L1.1

Differential Equations

Differential Equation (DE): an equation involving derivatives of a function(s)

DEs are central to most branches of applied maths and physics

We live in a rapidly changing world STFU

Example Let $\theta(t)$ denote temperature of a cup of coffee at time t
Let θ_0 denote air temperature

θ is a function of t; t is the argument

θ is the dependent variable, t the independent

Q: How does θ evolve in time?

Empirically: the rate of change of the temp. of a body is proportional to the difference between its temperature and that of its surroundings. (Newton's idea)

Mathematically $\dot{\theta} \propto (\theta - \theta_0)$; $\dot{\theta} = -k(\theta - \theta_0)$ where $k > 0$

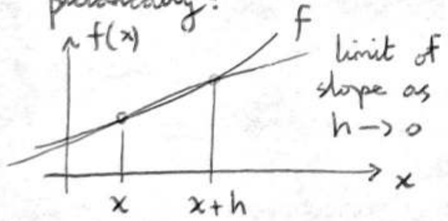
Before solving this, we will revisit derivatives and integration.

Derivatives

Here, define the derivative of a function $f(x)$ w.r.t. its argument x as the function

$$\frac{df}{dx} \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (*) \qquad \boxed{1.1}$$

pictorically:



Note: for derivative to exist, require:

$$\lim_{\substack{h \to 0^+ \\ (h>0)}} [*] = \lim_{\substack{h \to 0^- \\ (h<0)}} [*]$$

e.g. $|x|$ is not differentiable at $x = 0$
(it is elsewhere)

Notation $\frac{df}{dx} = f'(x) = \dot{f}(x) = D_x f$

Leibniz Lagrange Newton

L1.2 For sufficiently smooth functions, we can define derivatives recursively, e.g.

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = f''(x) = \ddot{f}(x) = D_x^2 f$$

n^{th} derivative: $\quad \frac{d^n f}{dx^n} = f^{(n)}(x) = D_x^n f$

<u>Order parameters</u>

Goal: compare the behaviour of functions

1. little oh: $\underline{o}$
2. big Oh: $\mathcal{O}$

I. Little oh: "much smaller than..."

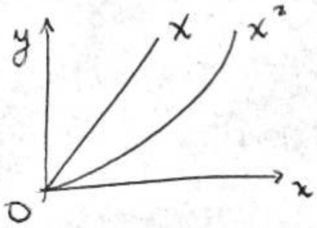
$f(x) = \underline{o}[g(x)]$ as $x \to x_0$

if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

e.g. $x^2 = \underline{o}[x]$ as $x \to 0$

let $f(x) = x^2$, $g(x) = x$

$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$ ✓



II. Big Oh: "can be bounded by..."

$f(x) = \mathcal{O}[g(x)]$ as $x \to x_0$ if

$x(mx-1) < 0$ for +ve x
$x(mx+1) < 0$ for −ve x

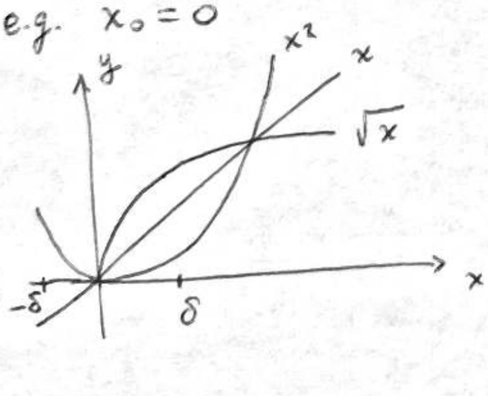
no matter m,
$|x| \overset{?}{>} mx^2$
will hold for
$|x| < \frac{1}{m}$

case 1: x_0 is finite

$\exists$ two +ve constants m and δ such that $\forall x$

where $|x - x_0| < \delta$, $|f(x)| \leq m|g(x)|$

e.g. $x_0 = 0$



$x \neq \mathcal{O}(x^2)$ as $x \to 0$

$x^2 = \mathcal{O}(x)$ as $x \to 0$

$x = \mathcal{O}(\sqrt{x})$ as $x \to 0$

in fact... $f(x) = \underline{o}[g(x)]$ at x_0

$\overset{?}{\Rightarrow}$ $f(x) = \mathcal{O}[g(x)]$ at x_0

I think so... for $\delta(\varepsilon)$, $\frac{f(x)}{g(x)} \in [\varepsilon, -\varepsilon]$

so $|f(x)| \overset{?}{\leq} \varepsilon |g(x)|$ yes, since we can assume $g(x) \neq 0$ in a neighbourhood ☺

Case 2: $x_0 = \infty$

$\exists$ two +ve constants M and x_1 s.t. $\forall\, x > x_1$, $|f(x)| \leq M|g(x)|$

e.g. $2x^3 + 4x + 12 = \mathcal{O}(x^3)$ as $x \to \infty$

Example using $\underline{o}$: Equation of a line tangent to $f(x)$ at $x = x_0$

from $\boxed{1.1}$ $\left.\frac{df}{dx}\right|_{x=x_0} = \left[\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} + \frac{\underline{o}(h)}{h} \right]$

added $\emptyset$ in a clever way

hence $f(x_0+h) = f(x_0) + \left(\left.\frac{df}{dx}\right|_{x_0}\right)h + \underline{o}(h)$ $\boxed{1.2}$ ← questionable as f

let $x = x_0 + h_*$, $y = f(x)$, $y_0 = f(x_0)$,

$m = \left.\frac{df}{dx}\right|_{x_0}$ $\quad y = y_0 + m(x - x_0) + \underline{o}(h)$

$$\lim_{h\to 0}\left[\frac{f(x_0+h) - f(x_0)}{h} - \left.\frac{df}{dx}\right|_{x_0} \right] \stackrel{?}{=} 0$$

It is!

This shit is true!

Rules for differentiation

Chain rule consider $f(x) = F[g(x)]$

$$\frac{df}{dx} = F'[g(x)]\frac{dg}{dx} = \frac{dF}{dg}\cdot\frac{dg}{dx} \qquad (2.1)$$

e.g. $F(x) = \sin x$, $g(x) = x^2 - x + 2$

$$f(x) = F(g(x)) = \sin(x^2 - x + 2)$$

$$\frac{df}{dx} = \left[\cos(x^2 - x + 2)\right](2x-1)$$

Product rule consider $f(x) = u(x)\,v(x)$

$$\frac{df}{dx} = u'(x)\,v(x) + u(x)\,v'(x) = u'v + uv' \qquad (2.2)$$

Leibniz rule

(follows from repeated application of product rule)

consider $f(x) = u(x)\,v(x)$

$$f'(x) = u'v + uv'$$

$$f''(x) = u''v + u'v' + u'v' + uv''$$

$$= u''v + 2u'v' + uv''$$

$$f'''(x) = u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$$

$$= u'''v + 3u''v + 3u'v'' + uv'''$$

Note similarity to Pascal's triangle

⋮

$$f^{(n)}(x) = u^{(n)}v + n\,u^{(n-1)}v' + \frac{n(n-1)}{2}u^{(n-2)}v'' + \dots + {}^nC_m\,u^{(n-m)}v^{(m)} + \dots + uv^{(n)} \qquad (2.3)$$

$$= \sum_{r=0}^{n}\binom{n}{r}u^{(n-r)}v^{(r)}$$

"n choose m"
binomial coefficients
(number of ways of picking m items from a set of n items if order doesn't matter)

Taylor series

Suppose we want to approximate a function $f(x)$ near $x=0$ with a polynomial of order n

i.e. $f(x) \simeq a_0 + a_1 x + \cdots + a_n x^n = P_n(x)$

How do we find the appropriate values for the a_i s?

note $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$

$f''(x) = 2a_2 + 3\cdot 2 a_3 x + \cdots + n(n-1) a_n x^{n-2}$

$\vdots$

Evaluate at $x=0$

$f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2$

$\cdots f^{(n)}(0) = n!\, a_n$

Therefore: (we can repeat the same process at $x = x_0$)

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad (2.4)$$

Taylor polynomial of degree n

(2.5)

write $f(x) = P_n(x) + E_n$

E_n: Error (or remainder)

recall Eq (1.2) $f(x+h) = f(x) + hf'(x) + \underline{o}(h)$ as $h \to 0$

this can be generalised, provided that the first n derivatives of $f(x)$ exist:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^n}{n!}f^{(n)}(x) + \underline{o}(h^n) \quad \text{as } h \to 0 \quad (2.6)$$

By comparison with (2.5), $E_n = \underline{o}(h^n)$
(where $h = x - x_0$)

Taylor's theorem

$E_n = \mathcal{O}(h^{n+1})$ as $h \to 0$ — this is a stronger statement than $\underline{o}(h^n)$

(provided that $f^{(n+1)}$ exists)

L2.3

e.g. h^{n+a} $\forall a \in (0,1)$ is $\underline{o}(h^n)$ as $h \to 0$

since $\lim_{h\to 0} \frac{h^{n+a}}{h^n} = \lim_{h\to 0} h^a = 0$ ✓

but $h^{n+a} \neq \mathcal{O}(h^{n+1})$

can't keep h^{n+a} smaller than Mh^{n+1} for const. M with h arbitrarily small

L'Hopital's Rule

Let $f(x)$ and $g(x)$ be differentiable at $x = x_0$

and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$.

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ provided $g'(x_0) \neq 0$ (2.7)

proof using Taylor series

$$f(x) = \cancel{f(x_0)}^{\,0} + (x-x_0)f'(x_0) + \underline{o}(x-x_0)$$
$$g(x) = \cancel{g(x_0)}^{\,0} + (x-x_0)g'(x_0) + \underline{o}(x-x_0)$$
as $x \to x_0$

$$\Rightarrow \frac{f}{g} = \frac{f'(x_0) + \frac{o(x-x_0)}{x-x_0}}{g'(x_0) + \frac{o(x-x_0)}{x-x_0}} \longrightarrow \frac{f'}{g'} \text{ as } x \to x_0$$

huh? $\lim_{x\to x_0} \frac{f'(x_0) + \frac{o(x-x_0)}{x-x_0}}{g'(x_0) + \frac{o(x-x_0)}{x-x_0}} \overset{?}{=} \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$

can generalise to higher orders

e.g. $f(x) = 3\sin x - \sin 3x$; $g(x) = 2x - \sin 2x$

$f(0) = g(0) = 0$; $f'(0) = g'(0) = 0$

$f''(0) = g''(0) = 0$; $g'''(0) \neq 0$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'''(0)}{g'''(0)} = \left.\frac{-3\cos x + 27\cos 3x}{8\cos 2x}\right|_{x=0} = 3$$

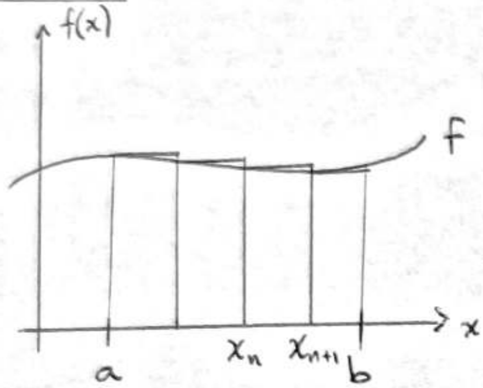
Integration

We will define the integral as an <u>infinite sum</u> and assume functions are sufficiently well-behaved such that the integral is well-defined.
(see IA Analysis)

$$\int_a^b f(x)\,dx \equiv \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} f(x_n)\,\Delta x \qquad (3.1)$$

where $\Delta x = \dfrac{b-A}{N}$, $x_n = a + n\,\Delta x$

<u>pictorically</u>



How does this relate to area under curve?
expand $f(x)$ in Taylor Series about $x = x_n$ and evaluate at x_{n+1}.

$$f(x_{n+1}) = f(x_n) + \Delta x\, f'(x_n) + \mathcal{O}(\Delta x^2)$$

or just $f(x_{n+1}) = f(x_n) + \mathcal{O}(\Delta x)$ $\xrightarrow{\text{as } \Delta x} 0$

(3.2)

Area under $f(x)$ from x_n to x_{n+1} is

$$\left[f(x_n) + \mathcal{O}(\Delta x)\right]\Delta x = \Delta x\, f(x_n) + \mathcal{O}(\Delta x^2) \qquad (3.3)$$

(prove using mean value theorem)

Therefore, area under curve from a to b is

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N-1} f(x_n)\,\Delta x + \underbrace{\mathcal{O}(N\Delta x^2)}_{=\underbrace{\mathcal{O}\left(\frac{(b-a)^2}{N}\right)}_{=0 \text{ applying limit}}}\right] = \int_a^b f(x)\,dx \quad \text{from } (3.1)$$

(Note $\Delta x \to 0$ as $N \to \infty$)

<u>Fundamental Theorem of Calculus (FTC)</u>

let $F(x) = \displaystyle\int_0^x f(t)\,dt$

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\left\{\underbrace{\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt}_{\text{from eq}^n (1.1)}\right\} = \lim_{h\to 0} \frac{1}{h}\left\{\int_x^{x+h} f(t)\,dt\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\underbrace{\left[f(x)h + \mathcal{O}(h^2)\right]}_{\text{from eq}^n (3.3)} = f(x) \quad \text{(applying limit)}$$

L3.3 - LO.1

$$\Rightarrow \boxed{\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad (3.4)}$$

Note $F(x)$ is the solution of the DE $\frac{dF}{dx} = f(x)$

Corollaries of FTC

$$\frac{d}{dx}\int_x^b f(t)\,dt = -f(x) \qquad (3.5)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx} F(g(x)) = \underbrace{\frac{dF}{dg}\cdot\frac{dg}{dx}}_{\text{(chain rule)}} = f(g(x))\cdot g'(x) \quad \text{from FTC}$$

Notation

Define the indefinite integral

$$\int f(x)\,dx = \int^x f(t)\,dt$$

Integration Techniques

1. Substitution

If an integrand contains a function of a function, it might help to sub. for the inner one.

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + c$$

e.g. $\int \frac{1-2x}{\sqrt{x-x^2}}\,dx = 2\sqrt{x-x^2} + c$ where $f(t) = \frac{1}{\sqrt{x}}$, $g(t) = x - x^2$

2. Trig substitutions

$\cos^2\theta + \sin^2\theta = 1 \longrightarrow \sqrt{1-x^2} \longrightarrow x = \sin\theta$

$1 + \tan^2\theta = \sec^2\theta \longrightarrow 1 + x^2 \longrightarrow x = \tan\theta$

$\cosh^2 x - \sinh^2 u = 1 \longrightarrow \sqrt{1+x^2} \longrightarrow x = \sinh u$

$\sqrt{x^2-1} \longrightarrow x = \cosh u$

$\tanh^2 x + \operatorname{sech}^2 x = 1 \longrightarrow 1 - x^2 \longrightarrow x = \tanh u$

Example: $\int \sqrt{2x-x^2}\,dx = \int \sqrt{1-(x-1)^2}\,dx = \int \cos^2\theta\,d\theta$ where $x - 1 = \sin\theta$

$$= \int\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + c$$

Learnin stuff boi

L3.3

3. Integration by parts

$$(uv)' = u'v + uv'$$

$$\Rightarrow \int uv'\,dx = uv - \int u'v\,dx$$

Fun Fact:

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

can be proved like this

L4.1

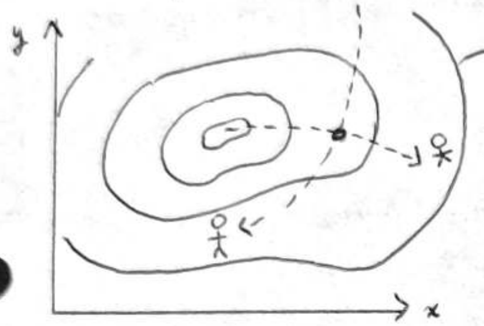
Partial differentiation

in many real-world applications, functions involve more than one independent variable.

Examples

- Elevation vs latitude, longitude
- Temperature vs $x, y, z, (t)$
- Ideal gas law: pressure vs V, T

Contour plot



$f(x,y) = \text{const.}$

slope of a function depends on direction

partial derivative: Derivative of a function of multiple independent variables wrt one variable while holding the others fixed

Mathematically, the partial derivative wrt x of $f(x,y)$ is

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \qquad (4.1)$$

Note use of "∂" instead of "d".

Similarly, $\left.\frac{\partial f}{\partial y}\right|_x \equiv \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$

Example $\quad f(x,y) = x^2 + y^3 + e^{xy^2}$

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + y^2 e^{xy^2}$$

can similarly calculate higher order partial derivatives

e.g. $\left.\left.\frac{\partial^2 f}{\partial x^2}\right|_y\right|_y = 2 + y^4 e^{xy^2}$

Now we can calculate cross-derivatives e.g. $\left.\frac{\partial}{\partial y}\left.\frac{\partial f}{\partial x}\right|_y\right|_x = 2y e^{xy^2} + 2xy^3 e^{xy^2}$

Notation gets cumbersome. Usually we omit $\Big|_y$ and use of "∂" on its own this implies that other variables are held fixed.

4.2

<u>Note</u> $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$ (4.2)

(works if they exist AND are well behaved?)

Be careful with notation, for example if $f = f(x,y,z)$

$$\frac{\partial f}{\partial x} \equiv \left.\frac{\partial f}{\partial x}\right|_{y,z} \quad \underline{\text{but}} \quad \frac{\partial f}{\partial x} \neq \left.\frac{\partial f}{\partial x}\right|_{y}$$ depends on path taken in x-z plane

Alternate notation:

$$\frac{\partial f}{\partial x} = f_x \;, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \;??$$

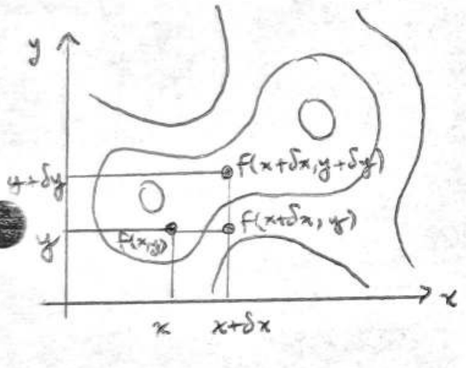
<u>Multi-chadiate chain rule</u>

chain rule applied to multiple functions/variables

e.g. $f(x(t), y(t))$

Define the <u>differential</u> of a function $f(x,y)$

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y) \qquad (4.3)$$



$$\delta f = f(x+\delta x, y+\delta y) - f(x+\delta x, y)$$
$$+ f(x+\delta y, y) - f(x,y)$$

Recall (1.2) : $f(x_0+h) = f(x_0) + h \left.\frac{df}{dx}\right|_{x_0} + \underline{o}(h)$

and using def^n's of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$,

$$\delta f = \cancel{f(x+\delta x, y)} + \delta y \left.\frac{\partial f}{\partial y}\right|_x (x+\delta x, y) + \underline{o}(\delta y) \cancel{- f(x+\delta x, y)}$$
$$\cancel{+ f(x,y)} + \delta x \left.\frac{\partial f}{\partial x}\right|_y (x,y) + \underline{o}(\delta x) \cancel{- f(x,y)}$$

$$\Rightarrow \delta f = \delta y \frac{\partial f}{\partial y}(x+\delta x, y) + \delta x \frac{\partial f}{\partial x}(x,y) + \underline{o}(\delta x, \delta y)$$

$$= \delta y \left[\frac{\partial f}{\partial y}(x,y) + \delta x \left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\right] + \underline{o}(\delta x)\right] + \delta x \frac{\partial f}{\partial x}(x,y) + \underline{o}(\delta x, \delta y)$$

4.3

$$\Rightarrow \delta f = \delta y \frac{\partial f}{\partial y}(x,y) + \delta x \frac{\partial f}{\partial x}(x,y) + \underbrace{\delta x \delta y \frac{\partial^2 f}{\partial x \partial y}(x,y)}_{\text{this } o(\delta y) \text{ as } \delta x \to 0} + o(\delta x, \delta y)$$

which, as $\delta x \to 0$ and $\delta y \to 0$, simplifies to

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \qquad (4.4)}$$

chain rule in diff. form

We can apply the chain rule by taking (4.4) by dividing by another differential before applying $\lim \delta \to d$

Example suppose $[x(t), y(t)]$ are coordinates and $f(x(t), y(t))$ is altitude

$$\frac{df}{dt} = \lim_{\substack{\delta t \to 0 \\ \delta x\, \delta y}} \left[\frac{\partial f}{\partial x} \frac{\delta x}{\delta t} + \frac{\partial f}{\partial y} \frac{\delta y}{\delta t} \right]$$

"ordinary" derivative of GPS recorded altitude $f(t)$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \qquad (4.5)}$$ multivariate chain rule

Similarly if $f = f(x, y(x))$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \overbrace{\cancel{\frac{dx}{dx}}}^{1} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

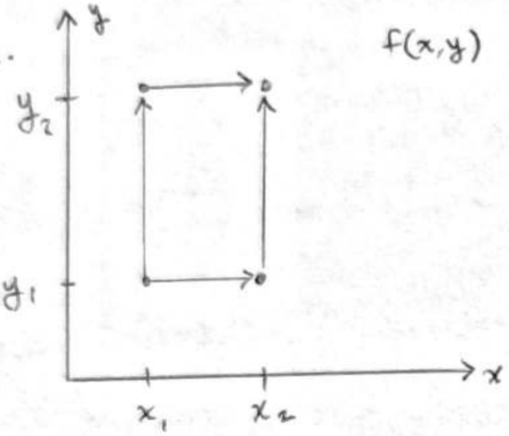
Chain rule in "integral form"

Integrate (4.4) to obtain the chain rule in <u>integral form</u>

$$\int df = \int \frac{\partial f}{\partial x}\,dx + \int \frac{\partial f}{\partial y}\,dy \qquad (5.1)$$

Note: need to integrate these along a given <u>path</u> (see Vector Calculus)

e.g.



$f(x,y)$

$$LHS = f(x_2,y_2) - f(x_1,y_1)$$
$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x,y_1)\,dx + \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_2,y)\,dy$$
$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x,y_2)\,dx + \int_{y_1}^{y_2} \frac{\partial f}{\partial x}(x_1,y)\,dy$$

<u>but</u> $f(x_2,y_2) - f(x_1,y_1) \neq \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x,y_1)\,dx + \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_1,y)\,dy$

e.g. $f(x,y) = x + y^2 + x^2 y$

integrate from $(0,0)$ to $(1,1)$

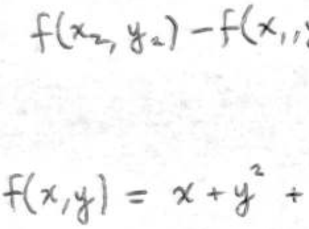
<u>Applications of multivariate chain rule</u>

1. Change of variables

It is often helpful to write a D.E. in a different coordinate system.

To do this we need to transform the derivatives from one coordinate system to another.

E.g.



first write $f(x,y) = f(x(r,\theta), y(r,\theta))$

then $$\left(\frac{\partial f}{\partial r}\right)_\theta = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_\theta + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial r}\right)_\theta$$

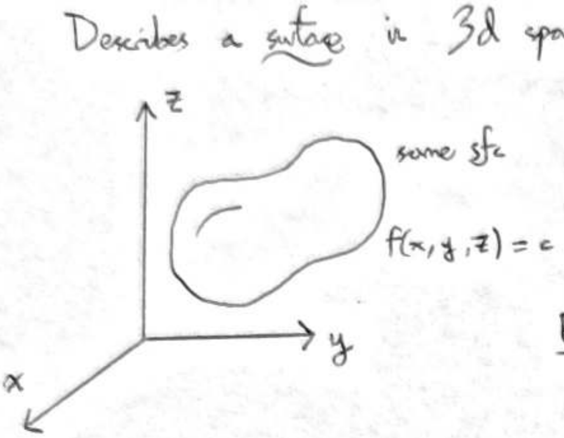
Then use $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \left.\frac{\partial f}{\partial r}\right|_\theta = \cos\theta \left.\frac{\partial f}{\partial x}\right|_y + \sin\theta \left.\frac{\partial f}{\partial y}\right|_x \qquad \left(\text{similar for } \left.\frac{\partial f}{\partial \theta}\right|_r\right)$$

Implicit differentiation

Consider $f(x,y,z) = c$, c = const.

Describes a surface in 3d space.



the surface implicitly defines
$z(x,y)$ or $x(y,z)$ or $y(z,x)$

Example for $xy^2+yz^2+z^2x = 5$ (*)

$$x = \frac{5-yz^2}{y^2+z^5}$$ can also solve for $y(z,x)$

cannot find $z(x,y)$ explicitly

However, we can find $\frac{\partial z}{\partial y}\Big|_x$ by differentiating (*)

$$\frac{\partial(*)}{\partial x}\Big|_y \iff y^2 + 2yz\frac{\partial z}{\partial x}\Big|_y + z^5 + 5xz^4\frac{\partial z}{\partial x}\Big|_y = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}\Big|_y = \frac{-y^2 \mp z^5}{2yz + 5xz^4}.$$

In general for a 3D function we can write

$$f(x,y,z(x,y)) = c$$

to find $\frac{\partial z}{\partial x}\Big|_y$ etc use multivariate chain rule

$$df = \frac{\partial f}{\partial x}\Big|_{yz} dx + \frac{\partial f}{\partial y}\Big|_{xz} dy + \frac{\partial f}{\partial z}\Big|_{xy} dz \qquad (5.2)$$

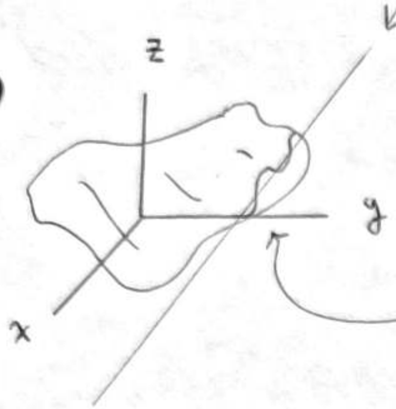
$$\cancel{\frac{\partial f}{\partial x}\Big|_y}^{\,0} = \frac{\partial f}{\partial x}\Big|_{yz}\cancel{\frac{\partial x}{\partial x}\Big|_y}^{\,1} + \frac{\partial f}{\partial y}\Big|_{xz}\cancel{\frac{\partial y}{\partial x}\Big|_y}^{\,0} + \frac{\partial f}{\partial z}\Big|_{xy}\frac{\partial z}{\partial x}\Big|_y$$

$$\frac{\partial f}{\partial x}\Big|_y = \frac{\partial f}{\partial x}\Big|_{yz} + \frac{\partial f}{\partial z}\Big|_{xy}\frac{\partial z}{\partial x}\Big|_y$$

"note x, z vary"

L5.3

Sketch



$\left.\frac{\partial f}{\partial x}\right|_{yz}$ ignores sfc $\neq 0$

$\left.\frac{\partial f}{\partial x}\right|_{y}$ keeps on sfc, so it is 0!

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{y} = \frac{-\partial f/\partial x|_{yz}}{-\partial f/\partial z|_{xy}}$$ see problem 11

Note: reciprocal rules apply to partial derivatives provided that the same variables are held fixed

e.g. for $(r,\theta) \to (x,y)$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial r}\right)_y} \quad \leftarrow \theta \text{ varies}$$

but: $$\frac{\partial r}{\partial x} \neq \frac{1}{\partial x/\partial r}$$

fixing y by default — fixed θ by default

Differentiation under the Integral Sign

Consider a (loving) family of functions $f(x;c)$ where c is a parameter e.g. $f = \log_c(x)$

Define $I(c) = \int_0^b f(x;c)\,dx$.
b

$$\left.\frac{\partial I}{\partial b}\right|_c = f(b;c) \quad \text{(from FTC)}$$

and

$$\left.\frac{\partial I}{\partial c}\right|_b = \lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_0^b f(x;c+\delta c)\,dx - \int_0^b f(x;c)\,dx\right]$$

$$= \lim_{\delta c\to 0} \int_0^b \frac{f(x;c+\delta c) - f(x;c)}{\delta c}\,dx$$

$$= \int_0^b \left.\frac{\partial f}{\partial c}\right|_x dx \quad \text{by pulling in limit}$$

Suppose $b = b(t)$ and $c = c(t)$

$I = I(b(t), c(t))$

$$\frac{dI}{dt} = \frac{\partial I}{\partial b}\frac{db}{dt} + \frac{\partial I}{\partial c}\frac{dc}{dt}$$

$$= f(b,c)\dot{b} + \dot{c}\int_0^b \frac{\partial f}{\partial c}\,dx$$

can also have lower limit change

L6.1

First order linear DEs

Terminology:

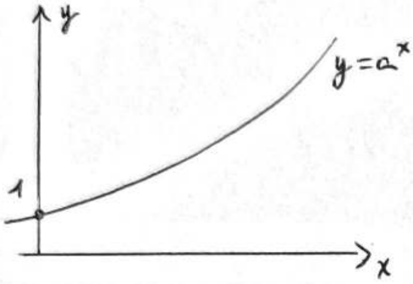
n^{th} order D.E.: highest order derivative is n

linear: dependent variable appears linearly

e.g. $x^2 y + y' = 0$: First order linear DE

Preamble: Exponential function

consider $f(x) = a^x$ with $a > 0$ is constant



$$\frac{df}{dx} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \underbrace{\lim_{h\to 0} \frac{a^h - 1}{h}}_{\lambda}$$

$$\therefore \frac{df}{dx} = \lambda a^x \qquad (6.1)$$

Here, define $\exp(x) = e^x$ as the solution to the following DE

$$\frac{df}{dx} = f(x) \quad \text{with} \quad f(0) = 1 \qquad (6.2)$$

Therefore "e" is the value of "a" such that $\lambda = \lim_{h\to 0} \frac{a^h - 1}{h} = 1$.

numerically $e = 2.718\ldots$

Natural logarithm $\ln(x)$ is the inverse of e^x such that $e^{\ln x} = x$

Notation

$$\ln x = \log_e x = \log x$$

return to (6.1) let $y(x) = a^x = (e^{\ln a})^x = e^{x \ln a}$

$$\frac{dy}{dx} = (\ln a) e^{x \ln a} = (\ln a) \cancel{e} a^x \Rightarrow \lambda = \ln a \quad \text{in (6.1)}$$

Exponential function plays an important role in DE because it is the <u>eigenfunction</u> of the <u>differential operator</u>

The eigenfunction of an operator is the function that is unchanged by the action of the operator up to scaling by the <u>eigenvalue</u>.

LG.2 Terms come from David Hilbert

eigen: German for "own"

consider $\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x}$

$e^{\lambda x}$ is the eigenfunction of $\frac{d}{dx}$

<u>Rules for linear DEs</u>

1. For any linear <u>homogeneous</u> ODE with <u>constant coefficients</u> has solutions of the form $e^{\lambda x}$

<u>homogeneous</u>: $y=0$ is a solution

<u>const. coeff.</u>: the independent variable doesn't appear explicitly

<u>Example</u>: $5y' + -3y = 0$ (6.3)

try $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$

$$5y' - 3y = 0 \iff 5\lambda e^{\lambda x} - 3e^{\lambda x} = 0$$
$$\iff 5\lambda - 3 = 0 \leftarrow \text{characteristic equation}$$
$$\iff \lambda = \frac{3}{5}$$
$$\iff y = e^{3x/5}$$

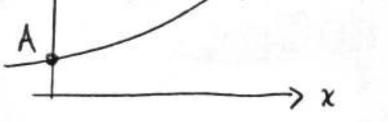
2. For <u>linear</u>, <u>homogeneous</u> DEs, any constant multiple of a solution is also a solution

e.g. $y = Ae^{3x/5}$ is a solution to (6.3) $\forall A$

3. Any n^{th} order linear DE has only n independent solutions

Therefore $y = Ae^{3x/5}$ is <u>the</u> general solution to (6.3)

Set A by applying initial conditions



e.g. $y = A$ at $x = 0$

L7.1

Definition

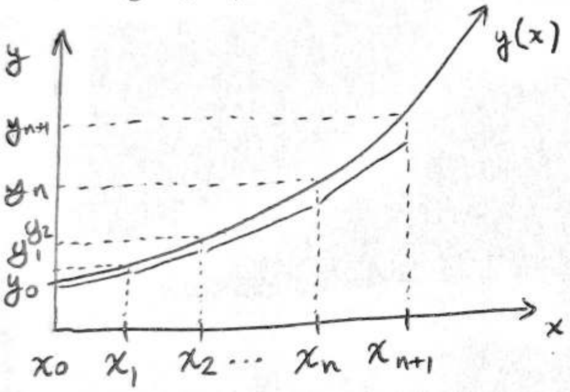
- Ordinary Differential Equation (ODE)
  - Equation in unknown function, involve <u>one</u> independent variable
- Partial Differential Equation (PDE)
  - More than one independent variable, partial derivatives

Discrete Equations

Sometimes it's useful to consider a function evaluated at a discrete set of points e.g. $y_0(x_0), y_1(x_1), \dots$

1. Numerical Integration

Consider $y_0(x_0)\dots$

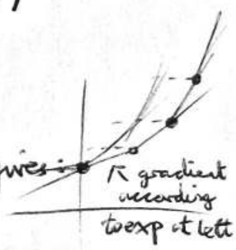


One approximation to $\left.\frac{dy}{dx}\right|_{x_n} \simeq \frac{y_{n+1}-y_n}{\underbrace{x_{n+1}-x_n}_{\text{also called } h}}$

(Forward Euler – not the best)

here $5y' - 3y = 0$ (6.3)

$\Rightarrow 5\,\frac{y_{n+1}-y_n}{h} - 3y_n = 0$ ← this eq^n gives:



$$y_{n+1} = \left(1+\frac{3h}{5}\right) y_n$$

This is a "recurrence relation". (note similarity to comp. interest)

Applying recurrence relation repeatedly,

$$y_n = \left(1+\frac{3h}{5}\right) y_{n-1} = \left(1+\frac{3h}{5}\right)^2 y_{n-2} = \left(1+\frac{3h}{5}\right)^n y_0$$

$$= \left(1+\frac{3x_n}{5n}\right)^n y_0 = \left(1+\frac{\left(\frac{3}{5}\right)x_n}{n}\right)^n y_0 \approx e^{\frac{3}{5}x_n} y_0 \text{ woah dude!}$$

Euler defined $\exp(x) \equiv \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

Here $\lim_{n\to\infty} y_n = y_0 e^{\frac{3}{5}x_n}$. Now for n finite $y_n < y(x)$.

L7.2 Interest rate applied daily (e.g. credit cards) > same interest rate applied monthly

2 Series Solutions

A powerful way to solve ODEs is to seek solutions in the form of an infinite power series, i.e. let $y = \sum_{n=0}^{\infty} a_n x^n$ plug into DE, solve for a_n

Example 6.3: $5y' - 3y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$$xy' = \sum_{n=0}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} n a_n x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} \underset{m \equiv n+1}{=} \sum_{m=1}^{\infty} a_{m-1} x^m = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Hence $5xy' - 3xy = 0 \Rightarrow 5\sum_{n=1}^{\infty}(n a_n x^n) - 3\sum_{n=1}^{\infty}(a_{n-1} x^n) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} (5n a_n - 3a_{n-1}) x^n = 0$$

To be true $\forall x$, need $5n a_n - 3a_{n-1} = 0$

$\Leftrightarrow a_n = \frac{3}{5n} a_{n-1}$ recurrence relation (discrete equation)

$$a_n = \left(\frac{3}{5}\right)^2 \frac{a_{n-2}}{n(n-1)} = \left(\frac{3}{5}\right)^n \frac{a_0}{n!}$$

Hence $y = a_0 \Big(\underbrace{1 + \frac{3}{5}x + \frac{(3x/5)^2}{2!} + \dots + \frac{(3x/5)^r}{r!} + \dots}_{e^{3x/5} \text{ converges } \forall x} \Big)$

$y = a_0 e^{3x/5}$ ✓

L7.3 Forced (inhomogeneous) first order ODEs with const. coeff.

- case: constant forcing

e.g. $5y' - 3y = 10$

Steps

1. Find steady (equilibrium) sol^n w/ $y' = 0$ e.g. $y = y_P = -\frac{10}{3}$

2. Write general sol^n as $y = y_P + y_C$

y_P: particular integral

y_C: complementary function

3. Insert into DE

$5y_C' - 3y_C = 0$

Note y_C is sol^n in homogeneous case

4. Solve for y_C

$y_C = Ae^{3x/5}$

5. Combine $y = y_P + y_C$

$y = Ae^{3x/5} - \frac{10}{3}$

Forced first-order ODEs with const. coeff. (cont.)

Case 2 Eigenfunction forcing

Example In a radioactive rock, isotope A decays into isotope B at a rate proportional to a, the number of nuclei of isotope A, while isotope B decays into C at a rate proportional to b, the number of nuclei of B. Find $b(t)$.

rate

qty: $\boxed{a} \xrightarrow{k_a a} \boxed{b} \xrightarrow{k_b b} \boxed{c}$

$\quad$ A $\qquad$ B $\qquad$ C

$$\frac{da}{dt} = -k_a a \Rightarrow a = a_0 \exp(-k_a t)$$

$$\frac{db}{dt} = k_a a - k_b b \iff \dot{b} + k_b b = \underbrace{k_a a_0 \exp(-k_a t)}_{\text{forcing is an eigenfunction of the diff. operator}}$$

try particular integral $\quad b_p = C\exp(-k_a t)$

$$\Rightarrow -k_a C + k_b C = k_a a_0 \Rightarrow C = \frac{k_a}{k_b - k_a} a_0 \qquad (\text{for } k_a \neq k_b)$$

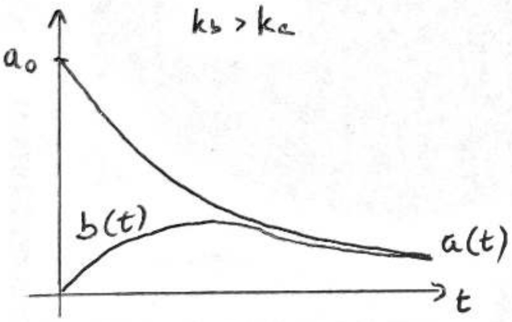
$\downarrow$ consider $Cte^{-k_a t}$

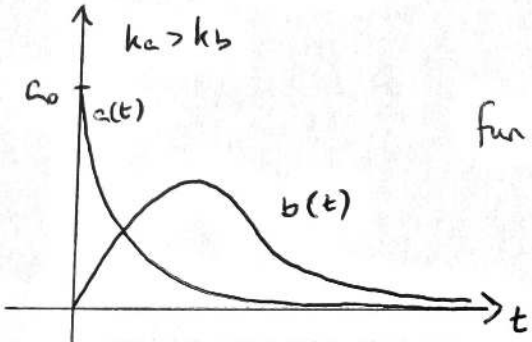
General solution: $\quad b = b_p + b_c$

$$\dot{b}_c + k_b b_c = 0 \Rightarrow \cancel{k}\ b_c = D\exp(-k_b t)$$

$$\therefore \quad b = \frac{a_0 k_a}{k_b - k_a} e^{-k_a t} + De^{-k_b t}$$

For I.C. $b = 0$ at $t = 0 \Rightarrow D = -C$, $\quad b = \dfrac{k_a a_0}{k_b - k_a}\left(e^{-k_a t} - e^{-k_b t}\right)$





fun fact: when $b - a \geq 1$ never cross, otherwise yes

$$\frac{b(t)}{a(t)} = \frac{k_a}{k_b - k_a}\left[1 - e^{(k_a - k_b)t}\right]$$

Allows rocks to be dated by measuring the ratios of certain isotopes.

L8.2

First order ODEs w/ non-constant coefficients

'General' form $a(x)y' + b(x)y = c(x)$

'standard' form $y' + p(x)y = f(x)$

Solve using Integrating factor (IF)

Multiply by $\mu(x)$ (IF) so that

$$\underbrace{\mu y' + (\mu p)y}_{= (\mu y)' \text{ if } \mu p = \mu' \text{ (prod. rule)}} = \mu f$$

$$p = \frac{\mu'}{\mu} \Rightarrow \int p(x)\,dx = \ln\mu \Rightarrow \mu = e^{\int p\,dx} \qquad (8.1)$$

Then $y' + py = f \Rightarrow (\mu y)' = \mu f \Rightarrow \mu y = \int \mu f\,dx$ integrate, solve for $y(x)$

Non-linear first order ODEs

General form: $Q(x,y)\frac{dy}{dx} + P(x,y) = 0$ (8.2)

Two special cases

1. Separable Equations

(8.2) is separable if it can be written in the form:

$$q(y)\,dy = p(x)\,dx \qquad (8.3)$$

Solve for $y(x)$ by integrating both sides.

2. Exact Equation

(8.2) is exact iff

$$Q(x,y)\,dy + P(x,y)\,dx \qquad (8.4)$$

is an exact differential of some function $f(x,y)$

i.e. $df = Q\,dy + P\,dx$

- If this holds (8.2) becomes $df = 0 \Rightarrow f = \text{const.}$ giving the solution

to check (8.4) and find $f(x,y)$ use the multivariate chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Comparing with (8.2) if (8.4) is an exact differential, $\exists\, f(x,y)$ s.t.

- $$\frac{\partial f}{\partial x} = P(x,y), \quad \frac{\partial f}{\partial y} = Q(x,y) \qquad (8.5)$$

So then $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial P}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial Q}{\partial x}$

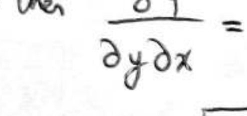
$\Rightarrow$ we need $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \quad (8.6)}$

Reverse implication

- If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout a <u>simply</u>-<u>connected</u> domain D, then $P\,dx + Q\,dy$ is an exact differential of some single-valued function $f(x,y)$ defined in D.

In 2D, domain without holes

i.e.



simply connected — <u>not</u> simply connected

Therefore use (8.6) to check for exact equation.
If (8.6) holds, find $f(x,y)$ by integrating (8.5).

<u>Example</u> $6y(y-x)\dfrac{dy}{dx} + (2x-3y^2) = 0$

Let $P = 2x-3y^2$, $Q = 6y(y-x) \Rightarrow \dfrac{\partial P}{\partial y} = -6y$, $\dfrac{\partial Q}{\partial x} = -6y \Rightarrow$ <u>exact</u>

- $\dfrac{\partial f}{\partial y} = 6y(y-x)$ and $\dfrac{\partial f}{\partial x} = 2x - 3y^2 \longrightarrow f = x^2 - 3xy^2 + c(y)$

$f = 2y^3 - 3xy^2 + c(x) \longrightarrow \boxed{f(x,y) = x^2 - 3xy^2 + 2y^3 + c = 0}$

L9.1

Isoclines and Solution Curves

Non-linear first order ODEs (cont.)

Non-linear equations are not guaranteed to have simple closed form solutions. However, we can analyse the behaviour of the solution to the DE w/o solving the DE.

For illustration, let's consider an equation that we can solve

e.g. $$\dot{y} = t\,(1-y^2) \qquad (9.1)$$

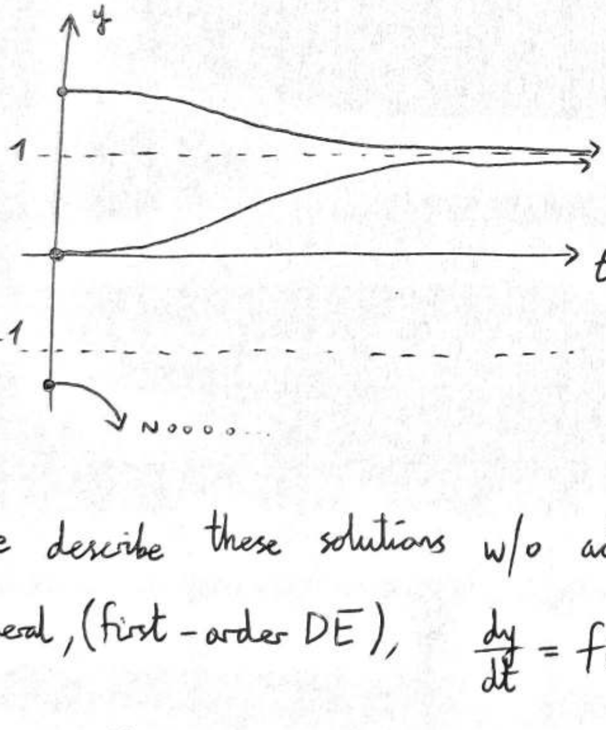
Separable: $$\frac{dy}{1-y^2} = t\,dt$$

After integrating: $\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \frac{t^2}{2} + c \implies \frac{1+y}{1-y} = Ae^{t^2}$

$\implies y = \dfrac{A-e^{-t^2}}{A+e^{-t^2}}$. This general solution gives us a family of solutions for different initial conditions, parametrized by A.

e.g. If $y(t=0) = y_0 \implies A = \dfrac{1+y_0}{1-y_0}$

Sketch



$y(0) = 0 \implies A = 1,\ y = \dfrac{1-e^{-t^2}}{1+e^{-t^2}}$

$y(0) = \frac{3}{2} \implies A = -5,\ y = \dfrac{-5-e^{-t^2}}{-5+e^{-t^2}}$

$y(0) = -\frac{3}{2} \implies A = -\frac{1}{5},\ y = \dfrac{-\frac{1}{5}-e^{-t^2}}{-\frac{1}{5}+e^{-t^2}}$

Can we describe these solutions w/o actually solving the D.E.?

In general, (first-order DE), $\dfrac{dy}{dt} = f(y,t)$

For (9.1), $f(y,t) = t(1-y^2)$. Note $\dot{y} = 0$ for $y \in \{1,-1\}$ and for $t=0$.

$\dot{y} < 0$ for $y > 1$ and $y < -1$ $(t>0)$

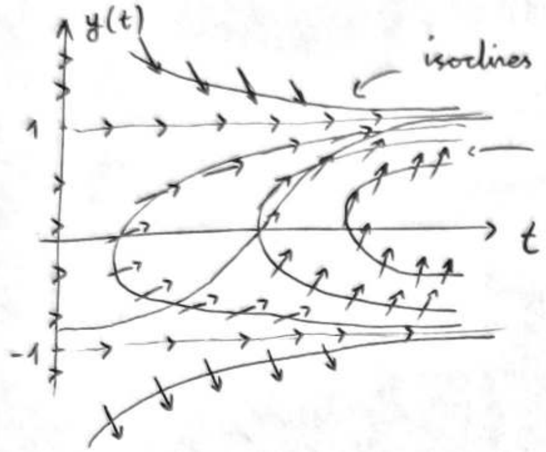
$\dot{y} > 0$ for $-1 < y < 1$

L9.2 Isoclines: curves along which $f = \dot{y} = \text{const.}$

$$t(1-y^2) = D \quad (\text{constant})$$

$$\Rightarrow y^2 = 1 - \frac{D}{t}$$

Sketch



Note: if $f(y,t)$ is a single-valued, solution curves can't cross

Fixed (equilibrium) points

points where $\frac{dy}{dt} = f(y,t) = 0 \ \forall t$

e.g. for (9.1) $y = -1, 1$ fixed pts

Note solution curves converge to $y=1$ and diverge from $y=-1$

this gives an indication of stability of the fixed points

Stability of fixed points

Stable (unstable) fixed points are points: solution curves in a small neighbourhood of the fixed point converge (diverge) to the fixed point.

Perturbation analysis

Let $y = a$ be a fixed point of $\frac{dy}{dt} = f(y,t)$ such that $f(a,t) = 0$.

perturb about the fixed point $\quad y = a + \varepsilon(t) \quad$ (for small ε)

$$\frac{d\varepsilon}{dt} = f(a+\varepsilon, t) = \underbrace{f(a,t)}_{0 \text{ by def}^n} + \varepsilon \frac{\partial f}{\partial y} + O(\varepsilon^2) \quad (\text{Taylor-series})$$

L9.3

$$\Rightarrow \frac{d\varepsilon}{dt} \simeq \varepsilon \frac{\partial f}{\partial y}(a,t) \qquad \text{for small } \varepsilon$$

Linear ODE for $\varepsilon(t)$

If $\begin{cases} \lim_{t\to\infty} \varepsilon = 0 \Rightarrow \text{stable fixed point} \\ \lim_{t\to\infty} \varepsilon = \pm\infty \Rightarrow \text{unstable fixed point} \end{cases}$

for (9.1) $\frac{\partial f}{\partial y} = -2yt = \begin{cases} -2t \text{ at } y=1 \\ 2t \text{ at } y=-1 \end{cases}$

$\underline{y=1}$ | $\dot{\varepsilon} = -2t\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{-t^2}$

$\lim_{t\to\infty} \varepsilon = 0 \Rightarrow$ stable

$\underline{y=-1}$ | $\dot{\varepsilon} = 2t\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{t^2}$

$\lim_{t\to\infty} \varepsilon = \pm\infty \Rightarrow$ unstable

Autonomous Systems

$\dot{y} = f(y)$ i.e. special case independent of t

In this case, near fixed point $y=a$

$$y = a + \varepsilon(t) \Rightarrow \dot{\varepsilon} \approx \varepsilon \underbrace{\frac{df}{dy}(a)}_{\text{constant } = k}$$

$$\Rightarrow \varepsilon = \varepsilon_0 e^{kt}$$

$\therefore$ for autonomous O.D.E.s

If $\begin{cases} f'(a) < 0 \Rightarrow \underline{\text{stable}} \\ f'(a) > 0 \Rightarrow \underline{\text{unstable}} \end{cases}$

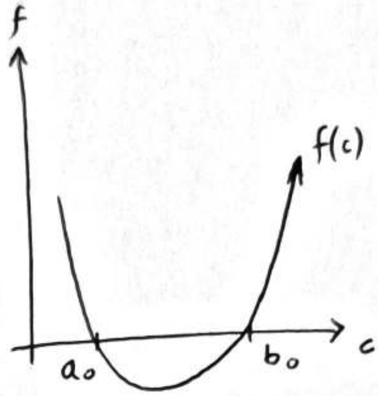
Phase portrait A geometrical representation of the solution to a DE

Example 1: Chemical reaction

drain cleaner + acid $\longrightarrow$ salt water

$$NaOH + HCl \longrightarrow H_2O + NaCl$$

# molecules	$a(t)$	$b(t)$	$c(t)$	$c(t)$
$t=0$	a_0	b_0	0	0



model for reaction: $\frac{dc}{dt} = \lambda ab$ (λ const.)

$a = a_0 - c$ and $b = b_0 - c$

$$\therefore \frac{dc}{dt} = \lambda(a_0 - c)(b_0 - c) = f(c)$$

check w/ perturbation analysis

Example 2: Population dynamics

let $y(t)$: population

αy: birth rate ($\alpha, \beta > 0$ const.)

βy: death rate

Linear model: $\frac{dy}{dt} = (\alpha - \beta) y$

$$y = y_0 e^{(\alpha-\beta)t}$$

If $\alpha > \beta$ $\lim_{t\to\infty} y = \infty$. Not realistic.

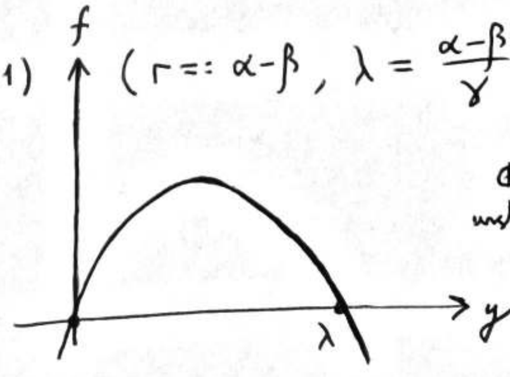
Non-linear model: $\frac{dy}{dt} = (\alpha - \beta) y - \gamma y^2$

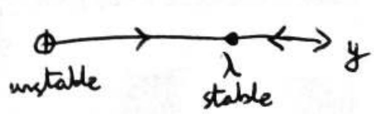
γy^2: models increased death rate for high population density

e.g. lack of food, etc.

alternatively $\dot{y} = r y (1 - \frac{y}{\lambda})$ (# 10.1) ($r =: \alpha - \beta$, $\lambda = \frac{\alpha - \beta}{\gamma}$)

$\uparrow$ differential logistic eq.





L10.2

Fixed points + stability in discrete equations

Consider a first-order discrete (difference) equation of form:

$$x_{n+1} = f(x_n)$$

Fixed points $\quad x_{n+1} = x_n \quad \Rightarrow \quad f(x_n) = x_n$

Stability of fixed points in discrete equations depends on slope of $f(x)$.

Let x_f be a fixed point. Expand $f(x)$ in Taylor series about x_f

$$f(x_f + \varepsilon) = f(x_f) + \varepsilon \left.\frac{df}{dx}\right|_{x_f} + O(\varepsilon^2)$$

$$\simeq x_f + \varepsilon \left.\frac{df}{dx}\right|_{x_f} \qquad \text{(small } \varepsilon\text{)}$$

$$x_f \text{ is } \begin{cases} \text{stable if } & \left| \left.\frac{df}{dx}\right|_{x_f} \right| < 1 \\ \text{unstable if } & \left| \left.\frac{df}{dx}\right|_{x_f} \right| > 1 \end{cases}$$

Example: discrete logistic equation

Increase in population might occur at fixed time intervals (e.g. births in spring).

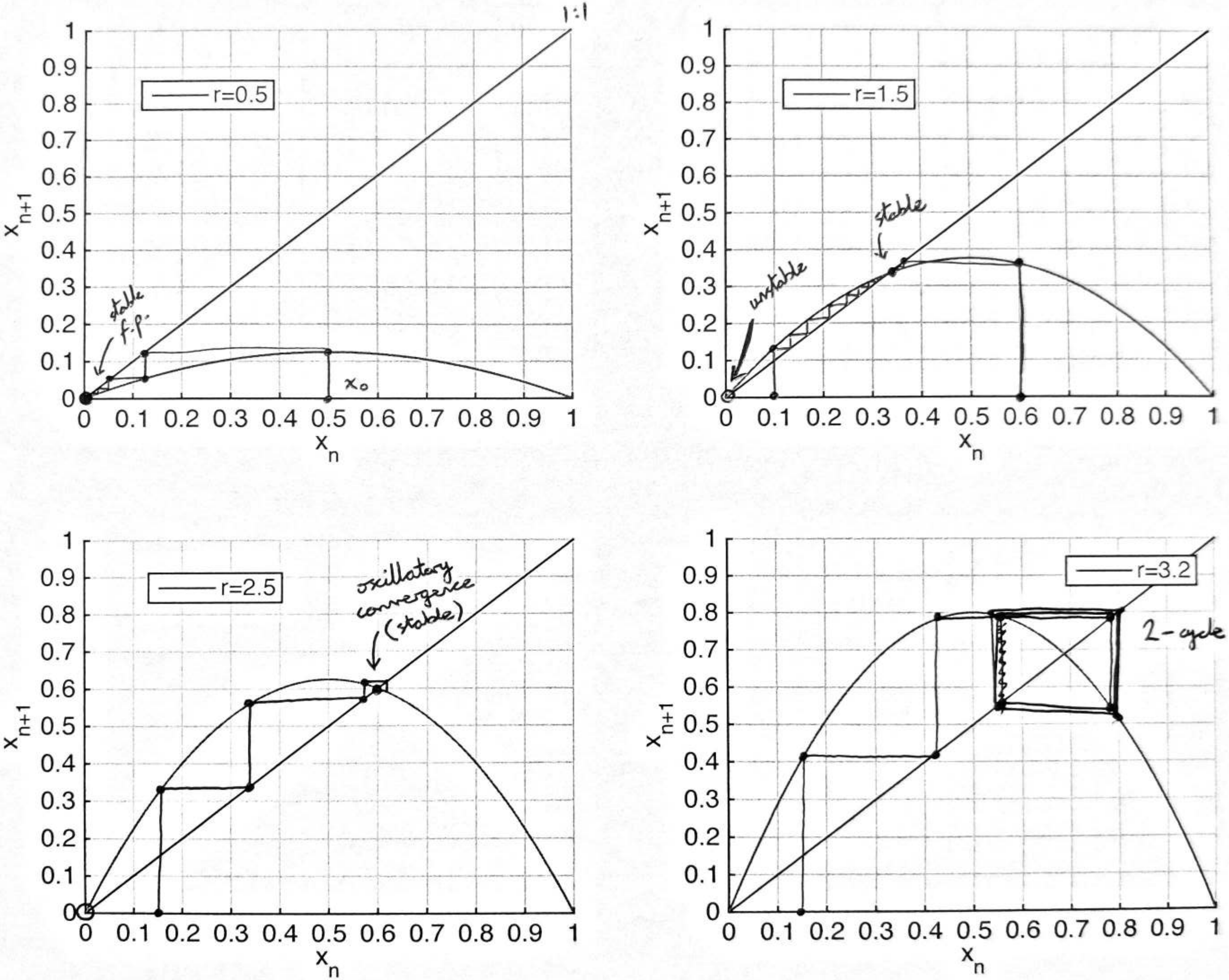
Non-linear model: $\quad \dfrac{x_{n+1} - x_n}{\Delta t} = \lambda x_n - \gamma x_n^2 \qquad$ discrete form of 10.1

$$x_{n+1} = (\lambda \Delta t + 1) x_n - \gamma \Delta t \, x_n^2 \qquad \text{c.f. interest rate}$$

Simpler version:

$$x_{n+1} = r x_n (1 - x_n) = f(x_n) \qquad \text{Discrete logistic equation}$$

logistic map



r=0.5
stable f.p.
x_0
1:1
r=1.5
unstable
stable
r=2.5
oscillatory convergence (stable)
r=3.2
2-cycle
x_n
x_{n+1}

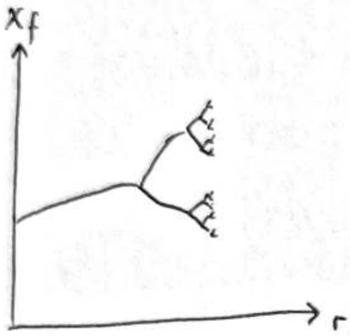
L10.3

fixed points: $f(x_n) = x_n$

$r x_n(1-x_n) = x_n \quad : \quad x_n = 0, \quad x_n = 1 - \frac{1}{r}$

$\frac{df}{dx} = r(1-2x)$ $\quad$ $\underline{x_n = 0}$ $\quad \left.\frac{df}{dx}\right|_{x=0} = r$ $\quad$ $0<r<1$: stable
$r>1$: unstable

$\underline{x_n = 1 - \frac{1}{r}}$ $\quad \left.\frac{df}{dx}\right|_{x_f} = 2 - r$ $\quad$ $0<r<1$: $x_f<0$ unphysical
$1<r<3$: stable
$r>3$: unstable



number of fixed points becomes weird

L11.1

Part 4 second order DEs

- closed form solutions to linear second order DEs do not always exist (unlike linear first order DEs?) does $y' = x^x$ have closed sol^n?

- however, there are special cases where solutions will exist

constant coefficients

general form for linear 2nd order ODE w/ const. coeff

$$ay'' + by' + cy = f(x) \qquad (11.1)$$

Exploit linearity: principple of superposition

For a linear differential operator $\mathcal{D}$ e.g. $\mathcal{D} = \left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c*\right]$

$$\mathcal{D}(y_1 + y_2) = \mathcal{D}(y_1) + \mathcal{D}(y_2)$$

Solve (11.1) in 3 steps

1. Find the complementary functions y_1, y_2 which satisfy the homogeneous eq.

$ay'' + by' + c = 0$ (11.2)

2. Find one particular solution (integral) which solves (11.1) y_P

3. If $y_1(x)$ and $y_2(x)$ are linearly indep. then $y = y_1 + y_P$ and $y = y_2 + y_P$ are two linearly indep. solutions to (11.1) check out ✓

Linearly independent functions

A set of N linearly dependent functions $f_1, f_2, \ldots, f_N$ satisfies

$$\sum_{i=1}^{N} c_i f_i(x) = 0 \quad \forall x \text{ for some non-zero } c_i$$

Otherwise our f_is are linearly independent (more later)

Recall that $e^{\lambda x}$ is an eigenfunction of d/dx; it is also an eigenfunction of d^2/dx^2, etc...

L11.2 $\frac{d^2}{dx^2}(e^{\lambda x}) = \lambda^2 e^{\lambda x}$

Therefore the complementary functions of (11.1) must have the form $y_c = e^{\lambda x}$.

Plug y_c into (11.2):

$$a\lambda^2 + b\lambda + c = 0 \qquad \text{characteristic (auxiliary) equation} \qquad (11.3)$$

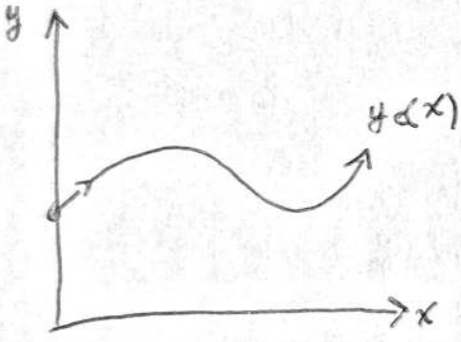
Let λ_1, λ_2 be roots of (11.3)

<u>case 1</u> $\lambda_1 \neq \lambda_2$ $\quad y_1 = Ae^{\lambda_1 x},\ Be^{\lambda_2 x} = y_2$

In this case y_1 and y_2 are linearly independent and <u>complete</u>; they form a <u>basis</u> of the solution space for (11.2). Any other solution to (11.2) can be written as a linear combination of y_1, y_2

e.g. $f(x) = c_1 y_1(x) + c_2 y_2(x)$

Here, general form of complementary function $y_c = Ay_1 + By_2 = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$



given $y(0) = y_0$
$y'(0) = y_0'$

L12.1 Homogeneous 2^{nd} order linear ODEs w/ non-constant coefficients

general form: $y'' + p(x)y' + q(x)y = 0$ (12.1)

Reduction of Order

Method to find second solution (y_2) given one solution (y_1)

try form: $y_2(x) = \underbrace{v(x)}_{\text{TBD}} y_1(x)$ $\quad y_2' = v'y_1 + vy_1'$

$y_2'' = v''y_1 + 2v'y_1' + vy_1''$

If y_2 is a solution, $y_2'' + py_2' + qy_2 = 0$

$$\Rightarrow v\underbrace{(y_1'' + py_1' + qy_1)}_{\text{zero}} + v'(2y_1' + py_1) + v''y_1 = 0$$

since y_1 is a solution to (12.1)

Let $u = v'$, so $u'y_1 + u(2y_1' + py_1) = 0$ is a separable first order ODE:

$\frac{u'}{u} = -\frac{2y_1' + py_1}{y_1}$. Integrate both sides to get $u(x)$

$\Rightarrow v(x)$ and hence $y_2(x)$

e.g. $y'' - 4y' + 4y = 0$ (11.4)

$y_1 = e^{2x}$ try $y_2 = v(x)e^{2x}$, $y_2' = (v' + 2v)e^{2x}$, $y_2'' = (v'' + 4v' + 4v)e^{2x}$

$\therefore v'' + \cancel{4v'} + \cancel{4v} - \cancel{4v'} - \cancel{8v} + \cancel{4v} = 0 \Rightarrow v'' = 0$, $v = Ax + B$

$y_2(x) = (A + Bx)e^{2x}$ (or just Axe^{2x})

Phase space A DE of order n defines the n^{th} derivative $y^{(n)}(x)$ in terms of $y(x), y'(x), \dots, y^{(n-1)}(x)$. Therefore, the state of the solution to the DE can be described by an n-dimensional solution vector

$$\underline{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}$$

L12.2

<u>Comments</u> · For each value of x, the solution vector corresponds to a single point in n-dimensional phase space.

· As x varies, $\underline{Y}(x)$ traces out a <u>trajectory</u> in phase space.

<u>Example</u> $y'' + 4y = 0$ (12.3)

(i.e. spring: $\ddot{y} = -\frac{k}{m} y$)

small θ pendulum

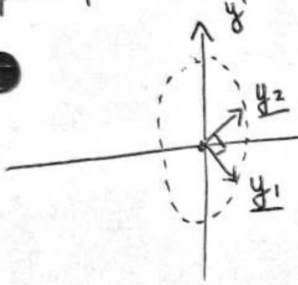
$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

$$y_1' = -2\sin 2x, \quad y_2' = 2\cos 2x$$

<u>solution vectors</u>

$$\underline{Y}_1 = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix} \qquad \underline{Y}_2 = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$$

2D phase space:



Linear independence of solution vectors
feat. <u>Wronskian</u>

· The general sol^{n}s $y_1, \dots, y_n$ to an n^{th} order DE are linearly independent if their solution vectors $\underline{Y}_1, \dots, \underline{Y}_n$ are also linearly independent

· The <u>Wronskian</u> $W(x)$ is the determinant of the fundamental matrix formed by placing $\underline{Y}_i$ in the i^{th} column.

$$W(x) = \begin{vmatrix} \uparrow & & \uparrow \\ \underline{Y}_1 & \cdots & \underline{Y}_n \\ \downarrow & & \downarrow \end{vmatrix} = \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

· The solution vectors are linearly independent if their wronskian is not $0(x)$.

· However, note that $W(x) \equiv 0$ does not imply linear dependence

Example: $y'' + 4y = 0$ (12.3)

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

$$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0$$

$\Rightarrow$ linearly independent

L12.3

Abel's Theorem   Niels Henrik Abel (1802-1829)

Consider a 2^{nd} order homogeneous ODE $y'' + p(x)y' + q(\cancel{y}x)y = 0$.

If $p(x)$ and $q(x)$ are continuous on an interval I, then the Wronskian W is either $\equiv 0$ or $\neq 0 \ \forall x \in I$.

Proof of Abel's theorem

- Let y_1 and y_2 be solutions to the 2^{nd} order homogeneous linear ODE

$y'' + p(x)y' + q(x)y = 0$. Then $y_2(y_1'' + py_1' + qy_1) = 0$ (13.2)

and $y_1(y_2'' + py_2' + qy_2) = 0$ (13.3) Subtracting one from the other gives

$$(y_2y_1'' - y_1y_2'') + p(y_2y_1' - y_1y_2') = 0 \qquad (13.4)$$

$W = y_1y_2' - y_2y_1' \Rightarrow W' = y_1y_2'' + \cancel{y_1'y_2'} - \cancel{y_2'y_1'} - y_2y_1''$

- (13.4) $\Rightarrow W' + pW = 0$ (13.5) $\Rightarrow$ $\boxed{W(x) = W_0 e^{-\int_{x_0}^{x} p(u)\,du}.\ (13.6)\ \text{Abel's identity}}$

Exponential function is never zero, hence

if $W_0 = 0$, $W(x) \equiv 0$, and

if $W_0 \neq 0$, $W(x) \neq 0\ \forall x$. $\square$

Corollary: If $p(x) = 0$, $W = \text{const}$.

Note: Abel's identity is particularly useful for analysing DEs w/ no closed form sol^n,

- we can use (13.6) to find $W(x)$

Example: Bessel's Equation

$x^2y'' + xy' + (x^2 - n^2)y = 0 \longrightarrow y'' + \frac{1}{x}y' + \frac{x^2-n^2}{x^2}y = 0$ $\quad P(x) = \frac{1}{x}$

(no closed form solutions)

so $W(x) = W_0 e^{\int^x -\frac{du}{u}} = W_0 e^{-\ln x} = \frac{W_0}{x}$

noice

Application: Abel's identity can be used to find a second solution y_2 given one known solution y_1 (13.6) $\Rightarrow y_1y_2' - y_2y_1' = W_0 e^{-\int_{x_0}^{x} p(u)\,du}$

- Linear first order ODE for y_2

L13.2 Generalization to higher order ODEs

Any linear n^{th} order ODE can be written in the form

- $\underline{Y}' + A(x)\underline{Y} = \underline{0}$ (will be shown later)

it can be shown then $W' + Tr(A)W = 0 \Rightarrow W = W_0 e^{-\int_{x_0}^{x} Tr(A)\, du}$

Abel's theorem still holds

Forced (inhomogeneous) second order ODEs

General form: $y'' + p(x)y' + q(x)y = f(x)$ (13.7)

- Methods to find particular integral y_P

Method 1: Guess work

$f(x)$	y_P
e^{mx}	$\longrightarrow Ae^{mx}$
$\left.\begin{matrix}\sin kx \\ \cos kx\end{matrix}\right\}$	$\longrightarrow A\sin kx + B\cos kx$
$\left.\begin{matrix}x^n \\ \underset{\text{degree } n}{P_n(x)}\end{matrix}\right\}$	$\longrightarrow \underset{\text{degree } n}{q_n(x)}$

- plug guess into (13.7) & solve for coeff.

Note: since (13.7) is linear, we can superpose terms

Example: $y'' - 5y' + 6y = 2x + e^{4x}$

try $y_P = ax + b + ce^{4x}$

$\Rightarrow y_P = \frac{1}{2}e^{4x} + \frac{x}{3} + \frac{5}{18}$

- Method 2: Variation of parameters

let $y_1(x)$ and $y_2(x)$ be linearly independent complementary functions with solution vectors $\underline{Y}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ and $\underline{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ which form a basis of

L13.3 our solution space.

$\xrightarrow[*]{*}$ Then, the solution vector for the particular integral satisfies

$$\underline{Y}_p(x) = \begin{pmatrix} y_p \\ y_p' \end{pmatrix} = u(x)\,\underline{Y}_1(x) + v(x)\,\underline{Y}_2(x) \qquad (13.8)$$

$(13.8) \Rightarrow \quad y_p = u(y_1) + v(y_2) \quad (a) \Rightarrow y_p' = u y_1' + y_1 u' + v y_2' + v' y_2 \quad (e)$

$\qquad\qquad y_p' = u y_1' + v y_2' \quad (b) \Rightarrow y_p'' = u y_1'' + u' y_1' + v' y_2' + v y_2'' \quad (c)$

$(c) + p(b) + q(a) \Rightarrow \boxed{u' y_1' + v' y_2' = f(x)} \quad (d)$

$(e) - (b) \Rightarrow \boxed{y_1 u' + y_2 v' = 0} \quad (f)$

(d) and (f) $\Rightarrow$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$\therefore\ u' = -\dfrac{y_2}{W} f$ and $v' = \dfrac{y_1}{W} f$

ergo $y_p = y_2 \displaystyle\int^x \frac{y_1(u) f(u)}{W(u)}\, du$

$\qquad \overset{\text{minus}}{+}\ y_1 \displaystyle\int^x \frac{y_2(u) f(u)}{W(u)}\, du$

 Forced second order ODEs (cont.)

We can find both complementary functions for some special cases.

- <u>Linear Equidimensional Equations</u>

General form: $ax^2y'' + bxy' + cy = f(x)$ (14.1)

(14.1) is equidimensional since LHS is unaffected by the change of variables

$$X = \alpha x \Rightarrow \cancel{\frac{dX}{}} \quad \frac{d}{dX} = \frac{1}{\alpha}\frac{d}{dx}, \quad \frac{d^2}{dX^2} = \frac{1}{\alpha^2}\frac{d^2}{dx^2}$$

Here $\frac{a}{\cancel{\alpha^2}} X^2 \frac{d^2y}{dX^2} + \frac{b}{\cancel{\alpha}} X \frac{dy}{dX} + cy = f\left(\frac{X}{\alpha}\right)$. ??? yes! since f "equidimensional"

- Complementary functions for linear equidimensional equations

1. Note $y = x^k$ is eigenfunction of operator $\left[x\frac{d}{dx}\right]$

To solve $ax^2y'' + bxy' + cy = 0$, try $x^k = y$. (14.2)

$xy' = kx^k$ & $x^2y'' = k(k-1)x^k$.

$\Rightarrow ak(k-1) + bk + c = 0$ solve for k_1, k_2.

then $y_c = Ax^{k_1} + Bx^{k_2}$. (14.3)

2. The transformation $z = \ln x$ turns (14.1) into an equation with const. coeff.

$$a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f(e^z) \quad \text{[exercise]}$$

Characteristic Eq $(y_c = e^{\lambda z})$

$a\lambda^2 + (b-a)\lambda + c = 0 \Rightarrow y_c = Ae^{k_1 z} + Be^{k_2 z}$

solve for $\lambda = k_1, k_2$ $\quad = Ax^{k_1} + Bx^{k_2}$ as in (14.3)

Note: for <u>repeated roots</u> $k_1 = k_2 = k$, $y_c = Ae^{kz} + Bze^{kz}$

$$= Ax^k + Bx^k(\ln x)$$

<u>Forced oscillating systems & resonance</u>

- <u>Example</u>: Consider a frictionless oscillator with amplitude y and frequency ω_0: $\ddot{y} + \omega_0^2 y = 0$ (14.4) $\quad y = A\sin\omega_0 t + B\cos\omega_0 t$. SHM

L14.2

Consider forcing at frequency ω_0:

$$\ddot{y} + \omega_0^2 y = \underset{\substack{\uparrow \\ \text{resonant}}}{\sin \omega_0 t} \qquad (14.5)$$

guess $y_p = C \sin \omega_0 t + D \cos \omega_0 t$ but $\ddot{y}_p + \omega_0^2 y_p = 0 \ \forall C, D$
so can't cancel forcing term

<u>Detuning</u> (revisited)

Consider $\ddot{y} + \omega_0^2 y = \sin \omega t \quad (\omega \neq \omega_0) \qquad (14.6)$

$$y_p = C \sin \omega t \Rightarrow C(-\omega^2 + \omega_0^2) = 1$$

$$\Rightarrow y_p = \frac{1}{\omega_0^2 - \omega^2} \sin \omega t$$ undefined for $\omega = \omega_0$

<u>Note</u>: since 14.6 is linear, we can add a const. multiple of y_c to y_p, e.g.

$$y_p = \frac{\sin \omega t}{\omega_0^2 - \omega} + A \sin \omega_0 t$$ so pick $A = -\frac{1}{\omega_0^2 - \omega^2}$ to get

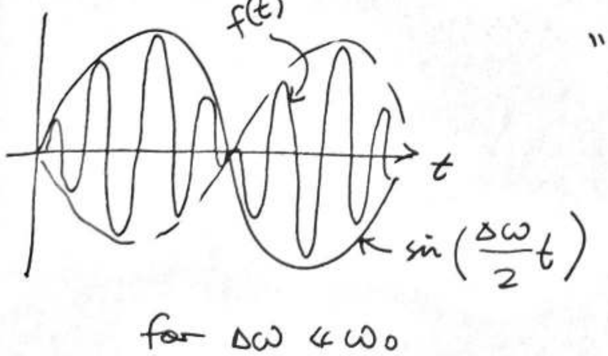
$$y_p = \frac{\sin \omega t - \sin \omega_0 t}{\omega_0^2 - \omega^2} = \frac{2}{\omega_0^2 - \omega^2}\left[\cos\left(\frac{\omega + \omega_0}{2}\right)t * \sin\left(\frac{\omega - \omega_0}{2}\right)t\right]$$

verify using angle addition formulae

let $\Delta\omega = \omega_0 - \omega$

$$\frac{\omega_0 + \omega}{2} = \omega_0 - \frac{(\omega_0 - \omega)}{2} = \omega_0 - \frac{\Delta\omega}{2}$$

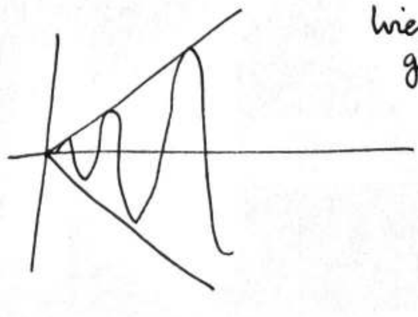
$$y_p = \frac{-2}{(\omega_0 + \omega)\Delta\omega}\underbrace{\left[\cos\left(\omega_0 - \frac{\Delta\omega}{2}\right)t \cdot \sin\left(\frac{\Delta\omega}{2}t\right)\right]}_{f(t)}$$



for $\Delta\omega \ll \omega_0$

"beating"

$$\lim_{\Delta\omega \to 0} \frac{\sin\left(\frac{\Delta\omega}{2}\right)t}{\frac{\Delta\omega}{2}t} = 1$$



linear growth in amplitude

L14.3

$$\lim_{\Delta\omega \to 0} \left(y_p = -\frac{2}{(\omega_0 + \frac{\Delta\omega}{2})\cancel{\Delta\omega}} (\cos \omega_0 t) \left(\frac{\cancel{\Delta\omega}}{2} t \right) \right)$$

$$= -\frac{t}{2\omega_0} \cos \omega_0 t \text{, as seen in graph}$$

<u>General rule</u>: For const. coeff. DEs if forcing term is a linear combi of cfs (resonant), then y_p is t times the non-resonant case.

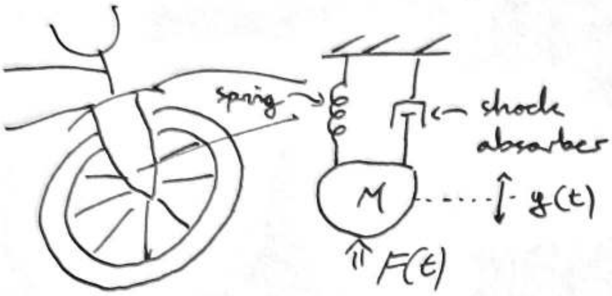
L15.1

Forced 2nd order ODEs (cont.)

Transients + Damping

Many physical systems have both a restoring force and damping.

Example Dirt bike suspension



N2 gives $m\ddot{y} = F(t) - Ky - L\dot{y}$ (15.1)

$\underbrace{Ky}_{\text{spring}}$, $\underbrace{L\dot{y}}_{\text{shock absorber}}$

$$\therefore \ddot{y} + \frac{L}{M}\dot{y} + \frac{K}{M}y = \frac{1}{M}F(t)$$

Let $\tau = \sqrt{\frac{K}{M}}\,t$, $y = y(\tau)$.

$$\Rightarrow \frac{d}{dt} = \sqrt{\frac{K}{M}}\frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = \frac{K}{M}\frac{d^2}{d\tau^2}$$

giving $\ddot{y} + 2\kappa\dot{y} + y = f(\tau)$

where $\dot{(\ )} = \frac{d}{d\tau}$, and

$$\kappa = \frac{L}{2\sqrt{KM}}, \quad f = \frac{F}{M}$$

System is described by one parameter κ

Free (natural) response: $f = 0$

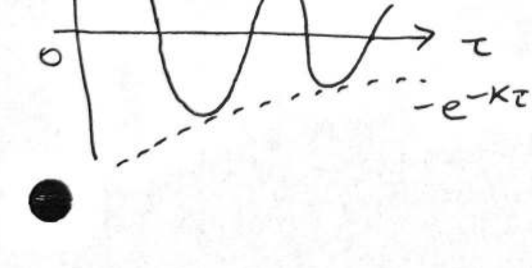
$$\ddot{y} + 2\kappa\dot{y} + y = 0, \quad y = e^{\lambda\tau}$$

$$\lambda^2 + 2\kappa\lambda + 1 = 0$$

$$\lambda = -\kappa \pm \sqrt{\kappa^2 - 1} \quad (= \lambda_1, \lambda_2)$$

Case 1 : $\kappa < 1$ Underdamped

λ_1, λ_2 complex : $y = e^{-\kappa\tau}\left(A \sin\sqrt{-\kappa^2+1}\,\tau + B\cos\sqrt{1-\kappa^2}\,\tau\right)$



Damped oscillator

period $= \frac{2\pi}{\sqrt{1-\kappa^2}}$ is longer

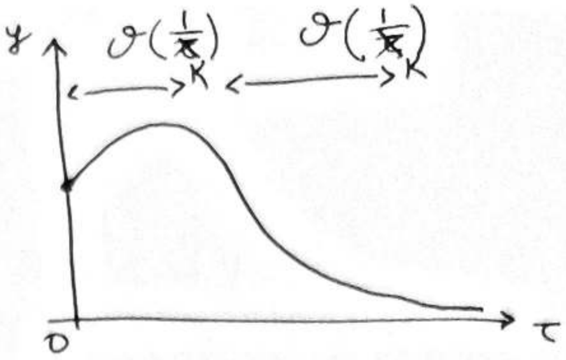
larger κ increases period

as $\kappa \to 1$, period $\to \infty$

Case 2 : $\kappa = 1$ Critically damped

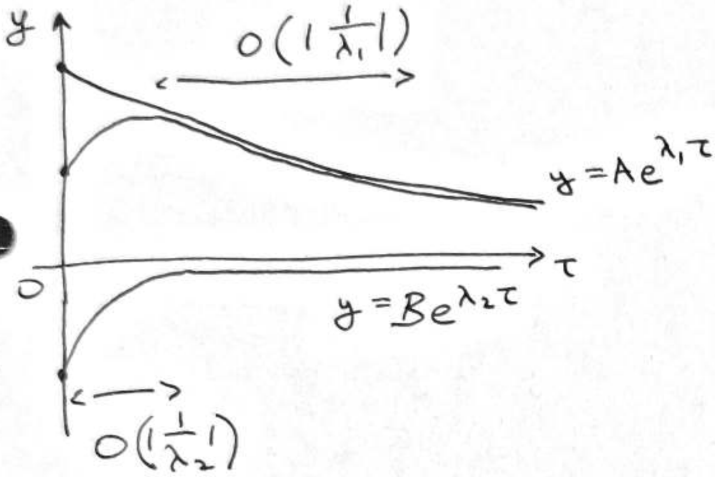
- $\lambda_1 = \lambda_2$ real (degenerate)

$$y = (A + B\tau)e^{-\kappa\tau}$$



Case 3 : $\kappa > 1$ Overdamped

$\lambda_1, \lambda_2 < 0$, real (let $\lambda_2 < \lambda_1$)



Note: unforced response decays

In general, for forced, damped systems, <u>cfs</u> give transient response to initial conditions, while the <u>pi</u> gives long time response.

Damping and resonance

Consider $\ddot{y} + \mu\dot{y} + \omega_0^2 y = \sin\omega t$ (c.f. 14.6 detuning)

$$y_p = A\sin\omega t + B\cos\omega t \;\cdots = \frac{1}{(\omega_0^2-\omega^2)^2 + \mu^2\omega^2}\left[(\omega_0^2-\omega^2)\sin\omega t - \mu\omega\cos\omega t\right]$$

In the limit $\omega \to \omega_0$,

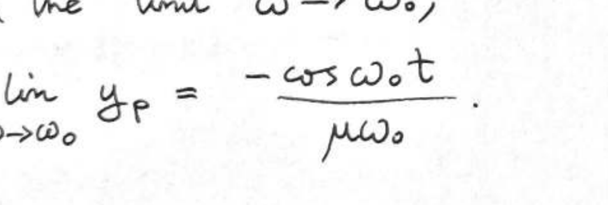
$$\lim_{\omega\to\omega_0} y_p = \frac{-\cos\omega_0 t}{\mu\omega_0}.$$

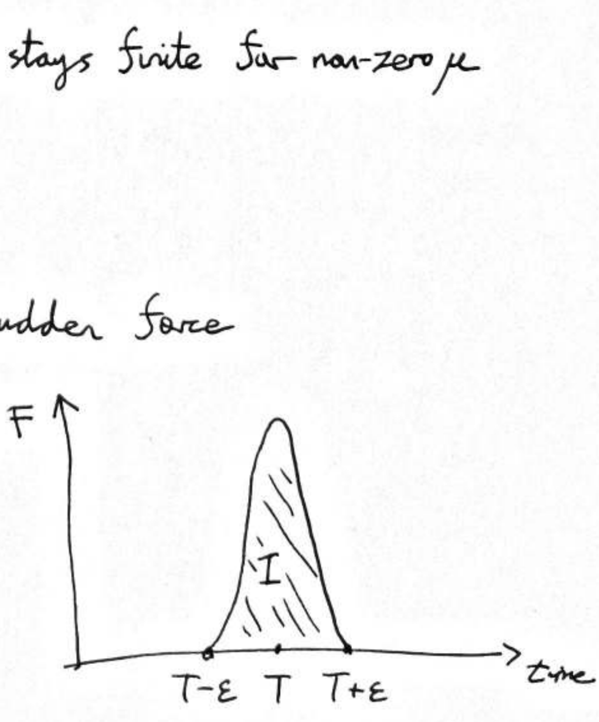
$|y_p|$ stays finite for non-zero μ

Impulses and point forces

Consider a system that experiences a sudden force

e.g. mountainbike riding over a curb





Consider $\lim_{\varepsilon \to 0}$: Force becomes a sudden instantaneous "impulse"

(15.1) $m\ddot{y} = F(t) - Ky - L\dot{y}$

Integrate w.r.t. time from $T-\varepsilon$ to $T+\varepsilon$ and take limit:

$$\lim_{\varepsilon \to 0}\left[m\int_{T-\varepsilon}^{T+\varepsilon} \ddot{y}\, dt = \underbrace{\int_{T-\varepsilon}^{T+\varepsilon} F(t)\, dt}_{\equiv I} - k\underbrace{\int_{T-\varepsilon}^{T+\varepsilon} y\, dt}_{\substack{\to 0 \\ \text{since } y \text{ finite}}} - L\int_{T-\varepsilon}^{T+\varepsilon} \dot{y}\, dt \right]$$

$$\lim_{\varepsilon \to 0}\left[M\left[\dot{y}\right]_{T-\varepsilon}^{T+\varepsilon} = I - \underbrace{L\left[y\right]_{T-\varepsilon}^{T+\varepsilon}}_{\substack{0 \text{ since } y \\ \text{is continuous}}} \right]$$

Hence $I \equiv \lim_{\varepsilon \to 0} \int_{T-\varepsilon}^{T+\varepsilon} F(t)\, dt = \lim_{\varepsilon \to 0} M\left[\dot{y}\right]_{T-\varepsilon}^{T+\varepsilon}$

Note: the only feature of $F(t;\varepsilon)$ that matters for the response of the system as $\varepsilon \to 0$ is the integral of the force.

L16.1 Impulses and Point Forces (cont.)

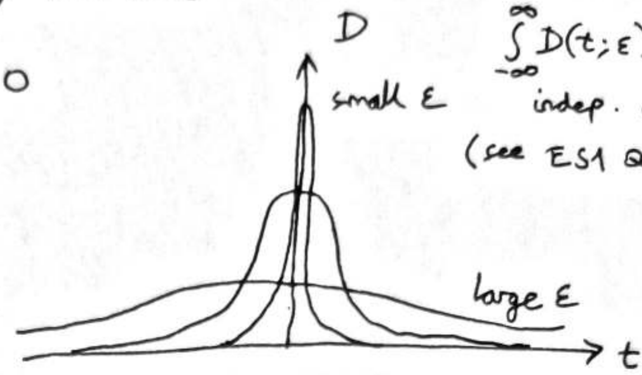
Dirac delta 'function'

- Paul Dirac 1902 – 1984
  Lucasian Prof 1932 – 1969

Consider a family of functions $D(t;\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} D(t;\varepsilon) = 0 \quad \text{for all } t \neq 0$$

& $$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} D(t;\varepsilon)\, dt = 1.$$



$\int_{-\infty}^{\infty} D(t;\varepsilon)\,dt$ indep. of ε (see ES1 Q14)

Example: $$D(t;\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}}\, e^{-t^2/\varepsilon^2}$$

- Define Dirac delta function $\delta(t)$ as $\lim_{\varepsilon \to 0} D(t;\varepsilon)$.

Note $\delta(t=0)$ is undefined, and technically δ is not a function, but rather a "functional distro."

Properties of δ:

1) $\delta(x) = 0 \ \forall\ x \neq 0$
2) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$
3) sampling property: For all functions $g(x)$ which are continuous at $x=0$,

$$\int_{-\infty}^{\infty} g(x)\,\delta(x)\, dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} g(x)\, D(x;\varepsilon)\, dx$$

$$= \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} g(x)\, D(x;\varepsilon)\, dx = g(0) \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} D(x;\varepsilon)\, dx$$

$$= g(0) \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} D(x;\varepsilon)\, dx = g(0).$$ ☺

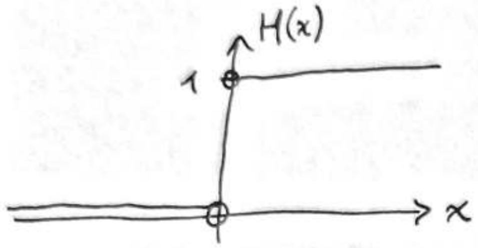
- More generally $$\int_a^b g(x)\,\delta(x-x_0)\, dx = \begin{cases} g(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases} \qquad (16.1)$$

Heaviside Step Function $H(x)$

Oliver Heaviside 1850–1925

- $$H(x) =: \int_{-\infty}^{x} \delta(t)\, dt \qquad (16.2)$$

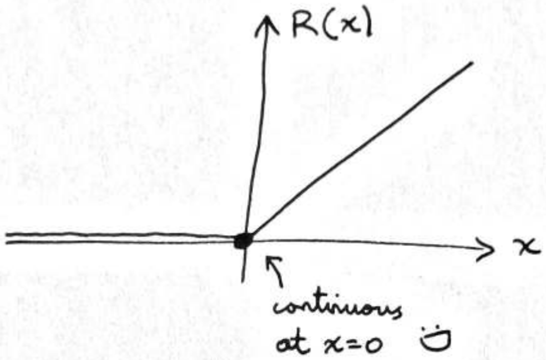
From properties of δ,
$H(x) = 0$ for $x < 0$
$H(x) = 1$ for $x > 0$
$H(x)$ is undefined at $x = 0$.



From FTC, $\dfrac{dH}{dx} = \delta(x)$.

- **Ramp function** $R(x)$

$$R(x) = \int_{-\infty}^{x} H(t)\, dt$$



Note: functions get smoother as we integrate

Delta function forcing

- Consider $y'' + p(x)y' + q(x)y = \delta(x - x_0) \qquad (16.3)$

for p, q continuous. $y(x)$ must satisfy $y'' + p(x)y' + q(x)y = 0$

for $x < x_0$ and $x > x_0$.

$y(x)$ must also satisfy two "jump conditions"

1. $y(x)$ is continuous at $x = x_0$ or $\left[y\right]_{x_0^-}^{x_0^+} = 0$.
2. $y'(x)$ has a jump of 1 at $x = x_0$ or $\left[y'\right]_{x_0^-}^{x_0^+} = 1$.

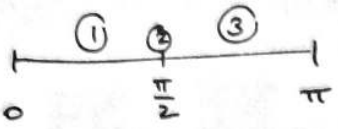
-

L16.3

Example $y'' - y = 3\delta(x - \frac{\pi}{2})$ (16.4)

& $y = 0$ at $x = 0$ and $x = \pi$

Consider 3 regions



① $0 \le x < \frac{\pi}{2}$: $y'' - y = 0$

$$y = Ae^{x} + Be^{-x}$$

$$= A\sinh x + B\cosh x \quad \text{(different } A, B\text{)}$$

$y = 0$ at $x = 0$

$\Rightarrow y = A\sinh x$

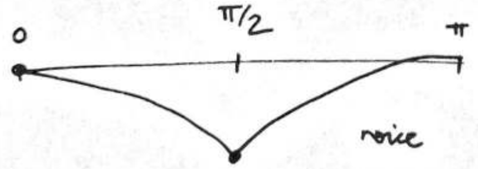
③ By symmetry, $y = C\sinh(\pi - x)$

② $[y]_{\frac{\pi}{2}-}^{\frac{\pi}{2}+} = 0 \Rightarrow A\sinh\frac{\pi}{2} = C\sinh\frac{\pi}{2} \Rightarrow A = C$

Integrate (16.4) : $\lim_{\varepsilon \to 0} \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} (16.4)\, dx$ KEK

$\Rightarrow [y']_{\frac{\pi}{2}-}^{\frac{\pi}{2}+} = 3 \Rightarrow -C\cosh\frac{\pi}{2} - A\cosh\frac{\pi}{2} = 3$

$\Rightarrow A = C = \dfrac{-3}{2\cosh\frac{\pi}{2}}$ so $y = \begin{cases} -\frac{3}{2}\frac{\sinh x}{\cosh \pi/2} & \text{for } 0 \le x \le \frac{\pi}{2}, \\ -\frac{3}{2}\frac{\sinh(\pi - x)}{\cosh \pi/2} & \text{for } \frac{\pi}{2} \le x \le \pi. \end{cases}$



Heaviside Step Function forcing

Consider $y'' + p(x)y' + q(x)y = H(x - x_0)$ (16.5)

for p, q continuous.

$y(x)$ satisfies $y'' + py' + qy = 0$ for $x < x_0$

and $y'' + py' + qy = 1$ for $x > x_0$

AND $[y']_{x_0^-}^{x_0^+} = [y]_{x_0^-}^{x_0^+} = 0$, i.e. y & y' are continuous at $x = x_0$

L17.1 Impulse forcing (cont.)

- jump conditions

Consider $y'' + p(x)y' + q(x)y = f(x)\,H(x-x_0)$ (17.1)

for p, q, f continuous.

Evaluate (17.1) on both sides of x_0:

$$[y'']_{x_0^-}^{x_0^+} + p(x_0)\underbrace{[y']_{x_0^-}^{x_0^+}}_{\text{zero}} + q(x_0)\underbrace{[y]_{x_0^-}^{x_0^+}}_{\text{zero}} = f(x_0)$$

$[y'']_{x_0^-}^{x_0^+} = f(x_0) \Rightarrow y'' \sim H(x)$ near x_0

- If $y'' \sim H \Rightarrow \underbrace{y' \sim R, \; y \sim \int r(x)}_{\text{continuous}}$

Use $[y]_{x_0^-}^{x_0^+} = [y']_{x_0^-}^{x_0^+} = 0$ along with ICs / BCs to solve 17.1

Example: switched circuit ($Q(t) = $ charge)

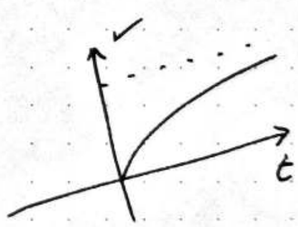
$$\dot{Q} + \frac{1}{RC} Q = \frac{V}{R} H(t) \quad (17.2)$$

with $Q = 0$ at $t = 0$

- near $t = 0$: $\dot{Q} \sim H$, $Q \sim R$ is continuous

jump condition: $[Q]_{0^-}^{0^+} = 0$

$t > 0$: $\dot{Q} + \frac{1}{RC} Q = \frac{V}{R} \Rightarrow Q = VC + Be^{-t/RC}$



$[Q]_{0^-}^{0^+} = 0 \Rightarrow Q(t = 0^+) = 0$

$\Rightarrow B = -VC, \quad Q = VC(1 - e^{-t/RC})$

Higher order discrete (difference) equations

- General form for m^{th} order linear discrete equation with constant coeffs: $a_m y_{n+m} + a_{m-1} y_{n+m-1} + \dots + a_0 y_n = f_n$ (17.3)

Closely related to higher order ODEs, use same principles

L17.2

$x_{n+2} + x_n = 0$

$x_n = A(i)^n + B(-i)^n$

need $A = B$

$i^n = (\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2})$

so $(A+B) \cos \frac{n\pi}{2}$ + something

<u>Difference operator</u> $D[y_n] = y_{n+1}$

has eigenfunction $y_n = k^n$ since

$D[k^n] = k^{n+1} = k \cdot k^n = k \cdot y_n$.

(17.3) is linear in y, so $y_n = y_n^{(c)} + y_n^{(p)}$ ← particular integral

↑ complementary function, $f=0$

<u>Example</u> $a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = f_n$

Homogeneous eq^n with $y_n = k^n$:

$a_2 k^{n+2} + a_1 k^{n+1} + a_0 k^n = 0$

$\Rightarrow a_2 k^2 + a_1 k + a_0 = 0$.

Solve quadratic for k_1, k_2, and then

$$y_n^{(c)} = \begin{cases} A k_1^n + B k_2^n & \text{when } k_1 \neq k_2, \\ (A + Bn) k^n & \text{if } k_1 = k_2 = k. \end{cases}$$

<u>Particular integral forms</u>

f_n	$y_n^{(p)}$	
k^n	$a \cdot k^n$	if $k \neq k_1, k_2$
k_1^n	$a n k_1^n$	may need higher powers
n^p	$A_p n^p + \cdots + A_0$	(p degree polynomial)

<u>Example: Fibonacci Sequence</u>

$1, 1, 2, \ldots$ $\quad y_n = y_{n-1} + y_{n-2}$, $y_0 = y_1 = 1$

standard form: $y_{n+2} - y_{n+1} - y_n = 0$

Try $y = k^n$: $k^2 - k + 1 = 0$, so $k = \frac{1 \pm \sqrt{5}}{2} = \phi, 1 - \phi = \psi$

Hence $y_n = A\varphi^n + B(-\frac{1}{\varphi})^n$. "Golden ratio"

$n=0: \quad y_0 = 1 = A + B$

$n=1: \quad y_1 = 1 = A\varphi - \frac{B}{\varphi}$

so $A = \frac{\varphi}{\sqrt{5}}, \quad B = \frac{\frac{1}{\varphi}}{\sqrt{5}}$.

so then $y_n = \dfrac{\varphi^{n+1} - \left(-\frac{1}{\varphi}\right)^{n+1}}{\sqrt{5}}$ is an integer #soy #blessed

Also $\lim_{n \to \infty} \frac{y_{n+1}}{y_n} = \varphi$.

Series Solutions for higher order ODEs

Method of Frobenius

When we can't find closed form solutions to linear higher order ODEs, series solutions can be very useful.

Method depends on properties of singular points

Consider $p(x)y'' + q(x)y' + r(x)y = 0$. (17.4)

The point $x = x_0$ is ordinary in (17.4) if $\frac{q}{p}$ and $\frac{r}{p}$ are analytic at $x = x_0$ (i.e. there is a Taylor series that works).

Otherwise $x = x_0$ is a singular point.

cool:

$Ar^n(\cos n\theta + i\sin n\theta)$

$+ Br^n(\cos n\theta - i\sin n\theta)$

$(A+B)r^n\cos n\theta$

$+ i(A-B)r^n\sin n\theta$

L18.1

Singular points + Series solutions

- If x_0 is a singular point, but (17.3) can be written as

$$P(x)(x-x_0)^2 y'' + Q(x)(x-x_0)y' + R(x)y = 0$$

where $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic then x_0 is a regular singular point.

Note $\frac{Q}{P} = (x-x_0)\frac{q}{p}$ and $\frac{R}{P} = (x-x_0)^2\frac{r}{p}$

Otherwise x_0 is an irregular singular point (☹)

- Examples 1. $(1-x^2)y'' - 2xy' + 2y = 0$

$\frac{q}{p} = -\frac{2x}{1-x^2}$ $\quad \lim_{x\to\pm1} \frac{q}{p} = \infty$ so singular points $1, -1$

$(x-1)\frac{q}{p} = \frac{2x\cancel{(1-x)}}{\cancel{(1-x)}(1+x)} = \frac{2x}{1+x}$ works at $x=1$ ✓

similar for $\frac{r}{p}$.

2. $(\sin x)y'' + (\cos x)y' + 2y = 0$

$\frac{q}{p} = \frac{\cos x}{\sin x}$, $\frac{r}{p} = \frac{2}{\sin x}$ so singular $x = n\pi$

using $\sin x \sim x$ note these are regular

3. $(1+\sqrt{x})y'' - 2xy + 2y = 0$

$\frac{q}{p} = \frac{-2x}{1+\sqrt{x}}$ has 0 as an irregular singular point
(try Taylor series) $\sqrt{x}$ not differentiable

Fuch's Theorem 1. If x_0 is an ordinary point, there are two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ convergent in some neighbourhood of x_0.

L18.2

2. If x_0 is a <u>regular singular point</u> there is at least one solution of the form $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\sigma}$

where σ is real and $a_0 \neq 0$.

<u>Example of case 1</u> $(1-x^2)y'' - 2xy' + 2y = 0$ (18.1)

Find series solutions about $x=0$.

$x=0$ is <u>ordinary</u> so try $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$x^2 * (18.1)$: $(1-x^2)x^2y'' - (2x^2)xy' + (2x^2)y = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n [n(n-1) - x^2 n(n-1)] x^n$$

$$- 2\sum_{n=1}^{\infty} a_n [nx^2] x^n + 2\sum_{n=0}^{\infty} a_n [x^2] x^n = 0$$

Equate coeffs. of same power (for $n \geq 2$) $\triangleleft$

$$a_n n(n-1) - a_{n-2}(n-2)(n-3) - 2a_{n-2}(n-2) + 2a_{n-2} = 0$$

<u>recursion relation</u> (discrete equation) $\triangle$

$$n(n-1)a_n = n(n-3)a_{n-2}$$

$$\Rightarrow a_n = \frac{n-3}{n-1} a_{n-2} \quad \text{for } n \geq 2 \qquad (18.2)$$

a_0 & a_1 aren't determined by recursion relation and are arbitrary constants, set from ICs $\triangleleft$

Consider odd terms in (18.3)

$n=3$: $a_3 = 0 \Rightarrow a_5 = 0 \Rightarrow \cdots \; a_{odd} = 0$

even terms

$$a_n = \frac{n-3}{n-1} a_{n-2} = \frac{\cancel{n-3}}{n-1} \cdot \frac{n-5}{\cancel{n-3}} a_{n-4} = -\frac{1}{n-1} a_0 \quad \ddot{\cup}$$

L18.3

Consider all terms:

- $y = a_1 x + a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \dots\right]$

$\log(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \dots$

$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$

so $y = a_0\left[1 - \frac{x}{2}\log\left(\frac{1+x}{1-x}\right)\right] + a_1 x$

Note: log only nice within $(-1, 1)$

- Case 2 Example

$4xy'' + 2(1-x^2)y' - xy = 0 \quad (18.3)$

$x=0$: regular singular point

try $y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$, $a_0 \neq 0$

$y' = \sum_{n=0}^{\infty} a_n(n+\sigma)x^{n+\sigma-1}$, $y'' = \sum_{n=0}^{\infty} a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-2}$

$x * (18.3)$: $4x^2y'' + 2(1-x^2)xy' - x^2y = 0$

- $\sum_{n=0}^{\infty} a_n\left[4\cancel{x}(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2\right]x^{n+\sigma} = 0$

Equate coeffs. of $x^{n+\sigma}$ $(n \geq 2)$ ◁

$[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma)]a_n$

$+ [-2(n-2+\sigma) - 1]a_{n-2} = 0$

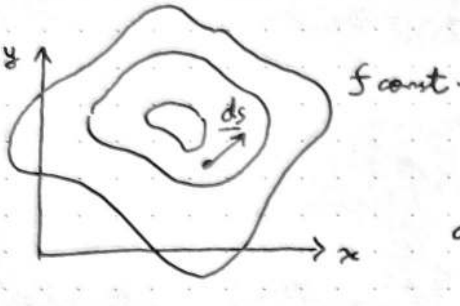
$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}$

recurrence relation ◁

-

Directional derivatives + gradient vector

- Consider $f(x,y)$



Consider a vector displacement

$$\underline{ds} = (dx, dy)$$

change in $f(x,y)$ along $\underline{ds}$ is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \text{(chain rule)}$$

$$= (dx, dy) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$= \underline{ds} \cdot \underline{\nabla} f,$$

where $\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ gives the gradient vector $\underline{\nabla} f$.

If we write $\underline{ds} = ds \cdot \hat{s}$ where $|\hat{s}| = 1$, then

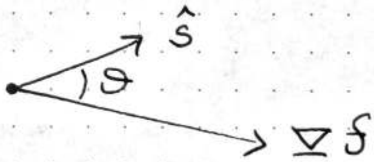
$$\frac{df}{ds} = \hat{s} \cdot \underline{\nabla} f \quad (20.1)$$ is the definition

of the directional derivative ☺

Properties of $\underline{\nabla} f$

1. The magnitude of $\underline{\nabla} f$ is the maximum rate of change of $f(x,y)$.

i.e. $\max_{\forall \theta} \left(\frac{df}{ds} \right) = |\underline{\nabla} f|$



2. The direction of $\underline{\nabla} f$ is the direction in which f increases most rapidly. $\left| \frac{df}{ds} \right| = \overset{1}{\cancel{|\hat{s}|}} \, |\underline{\nabla} f| \cos\theta$

3. If $\underline{ds}$ $\parallel$ to contour of f then $\frac{df}{ds} = 0$

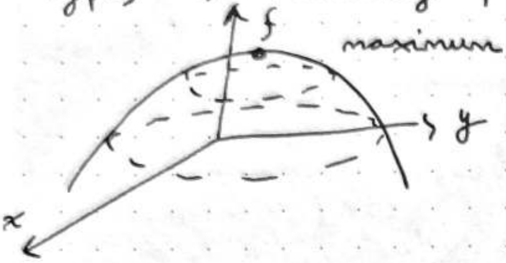
$\Rightarrow \underline{\hat{s}} \cdot \underline{\nabla} f = 0$ from (20.1)

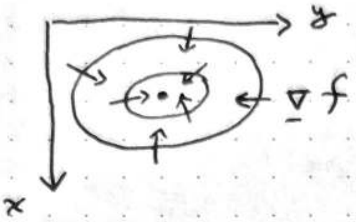
- Hence $\underline{\nabla} f$ is $\perp$ to contours of f and $|\underline{\nabla} f|$ is the slope in the "uphill" direction.

Stationary points There is always one direction where $\frac{df}{ds} = 0$ (parallel to contour of f).

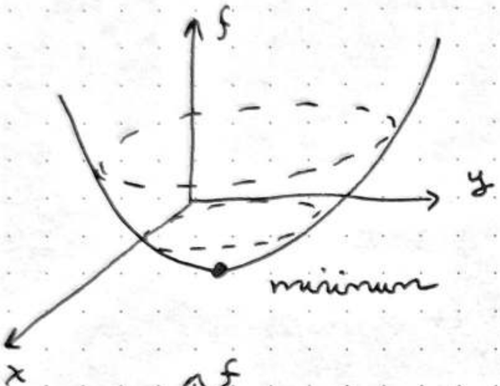
Stationary points have $\frac{df}{ds} = 0$ for all directions.

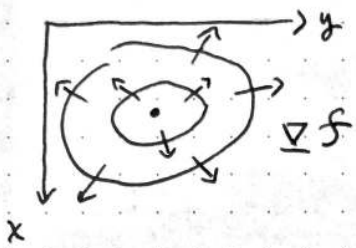
types of stationary points



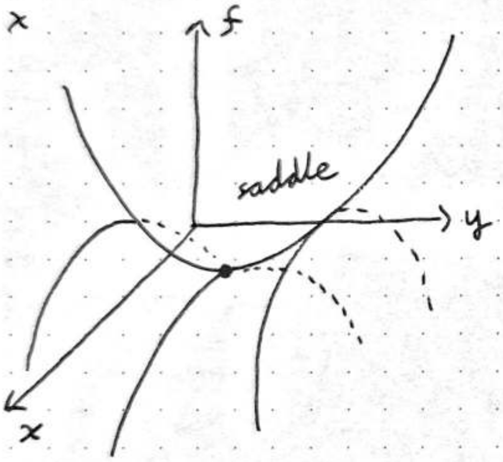


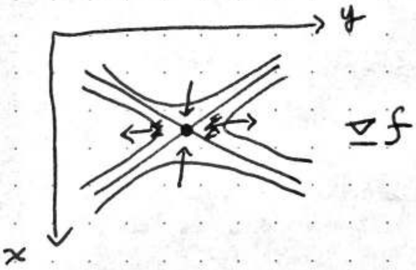
- Near max/min contours of f are elliptic





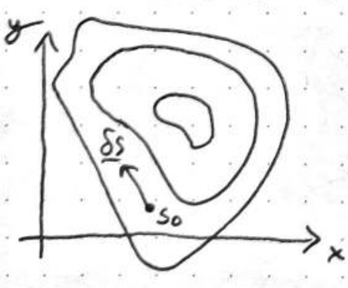
- Near saddle points they are hyperbolic
- Saddle points are the only place where contours cross





Taylor Series for multichadiate functions

Consider $f(x,y)$



Consider a displacement $\underline{\delta s}$ away from so.

$\underline{\delta s} = (\delta x, \delta y)$.

$$\delta s \frac{d}{ds} = \underline{\delta s} \cdot \underline{\nabla} \quad \text{since} \quad \frac{d}{ds} = \hat{s} \cdot \underline{\nabla}$$

The Taylor series along $\underline{\delta s}$ is

$$f(s_0 + \delta s) = f(s_0) + \delta s \frac{df}{ds} + \frac{1}{2}(\delta s)^2 \frac{d^2 f}{ds^2} + \cdots$$

$$= f(s_0) + \underbrace{\underline{\delta s} \cdot \underline{\nabla} f}_{①} + \underbrace{\frac{1}{2}(\delta s)^2 (\hat{s} \cdot \underline{\nabla})(\hat{s} \cdot \underline{\nabla}) f}_{②} + \cdots$$

In Cartesian coordinates,

$$① = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$$

$$② = (\delta s)^2 \left[\hat{s}_x \frac{\partial}{\partial x} + \hat{s}_y \frac{\partial}{\partial y} \right] \left[\hat{s}_x \frac{\partial}{\partial x} + \hat{s}_y \frac{\partial}{\partial y} \right] f$$

$$= \delta x^2 f_{xx} + \delta x \delta y (f_{yx} + f_{xy}) + \delta y^2 f_{yy}$$

$$= (\delta x \quad \delta y) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

<u>Hessian matrix</u>

$$\underline{\underline{H}} = [\underline{\nabla}(\underline{\nabla} f)] = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

<u>Multivariate Taylor Series</u>

2D Cartesian

$$f(x,y) = f(x_0, y_0) + (x - x_0) f_x + (y - y_0) f_y$$
$$+ \frac{1}{2}\left[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0) f_{xy} + (y-y_0)^2 f_{yy} \right]$$
$$+ \cdots \qquad (20.2)$$

General, coordinate independent form (N-dimensions)

$$f(\underline{x}) = f(\underline{x}_0) + \underline{\delta x} \cdot \underline{\nabla} f(\underline{x}_0) + \frac{1}{2} \underline{\delta x} \cdot [\underline{\nabla}\underline{\nabla} f] \cdot \underline{\delta x} + \cdots$$

L20.4

Classification of stationary points

Since $\underline{\nabla} f = \underline{0}$ at a stat. point x_0,

Taylor series $f(x) \simeq f(x_0) + \frac{1}{2} \underline{\delta x} \cdot [\underline{\nabla}\underline{\nabla} f] \cdot \underline{\delta x}$

3 cases: 1 minimum $\underline{\delta x} \cdot \underline{\underline{H}} \cdot \underline{\delta x} > 0 \quad \forall \underline{\delta x}$

$\Rightarrow \underline{\underline{H}}$ is positive definite (yay!)

2. maximum $\underline{\delta x} \cdot \underline{\underline{H}} \cdot \underline{\delta x} < 0 \ \forall \underline{\delta x}$

$\Rightarrow \underline{\underline{H}}$ is -ve definite (woo!)

3. saddle $\underline{\underline{H}}$ is indefinite (hype!)

L21.1

Multiple independent variables (cont.)

- Eigenvalues of Hessian matrix

Since $\underline{\underline{H}}$ is symmetric, it can be diagonalised w.r.t. to its principal axes.

Consider $f(x_1, \dots, x_n)$

$$\underline{\underline{H}} = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & & \ddots & \vdots \\ \vdots & & & \\ f_{x_nx_1} & \cdots & & f_{x_nx_n} \end{bmatrix} \quad \text{i.e.} \quad \underline{\underline{H}}_{ij} = f_{x_ix_j}$$

-

$\underline{\delta x}^T \underline{\underline{H}}\, \underline{\delta x}$ can be written

$$(\delta x_1 \cdots \delta x_n) \; \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix}$$

where λ_i are eigenvalues of $\underline{\underline{H}}$ and δx_i are components of $\underline{\delta x}$ along principal axes (eigenvectors).

Hence $\underline{\delta x}^T \underline{\underline{H}}\, \underline{\delta x} = \lambda_1 \delta x_1^2 + \dots + \lambda_n \delta x_n^2$.

- At maximum $\underline{\delta x}^T \underline{\underline{H}}\, \underline{\delta x} < 0 \;\; \forall \; \underline{\delta x}$

  $\Rightarrow \lambda_i < 0 \;\; \forall i$, i.e. negative definite

At minimum $\underline{\delta x}^T \underline{\underline{H}}\, \underline{\delta x} > 0 \;\; \forall \; \underline{\delta x}$

$\Rightarrow \lambda_i > 0 \;\; \forall i$, i.e. positive definite

Signature of Hessian

The signature of $\underline{\underline{H}}$ is the pattern of the signs of the subdeterminants:

- $$|f_{x_1x_1}|, \quad \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}, \dots, |\underline{\underline{H}}|$$

$$|\underline{\underline{H}}_1|, \quad |\underline{\underline{H}}_2|, \quad \dots, \quad |\underline{\underline{H}}_n| = |\underline{\underline{H}}|.$$

If $\underline{\underline{H}}$ is positive (neg.) definite, then each of the sub-Hessians $\underline{\underline{H}}_1, \underline{\underline{H}}_2, \ldots, \underline{\underline{H}}_{n-1}$ is positive (neg.) definite, since ~~Therefore~~ a maximum point of a function in n-variables is also a maximum in any subspace.

Signatures — for pos. def. $\underline{\underline{H}}$: $+ \; + \; \cdots \; +$

— for neg. def. $\underline{\underline{H}}$: $- \; + \; - \; + \; \cdots \; (-1)^n$

odd terms $-$

— otherwise $\underline{\underline{H}}$ is indefinite

Note: if $|\underline{\underline{H}}| = 0$ stationary point is degenerate : cannot determine the nature of stat. point at this order

Contours of $f(x,y)$

Consider a coordinate system aligned with the principal axes of $\underline{\underline{H}}$: $\underline{\underline{H}} = \text{diag}(\lambda_1, \lambda_2)$ let $\underline{\delta x} = (\underline{x} - \underline{x}_0) = (\xi, \eta)$ where $\underline{x}_0$ is stat. point.

In a small region near stat. point, contours satisfy

$$f = \text{constant} \simeq f(x_0) + \tfrac{1}{2} \underline{\delta x}^T \underline{\underline{H}} \, \underline{\delta x} \quad \text{(Taylor series)}$$

$$\Rightarrow \lambda_1 \xi^2 + \lambda_2 \eta^2 = \text{const.} \quad (21.1)$$

At minima and maxima, λ_1, λ_2 have same sign so that (21.1) is equation for ellipse.

At saddle, λ_1, λ_2 have opposite sign so that (21.1) is the equation of a hyperbola.

Example: Classify stat. points of $f(x,y) = 4x^3 - 12xy + y^2 + 10y$ and sketch contours.

$f_x = 12x^2 - 12y \qquad f_{xx} = 24x$

$f_y = -12x + 2y + 10 \qquad f_{xy} = -12$

$f_{yy} = 2$

stat. points: $f_x = f_y = 0 \Rightarrow y = x^2$

$-12x + 2y + 10 = 0 \Rightarrow (x,y) = (1,1)$ or $(5,25)$

$(1,1)$: $\underline{\underline{H}} = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix}$ $\quad |\underline{\underline{H}}_1| = 24$

$|\underline{\underline{H}}_2| = 48 - 144$

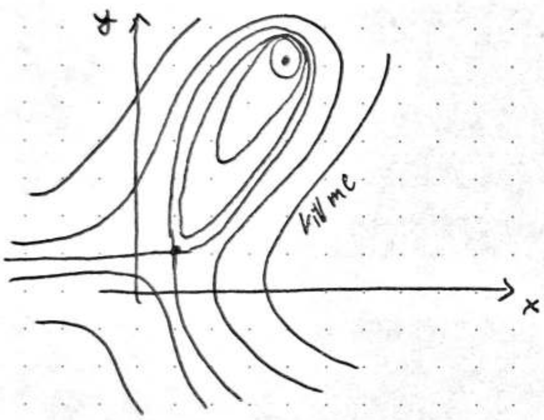
signature is $+ \; -$ $\Rightarrow$ saddle

$(5,25)$: $\underline{\underline{H}} = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix}$ $\quad |\underline{\underline{H}}_1| = 120$

$|\underline{\underline{H}}_2| = 240 - 144$

signature is $+ \; +$ $\Rightarrow$ minimum point

rough sketch



V Part 2: systems of ODEs

Consider two dependent variables $y_1(t)$, $y_2(t)$ which satisfy a system of linear coupled ODEs

e.g. $$\left.\begin{aligned} \dot{y}_1 &= a y_1 + b y_2 + f_1(t) \\ \dot{y}_2 &= c y_1 + d y_2 + f_2(t) \end{aligned}\right\} \quad (21.2)$$

In vector form: $\dot{\underline{Y}} = \underline{\underline{M}}\,\underline{Y} + \underline{F}$

where $\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underline{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Any n^{th} order ODE can be written as a system of n first order ODEs.

e.g. $\ddot{y} + a\dot{y} + by = f \quad (21.4)$

Let $y_1 = y$, $y_2 = \dot{y}$, $\underline{Y} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

hence $$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 + a y_2 + b y_1 &= f \end{aligned} \Rightarrow \dot{\underline{Y}} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \underline{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Systems of <u>linear</u> ODEs (cont.)

- $\dot{\underline{Y}} - \underline{\underline{M}}\,\underline{Y} = \underline{F}$ (22.1)

Solution via matrix methods:

<u>Steps</u>: 1. solve for complementary functions $\underline{Y}_c$

$$\dot{\underline{Y}}_c - \underline{\underline{M}}\,\underline{Y}_c = \underline{0} \qquad (22.2)$$

seek solutions of form $\underline{Y}_c = \underline{v}\,e^{\lambda t}$

for constant vector $\underline{v}$

$(22.2) \Rightarrow \lambda \underline{v} - \underline{\underline{M}}\,\underline{v} = \underline{0}$ or $\underline{\underline{M}}\,\underline{v} = \lambda \underline{v}$

Hence $\underline{v}$ is an eigenvector of $\underline{\underline{M}}$ with eigenvalue λ

2. find particular integral $\underline{Y}_p$ inspired by form of $\underline{F}$

<u>Example</u> $\dot{\underline{Y}} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} \underline{Y} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$ (22.3)

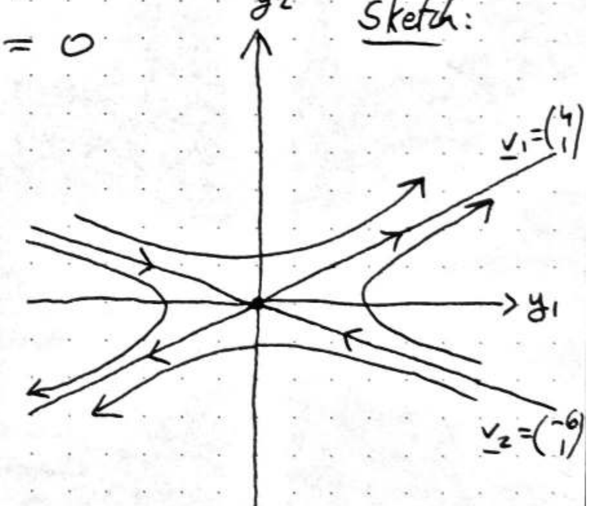
$\underline{Y}_c = \underline{v}\,e^{\lambda t}$ To find λ: $\begin{vmatrix} -4-\lambda & 24 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$\Leftrightarrow 8 + 6\lambda + \lambda^2 - 24 = 0$

$\Leftrightarrow (\lambda + 8)(\lambda - 2) = 0$

so $\lambda = 2$ or -8

Sketch:



$\lambda = 2$: $\begin{pmatrix} -6 & 24 \\ 1 & -4 \end{pmatrix}$ has $\underline{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda = -8$: $\begin{pmatrix} 4 & 24 \\ 1 & 6 \end{pmatrix}$ has $\underline{v} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

Hence $\underline{Y}_c = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t}$

Try $\underline{Y}_p = \underline{u}\,e^t$ w/ $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\therefore \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 5 & -24 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

General solution: $\underline{Y} = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$.

phase - plane portraits

For complementary function $\dot{\underline{Y}}_c = \underline{\underline{M}}\,\underline{Y}_c$,

$$\underline{Y}_c = \underline{v_1} e^{\lambda_1 t} + \underline{v_2} e^{\lambda_2 t}$$

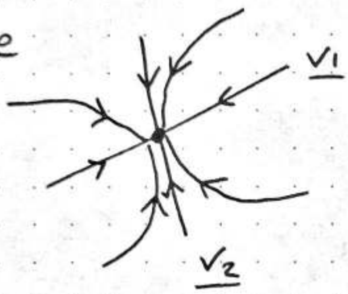
3 cases (assume $\lambda_1, \lambda_2 \neq 0$ nondegenerate)

1. $\lambda_1, \lambda_2 \in \mathbb{R}$ with opposite sign

   e.g. $\lambda_1 > 0$, $\lambda_2 < 0$ 

"saddle" node

2. $\lambda_1, \lambda_2 \in \mathbb{R}$ with same sign

   a) both -ve 

e.g. $|\lambda_1| > |\lambda_2|$

   faster collapse in $\underline{v_1}$ direction

   "stable" node

   b) both +ve give a "non-stable" node

   same sketch with arrows reversed ...?

3. ~~at~~ λ_1, λ_2 complex conjugate pair

   a) $Re(\lambda_1, \lambda_2) < 0$

   stable spiral (could have either handedness)

   b) $Re(\lambda_1, \lambda_2) > 0$ gives unstable spiral

   c) $Re(\lambda_1, \lambda_2) = 0$ gives ellipses  "centre"

L22.3

Non-linear systems of ODEs

- Consider 2nd order $\underbrace{\text{autonomous}}_{f,g \text{ indep. of } t}$ systems

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

Equilibrium (fixed) points (x_0, y_0) have $\dot{x} = \dot{y} = 0$.

$$\therefore \begin{cases} f(x_0,y_0) = 0 \\ g(x_0,y_0) = 0 \end{cases} \quad \text{solve equations for } x_0, y_0$$

- Stability of equilibrium points

Consider a small displacement $\begin{pmatrix} \xi(t), \\ \eta(t) \end{pmatrix}$ away from $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Then $\dot{\xi} = f(x_0 + \xi, y_0 + \eta)$

$$= \underbrace{f(x_0,y_0)}_{=0} + \xi f_x(x_0,y_0) + \eta f_y(x_0,y_0) + O(\xi^2, \eta^2)$$

and $\dot{\eta} = \underbrace{g(x_0,y_0)}_{=0} + \xi g_x(x_0,y_0) + \eta g_y(x_0,y_0) + O(\xi^2, \eta^2)$

$$\therefore \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{in a first-order approximation}$$

2^{nd} order linear system

L23.1

Non-linear systems of ODEs (cont.)

Example: predator-prey model

prey $\quad \dot{x} = f(x,y) = \alpha x - \beta xy$

predator $\quad \dot{y} = g(x,y) = \delta xy - \gamma y$

where $\alpha, \beta, \gamma, \delta > 0$ constant

<u>Fixed points</u> $\quad \dot{x} = 0 \Rightarrow x = 0$ or $y = \dfrac{\alpha}{\beta}$

$\dot{y} = 0 \Rightarrow y = 0$ or $x = \dfrac{\gamma}{\delta}$

$(x_0, y_0) = (0,0)$ and $\left(\dfrac{\gamma}{\delta}, \dfrac{\alpha}{\beta}\right)$

<u>Stability</u>

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}$$

$\underline{(0,0)}$ $\quad \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

eigenvalues α and $-\gamma$ with standard basis as evecs

This fixed point is a saddle

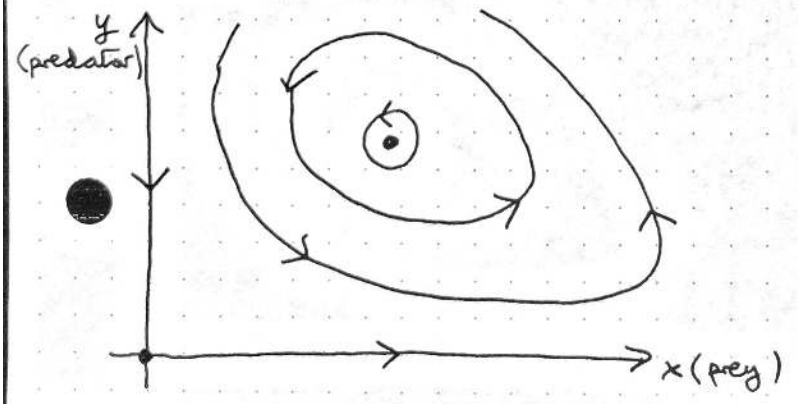
$\underline{\left(\dfrac{\gamma}{\delta}, \dfrac{\alpha}{\beta}\right)}$ $\quad \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{\beta\gamma}{\delta} \\ \dfrac{\delta\alpha}{\beta} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

eigenvalues : $\lambda^2 + \alpha\gamma = 0$ so $\lambda = \pm i\sqrt{\alpha\gamma}$

$\Rightarrow$ <u>center</u>, oscillatory behaviour

$\dot{\xi} = -\dfrac{\beta\gamma}{\delta}\eta$

so $\eta > 0$, $\dot{\xi} < 0$



Partial Differential Equations (PDEs)

DE with more than one independent variable

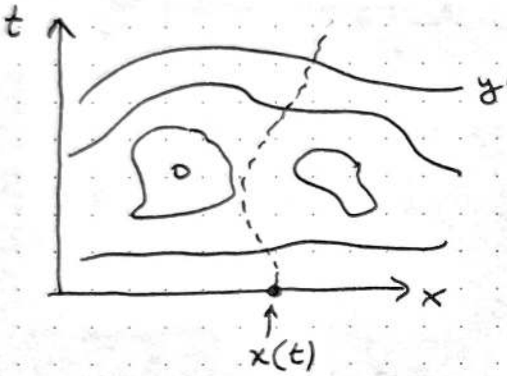
Here, 3 examples

1. First order wave equation

Consider $y = y(x,t)$

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0 \quad \text{where } c = \text{constant} \qquad (23.1)$$

Solve w/ method of characteristics.



Imagine sampling y along some path given by $x(t)$.

$y = y(x(t), t)$ along path

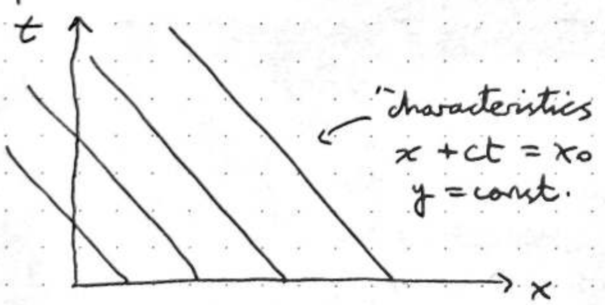
Multivariate chain rule:

$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial t}$$

So we compare with (23.1) and so if we make $\frac{dx}{dt} = -c$ then we see $\frac{dy}{dt} = 0$ along $x = x_0 - ct$.

Therefore $y = f(x_0) = f(x+ct)$.

updated sketch



Example 1 unforced wave

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0 \quad \text{w/ } y(x,0) = x^2 - 3$$

then $y(x,t) = (x+ct)^2 - 3$

for $t \geq 0$

$$(\nabla y)\cdot\begin{pmatrix}1\\-c\end{pmatrix} = 0$$

Example 2 forced wave

$$\frac{\partial y}{\partial t} + 5\frac{\partial y}{\partial x} = e^{-t} \quad \text{with} \quad y(x_0,0) = e^{-x_0^2}$$

$$\frac{dy}{dt} = e^{-t} \quad \text{along paths with} \quad \frac{dx}{dt} = 5$$

hence $y = A - e^{-t}$ along $x = x_0 + 5t$

at $t=0$, $y = A - 1$ so $A = 1 + e^{-x_0^2}$

$\therefore\ y = 1 + e^{-(x-5t)^2} - e^{-t}$

2. Second order wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0 \qquad (23.2)$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) y = 0$$

Note: operators commute by smoothness

Here from (23.1) $f(x+ct)$, $g(x-ct)$ are both solutions.
Since (23.2) is linear get

$$\boxed{y = f(x+ct) + g(x-ct)}$$

Check 23.2

$$c^2 f'' + c^2 g'' - c^2 f'' - c^2 g'' = 0$$

Example $y_{tt} - \cancel{c}^{x^2} y_{xx} = 0$

subject to $y = \frac{1}{1+x^2}$ at $\cancel{x}$ $t=0$

$y_t = 0$ at $t=0$

and $y \to 0$ as $x \to \pm\infty$

L23.4

solution form: $y = f(x+ct) + g(x-ct)$

at $t=0$ $\quad f(x) + g(x) = \dfrac{1}{1+x^2}$

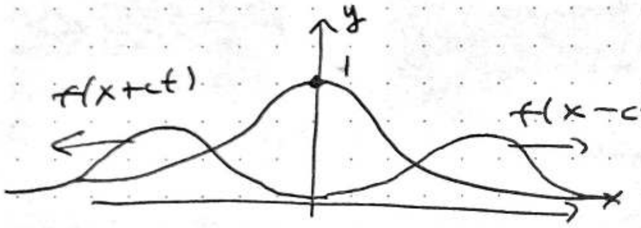
$$c f'(x) - c g'(x) = 0$$

$$\Rightarrow f' = g' \Rightarrow f = g + \text{const.}$$

$$\Rightarrow f = g \text{ since } y \to 0 \text{ as } x \to \pm\infty$$

$$f(x) = g(x) = \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

$$y(x,t) = \frac{1}{2}\left[\underbrace{\frac{1}{1+(x+ct)^2}}_{\text{moves left}} + \underbrace{\frac{1}{1+(x-ct)^2}}_{\text{moves right}}\right] \qquad (c>0)$$



$t=0$

Diffusion Equation (Heat Equation?)

- Uses:
  - pollution transport
  - heat conduction
  - movement of microbes

Derivation: (via random walk)

let $c(x,t)$ be the number of particles at position x at time t. After discrete time Δt

- probability of moving one step left $= p$ $\quad (p < \frac{1}{2})$
- " " " " " right $= p$
- " " staying at $x = 1 - 2p$

Then
$$\underbrace{c(x, t+\Delta t)}_{c(x,t) + \frac{\partial c}{\partial t}\Delta t} = (1-2p)\,c(x,t) + p\Big[\underbrace{c(x+\Delta x, t)}_{c(x,t) + \frac{\partial c}{\partial x}\Delta x + \frac{\partial^2 c}{\partial x^2}\frac{\Delta x^2}{2}} + c(x-\Delta x\cancel{\mp}, t)\Big]$$

$(+O(t^2))$ $\qquad$ with $-$, get $-$ $\quad$ $+$

(24.1) becomes
$$\cancel{c} + \Delta t \frac{\partial c}{\partial t} = (\cancel{1} - \cancel{2p})\,c + p\left[\cancel{2}c + \frac{\partial^2 c}{\partial x^2}\Delta x^2\right] + O(\Delta x^4)$$

$$\therefore \frac{\partial c}{\partial t} = \frac{p\,\Delta x^2}{\Delta t\cancel{{}^2}}\frac{\partial^2 c}{\partial x^2} + O\left(\frac{\Delta x^4}{\Delta t}\right) \quad \left[\text{all with } O(t^2),\ O(x^3),\ O\left(\frac{x^3}{t}\right) \text{ everywhere}\right]$$

Let $\Delta x, \Delta t \to 0$ while holding $\frac{\Delta x^2}{\Delta t\cancel{{}^2}}$ constant

Hence
$$\underbrace{\frac{\partial c}{\partial t} = K\frac{\partial^2 c}{\partial x^2}}_{\text{diffusion eq}^n}, \quad \underbrace{K}_{\text{diffusion coeff.}} = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x^2}{\Delta t}p \qquad (24.2)$$

Example: solve $\frac{\partial c}{\partial t} = K\frac{\partial^2 c}{\partial x^2}$, K const.

subject to $c(x,0) = \delta(x)$ and $c = 0$ as $x \to \pm\infty$

Define $\eta \equiv \frac{x^2}{4\kappa t}$ <u>similarity variable</u>

Look for solutions of form $c(x,t) = t^{-\alpha} f(\eta)$

where α and f are unknown.

$$\frac{\partial \eta}{\partial x} = \frac{2x}{4\kappa t} = \frac{2\eta}{x}, \quad \frac{\partial \eta}{\partial t} = -\frac{x^2}{4\kappa t^2} = -\frac{\eta}{t}$$

$$\frac{\partial c}{\partial t} = -\alpha t^{-\alpha-1} f + t^{-\alpha} \frac{df}{d\eta} \cdot \frac{\partial \eta}{\partial t}$$

$$= -\alpha t^{-\alpha-1} f - \eta t^{-\alpha-1} f', \quad \text{and also}$$

$$\frac{\partial c}{\partial x} = t^{-\alpha} \frac{df}{d\eta} \cdot \frac{\partial \eta}{\partial x} = \frac{x}{2\kappa t} t^{-\alpha} f'$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{1}{2\kappa} t^{-\alpha-1} f' + \frac{x}{2\kappa} t^{-\alpha-1} f'' \frac{2\eta}{x}$$

so that $c_t = \kappa c_{xx} \Rightarrow -\alpha f - \eta f' = \frac{1}{2} f' + \eta f''$

$$\Rightarrow \eta \frac{d}{d\eta}(f + f') + \frac{1}{2}(f' + 2\alpha f) = 0$$

(converted PDE into ODE)

Also note α is arbitrary

Let $\alpha = \frac{1}{2}$. One sol^n would be $f + f' = 0 \ \forall \eta$

$\Rightarrow f = Ae^{-\eta}$ and hence $c(x,t) = A t^{-\frac{1}{2}} e^{-x^2/4\kappa t}$

Recall from δ function lecture $D(t;\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-t^2/\varepsilon^2}$

and $\int_{-\infty}^{\infty} D(t;\varepsilon)\, dt = 1$. Let t be x, ε^2 be $\tilde{t}$.

$D(x;\tilde{t}) = \frac{1}{\sqrt{\pi}} \tilde{t}^{-\frac{1}{2}} e^{-x^2/\tilde{t}}$ Let $\tilde{t}$ be $4\kappa t$

$$\int_{-\infty}^{\infty} D(x;\tilde{t})\, dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa}} t^{-\frac{1}{2}} e^{-x^2/4\kappa t}\, dx = 1$$

L 24.3

Hence $A = \frac{1}{\sqrt{4\pi\kappa}}$, $c(x,t) = \frac{1}{\sqrt{4\pi\kappa}}\, t^{-\frac{1}{2}} e^{-x^2/4\kappa t}$.