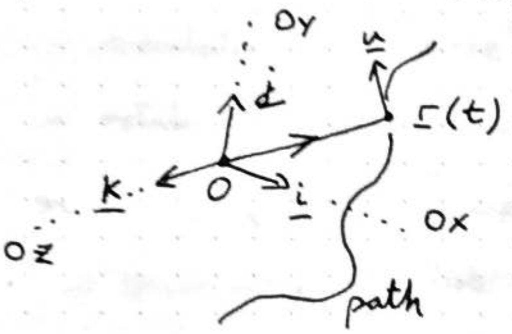
L1.1

Dynamics & Relativity

1: Newtonian Dynamics - basic concepts

Preliminaries: A particle is an object with insignificant size.
It has a mass $m > 0$ and perhaps an electric charge q
The position of the particle at time t is described by the position vectors $\underline{r}(t)$ or $\underline{x}(t)$ with respect to the origin O. [if particle is at origin then $\underline{r} = \underline{0}$]

The Cartesian components $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = (x, y, z)$ defined with respect to unit vectors $\underline{i}, \underline{j}, \underline{k}$ (or $\hat{x}, \hat{y}, \hat{z}$) parallel to axes Ox, Oy, Oz



Choice of origin and axes implies a frame of reference S

The velocity of the particle is
$$\underline{u} = \dot{\underline{r}} = \frac{d\underline{r}}{dt}$$

$\underline{u}$ tangent to path / trajectory

The momentum of the particle is $\underline{p} = m\underline{u} = m\dot{\underline{r}}$

Acceleration of particle is $\underline{a} = \dot{\underline{u}} = \ddot{\underline{r}} = \dfrac{d^2\underline{r}}{dt^2}$

Note time derivative of vector $\underline{u}(t)$ defined by
$$\dot{\underline{u}}(t) = \lim_{h \to 0} \frac{\underline{u}(t+h) - \underline{u}(t)}{h}$$

Limits of vectors defined by $\underline{u} \to \underline{u}_0$ iff $|\underline{u} - \underline{u}_0| \to 0$

Note - can consider components with respect to constant basis vectors and differentiate component-wise $\dot{\underline{r}} = (\dot{x}, \dot{y}, \dot{z})$

Differentiation rules for vectors: $\frac{d}{dt}(f\underline{g}) = \dot{f}\underline{g} + f\dot{\underline{g}}$

$\frac{d}{dt}(\underline{f} \cdot \underline{g}) = \dot{\underline{f}} \cdot \underline{g} + \underline{f} \cdot \dot{\underline{g}}$ $\quad \frac{d}{dt}(\underline{f} \times \underline{g}) = \dot{\underline{f}} \times \underline{g} + \underline{f} \times \dot{\underline{g}}$

Newton's laws of motion

Law 1: $\exists$ inertial frames (of reference) in which a particle remains at rest or moves with constant velocity unless it is acted on by a force. (Galileo's Law of Inertia)

Law 2: In an inertial frame, $\dot{\underline{p}}$ = force on particle

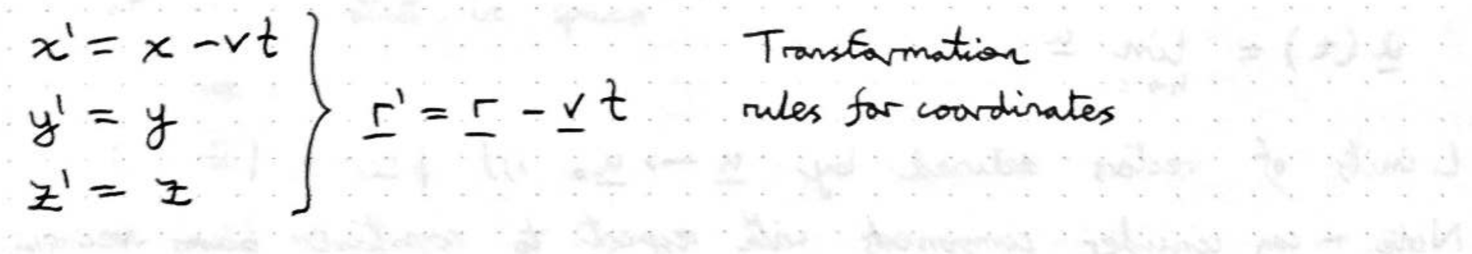
Law 3: To every action there is an equal and opposite reaction i.e. the force A exerts on B, $\underline{F}_{AB}$, is equal and opposite to the force B exerts on A, $\underline{F}_{BA}$

Laws apply to both "particles" and "bodies"

Inertial frames ~ Galilean transformations

In an inertial frame $\ddot{\underline{r}} = \underline{0}$ (no acceleration) if there is no force acting. Inertial frames are not unique.

If S is an inertial frame, then any other frame S' which is in uniform motion relative to S is also an inertial frame



$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}\right\} \underline{r}' = \underline{r} - \underline{v}t$$

Transformation rules for coordinates

More generally, consider S' defined relative to S by

$$\underline{r}'(t) = \underline{r}(t) - \underline{v}t$$

$$t' = t \quad \text{< relevant later >}$$

where $\underline{v}$ is the constant velocity of S' relative to S.

This type of Galilean transformation is called a "boost".

For a particle with position vector $\underline{r}(t)$ in S, and $\underline{r}'(t')$ in S', the velocity $\dot{\underline{r}}$ and acceleration $\ddot{\underline{r}}$ transform as:

$$\underline{u}' = \underline{u} - \underline{v}, \quad \underline{a}' = \underline{a}$$

A general Galilean transformation combines a boost with:

- translations of space $\underline{r}' = \underline{r} - \underline{r}_0$ ($\underline{r}_0$ const.)
- translations of time $t' = t - t_0$ (t_0 ")
- rotations (and reflections) of space } $\underline{r}' = R^{-1}\underline{r}$

  where $RR^T = I$ and R is constant

These generate the "Galilean group".

In any frame generated by a Galilean transformation from another inertial frame, $\ddot{\underline{r}} = \underline{0} \Rightarrow \ddot{\underline{r}}' = \underline{0}$ i.e. S' inertial too

<u>Galilean relativity</u> laws of Newtonian dynamics are the same in all inertial frames.

This implies the laws of physics are the same

- at every point in space
- at all times
- in all directions
- at all velocities

So equations of motion of Newtonian dynamics must be invariant under Galilean transformations.

- Note: velocity is relative, not absolute

  acceleration is absolute

L2.2

Newton's 2^{nd} Law $\rightarrow$ Equations of motion

$$\frac{d}{dt}(m\underline{\dot{r}}) = \underline{F} \quad \text{where } \underline{F} \text{ is the force on the particle}$$

momentum of particle

Assuming m is contstant, N2 becomes $\underline{F} = m\underline{\ddot{r}}$.

m is a measure of the reluctance to accelerate of a particle

If $\underline{F}$ is a function of $\underline{r}$, $\underline{\dot{r}}$, t we have a vector second order differential equation: $m\underline{\ddot{r}} = \underline{F}(\underline{r}, \underline{\dot{r}}, t)$

To specify solution we need two initial conditions, typically $\underline{r}(t_0)$ and $\underline{\dot{r}}(t_0)$. The trajectory of the particle is determined both forwards and backwards in time.

<u>EX</u> 1) Gravitational force

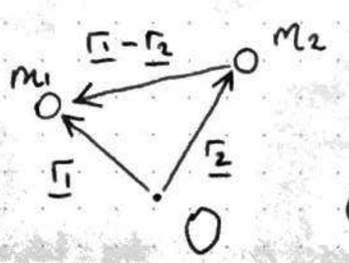
Consider two particles with forces acting on, masses, positions denoted by the subscripts 1 and 2.

Newton's law of gravitation specifies gravitational force on a particle:

$$\underline{F_1} = -\frac{Gm_1m_2(\underline{r_1} - \underline{r_2})}{|\underline{r_1} - \underline{r_2}|^3} = -\underline{F_2}$$

G is Newton's gravitational constant (dimension $L^3M^{-1}T^{-2}$)

$|\underline{F_1}| = |\underline{F_2}| \propto 1/|\underline{r_1} - \underline{r_2}|^2$ "inverse square law"



Notice force is attractive

$\underline{F_1}$ towards m_2 and vice-versa

(see later on in course)

2) Electromagnetic forces

Consider a particle with charge q, and specified electric and magnetic fields $\underline{E}(\underline{r}, t)$ and $\underline{B}(\underline{r}, t)$.

L2.3 Then $\underline{F}(\underline{r},\dot{\underline{r}},t) = q(\underline{E}(\underline{r},t) + \dot{\underline{r}} \times \underline{B}(\underline{r},t))$.

(fields $\underline{E}$ and $\underline{B}$ may have to be determined)

"Lorentz force law"

Say $\underline{E} = \underline{0}$ and $\underline{B}$ = const. to give

$$m\ddot{\underline{r}} = q\dot{\underline{r}} \times \underline{B}$$

Choose axes s.t. $\underline{B} = B\hat{\underline{z}}$

$\ddot{\underline{r}} \cdot \dot{\underline{r}} = 0$

$q(\dot{\underline{r}} \times \underline{B}) \times \ddot{\underline{r}} = 0$

$\Rightarrow \underline{B} \cdot \ddot{\underline{r}} = 0$

$\ddot{\underline{r}} \cdot \underline{B} = 0$

Note by dotting with $\underline{B}$, $\ddot{\underline{r}} \cdot \underline{B} = 0$ and so $\frac{d^2}{dt^2}(\underline{r} \cdot \underline{B}) = 0$ and hence $\underline{r} \cdot \underline{B}$ is linear function of time, i.e. $z = z_0 + ut$

For the x, y components,

$$\begin{cases} m\ddot{x} = qB\dot{y} \\ m\ddot{y} = -qB\dot{x} \end{cases} \quad \text{define } \omega = \frac{qB}{m}$$

$$\Rightarrow \quad \begin{aligned} x &= x_0 - \alpha\cos(\omega(t-t_0)) \\ y &= y_0 + \alpha\sin(\omega(t-t_0)) \end{aligned}$$

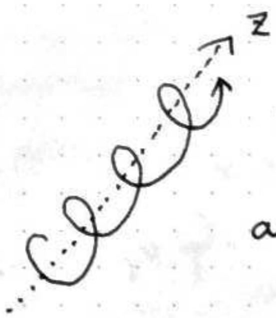
$m\dot{\underline{V}} = \begin{bmatrix} 0 & qB \\ -qB & 0 \end{bmatrix} \underline{V}$

$\dot{\underline{V}} = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{V}$

$\Rightarrow \underline{V} = \alpha\omega \begin{pmatrix} \sin \\ \cos \end{pmatrix}$

depends on $z_0, u, x_0, y_0, \alpha, t_0$

Constant velocity in z direction, and ~~ct~~ circular motion in the x, y plane.



helical motion

axis of helix parallel to $\underline{B}$

$\dot{\underline{V}} = A\underline{V}$ where $A^2 = -\omega^2$

i.e. $\frac{A}{\omega} = i$, so then

$\exp(At) = \exp\left(\frac{A}{\omega}(\omega t)\right)$

$= \cos\omega t + \frac{A}{\omega}\sin\omega t$

§2. Dimensional Analysis

For many problems in dynamics there are three dimensional quantities: length L mass M time T

Dimensions of a physical quantity $[x]$ are expressible uniquely in terms of M, L, T.

$[\text{density}] = ML^{-3}$ $\qquad [\text{force}] = MLT^{-2}$

Only powers of basic dimensional quantities can occur i.e. there are no quantities that require more complicated expressions

Units - introduce units for basic physical quantities M, L, T and obtain units for other quantities

c.f. units change

e.g. SI units

M - kilogram kg

L - metre m

T - second s

consider gravitational constant G in

$$F = G\frac{m_1 m_2}{r^2}$$

so that $[G] = \frac{ML}{T^2} \cdot \frac{L^2}{M^2} = \frac{L^3}{MT^2}$,

in SI units $G = 6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

— — — — —

Equations in physics/dynamics must work in any system of units

Suppose physical quantity Y depends on $X_1, X_2, \dots, X_n$

Let dimensions be $[Y] = L^a M^b T^c$, $[X_i] = L^{a_i} M^{b_i} T^{c_i}$

Consider $n \leq 3$: $Y = C X_1^{p_1} X_2^{p_2} X_3^{p_3}$ and then determine p_1, p_2, p_3 by considering dimensions

$$\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right.$$

with """unique""" solution for p_1, p_2, p_3

not unique if dimensions of X_i linearly dependent

E3.2

If $n \geq 4$ the dimensions of X_i are necessarily dependent.
Choose X_1, X_2, X_3 (indep.) and form $n-3$ dimensionless groups

$$\lambda_1 = \frac{X_4}{X_1^{q_{11}} X_2^{q_{12}} X_3^{q_{13}}}$$

$\sim$ basis for Kernel ??

$$\vdots$$

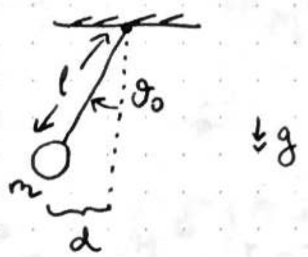
$$\lambda_{n-3} = \frac{X_n}{X_1^{q_{n1}} X_2^{q_{n2}} X_3^{q_{n3}}}$$

q_{ij} chosen so λ_r dimensionless

Now expression for Y becomes

$$Y = \underbrace{C(\lambda_1, \dots, \lambda_{n-3})}_{\text{unknown dimensionless function}} X_1^{p_1} X_2^{p_2} X_3^{p_3}$$

Pendulum



How does period p depend on l, d, g, m?

$[p] = T$
$[l] = [d] = L$
$[m] = M$
$[g] = \cancel{M} L^{+1} T^{-2}$

Choose m, l, g as dimensionally independent quantities

Dimensional group d/l

Here $p = f(d/l)\, m^{p_1} l^{p_2} g^{p_3}$

$$T = M^{p_1} L^{p_2} L^{p_3} T^{-2p_3}$$

(M) : $p_1 = 0$
(L) : $p_2 + p_3 = 0$ $\Rightarrow$ $p = f(d/l)\sqrt{\frac{l}{g}}$
(T) : $2p_3 = -1$

Not all information, but there is some useful info. here

$d \to 2d$, $l \to 2l$ then $p \to \sqrt{2}\, p$ but $d \to 2d$, $l \to l$ then $p \to$?

E3.3 P independent of m

Again, $f(d/l)(l/g)^{1/2}$ is much simpler than $F(m, l, d, g)$

«flippant remark»

Taylor's (1948) estimate of energy released in first atomic explosion using some pictures of atomic blast

radius R	$[R] = L$
time since explosion t	$[t] = T$
density of air ρ_0	$[\rho_0] = ML^{-3}$
energy of explosion E	$[E] = ML^2T^{-2}$

Hence $R = C t^\alpha \rho_0^\beta E^\gamma$

$$L = T^\alpha M^\beta L^{-3\beta} M^\gamma L^{2\gamma} T^{-2\gamma}$$

$\therefore \alpha = 2\gamma, \ \beta = -\gamma, \ 2\gamma = 3\beta + 1$

and so $R = \left(\frac{t^2 E}{\rho_0}\right)^{1/5} \cdot C$

After checking $R \propto t^{2/5}$, rearrange into

$$E = \frac{\rho_0 R^5}{C^5 t^2}$$

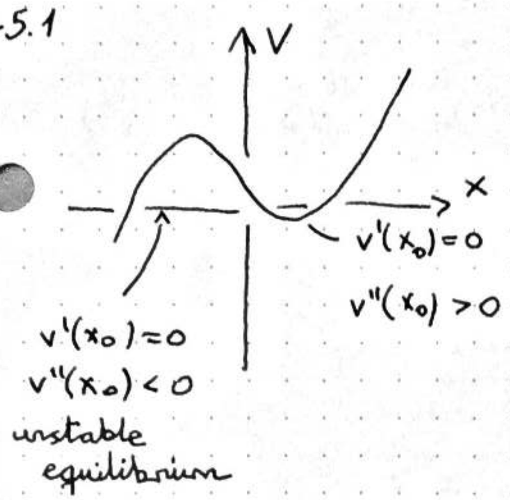
$R^5/t^2 \sim 6.67 \times 10^{13}\ m^5 s^{-2}$

$\rho_0 \sim 1.25\ kg\ m^{-3}$

If $C \sim 1$ then $E \sim 10^{14}\ J \sim$ tn made

backed by evidence

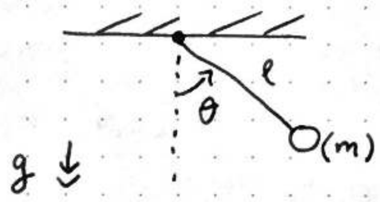
L5.1



stable equilibrium motion about this described by harmonic oscillator equation

Harmonic oscillator important because it approximates motion about a point of stable equilibrium.

<u>Ex:</u> pendulum

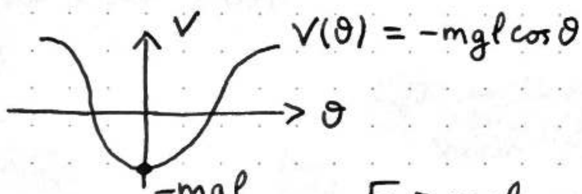


Equation of motion

$$m\ell\ddot{\theta} = -mg\sin\theta$$
$$= -\frac{\partial}{\partial\theta}(-mg\cos\theta)$$

$$E = T + V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and $\dot{E} = 0$. [note that $\underline{v} = (\ell\dot{\theta}\cos\theta, \ell\dot{\theta}\sin\theta)$]



Stable equilibrium at $\theta = 0, 2\pi, \ldots$

Unstable equilibrium at $\theta = \pi, -\pi, \ldots$

$E > mg\ell \Rightarrow \dot{\theta} \neq 0$ for all θ

pendulum bob moves in circles

$-mg\ell < E < mg\ell \Rightarrow \dot{\theta} = 0$ at $\theta = \pm\theta_0$

where $E = -mg\ell\cos\theta_0$

Period of oscillation $\theta_0 \to -\theta_0 \to \theta_0$?

$$P = 4\int_0^{\theta_0} \frac{d\theta}{\left(\frac{1}{\ell^2}(2g\ell\cos\theta - 2g\ell\cos\theta_0)\right)^{1/2}}$$

$$= 4\sqrt{\frac{\ell}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}$$

$$= 4\sqrt{\frac{\ell}{g}}\, f(\theta_0)$$ as done in Dimensional Analysis

L5-2 When $\theta_0 \ll 1$,

$$P \simeq 4\sqrt{\frac{l}{g}}\int_0^{\theta} \frac{d\theta}{\sqrt{\theta_0^2-\theta^2}} = 2\pi\sqrt{\frac{l}{g}}$$

... links with $V \simeq -mgl + mgl\left(\frac{1}{2}\theta^2\right)$ so $\omega = \sqrt{\frac{mgl}{ml^2}}$

§ 3.2 Force and potential energy in 3 dimensions

Equation of motion $m\ddot{\underline{r}} = \underline{F}$.

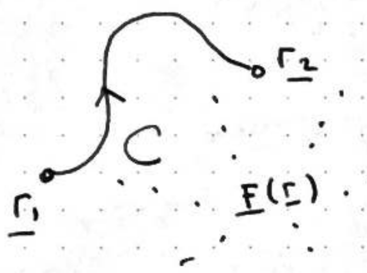
Kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\underline{r}}^2 = \frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}}$

$$\frac{dT}{dt} = m\dot{\underline{r}}\cdot\ddot{\underline{r}} = \dot{\underline{r}}\cdot\underline{F}$$ "power", rate of working

Total work done by force on particle as it moves from $\underline{r}=\underline{r}_1$ to $\underline{r}=\underline{r}_2$ along a path C. Say $\underline{r}_1 = \underline{r}(t_1)$, $\underline{r}_2 = \underline{r}(t_2)$. Then

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \frac{dT}{dt}\,dt = \int_{t_1}^{t_2} \dot{\underline{r}}\cdot\underline{F}\,dt = \int_C \underline{F}\cdot d\underline{r}$$

"line integral" from Vec Calc



Suppose $\underline{F} = \underline{F}(\underline{r})$, i.e. force does not depend on t, $\dot{\underline{r}}$.

A conservative force is one that satisfies

$$\underline{F} = -\nabla V(\underline{r}) \quad \text{i.e.} \quad F_i = -\frac{\partial}{\partial x_i}V$$

where $V(\underline{r})$ is a potential energy function.

In such a case,

$$\int_C \underline{F}\cdot d\underline{r} = -\int_C \nabla V\cdot d\underline{r} = -\int_C dV = V(\underline{r}_1) - V(\underline{r}_2)$$

and hence $E = T + V$ is conserved ☺

Work independent of path <=> Energy conserved

Iff $\underline{F} = -\nabla V$ for some $V(\underline{r})$.

Contrast with 1D where all $\underline{F}$ had a potential.

In 3D, only forces with $\nabla\times\underline{F}$ zero have a potential.

L6.1 If $\underline{F}$ is conservative the the energy

$$E = T + V = \tfrac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + V(\underline{r}) \text{ is conserved.}$$

Proof: $\dot{E} = m \dot{\underline{r}} \cdot \ddot{\underline{r}} + \dot{\underline{r}} \cdot \underline{\nabla} V(\underline{r})$ chain rule, see VC

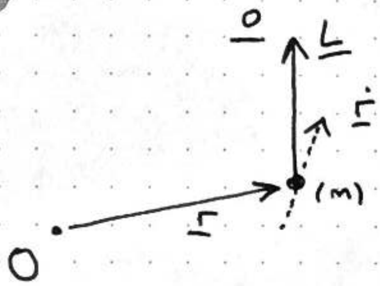
$$= \dot{\underline{r}} \cdot \underbrace{(m \ddot{\underline{r}} + \underline{\nabla} V(\underline{r}))}_{\underline{0} \text{ by N2}} = 0$$

Angular momentum

another important quantity

$$\underline{L} = \underline{r} \times \underline{p} = m \underline{r} \times \dot{\underline{r}}$$

$$\dot{\underline{L}} = \underbrace{m \dot{\underline{r}} \times \dot{\underline{r}}}_{\underline{0}} + m \underline{r} \times \ddot{\underline{r}} = \underline{r} \times \underline{F} = \underline{G} \leftarrow \text{"torque"/"moment of force"}$$



$\underline{L}$ and $\underline{G}$ depend on the origin, so talk about "angular momentum about O" or "torque about O"

If $\underline{r} \times \underline{F} = \underline{0}$ then $\dot{\underline{L}} = \underline{0}$ so angular momentum is conserved

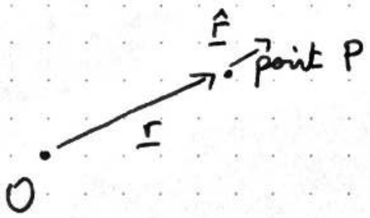
So if force is central, $\dot{\underline{L}} = \underline{0}$ with $\underline{L}$ about O.

Central forces

Special class of conservative forces

$$V(\underline{r}) = V(|\underline{r}|)$$

Then $\underline{F} = -\underline{\nabla} V(r) = -\dfrac{dV}{dr} \hat{\underline{r}}$ with unit vector $\hat{\underline{r}} = \dfrac{\underline{r}}{r}$



[Check: $\underline{\nabla} V(\underline{r}) = \left(\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}, \dfrac{\partial V}{\partial z}\right)$

$$= \frac{\partial V}{\partial r}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)$$

$$= \frac{dV}{dr}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{dV}{dr}\hat{\underline{r}}$$]

Note $\underline{G} = -\dfrac{dV}{dr}(\underline{r} \times \hat{\underline{r}})$

$= \underline{0}$,

as claimed before.

L6.2

§ 3.3 Gravity

We have seen that gravitational force on mass m at point $\underline{r}$ due to mass M at O is

$$\underline{F} = -\frac{GMm}{r^3}\underline{r} = -\frac{GMm}{r^2}\hat{\underline{r}}$$

This is a central force, and in particular

$$V = -\frac{GMm}{r}, \quad \text{as then } -\frac{dV}{dr} = -\frac{GMm}{r^2}.$$

Often define gravitational potential and gravitational field

$$\Phi_g(\underline{r}) = -\frac{GM}{r}, \quad \underline{g} = -\underline{\nabla}\Phi_g(\underline{r}) = -\frac{GM}{r^2}\hat{\underline{r}}$$

"gravitational force per unit mass"

Then for mass m,

$$\underline{F} = m\underline{g} \qquad V = m\Phi_g$$

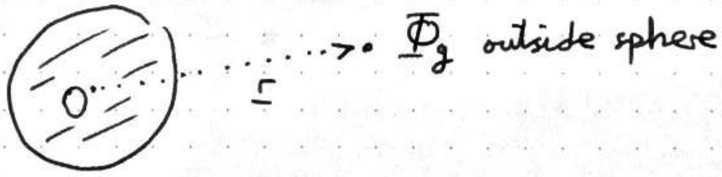
We can generalise to the gravitational field due to point masses M_i at position $\underline{r}_i$

$$\Phi_g(\underline{r}) = -\sum_i \frac{GM_i}{|\underline{r}-\underline{r}_i|} \qquad \underline{g} = -\sum_i \frac{GM_i}{|\underline{r}-\underline{r}_i|^3}(\underline{r}-\underline{r}_i)$$

We can generalise to continuous distributions of mass, sums become integrals. In particular for a ~~uniform sphere~~ spherically symmetric distribution of mass M, outside the object

$$\Phi_g(\underline{r}) = -\frac{GM}{r}$$

i.e. it is as if all the mass in the sphere were concentrated at O.



L6.3

Inertial mass vs Gravitational mass

Newton's 2nd law: $m_I \ddot{\underline{r}} = \underline{F}$

inertial mass
"reluctance to accelerate under a force"

Newton's law of gravitation: $\underline{F} = -\dfrac{GMm_G}{r^2}\hat{\underline{r}}$

gravitational mass
"strength of gravitational force on object"

m_I and m_g play very different roles, but are known to be the same to 1 part in 10^{12}

For a better understanding, see Einstein's theory of general relativity

Some simple results on gravity

1) Consider a small height z above the surface of planet radius R
i.e. $z \ll R$

$$V(R+z) = -\frac{GMm}{R+z} = -\frac{GMm}{R}\left(1 - \frac{z}{R} + \cdots\right)$$

$$= -\frac{GMm}{R} + \frac{GMm}{R^2}z + O\left(\frac{z^2}{R^2}\right)$$

$$\simeq \text{const.} + mgz + \cdots$$

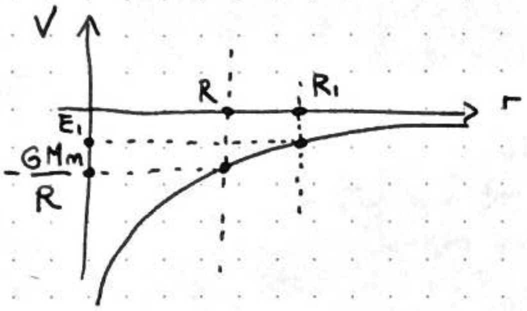
looks like potential energy in uniform g

In Earth, $g = \dfrac{GM}{R^2} \simeq 9.8\ \mathrm{ms^{-2}}$

2) Escape velocity

particle projected radially outwards with speed v

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{R}$$



To escape, need $E \geq 0$

i.e. $v \geq \sqrt{\dfrac{2GM}{R}} = v_{esc}$

If $E = E_1$, $r \leq R_1$.

L7.1

3.4 Electromagnetic forces

Recall force on a particle with electric charge q is

$$\underline{F} = q(\underline{E} + \dot{\underline{r}} \times \underline{B}) \qquad \text{Lorentz force}$$

where $\underline{E}(\underline{r}, t)$ & $\underline{B}(\underline{r}, t)$ are the electric and magnetic fields.

We consider time independent fields, & then

$\underline{E} = -\underline{\nabla}\,\Phi_e$ where $\Phi_e(\underline{r})$ is the electrostatic potential, and the electrostatic force is conservative.

Claim: $E = T + V = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + q\,\Phi_e$ is conserved if as above

Proof:
$$\dot{E} = m\dot{\underline{r}} \cdot \ddot{\underline{r}} + q(\underline{\nabla}\Phi_e) \cdot \dot{\underline{r}}$$
$$= \dot{\underline{r}} \cdot (m\ddot{\underline{r}} + q(-\underline{E}))$$

so from $\underline{F} = m\ddot{\underline{r}}$ we have

$$\dot{E} = \dot{\underline{r}} \cdot (q\,\dot{\underline{r}} \times \underline{B}) = 0, \text{ as claimed.}$$

[Note use of Chain Rule for $\Phi_e(\underline{r}(t))$]

Note $\Phi_e = V/q$ is potential energy per unit charge, and rate of working of magnetic force $q\,\dot{\underline{r}} \times \underline{B}$ zero since it is orthogonal to the velocity $\dot{\underline{r}}$.

Point charge

A particle of charge Q fixed at the origin produces an electrostatic potential $\Phi_e = \dfrac{Q}{4\pi\varepsilon_0 r}$ and a corresponding electric field

$$\underline{E} = -\underline{\nabla}\Phi_e = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}.$$

$\varepsilon_0 \simeq 8.85 \times 10^{-12}\, m^{-3}\, kg^{-1}\, C^2 s^2$ «electric constant»

The resulting force on a particle of charge q at position $\underline{r}$ is

$$\underline{F} = q\underline{E} = -q\underline{\nabla}\Phi_e = \frac{Qq}{4\pi\varepsilon_0} \cdot \frac{\hat{\underline{r}}}{r^2} \qquad \text{«Coulomb force law»}$$

Electrostatic forces are very similar in mathematical form to gravitational forces but they can be

repulsive		attractive
$Qq > 0$	or	$Qq < 0$
like signs		opposite signs

L7.2 Motion in uniform EM fields (not point sources) can be analysed quite easily e.g. $\underline{E}$ const., $\underline{B} = \underline{0}$ => constant acceleration or $\underline{E} = \underline{0}$, $\underline{B}$ const. => particle follows helical path (seen before)

See $\underline{B}$ does no work, as speed constant.

showed this by choosing suitable axis & considering components [could also have kept vector notation if convenient for problem]

3.5 Friction

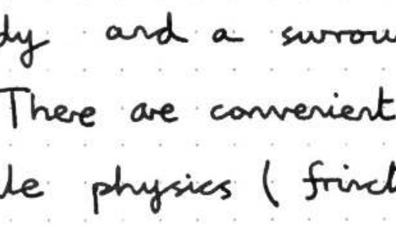
Friction is a contact force (e.g. between two solid bodies or between a solid body and a surrounding fluid).

There are convenient ways of describing these complicated micro-scale physics (friction not a fundamental force).

Dry friction

Consider solid objects in contact - a normal force $\underline{N}$ (perpendicular to contact surface) prevents them from merging, and a frictional force $\underline{F}$ (tangential to contact surface) restricts tangential relative motion (sliding / slipping).

Consider a solid block on an inclined plane, as shown



static friction: if no sliding occurs then $|\underline{F}| \leq \mu_s |\underline{N}|$ where μ_s is "coefficient of static friction"

The block can rest on the inclined plane without moving if $\alpha \leq \arctan(\mu_s)$

kinetic friction: if sliding occurs then $|\underline{F}| = \mu_K |\underline{N}|$ where μ_K is the "coefficient of kinetic friction"

Values of coeffs. μ depend on the two materials,

e.g. rubber / asphalt ≈ 0.8

teflon / teflon ≈ 0.04

Expect $\mu_s > \mu_k > 0$.

Hypothetical perfectly smooth surface has $\mu_s = \mu_k = 0$.

Fluid drag

A solid body moving through a fluid experiences a drag force.
Two important regimes:

- linear drag (small objects, low speed, high viscosity)

$$\underline{F} = -k_1 \underline{v}$$

where $\underline{v}$ is the velocity of the object relative to the fluid.
e.g. Stokes' Law: sphere in a viscous fluid

$k_1 = 6\pi\eta R$ where R is sphere radius, η viscosity.

Applies to bacterium or microorganism in water or small ice / aerosol particles in air.

- quadratic drag (large objects, high speed, low viscosity)

$$\underline{F} = -k_2 |\underline{v}| \underline{v}$$

k_2 a different constant from k_1

Typically $k_2 \sim \rho_{fluid} C_D R^2$ where R is length scale, and C_D is the drag coefficient.

e.g. fish in water or car / aircraft in air

The object loses energy as a result of the drag force – rate of working $\underline{F} \cdot \underline{v} = -k_1 |\underline{v}|^2$ (linear drag)

or $-k_2 |\underline{v}|^3$ (quadratic)

The fluid gains energy – in what form does this appear?
Problem in fluid dynamics.

L8.1 Evans said "Friction"

Ex: Simple harmonic motion w/ linear drag (c.f. DEs)

Ex: Projectile moving under uniform gravity with linear drag

$t=0, \quad \underline{x} = \underline{0}, \quad \dot{\underline{x}} = \underline{u}$

start at origin initial velocity

N2 gives, in terms of $\underline{v} = \dot{\underline{x}}$

$$m\frac{d\underline{v}}{dt} = m\underline{g} - k\underline{v}$$ solve for $\underline{v}(t)$, and then $\underline{x}(t)$

$$\Rightarrow \frac{d}{dt}\left(e^{(k/m)t}\underline{v}\right) = e^{(k/m)t}\underline{g} \quad \text{(integrating factor)}$$

$$\Rightarrow \underline{v} = \frac{m}{k}\underline{g} + \underline{C}e^{-kt/m} = \frac{m}{k}\underline{g} + \left(\underline{u} - \frac{m}{k}\underline{g}\right)e^{-kt/m} \quad \text{(using ICs)}$$

$$\Rightarrow \underline{x} = \frac{m}{k}\underline{g}t - \frac{m}{k}\left(\underline{u} - \frac{m}{k}\underline{g}\right)e^{-kt/m} + \underline{D}$$

$$= \frac{m}{k}\underline{g}t + \frac{m}{k}\left(\underline{u} - \frac{m}{k}\underline{g}\right)\left(1 - e^{-kt/m}\right) \quad \text{(ICs again)}$$

Now consider components (x, y, z) with

$\underline{u} = (u\cos\theta, 0, u\sin\theta) \qquad \underline{g} = (0, 0, -g)$

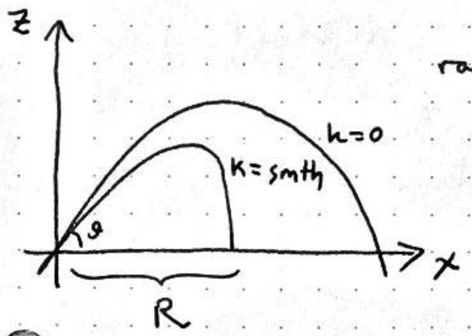
$v_x = u\cos\theta\, e^{-kt/m} \qquad v_y = 0 \qquad v_z = \left(u\sin\theta + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k}$

Note $v_z \to -\frac{mg}{k}$ "terminal velocity"

relevant for $t \gg \frac{m}{k}$. Look at trajectory:

$$x = \frac{m}{k}u\cos\theta\left(1 - e^{-kt/m}\right) \qquad y = 0$$

$$z = -\frac{mg}{k}t + \frac{m}{k}\left(u\sin\theta + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right)$$



range $R(\theta, u, k, m, g)$

$$R = \frac{mu}{k} f\left(\theta, \frac{mg}{ku}\right)$$

Note regimes: $ku/mg \ll 1$ "friction small"

$ku/mg \gg 1$ "friction large"

weak friction gives

$x = u\cos\theta\, t, \quad z = u\sin\theta\, t - \frac{gt^2}{2}$ (do Taylor expansion)

L8.2 strong friction gives $R = \frac{m u F}{K}\cos\theta$

§ Orbits: 4

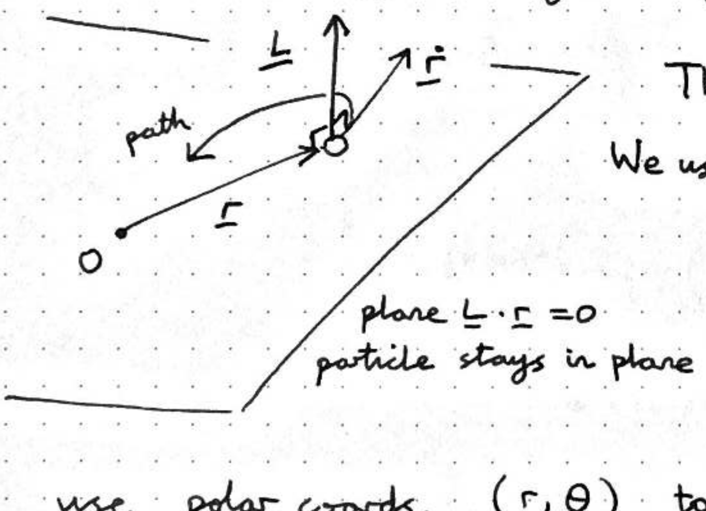
Motivated by motion of plantes, comets, etc. i.e. masses moving under the effect of attractive gravitational forces from distant massive objects. Typically a star, like the Sun. We also consider related problems for charged particles (Rutherford scattering).

Have particle with position $\underline{r}(t)$ in 3D, moving under the influence of a central force:

$$m\ddot{\underline{r}} = -\nabla V(r) \qquad (*)$$

Will regard body producing $V(r)$ as fixed at the origin. This is actually equivalent to having both bodies move, and provides a good approximation when $M \gg m$.

We have seen that $\underline{L} = m\underline{r}\times\dot{\underline{r}}$ is constant in time. Moreover $\underline{L}\cdot\underline{r} = 0$ for all time, so particle can only move in plane with normal $\underline{L}$ through origin.



This reduces problem to 2D.
We use polar coordinates.

§ 4.1 Polar coords. in 2D

Choose axes so that $\underline{L}\cdot\underline{r} = 0$ is $z = 0$ and use polar coords. (r,θ) to describe motion in (x,y) plane.

$x = r\cos\theta$, $y = r\sin\theta$, $\underline{e}_r = (\cos\theta, \sin\theta)$, $\underline{e}_\theta = (-\sin\theta, \cos\theta)$
(see Vector Calculus)

$$\frac{d^2}{dt^2}(\underline{r}) = \frac{d}{dt}\left(\frac{d}{dt}(r\underline{e}_r)\right) = \frac{d}{dt}\left(\dot r\underline{e}_r + r\underline{e}_\theta\dot\theta\right)$$

$$= \ddot r\underline{e}_r + 2\dot r\underline{e}_\theta\dot\theta - r\underline{e}_r\dot\theta^2 = \cancel{-K\underline{e}_r}\ K(r)\underline{e}_r$$

$$\ddot r - r\dot\theta^2 = -k(r), \quad 2\dot r\dot\theta = \cancel{0} - r\ddot\theta$$

$r^2\dot\theta$ const

have missed $r\underline{e}_\theta\ddot\theta$

L9.1 End of last lecture:

unit vectors for 2D polars

$$\underline{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \underline{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

hence $\dfrac{d\underline{e}_r}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \underline{e}_\theta$, $\dfrac{d\underline{e}_\theta}{d\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\underline{e}_r$.

For a particle in motion, $\underline{e}_r$ and $\underline{e}_\theta$ will vary:

$$\frac{d}{dt}\underline{e}_r = \frac{d}{d\theta}\underline{e}_r\,\dot\theta = \dot\theta\,\underline{e}_\theta \ , \quad \frac{d}{dt}\underline{e}_\theta = \frac{d}{d\theta}\underline{e}_\theta\,\dot\theta = -\dot\theta\,\underline{e}_r .$$

Now consider velocity and acceleration, in terms of $r, \theta, \underline{e}_r, \underline{e}_\theta$.

$$\underline{r}(t) = r\,\underline{e}_r$$

$$\Rightarrow \dot{\underline{r}}(t) = \underbrace{\dot r\,\underline{e}_r}_{\text{radial component}} + \underbrace{r\dot\theta\,\underline{e}_\theta}_{\text{angular component}}$$

$$\Rightarrow \ddot{\underline{r}}(t) = \ddot r\,\underline{e}_r + \dot r\dot\theta\,\underline{e}_\theta + \dot r\dot\theta\,\underline{e}_\theta + r\ddot\theta\,\underline{e}_\theta - r\dot\theta^2\underline{e}_r$$

$$= \underbrace{(\ddot r - r\dot\theta^2)\,\underline{e}_r}_{\text{radial}} + \underbrace{(2\dot r\dot\theta + r\ddot\theta)\,\underline{e}_\theta}_{\text{angular}}$$

Motion in a circle with $\dot\theta = \omega = \text{const.}$ has $\dot{\underline{r}} = r\omega\,\underline{e}_\theta$ and gives $\ddot{\underline{r}} = -r\omega^2\underline{e}_r$ « centripetal force »

§4.2 motion in a central force field

We have $m\ddot{\underline{r}} = \underline{F} = -\underline{\nabla}V(r) = -\dfrac{dV}{dr}\underline{e}_r$. Use components above:

$$\begin{cases} m(\ddot r - r\dot\theta^2) = -\dfrac{dV}{dr}(r) \\ 2\dot r\dot\theta + r\ddot\theta = 0 \end{cases}$$

$$\longrightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = 0$$ which gives $l = r^2\dot\theta$ const.

recall angular momentum $\underline{L} = m\underline{r}\times\dot{\underline{r}} = mr(r\dot\theta)(\underline{e}_r\times\underline{e}_\theta)$

$$= mr^2\dot\theta\,\underline{e}_z$$

$\dot{\text{O}}$

$mr^2\dot{\theta} = L$ magnitude of angular momentum constant
(used direction of $\underline{L}$ constant before)

introduce $h = \frac{L}{m} = r^2\dot{\theta}$

Recall radial component

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}$$

$$\Rightarrow m\ddot{r} = -\frac{dV}{dr} + mr\dot{\theta}^2 = -\frac{dV}{dr} + \frac{mh^2}{r^3} \underset{\text{suggestive}}{\overset{\uparrow}{=}} -\frac{dV_{eff}}{dr}$$

where $V_{eff} = V + \frac{mh^2}{2r^2}$

Ergo radial motion given by 1D potential V_{eff} ☺

Energy of particle

$$\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

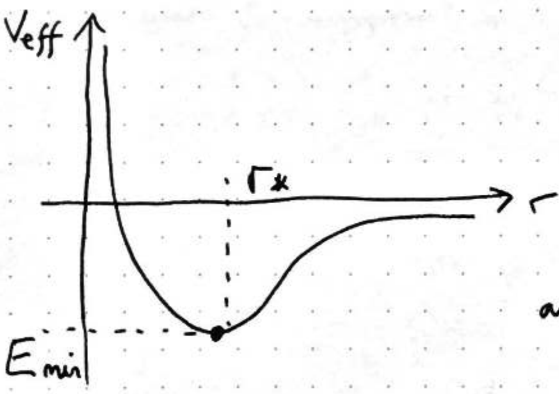
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{h^2}{r^2} + V(r)$$

$$= \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

$$\frac{GM}{r^2} = \frac{\cancel{m}h^2}{r^3}$$

Ex: inverse square law

$$V(r) = -\frac{GMm}{r} \Rightarrow V_{eff}(r) = -\frac{GMm}{r} + \frac{1}{2}\frac{mh^2}{r^2}$$



r_* verifies $V'_{eff}(r_*) = 0$

$$\Rightarrow r_* = \frac{h^2}{GM}$$

and $E_{min} = -m\frac{(GM)^2}{2h^2}$

If $E = E_{min}$, $r(t) = r_*$ for all t, $\dot{\theta} = \frac{h}{r_*^2}$ « circular orbit »

E9.3

If $E_{min} < E < 0$, $r(t)$ oscillates « ellipse »

between r_{min} = periapsis, r_{max} = apoapsis

r varies $\Rightarrow$ $\dot{\vartheta} = \frac{h}{r^2}$ varies too

If $E > 0$, come in, leave to ∞ « hyperbola »

Stability of circular orbits

stable at $V_{eff}'(r_*) = 0$ if $V_{eff}''(r_*) > 0$

In general $V_{eff} = V + \frac{mh^2}{2r^2}$ (assume $h \neq 0$)

So orbit at r_* if $\left.\frac{dV}{dr}\right|_{r_*} = \frac{mh^2}{r_*^3}$ $(m\omega^2 r)$

and stable if $V_*'' + \frac{3mh^2}{r^4_*} > 0$

i.e. $V_*'' + \frac{3}{r_*} V_*' > 0$.

Unstable if < 0.

L10.1 Stability of circular orbits

general $V(r) \longrightarrow F(r) = -V'(r)$

circular orbit $r(t) = r_*$ with $V_{eff}'(r_*) = 0$

stable if $V''(r_*) + \frac{3}{r_*} V'(r_*) > 0$

$$F'(r_*) + \frac{3}{r_*} F(r_*) < 0$$

<u>Ex</u>: $V(r) = -\frac{km}{r^p}$ «attractive central force» $k > 0,\ p > 0$

$F(r) = -\frac{pkm}{r^{p+1}}$

Circular orbit of radius r_*

$$\frac{pkm}{r_*^{p+1}} = \frac{mh^2}{r_*^3} \Rightarrow r_* = \left(\frac{pk}{h^2}\right)^{1/p-2}$$

have orbit for $p \neq 2$

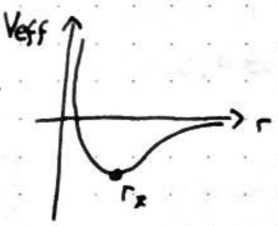
$$V''(r_*) + \frac{3}{r_*} V'(r_*) = -\frac{pkm(p+1)}{r_*^{p+2}} + \frac{3kmp}{r_*^{p+2}}$$

$$= p(2-p)\frac{mk}{r_*^{p+2}}$$ stable if $0 < p < 2$, unstable if $p > 2$

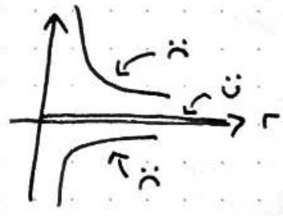
Consider shape of $V_{eff}(r)$ for $p=1$, $p=3$

$$V_{eff}(r) = V(r) + \frac{mh^2}{2r^2} = -\frac{km}{r^p} + \frac{mh^2}{2r^2}$$

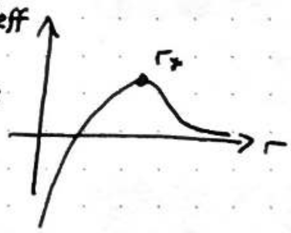
$p=1$:



$p=2$:



$p=3$:



<u>§4.4 Orbit equation</u>

Shape of orbit given by joint variation of $r(t), \theta(t)$. In principle could do this:

$$E = \tfrac{1}{2}m\dot{r}^2 + \underbrace{\frac{mh^2}{2r^2} + V(r)}_{V_{eff}(r)} \Rightarrow t = \int \pm \frac{dr}{\sqrt{\frac{2}{m}(E - V_{eff})}}$$

hence $r(t)$, obtain $\dot{\theta} = h/r^2$ for $\theta(t)$. Integral usually hard.

L10.2

A better approach is to use θ as independent variable, by writing

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta} \qquad \text{apply to N2}$$

$$\Rightarrow \frac{mh}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) - \frac{mh^2}{r^3} = F(r)$$

$$\rightarrow \; du = -\frac{dr}{r^2}$$

The $\frac{1}{r^2}dr$ suggests $u = \frac{1}{r}$. This gives

$$-mh^2u^2\frac{d^2u}{d\theta^2} - mh^2u^3 = F\left(\tfrac{1}{u}\right)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}F\left(\frac{1}{u}\right)$$

« Orbit equation »
Binet's equation

Can solve for $u(\theta)$, then obtain time evolution for $\dot\theta = hu^2$.

<u>Kepler problem</u>

Solve in case of gravitational central force: $F(r) = -\frac{mk}{r^2}$

$$u'' + u = -\frac{1}{\cancel{m}h^2\cancel{u^2}}\cdot -\cancel{m}k\cancel{u^2} = \frac{k}{h^2} \qquad \text{(SHM ü)}$$

solution: $u = \frac{k}{h^2} + A\cos(\theta - \theta_0)$ where A, θ_0 constants of $\int$

can rotate so $\theta_0 = 0$ i.e. apsides at $\theta = 0, \pi$

$$r = \frac{1}{u} = \frac{1}{k/h^2 + A\cos\theta} = \frac{\ell}{1 + e\cos\theta} \qquad \text{where } \ell = h^2/k, \quad e = Ah^2/k$$

Conic section with one focus at origin

$0 \le e < 1$ ellipse

$e = 1$ parabola

$e > 1$ hyperbola

e: eccentricity

rewrite in Cartesians: $r + re\cos\theta = \ell \Rightarrow r = \ell - ex$

$$\Rightarrow x^2 + y^2 = \ell^2 - 2\ell ex + e^2x^2$$

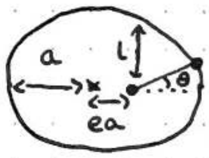
$$\Rightarrow (1-e^2)x^2 + 2\ell ex + y^2 = \ell^2 \qquad (\dagger)$$

L10.3

$0 \leq e < 1$ gives r bounded

<u>ellipse</u> $\qquad \frac{\cancel{e}\,\ell}{1+e} \leq r \leq \frac{\ell}{1-e}$

(†) rewritten as $\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a = \frac{\ell}{1-e^2}$, $b = \frac{\ell}{\sqrt{1-e^2}}$



ellipse with shifted origin

If $e=0$ then orbit is a circle ☺

$e > 1$ hyperbola : $r \to \infty$ as $\theta \to \pm\alpha$

where $\cos\alpha = -\frac{1}{e}$

so $\alpha > \frac{\pi}{2}$

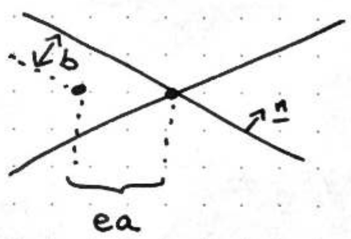
$$(1-e^2)x^2 + 2elx + y^2 = l^2 \qquad (\dagger)$$

gives $\dfrac{(x-ae)^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a = \dfrac{l}{e^2-1}$, $b = \dfrac{l}{\sqrt{e^2-1}}$

« incoming mass deflected by gravitational force »

asymptotes : $y = \pm \frac{b}{a}(x - ea) \Rightarrow bx \mp ay = eba$

Consider perpendicular distance between asymptote and central mass



$$\underline{r} \cdot \underline{n} = (x, y) \cdot \frac{(b, \mp a)}{\sqrt{b^2 + a^2}}$$

"impact parameter"

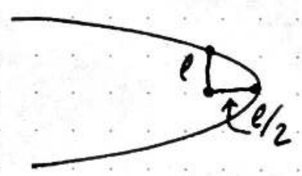
$$= \frac{bx \mp ay}{\sqrt{b^2 + a^2}} = \frac{eba}{\sqrt{a^2+b^2}} = b$$

$e = 1$ parabola

$$r = \frac{l}{1 + \cos\theta} \qquad r \to \infty \text{ as } \theta \to \pm\pi$$

« marginal case between ellipse and hyperbola »

$(\dagger)$ reads $y^2 = l(l - 2x)$



Energy and eccentricity

recall $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{mk}{r}$

$$= \frac{1}{2}mh^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - mku \qquad \left(u = \frac{1}{r},\ \dot{r} = -h\frac{du}{d\theta}\right)$$

$$= \frac{1}{2}m\frac{h^2}{l^2}\left(e^2\sin^2\theta + (1+e\cos\theta)^2\right) - \frac{mk}{l}(1 + e\cos\theta)$$

$$= \frac{mk}{2l}(e^2 - 1)$$ Jesus Christ

Bound orbits have $e < 1 \Rightarrow E < 0$ (recall discussion of V_{eff})

Unbound orbits have $e > 1 \Rightarrow E > 0$

Parabolic orbit has $e = 1 \Rightarrow E = 0$

Kepler's Laws of Planetary Motion

$\boxed{a = \frac{l}{1-e^2}}$

$\boxed{l = \frac{h^2}{k}}$

I Orbit of planet is ellipse with Sun at focus

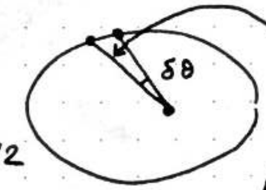
II Line between Sun sweeps equal area in equal time

III Square of period $\propto$ cube of semi-major axis

Have deduced I from orbit eq^n.

II is equivalent to conservation of angular momentum

III follows from I and II:



$\delta A = \frac{1}{2} r^2 \delta\theta$

$\Downarrow$

$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{h}{2}$

$$T = \frac{\pi a b}{h/2} = \frac{\pi a^2 \sqrt{1-e^2}}{\sqrt{ka(1-e^2)}/2} = \frac{2\pi}{\sqrt{k}} a^{3/2}$$

§4.5 Rutherford Scattering

Consider motion under repulsive inverse square law force

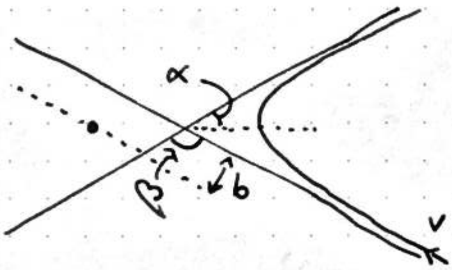
$$V(r) = \frac{mk}{r} \qquad F(r) = \frac{mk}{r^2} \qquad k = \frac{Qq}{4\pi\varepsilon_0 m} > 0$$

Solution to orbit eq^n $\quad u = -\frac{k}{h^2} + A\cos\theta$

$$\Rightarrow r = \frac{1}{u} = \frac{l}{e\cos\theta - 1} \quad \text{where } l = \frac{h^2}{k}, \quad e = \frac{Ah^2}{k}$$

Require $e > 1$ so that $r > 0$ sometimes.

$r \to \infty$ as $\theta \to \pm\alpha$ where $\cos\alpha = \frac{1}{e}$ so $\alpha < \frac{\pi}{2}$



$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with } a = \frac{l}{e^2-1}, \quad b = \frac{l}{\sqrt{e^2-1}}$$

deflection $\beta = \pi - 2\alpha$

$h = bv$

$E = \frac{1}{2}mv^2$

$\cancel{E = \frac{mk}{2b}(e^2-1) \Rightarrow \frac{1}{2}mv^2 = \frac{mk}{2bv}(e^2-1) \Rightarrow e^2-1 = \frac{bv^3}{2k}}$

$b = \frac{l}{\tan\alpha}$

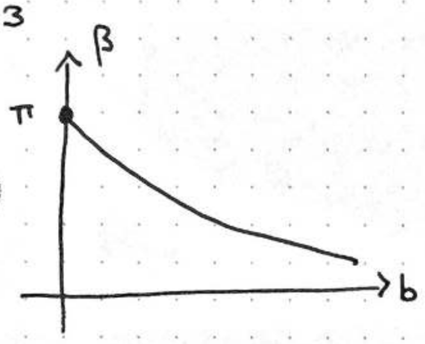
$\cancel{e = \sec\alpha \Rightarrow e^2 - 1 = \tan^2\alpha = \frac{bv^3}{2k}}$

$= \frac{h^2}{k}\tan\beta/2$

$\cancel{\tan\frac{\beta}{2} = 1/\tan 2\alpha = \frac{1-\tan^2\alpha}{2\tan\alpha}}$

$$\beta = 2\tan^{-1}\left(\frac{k}{bv^2}\right)$$

L11.3



$\beta \rightarrow \pi$ as $b \rightarrow 0$

Rutherford experiment on ~~resulting~~ scattering of α particles by gold leaf

- deduce positive charges in gold leaf concentrated in nuclei

L13.1

Coriolis Force

$$-2m\underline{\omega}\times\left(\frac{d\underline{r}}{dt}\right)_{S'} = -2m\underline{\omega}\times\underline{v}$$ proportional to $\underline{v}$ in rotating frame

Lorentz force $q\underline{v}\times\underline{B}$

Coriolis force is also $\perp$ to $\underline{v}$ and hence does no work on particle

Consider particular example of motion tangential to surface of rotating planet - configuration introduced earlier - same notation:

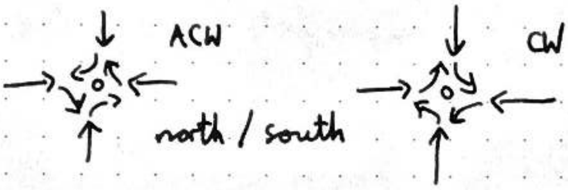
$\underline{v} = v_x\hat{\underline{x}} + v_y\hat{\underline{y}}$ $\qquad$ $\underline{\omega} = \omega(\cos\lambda\,\hat{\underline{y}} + \sin\lambda\,\hat{\underline{z}})$

Coriolis force $\quad -2m\underline{\omega}\times\underline{v} = 2m\sin\lambda(v_y\hat{\underline{x}} - v_x\hat{\underline{y}})\times\omega$ (horizontal)

$\qquad + 2m\omega\cos\lambda\, v_x\hat{\underline{z}}$ (vertical)

The Coriolis force causes acceleration to the right of the motion in the northern hemisphere, to the left in southern hemisphere.

c.f. hurricanes



Coriolis force ends up balanced by pressure gradient

Example problem: ball falling from tower

In rotating frame:

$$\ddot{\underline{r}} = \underline{g} - 2\underline{\omega}\times\dot{\underline{r}} - \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

Rotation "slow" i.e. $\omega^2 R \ll g$ with R radius of Earth

So work to first order in ω:

$$\ddot{\underline{r}} = \underline{g} - 2\underline{\omega}\times\dot{\underline{r}}$$

integrate $\Rightarrow$ $\dot{\underline{r}} = \underline{g}t - 2\underline{\omega}\times(\underline{r}-\underline{r}_0)$ (start at ~~origin~~ $\underline{r}_0$)

substitute $\Rightarrow$ $\ddot{\underline{r}} = \underline{g} - 2\underline{\omega}\times\underline{g}t + O(\omega^2)$

integrate $\Rightarrow$ $\dot{\underline{r}} = \underline{g}t - \cancel{2}\underline{\omega}\times\underline{g}t^2 + O(\omega^2)$

$\Rightarrow$ $\underline{r} = \underline{r}_0 + \frac{1}{2}\underline{g}t^2 - \frac{1}{3}(\underline{\omega}\times\underline{g})t^3 + O(\omega^2)$

Now consider particular example (in Equator)

$\underline{g} = (0, 0, -g)$ $\qquad$ $\underline{\omega} = (0, \omega, 0)$ $\qquad$ $\underline{r}_0 = (0, 0, R+h)$

L13.2

$$\underline{r} = \left(\tfrac{1}{3}\omega g t^3,\ 0,\ R + h - \tfrac{1}{2} g t^2 \right)$$

particle hits ground when $t = \sqrt{\frac{2h}{g}}$

at this time

$$\underline{r} = \left(\tfrac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2},\ 0,\ 0 \right) \qquad \text{i.e. East displacement}$$

Foucalt's pendulum

Consider pendulum at North pole - swings in plane which is fixed in inertial frame. Observer on rotating Earth sees plane of rotation changing orientation.

At Latitude λ rotation rate is $\omega \sin\lambda$ so period $= \dfrac{24\,\text{hrs}}{\sin\lambda}$

See Tong.

§6 Systems of particles

So far have considered motion of a single particle under specified force. Now we extend this to multi-particle systems.

Consider N particles, labelled $i = 1, \dots, N$, mass m_i, position $\underline{r}_i(t)$

$\Rightarrow$ momentum $\underline{p}_i = m_i \dot{\underline{r}}_i(t)$

Newton's 2nd law for particles gives

$$m_i \ddot{\underline{r}}_i = \dot{\underline{p}}_i = \underline{F}_i \qquad (\text{net force on } i^{th} \text{ particle})$$

where $\underline{F}_i = \underline{F}_i^{ext} + \sum_{j=1}^{N} \underline{F}_{ij}$ $\quad$ $\underline{F}_{ij}$ is force j exerts on i

($\underline{F}_{ii} = \underline{0}$ $\quad$ ith particle exerts no force on itself) $\quad$ Newton's 3rd law gives more general

$$\underline{F}_{ij} = -\underline{F}_{ji}$$

(notice how this is true for gravitational forces)

$$\underline{F}_{ij} = -G m_i m_j \frac{(\underline{r}_i - \underline{r}_j)}{|\underline{r}_i - \underline{r}_j|^3} = -\underline{F}_{ji}$$

Total momentum

$$\underline{P} = \sum_{i=1}^{N} m_i \dot{\underline{r}}_i = M \dot{\underline{R}}$$

equivalent to momentum of single particle

§6.1 Motion of centre of mass

Total mass $= \sum_{i=1}^{N} m_i = M$

Centre of mass located at $\underline{R} = \dfrac{1}{M} \sum_{i=1}^{N} m_i \underline{r}_i$

Consider rate of change of total momentum $\underline{P}$

$$\dot{\underline{P}} = M\ddot{\underline{R}} = \sum_i \dot{\underline{p}}_i = \sum_i \underline{F}_i^{ext} + \underbrace{\sum_i \sum_j \underline{F}_{ij}}_{\text{zero by antisymmetry (N3)}}$$

$$\therefore \quad \dot{\underline{P}} = \sum_i \underline{F}_i^{ext} = \text{total external force}$$

Centre of mass moves like particle of mass M under $\sum_i \underline{F}_i^{ext}$

Implication is that N2 generalises from particles to macroscopic bodies

Momentum conserved <=> $\sum_i \underline{F}_i^{ext} = \underline{0}$

If so, centre of mass frame is inertial $\ddot{\smile}$

Angular Momentum

Total ang. momentum is $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

Then
$$\dot{\underline{L}} = \sum_i \underbrace{\cancel{\dot{\underline{r}}_i \times \underline{p}_i}}_{\text{zero}} + \sum_i \underline{r}_i \times \dot{\underline{p}}_i$$

$$= \sum_i \underline{r}_i \times \underline{F}_i^{ext} + \underbrace{\sum_i \sum_j \underline{r}_i \times \underline{F}_{ij}}_{\sum_{i<j} (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ij}}$$

$$= \underline{G}^{ext} + \text{effect of internal forces}$$

Then if $\underline{F}_{ij}$ is parallel to $\underline{r}_i - \underline{r}_j$, internal contribution zero so

$$\dot{\underline{L}} = \underline{G}^{ext} = \sum_i \underline{G}_i^{ext} = \sum_i \underline{r}_i \times \underline{F}_i^{ext}$$

6.2 Motion relative to the centre of mass

Let $\underline{r}_i = \underline{R} + \underline{s}_i$. $\underline{s}_i$ relative position

$$\sum_i m_i \underline{s}_i = \sum_i m_i \underline{r}_i - \sum_i m_i \underline{R} = \underline{0}$$

$$\Rightarrow \sum_i m_i \dot{\underline{s}}_i = \underline{0}$$

Total momentum $\underline{P} = \sum_i m_i (\dot{\underline{R}} + \dot{\underline{s}}_i) = M\dot{\underline{R}}$ by above

Total angular momentum about origin

$$\underline{L} = \sum_i m_i (\underline{R} + \underline{s}_i) \times (\dot{\underline{R}} + \dot{\underline{s}}_i)$$

$$= M\underline{R} \times \dot{\underline{R}} + \overbrace{\cancel{\left(\sum_i m_i \underline{s}_i\right)}}^{\text{zero}} \times \dot{\underline{R}}$$

$$+ \underline{R} \times \overbrace{\cancel{\left(\sum_i m_i \dot{\underline{s}}_i\right)}}^{\text{zero}} + \sum_i m_i \underline{s}_i \times \dot{\underline{s}}_i$$

$$= \underline{R} \times \underline{P} + \sum_i m_i \underline{s}_i \times \dot{\underline{s}}_i$$

angular momentum of mass M at $\underline{R}$ with momentum $\underline{P}$

angular momentum in COM frame relative to COM

Total kinetic energy

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \frac{1}{2} \sum_i m_i (\dot{\underline{R}} + \dot{\underline{s}}_i) \cdot (\dot{\underline{R}} + \dot{\underline{s}}_i)$$

$$= \frac{1}{2} M\dot{R}^2 + \dot{\underline{R}} \cdot \overbrace{\cancel{\left(\sum_i m_i \dot{\underline{s}}_i\right)}}^{\text{zero}} + \frac{1}{2}\sum_i m_i \dot{s}_i^2$$

$$= \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\sum_i m_i \dot{s}_i^2$$

kinetic energy of mass M at $\underline{R}$

kinetic energy in COM frame

How to make energy conserved?

$V_{ij}(\underline{r}_i - \underline{r}_j) = V_{ji}(\underline{r}_j - \underline{r}_i)$

If $\underline{F}_i^{ext} = -\underline{\nabla}_i V_i^{ext}(\underline{r}_i)$

$$\underline{F}_{ij} = -\underline{\nabla}_i V_{ij}(\underline{r}_i - \underline{r}_j) \quad \text{where } V_{ij} = V_{ji}$$

then total energy E conserved

where $E = T + V$

$$= T + \sum_i V_i^{ext}(\underline{r}_i) + \sum_{i<j} V_{ij}(\underline{r}_i - \underline{r}_j)$$

6.3 Two body problem

Consider two particles with no external forces.

Centre of mass is at $\underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2)$.

Define separation vector $\underline{r} = \underline{r}_1 - \underline{r}_2$.

Then

$$\underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}, \quad \underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}.$$

Since $\underline{F}^{ext} = \underline{0}$, have $\ddot{\underline{R}} = \underline{0}$, easy motion.

Evolution of $\underline{r}$ given by

$$\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \frac{1}{m_1}\underline{F}_{12} - \frac{1}{m_2}\underline{F}_{21} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\underline{F}_{12}$$

$\Rightarrow \mu \ddot{\underline{r}} = \underline{F}_{12}$ where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

Assume $\underline{F}_{12}$ only depends on $\underline{r}$ (perfectly reasonable).

Obtain equation for a particle of mass μ under $\underline{F}_{12}$.

For example, if $\underline{F}_{12}$ is gravitational:

$$\mu \ddot{\underline{r}} = -\frac{G m_1 m_2}{r^3}\underline{r} \Rightarrow \cancel{\mu}\ \ddot{\underline{r}} = -\frac{GM}{r^3}\underline{r}$$

For Earth & Sun, have that they rotate about COM.

Shape of orbit is that of $\underline{r}$ up to scaling.

COM very close to centre of Sun though so Sun barely moves.

L15.1

Return to general two body problem

$$\underline{L} = M\underline{R} \times \dot{\underline{R}} + \mu \underline{r} \times \dot{\underline{r}}$$

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu \dot{r}^2$$

#epic

(When have 3 body problem becomes hard)

$$m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2$$

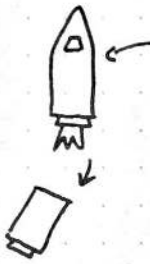
$$\underline{r} = \frac{M}{m_1}\underline{r}_1 = -\frac{M}{m_2}\underline{r}_2$$

$$\frac{1}{M^2}\cancel{M}\left(\frac{m_1}{m_2^{-2}} + \frac{m_2}{m_1^{-2}}\right)$$

$$= \frac{m_1 m_2}{M^2}(m_1 + m_2) = \mu$$

§ 6.4 Variable mass problems

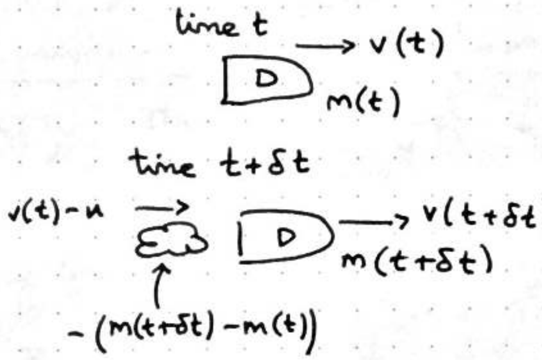
Motivation - the rocket problem



mass changes as fuel used say $m(t)$

$$\frac{dp}{dt} = +\overset{(v-u)}{\cancel{u}}\frac{dm}{dt}$$

$$\dot{m}v + m\dot{v} = v\dot{m} - u\dot{m}$$



$$\delta p = m(t+\delta t)v(t+\delta t) - m(t)v(t) + (m(t) - m(t+\delta t))(v(t)-u)$$

$$\approx (m\dot{v} + \dot{m}u)\delta t$$

If no external force,

$$\frac{\delta p}{\delta t} = 0 \Rightarrow m\dot{v} + \dot{m}u = 0$$

If external force F, then

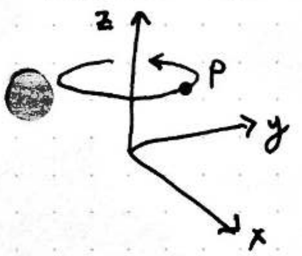
$$m\dot{v} + \dot{m}u = F \quad \text{"Rocket eq}^{n}\text{"}$$

Ex: if $F=0$ $\quad m\frac{dv}{dt} = -u\frac{dm}{dt} \Rightarrow v = v_0 + u\log\left(\frac{m_0}{m(t)}\right)$

§ 7: Rigid bodies

A rigid body is an extended object that can be considered as a multi-particle system such that $|\underline{r}_i - \underline{r}_j|$ is fixed in time. The possible motion of the rigid body given by isometries of $\mathbb{R}^3$. These are translations and rotations (see Geometry)

§ 7.1 Angular velocity



Spose rotation about z-axis with angular speed ω

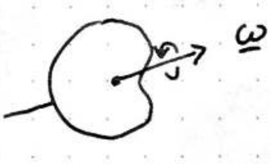
position $(a\cos\theta, a\sin\theta, z)$

velocity $(-a\sin\theta \cdot \dot{\theta}, a\cos\theta \cdot \dot{\theta}, 0)$

$\Rightarrow \underline{v} = \underline{\omega} \times \underline{r}$

L15.2

Can generalise to arbitrary $\underline{\omega}$



Kinetic energy $\frac{1}{2}m\dot{\underline{r}}^2 = \frac{1}{2}m(\underline{\omega}\times\underline{r}|^2$
$= \frac{1}{2}m r_\perp^2\omega^2$

Then $T = \frac{1}{2}I\omega^2$ where $I = m r_\perp^2$ is moment of inertia about axis

§7.2 Moment of inertia for a rigid body

Now rigid body rotates with $\underline{\omega}$ about some axis through O

$\dot{\underline{r}}_i = \underline{\omega}\times\underline{r}_i$

Consistent? $\frac{d}{dt}((\underline{r}_i-\underline{r}_j)\cdot(\underline{r}_i-\underline{r}_j)) = 2(\underline{r}_i-\underline{r}_j)\cdot(\dot{\underline{r}}_i-\dot{\underline{r}}_j)$
$= 2(\underline{r}_i-\underline{r}_j)\cdot[\underline{\omega}\times(\underline{r}_i-\underline{r}_j)]$

is zero so indeed $|\underline{r}_i-\underline{r}_j|$ remains constant

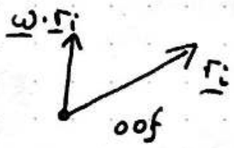
- Rotational kinetic energy

$$T = \sum_i \frac{1}{2}m_i\dot{r}_i^2 = \frac{1}{2}\sum_i m_i|\underline{\omega}\times\underline{r}_i|^2 = \frac{1}{2}\left(\sum_i m_i r_{i\perp}^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

$\therefore I = \sum_i m_i r_{i\perp}^2$ is the moment of inertia of the rigid body about axis

- Angular momentum of body

$$\underline{L} = \sum_i m_i\underline{r}_i\times\dot{\underline{r}}_i = \sum_i m_i\underline{r}_i\times(\underline{\omega}\times\underline{r}_i)$$



$$= \sum_i m_i\left[\underline{\omega}(r_i^2) - \underline{r}_i(\underline{\omega}\cdot\underline{r}_i)\right] \cancel{\neq \sum_i m_i}$$

Consider

$$\underline{L}\cdot\underline{n} = \sum_i m_i\left[\omega r_i^2 - \omega r_{i\parallel}^2\right] = \omega\sum_i m_i r_{i\perp}^2 = I\omega.$$

But $\underline{L}$ need not be parallel to $\underline{\omega}$!

In components

$$L_a = \sum_i m_i(\omega_a r_{ic}r_{ic} - r_{ia}\omega_b r_{ib})$$

$$= I_{ab}\omega_b \quad \text{where } I_{ab} = \sum_i m_i(\delta_{ab}r_{ic}r_{ic} - r_{ia}r_{ib})$$

is the inertia tensor

Calculation of moments of inertia

$$I = \sum_i m_i r_{i\perp}^2 \longrightarrow \int_V \rho(\underline{x})\, dV\, (\underline{x})_\perp^2 = \int_V \rho(\underline{x})\, dV\, |\underline{n} \times \underline{x}|^2$$

Replace summation with a mass-weighted integral

$$M = \int_V \rho(\underline{x})\, dV \qquad M\underline{R} = \int_V \rho(\underline{x})\, dV\, \underline{x}$$

We now evaluate I for some simple examples

1) (Uniform) thin ring, axis through centre $\perp$ plane of ring

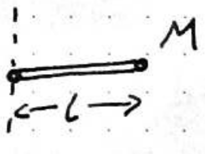
$$\Rightarrow \quad I = Ma^2$$

Every point a distance a from the axis

For thin bodies reduce volume integral to line integral.

$$I = \int_0^{2\pi} \eta(\theta)\, a^2\, d\theta = a^2 \int_0^{2\pi} \eta(\theta)\, d\theta = Ma^2$$

2) Uniform rod, axis through end-point $\perp$ to rod



$$I = \int_0^l \eta(x)\, x^2\, dx = \eta\left(\tfrac{1}{3} l^3\right) = \tfrac{1}{3} M l^2$$

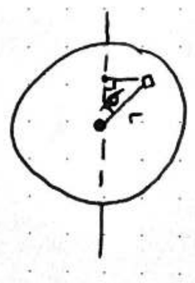
since η is constant, $\eta l = M$

3) Uniform disc, axis through centre $\perp$ disc

$$\int_0^{2\pi} d\phi \int_0^a dr \cdot r^2 \cdot r\sigma = 2\pi\sigma\left(\tfrac{1}{4} a^4\right) = \tfrac{1}{2} M a^2$$

since σ is constant, $\pi a^2 \sigma = M$

4) Same disc, axis through centre $\parallel$ disc



$$\int_0^{2\pi} d\phi \int_0^a dr \cdot r^2 \sin^2\phi \cdot r\sigma$$

$$= \pi\sigma\left(\tfrac{1}{4} a^4\right) = \tfrac{1}{4} M a^2$$

$I_z = I_x + I_y$ since $r^2 = x^2 + y^2$ (see later)

L16.2

5) Solid uniform sphere, axis through centre

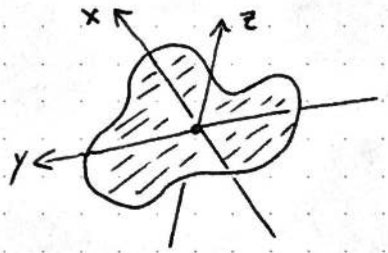
$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{a} dr \cdot r^2 \sin^2\theta \cdot r^2 \sin\theta \, \rho$$

$$= 2\pi\rho \cdot \left(\tfrac{1}{5}a^5\right) \cdot \frac{4}{3} = \frac{2}{5} M a^2$$

since ρ is constant, $\rho \cdot \frac{4\pi}{3} a^3 = M$

Some useful results:

- Perpendicular axes theorem



For a <u>lamina</u> in the x-y plane

$$I_z = I_x + I_y$$

Indeed,

$$I_z = \int_D \rho(x,y)(x^2+y^2)\, dA$$

$$= \int_D \rho(x,y)\, x^2\, dA + \int_D \rho(x,y)\, y^2\, dA$$

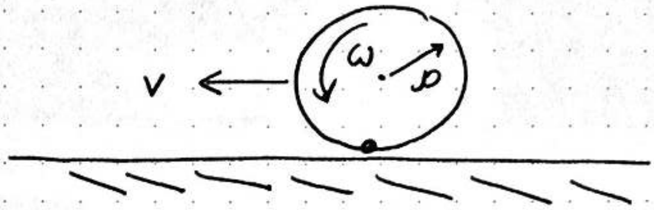
$$= I_y + I_x.$$

Note that if body has symmetry $x \leftrightarrow y$ then $I_x = I_y = \frac{1}{2} I_z$

L18.1

Sliding vs Rolling

Consider a cylinder or sphere moving along a stationary horizontal surface.



P point of contact
(fixed in cylinder or sphere)

General motion is translation of centre of mass (with velocity v) together with rotation about centre of mass (with angular velocity ω).

The horizontal velocity of the point of contact is $v_{slip} = v - a\omega$

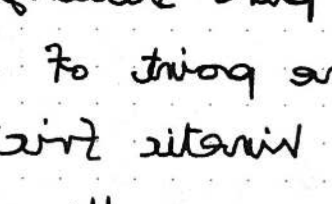
A pure sliding motion is $\omega = 0 \Rightarrow v_{slip} = v \neq 0$
The point of contact slips and there may be a kinetic frictional force.

A pure rolling motion is $v \neq 0$, $\omega \neq 0$ s.t. $v_{slip} = v - a\omega = 0$ (a no slip condition) The point of contact is stat.

A rolling body can be described instantaneously as rotating about the point of contact, with angular velocity ω.

Example: rolling downhill

cylinder or sphere mass M, radius a, rolling downhill a rough plane inclined at an angle α to the horizontal



L8.2 ☹

No slip: $v - a\omega = 0$

Energy: $T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}\left(M + \frac{I}{a^2}\right)\cancel{\omega} v^2$$

$$T + V = \frac{1}{2}\left(M + \frac{I}{a^2}\right)\dot{x}^2 - Mgx\sin\alpha$$

Total energy conserved:

$$\left(M + \frac{I}{a^2}\right)\ddot{x} = Mg\sin\alpha$$

$$\Rightarrow \ddot{x} = g\sin\alpha \cdot \frac{1}{1 + \frac{I}{Ma^2}}$$

If cylinder, $I = \frac{1}{2}Ma^2$ so $\ddot{x} = \frac{2}{3}g\sin\alpha$

In terms of forces + torques

Linear momentum equation: $M\dot{v} = Mg\sin\alpha - \cancel{\mu} F$

Angular momentum eq^n about CoM: $I\dot{\omega} = aF$

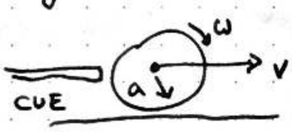
Rolling condition $\dot{v} = a\dot{\omega} \Rightarrow \frac{I}{a}\dot{v} = aF$

$$\therefore M\dot{v} = Mg\sin\alpha - \frac{I}{a^2}\dot{v} \Rightarrow \dot{v} = \frac{g\sin\alpha}{1 + \frac{I}{Ma^2}}$$ as before

Frictional force does no work: point of contact is stationary

Example: sliding to rolling transition

e.g. snooker ball hit by cue



initially $v = v_0$, $\omega = 0$

have initial sliding

kinetic friction force $F = \mu Mg$

linear momentum: $M\dot{v} = -F$

angular momentum: $I\dot{\omega} = aF$

$\therefore v = v_0 - \mu g t$, $\omega = \frac{aF}{I}t = \frac{5}{2a}\mu g t$ for $t < t_0$ so slipin

t_{roll}: when slip velocity $= 0$

L8.3

$a v_0 = \frac{7}{5} a v_{roll}$
via ang. momentum

$t_{roll} = \frac{2v_0}{7\mu g}$ $\quad v_{roll} = \frac{5}{7} v_0$

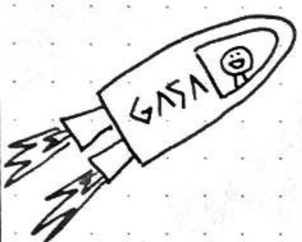
After t_{roll}, begins to roll w/out slippin

KE lost to friction
At $t = t_{roll}$, $\frac{5}{7} \cdot \frac{1}{2} M v_0^2$
During rolling, friction does no work
Work done by friction explicitly

$$\int_0^{t_{roll}} F v_{slip} \, dt = \int_0^{t_{roll}} F \cdot \left(v_0 - \tfrac{7}{2}\mu_k g t\right) dt = \frac{1}{7} M v_0^2 \checkmark$$

NOT COOL

§8: Special relativity over with Tong



Two postulates which kinda make sense
1: Laws of physics the same in all inertial frames
2: speed of light constant c

Second postulate confirmed by experiments
2^{nd} postulate incompatible with Galilean transformations, in particular the concept of absolute time

§ 8.1 Lorentz transformations

Consider inertial frames S and S'.

Suppose spatial origins coincide at $t = 0$.

Say S' moves with velocity $\underline{v}$ in the $+x$ direction relative to S.

< Suppose $y \to y'$ and $z \to z'$ are trivial >

Postulate 1: constant velocity paths map to constant velocity paths

These paths are straight lines in (x, t) plane.

So $(x, t) \to (x', t')$ must be linear.

$x - vt = 0$ must map to $x' = 0$, ergo

$x' = \gamma(x - vt)$ where γ could depend on $v = |\underline{v}|$

Conversely, $x = \gamma(x' + vt')$ by symmetry.

Postulate 2: $x - ct = 0$ must map to $x' - ct' = 0$

$$x' = \gamma(c - v)t \qquad \text{while} \qquad x = \gamma(c + v)t'$$
$$= ct' \qquad\qquad\qquad\qquad = ct$$

For compatibility, must have

$$\gamma\left(1 - \frac{v}{c}\right) = \frac{1}{\gamma\left(1 + \frac{v}{c}\right)} \implies \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{« Lorentz factor »}$$

$$x = \gamma(x' + vt') \longrightarrow t' = \frac{1}{v}\left(\frac{x}{\gamma} - x'\right) = \frac{1}{v}\left(\frac{x}{\gamma} - \gamma(x - vt)\right)$$

$$x' = \gamma(x - vt) \qquad = \gamma\left(t - x \cdot \left(\frac{1}{v} - \frac{1}{v\gamma^2}\right)\right) = \gamma\left(t - \frac{vx}{c^2}\right)$$

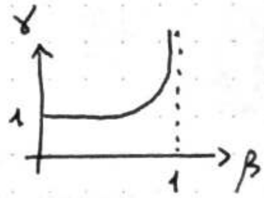
So, finally,

$$\left\{\begin{aligned} x' &= \gamma(x - vt) = \gamma\left(x - \frac{v}{c}ct\right) \\ ct' &= \gamma\left(ct - \frac{v}{c}x\right) \end{aligned}\right\} \quad \text{Lorentz boost}$$

Can trivially invert to get $(x', ct') \to (x, ct)$.

L19.2

Note $\gamma_v \geq 1$, it's an increasing function of v, and $\lim_{v \to c} \gamma_v = \infty$. For $v \simeq c$, get $\gamma \sim \frac{1}{\sqrt{2}} \cdot \frac{1}{(1-\frac{v}{c})^{1/2}}$.



Check constancy of light speed:

$x = ct$, $y = 0$, $z = 0$

seen in S' as

$x' = \gamma_v (x - vt) = \gamma_v (c - v) t$

$t' = \gamma_v (t - \frac{vx}{c^2}) = \gamma_v (1 - \frac{v}{c}) t$

so $x' = ct'$, $y' = 0$, $z' = 0$ as required.

As for in another direction:

$x = 0$, $y = ct$, $z = 0$

seen in S' as

$x' = \gamma_v (x - vt) = -v \gamma_v t$ speed is $\left(\frac{x'}{t'}\right)^2 + \left(\frac{y'}{t'}\right)^2$

$y' = y = ct$ $= v^2 + \left(\frac{c}{\gamma_v}\right)^2 = c^2$ ⍤

$t' = \gamma_v (t - \frac{vx}{c^2}) = \gamma_v t$

direction of light ray changed, speed did not

Lorentz transformations preserve

$c^2 t^2 - x^2 - y^2 - z^2$

$X^\mu X_\mu$ Ö

$\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$

Indeed, $y' = y$, $z' = z$, and

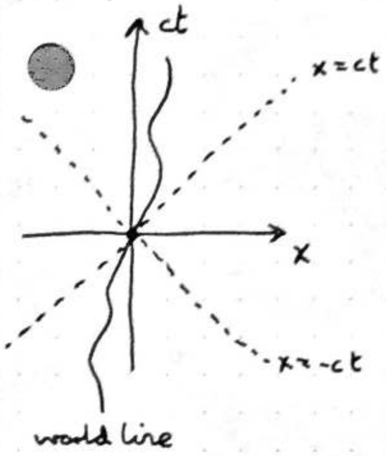
$$c^2 t'^2 - x'^2 = c^2 \gamma^2 (t - \frac{vx}{c^2})^2 - \gamma^2 (x - vt)^2$$
$$= c^2 \gamma^2 t^2 - \cancel{2\gamma^2 t v x} + \gamma^2 \frac{v^2 x^2}{c^2}$$
$$- \gamma^2 x^2 + \cancel{2\gamma^2 x v t} - \gamma^2 v^2 t^2$$
$$= \gamma^2 (c^2 - v^2) t^2 - \gamma^2 \left(1 - \frac{v^2}{c^2}\right) x^2$$
$$= c^2 t^2 - x^2.$$

WooH!

So light rays always move at c.

L19.3

Spacetime diagrams ~ Minkowski spacetime



← points in diagram are "events", given coords (ct, x)

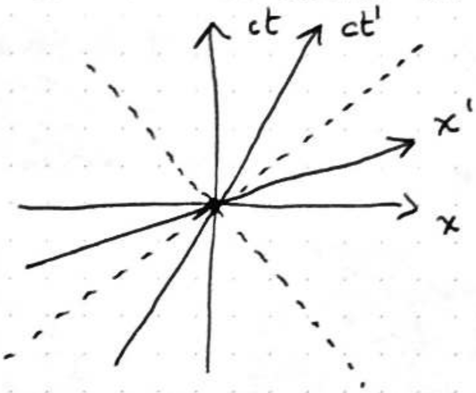
coordinates of events differ in other reference frames

path of particle : world-line

light rays move at 45° angles

Particles <cannot> travel faster than light so world-line must be steeper than 45°

Lorentz transforms on spacetime diagram



Indeed, $x'=0$ is $x-vt=0$ while $t'=0$ is $t-\frac{vx}{c^2}=0$.

Note equal angles:

$x-\frac{v}{c}(ct)=0, \quad (ct)-\frac{v}{c}x=0$

Also, lines of $x'=$ const. parallel to ct' axis

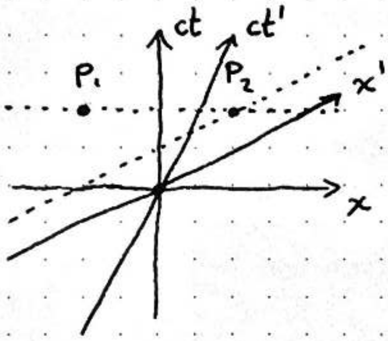
" " $ct'=$ const. " " x' axis

by similar reasoning.

L20.1

§ 8.2 Relativistic physics

Two events P_1, P_2 are simultaneous in frame S if their ct coordinates are the same.

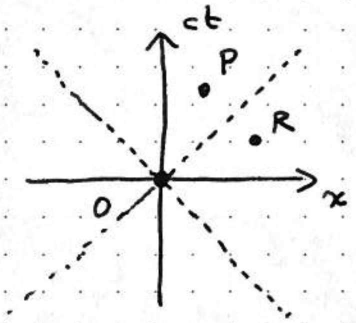


In S', P_2 happens before P_1.
More quantitatively, if distance from P_1 to P_2 is $\Delta x > 0$, time difference in S' can be obtained via $c^2\Delta t'^2 - \Delta x'^2 = -\Delta x^2$ and $\Delta x' = \gamma \Delta x$.

Simultaneity is frame dependant.

Causality

However, if the spacetime interval between two events $\Delta S^2 > 0$, the events are not simultaneous in any frame, and there is an ordering of the events which all observers agree upon.



Via light cone see when an event occurs after another in all frames.
Hence O can influence events only within its future light cone. Otherwise, in some frame O would seem to influence events in the past!

Likewise, O can only be influenced by events in its past light-cone.

Time dilation

A clock which ticks at $\Delta t'$ in S', a frame at rest with the clock. Then in a frame S which sees the clock move at speed v, $\Delta t = \gamma \Delta t'$. Moving clocks run slower.

$\boxed{\text{Time between events minimal when events occur in same place.}}$

Proper time τ is the time experienced by an object in a frame at ~~reference~~ rest with it.

L20.2

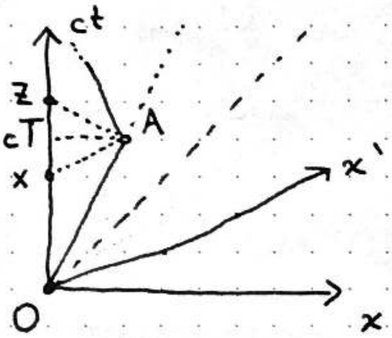
Twin paradox

Two twins, Peter and David.

Peter stays on Earth.

David flies over to Mars to do some work.

After, David comes back.



Say Peter sees David arrive after time $t = T$.

By time dilation, Peter experiences a time T/γ on his outward journey.

Likewise on the return journey.

So at the end, Peter is older than David:

$$2T > 2T/\gamma.$$

Paradox: situation is "symmetric" between Peter and David, right?

Well no, acceleration is not relative, and David had to accelerate in order to return from Mars.

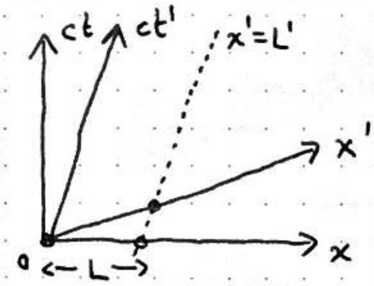
Moreover, over each leg of the journey, there is symmetry.

So as David reaches Mars, he thinks Peter sees a time $\frac{cT}{\gamma^2} < \frac{cT}{\gamma}$ so is younger than him.

However, as David turns, it seems to him that Peter rapidly ages from X to Z. Acceleration is bad fam.

Length contraction

Consider a rod of length L' as measured in a frame S' stationary with it. How long is it in S?



Length is the distance between two ends of the rod at the same time.

Want intersection of $x' = L'$ with $ct = 0$.

$L' = \gamma(x - vt) \cap ct = 0 \Rightarrow x = L'/\gamma$

So rod appears shorter in S by a factor of γ.

As before, can define "proper length" (quite useless).

L21.1

"Addition" of velocities

A particle moves with speed u' in the +ve x direction as seen in frame S'.

S' moves in the +ve x direction with speed v relative to S.

Assume particle starts at common spatial origin.

Then worldline is $x' = u't'$ or $x = ut$.

$$x = \gamma_v (x' + vt') = \gamma_v (u'+v) t'$$

$$t = \gamma_v \left(t' + \frac{vx'}{c^2} \right) = \gamma_v \left(1 + \frac{u'v}{c^2} \right) t'$$

$$\Rightarrow u = \frac{u'+v}{1+\frac{u'v}{c^2}} \quad \ddot{\smile}$$

4 vector $U^\mu = \gamma_v \begin{pmatrix} c \\ \underline{v} \end{pmatrix}$

$$\gamma_u u = \gamma_v (u'+v) \gamma_{u'}$$

$$\gamma_u c = \gamma_v \left(c + \frac{vu'}{c} \right) \gamma_{u'}$$

Obtain reverse transformation via $v \to -v$

Easy to see $|u'| < c, |v| < c \Rightarrow |u| < c$ via tanh

§8.3 Geometry of spacetime

Recall invariant interval $\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

Lorentz transformations characterised by preserving Δs^2

Introduce 4-vectors $X^\mu = (ct, x, y, z)$ and metric $\eta = \mathrm{diag}(1,-1,-1,-1)$

Then Lorentz transformations are to η what rotations are to I

$X^\mu X^\nu \eta_{\mu\nu}$ preserved since $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$

In general should use $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

But since η constant, transformations linear, no biggie

Spacetime is $\mathbb{R}^4$ endowed with a pseudo-Riemannian metric

Minkowski

Say it has $(1+3)$ dimensions

If $\Delta s^2 > 0$, call interval time-like

If $\Delta s^2 < 0$, call it space-like

If $\Delta s^2 = 0$, call it light-like, or null

Indeed, Δs^2 not positive definite

L21.2

Lorentz group consists of those transformations preserving η. i.e. $\Lambda^{\mu}{}_{\nu} \Lambda^{\rho}{}_{\sigma} \eta_{\mu\rho} = \eta_{\nu\sigma}$ or $\Lambda \eta \Lambda^{T} = \eta$

Λ tells you how coordinates of 4-vectors change

For 3d rotation

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

For x-boost

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v/c & 0 \\ -\gamma v/c & \gamma & \\ 0 & & I \end{pmatrix}$$

Two sorts of Λ

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$ where R is orthogonal i.e. rotations / reflections

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & 0 \\ & 0 & 1 \end{pmatrix}$$ boost in the x-direction

Set of all Lorentz transformations form Lorentz group $O(3,1)$

It includes reflections :(and time reversals :(

Taking those w/ unit det gives $SO(3,1)$, proper Lorentz group

But can also time reverse :(

So restrict to proper, orthocronous Lorentz transformations

$SO^+(3,1)$, restricted Lorentz group

Generated by boosts and spatial rotations

<u>Rapidity</u> Focus on $\Lambda[u] = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$

Combine two boosts:

$$\Lambda[u]\,\Lambda[v] = \begin{pmatrix} \gamma_u & -\gamma_u u/c \\ -\gamma_u u/c & \gamma_u \end{pmatrix} \begin{pmatrix} \gamma_v & -\gamma_v v/c \\ -\gamma_v v/c & \gamma_v \end{pmatrix}$$

$$= \Lambda\left[\frac{u+v}{1+\frac{uv}{c^2}}\right]$$

cf velocity addition

Perhaps more nicely

$$\Lambda[\beta_1]\,\Lambda[\beta_2] = \Lambda\left[\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right]$$

In search of niceness introduce rapidity φ with

$$\tanh\varphi = \beta \Rightarrow \cosh\varphi = \gamma$$

$$\Lambda[\beta] = \Lambda[\varphi] = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \Rightarrow \Lambda[\varphi_1]\,\Lambda[\varphi_2] = \Lambda[\varphi_1+\varphi_2]$$

§8.4 Relativistic kinematics

Suppose a particle moves along $\underline{x}(t)$ as seen in S.

Its velocity is $\underline{v} = \frac{d}{dt}\underline{x}$.

But this parametrisation does not transform nicely.

Introduce proper time τ, the time experienced by particle

$$\underbrace{\Delta\tau = \frac{1}{c}\Delta s}_{\text{see via rest frame of particle}} = \frac{1}{c}\sqrt{c^2\Delta t^2 - \Delta x^2} = \sqrt{1-\beta^2}\,\Delta t$$

$\boxed{\Delta\tau \text{ invariant}}$

$$\Rightarrow \frac{dt}{d\tau} = \gamma$$

Parametrise path via $X(\tau) = \begin{pmatrix} ct(\tau) \\ \underline{x}(\tau) \end{pmatrix}$

So everyone agrees where particle is at given τ

Can compute $\Delta\tau = \int d\tau = \int \frac{dt}{\gamma}$ ooh

4-velocity

$$U^\mu = \frac{d}{d\tau}X^\mu = \gamma\begin{pmatrix} c \\ \underline{u} \end{pmatrix} \quad \text{where } \underline{u} = \frac{d}{dt}\underline{x}$$

this is a 4-vector tangent to path of constant modulus

$$U^\mu U_\mu = \gamma^2(c^2 - u^2) = c^2$$

So U transforms via Lorentz transformations

[4-vectors are anything which has components that transform via Λ]

Velocity addition via U

Spose $U = \gamma_u(c, \underline{u})$ in S'. Boost by $-v$ to go to S.

Recover velocity addition $u' = \dfrac{u+v}{1+\frac{uv}{c^2}}$.

In general if

$$U = \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}\gamma_u \quad \text{transforms nicely.}$$

L23.1

4-velocity in S

$$U = \begin{pmatrix} \gamma_u c \\ \gamma_u u \cos\theta \\ \gamma_u u \sin\theta \\ 0 \end{pmatrix}$$

4-velocity in S'

$$U' = \begin{pmatrix} \gamma_v \left(\gamma_u c - \frac{v}{c} \gamma_u u \cos\theta \right) \\ \gamma_v \left(\gamma_u u \cos\theta - \frac{v}{c} c \gamma_u \right) \\ \gamma_u u \sin\theta \\ 0 \end{pmatrix}$$

$$\frac{u' \cos\theta'}{c} = \frac{u\cos\theta - v}{c - \frac{uv}{c}\cos\theta} \Rightarrow u'\cos\theta' = \frac{u\cos\theta - v}{1 - \frac{uv}{c^2}\cos\theta}$$ divide (1) by (0)

$$\tan\theta' = \frac{u \sin\theta}{\gamma_v (u\cos\theta - v)}$$ divide (2) by (1)

cf Stellar Aberration

4-momentum

If rest mass is m then 4-momentum is

$$P = mU = \begin{pmatrix} \gamma_u m c \\ \gamma_u m \underline{u} \end{pmatrix}$$

It is a 4-vector since m is a scalar.

As in Newtonian case, P conserved if no "external forces"

Call spatial components the relativistic 3-momentum $\gamma_u m \underline{u} = \underline{p}$

As for $P^0 = mc\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = mc\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right)$

$$= \frac{1}{c}\Big(\underset{\substack{\uparrow \\ \text{rest mass} \\ \text{energy}}}{mc^2} + \underset{\substack{\uparrow \\ \text{K.E.}}}{\tfrac{1}{2}mu^2} + \cdots \Big)$$

P^0 related to energy via Noether too ü $\qquad P^0 = \frac{E}{c}$

$E \to \infty$ as $u \to c$

Mass is energy $\quad E = mc^2$ at rest

Via $P \cdot P$ get $\quad E^2 = m^2c^4 + p^2c^2$

Massless particles i.e. photons

Since $m = 0$, $P \cdot P = 0$.

But have non-zero E and p: $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$

with $E = pc$.

Along such a particle's path, $\Delta\tau = 0$.

Revised N2

$$\frac{dP}{d\tau} = F \leftarrow \text{four-force} \qquad F = \begin{pmatrix} \underline{f}\cdot\underline{u}/c \\ \underline{f} \end{pmatrix} \gamma_u$$

Motivate by going into rest frame

$$\frac{d\underline{p}}{d\tau} = \gamma_u \underline{f} \Rightarrow \frac{d\underline{p}}{dt}\cancel{\gamma_u} = \cancel{\gamma_u}\underline{f}$$

$$\frac{dE}{d\tau} = \gamma_u \underline{f}\cdot\underline{u} \Rightarrow \frac{dE}{dt} = \underline{f}\cdot\underline{u}$$

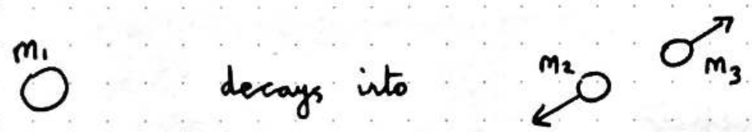
4-acceleration

$$A = \frac{dU}{d\tau} = \gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \underline{v} \end{pmatrix}$$

Particle physics

Conservation of 4-momentum v. useful

Centre of momentum frame cool



$$P_3 = \begin{pmatrix} E_3/c \\ -\underline{p} \end{pmatrix}$$

$$P_1 = \begin{pmatrix} m_1 c \\ \underline{0} \end{pmatrix} \qquad P_2 = \begin{pmatrix} E_2/c \\ \underline{p} \end{pmatrix}$$

$$m_1 c = \sqrt{m_2^2 c^2 + p^2} + \sqrt{m_3^2 c^2 + p^2} \geq m_2 c + m_3 c$$

So mass after $\leq$ mass before

Lose mass, conserve energy

L24.1

Example: $h \to \gamma\gamma$ ← photons

(Higgs particle → h)

$$P_h = P_{\gamma_1} + P_{\gamma_2}$$

In initial rest frame

$$P_h = \begin{pmatrix} m_h c \\ \underline{0} \end{pmatrix} = \begin{pmatrix} E_1/c \\ \underline{p}_1 \end{pmatrix} + \begin{pmatrix} E_2/c \\ \underline{p}_2 \end{pmatrix}$$

(← p_1, ← p_2)

so $E_1 = E_2 = \frac{1}{2} m_h c^2$

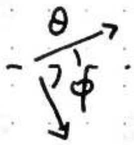
all mass turned into energy

Particle scattering - 2 particles collide

$$\underbrace{P_1 + P_2}_{\text{initial}} = \underbrace{P_3 + P_4}_{\text{final}}$$

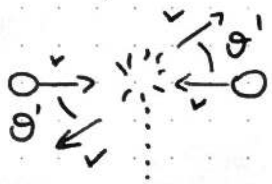
In lab frame, particle 2 at rest while particle 1 has speed u

$$P_1 = \begin{pmatrix} \gamma_u mc \\ \gamma_u mu \\ 0 \\ 0 \end{pmatrix}$$



What is relation between θ, ϕ?

Centre of momentum frame



Speeds after same via energy (v)
But angle essentially a parameter

$$\frac{2v}{1+\frac{v^2}{c^2}} = u$$

$$P_3' = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v\cos\theta' \\ m\gamma_v v\sin\theta' \\ 0 \end{pmatrix} \qquad P_4' = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v\cos\theta' \\ -m\gamma_v v\sin\theta' \\ 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \gamma & \gamma v/c & 0 \\ \gamma v/c & \gamma & \\ 0 & & 1 \end{pmatrix} \quad \text{so} \quad P_3 = \begin{pmatrix} m\gamma^2\left(c + \frac{v^2}{c}\cos\theta'\right) \\ m\gamma^2(v + v\cos\theta') \\ m\gamma v\sin\theta' \\ 0 \end{pmatrix}$$

$\cos\theta =$ ☹

$$\tan\theta = \frac{\cancel{m\gamma v}\sin\theta'}{\cancel{m\gamma^2 v}(1+\cos\theta')} = \frac{1}{\gamma}\tan\left(\theta'/2\right)$$

Similarly $\tan\theta' = \frac{1}{\gamma}\cot\left(\theta'/2\right)$

$$\therefore \tan\theta \cdot \tan\theta' = \frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = \text{some function of } u$$

L24.2

Smash particles hard enough to form new ones

$O \xrightarrow{v} \quad \xleftarrow{v} O$

$$P_1 + P_2 = \begin{pmatrix} 2m\gamma_v c \\ \underline{0} \end{pmatrix} \qquad P_3 + P_4 + P_5 = \begin{pmatrix} E_1/c + E_2/c + E_3/c \\ \underline{0} \end{pmatrix}$$

Hence $2m\gamma_v c^2 = E_1 + E_2 + E_3 \geqslant 2mc^2 + Mc^2$

possible if $\gamma_v \geqslant \left(1 + \dfrac{M}{2m}\right)$

Now transform to frame where one at rest

$O \xrightarrow{u} \quad O$

$u = \dfrac{2v}{1 + \frac{v^2}{c^2}} \Rightarrow \gamma_u = 2\gamma_v^2 - 1$

« cosh »

So $\gamma_u \geqslant 1 + \dfrac{2M}{m} + \dfrac{M^2}{2m^2}$

more energy needed