Let $s_n = \sum_{i=1}^n a_i$ and $t_k = \sum_{i=0}^k 2^i a_{2^i}$.

$a_i \geq 0 \Rightarrow$ both (s_n) and (t_k) are MI

so they converge iff they are bounded above, by MST

Claim 1: $s_{2^k-1} \leq t_{k-1}$

Proof: Induct on k

$k=1$ $\quad s_1 = a_1 = t_0$

In general
$$s_{2^{k+1}-1} = s_{2^k-1} + \overbrace{a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}}^{2^k \text{ terms}}$$
$$\leq t_{k-1} + a_{2^k} + a_{2^k} + \cdots + a_{2^k} = t_k. \qquad \square$$

Claim 2: $s_{2^k} \geq \frac{1}{2} t_{k}$

Proof: $k=1$ $\quad s_2 = a_1 + a_2 \geq \frac{1}{2}a_1 + a_2 = \frac{1}{2}t_1$

In general
$$s_{2^{k+1}} = s_{2^k} + \underbrace{a_{2^k+1} + \cdots + a_{2^{k+1}}}_{2^k \text{ terms}}$$
$$\geq \tfrac{1}{2}t_k + a_{2^{k+1}} + \cdots + a_{2^{k+1}}$$
$$= \tfrac{1}{2}t_k + 2^k a_{2^{k+1}} = \tfrac{1}{2}t_{k+1}. \qquad \square$$

If $\sum_n a_n$ converges, there is an S such that $s_n \leq S \ \forall n$.

Then $t_k \leqslant 2S_{2^k} \leqslant 2S$ for all k

$\Rightarrow (t_k)$ bounded above

$\Rightarrow (t_k)$ converges by MST

If $\sum_k 2^k a_{2^k}$ converges, there is T such that $t_k \leqslant T$ for all k

$\Rightarrow s_n \leqslant s_{2^n-1} \leqslant t_{n-1} \leqslant T$

$\Rightarrow (s_n)$ is bounded above

$\Rightarrow (s_n)$ converges by MST $\square$

1.3) Subsequences

Def.^{n} Spose $f: \mathbb{N} \to \mathbb{N}$ satisfies $f(k+1) > f(k)$ for all k. If (x_n) is a sequence in X, we say the sequence $(x_{f(k)})$ is a subsequence of (x_n).

Notation: Write $n_k = f(k)$ and say that (x_{n_k}) is a subsequence of (x_n).

Picture: $\cancel{x_1}, x_2, \cancel{x_3}, x_4, x_5, \cancel{x_6}, \ldots$

$$\downarrow$$

$$x_2, x_4, x_5, \ldots$$

Ex: If $(x_n) = ((-1)^n)$

then $((-1)^{2n}) = (1)$ is a subsequence

$((-1)^{3n}) = ((-1)^n)$ is a subsequence

Lemma: Spose (x_n) is a sequence in $\mathbb{R}$ and $x_n \to x \in \mathbb{R}$. If (x_{n_k}) is a subsequence of (x_n), then $x_{n_k} \to x$.

Proof: $n_k = f(k)$ where $f(k+1) > f(k)$ i.e. $f(k+1) \geqslant f(k)+1$

$\Rightarrow f(k) \geqslant k$

Given $\varepsilon > 0$, choose N s.t. $|x_n - x| < \varepsilon$ $\forall n > N$. Then for $k > N$, $n_k \geqslant k > N$, so $|x_{n_k} - x| < \varepsilon$. ~~for~~ Thus $x_{n_k} \to x$. $\square$

But a sequence which diverges may have a convergent subsequence.

Ex: (1) is a subsequence of $((-1)^n)$

(n) has no convergent subsequence

Def: If (x_n) is a sequence in $\mathbb{R}$, say (x_n) is bounded if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above and below.

Theorem (Bolzano - Weierstrass): Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Lemma: Spose (x_n) is a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. Then (x_n) has a subsequence $(x_{n_k}) \to x$ iff:

⊛ For every $\varepsilon > 0$, the set

$$C_\varepsilon = \{n \in \mathbb{N} : |x_n - x| < \varepsilon\}$$

is infinite.

Proof: Spose $x_{n_k} \to x$. Given $\varepsilon > 0$, we can find a N s.t. $|x_{n_k} - x| < \varepsilon$ for all $k > N$.

Then $\{n_k : k > N\} \subseteq C_\varepsilon \Rightarrow C_\varepsilon$ is infinite ✓

If ⊛ holds, then $C_{1/k}$ is infinite for all k.

Define n_k inductively by

n_k = smallest element of $C_{1/k}$ which is $> n_{k-1}$

Then $n_k > n_{k-1} \Rightarrow (x_{n_k})$ is a subsequence and

$$|x_{n_k} - x| < \frac{1}{k} \Rightarrow x_{n_k} \to x. \qquad \square$$

Proof of BW:

Step 1: Find $y \in \mathbb{R}$ that a subsequence should converge to

Spose (x_n) is a bounded sequence in $\mathbb{R}$. Then for each n, the set $Y_n = \{x_k : k \geq n\}$ is non-empty and bounded above.

By LUBP, it has a supremum $y_n = \sup Y_n$.

Note $Y_{n+1} \subseteq Y_n$, so y_n is an upper bound for $Y_{n+1} \Rightarrow y_{n+1} \leq y_n$

So (y_n) is monotone decreasing. Since (x_n) is bounded below $x_n \geq m$ for some m, $\forall n$. $\Rightarrow y_n \geq m \ \forall n$

$\Rightarrow (y_n)$ is bounded below.

By MST (applied to $(-y_n)$) we know $y_n \to y$ for some $y \in \mathbb{R}$.

LS.4 <u>Step 2</u>: Show there is a subsequence $\to y$

Proof by contradiction. If there is no such subsequence, then by Lemma, there is $\varepsilon > 0$ s.t. $\{n : |x_n - y| < \varepsilon\}$ is finite

<u>2 Cases</u> 1) Suppose $\{n : \exists\, x_n \geq y + \varepsilon\}$ is finite. Then there are only finitely many m s.t. $x_n > y - \varepsilon$, so $\exists N$ s.t. $x_n \leq y - \varepsilon$ for all $n > N$.

Then for $n > N$, $y - \varepsilon$ is an upper bound for $Y_n \Rightarrow y_n < y - \varepsilon$ for all $n > N$. This contradicts $y_n \to y$. ※

2) Spose $\{n : \exists\, x_n \geq y + \varepsilon\}$ is infinite. Then Y_n contains an element $\geq y + \varepsilon$ for all n.

$\Rightarrow y_n = \sup Y_n \geq y + \varepsilon$ for all n

Again, this contradicts $y_n \to y$. ※

Both cases give a contradiction, so a subsequence converging to y must exist. $\square$

1.3) Cont'd

Theorem (Bolzano-Weierstrass)

A bounded real sequence has a convergent subsequence.

Def: A complex sequence (z_n) is bounded if $(|z_n|)$ is bounded.
i.e. $\exists M \in \mathbb{R}$ s.t. $|z_n| \leq M$ for all n.

Cor: A bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Proof: Spose (z_n) is bounded, write $z_n = x_n + iy_n$, $x_n, y_n \in \mathbb{R}$.
Since $(|z_n|)$ is bounded, $\exists M$ s.t. $|z_n| \leq M$ for all n, so
$|x_n| \leq |z_n| \leq M$ and $|y_n| \leq |z_n| \leq M$ for all n.
i.e. (x_n) and (y_n) are bounded real sequences.
By BW (x_n) has a subsequence $x_{n_k} \to x \in \mathbb{R}$. Now consider the sequence (y_{n_k}). By BW, it has a convergent subsequence
$y_{n_{k_j}} \longrightarrow y \in \mathbb{R}$. Now $(x_{n_{k_j}})$ is a subsequence of (x_{n_k}) and $x_{n_k} \to x$, so $x_{n_{k_j}} \longrightarrow x$.
Hence $z_{n_{k_j}} = x_{n_{k_j}} + i y_{n_{k_j}} \longrightarrow x + iy$
so (z_n) has a convergent subsequence. □

Cauchy Sequences

Def: A complex sequence (z_n) is Cauchy if for every $\varepsilon > 0$,
$\exists N \in \mathbb{N}$ s.t. $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Lemma: If (z_n) is convergent, then (z_n) is Cauchy.

Proof: Spose $z_n \to z \in \mathbb{C}$.
Estimate $|z_n - z_m| \leq |z_n - z| + |z - z_m|$.
Given $\varepsilon > 0$, choose N s.t. $|z_n - z| < \varepsilon/2$ for $n > N$. Then for $n, m > N$,
$|z_n - z_m| \leq |z_n - z| + |z - z_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Theorem: If (z_n) is Cauchy, then $z_n \to z \in \mathbb{C}$.

N.B: Lemma works if we consider sequences in $\mathbb{Q}$, but Thm does not, e.g. $3, 3.1, 3.14, 3.141, \ldots$
is Cauchy but does not converge in $\mathbb{Q}$.

In fact, knowing Cauchy sequences converge in $\mathbb{R}$ can be used to prove LUBP.

Proof of Thm:

Step 1: Find a convergent subsequence.

Want to show (z_n) is bounded.

Choose N s.t. $|z_n - z_m| < 1$ for all $n, m > N$. (since (z_n) Cauchy)

Then for $n > N$, $|z_n - z_{N+1}| < 1$ i.e. $|z_n| \leq 1 + |z_{N+1}|$.

So if $M = \max\{|z_1|, |z_2|, \dots, |z_N|, |z_{N+1}| + 1\}$ then $|z_n| < M$ for all n. Hence (z_n) is bounded.

By BW, there is a subsequence $z_{n_k} \longrightarrow z \in \mathbb{C}$.

Step 2: Show $z_n \longrightarrow z$.

Estimate $|z_n - z| \leq |z_n - z_{n_R}| + |z_{n_R} - z|$.

Since (z_n) is Cauchy, can choose N_1 s.t. $|z_n - z_m| < \varepsilon/2$ for $n, m > N_1$.

Since $z_{n_k} \to z$, can choose N_2 s.t. $|z_{n_k} - z| < \varepsilon/2$ for $k > N_2$.

Pick $R > \max\{N_1, N_2\}$.

Then $|z_{n_R} - z| < \varepsilon/2$, and ~~for $n >$~~ $n_R \geq R > N_1$, so if $n > N_1$, then $|z_n - z_{n_R}| < \varepsilon/2$.

So for $n > N_1$, we have

$$|z_n - z| \leq |z_n - z_{n_R}| + |z_{n_R} - z| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

I.e. $z_n \to z$. $\square$

Cor: (General principle of convergence)

Suppose (a_n) is a sequence in $\mathbb{C}$. Then $\sum_n a_n$ converges if and only if $\circledast$: for every $\varepsilon > 0$, there is an N s.t.

$$\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon \quad \text{for all } m > n > N.$$

Proof: Let $s_n = \sum_{i=1}^{n} a_i$. Then $s_m - s_n = \sum_{i=n+1}^{m} a_i$, so (s_n) is Cauchy iff $\circledast$ holds. By Theorem, this is equivalent to (s_n) converging. $\square$

N.B: To check that (s_n) is Cauchy it does not suffice to check $|s_{n+1} - s_n| < \varepsilon$ for all $n > N$.
Since if $s_n = \sum_{k=1}^{n} 1/k$, $s_{n+1} - s_n \to 0$ but s_n diverges.

Prop: (Alternating Series Test)
Suppose $a_n \geq a_{n+1} \geq 0$ for all n.
Then $\sum_n (-1)^n a_n$ converges iff $a_n \to 0$.

Proof: Use GPC. Let

$$s_{n,m} = a_n - a_{n+1} + a_{n+2} - \cdots \pm a_m$$

Claim 1: $s_{n,m} \geq 0$

Proof: if $m = n + 2k-1$, then

$$s_{n,m} = (a_n - a_{n+1}) + (a_{n+2} - a_{n+3}) + \cdots + (a_{n+2k-2} - a_{n+2k-1}) \geq 0$$

since (a_n) is decreasing.
if $m = n+2k$, then
$s_{n,m} = s_{n,m-1} + a_{n+2k} \geq 0$ by previous case. $\square$

Claim 2: $s_{n,m} \leq a_n$

Proof: $s_{n,m} = a_n - s_{n+1,m} \leq a_n$ by Claim 1. $\square$

Summary: $|s_{n,m}| \leq a_n$

Given $\varepsilon > 0$, choose N s.t. $|a_n| < \varepsilon$ for $n > N$. Then for $m > n > N$, $\left| \sum_{k=n+1}^{m} (-1)^k a_k \right| = |s_{n+1,m}| \leq a_n < \varepsilon$,
so by GPC, $\sum (-1)^n a_n$ converges. $\square$

Ex: The ~~sequence~~ series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges since $1/n \to 0$ but does not absolutely.

Def: If $\sum_n a_n$ converges but $\sum_n |a_n|$ does not, say $\sum_n a_n$ converges conditionally.

L6.4 How to construct $\mathbb{R}$ from $\mathbb{Q}$ (Sketch)

Definitions of convergence, Cauchy work using $\varepsilon > 0$, $\varepsilon \in \mathbb{Q}$.

Let $\mathcal{R} = \{ \vec{x} : \vec{x} = (x_n) \text{ is a Cauchy sequence in } \mathbb{Q} \}$.

Say that $(x_n) \sim (y_n)$ if $x_n - y_n \longrightarrow 0$.

This is an equivalence relation.

Define $\mathbb{R} = \mathcal{R} / \sim$.

1) $\mathbb{Q} \subset \mathbb{R}$

$\quad q \longrightarrow (q)$

2) Field operations

$(x_n) + (y_n) = (x_n + y_n)$

$(x_n) \cdot (y_n) = (x_n y_n)$

check respects $\sim$, well defined

3) Order

$(x_n) > 0$ if for ~~every~~ some $\varepsilon > 0$ $\exists N$ s.t. $x_n > \cancel{0}\ \varepsilon$ for all $n > N$

$(x_n) > (y_n)$ if $(x_n - y_n) > 0$.

4) <u>Them</u>: If $(\vec{x}_n)$ is a Cauchy sequence in $\mathbb{R}$, then we have

$\vec{x}_n = (x_{n,k})$

$(\vec{x}_n) \longrightarrow \vec{x}$ where $\vec{x} = (x_{n,n})$.

L7.1 Alternating Series Test

If $a_n \geqslant a_{n+1} \geqslant 0 \ \forall n$, then

$\sum_n (-1)^n a_n$ converges $\Leftrightarrow a_n \to 0$

Proved $\Leftarrow$ on Wed.

$\Rightarrow$ follows from n^{th} term test.

II) Limits & Continuity

2.1) Limits

To do calculus, I need to take limits of functions $g: \mathbb{R} \to \mathbb{R}$

e.g. $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

Def: Spose $A \subset \mathbb{C}$ and $a \in \mathbb{C}$. We say a is a limit point of A if there's a sequence $z_n \to a$ s.t. $z_n \in A \setminus \{a\} \ \forall n$.

N.B: a may or may not be an element of A.

Ex: 1) $A = \mathbb{Z} \subset \mathbb{C}$ has no limits

2) $A = \mathbb{Q}$ has all $x \in \mathbb{R}$ as limit points

Proof: Let $x_n = a/n$ be the rational number with denominator n which is closest but not equal to x. Then $|x_n - x| < \frac{1}{n}$, so indeed $x_n \to x$. $\square$

Lemma: Spose A and a are as above, and then a is a limit point of A iff

(*) for every $\varepsilon > 0$ there is some $z \in A$ with $0 < |z - a| < \varepsilon$.

Proof: Spose a is a limit point of A. Then we can find $z_n \to a$, $z_n \in A \setminus \{a\}$ for all n. So given $\varepsilon > 0$, choose N s.t. $|z_n - a| < \varepsilon$ for $n > N$. So $z_{N+1} \in A \setminus \{a\}$ and $|z_{N+1} - a| < \varepsilon$, so (*) holds.

Conversely, if (*) holds, then for every $n \in \mathbb{N}$ we can find $z_n \in A \setminus \{a\}$ such that $|z_n - a| < 1/n$. So $z_n \to a$ i.e. a is a limit point of A $\square$

Def: Suppose $A \subset \mathbb{C}$, a is a limit point of A, $f : A \to \mathbb{C}$, and $c \in \mathbb{C}$. We say

$$\lim_{z \to a} f(z) = c \quad \text{or} \quad f(z) \to c \text{ as } z \to a$$

if for every sequence (z_n) in $A \setminus \{a\}$ with $z_n \to a$, the sequence $f(z_n) \to c$.

NB: The condition that a is a limit point of A means such sequences (z_n) exists.

Ex: 1) $\lim_{z \to a} z = a$

2) $f : \mathbb{R} \to \mathbb{R}$ $\qquad \lim_{x \to 0} f(x) = 1$

$$x \to \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

3) $f : \mathbb{R} \to \mathbb{R}$

$$x \to \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\lim_{x \to 0} f(x)$ does not exist since

$1/n \to 0$, $f(\frac{1}{n}) = 1 \; \forall n \to 1$

$-1/n \to 0$, $f(-\frac{1}{n}) = 0 \; \forall n \to 0$

Proposition: Spose A, a, c, f are as in definition. Then

$$\lim_{z \to a} f(z) = c \iff (\mathrm{A})$$

(A) For every $\varepsilon > 0$, there's a $\delta > 0$ s.t. $0 < |z - a| < \delta \Rightarrow |f(z) - c| < \varepsilon$.

I give you a and ε, you find δ.

Proof: Suppose (A) holds. We must show that $f(z_n) \to c$ whenever $z_n \to a$ and $z_n \in A \setminus \{a\}$ for all n.

Suppose we're given such a sequence $z_n \to a$, and $\varepsilon > 0$. Must find N s.t. $|f(z_n) - c| < \varepsilon$ for all $n > N$. By (A), can find $\delta > 0$ s.t. $|f(z) - c| < \varepsilon$ whenever $0 < |z - a| < \delta$. Since $z_n \to a$, can choose N s.t. $|z_n - a| < \delta$ for all $n > N$. Since $z_n \neq a$, L7.3,

$0 < |z_n - a| < \delta$ for $n > N \Rightarrow |f(z_n) - c| < \varepsilon$ for $n > N$

i.e. $f(z_n) \longrightarrow c$. One direction done ✓

Conversely, spose (A) is falls. Then there is some $\varepsilon > 0$ s.t. there is no $\delta > 0$ s.t. $|f(z) - c| < \varepsilon$ for all $z \in A$ with $0 < |z-a| < \delta$. In particular, $\delta = \frac{1}{n}$ doesn't work, so $\exists\, z_n \in A$ with $0 < |z_n - a| < \frac{1}{n}$ and $|f(z_n) - c| \geq \varepsilon$. Then $z_n \in A$, $z_n \neq a$, and

$|z_n - a| < 1/n \Rightarrow z_n \longrightarrow a$. But $|f(z_n) - c| \geq \varepsilon > 0$

$\Rightarrow f(z_n) \not\longrightarrow c$.

i.e. we found a sequence $z_n \to a$ with $f(z_n) \not\longrightarrow c$, so

$\lim_{z \to a} f(z) \neq c$. $\square$

Law of limits

Prop Spose $f, g : A \longrightarrow \mathbb{C}$, and that

$$\lim_{z \to a} f(z) = c_1 \quad , \quad \lim_{z \to a} g(z) = c_2 .$$

Then 1) $\lim_{z \to a} f(z) + g(z) = c_1 + c_2$

2) $\lim_{z \to a} f(z) \cdot g(z) = c_1 c_2$

3) If $c_1 \neq 0$, then $\lim_{z \to a} \frac{1}{f(z)} = \frac{1}{c_1}$

Proof: These all follow directly from LOL for sequences.

e.g. 1) Spose $z_n \to a$, $z_n \in A \setminus \{a\}$.

Then $f(z_n) \longrightarrow c_1$

$g(z_n) \longrightarrow c_2$.

By LOL for sequences, $f(z_n) + g(z_n) \longrightarrow c_1 + c_2$.

This exactly says that $\lim_{z \to a} f(z) + g(z) = c_1 + c_2$, since z_n was arbitrary.

2) + 3) are the exact same, except for 3) have to restrict domain of f so $1/f(z)$ defined.

actually... better if $f(z) \neq 0$

L7.4

<u>Prop</u>: Spose $A \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $f, g : A \to \mathbb{R}$

and $f(x) \to c_1$ as $x \to a$.

$g(x) \to c_2$

Then: 1) If $f(x) \geq g(x)$ for all $x \in A$, then $c_1 \geq c_2$.

2) (Squeeze Rule): If $f(x) \geq h(x) \geq g(x)$ for all $x \in A$

and $c_1 = c_2 = c$, then $\lim_{x \to a} h(x) = c$.

Again, proof follows from LOL for sequences.

<u>Variants</u>:

<u>Def</u>: Suppose $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$.

We say $\lim_{x \to +\infty} f(x) = c$ if for every $\varepsilon > 0$ there is $M \in \mathbb{R}$ s.t.

$|f(x) - c| < \varepsilon$ whenever $x > M$.

<u>One-sided limits</u>: say $\lim_{x \to a^+} f(x) = c$ if

$$\lim_{x \to a} f\big|_{A \cap (a, \infty)}(x) = c.$$

Similarly for $\lim_{x \to a^-} f(x) = c$ use $f\big|_{A \cap (-\infty, a)}$.

L8.1

2.2) Continuous Functions

<u>Setup</u>: $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$

<u>Recall</u>: If a is a limit point of A

$$\lim_{z \to a} f(z) = c$$

iff whenever (z_n) is a sequence in $A \setminus \{a\}$ and $z_n \to a$, $f(z_n) \to c$,

or equivalently if for every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(z) - c| < \varepsilon$ whenever $0 < |z - a| < \delta$.

but $\neq a$

<u>Idea</u>: if z is very close to a, then $f(z)$ is very close to $\cancel{\varepsilon}$ c

$$\boxed{\begin{array}{l}\underline{\text{Def}}: f : A \to \mathbb{C} \text{ is } \underline{\text{continuous at } a} \in A \text{ if for every } \varepsilon > 0, \\ \exists\, \delta > 0 \text{ s.t. } |f(z) - f(a)| < \varepsilon \text{ whenever } |z - a| < \delta.\end{array}}$$

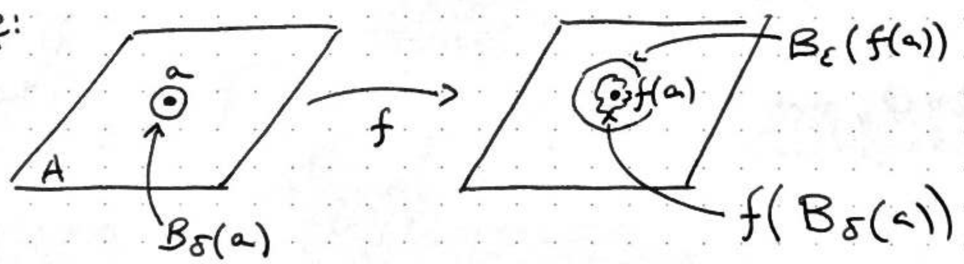
N.B: $a \in A$, but it doesn't have to be a limit point of A. If it's not, f is automatically continuous at a.

<u>Proof</u>: a is not a limit point, so $\exists\, \delta > 0$ s.t. $\{z \in A : |z - a| < \delta\} = \{\cancel{z}\, a\}$. This δ works for any $\varepsilon > 0$. $\square$

<u>Ex</u>: $f : \mathbb{Z} \to \mathbb{C}$ is continuous for any f

<u>Notation</u>: $B_\varepsilon(c) = \{\cancel{\emptyset}\, w \in \mathbb{C} : |\cancel{z}\, w - c| < \varepsilon\}$

<u>Picture</u>:



<u>Lemma</u>: f is continuous at a iff

either a is not a limit point of A

or $\lim_{z \to a} f(z) = f(a)$

<u>Proof</u>: If a is not a limit point, done by the above.

If a is a limit point, then $|z - a| = 0 \Rightarrow z = a \Rightarrow f(z) = f(a)$

so ① For every $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ when $0 < |z-a| < \delta$

② For every $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ when $|z-a| < \delta$

are equivalent.

But ① $\iff \lim_{z \to a} f(z) = f(a)$

② $\iff$ f is continuous at a. □

<u>Def</u>: We say $f : A \to \mathbb{C}$ is <u>continuous</u> if it is continuous at a for all $a \in A$.

<u>Ex</u>: 1) $f : \mathbb{C} \to \mathbb{C}$ is continuous.
$z \to c$

Need to check that for every $a \in \mathbb{C}$ and $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ when $|z-a| < \delta$. But $|f(z) - f(a)| = |c - c| < \varepsilon$. So any δ will do.

2) $f : \mathbb{C} \to \mathbb{C}$ is continuous.
$z \to z$

Need to check that for every $a \in \mathbb{C}$ and $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ when $|z-a| < \delta$. But $|f(z) - f(a)| = |z-a|$, so taking $\delta = \varepsilon$ will do.

3) $f : [0, \infty) \to [0, \infty)$ is continuous
$x \to \sqrt{x}$

$a = 0$. Given $\varepsilon > 0$, must find δ s.t. $|f(x) - f(0)| < \varepsilon$ when $|x - 0| < \delta$.

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon$$

if $|x| < \varepsilon^2$ so take $\delta = \varepsilon^2$.

$a \neq 0$. Must estimate $|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = |\sqrt{x} - \sqrt{a}| \cdot \dfrac{|\sqrt{x} + \sqrt{a}|}{|\sqrt{x} + \sqrt{a}|}$

$$= \frac{|x-a|}{|\sqrt{x} + \sqrt{a}|} \leqslant \frac{|x-a|}{\sqrt{a}}$$

so if $|x - a| < \delta = \varepsilon\sqrt{a}$, then $|f(x) - f(a)| < \varepsilon$. ☺

<u>Prop</u>: $f: A \to \mathbb{C}$ is continuous at a iff $[f(z_n) \to f(a)$ whenever $z_n \to a$ is a sequence in $A.]$ (*)

<u>Proof</u>: Spose (*) holds. If a is not a limit point, both conditions are true. If a is a limit point, then (*) implies that $\lim_{z \to a} f(z) = f(a)$, so by Lemma, f is continuous at a.

Conversely, spose f is continuous ~~f is continuous~~ at a and $z_n \to a$, $z_n \in A$. Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ when $|z - a| < \delta$. Since $z_n \to a$, can choose N s.t. $|z_n - a| < \delta$ when $n > N$. Then for $n > N$, $|f(z_n) - f(a)| < \varepsilon$, so $f(z_n) \to f(a)$. $\square$

<u>Prop</u>: Spose $f, g: A \to \mathbb{C}$ are continuous at a. Then $f+g$, fg are continuous at a. If f is never zero, $1/f$ is continuous at a.

<u>Proof</u>: Obviously true if a is not a limit point. If a is a limit point, then

$$\begin{aligned} \lim_{z \to a} f(z) &= f(a) \\ \lim_{z \to a} g(z) &= g(a) \end{aligned} \implies \begin{aligned} \lim_{z \to a} f(z) + g(z) &= f(a) + g(a) \\ \lim_{z \to a} f(z) g(z) &= f(a) g(a) \end{aligned}$$

by LOC. Similarly if $f(a) \neq 0$, $\lim_{z \to a} 1/f(z) = 1/f(a)$. $\square$

<u>Cor</u>: If $f, g: A \to \mathbb{C}$ are continuous then $f+g: A \to \mathbb{C}$, $fg: A \to \mathbb{C}$ are continuous and so is $1/f: A \to \mathbb{C}$ if f is never zero on A.

<u>Ex</u>: 1) Polynomials $p(z) = \sum_{i=0}^{n} a_i z^i$ are continuous maps $\mathbb{C} \to \mathbb{C}$.

2) Rational functions $f(z) = \dfrac{p(z)}{q(z)}$ p, q polynomials

$f: A' \to \mathbb{C}$, $A' = \{z \in \mathbb{C} : q(z) \neq 0\}$ is continuous.

<u>Prop</u>: Suppose $A, B, C \subset \mathbb{C}$, $f : A \to B$, $g : B \to C$.
If f is continuous at a and g is continuous at $f(a)$, then $g \circ f : A \to C$ is continuous at a.

<u>Proof</u>: We'll use characterisation of continuity in terms of sequences. If $z_n \to a$, $z_n \in A$, then $f(z_n) \to f(a)$ since f is continuous at a.
So then $g(f(z_n)) \to g(f(a))$ since g is continuous at $f(a)$.
It follows that ~~$g \circ f$ is continuous~~ if $z_n \to a$, $g \circ f(z_n) \to g \circ f(a)$
so $g \circ f$ is continuous at a. □

<u>Cor</u>: If $f : A \to B$, $g : B \to C$ are continuous, then $g \circ f : A \to C$ is continuous.

<u>Ex</u>: $f : \mathbb{R} \to \mathbb{R}$
$f(x) = \sqrt{x^2+1}$ is continuous
since $f = g \circ h$ where $h : \mathbb{R} \to [0, \infty)$, $x \to x^2+1$ and $g : [0, \infty) \to \mathbb{R}$, $x \to \sqrt{x}$
are continuous.

L9.1 2.2) Cont'd

Recall that $f: A \to \mathbb{C}$ is continuous at a iff for every $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ whenever $|z - a| < \delta$, or, equivalently, $\boxed{\text{if every sequence } (z_n) \text{ in } A \text{ converging to } a, \ f(z_n) \text{ converges to } f(a).}$

Today: $f: [a,b] \to \mathbb{R}$

where $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for $a, b \in \mathbb{R}$.

Bolzano Weierstrass: A bounded sequence of real numbers has a convergent subsequence.

Cor: If (x_n) is a sequence in $[a,b]$, it has a subsequence $x_{n_k} \to c \in [a,b]$.

Proof: $x_n \in [a,b]$, so $a \leq x_n \leq b \ \forall n$, i.e. (x_n) is bounded.

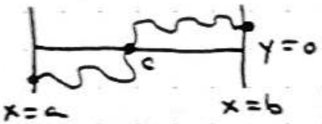
By BW, there's a subsequence $x_{n_k} \to c \in \mathbb{R}$.

Now, $a \leq x_{n_k} \leq b$, so by LOL(II) know $a \leq c \leq b$, i.e. $c \in [a,b]$. $\square$

Theorem (Intermediate Value Theorem)

If $f: [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0$, $f(b) > 0$, then $\exists\, c_{\in [a,b]}$ with $f(c) = 0$.

Picture:



Notes: 1) c need not be unique

2) This is false for $[a,b] \cap \mathbb{Q}$ instead of $[a,b]$.

Ex: $f: [0,2] \cap \mathbb{Q} \to \mathbb{R}$

$$x \longrightarrow x^2 - 2$$

3) If $f(a) > 0$, $f(b) < 0$, same statement true; just consider $-f$

Idea of Proof: Find sequences (x_n) and (y_n) with $f(x_n) \leq 0$ and $f(y_n) > 0$ and $x_n \to c$, $y_n \to c$. Hence $f(x_n) \to f(c)$ and $f(y_n) \to c$ so $f(c) \leq 0$, $f(c) \geq 0$ i.e. $f(c) = 0$. ☺

Proof: First assume $[a,b]=[0,1]$. For $n \in \mathbb{N}$, let

$$Q_n = \left\{ \frac{a}{n} : a \in \{0,1,\dots,n\}, f\left(\frac{a}{n}\right) \leq 0 \right\}$$

Notice $0/n \in Q_n$ since $f(0)<0 \Rightarrow Q_n \neq \emptyset$

$n/n \notin Q_n$ since $f(1)>0$.

Let $x_n = \max Q_n$. Then $x_n \in [0, \frac{n-1}{n}]$ and $f(x_n) \leq 0$ for all n.

Let $y_n = x_n + \frac{1}{n}$. $y_n \in [\frac{1}{n}, 1]$ and $f(y_n) > 0$ (else $y_n \in Q_n$)

Now (x_n) is a sequence in $[0,1]$, so by Cor to BW, it has a subsequence $x_{n_k} \to c \in [0,1]$.

Note that $\frac{1}{n} \to 0$, $\frac{1}{n_k}$ is a subsequence of $\frac{1}{n}$, so $\frac{1}{n_k} \to 0$, hence

$$y_{n_k} = x_{n_k} + \frac{1}{n_k} \to c + 0 = c \quad \text{by LOL.}$$

f is continuous, so $f(x_{n_k}) \to f(c)$

$f(y_{n_k}) \to f(c)$

and since $\left.\begin{array}{l} f(x_{n_k}) \leq 0 \Rightarrow f(c) \geq 0 \\ f(y_{n_k}) > 0 \Rightarrow f(c) < 0 \end{array}\right\}$ by LOL 2

we must have $f(c)=0$, which is what we wanted.

If $[a,b] \neq [0,1]$, consider $h:[0,1] \to [a,b]$ given by $h(x) = a(1-x)+bx$. Then h is continuous (it's a polynomial) and $h(0)=a$, $h(1)=b$.

So $f \circ h : [0,1] \to \mathbb{R}$ satisfies hypotheses, so $\exists\, c' \in [0,1]$ with $f\circ h(c') = 0$ i.e. $f(h(c'))=0$ so take $c = h(c')$. $\square$

Corollary: If $p(x) = x^{2n+1} + a_{2n}x^{2n} + \dots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ is an odd degree real polynomial, then p has a root in $\mathbb{R}$.

Proof: For $x \geq 1$, have

$$\left| \sum_{i=0}^{2n} a_i x^i \right| \leq \sum_{i=0}^{2n} |a_i| x^{2n}$$

so if $M > \sum_{i=0}^{2n} |a_i|$, and $M \geq 1$, $\left| \sum_{i=0}^{2n} a_i M^i \right| < M \cdot M^{2n} = M^{2n+1}$

$\Rightarrow P(M) > 0$. Similarly $P(-M) < 0$.

p is continuous (it's a polynomial) so by Int. Value Theorem $\exists\, c \in [-M, M]$ with $p(c) = 0$.

<u>Theorem</u> (Maximum Value Theorem)
If $f : [a,b] \to \mathbb{R}$ is continuous, $\exists\, c \in [a,b]$ with $f(c) \geqslant f(x)$ for all $x \in [a,b]$.

<u>Note</u>: 1) This is a theorem about $\mathbb{R}$
e.g. $f(x) = x - \cancel{3}x^3$ does not ~~allat~~ attain a maximum on $[0,2] \cap \mathbb{Q}$
2) It's important that we used $[a,b]$ and not (a,b)
<u>Ex</u>: $f : (0,1) \to \mathbb{R}$
$f(x) = 1/x$ does not attain a maximum on $(0,1)$
3) Analogous minimum value theorem obtained by considering $-f$.

<u>Idea of Proof</u>: Find a sequence x_n with $f(x_n) \to \sup \operatorname{im} f$. If x_{n_k} is a convergent subsequence $x_{n_k} \to c$ then $f(x_{n_k}) \to f(c)$ i.e. $f(c) = \sup \operatorname{im} f$.

<u>Proof</u>: Suppose $\operatorname{im} f$ is not bounded above. Then for every $n \in \mathbb{N}$ ~~I~~ can find $x_n \in [a,b]$ with $f(x_n) > n$. Then $f(x_n) \to \infty$.
By Cor to BW, (x_n) has a subsequence $x_{n_k} \to c \in [a,b]$.
Since f is continuous, $f(x_{n_k}) \to f(c)$. But $f(x_{n_k}) \to \infty$
$\Rightarrow f(x_{n_k})$ diverges. This contradicts $f(x_{n_k}) \to f(c)$.
Conclude that $\operatorname{im} f$ is bounded above.

Let $J = \sup \operatorname{im} f$. If $n \in \mathbb{N}$, then I can find $x_n \in [a,b]$ with $f(x_n) > J - 1/n$. (otherwise $J - 1/n$ upper bound for $\operatorname{im} f$) By Cor to BW, (x_n) has a convergent subsequence $x_{n_k} \to c \in [a,b]$. Since f is continuous, $f(x_{n_k}) \to f(c)$. I know $J - \frac{1}{n_k} < f(n_k) \leqslant J$.
$J - 1/n_k \to J$ so by squeeze rule, $f(n_k) \to J$. But $f(n_k) \to f(c)$ so by uniqueness of limits, conclude that $f(c) = J = \sup \operatorname{im} f$, so $f(c) \geqslant f(x)$ for all $x \in [a,b]$. $\square$

L10.1

III The Derivative

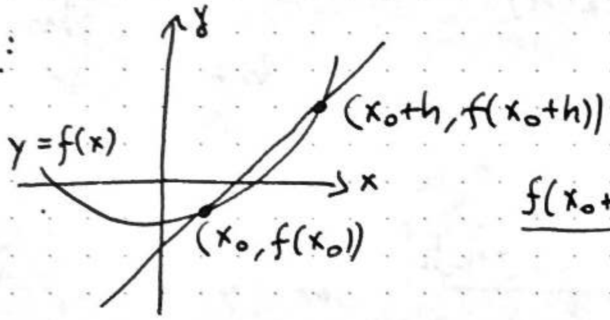
3.1) Differentiation Rules

Until further notice, we'll think about $f: I \to \mathbb{R}$ where I is an interval $I = [a,b]$ $a,b \in \mathbb{R}$
or $I = (a,b)$ $a,b \in \mathbb{R} \cup \{\pm\infty\}$.

<u>Defn</u>: If $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = m \in \mathbb{R}$ we say f is <u>differentiable at $x_0 \in I$</u> and $f'(x_0) = m$.
If is diff'ble at all $x \in I$, we say f is diff'ble.

Strictly speaking should define $\{h \in \mathbb{R} : x_0 + h \in I\} = D$ and do $F: D \setminus \{0\} \to \mathbb{R}$, $h \to \frac{f(x_0+h)-f(x_0)}{h}$ and take $\lim_{h\to 0} F(h)$

<u>Picture</u>:



$\frac{f(x_0+h)-f(x_0)}{h}$ is slope of secant

Warning: f diff'ble on $I \not\Rightarrow f'$ continuous.

<u>Examples</u>: 1) $f: \mathbb{R} \to \mathbb{R}$ $f(x) \equiv c$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

so f is diff'ble and $f'(x) \equiv 0$.

2) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

so f is diff'ble and $f'(x) \equiv 1$.

3) $f: (0,\infty) \to \mathbb{R}$ $f(x) = \frac{1}{x}$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x_0+h} - \frac{1}{x_0}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x_0 - x_0 - h}{x_0(x_0+h)}\right) = -\lim_{h\to 0} \frac{1}{x_0(x_0+h)} = -\frac{1}{x_0^2}$$

use continuity of $1/x$

so f is diff'ble and $f'(x) = -\frac{1}{x^2}$.

4) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = |x|$
is not diff'ble at 0, since

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} \text{ does not exist.}$$

Alternative Characterisation of diff'tiability

Lemma: $f'(x_0) = m$

$$\Updownarrow$$

$$f(x_0 + h) = f(x_0) + mh + h\alpha(h)$$ for some $\alpha : D \to \mathbb{R}$

where $\alpha(0) = 0$ and α is continuous at $h = 0$.

Note: Given m, α determined

$$\alpha(h) = \frac{f(x_0+h) - f(x_0) - mh}{h} \quad \text{for } h \neq 0$$

$$\alpha(0) = 0$$

Proof: Take α as given by formula above.

$$f'(x_0) = m \iff \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} - m = \lim_{h \to 0} \frac{mh}{h}$$

$$\iff \lim_{h \to 0} \frac{f(x_0+h) - f(x) - mh}{h} = 0$$

$$\iff \lim_{h \to 0} \alpha(h) = 0$$

$$\iff \alpha \text{ is continuous at } 0 \text{ since } \alpha(0) = 0. \quad \square$$

Recall: 1) If f, g are continuous at x_0, then $f+g$ and fg are too.
2) If f is continuous at x_0 and g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0.

L10.3

<u>Cor</u>: If f is diff'ble at x_0, it is continuous at x_0.

<u>Proof</u>: By Lemma we know

$$f(x_0+h) = f(x_0) + hf'(x_0) + h\alpha(h)$$

where $\alpha(0)=0$ and α is continuous at 0.

Take $h = x - x_0$, then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + (x-x_0)\alpha(x-x_0).$$

$f(x_0)$, $f'(x_0)$ are constant

$(x-x_0)$ is a polynomial

$\alpha(x-x_0)$ is composition of continuous functions (at x_0)

$\Rightarrow$ By rules of continuity, f continuous at x_0. $\square$ #dab

<u>Prop</u>: Spose f, g are diff'ble at x_0 ~~and~~ Then

1) $f+g$ is diff'ble at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

2) $f \cdot g$ is diff'ble at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ (Product Rule)

We prove these using alt. characterisation.

<u>Proof</u>: Can write

$$\cancel{f(x_0)}\ f(x_0+h) = f(x_0) + hf'(x_0) + h\alpha(h)$$
$$g(x_0+h) = g(x_0) + hg'(x_0) + h\beta(h)$$

where $\alpha(0) = \beta(0) = 0$ and α, β continuous at 0.

(i) $(f+g)(x_0+h) = f(x_0+h) + g(x_0+h)$
$$= f(x_0) + g(x_0) + (f'(x_0) + g'(x_0))h + h\gamma(h)$$

where $\gamma(h) = \alpha(h) + \beta(h)$. Then by alt. characterisation, to see that $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ it's enough to check that $\gamma(0) = 0$ and γ is continuous at ~~x_0~~ 0.

$\gamma(0) = \alpha(0) + \beta(0) = 0$ and α, β continuous at 0 $\Rightarrow$ γ continuous at 0.

So ok.

L10.4

2) $(f \cdot g)(x_0+h) = [f(x_0) + hf'(x_0) + h\alpha(h)]$
$\cdot [g(x_0) + hg'(x_0) + h\beta(h)]$

$= f(x_0)g(x_0) + h[f(x_0)g'(x_0) + f'(x_0)g(x_0)] + h\gamma(h)$

where $\gamma(h) = \alpha(h)[g(x_0) + hg'(x_0) + h\beta(h)]$
$+ f'(x_0)[hg'(x_0) + h\beta(h)]$
$+ f(x_0)\beta(h)$

Need to check $\gamma(0) = 0$, γ continuous at 0.
Each term in γ is a product of continuous factors at $h=0$, and at least one factor vanishes at $h=0$. So done. $\square$

<u>Proposition</u> If f is diff'ble at x_0, g is diff'ble at $f(x_0)$, then $g \circ f$ is diff'ble at x_0 and
$(g \circ f)'(x_0) = f'(x_0)\, g'(f(x_0))$.

<u>Proof</u>: $f(x_0+h) = f(x_0) + hf'(x_0) + h\alpha(h)$
$g(f(x_0) + \tilde{h}) = g(f(x_0)) + \tilde{h} g'(f(x_0)) + \tilde{h}\beta(\tilde{h})$
where α, β as seen before.
Then
$$g(f(x_0+h)) = g(f(x_0) + \overbrace{hf'(x_0) + h\alpha(h)}^{\tilde{h}})$$
$$= g(f(x_0)) + g'(f(x_0)) \times hf'(x_0)$$
$$+ g'(f(x_0)) \times h\alpha(h) + \tilde{h}\beta(\tilde{h})$$
$$= g(f(x_0)) + hf'(x_0)g'(f(x_0)) + h\gamma(h)$$
where $\gamma(h) = \alpha(h)g'(f(x_0)) + [f'(x_0) + \alpha(h)]\beta(\varphi(h))$
where $\varphi(h) = hf'(x_0) + h\alpha(h)$.

Now φ is continuous at 0, $\varphi(0) = 0$.
So $\beta \circ \varphi$ is continuous at 0, $\beta \circ \varphi(0) = 0$.
So we see that $\gamma(0) = 0$, γ continuous at 0.
By alt. characterisation, we done bbys. $\square$

3.1) Cont'd

Last Time: proved sum, product and chain rules

Also showed that if $f(x) = \frac{1}{x}$, then f is diff'ble for all $x \neq 0$ and $f'(x) = -\frac{1}{x^2}$.

Cor: 1) If f is diff'ble at x_0 and $f(x_0) \neq 0$, then $g(x) = 1/f(x)$ is diff'ble at x_0 and $g'(x_0) = -\dfrac{f'(x_0)}{f(x_0)^2}$

2) If f, g are diff'ble at x_0 and $g(x_0) \neq 0$, then f/g is diff'ble at x_0 and

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof: 1) $h(x) = 1/x$, then $g(x) = (h \circ f)(x)$. Apply Chain Rule

2) Follows from 1) by Product Rule applied to $f \cdot (1/g)$.

Cor: Polynomials are diff'ble

Rational functions $\dfrac{P(x)}{Q(x)}$ are diff'ble where defined

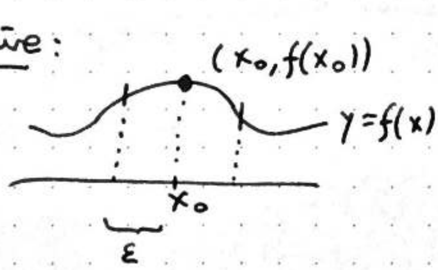
Proof: Apply Sum, Product, Quotient Rules.

3.2) Mean Value Theorem

Local Maxima: $f: I \to \mathbb{R}$

Def: Say $x_0 \in I$ is a local maximum for f if $\exists\, \varepsilon > 0$ s.t. $f(x_0) \geq f(x)$ for all x with $|x_0 - x| < \varepsilon$.

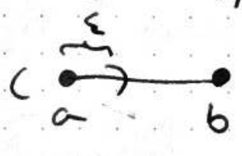
Picture:



Say $x_0 \in I$ is an interior point of I if $\exists\, \varepsilon > 0$ s.t. $x \in I$ whenever $|x_0 - x| < \varepsilon$.

Ex: Every $x_0 \in (a, b)$ is an interior point $\left(\varepsilon = \frac{1}{2}\min\{|x_0 - a|, |x_0 - b|\}\right)$

But a, b are not interior points of $[a, b]$



<u>Prop</u>: If $f : [a,b] \to \mathbb{R}$ is diff'ble at x_0, x_0 is an interior point of $[a,b]$, and x_0 is a local maximum for f, then $f'(x_0) = 0$.

<u>Note</u>: 1) hypothesis that x_0 is interior necessary
e.g. $f : [0,1] \to \mathbb{R}$, $x \to x$ has a local max at $x=1$, but $f'(1) \neq 0$

2) Similar theorem for local minima (consider $-f$)

<u>Proof</u>: Let $\varphi(h) = \dfrac{f(x_0+h)-f(x_0)}{h}$, where $\varphi : A \to \mathbb{R}$ and $A = \{h : x_0 + h \in [a,b], h \neq 0\}$

Then $\lim_{h \to 0} \varphi(h) = f'(x_0)$.
So if h_n is a sequence in A with $h_n \to 0$, then $\varphi(h_n) \to f'(x_0)$.

Since x_0 is a local maximum, $\exists\, \varepsilon > 0$ s.t. $f(x_0) \geq f(x)$ if $|x_0 - x| < \varepsilon$.
Since x_0 is an interior point, $\exists\, \eta > 0$ s.t. $x \in [a,b]$ whenever $|x - x_0| < \eta$ i.e. $h \in A$ whenever $h \in (-\eta, \eta) \setminus \{0\}$.

Take $M = \max\{1/\varepsilon, 1/\eta\}$ and consider the sequence $(h_n^+) = \left(\frac{1}{n}\right)_{n > M}$. Then (h_n^+) is a sequence in A and $h_n^+ \to 0$ and $\varphi(h_n^+) = \dfrac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} \leq 0 \quad \forall n$.

So $\varphi(h_n^+) \to f'(x_0)$, so by LOL (II), $f'(x_0) \leq 0$.

Similarly, if $(h_n^-) = \left(-\frac{1}{n}\right)_{n>M}$, then (h_n^-) is a sequence in A, $h_n^- \to 0$ and $\varphi(h_n^-) = \dfrac{f(x_0 - \frac{1}{n}) - f(x_0)}{-1/n} \geq 0 \quad \forall n$.

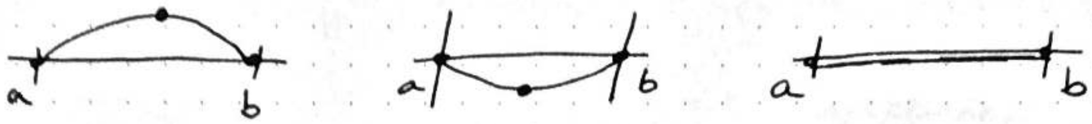
So $\varphi(h_n^-) \to f'(x_0)$, so $f'(x_0) \geq 0$.

Hence $f'(x_0) = 0$. $\square$

L11.3

Theorem (Rolle's Theorem): Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, diff'ble on (a,b), and $f(a) = f(b) = 0$. Then $\exists\, c \in [a,b]$ with $f'(c) = 0$. $(a < b)$

Picture:



Proof: By Maximum / Minimum Value Theorem, there exist $c_+, c_- \in [a,b]$ such that $f(c_-) \leq f(x) \leq f(c_+)\ \forall\, x \in [a,b]$.
If c_+ is an interior point of $[a,b]$, then $f'(c_+)$ is zero by proposition. Similarly, c_- interior $\Rightarrow f'(c_-) = 0$.

Otherwise, $c_+, c_- \in \{a, b\}$ so $f(c_+) = f(c_-) = 0$.
$\Rightarrow f(x) \equiv 0$ so $f'(x) \equiv 0\ \forall\, x \in [a,b]$. $\square$

Thm (Mean Value Theorem):
Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and diff'ble on (a,b). Then $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

Theorem says if I drive my car from time $t = a$ to $t = b$, ~~then~~ so my average speed is $\dfrac{f(b) - f(a)}{b-a}$, then $\exists$ some time c where my spedometer reading was average speed.

Proof: Consider $g(x) = f(x) - \dfrac{1}{b-a}\left[(x-a)f(b) + (b-x)f(a)\right]$

We have $g(a) = f(a) - f(a) = 0$, $g(b) = f(b) - f(b) = 0$.
Second part of $g(x)$ is a polynomial, hence diff'ble. So g is continuous on $[a,b]$ and diff'ble on (a,b). By Rolle's Theorem applied to g, $\exists$ some $c \in (a,b)$ with $g'(c) = 0$ i.e.

$$f'(c) - \frac{1}{b-a}\left[f(b) - f(a)\right] = 0$$ 😃

$\square$

L11.4

~~Cor~~ **Def**: Say $f: I \to \mathbb{R}$ is *increasing* if $f(x) \geq f(y)$ whenever $x \geq y$ and *strictly increasing* if $f(x) > f(y)$ whenever $x > y$.

Cor: If $f: [a,b] \to \mathbb{R}$ is diff'ble and ① $f'(x) \geq 0$ $\forall x \in [a,b]$, then f is increasing. ② $f'(x) > 0$ for all $x \in [a,b]$, then f is strictly increasing.

Proof: By MVT applied to $[y,x]$, if $a \leq y \leq x \leq b$, we have

$$\frac{f(x)-f(y)}{x-y} = f'(c) \text{ for some } c \in (y,x)$$
$$\geq 0$$

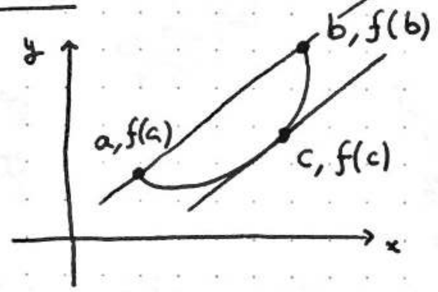
$$\Rightarrow f(x) - f(y) \geq 0 \quad \text{i.e. } f \text{ increasing.} \qquad \square$$

3.2) Cont'd

Mean Value Theorem: If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f(b)-f(a) = (b-a)f'(c)$$

Picture:



$m_{sec} = \dfrac{f(b)-f(a)}{b-a}$

$m_{tan} = f'(c)$

$\exists\, c \in (a,b)$ s.t.

$m_{sec} = m_{tan}$

Cor: 1) If $f: I \to \mathbb{R}$ is diff'ble and $f'(x)=0$ for all $x \in I$, then f is constant

2) If $f,g: I \to \mathbb{R}$ are diff'ble and $f' \equiv g'$, then $f \equiv g + c$ for some $c \in \mathbb{R}$

N.B. Important that domain of f is an interval, e.g.

$f: [0,1] \cup [2,3] \to \mathbb{R}$

$x \to \begin{cases} 0 & \text{if } x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$ Then $f' \equiv 0$ but f is not constant

Proof: 1) Given $x, y \in I$, ~~x≠~~ $y < x$, then $[y,x] \subset I$, so by MVT $\exists\, c \in [y,x]$ with $f(x)-f(y) = (x-y)f'(c) = (x-y)\cdot 0 = 0$

i.e. $f(x) = f(y) \quad \forall\, x,y \in I$

2) Apply 1) to $f-g$. $\square$

This solves the subtle problem from Lecture 1. Recap of Proof:

LUBP $\to$ MST $\to$ BW $\to$ Max VT $\to$ Rolle $\to$ MVT

Q: $f: [0,2] \cap \mathbb{Q} \to \mathbb{R}$

$$x \to \begin{cases} 0 & \text{if } x^2 < 2 \\ 1 & \text{if } x^2 \geq 2 \end{cases}$$

What goes wrong if I try to apply proof of MVT to f?

L12.2

Inverse Function Theorem

Recall: if $f: I \to \mathbb{R}$ is diff'ble and $f'(x) > 0 \ \forall x \in I$, then f is strictly increasing. For if $x > y$, $x, y \in I$, applying MVT to $[y, x]$ gives $f(x) - f(y) = (x-y) f'(c) > 0$ for some $c \in I$.

Lemma 1: if $f: I \to \mathbb{R}$ has $f'(x) > 0 \ \forall x \in I$ then $f: I \to f(I)$ is a bijection and $f(I)$ is an interval.

Proof: If $x > y$, $f(x) > f(y) \Rightarrow f$ injective $\Rightarrow f: I \to f(I)$ bijective

If $A = f(a)$, $B = f(b)$, $a, b \in I$, $a < b$, ~~then~~ $C \in [A, B]$ then by the Intermediate Value Theorem applied to $f - C$, $\exists \ c \in [a, b]$ with $f(c) - C = 0$ i.e. $C \in f(I)$. (f is diff'ble, hence continuous on I, so IVT applies.) So if $A, B \in f(I)$, $A \leq C \leq B$, then $C \in f(I) \Rightarrow f(I)$ is an interval. $\square$

$f: I \to f(I)$ is a bijection, so have $f^{-1}: f(I) \to I$

Lemma 2: If $f: I \to f(I)$ is a continuous bijection, then $f^{-1}: f(I) \to I$ is continuous

Proof: ES2 #6, 7 — not too bad, especially if given that it's increasing

Theorem: (Inverse Function Theorem)

If $f: I \to f(I)$ is diff'ble and $f'(x) > 0$ for all $x \in I$ then f^{-1} is diff'ble and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

EX: 1) $f: (0, \infty) \to (0, \infty)$, $x \to x^n$ $\Rightarrow f'(x) = nx^{n-1}$

$f^{-1}(y) = y^{1/n}$ (n^{th} root)

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} y^{\frac{1}{n} - 1}$$

2) $f : (-\pi/2, \pi/2) \longrightarrow (-\infty, \infty)$ $\quad \Rightarrow f'(\theta) = \sec^2\theta$

$\theta \longrightarrow \tan\theta$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{\sec^2(\arctan y)} = \frac{1}{y^2+1}$$

3) Observe $f(f^{-1}(y)) = y$.

$(f^{-1})'(y) \cdot f'(f^{-1}(y)) = 1$ which is same as before.

Q: Is this a proof of IFT? No. To apply Chain Rule, need to know f^{-1} is diff'ble.

<u>Proof of IFT</u>: Want to show $\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$

where $f(x_0) = y_0$}. Enough to show that if $y_n \to y_0$ where $y_n \in f(I) \setminus \{y_0\}$, then $\frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} \longrightarrow \frac{1}{f'(x_0)}$.

Let $x_n = f^{-1}(y_n)$. By Lemma 2, f^{-1} is continuous, so $x_n \longrightarrow f^{-1}(y_0) = x_0$. Also f^{-1} is injective and $y_n \neq \cancel{0}y_0 \Rightarrow x_n \neq x_0$.

Since f is diff'ble, $\frac{f(x_n) - f(x_0)}{x_n - x_0} \longrightarrow f'(x_0) \neq 0$

$$\Rightarrow \frac{x_n - x_0}{f(x_n) - f(x_0)} \longrightarrow \frac{1}{f'(x_0)}$$

i.e. $$\frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} \longrightarrow \frac{1}{f'(x_0)}.$$

So done. ☺ $\square$

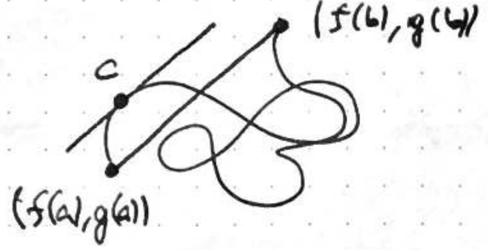
L12.4

L'Hôpital's Rule

<u>Prop</u>: (Cauchy's MVT): Spose $f, g : [a,b] \to \mathbb{R}$ are diff'ble.
Then $\exists\ c \in (a,b)$ with

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(\overset{c}{\cancel{x}}).$$

<u>Picture</u>: Let $\gamma(t) = (f(t), g(t))$ be a parametrised curve.
Then $\gamma'(t) = (f'(t), g'(t))$.



$$m_{sec} = \frac{g(b) - g(a)}{f(b) - f(a)}$$

$$m_{tan} = \frac{g'(c)}{f'(c)}$$

$\exists\ c \in (a,b)$ s.t. $m_{sec} = m_{tan}(c)$.

<u>Proof</u>: consider

$$h(t) = \cancel{t}\,(f(b) - f(a))\, g(t) - (g(b) - g(a)) f(t)$$

Then $h(a) = f(b)g(a) - g(b)f(a) = h(b)$.
So by MVT, $\exists\ c \in (a,b)$ with
$0 = h(b) - h(a) = (b-a)\, h'(c)$ i.e. $h'(c) = 0$

So $(f(b) - f(a))\, g'(c) - (g(b) - g(a))\, f'(c) = 0$ $\square$

L13.1

3.2) Cont'd

Cauchy MVT: If $f, g : [a,b] \to \mathbb{R}$ are diff'ble, then $\exists\, c \in (a,b)$ with

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c).$$

Prop: (Simple L'Hôpital): If $f, g : I \to \mathbb{R}$, $a \in I$ with $g(a) = f(a) = 0$, and f, g are diff'ble, and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = k$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = k$ too.

note g' is never 0 on I. else def^n botched. also $\Rightarrow$ g only zero at a by MVT

Ex: $$\lim_{x \to 0} \frac{1 - (1-2x)^{1/3}}{1 - (1+x)^{1/5}} = \lim_{x \to 0} \frac{\frac{2}{3}(1-2x)^{-2/3}}{-\frac{1}{5}(1+x)^{1/5}} = -\frac{10}{3}$$

Proof: Must check that if $x_n \to a$, $x_n \neq a$, then $f(x_n)/g(x_n) \longrightarrow k$.

By Cauchy MVT

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}$$ where $c_n \in (a, x_n)$ or (x_n, a) $\Rightarrow 0 < |c_n - a| < |x_n - a|$.

$x_n \to a \Rightarrow |x_n - a| \to 0$ so by squeeze rule $|c_n - a| \to 0$ ie. $c_n \to a$.

Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = k$, $\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)} \to k$

which is what we wanted. □

3.3) Taylor Series

Higher derivatives

Def: We say $f : I \to \mathbb{R}$ is k-times diff'ble ($k \in \mathbb{N}$) with k^{th} derivative $f^{(k)}$ if $f^{(k-1)}$ is diff'ble with derivative $(f^{(k-1)})' =: f^{(k)}$.

We say $f \in C^k$ if f is k-times diff'ble and $f^{(k)}$ is continuous.

Warning: f is diff'ble does not imply f' is continuous

Ex: $$f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h \sin 1/h = 0$$

so f is diff'ble with

$$f'(x) = \begin{cases} 2x \sin 1/x - \cos 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

but not C^1_I since $\lim_{x\to 0} \cos 1/x$ does not exist.

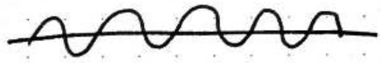
<u>Exercise</u>: If $f: I \to \mathbb{R}$ is C^1, a is an interior point of I and $f'(a) > 0$, then $\exists\, \varepsilon > 0$ s.t. f is increasing on $(a-\varepsilon, a+\varepsilon)$.

Q: If $f: I \to \mathbb{R}$ is C^1, what's the best linear function approximating f?

A: This depends on how you define best.

E.g. $f: [-10\pi, 10\pi] \to \mathbb{R}$

$x \longrightarrow \sin x$



best approximation is probably $p_1(x) = 0$ not the tangent line at any point on the graph.

Tangent line at x_0:

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

is best linear approximation $\boxed{\text{near } x_0}$ in the sense that

$$f(x) = p_1(x) + (x - x_0)\alpha(x - x_0)$$

where $\alpha(h) \to 0$ as $h \to 0$.

If $f(x) = q_1(x) + (x - x_0)\beta(x - x_0)$ where $\beta(h) \to 0$ as $h \to 0$, then

$$p_1(x) - q_1(x) = a_0 + a_1(x - x_0) = (x - x_0)(\alpha - \beta)(x - x_0)$$

$x = x_0$: $a_0 = 0$

$x \neq x_0$: $a_1 = (\alpha - \beta)(x - x_0) \Rightarrow a_1 = 0$ i.e. $p_1 = q_1$

<u>Goal</u>: Find a degree k polynomial p_k which is the best approximation to $f(x)$ near x_0.

(Now assume that $f: I \to \mathbb{R}$ is C^k)

In the sense that $f(x) = p_k(x) + (x-x_0)^k \alpha(x-x_0)$

where $\alpha(h) \to 0$ as $h \to 0$.

$$f(x) - p_k(x) = (x-x_0)^k \alpha(x-x_0)$$

$$\frac{d^i}{dx^i}\left[f(x)-p_k(x)\right]\Big|_{x=x_0} = \frac{d^i}{dx^i}\left[(x-x_0)^k \alpha(x-x_0)\right]\Big|_{x=x_0} = 0$$

for $0 \leq i \leq k$. (exercise)

i.e. $f^{(i)}(x_0) = p_k^{(i)}(x_0) \quad 0 \leq i \leq k$

$$p_k(x) = \sum_{i=0}^{k} a_i (x-x_0)^i$$

Then $p_k^{(i)}(x) = i!\, a_i + (x-x_0)(\text{stuff})$

$\Rightarrow p_k^{(i)}(x_0) = i!\, a_i$

i.e. $f^{(i)}(x_0) = i!\, a_i$

so $a_i = \dfrac{f^{(i)}(x_0)}{i!}$

so if such a p_k exists it's determined by $f(x_0), \dots, f^{(k)}(x_0)$.

<u>Theorem</u> (Taylor's Theorem):

If $f: I \to \mathbb{R}$, $f \in C^{k+1}$ and $x_0, x \in I$, then $\exists\, c \in \begin{matrix}(x_0, x)\\(x, x_0)\end{matrix}$

s.t. $f(x) - p_k(x) = \dfrac{f^{(k+1)}(c)}{(k+1)!}(x-x_0)^{k+1}$

i.e.
$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + \frac{f^{(k+1)}(c)}{(k+1)!}(x-x_0)^{k+1}$$

<u>Note</u>: $k=0$ is MVT

L13.4

Proof: Let $F(x) = f(x) - p_k(x)$

$G(x) = (x - x_0)^{k+1}$

Then $F^{(i)}(x_0) = 0$ for $0 \le i \le k$.

$G^{(i)}(x_0) = 0$ " "

So $$\frac{F(x)}{G(x)} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \frac{F'(x_1)}{G'(x_1)}$$ by Cauchy MVT

where $x_1 \in (x_0, x)$.

$$= \frac{F'(x_1) - F'(x_0)}{G'(x_1) - G'(x_0)} = \frac{F^{(2)}(x_2)}{G^{(2)}(x_2)}$$ where $x_2 \in (x_0, x_1)$

$$\vdots$$

$$= \frac{F^{(k+1)}(x_{k+1})}{G^{(k+1)}(x_{k+1})}$$ where $x_{k+1} \in (x_0, x_k) \subset (x_0, x)$

Let $c = x_{k+1}$. Then

$$\frac{F(x)}{G(x)} = \frac{f(x) - p_k(x)}{(x - x_0)^{k+1}} = \frac{f^{(k+1)}(c) - \cancel{p_k^{(k+1)}(c)}^{\text{zero}}}{(k+1)!}$$

i.e. $$f(x) - p_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$ $\square$

Corollary: If $f: I \to \mathbb{R}$ is C^{k+1}, then

$$f(x) = p_k(x) + (x - x_0)^k \alpha(x - x_0)$$

where $\alpha(h) \to 0$ as $h \to 0$.

Proof: Take $$\alpha(x - x_0) = \frac{(x - x_0)}{(k+1)!} f^{(k+1)}(c(x))$$

where $c(x)$ is as in the theorem.

Since $f^{(k+1)}$ is continuous on $[x_0, x]$, then by MaxVT

$|f^{(k+1)}(c(x))| \le M \Rightarrow \alpha(h) \to 0$ as $h \to 0$. $\square$

Clarification / Simplification

Simple L'Hôpital: Suppose $f, g : I \to \mathbb{R}$ are diff'ble and $f(a) = g(a) = 0$. Then

if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = K$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = K$.

(Suppose g' never vanishes on I)

3.3) Cont'd

If $f : I \to \mathbb{R}$ is in C^k then the k^{th} Taylor polynomial of f centred at $x_0 \in I$ is

$$p_k(x) = \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Chosen so $f^{(i)}(x_0) = p_k^{(i)}(x_0)$ $\quad 0 \leq i \leq k$.

p_k are partial sums of the Taylor series for f centred at x_0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad \text{(assume smoothness)}$$

Taylor's Thm: If $f : I \to \mathbb{R}$ is in C^{k+1} and $x_0, x \in I$, then $\exists$ $c \in (x_0, x)$ [or (x, x_0)] with

$$f(x) - p_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}$$

Sometimes, if we know enough about f, we can use Taylor's Thm to show that Taylor series converges to $f(x)$.

Ex: Given $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

Study Taylor series of $\sin x$ centred at $x_0 = 0$.

So Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Taylor's Thm says: $\exists$ $c \in (x_0, x)$ with

$$f(x) - P_{2k+1}(x) = \frac{(-1)^k \sin(c)}{(2k+2)!} x^{2k+2}$$

$$\Rightarrow |f(x) - p_{2k+1}(x)| \leqslant \frac{|x|^{2k+2}}{(2k+2)!} \quad \text{since } |\sin(c)| \leqslant 1.$$

<u>Lemma</u>: If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\lim_{n\to\infty} a_n = 0$.

<u>Proof</u>: By ratio test, $\sum_n a_n$ converges.
By n^{th} term test, $a_n \longrightarrow 0$. □

If $a_{k+1} = \frac{|x|^{2k+2}}{(2k+2)!}$, then

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{|x|^{2k+2}}{|x|^{2k}} \cdot \frac{(2k)!}{(2k+2)!} = \frac{|x|^2}{(2k+2)(2k+1)} \longrightarrow 0$$

$$\Rightarrow \frac{|x|^{2k+2}}{(2k+2)!} \longrightarrow 0$$

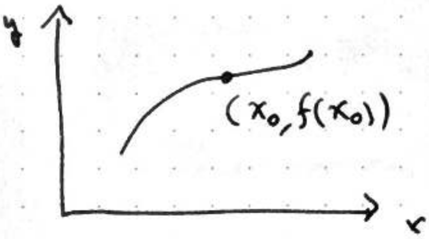
By squeeze rule, $|f(x) - p_{2k+1}(x)| \longrightarrow 0$
i.e. $p_{2k+1}(x) \longrightarrow f(x)$

boi

i.e. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+1)!}$.

In general, even if Taylor series converges, it may have NOTHING to do with $f(x)$ for $x \neq x_0$.

<u>Thought experiment</u>:



$y = f(x)$

Taylor series at x_0 determined by derivatives of f at x_0, i.e. f around $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Why should the values of f near x_0 determine the value of f far away?

$e^{-\frac{1}{x^2}}$

<u>Mathematics</u> $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then f is C^∞ and $f^{(k)}(0) \equiv 0$.

$\Rightarrow$ Taylor series at 0 is zero

But $f(x) \neq 0$ for $x \neq 0$

$\Rightarrow$ Taylor series converges everywhere but only to $f(x)$ at 0

~ Proof by example sheet

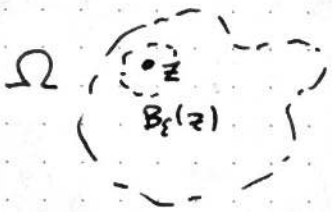
3.4) Complex differentiation

Def^n: If $a \in \mathbb{C}$ and $r \in \mathbb{R}^+$, then

$$B_r(a) = \{ z \in \mathbb{C} : |z-a| < r \}$$

is the open ball of radius r centred at a.

If $\Omega \subseteq \mathbb{C}$, we say Ω is open if for all $z \in \Omega$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(z) \subseteq \Omega$.



Def^n: Spose $\Omega \subseteq \mathbb{C}$ is open, $f: \Omega \to \mathbb{C}$. If

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0+h) - f(z_0)}{h} = c \in \mathbb{C}$$

then we say f is complex diff'ble at z_0 and write $f'(z_0) = c$.

Ω open means that h can approach 0 from any direction in $\mathbb{C}$.
Everything we did in 3.1) applies verbatim in this case.

Write $z = x + iy$, $x, y \in \mathbb{R}$.

$f(z) = U(x,y) + iV(x,y)$

where $U, V : \underset{\mathbb{R}^2}{\underset{\cap}{\Omega}} \to \mathbb{R}$

Partial derivatives: Spose $G: \mathbb{R}^2 \to \mathbb{R}$.

Define $\left.\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} = g_1'(x_0)$ where $g_1(x) = G(x, y_0)$

$\left.\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} = g_2'(y_0)$ where $g_2(y) = G(x_0, y)$

L14.4

Prop: (Cauchy - Riemann)

If $f(x+iy) = U(x,y) + iV(x,y)$ is complex diff'ble at $z_0 = x_0 + iy_0$ with $f'(z_0) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ then

$$\frac{\partial U}{\partial x}\Big|_{(x_0,y_0)} = \alpha = \frac{\partial V}{\partial y}\Big|_{(x_0,y_0)}$$

$$\frac{\partial V}{\partial x}\Big|_{(x_0,y_0)} = \beta = -\frac{\partial U}{\partial y}\Big|_{(x_0,y_0)}$$

Proof: Spose $h_n \to 0$, $h_n \in \mathbb{R}\setminus\{0\}$.

Then $$\frac{f(z_0+h_n) - f(z_0)}{h_n} \longrightarrow \alpha + i\beta$$

i.e. $$\frac{U(x_0+h_n, y_0) - U(x_0,y_0)}{h_n} + i\,\frac{V(x_0+h_n,y_0) - V(x_0,y_0)}{h_n} \longrightarrow \alpha + i\beta$$

$$\Rightarrow \frac{U(x_0+h_n, y_0) - U(x_0,y_0)}{h_n} \longrightarrow \alpha, \quad \frac{V(x_0+h_n, y_0) - V(x_0,y_0)}{h_n} \longrightarrow \beta$$

i.e. $$\underbrace{\frac{\partial U}{\partial x}\Big|_{(x_0,y_0)}}_{=\alpha} \qquad \frac{\partial V}{\partial x}\Big|_{(x_0,y_0)} = \beta.$$

But also $ih_n \to 0$, so $$\frac{f(z_0+ih_n) - f(z_0)}{ih_n} \longrightarrow \alpha + i\beta$$

$$\frac{V(x_0, y_0+h_n) - V(x_0,y_0)}{h_n} - i\,\frac{U(x_0, y_0+h_n) - U(x_0,y_0)}{h_n} \longrightarrow \alpha + i\beta$$

i.e. $$\frac{\partial V}{\partial y}\Big|_{(x_0,y_0)} = \alpha, \quad \frac{\partial U}{\partial y}\Big|_{(x_0,y_0)} = -\beta$$

$\square$

3.4) Cont'd

$\Omega \subseteq \mathbb{C}$ open

$f: \Omega \to \mathbb{C}$ given by $f(x+iy) = U(x,y) + iV(x,y)$

<u>Cauchy-Riemann</u>: if f is complex diff'ble at $z_0 = x_0 + iy_0 \in \Omega$ with $f'(z_0) = \alpha + i\beta$, then

$$\left.\frac{\partial U}{\partial x}\right|_{(x_0,y_0)} = \alpha = \left.\frac{\partial V}{\partial y}\right|_{(x_0,y_0)}$$

$$-\left.\frac{\partial U}{\partial y}\right|_{(x_0,y_0)} = \beta = \left.\frac{\partial V}{\partial x}\right|_{(x_0,y_0)}$$

<u>Exercise</u>: if f is complex diff'ble then U, V satisfy Laplace's equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$.

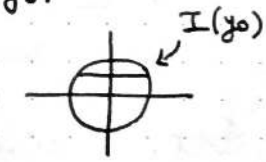
<u>Lemma</u>: Suppose $G: \cancel{\Omega}\, B_r(0) \to \mathbb{R}$ satisfies $\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} \equiv 0$.

Then G is constant.

<u>Proof</u>: Fix $y_0 \in (-r, r)$ and consider $g_1(x): I(y_0) \to \mathbb{R}$

$x \to G(x, y_0)$

where $I(y_0) = (-\sqrt{r^2 - y_0^2}, \sqrt{r^2 - y_0^2})$.



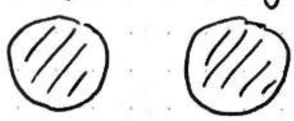
Then $g_1'(x) = \left.\frac{\partial G}{\partial x}\right|_{(x,y_0)} = 0$.

So g_1 is constant, so $G(x, y_0) = G(0, y_0)$.

Similarly, I see that $G(x_0, y) = G(x_0, 0)$.

$\Rightarrow G(x,y) = G(x,0) = G(0,0)$ $\square$

Q: What goes wrong if domain of G = union of two disjoint discs?



<u>Cor</u>: If $f: B_r(0) \to \mathbb{C}$ is complex diff'ble at all $z \in B_r(0)$ and $f'(z) \equiv 0$, then f is constant.

<u>Proof</u>: By CR, $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$,

so by Lemma, U and V are constant. $\square$

<u>Amazing fact:</u>

1) If $f : \Omega \to \mathbb{C}$ is cx diff'ble then so is f' $\Rightarrow$ f is C^∞

2) If $f : B_r(a) \to \mathbb{C}$ is cx diff'ble then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \longrightarrow f(z)$$

$\forall\, z \in B_r(a)$.

$$\boxed{\begin{array}{c}\text{Toy Ex: If } f = f' \\ \Rightarrow f \text{ is } C^\infty\end{array}}$$

IV Exp, sin, log + all that

4.1) Power series

Recall from section 1.2)

<u>n^{th} term test</u>: If $\sum_n c_n$ converges then $c_n \longrightarrow 0$.

<u>Strong comparison</u>: If $|c_n| < a_n$ for all n, and $\sum_n a_n$ converges then $\sum_n c_n$ converges too.

<u>Ratio Test</u>: If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$,

then $\sum_n a_n$ converges if $r < 1$,
diverges if $r > 1$.

<u>Def</u>: A series of the form

$$\sum_{n=0}^{\infty} c_n (z-a)^n, \quad a, c_n \in \mathbb{C}$$

is called a <u>power series</u>.

<u>Ex</u>: $\sum_{n=0}^{\infty} z^n$ (geometric series)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sin(z)$$

any Taylor series.

Q: For what values of z does the series converge?

Ex: If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = r$, then

r=0 ⇒ converges always

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n}\right| = r|z-a|$$

So by ratio test, series converges if $|z-a| < 1/r$,
diverges if $|z-a| > 1/r$.

series converges inside
diverges outside

Lemma: (Key Estimate)

If $\sum_n c_n(z_0-a)^n$ converges, with $z_0 \neq a$, then $\exists M \in \mathbb{R}$ such that

$$\boxed{|c_n| \leq \frac{M}{|z_0-a|^n}} \text{ for all } n.$$

Proof: By n^{th} term test, $c_n(z_0-a)^n \longrightarrow 0$.

So $\exists N \in \mathbb{N}$ s.t. $|c_n(z_0-a)^n| < 1$, for all $n > N$.

Let $M = \max\{|c_0|, |c_1(z_0-a)|, \ldots, |c_N(z_0-a)^N|, 1\}$.

Then $|c_n(z_0-a)^n| \leq M$ for all $n \in \mathbb{N}$.

$\Rightarrow |c_n| \leq M/|z_0-a|^n$. $\square$

Theorem: Suppose $\sum_n c_n(z_0-a)^n$ converges, $z_0 \neq a$.

If $|z-a| < |z_0-a|$ then $\sum_n c_n(z-a)^n$ also converges.

Proof: Let $\alpha = \frac{|z-a|}{|z_0-a|}$ so $0 \leq \alpha < 1$.

Then $\sum_n \alpha^n$ is a geometric series with ratio < 1, so converges.

By the Key Lemma, $\exists M$ s.t. $|c_n| \leq M/|z_0-a|^n$

$$\Rightarrow |c_n(z-a)^n| \leq M\left|\frac{z-a}{z_0-a}\right|^n = M\alpha^n.$$

$\sum_n \alpha^n$ converges $\Rightarrow \sum_n M\alpha^n$ converges

so by strong comparison

$\sum_n c_n(z-a)^n$ converges. $\square$

<u>Rephrase</u>: Let $C = \{ |z-a| : \sum_n c_n (z-a)^n \text{ converges} \}$ for some z ??

Thm says that if $r \in C$ and $\cancel{\forall}$ $|z-a| < r$ then $\sum_n c_n (z-a)^n$ converges. So if $r \in C$ and $0 \leq r' < r$, then $r' \in C$.

<u>Def</u>: The <u>radius of convergence</u> of $\sum_n c_n (z-a)^n$ is

$\sup C$ if this exists, and ∞ otherwise.

<u>Cor</u>: Suppose $\sum_n c_n (z-a)^n$ has radius of convergence R.
Then $\sum_n c_n (z-a)^n$ converges if $|z-a| < R$,
diverges if $|z-a| > R$.

<u>Proof</u>: Suppose $|z-a| < R = \sup C$.
Then $\exists\, r'$ with $|z-a| < r' < R$ with $r' \in C$.
Then by the Thm, $\sum_n c_n (z-a)^n$ converges.
If $|z-a| > R$, then $|z-a| \notin C$ as R an upper bound for C.
$\Rightarrow \sum_n c_n (z-a)^n$ diverges. $\square$

To find radius of convergence, try the ratio test.

<u>Ex</u>: $p \in (0, \infty)$, consider $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$.

Ratio Test: $\lim_{n \to \infty} \left| \frac{z^{n+1}/(n+1)^p}{z^n/(n)^p} \right| = \lim_{n \to \infty} |z| \left| 1 + \frac{1}{n} \right|^p = |z|$

so series converges if $|z| < 1$, diverges if $|z| > 1$.

On the boundary, $|z| = 1$, series is more interesting.

If $p > 1$, and $|z| = 1$, then $\sum_n z^n/n^p$ converges by comparison with $\sum_n 1/n^p$, which converges by Cauchy condensation.

If $p \leq 0$ and $|z| = 1$, then $\sum_n z^n/n^p$ diverges by n^{th} term test.

If $0 < p \leq 1$, some points converge, some diverge.

E.g. $\sum \frac{1^n}{n}$ diverges but $\sum \frac{(-1)^n}{n}$ converges

L16.1

4.1) Cont'd

Derivatives of power series

Recall: if $\sum_{n=0}^{\infty} c_n(z-a)^n$ is a power series, its radius of convergence is $R = \sup\{|z-a| : \sum_{n=0}^{\infty} c_n(z-a)^n \text{ converges}\}$.

If $|z-a| < R$, then the series converges, and if $|z-a| > R$, then it diverges.

So we define $f: B_R(a) \longrightarrow \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

Key Estimate: If $r < R$, then $\exists M$ (depending on r) s.t. $|c_n| \leq M/r^n$ for all $n \in \mathbb{N}$.

Ex: The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$ (ratio test).

So $R = \infty$ and we can define $\exp: \mathbb{C} \to \mathbb{C}$ by $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$.

We'd like to differentiate term-by-term to see that $\exp'(z) \stackrel{?}{=} \exp(z)$

Have to justify this. Warning: $f_n(x) \to f(x)$ does not imply that $f_n'(x) \to f'(x)$. Ex: $f_n(x) = \frac{1}{n}\sin(nx)$

Setup: Spose $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence R. Then $f: B_R(0) \longrightarrow \mathbb{C}$ given by $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a thing.

Want to show that $f(z)$ is complex diff'ble on $B_R(0)$ and $f'(z) = \sum_{n=0}^{\infty} n c_n z^{n-1}$.

Lemma 1: If $z_0 \in B_R(0)$, then $\sum_{n=0}^{\infty} n c_n z_0^{n-1}$ converges.

Proof: Let $r_1 = |z_0| < R$. Pick r_2 with $r_1 < r_2 < R$. Then by Key Estimate, $\exists M$ s.t. $|c_n| \leq M/r_2^n$ for all n.

$\Rightarrow |\cancel{} n c_n z_0^{n-1}| \leq \frac{nM}{r_2}\left(\frac{r_1}{r_2}\right)^{n-1} =: a_n$

$\left(\frac{r_1}{r_2}\right) < 1$, so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \left(\frac{r_1}{r_2}\right) = \left(\frac{r_1}{r_2}\right) < 1$

L16.2

So $\sum_n a_n$ converges by the ratio test.

Now $|nc_n z_0^{n-1}| \leq a_n$, so $\sum_n nc_n z_0^{n-1}$ converges by strong comparison. $\square$

Define $g : B_R(0) \to \mathbb{C}$ by $g(z) = \sum_{n=0}^{\infty} nc_n z^{n-1}$.

Fix $z_0 \in B_R(0)$. To show that f is diff'ble at z_0 with $f'(z_0) = g(z_0)$, it's enough to show that

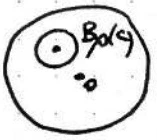
$$f(z_0+h) = f(z_0) + hg(z_0) + h\alpha(h)$$

where $\alpha(h) \to 0$ as $h \to 0$.

Let $r_1 = |z_0| < R$. $\rho = \frac{1}{2}(R - r_1) > 0$.

For $h \in B_\rho(0)$, $|z_0 + h| \leq |z_0| + |h| < \cancel{R}\, r_1 + \frac{1}{2}(R - r_1) = \frac{1}{2}(R + r_1) < R$

<u>Picture</u>:



For $h \in B_\rho(0)$, $f(z_0+h) - f(z_0) - hg(z_0)$

$$= \sum_{n=0}^{\infty} c_n (z_0+h)^n - \sum_{n=0}^{\infty} c_n z_0^n - h \sum_{n=0}^{\infty} nc_n z_0^{n-1}$$

$$= \sum_{n=\cancel{0}2}^{\infty} c_n q_n(h)$$

where $q_n(h) = (z_0 + h)^n - z_0^n - nhz_0^{n-1}$.

By binomial thm,

$$q_n(h) = \sum_{i=2}^{n} \binom{n}{i} h^i z_0^{n-i} = h^2 \sum_{i=2}^{n} \binom{n}{i} h^{i-2} z_0^{n-i}$$

$$= h^2 \tilde{q}_n(h).$$

<u>Summary</u>: $f(z_0+h) - f(z_0) - hg(z_0) = h^2 \sum_{n=\cancel{0}2}^{\infty} c_n \tilde{q}_n(h) = h\alpha(h)$

where $\alpha(h) = h \sum_{n=0}^{\infty} c_n \tilde{q}_n(h)$.

<u>Lemma 2</u>: If $h \in B_\rho(0)$, then

$$|\tilde{q}_n(h)| \leq \binom{n}{2} r_2^{\,n-2} \qquad \leftarrow r_2 = \tfrac{1}{2}(R+r_1)$$

<u>Proof</u>: $$|q_n(h)| = \left| \sum_{i=2}^{n} \binom{n}{i} h^i z_0^{\,n-i} \right|$$
$$\leq \sum_{i=2}^{n} \binom{n}{i} |h|^i |z_0|^{n-i}$$
$$= (|z_0|+|h|)^n - |z_0|^n - |h|\cdot n \cdot |z_0|^{n-1}$$

By Taylor's Thm applied to $F(x) = x^n$ on $[|z_0|, |z_0|+|h|]$, we have

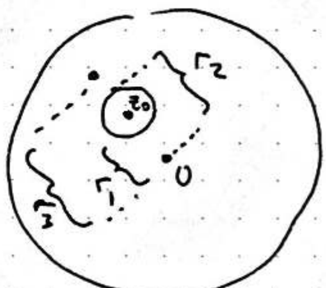
$$|q_n(h)| = \frac{n(n-1)}{2} \cdot c^{n-2} \cdot |h|^2 \text{ for some } c \in (|z_0|, |z_0|+|h|) \quad .$$
$$\leq \binom{n}{2} (|z_0|+|h|)^{n-2} |h|^2$$
$$\leq \binom{n}{2} r_2^{\,n-2} |h|^2$$

But $|q_n(h)| = |h|^2 |\tilde{q}_n(h)|$ => done. □

<u>Corollary</u>: $\exists\, C \in \mathbb{R}$ s.t. $\left|\sum_{n=2}^{\infty} c_n \tilde{q}_n(h)\right| \leq C$ for all $h \in B_\rho(0)$.

<u>Proof</u>: Pick r_3 with $r_2 < r_3 < R$.

By the key estimate $\exists\, M$ s.t. $|c_n| \leq M/r_3^{\,n}$? yes



$B_R(0)$

so $$|c_n \tilde{q}_n(h)| \leq \frac{M}{r_3^{\,n}} \cdot \binom{n}{2} r_2^{\,n-2} =: a_n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(n)}{n(n-1)} \frac{r_2}{r_3} = \frac{r_2}{r_3} < 1$$

By Ratio test, $\sum_n a_n$ converges.

By strong comparison, $\sum_n c_n \tilde{q}_n(h)$ converges and

$$\left|\sum_{n=0}^{\infty} c_n \tilde{q}_n(h)\right| \leq \sum_{n=0}^{\infty} a_n =: C \qquad □$$

<u>Thm</u>: If $\sum_{n=0}^{\infty} c_n z^n$ has roc R,

$z_0 \in B_R(0)$ and $h \in B_\rho(0)$ as above, then $f(z) = \sum_{n=0}^{\infty} c_n z^n$

L16.4

is diff'ble at z_0 and $f'(z) = \sum_{n=0}^{\infty} n c_n z^{n-1} = g(z)$.

<u>Proof</u>: We've seen that

$f(z_0+h) - f(z_0) - h g(z_0) = h\alpha(h)$

where $\alpha(h) = h \sum_n c_n \tilde{q}_n(h) \Rightarrow |\alpha(h)| \leq |h| C$ from Corollary

$\Rightarrow \alpha(h) \to 0$ as $h \to 0$ by Sandwich Thm

$\square$

L17.1

4.1 Cont'd

Last time: If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $z \in B_R(0)$, then f is cx diff'ble on $B_R(0)$ and $f'(z) = \sum_{n=0}^{\infty} n c_n z^{n-1}$.

Cor 1: If $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ converges on $B_R(a)$, then g is cx diff'ble on $B_R(a)$ and $g'(z) = \sum_{n=0}^{\infty} n c_n (z-a)^{n-1}$.

Proof: $g = f \circ h$ where $h(z) = z - a$. Apply chain rule. $\square$

Cor 2: g as above is infinitely diff'ble

Proof: Apply Cor 1 to $g'(z)$, induce

Ex: Evaluate $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad |z| < 1$$

$$\sum_{n=0}^{\infty} n z^n = z \cdot \frac{d}{dz}\left(\frac{1}{1-z}\right) \quad |z|<1$$
$$= z/(1-z)^2$$

$$\sum_{n=0}^{\infty} n^2 z^n = z \cdot \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) \quad |z|<1$$
$$= \frac{(z)(1+z)}{(1-z)^3}$$

So for $z = \frac{1}{2}$: $\frac{\frac{1}{2}(\frac{3}{2})}{(\frac{1}{2})^3} = 6$

4.2) Exp and log

$\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges $\forall z \in \mathbb{C}$

Proof: by ratio test, as $\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} \to 0$. $\square$

Define: $\exp: \mathbb{C} \to \mathbb{C}$ by $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

By Friday's Thm

① $\exp'(z) = \sum_{n=0}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z)$

$k = n-1$

② $\exp(0) = 1$

L17.2

Would like to prove that

3 $\exp(z+w) = \exp(z)\exp(w)$

Idea: Fix w, set $f(z) = \exp(z+w)$, $g(z) = \exp(w)\exp(z)$.

Then $f'(z) = f(z)$, $g'(z) = g(z)$, so if

$$h(z) = \frac{f(z)}{g(z)} \quad \text{then} \quad h' = \frac{f'g - g'f}{g^2} = \frac{fg - gf}{g^2} = 0$$

$\Rightarrow h \equiv C \Rightarrow \exp(z+w) = C\exp(w)\exp(z)$

Problem: Need $g(z)$ never to be 0.

Lemma 1: $\exists\, \rho > 0$ s.t. $\exp(z) \neq 0$ for $z \in B_\rho(0)$

Proof: exp is diff'ble, hence continuous. So $\exists\, \rho > 0$ s.t.

$|\exp(z) - \exp(0)| < 1$ if $|z - 0| < \rho$

$\Rightarrow |\exp(z) - 1| < 1$ if $z \in B_\rho(0)$

$\Rightarrow 0 < |\exp(z)|$ if $z \in B_\rho(0)$. $\square$

Lemma 2: If $\exp(z_0) \neq 0$ then $\exp(z) \neq 0$ $\forall\, z \in B_\rho(z_0)$ where ρ is as in Lemma 1.

N.B. ρ does not depend on z_0

Proof: Define $f, g : B_\rho(0) \longrightarrow \mathbb{C}$ by $f(w) = \exp(w + z_0)$, $g(w) = \exp(z_0)\exp(w)$

Then $f' = f$ and $g' = g$ on $B_\rho(0)$ and, by Lemma 1, $g(w)$ is never 0

So $h : B_\rho(0) \longrightarrow \mathbb{C}$ given by $h(w) = \frac{f(w)}{g(w)}$ is diff'ble, and

$$h'(w) = \frac{f'g - g'f}{g^2} \equiv 0 \quad \text{on } B_\rho(0)$$

So $h(w) \equiv C$ for some $C \in \mathbb{C}$ $\Rightarrow$ $\exp(w + z_0) = C\exp(z_0)\exp(w)$

$w = 0$ gives $\exp(z_0) = \exp(z_0) \cdot C \Rightarrow C = +1$

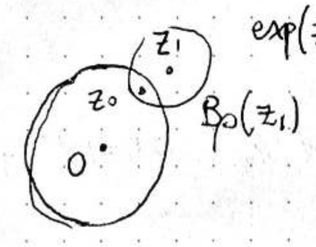
i.e. $\exp(w + z_0) = \exp(w)\exp(z_0)$ for $w \in B_\rho(0)$. If $z \in B_\rho(z_0)$, $w = z - z_0 \in B_\rho(0)$, so $\exp(z) = \exp(z_0)\exp(w) \neq 0$. $\square$

Corollary: $\exp(z)$ is never 0

Proof: Let $A = \{ |z| : \exp(z) = 0 \}$

If $A \neq \emptyset$, A is bounded below by ρ, so it has a infimum r.

L17.3



$\exp(z_1) = 0$

Since $r = \inf(A)$, $\exists\, z_1 \in \mathbb{C}$ with $\exp(z_1) = 0$, $|z_1| < \rho/2 + r$

Let $z_0 = (r - \rho/2) \cdot \frac{z_1}{|z_1|}$. Then $|z_0| = r - \rho/2 \notin A$ and so $\exp(z_0) \neq 0$.

But $|z_1 - z_0| = |z_1| \cdot \left(1 - \frac{|z_0|}{|z_1|}\right) = \big||z_1| - (r - \rho/2)\big| < \rho$

i.e. $z_1 \in B_\rho(z_0)$. By Lemma 2, $\exp(z_1) \neq 0$. ※

So $A = \emptyset$.

<u>Prop</u>: $\exp(z+w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$

<u>Proof</u>: Argue exactly as in Lemma 2. □

<u>Corollary</u>: If $n \in \mathbb{Z}$, $\exp(nz) = \exp(z)^n$

<u>Summary</u>:
1) $\exp(0) = 1$
2) exp is diff'ble and $\exp' = \exp$
3) $\exp(z+w) = \exp(z)\exp(w)$
4) $\exp(nz) = \exp(z)^n$

proof by rearranging neater

<u>$\exp: \mathbb{R} \to \mathbb{R}$</u>

If $x \in \mathbb{R}$, $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in \mathbb{R}$

<u>Lemma</u>: $\exp(\mathbb{R}) = (0, \infty)$

<u>Proof</u>: If $x \geq 0$, $\exp(x) \geq 1 + x \geq 1$

If $M \in [1, \infty)$, then $\exp(M) \geq 1 + M > M$

and $\exp(0) = 1$, so by the intermediate value theorem $\exists\, c \in (0, \infty)$ s.t. $\exp(c) = M \Rightarrow \exp([0, \infty)) = [1, \infty)$

Since $\exp(-x) = 1/\exp(x)$, $\exp((-\infty, 0]) = (0, 1]$. □

If $x \in \mathbb{R}$, $\exp'(x) = \exp(x) > 0$

$\Rightarrow$ exp is strictly increasing

L17.4

$\Rightarrow \exp: \mathbb{R} \to (0,\infty)$ is a bijection

Def: $\log: (0,\infty) \to \mathbb{R}$ is the inverse of exp (called the (natural) logarithm)

Properties:

1) a) $\log(ab) = \log(a) + \log(b)$

b) $\log(a^n) = n\log(a)$ for $n \in \mathbb{Z}$

Proof: a) $\log(ab) = \log(\exp(\log(a))\exp(\log(b)))$

$= \log(\exp(\log a + \log b))$

$= \log a + \log b$

b) is similar $\square$

2) log is diff'ble and $\log'(x) = \frac{1}{x}$

Proof: This follows immediately from inverse function theorem

$$\frac{d}{dt}\log(t) = \frac{1}{\exp'(\log(t))} = \frac{1}{\exp(\log(t))} = \frac{1}{t}$$ $\square$

Exponentials with other bases: If $a \in (0,\infty)$ define

$$p_a(x) = \exp(x \log a)$$

If $x \in \mathbb{Q}$, $x = p/q$, then

$$(p_a(x))^q = \exp(p/q \log(a))^q = \exp(p\log(a))$$
$$= \exp(\log(a))^p = a^p$$

i.e. $p_a(x) = a^{p/q}$.

For $x \in \mathbb{R}$, define a^x to be $p_a(x) = \exp(x \log a)$

This agrees with our previous definition for $x \in \mathbb{Q}$ and is a continuous extension of that map to $\mathbb{R}$.

a^x is diff'ble with $\frac{d}{dx}(a^x) = \log(a)\, a^x$

L18.1

4.2) Cont'd

Proposition: $\exp(z) \neq 0$ for $z \in \mathbb{C}$

Easy Proof: Define $f(z) = \exp(z)\exp(-z)$.

Then $f'(z) = \exp(z)\exp(-z) - \exp(z)\exp(-z) = 0$

$\Rightarrow f$ is constant

$f(0) = \exp(0) \cdot \exp(0) = 1$

$\Rightarrow \exp(z)\exp(-z) \equiv 1 \Rightarrow \exp(z) \neq 0.$ $\square$

Def: $e = \exp(1) \in (0, \infty)$

Recall that if $a \in (0, \infty)$, we defined

$a^x = \exp(x \log a)$

In particular $\boxed{\exp(x) = \exp(x \log e) = e^x.}$

4.3) sin and cos

Def: Will define $\sin, \cos : \mathbb{C} \to \mathbb{C}$ by

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \quad \cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Properties: 0) $\sin(0) = 0$, $\cos(0) = 1$

1) $\sin(-z) = -\sin(z)$, $\cos(-z) = \cos(z)$

2) $e^{iz} = \cos(z) + i\sin(z)$

3) $\sin^2(z) + \cos^2(z) = 1 \quad \forall z \in \mathbb{C}$

4) $$\left.\begin{aligned} \sin(z+w) &= \sin(z)\cos(w) + \cos(z)\sin(w) \\ \cos(z+w) &= \cos(z)\cos(w) - \sin(z)\sin(w) \end{aligned}\right\} \forall z, w \in \mathbb{C}$$

5) sin and cos are cx diff'ble with

$$\frac{d}{dz}\sin(z) = \cos(z), \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

6) We have

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

7) If $x \in \mathbb{R}$, $\sin x, \cos x \in \mathbb{R}$

L18.2

<u>Proofs:</u> 0), 1) Obvious.

2) $\cos z + i \sin z = \frac{1}{2}(e^{iz}+e^{-iz}) + \frac{1}{2}(e^{iz}-e^{-iz}) = e^{iz}$ ✓

3) $1 = e^{iz} \cdot e^{-iz} = (\cos z + i\sin z)(\cos z - i \sin z) = \cos^2 z + \sin^2 z$

use odd, even

4) $\sin z \cos w + \cos z \sin w$

$$= \frac{1}{4i}\left[(e^{iz}-e^{-iz})(e^{iw}+e^{-iw}) + (e^{iz}+e^{-iz})(e^{iw}-e^{-iw})\right]$$

$$= \frac{1}{4i}\left[2e^{i(z+w)} - 2e^{-i(z+w)}\right] = \sin(z+w)$$

Proof for cos very similar.

5) $(\sin z)' = \frac{1}{2i}(e^{iz}-e^{-iz})' = \frac{1}{2i}(ie^{iz}+ie^{-iz}) = \cos z$

similarly for cos.

6) $e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \qquad e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!}$

$\Rightarrow \sin z = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{z^k}{k!}\left[i^k - (-i)^k\right]$

$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \cdot (-1)^n$

$k = 2n+1$

k	$i^k - (-i)^k$
0	0
1	$2i$
2	0
3	$-2i$
4	0
⋮	

Proof for cos similar.

7) Follows immediately from 6)

<u>Real Values</u> Now study $\sin, \cos : \mathbb{R} \to \mathbb{R}$

1) For $x \in \mathbb{R}$, $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$

<u>Proof:</u> Use 2) above.

2) <u>Proposition:</u> $\exists\, \kappa \in \mathbb{R}$ s.t.

1) $\kappa > 0$

2) $\cos \kappa = 0$, $\sin \kappa = 1$

3) $0 < \cos x < 1$, $0 < \sin x < 1$ for $x \in (0, \kappa)$

L8.3

Proof: Observe that for $n>1$, $\frac{2^{2n+1}}{(2n+2)!} < \frac{2^n}{(2n)!}$ since

$\frac{2^2}{(2n+1)(2n+2)} < 1$. Consider

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\cos(2) = \left(1 - \frac{2^2}{2!} + \frac{2^4}{4!}\right) + \left(-\frac{2^6}{6!} + \frac{2^8}{8!}\right) + \cdots$$

$$= \left(1 - 2 + \frac{2}{3}\right) + \left(-\frac{2^6}{3!} + \frac{2^8}{8!}\right) + \cdots$$

Each term in brackets is less than 0, so $\cos(2) < 0$.

$\cos(0) = 1$ and cos is continuous, so by the intermediate value theorem, $\exists\ c \in (0,2)$ with $\cos(c) = 0$.

Let $A = \{x \in [0,\infty) : \cos x = 0\}$.

A is non-empty for $c \in A$. It's trivially bdd below.

Define $\kappa = \inf A$.

Clearly $\kappa \geq 0$. Since $\kappa = \inf A$, we can find $x_n \in A$ with $x_n \to \kappa$.

cos is continuous so $\cos(x_n) \to \cos(\kappa) \Rightarrow \cos(\kappa) = 0$.

$\cos(0) = 1$, ~~so by continuity~~ so $\kappa > 0$.

If $x \in (0,\kappa)$ with $\cos(x) \leq 0$, then by IVT $\exists\ c' \in (0,x)$ with $\cos(c') = 0$, contradicting $\kappa \leq c'$.

Hence $\cos(x) > 0$ for $x \in (0,\kappa)$.

Observe that $(\sin x)' = \cos x > 0$ for $x \in (0,\kappa)$. So sin is strictly increasing on $[0,\kappa] \Rightarrow \sin x > \sin(0) = 0$ for $x \in (0,\kappa]$.

Then $\sin(\kappa) > 0$, and since $\sin^2(\kappa) + \cos^2(\kappa) = 1$, get $\sin(\kappa) = 1$.

$\sin x, \cos x > 0$ for $x \in (0,\kappa) \Rightarrow \sin x, \cos x < 1$ for $x \in (0,\kappa)$. □

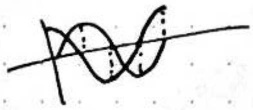
Def: $\pi = 2\kappa$ i.e. $\kappa = \frac{\pi}{2}$

3) Then $\sin(x + \frac{\pi}{2}) = \sin x \cos(\frac{\pi}{2}) + \cos x \sin(\frac{\pi}{2}) = \cos(x)$

$\cos(x + \frac{\pi}{2}) = \cos x \cos(\frac{\pi}{2}) - \sin x \sin(\frac{\pi}{2}) = -\sin(x)$

L8.4

	0		$\pi/2$		π		$3\pi/2$		2π
$\sin x$	0	>0	1	>0	0	<0	-1	<0	0
$\cos x$	1	>0	0	<0	-1	<0	0	>0	1



Addition formulae

<u>Cor:</u> 1) $\sin(x+2\pi) = \sin(x)$, $\cos(x+2\pi) = \cos(x)$

2) $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

3) $\sin x = 0$ iff $x = (n + \cancel{\tfrac{1}{2}})\pi$ for some $n \in \mathbb{Z}$

$\cos x = 0$ iff $x = (n+\tfrac{1}{2})\pi$ for some $n \in \mathbb{Z}$

4) $e^{x+iy} = 1 \iff y = 2n\pi$ for some $n \in \mathbb{Z}$, $x = 0$

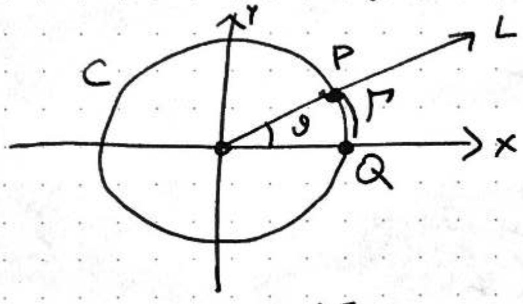
<u>Proof of 4)</u> $e^{x+iy} = e^x(\cos y + i \sin y) = 1$

Take norms: $|e^{x+iy}| = e^x = 1 \Rightarrow x = 0$ exp is injective on $\mathbb{R}$

$e^{iy} = 1 \Rightarrow \cos y = 1$, $\sin y = 0$ follows from previous. □

<u>Relation to triangles</u>

$C = \{(x,y) : x^2 + y^2 = 1\}$



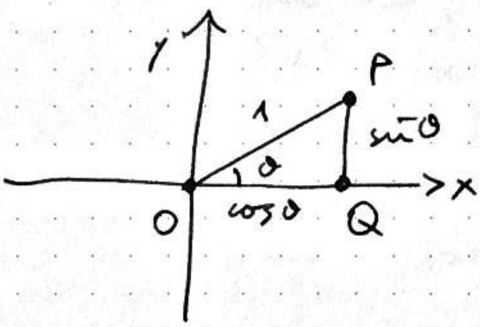
L makes an angle θ with positive x axis if the arc Γ has length θ.

$Q = (1,0)$, so Γ is parametrised by $\Gamma(t) = (\cos t + \sin t)$

Length of Γ for $t \in [0, \theta]$ is

$$\int_0^\theta \|\Gamma'(t)\|\, dt = \theta$$

i.e. if Γ has length θ the point P is $(\cos\theta, \sin\theta)$



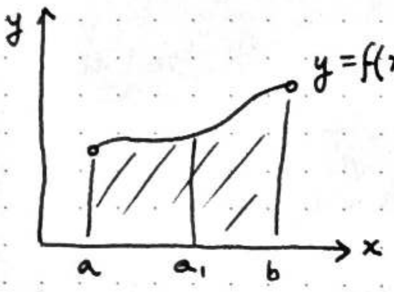
<u>It all works out.</u>

V) Integration

5.1) The Riemann Integral

Suppose $a, b \in \mathbb{R}$, $a < b$, and $f : [a,b] \longrightarrow \mathbb{R}$.

$\int_a^b f(x)\,dx$ should be "the area under graph $y = f(x)$"



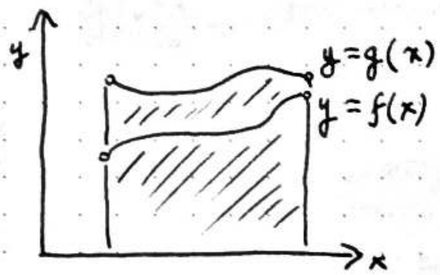
Wish List

$\int_a^b f(x)\,dx$ should satisfy

1) $\int_a^b c\,dx = c(b-a)$ <rectangle>

2) If $a_0 < a_1 < a_2$, then
$$\int_{a_0}^{a_2} f(x)\,dx = \int_{a_0}^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx$$

3) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

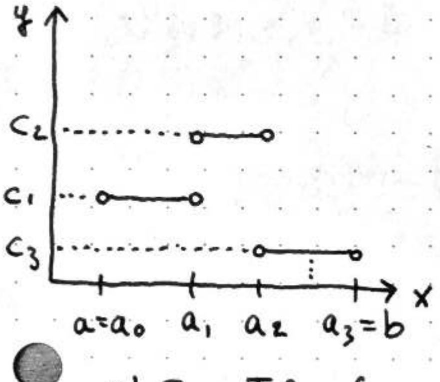


Piece-wise constant functions

Suppose $A \subset [a,b]$ of the form
$$A = \{a_0, a_1, \dots, a_n\}$$
where $a = a_0 < a_1 < \dots < a_n = b$.

(A is sometimes called a <u>dissection</u> of $[a,b]$)

<u>Def</u>: $f : [a,b] \longrightarrow \mathbb{R}$ is <u>piecewise constant wrt A</u> if $\exists\, c_1, \dots, c_n \in \mathbb{R}$ s.t. $f(x) = c_i$ for $x \in (a_{i-1}, a_i)$.



(No restriction on $f(a_i)$)

If so, say $f \in \mathcal{P}_A([a,b])$.

We say f is <u>piecewise constant</u> if it is piece-wise constant wrt some A.

We write $f \in \mathcal{P}([a,b])$.

<u>N.B.</u> If $f \in \mathcal{P}_A([a,b])$ and $A' \subset [a,b]$ is finite, then $f \in \mathcal{P}_{A \cup A'}([a,b])$.

<u>Shorthand</u> In this lecture, write $\mathcal{P}_A$ for $\mathcal{P}_A([a,b])$ and $\mathcal{P}$ for $\mathcal{P}([a,b])$.

<u>Def</u>: If $f \in \mathcal{P}_A$, then
$$I_A(f) = \sum_{i=1}^{n} c_i (a_i - a_{i-1})$$
where $f(x) = c_i$ for $x \in (a_{i-1}, a_i)$.

(Note: We want $I_A(f) = \int_a^b f(x)\,dx$. This formula is forced on us by properties 1+2 on the Wish List.)

<u>Lemma</u>: Suppose $f \in \mathcal{P}_A$ and $A' \subset [a,b]$ is finite. Then
$$I_A(f) = I_{A \cup A'}(f).$$

<u>Proof</u>: Spose $A' = \{a'\}$ where $a_{j-1} < a' < a_j$. Then
$$\begin{aligned} I_{A\cup A'}(f) &= \sum_{i \neq j} c_i (a_i - a_{i-1}) + c_j (a' - a_{j-1}) + c_j (a_j - a') \\ &= \sum_{i \neq j} c_i (a_i - a_{i-1}) + c_j (a_j - a_{j-1}) \\ &= I_A. \end{aligned}$$

General case follows by induction on $|A'|$. $\square$

<u>Cor</u>: If $f \in \mathcal{P}_A$ and $f \in \mathcal{P}_{A'}$, then
$$I_A = I_{A'}.$$

<u>Proof</u>: $I_A(f) = I_{A \cup A'}(f) = I_{A'}(f)$.

<u>Def</u>: If $f \in \mathcal{P}$, we define
$$I(f) = I_A(f) \text{ where } f \in \mathcal{P}_A.$$

Cor $\Rightarrow$ all choices of A give same answer.

<u>Notation</u>: If $f, g : [a,b] \to \mathbb{R}$, we say $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [a,b]$.

$\leq$ is a partial order: if $f \leq g$ and $g \leq h$ then $f \leq h$

<u>Lemma</u>: If $f, g \in \mathcal{P}_A$ and $f \leq g$, then
$$I_A(f) \leq I_A(g).$$

<u>Proof</u>: Say $f(x) = c_i$, $g(x) = d_i$ for $x \in (a_{i-1}, a_i)$

Then $f \leq g \Rightarrow c_i \leq d_i$ for all i.

Then $c_i(a_i - a_{i-1}) \leq d_i(a_i - a_{i-1})$

$$\Rightarrow I_A(f) = \sum_i c_i(a_i - a_{i-1}) \leq \sum_i d_i(a_i - a_{i-1}) = I_A(g). \qquad \square$$

<u>Cor</u>: If $f, g \in \mathcal{P}$ and $f \leq g$, then $I(f) \leq I(g)$.

<u>Proof</u>: Spose $f \in \mathcal{P}_A$, $g \in \mathcal{P}_{A'}$. Then

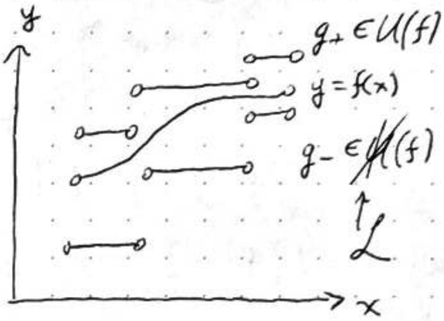
$$I(f) = I_A(f) = I_{A \cup A'}(f) \leq I_{A \cup A'}(g) = I(g). \qquad \square$$

<u>The integral</u>:

<u>Def</u>: $f : [a, b] \to \mathbb{R}$ is <u>bounded</u> if $\exists M \in \mathbb{R}$ s.t.

$-M \leq f(x) \leq M$ for all $x \in [a, b]$

i.e. $g_{-M} \leq f \leq g_M$ where $g_{\pm M}(x) \equiv \pm M$.



If $f : [a, b] \to \mathbb{R}$ is bounded, let

$\mathcal{L}(f) = \{ g \in \mathcal{P} : g \leq f \}$

$\mathcal{U}(f) = \{ g \in \mathcal{P} : f \leq g \}$

Note $g_{-M} \in \mathcal{L} \neq \emptyset$,

$g_{+M} \in \mathcal{U}(f) \neq \emptyset$.

Now let $L(f) = \{ I(g) : g \in \mathcal{L}(f) \} \neq \emptyset$,

$U(f) = \{ I(g) \neq : g \in \mathcal{U}(f) \} \neq \emptyset$.

<u>Note</u>: if $g_- \in \mathcal{L}(f)$, $g_+ \in \mathcal{U}(f)$, then

$g_- \leq f \leq g_+ \Rightarrow g_- \leq g_+$

$\Rightarrow I(g_-) \leq I(g_+)$

i.e. if $\alpha \in L(f)$ and $\beta \in U(f)$ then $\alpha \leq \beta$

<u>Def</u>: If $f : [a, b] \to \mathbb{R}$ is bounded,

$\ell(f) = \sup L(f)$

$u(f) = \inf U(f)$

are called the <u>lower</u> and <u>upper</u> integrals of f.

<u>Idea</u>: if $I(f) = \int_a^b f(x)\,dx$ makes sense, then property 3 on wish list

$\Rightarrow$ if $g_- \in \mathcal{L}(f)$, $g_+ \in \mathcal{U}(f)$ so $g_- \leq f \leq g_+$ then

$$\int_a^b g_-\,dx \leq \int_a^b f\,dx \leq \int_a^b g_+\,dx$$

L19.4

$I(g_-) \leq I(f) \leq I(g_+)$

$\Rightarrow \alpha \in L(f), \beta \in U(f)$, then

$\alpha \leq I(f) \leq \beta$

$\Rightarrow I(f)$ is an upper bound for $L(f)$

" lower " $U(f)$

$\Rightarrow \ell(f) \leq I(f) \leq u(f)$

Lemma: $\ell(f) \leq u(f)$

Proof: Choose $x_n^- \in L(f)$ with $x_n^- \longrightarrow \ell(f)$,

$x_n^+ \in U(f)$ " $x_n^+ \longrightarrow u(f)$.

Then $x_n^+ - x_n^- \geq 0$ and $x_n^+ - x_n^- \longrightarrow u(f) - \ell(f)$

$\Rightarrow u(f) - \ell(f) \geq 0$. $\square$

Def: If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded, we say f is Riemann integrable if $u(f) = \ell(f)$. In this case we define

$$\int_a^b f(x)\,dx = u(f) = \ell(f).$$

5.2) Properties of the Integral

Recap: $\mathcal{P} = \mathcal{P}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ is piecewise constant} \}$

If $f : [a,b] \to \mathbb{R}$ is bounded

$\mathcal{L}(f) = \{ g \in \mathcal{P} : g \leq f \} \subset \mathcal{P}$

$\mathcal{U}(f) = \{ g \in \mathcal{P} : g \geq f \} \subset \mathcal{P}$

$L(f) = \{ I(g) : g \in \mathcal{L} \} \subset \mathbb{R}$

$U(f) = \{ I(g) : g \in \mathcal{U} \} \subset \mathbb{R}$

$\ell(f) = \sup L(f) \in \mathbb{R}$

$u(f) = \inf U(f) \in \mathbb{R}$

Lemma: $\ell(f) \leq u(f)$

Def: f is integrable if $\ell(f) = u(f)$

if so $\int_a^b f(x)\,dx = \ell(f) = u(f)$.

Ex: If $f \in \mathcal{P}$, then

$f \in \mathcal{L}(f) \Rightarrow I(f) \in L(f) \Rightarrow \ell(f) \geq I(f)$

$f \in \mathcal{U}(f) \Rightarrow I(f) \in U(f) \Rightarrow u(f) \leq I(f)$

so $\ell(f) \geq I(f) \geq u(f) \geq \ell(f)$

$\Rightarrow$ all are equalities

so f is integrable and $\int_a^b f(x)\,dx = I(f)$.

Ex: $f : [0,1] \to \mathbb{R}$

$$x \to \begin{cases} 0 \text{ if } x \in \mathbb{Q} \\ 1 \text{ if } x \notin \mathbb{Q} \end{cases}$$

If $g \in \mathcal{P}$, $g \in \mathcal{P}_A$ i.e. $g(x) = c_i$ for $x \in (a_{i-1}, a_i)$

If $g \leq f$, since (a_{i-1}, a_i) contains a rational, then $c_i \leq 0$

i.e. $I(g) = \sum c_i (a_i - a_{i-1}) \leq 0$.

So $\ell(f) \leq 0$.

Similarly, if $g \geq f$, (a_{i-1}, a_i) contains an irrational, so $c_i \geq 1$

i.e. $I(g) \geq I(1) = 1$ so $u(f) \geq 1$.

$\Rightarrow f$ is not Riemann integrable

L20.2

Criteria for integrability

Lemma 1: If $f:[a,b] \to \mathbb{R}$ is bounded, then

a) f is integrable and $\int_a^b f(x)\,dx = \alpha$

if and only if

b) for every $\varepsilon > 0$ we can find $f_- \in \mathcal{L}(f)$, $f_+ \in \mathcal{U}(f)$ such that $I(f_-) > \alpha - \varepsilon$ and $I(f_+) < \alpha + \varepsilon$.

Proof: If a) is true, given $\varepsilon > 0$, since

$\alpha = \ell(f) = \sup\{I(g) : g \in \mathcal{L}(f)\}$

$\Rightarrow \exists\, f_- \in \mathcal{L}(f)$ s.t. $I(f_-) > \alpha - \varepsilon$

otherwise $\alpha - \varepsilon$ would be an upper bound

Similarly get $f_+ \in \mathcal{U}(f)$ s.t. $I(f_+) < \alpha + \varepsilon$, so b) holds.

If b) holds, then for any $\varepsilon > 0$,

$$\begin{aligned} \ell(f) &\geq \alpha - \varepsilon \\ u(f) &\leq \alpha + \varepsilon \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ell(f) &\geq \alpha \\ u(f) &\leq \alpha \end{aligned}$$

so $\ell(f) \geq \alpha \geq u(f) \geq \ell(f)$

$\Rightarrow \ell(f) = u(f) = \alpha$ i.e. f is integrable with $\int_a^b f(x)\,dx = \alpha$. $\square$

Lemma 1': If $f:[a,b] \to \mathbb{R}$ is bounded, then

a) f is integrable

if and only if

b) for every $\varepsilon > 0$, $\exists\, f_- \in \mathcal{L}(f)$, $f_+ \in \mathcal{U}(f)$ such that $I(f_+) - I(f_-) < \varepsilon$.

Proof: If a) holds, Let $\alpha = \int_a^b f(x)\,dx$.

By Lemma 1, $\exists\, f_- \in \mathcal{L}(f)$, $f_+ \in \mathcal{U}(f)$ such that

$I(f_-) > \alpha - \varepsilon/2$, $I(f_+) < \alpha + \varepsilon/2$

$\Rightarrow I(f_+) - I(f_-) < \varepsilon$ so b) holds.

If b) holds, then given $\varepsilon > 0$, choose f_+ and f_- as in statement

Then $\ell(f) \geq I(f_-)$ and $u(f) \leq I(f_+)$

so $u(f) - \ell(f) \leq I(f_+) - I(f_-) < \varepsilon$.

Since ε was arbitrary, $u(f) = \ell(f)$. $\square$

L20.3

<u>Prop</u>: If $f:[a,b] \to \mathbb{R}$ is increasing, then f is integrable.

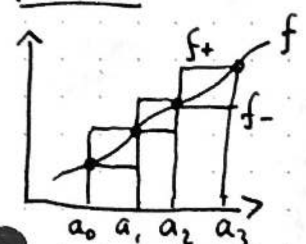
<u>Proof</u>: Suppose A is a dissection of $[a,b]$. Define $f_-:[a,b] \to \mathbb{R}$ by $f_-(x) = f(a_{i-1})$ if $x \in [a_{i-1}, a_i)$.

Clearly, $f_- \in \mathcal{P}$ and $f_- \leq f$ since f is increasing.

Similarly, define $f_+ \in \mathcal{P}$ by $f_+(x) = f(a_i)$ if $x \in (a_{i-1}, a_i]$.

Then $f_+ \geq f$ by same reasoning.

<u>Picture</u>:



So $I(f_+) - I(f_-) = \sum_{i=1}^{n} \left[f(a_i) - f(a_{i-1})\right](a_i - a_{i-1})$

Let $\delta = \max\{a_i - a_{i-1}\}$. Then

$$I(f_+) - I(f_-) \leq \sum_{i=1}^{n} \left[f(a_i) - f(a_{i-1})\right]\delta$$

$$= \delta(f(a_n) - f(a_0))$$

i.e. $I(f_+) - I(f_-) \leq \delta(f(a_n) - f(a_0))$

Pick A such that $\delta < \dfrac{\varepsilon}{f(a_n) - f(a_0)}$ (assuming f not constant)

where $\varepsilon > 0$ is given.

Done by Lemma 1'. $\square$

<u>Properties of piece-wise constant functions</u>

<u>Lemma</u>: Suppose $f, g \in \mathcal{P}$. Then

1) $f + g \in \mathcal{P}$ and $I(f) + I(g) = I(f+g)$,
2) if $c \in \mathbb{R}$, then $cf \in \mathcal{P}$ and $cI(f) = I(cf)$,
3) if $h(x) = \max\{0, f(x)\}$ then $h \in \mathcal{P}$,
4) if $J \subset [a,b]$ is a closed interval, then $f|_J \in \mathcal{P}(J)$. If $f \geq 0$, $I(f) \geq I(f|_J)$,
5) if $a < a' < b$, then $I(f) = I(f|_{[a,a']}) + I(f|_{[a',b]})$.

<u>Proof</u>: If $f \in \mathcal{P}_A$, $g \in \mathcal{P}_{A'}$, then $f, g \in \mathcal{P}_{A \cup A'}$ so wlog $f, g \in \mathcal{P}_A$. Then

$f(x) = c_i$ if $x \in (a_{i-1}, a_i)$

$g(x) = d_i$ " "

L20.4

$\Rightarrow (f+g)(x) = c_i + d_i$ if $x \in (a_{i-1}, a_i]$

$\Rightarrow f+g \in \mathcal{P}_A$, and

$$I_A(f+g) = \sum (c_i + d_i)(a_i - a_{i-1})$$
$$= \sum c_i (a_i - a_{i-1}) + \sum d_i (a_i - a_{i-1})$$
$$= I_A(f) + I_A(g).$$

2) is very similar to 1)

3) If $f \in \mathcal{P}_A$ as above, then

$\cancel{g}\, h(x) = \max(0, c_i)$ if $x \in (a_{i-1}, a_i]$

$\Rightarrow \cancel{g}\, h \in \mathcal{P}_A \Rightarrow h \in \mathcal{P}$

4) Obvious that $f|_J \in \mathcal{P}(J)$ and wlog $f \in \mathcal{P}_A$, where $J = [a_j, a_k]$. Then $I_A(f) = \sum_{i=1}^{n} c_i (a_i - a_{i-1})$.

If $f \geq 0$, $c_i \geq 0$ so $I_A(f) \geq \sum_{i=j+1}^{k} c_i (a_i - a_{i-1}) = I(f|_J)$.

5) wlog $a' \in A$. Then $a' = a_j$.

$$I_A(f) = \sum_{i=1}^{n} c_i (a_i - a_{i-1}) = \sum_{i=1}^{j} c_i (a_i - a_{i-1}) + \sum_{i=j+1}^{n} c_i (a_i - a_{i-1})$$
$$= I(f|_{[a,a']}) + I(f|_{[a',b]}).$$

$\square$

5.2) Cont'd

Recall:

Lemma 1': $f:[a,b] \to \mathbb{R}$ is integrable iff for every $\varepsilon > 0$, there exist $f_- \in \mathcal{L}(f)$ and $f_+ \in \mathcal{U}(f)$ with $I(f_+) - I(f_-) < \varepsilon$.

If $f:[a,b] \to \mathbb{R}$ is continuous, f is bounded by the maximum/minimum value theorem.

Goal: prove f is integrable, using Lemma 1'.

Def: If A is a dissection of $[a,b]$ define $f_A^{\pm} \in \mathcal{P}_A$ by
$$f_A^{\pm}(x) = c_i^{\pm} \quad \text{for } x \in (a_{i-1}, a_i)$$
where $c_i^- = \min\{f(x) : x \in [a_{i-1}, a_i]\}$ $\qquad [f_A^{\pm}(a_i) = f(a_i)]$

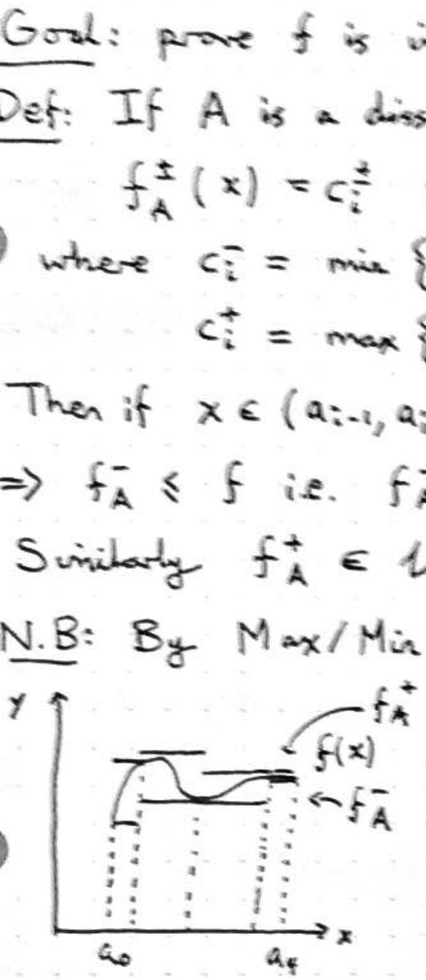
$c_i^+ = \max\{f(x) : x \in [a_{i-1}, a_i]\}$.

Then if $x \in (a_{i-1}, a_i)$, $f(x) \geq c_i^-$

$\Rightarrow f_A^- \leq f$ i.e. $f_A^- \in \mathcal{L}(f)$.

Similarly $f_A^+ \in \mathcal{U}(f)$.

N.B: By Max/Min Value Thm, $c_i^{\pm} = f(x_i^{\pm})$ for $x_i^{\pm} \in [a_{i-1}, a_i]$



f_A^- is best estimate in $\mathcal{L}(f) \cap \mathcal{P}_A$

f_A^+ " " " " $\mathcal{U}(f) \cap \mathcal{P}_A$

$$I(f_A^+) - I(f_A^-) = \sum_{i=1}^{n} (c_i^+ - c_i^-)(a_i - a_{i-1}) \quad (*)$$

Idea: If $\delta(A) := \max\{a_i - a_{i-1}\}$ is small enough, then $c_i^+ - c_i^-$ will be arbitrarily small.

Theorem (Uniform Continuity): If $f:[a,b] \to \mathbb{R}$ is continuous, then for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

f continuous: I give you $\varepsilon > 0$ and x, you give me δ

f uniformly cts: I give you $\varepsilon > 0$, you give me δ (works everywhere)

Proof: By contradiction. If conclusion of theorem doesn't hold, it means that there is some $\varepsilon_0 > 0$ such that no $\delta > 0$ works.

In particular, $\delta = \frac{1}{n}$ doesn't work for any n.

So $\forall n \in \mathbb{N}$, there are $x_n, y_n \in [a,b]$ s.t. $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

By Bolzano-Weierstrass, (x_n) has a convergent subsequence $x_{n_k} \longrightarrow x \in [a,b]$.

Now $x_n - y_n \longrightarrow 0 \Rightarrow x_{n_k} - y_{n_k} \longrightarrow 0$, so $y_{n_k} \longrightarrow x$ as well.

f is continuous, so $f(x_{n_k}) \longrightarrow f(x)$ and $f(y_{n_k}) \longrightarrow f(x)$.

So $f(x_{n_k}) - f(y_{n_k}) \longrightarrow 0$. This contradicts $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon_0 \ \forall k$. $\square$

<u>Theorem</u>: If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then f is integrable.

<u>Proof</u>: Use Lemma 1'. Given $\varepsilon > 0$, find a $\delta > 0$ with $|f(x) - f(y)| < \varepsilon/_{b-a}$ whenever $|x-y| < \delta$. (Thm)

Pick a dissection A with $\delta(A) < \delta$.

Then if $f_A^{\pm}$ are as above, and $c_i^{\pm} = f(x_i^{\pm})$ for $x_i^{\pm} \in [a_{i-1}, a_i]$, then $|x_i^+ - x_i^-| \leq |a_i - a_{i-1}| < \delta$ so $|c_i^+ - c_i^-| < \varepsilon/_{b-a}$.

By $(*)$,
$$I(f_A^+) - I(f_A^-) = \sum_{i=1}^{n} (c_i^+ - c_i^-)(a_i - a_{i-1})$$
$$\leq \frac{\varepsilon}{b-a} \sum_{i=1}^{n} a_i - a_{i-1}$$
$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon. \quad \square$$

<u>Properties of the Integral</u>

Notation: $\mathcal{I} = \{ f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ integrable} \}$.

<u>Prop</u>: If $f, g \in \mathcal{I}$, then

1) If $f \leq g$, $\int_a^b f\,dx \leq \int_a^b g\,dx$

2) $f+g \in \mathcal{I}$ and $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$

3) If $c \in \mathbb{R}$, then $cf \in \mathcal{I}$ and $\int_a^b cf\,dx = c\int_a^b f\,dx$

4) If $h(x) := \max(0, f(x))$, then $h \in \mathcal{I}$

5) ~~If~~ $|f| \in \mathcal{I}$ and $\int_a^b |f|\,dx \geq \left|\int_a^b f\,dx\right|$.

L21.3

<u>Proof</u> Let $\alpha = \int_a^b f\,dx$, $\beta = \int_a^b g\,dx$.

1) $f \leq g$, so if $f_- \in \mathcal{L}(f)$, $f_- \in \mathcal{L}(g)$

$$\Rightarrow L(f) \subset L(g)$$
$$\Rightarrow \ell(f) \leq \ell(g)$$
$$\Rightarrow \int_a^b f\,dx = \int_a^b g\,dx.$$

2) By Lemma 1, we can find, given $\varepsilon > 0$,

$f_- \in \mathcal{L}(f)$ with $I(f_-) > \alpha - \varepsilon/2$

$f_+ \in \mathcal{U}(f)$ with $I(f_+) < \alpha + \varepsilon/2$.

Define g_-, g_+ analogously.

So $f_- + g_- \in \mathcal{L}(f+g)$ and $I(f_- + g_-) = I(f_-) + I(g_-) > \alpha + \beta - \varepsilon$

Similarly, $f_+ + g_+ \in \mathcal{U}(f+g)$ and $I(f_+ + g_+) < \alpha + \beta + \varepsilon$

$\Rightarrow \int_a^b (f+g)\,dx = \alpha + \beta$ by Lemma 1.

3) Is similar, but easier.

4) Choose f_- and f_+ as in 2).

Let ~~h_- and~~ $h_\pm$ be $\max(0, f_\pm)$.

Then $h_+(x) - h_-(x) \leq f_+(x) - f_-(x)$; CHECK!

$h_+ \in \mathcal{U}(h)$, $h_- \in \mathcal{L}(h)$,

and $I(h_+) - I(h_-) = I(h_+ - h_-) \leq I(f_+ - f_-) = \underbrace{I(f_+) - I(f_-)}_{<\varepsilon}$

So $h \in \mathcal{I}$ by Lemma 1'.

5) Let $h_1 = \max(0, f)$, $h_2 = \max(0, -f)$.

Then $|f| = h_1 + h_2$, $f = h_1 - h_2$

$h_1, h_2 \in \mathcal{I}$ by 4) + 3), so $|f| \in \mathcal{I}$ by 2),

and
$$\int_a^b |f|\,dx = \int_a^b h_1\,dx + \int_a^b h_2\,dx$$
$$\geq \left| \int_a^b h_1\,dx - \int_a^b h_2\,dx \right|$$
$$= \left| \int_a^b f\,dx \right|$$

follows from $h_1, h_2 \geq 0$ so $\int h_1, \int h_2 \geq 0$ by 1)

using 2)

$\square$

5.2) Cont'd

Spose $f : [a,b] \to \mathbb{R}$.

Let $J = [a', b'] \subset [a,b]$.

Have restriction

$f|_J : [a', b'] \to \mathbb{R}$ given by $f|_J(x) = f(x)$.

<u>Lemma:</u> If $f : [a,b] \to \mathbb{R}$ is integrable, then so is $f|_J$.

<u>Proof:</u> Given $\varepsilon > 0$, can find

$f_- \in \mathcal{L}(f)$, $f_+ \in \mathcal{U}(f)$ s.t. $I(f_+) - I(f_-) < \varepsilon$.

Then $f_-|_J \in \mathcal{L}(f|_J)$, $f_+|_J \in \mathcal{U}(f|_J)$ and

$f_+ - f_- \geq 0$, so

$$\begin{aligned} I(f_+|_J) - I(f_-|_J) &= I((f_+ - f_-)|_J) \\ &\leq I(f_+ - f_-) \\ &< \varepsilon, \end{aligned}$$

so $f|_J$ is integrable by Lemma 1'. $\square$

Now suppose that $a < a' < b$.

Let $J_1 = [a, a']$, $J_2 = [a', b]$.

<u>Prop:</u> $f : [a,b] \to \mathbb{R}$ is integrable

$\Leftrightarrow$ $f|_{J_1}$ and $f|_{J_2}$ are integrable.

If so $\int_a^b f \, dx = \int_a^{a'} f \, dx + \int_{a'}^b f \, dx$.

<u>Proof:</u> If f is integrable, $f|_{J_i}$ are integrable by Lemma.

Conversely, spose $f|_{J_1}$, $f|_{J_2}$ are integrable, and let

$\alpha_i = \int_{J_i} f \, dx$.

By Lemma 1, given $\varepsilon > 0$, can find $f_-^i \in \mathcal{L}(f|_{J_i})$ and $f_+^i \in \mathcal{U}(f|_{J_i})$ s.t. $I(f_-^i) > \alpha_i - \varepsilon/2$ and $I(f_+^i) < \alpha_i + \varepsilon/2$.

Define $f_\pm(x) = \begin{cases} f_\pm^1(x) : x \leq a', \\ f_\pm^2(x) : x > a'. \end{cases}$

Then $f_- \in \mathcal{L}(f)$ and $f_+ \in \mathcal{U}(f)$.

Have $I(f_-) = I(f_-^1) + I(f_-^2) > \alpha_1 + \alpha_2 - \varepsilon$ and

L22.2

$I(f_+) = I(f_+^1) + I(f_+^2) < \alpha_1 + \alpha_2 + \varepsilon$.

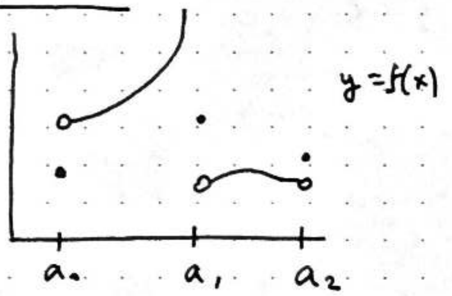
So f is integrable and

$$\int_a^b f \, dx = \alpha_1 + \alpha_2$$

by Lemma 1. $\square$

<u>Def</u>: $f : [a,b] \to \mathbb{R}$ is piecewise continuous if there's a dissection $a = a_0 < a_1 < \cdots < a_n = b$ of $[a,b]$ such that $f|_{(a_{i-1}, a_i)}$ is continuous for $i = 1, \ldots, n$.

<u>Picture</u>:



Note such an f need not be bounded.

<u>Prop</u>: If $f : [a,b] \to \mathbb{R}$ is bounded and piecewise continuous, then f is integrable.

<u>Proof</u>: By the previous proposition, suffices to prove that $f|_{[a_{i-1}, a_i]}$ is integrable for each i.

Given $\delta > 0$, let $a' = a_{i-1} + \delta$, $b' = a_i - \delta$.

Then $f|_{[a', b']}$ is continuous, hence integrable.

So, given $\eta > 0$ we can find $f_- \in \mathcal{L}(f|_{[a',b']})$ and $f_+ \in \mathcal{U}(f|_{[a',b']})$ with $I(f_+) - I(f_-) < \eta$. (Lemma 1')

f is bounded, so there exists some M s.t. $-M \leq f(x) \leq M$ for all $x \in [a_{i-1}, a_i]$.

Define $\tilde{f}_\pm(x) = \begin{cases} f_\pm(x) & \text{if } x \in [a', b'], \\ \pm M & \text{if } x \notin [a', b']. \end{cases}$

Then $\tilde{f}_- \cancel{(x)} \in \mathcal{L}(f|_{[a_{i-1}, a_i]})$, $\tilde{f}_+ \cancel{f} \in \mathcal{U}(f|_{[a_{i-1}, a_i]})$ and $I(\tilde{f}_+) - I(\tilde{f}_-) \leq \eta + 4\delta M$.

Given $\varepsilon > 0$, choose $\delta < \varepsilon / 8M$ then choose $\eta < \varepsilon/2$.

So f is integrable by Lemma 1'. $\square$

5.3) Fundamental Thm of Calculus

Suppose $f:[a,b]\to\mathbb{R}$ is continuous.

Then if $a\le x\le b$, $f:[a,x]\to\mathbb{R}$ is continuous, so integrable.

Define $F(x)=\int_a^x f(t)\,dt$.

<u>Prop:</u> F is diff'ble at all $x\in[a,b]$ and $F'(x)=f(x)$.

<u>Proof:</u> Fix x_0 and consider $F(x_0+h)-F(x_0)$. If $h>0$,

$$F(x_0+h)-F(x_0)=\int_a^{x_0+h}f(t)\,dt-\int_a^{x_0}f(t)\,dt$$

$$=\int_a^{x_0}f(t)\,dt+\int_{x_0}^{x_0+h}f(t)\,dt-\int_a^{x_0}f(t)\,dt$$

$$=\int_{x_0}^{x_0+h}f(t)\,dt.$$

Let $M(h)=\max\{f(t):t\in[x_0,x_0+h]\}$,

$m(h)=\min\{f(t):t\in[x_0,x_0+h]\}$.

Then $m(h)\le f(t)\le M(h)$ for $t\in[x_0,x_0+h]$.

So
$$\underbrace{\int_{x_0}^{x_0+h}m(h)\,dt}_{=\,hm(h)}\le\int_{x_0}^{x_0+h}f(t)\,dt\le\underbrace{\int_{x_0}^{x_0+h}M(h)\,dt}_{=\,hM(h)}$$

i.e. $$m(h)\le\frac{F(x_0+h)-F(x_0)}{h}\le M(h)$$

Given $\varepsilon>0$, since f cts at x_0, can find $\delta>0$ s.t. $|f(y)-f(x)|<\varepsilon$ whenever $|x-y|<\delta$, i.e. if $0<h<\delta$,

$f(x_0)-\cancel{\delta}\varepsilon<m(h)$ and $M(h)<f(x_0)+\varepsilon$.

$\Rightarrow$ if $0<h<\delta$, then
$$f(x_0)-\varepsilon\le\frac{F(x_0+h)-F(x_0)}{h}<f(x_0)+\varepsilon.$$

A similar argument works for $h<0$.

i.e. $$\lim_{h\to0}\frac{F(x_0+h)-F(x_0)}{h}=f(x_0).$$

$\square$

L22.4

<u>Thm (Fund Thm of Calculus)</u>: If $f:[a,b] \to \mathbb{R}$ is continuous, $F:[a,b] \to \mathbb{R}$ is diff'ble and $f(x) = F'(x)$ $\forall$ $x \in [a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>Proof</u>: Define

$$\bar{F}(x) = \int_a^x f(x)\,dx \quad \text{as above.}$$

So $\bar{F}'(x) = f(x) = F'(x)$.

So by Cor to MVT, $F(x) = \bar{F}(x) + C$ for a constant C.

Setting $x = a$ gives

$$F(a) = \int_a^a f(x)\,dx + C = C$$

and setting $x = b$ gives

$$F(b) = \int_a^b f(x)\,dx + C = \int_a^b f(x)\,dx + F(a). \qquad \square$$

L23.1

5.3) Cont'd

<u>Convention</u>: If $a < b$, define

$$-\int_a^b f(t)\,dt = \int_b^a f(t)\,dt$$

$$\int_a^a f(t)\,dt = 0$$

With this convention

$$\int_a^{a'} f(t)\,dt + \int_{a'}^b f(t)\,dt = \int_a^b f(t)\,dt$$

regardless of a, a' and b.

<u>Ex</u>: if $a < b < a'$, then

$$\int_a^{a'} f(t)\,dt = \int_a^b f(t)\,dt + \int_b^{a'} f(t)\,dt$$

$$= \int_a^b f(t)\,dt - \int_{a'}^b f(t)\,dt.$$

<u>Def</u>: If $f, F: [a,b] \longrightarrow \mathbb{R}$, we say F is an <u>antiderivative</u> of f if F is diff'ble and $F' = f$.

<u>Last time</u>: if f is continuous, it has an antiderivative, namely

$$F(x) = \int_a^x f(t)\,dt \qquad (1)$$

Mean Value Thm $\Rightarrow$ antiderivatives are unique up to a constant

$\therefore$ FTC: If $F: [a,b] \longrightarrow \mathbb{R}$ is C^1, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

<u>Cor 1</u>: Spose $f: [a,b] \longrightarrow \mathbb{R}$ is continuous, $g: [\overset{c}{\cancel{a}}, \overset{d}{\cancel{b}}] \longrightarrow \mathbb{R}$ is C^1 and $g(c) = a$, $g(d) = b$. Then

$$\int_a^b f(t)\,dt = \int_c^d f(g(s))\,g'(s)\,ds.$$

need f to be defined & cts on $g([c,d])$

<u>Proof</u>: Let F be an antiderivative of f. Then $F \circ g$ is diff'ble and $(F \circ g)'(s) = f(g(s))\,g'(s)$ by chain rule.

f, g, g' are continuous $\Rightarrow$ $F \circ g$ is C^1.

FTC $\Rightarrow$ $\int_c^d f(g(s))\,g'(s)\,ds = (F \circ g)(d) - (F \circ g)(c) = F(b) - F(a)$. $\square$

<u>NB</u>: Exact same statement true if $g(c) = b$, $g(d) = a$.

<u>Cor 2</u>: (Integration by parts) If $f, g : [a,b] \to \mathbb{R}$ are both C^1, then $\int_a^b f'(t) g(t)\, dt = f(b)g(b) - f(a)g(a) - \int_a^b f(t) g'(t)\, dt$.

<u>Proof</u>: $F(t) = f(t)g(t)$ is C^1 with $F'(t) = f'(t)g(t) + f(t)g'(t)$ by the product rule. By FTC,

$$\int_a^b [f'g + fg']\, dt = F(b) - F(a). \quad \text{Use linearity.}$$

$\square$

<u>Taylor series (Reprise)</u> Suppose $f : [a,b]$ is C^k. Then

$$p_{k-1}(x) = \sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

is the $(k-1)$st Taylor polynomial of f centred at a.

<u>Taylor's Thm (Integral Form)</u> Suppose $f : [a,b] \to \mathbb{R}$ is C^k and that $x \in [a,b]$. Then

$$f(x) = p_{k-1}(x) + R_k(x) = p_{k-1}(x) + \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\, dt$$

<u>Proof</u>: By induction on k.

$$k=1: \quad f(x) \stackrel{?}{=} \underbrace{f(a)}_{p_0(x)} + \underbrace{\int_a^x f'(t)\, dt}_{R_1(x)} \quad \text{is true by FTC}$$

In general, suppose statement holds for k and $f \in C^{k+1}$. Then $f \in C^k$, so by induction

$$f(x) = p_{k-1}(x) + R_k(x)$$

$$R_k(x) = \int_a^x \underbrace{\frac{(x-t)^{k-1}}{(k-1)!}}_{a'} \underbrace{f^{(k)}(t)}_{b}\, dt \quad \text{use integration by parts}$$

$$= \frac{1}{(k-1)!} \cdot \frac{1}{k}\left[-(x-x)^k f^{(k)}(x) + (x-a)^k f^{(k)}(a)\right]$$

$$+ \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt$$

L23.3

Hence $f(x) = p_{k-1}(x) + \frac{1}{k!}(x-a)^k f^{(k)}(a) + R_{k+1}(x)$

$$= p_k(x) + R_{k+1}(x).$$

The theorem follows by induction. $\square$

<u>Cor</u>: If $f \in C^k$ as above, then

$$|f(x) - p_{k-1}(x)| \leq M|x-a|^n$$

for some $M \in \mathbb{R}$ and all $x \in [a,b]$.

<u>Proof</u>: $f \in C^k$, so $\exists\, M' \in \mathbb{R}$ s.t. $|f^{(k)}(x)| \leq M'$ for all $x \in [a,b]$.

Also $|x-t| \leq |x-a|$ for $t \in [a,x]$, so

$$|f(x) - p_{k-1}(x)| = |R_k(x)| \leq \int_a^x \left| \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) \right| dt$$

$$\leq \int_a^x \frac{|x-a|^{k-1}}{(k-1)!} \underbrace{\cancel{f^{(k)}(t)}}_{M'} dt = \frac{M'}{(k-1)!}|x-a|^k. \quad \square$$

<u>Newton's Binomial Thm</u>

If $x > 0$ and $s \in \mathbb{R}$, let $h(x) = x^s = \exp(s \log x)$.

Then $h'(x) = \frac{s}{x}\exp(s\log x) = s x^{s-1}$.

Consider $f(x) = (1+x)^s$. Then

$$\frac{1}{k!} f^{(k)}(x) = \frac{s(s-1)\cdots(s-k+1)}{k!}(1+x)^{s-k}$$

so Taylor series at zero is

$$\sum_{k=0}^{\infty} \binom{s}{k} x^k \quad \text{where} \quad \binom{s}{k} := \frac{s(s-1)\cdots(s-k+1)}{k!}$$

<u>Lemma</u>: 1) $k\binom{s}{k} = s\binom{s-1}{k-1}$

2) $\binom{s}{k-1} + \binom{s}{k} = \binom{s+1}{k}$

<u>Proof</u>: 1) $k\binom{s}{k} = k \cdot \frac{s(s-1)\cdots(s-k+1)}{k!} = s\,\frac{(s-1)\cdots(s-1-k)}{(k-1)!} = s\binom{s-1}{k-1}$

2) $\binom{s}{k-1} + \binom{s}{k} = \binom{s}{k-1}\cdot\left[1 + \frac{s-k+1}{k}\right] = \binom{s}{k-1}\cdot\frac{s+1}{k} = \binom{s+1}{k}$ by 1)

$\square$

L23.4

Consider $\sum_{k=0}^{\infty} \binom{s}{k} x^k$.

The Ratio Test gives

$$\lim_{k\to\infty} \left| \frac{\binom{s}{k+1} x^{k+1}}{\binom{s}{k} x^k} \right| = |x| \lim_{k\to\infty} \left| \frac{s-k}{k+1} \right| = |x|$$

so series converges for $x \in (-1,1)$.

Theorem (Newton): For $x \in (-1,1)$

$$\sum_{k=0}^{\infty} \binom{s}{k} x^k = (1+x)^s$$

Proof: Let $g(x) = \sum_{k=0}^{\infty} \binom{s}{k} x^k$ for $x \in (-1,1)$. Then

$g'(x) = \sum_{k=0}^{\infty} \binom{s}{k} k x^{k-1}$. So

$$\begin{aligned}
(1+x)\, g'(x) &= \sum_{k=0}^{\infty} \left[(k+1) \binom{s}{k+1} + k \binom{s}{k} \right] x^k \quad \swarrow \text{Lemma} \\
&= s \sum_{k=0}^{\infty} \left[\binom{s-1}{k} + \binom{s-1}{k-1} \right] x^k \quad \swarrow \text{Lemma} \\
&= s \sum_{k=0}^{\infty} \binom{s}{k} x^k \\
&= s\, g(x).
\end{aligned}$$

If we let $G(x) = (1+x)^{-s} g(x)$, then

$$G'(x) = -s(1+x)^{-s-1} g(x) + (1-x)^{-s} g'(x) = 0$$

$\Rightarrow G(x)$ is constant, equal to $G(0) = (1+0)^{-s} g(0) = 1$. $\square$

L24.1

<u>Substitution</u>: If $f:[a,b]\to\mathbb{R}$ is continuous, $g:[c,d]\to[a,b]$ is C^1, and $g(c)=a$, $g(d)=b$, then

$$\int_a^b f(t)\,dt = \int_c^d f(g(s))\,g'(s)\,ds.$$

5.4) Improper integrals

Recall that to define $\int_a^b f(t)\,dt$, we required

1) $a,b\in\mathbb{R}$, 2) f is bounded.

Suppose $f:[a,b)\to\mathbb{R}$ is continuous.

Then $F(x)=\int_a^x f(t)\,dt$ is defined for all $x\in[a,b)$.

<u>Def</u>: We say the improper integral

$$\int_a^b f(t)\,dt = \lim_{x\to b^-}\int_a^x f(t)\,dt \quad \text{if the RHS exists.}$$

If so, we say the integral converges.

If f is actually defined and continuous on $[a,b]$, then F is diff'ble, hence continuous on $[a,b]$, so

$$\lim_{x\to b^-}\int_a^x f(t)\,dt = \lim_{x\to b^-}F(x) = F(b) = \int_a^b f(t)\,dt.$$

So definition agrees with previous one.

<u>Ex</u>: $\int_1^\infty x^{-s}\,dx$ with $s\neq 1$

$$\int_1^r x^{-s}\,dx = \frac{1}{1-s}x^{1-s}\Big|_1^r = \frac{1}{s-1}\left(1-r^{1-s}\right)$$

which tends to $\frac{1}{s-1}$ if $s>1$ and diverges if $s<1$

$$\lim_{r\to\infty}\int_1^r t^{-1}\,dt = \lim_{r\to\infty}\log r \quad \text{diverges}$$

so $$\int_1^\infty x^{-s}\,dx = \begin{cases}(s-1)^{-1} & \text{if } s>1,\\ \text{diverges} & \text{if } s\leq 1.\end{cases}$$

Similarly, if $f:(a,b]$ is continuous define

$$\int_a^b f(t)\,dt = \lim_{x\to a^+}\int_x^b f(t)\,dt \quad \text{if RHS exists}$$

<u>Ex</u>: $$\int_0^1 x^{-s}\,dx = \begin{cases}(1-s)^{-1} & \text{if } s<1\\ \text{diverges} & \text{if } s\geq 1\end{cases}$$

<u>Def</u>: If $f:(a,b) \longrightarrow \mathbb{R}$ is continuous, define

$$\int_a^b f(t)\,dt = \int_a^{a'} f(t)\,dt + \int_{a'}^b f(t)\,dt$$

where $a < a' < b$, if both terms on RHS exist.

<u>Exercise</u>: If RHS is defined, it does not depend on a'

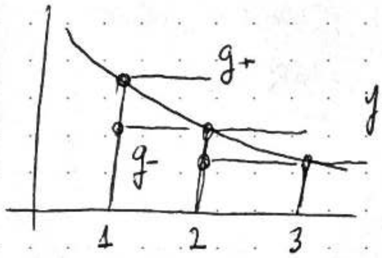
<u>Prop</u> (Integral Test): If $f: [1, \infty) \longrightarrow \mathbb{R}^+$ is continuous, decreasing and positive, then $\int_1^\infty f(t)\,dt$ converges $\Leftrightarrow \sum_{n=1}^{\infty} f(n)$ converges

<u>Proof</u>: Let $g_-(t) = f(\lceil t \rceil)$ and $g_+(t) = f(\lfloor t \rfloor)$.

Now $\lfloor t \rfloor \leq t$, $\lceil t \rceil \geq t$ and g is decreasing, so

$$g_-(t) \leq f(t) \leq g_+(t).$$

<u>Picture</u>:



$$\text{So} \quad \int_1^n g_-(t)\,dt \leq \int_1^n f(t)\,dt \leq \int_1^n g_+(t)\,dt$$

$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(t)\,dt \leq \sum_{i=1}^{n-1} f(i)$$

Let $s_n = \sum_{i=1}^{n} f(i)$, $F(x) = \int_1^x f(t)\,dt$.

So $s_n - s_1 \leq F(n) \leq s_{n-1}$. ⊛

$f \geq 0 \Rightarrow F$ is increasing $\Rightarrow \lim_{x\to\infty} F(x)$ exists $\Leftrightarrow \{F(x) : x \geq 1\}$ bdd above

$\Leftrightarrow (F(n))$ bdd above

By ⊛ $\Leftrightarrow (s_n)$ bdd above

$\Leftrightarrow \sum f_i$ converges. □

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s>1$ since $\int_1^\infty x^{-s}$ does.

<u>Theorem</u>: Suppose $f, g : [a,b) \longrightarrow \mathbb{R}$ are continuous and $|g| \leq f$, and $\int_a^b f(t)\,dt$ converges. Then $\int_a^b g(t)\,dt$ converges.

L.24.3

Recall: A sequence (x_n) is Cauchy if for every $\varepsilon > 0$, $\exists N$ s.t. $|x_n - x_m| < \varepsilon$ wherever $n, m > N$.

$\Leftrightarrow$ (x_n) converges

Proof: 1) Let $\alpha(x) = \int_a^x f(t)\,dt$ and $\beta(x) = \int_a^x g(t)\,dt$.

Then $\underset{y \le x}{\uparrow}$ $|\beta(x) - \beta(y)| = \left|\int_y^x g(t)\,dt\right| \le \int_y^x |g(t)|\,dt \le \int_y^x f(t)\,dt = \alpha(x) - \alpha(y)$ (*)

2) Suppose $x_n \to b^-$. Then

$\int_a^b f(t)\,dt$ converges $\Rightarrow$ $\alpha(x_n)$ converges

$\Rightarrow$ $\alpha(x_n)$ is Cauchy

By (*), $\Rightarrow$ $\beta(x_n)$ is Cauchy

$\Rightarrow$ $\beta(x_n)$ converges

3) Suppose $x_n \to b^-$, $x_n' \to b^-$.

Then $\left.\begin{array}{l} \alpha(x_n) \longrightarrow \alpha = \int_c^b f(t)\,dt \\ \alpha(x_n') \longrightarrow \alpha \end{array}\right\}$ meh

By 2) $\beta(x_n) \to \beta$

$\beta(x_n') \to \beta'$

Consider the sequence $x_1, x_1', x_2, x_2', \dots = (y_i)$

Then by 2) $\beta(y_i) \to \beta''$, but (x_n) and (x_n') are subsequences of (y_i), so $\beta = \beta' = \beta''$, and we are done. $\square$

Ex: 1) $\int_1^\infty x^{s-1} e^{-x}\,dx$ where $s \in \mathbb{R}$

Choose $N \in \mathbb{Z}^+$, $N > s$. Then for $x \ge 1$,

$e^{+x} \ge x^N / N!$, so $x^{s-1} e^{-x} \le N!\, x^{s-1-N}$

Now $\int_1^\infty x^{s-1-N}\,dx$ converges since $N > s$, so

$\int_1^\infty x^{s-1} e^{-x}\,dx$ converges by comparison.

2) $\int_0^1 x^{s-1} e^{-x}\,dx$ (*)

$x^{s-1} \ge x^{s-1} e^{-x} \ge e^{-1} x^{s-1}$ for $x \in [0, 1]$ so by comparison (*) converges iff $\int_0^1 x^{s-1}\,dx$ converges $\Leftrightarrow s > 0$.

L24.4

Summary: $\int_0^\infty x^{s-1}e^{-x}dx$ converges $\Leftrightarrow s>0$

Define $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ for $s>0$.

Prop: If $s>0$, $\Gamma(s+1) = s\Gamma(s)$

Proof: Integrate $\int_a^b x^{s-1}e^{-x}dx$ by parts to get

$$\frac{1}{s}x^s e^{-x}\Big|_b^a + \int_a^b \frac{x^s}{s}e^{-x}dx$$

$$\lim_{b\to\infty} b^s e^{-b} = 0 = \lim_{a\to 0} a^s e^{-a}$$

so $\int_0^\infty x^{s-1}e^{-x}dx = \frac{1}{s}\int_0^\infty x^s e^{-x}dx$, as desired. $\square$

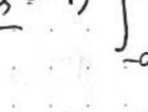
$$\boxed{V_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}}$$

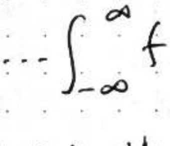
Corollary: $\Gamma(s+1) = s!$ for $s\in\mathbb{N}$

Proof: $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$ (check)

Then use induction via Prop.

Def: $S^{n-1} = \{\vec{x}\in\mathbb{R}^n \text{ s.t. } |\vec{x}|=1\}$

S^1  circle

S^2  sphere

$V_1 = 2\pi$ $\qquad V_2 = 4\pi$

$$V_{n-1} = \frac{2(\Gamma(\frac12))^n}{\Gamma(\frac{n}{2})}$$

$n=2$: $V_1 = 2\pi = \frac{2(\Gamma(\frac12))^2}{\Gamma(1)}$

$\Rightarrow \Gamma(\frac12) = \sqrt{\pi}$

Facts: 1) $\int_{-\infty}^\infty f_1(x_1)dx_1 \cdots \int_{-\infty}^\infty f_n(x_n)dx_n = \int_{\mathbb{R}^n} f_1(x_1)\cdots f_n(x_n)dx_1\cdots dx_n$

2) If $F(\vec{x}) = f(|\vec{x}|)$, then

$$\int_{\mathbb{R}^n} F(\vec{x}) = V_{n-1}\int_0^\infty r^{n-1}f(r)\,dr$$

Now consider

$$\int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx \overset{\sqrt{u}=x}{=} 2\int_0^\infty \frac12 u^{-1/2}e^{-u}du = \Gamma(\tfrac12)$$

Hence $\int_{-\infty}^\infty e^{-x_1^2}dx_1\cdots\int_{-\infty}^\infty e^{-x_n^2}dx_n = (\Gamma(\frac12))^n$

$$= \int_{\mathbb{R}^n} e^{-|\vec{x}|^2}dx_1\cdots dx_n = V_{n-1}\int_0^\infty r^{n-1}e^{-r^2}dr \overset{r=\sqrt{u}}{=} V_{n-1}\int_0^\infty \frac12 u^{\frac{n}{2}-1}e^{-u}du = \frac12 V_{n-1}\Gamma(\tfrac{n}{2})$$