L1.1

Vector Calculus

§1 Differential Geometry of curves

§1.1 Parametrised curves + arc length

continuous & invertible

A curve C is just the image of a map $[a,b] \ni t \longmapsto \underline{x}(t)$.

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3 \right]$$

Say C is differentiable if each component $x_i(t)$ is differentiable. Say C is regular if $|\underline{x}'(t)| \neq 0$.

$\boxed{\text{equivalent to } \lim_{h \to 0} \frac{\underline{x}(t+h) - \underline{x}(t)}{h}}$

$$\underline{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$$

Say C is smooth if both differentiable and regular.

Recall $x_i(t)$ is differentiable at t if $x_i(t+h) = x_i(t) + h x_i'(t) + o(h)$ as $h \to 0$, i.e. $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

Equivalently,

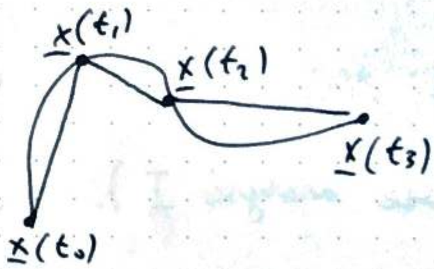
$$\underline{x}(t+h) = \underline{x}(t) + h\underline{x}'(t) + \underline{o}(h).$$

How to define length of C?

Define partition P of interval $[a,b]$ by $\{t_i\}_{i=0}^{n}$, with $t_0(a)$, $t_n(b)$, and $t_0 < t_1 < \cdots < t_n$.

Set $\Delta t = \max_i (t_{i+1} - t_i)$. Define

$$\ell(C, P) = \sum_{i=0}^{n-1} |\underline{x}(t_{i+1}) - \underline{x}(t_i)|$$



Expect this to become more accurate as $\Delta t \to 0$. DEFINE length of C by

$$\ell(C) = \lim_{\Delta t \to 0} \ell(C, P) \qquad \text{*if it exists}$$

Assume that C is continuously differentiable, i.e. $\underline{x}'(t)$ exists and is continuous, on $[a,b]$. Set $\Delta t_i = t_{i+1} - t_i$.

$$\underline{x}(t_{i+1}) = \underline{x}(t_i + \Delta t_i) = \underline{x}(t_i) + \Delta t_i\, \underline{x}'(t_i) + \underline{o}(\Delta t_i)$$

L1.2

Here $|\underline{x}(t_{i+1}) - \underline{x}(t_i)| = |\underline{x}'(t_i)|\Delta t_i + o(\Delta t_i)$, so

$$\ell(C,P) = \sum_{i=0}^{n-1} |\underline{x}'(t_i)|\Delta t_i + \sum_{i=0}^{n-1} o(\Delta t_i).$$

Since $\frac{o(\Delta t_i)}{\Delta t_i} \to 0$ as $\Delta t_i \to 0$. So for any $\varepsilon > 0$, can choose $\Delta t = \max_i(\Delta t_i)$ sufficiently small so that $\frac{|o(\Delta t_i)|}{\Delta t_i} < \varepsilon$.

So for Δt sufficiently small, $|o(\Delta t_i)| < \varepsilon \Delta t_i$ (for any $\varepsilon > 0$).

By the triangle inequality,

$$\left| \ell(C,P) - \sum_{i=1}^{n-1} |x_i'(t_i)|\Delta t_i \right| \leq \sum_{i=0}^{n-1} |o(\Delta t_i)|$$

$$< \varepsilon \sum_{i=0}^{n-1} \Delta t_i \qquad \boxed{\text{uniform continuity}}$$

$$= \varepsilon(b-a).$$

So, since $\varepsilon > 0$ was arbitrary, find that

$$\lim_{\Delta t \to 0} \sum_{i=0}^{n-1} o(\Delta t_i) = 0$$ and hence

$$\ell(C) = \lim_{\Delta t \to 0} \ell(C,P) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} |x'(t_i)|\Delta t_i.$$

If $\underline{x}'(t)$ is continuous then $\lim_{\Delta t \to 0} \sum_{i=0}^{n-1} |x'(t_i)|\Delta t_i = \int_a^b |x'(t)|\, dt$ (see analysis I). So have

$$\ell(C) = \int_C ds = \int_a^b |\underline{x}'(t)|\, dt$$

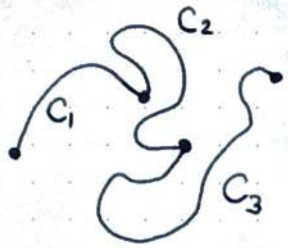
for $C: [a,b] \ni t \to \underline{x}(t)$. Also define

$$\int_C f(\underline{x})\, ds = \int_a^b f(\underline{x}(t))\, |\underline{x}'(t)|\, dt.$$

arc length element

$ds = |x'(t)|\, dt$ i.e. $ds^2 = dx^2 + dy^2 + dz^2$

L1.3 If C is made from m smooth curves $C_1, \dots, C_m$



$$\int_C f(\underline{x})\, ds = \sum_{i=1}^{m} \int_{C_i} f(\underline{x})\, ds$$

<u>Example</u> Circle of radius $r > 0$

$$\underline{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix} \quad \text{for } t \in [0, 2\pi]$$

$$\underline{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix} \quad \text{"}$$

$$\ell(\text{circle}) = \int_0^{2\pi} \left[r^2 \sin^2 t + r^2 \cos^2 t \right]^{1/2} dt = 2\pi r$$

Also : $\displaystyle \int_C x^2 y \, ds = \int_0^{2\pi} (r\cos t)^2 r \sin t \cdot r \, dt = 0$

Is $\ell(C)$ dependent on the parametrisation?

E.g. circle: $\underline{x} = \underline{x}_1(t) = (R\cos t, R\sin t, 0) \quad t \in [0, 2\pi]$

$\underline{x} = \underline{x}_2(\tau) = (R\cos 2t + R\sin 2\tau, 0) \quad t \in [0, \pi]$

Suppose C has two different parametrisations

$\underline{x} = \underline{x}_1(t), \quad t \in [a, b]$

$\underline{x} = \underline{x}_2(t), \quad \tau \in [\alpha, \beta]$

Must have a function $t = t(\tau)$ such that $\underline{x}_2(\tau) = \underline{x}_1(t(\tau))$.

Assume $dt/d\tau \neq 0$ so relationship invertible. Note

$$\frac{d}{d\tau} \underline{x}_2(\tau) = \frac{dt}{d\tau} \underline{x}_1'(t(\tau))$$

By definition, $\ell(C) = \int_c^b |\underline{x}_1'(t)| \, dt$

Make sub $t = t(\tau)$, so that (assuming $dt/d\tau > 0$)

$$\ell(C) = \int_\alpha^\beta |\underline{x}_1'(t(\tau))| \frac{dt}{d\tau} d\tau$$

$$= \int_\alpha^\beta |\underline{x}_2'(\tau)| \, d\tau \quad \text{, so it's consistent!}$$

If $\frac{dt}{d\tau} < 0$, still works (flip α & β)

For parametrised curve C, $[a, b] \ni t \to \underline{x}(t)$, have

$\ell(C) = \int_C ds = \int_a^b |\underline{x}'(t)| \, dt$ independent of parametrisation.

L2

Define arc-length function

$$s(t) = \int_a^t |\underline{x}'(\tau)|\, d\tau$$

So $s(a) = 0$, $s(b) = \ell(C)$. Also

$$\frac{ds}{dt} = |\underline{x}'(t)| \geq 0.$$

If parametrisation regular, $ds/dt > 0$. So can invert to get $t = t(s)$.
By chain rule $\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\underline{x}'(t)|}$, and so

we define $\underline{r}(s) = \underline{x}(t(s))$. Then

$$\underline{r}'(s) = \frac{d}{ds}\underline{x}(t(s)) = \frac{dt}{ds}\underline{x}'(t(s)) = \frac{\underline{x}'(t(s))}{|\underline{x}'(t(s))|}$$

i.e. $|\underline{r}'(s)| = 1$. This is consistent with

$$\ell(C) = \int_0^{\ell(C)} |\underline{r}'(s)|\, ds = \int_0^{\ell(C)} ds = \ell(C)$$

§1.2 Curvature + Torsion

Throughout this section, curve parametrised by arc-length $s \to \underline{r}(s)$.
Define tangent vector $\underline{t}(s) = \underline{r}'(s)$.

Note $|\underline{t}| = 1$. Consider $\underline{t}'(s) = \underline{r}''(s)$. Since $\underline{t}$ only changes in direction, $|\underline{t}'(s)|$ gives idea of how fast curve changes direction.

Define curvature $\kappa(s) = |\underline{t}'(s)|$.

Also define normal to curve, $\underline{n}$, $\underline{t}' = \kappa \underline{n}$ (so $|\underline{n}| = 1$)

Since $|\underline{t}|^2 = 1$, $0 = (\underline{t}\cdot\underline{t})' = 2\underline{t}\cdot\underline{t}' \Rightarrow \underline{n} \perp \underline{t}$.

Since $\{\underline{t}, \underline{n}\}$ orthonormal, define $\underbrace{\underline{b}}_{\text{binormal}} = \underline{t} \times \underline{n}$ (so $|\underline{b}| = 1$).

So $\{\underline{t}, \underline{n}, \underline{b}\}$ orthonormal basis for $\mathbb{R}^3$.

Note that $0 = \underline{b}\cdot\underline{b}'$. Also $\underline{t}\cdot\underline{n} = \underline{b}\cdot\underline{n} = 0$. (As is $\underline{t}\cdot\underline{b}$.)

$$0 = (\underline{t}\cdot\underline{b})' = \underline{t}'\cdot\underline{b} + \underline{t}\cdot\underline{b}' = \kappa\underbrace{\underline{n}\cdot\underline{b}}_{\text{zero}} + \underline{t}\cdot\underline{b}'$$

So $\underline{b}' \perp \underline{t}$, $\underline{b}' \perp \underline{b}$. So $\underline{b}' \parallel \underline{n}$.

Define torsion: $\underline{b}'(s) = -\tau(s)\underline{n}(s)$. $\quad (\underline{b} = \underline{t} \times \underline{n})$

In summary:

$$\underline{t}' = \kappa \underline{n}, \quad \underline{b}' = -\tau \underline{n}$$

<u>Proposition</u> Curvature (κ) and torsion (τ) define smooth curves in $\mathbb{R}^3$ up to translation and orientation.

<u>Proof</u> Since $\underline{n} = \underline{b} \times \underline{t}$, have

$$\underline{t}' = \kappa(\underline{b} \times \underline{t})$$
$$\underline{b}' = -\tau(\underline{b} \times \underline{t}).$$

Six eq^{n}s, six unknowns. Solve for $\underline{b}, \underline{t}$ given $\{\underline{t}(0), \underline{b}(0), \kappa(s), \tau(s)\}$. (See Analysis 2)

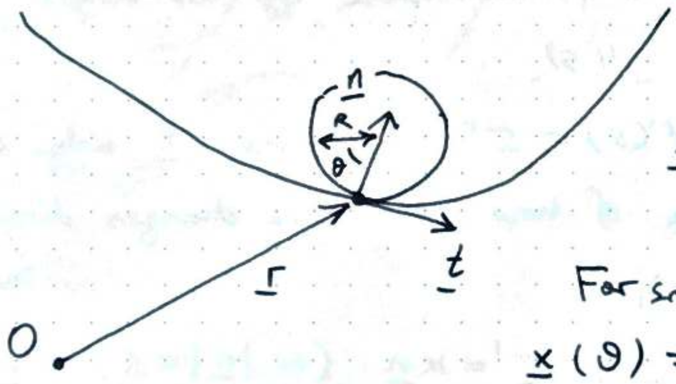
§1.3 Radius of Curvature

Small s Taylor expansion of generic curve $s \to \underline{r}(s)$.

$$\underline{r}(s) = \underline{r}(0) + s\underline{r}'(0) + \tfrac{1}{2}s^2\underline{r}''(0) + \underline{o}(s^2)$$

Set $\underline{r}(0) = \underline{r}$, $\underline{t}(0) = \underline{t}$ etc.

$$\Rightarrow \underline{r}(s) = \underline{r} + s\underline{t} + \tfrac{1}{2}s^2\kappa\underline{n} + \underline{o}(s^2) \qquad (\dagger)$$



Eq^n of circle

$$\underline{x}(\theta) = \underline{r} + R(1-\cos\theta)\underline{n} + R\sin\theta\,\underline{t}$$

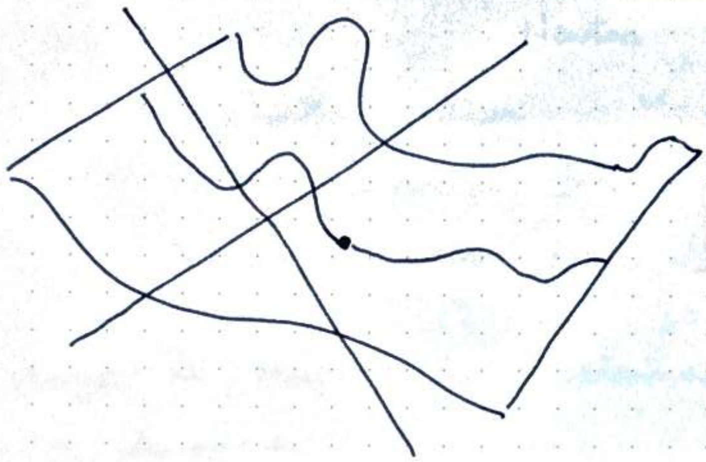
For small θ, we get

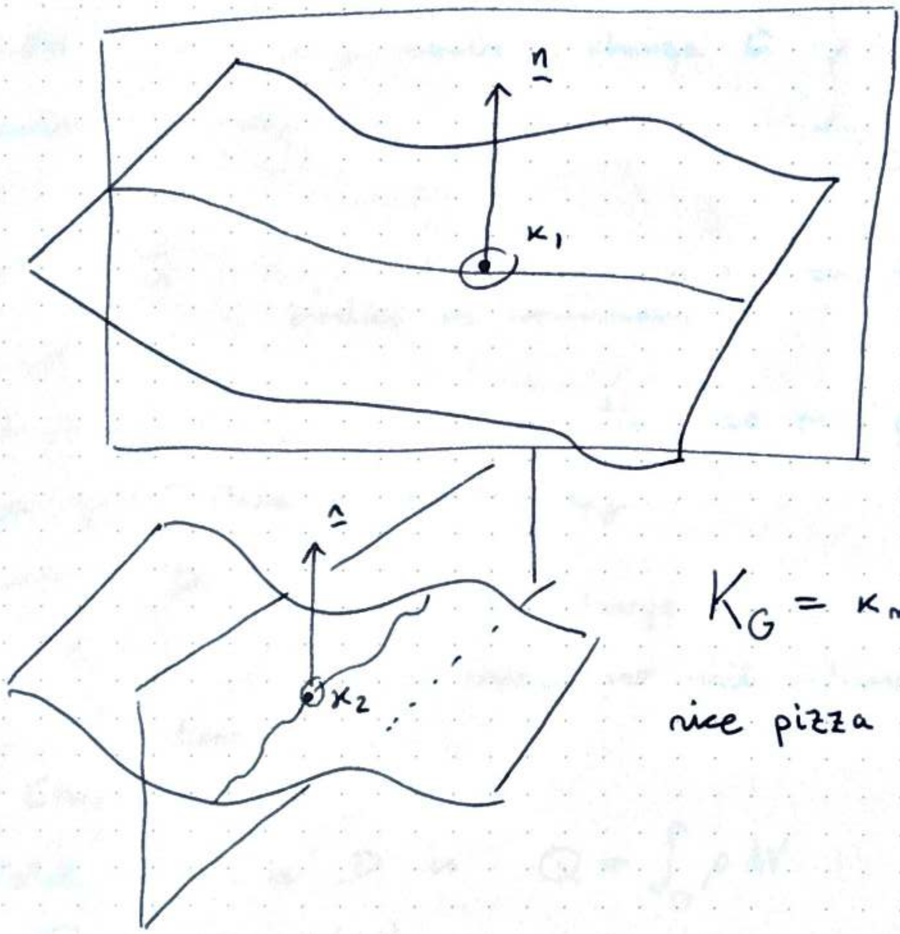
$$\underline{x}(\theta) = \underline{r} + R\theta\underline{t} + \tfrac{1}{2}R\theta^2\underline{n} + \underline{o}(\theta^2)$$

Using $s = R\theta$ on circle, $\underline{x}(\theta) = \underline{r} + s\underline{t} + \frac{1}{2}\frac{1}{R}s^2\underline{n} + \underline{o}(s^2)$

Compare to $(\dagger)$. See that $R = \frac{1}{\kappa}$ for best fit. More generally, define <u>radius of curvature</u> $R(s) = \dfrac{1}{\kappa(s)}$.

Surfaces





plane containing $\underline{n}$

$$K_G = \kappa_{min}\,\kappa_{max}$$

nice pizza

L3.1

§2 Coordinates, differentials + gradients

§2.1 Differentials and first-order changes

Recall if $f = f(u_1, u_2, \dots, u_n)$ then

$$df = \frac{\partial f}{\partial u_i} du_i \qquad \text{(summation convention)}$$

Then $\{du_i\}$ are called differential forms, defined to be linearly independent if $\{u_i\}$ are. $[\alpha_i du_i = 0 \Rightarrow \alpha_i = 0 \ \forall i]$ Similarly, if $\underline{x} = \underline{x}(u_1, \dots, u_n)$ then

$$d\underline{x} = \frac{\partial \underline{x}}{\partial u_i} du_i$$

By ordinary calculus

$$f(u_1 + \delta u_1, \dots, u_n + \delta u_n) - f(u_1, \dots, u_n)$$
$$= \frac{\partial f}{\partial u_i} \cancel{du_i}\, \delta u_i + o(\delta \underline{u})$$

where $\delta \underline{u} = (\delta u_1, \dots, \delta u_n)$ and $\dfrac{o(\delta \underline{u})}{|\delta \underline{u}|} \to 0$ as $|\delta \underline{u}| \to 0$.

i.e. up to first order

$$\delta f \simeq \frac{\partial f}{\partial u_i} \delta u_i$$

similarly if $\underline{x} = \underline{x}(u_1, \dots, u_n)$

$$\delta \underline{x} \simeq \frac{\partial \underline{x}}{\partial u_i} \delta u_i$$

<u>E.g.</u> $f = f(u, v, w) = u^2 + w \sin v$

$$\Rightarrow df = 2u\, du + w \cos v\, dv + \sin v\, dw$$

$$\underline{x} = \underline{x}(u, v, w) = \begin{pmatrix} u^2 - v^2 \\ e^v \\ w \end{pmatrix}$$

$$d\underline{x} = \begin{pmatrix} 2u \\ 0 \\ 0 \end{pmatrix} du + \begin{pmatrix} -2v \\ e^v \\ 0 \end{pmatrix} dv + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dw$$

$[\, F(u_1, \dots, u_n) = f(x_1, \dots, x_n), \quad x_i = x_i(u_1, \dots, u_n)$

$$\frac{\partial F}{\partial u_i} du_i = dF = df = \frac{\partial f}{\partial x_j} dx_j = \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial u_i} du_i$$

$$\Rightarrow \frac{\partial F}{\partial u_i} = \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial u_i} \quad]$$

L3.2

§2.2 Coordinates and line elements

Recall polar coords. (r, θ) defined by relationship to Cartesian coords (x, y)

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$0 \leq r < \infty, \quad 0 \leq \theta < 2\pi$$

These relations are invertible (almost) everywhere.

In general (u,v) coords for $\mathbb{R}^2$ are defined by their relationship to (x,y): $\quad x = x(u,v), \quad y = y(u,v)$

so that $u = u(x,y), \quad v = v(x,y)$ are smooth

[i.e. relationship is invertible]

Similarly (u,v,w) coords for $\mathbb{R}^3$ defined by

$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w)$ which can be inverted to give (u,v,w) in terms of smooth functions of (x,y,z).

For (x,y) Cartesian coords

$$\underline{x} = \underline{x}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\,\underline{e}_x + y\,\underline{e}_y$$

Note $\underline{e}_x$ = "direction of change of $\underline{x}$ when x changes"

$$= \frac{\partial}{\partial x}\underline{x}(x,y) \Big/ \left|\frac{\partial}{\partial x}\underline{x}(x,y)\right| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly $\underline{e}_y$ = "... y changes"

$$= \frac{\partial}{\partial y}\underline{x}(x,y) \Big/ \left|\frac{\partial}{\partial y}\underline{x}(x,y)\right| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Look at $d\underline{x}$ (line element)

$$d\underline{x} = \frac{\partial \underline{x}}{\partial x}dx + \frac{\partial \underline{x}}{\partial y}dy = \underline{e}_x\,dx + \underline{e}_y\,dy$$

i.e. if $x \to x + \delta x$ then to first order

$$\underline{x} \longrightarrow \underline{x} + \delta x\,\underline{e}_x$$

L3.3 This is <u>SPECIAL</u>.

For Polar coords (r, θ),

$$\underline{x}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Let us <u>define</u> $\underline{e_r} = \frac{\partial \underline{x}}{\partial r} \Big/ \left|\frac{\partial \underline{x}}{\partial r}\right|$ and $\underline{e_\theta} = \frac{\partial \underline{x}}{\partial \theta} \Big/ \left|\frac{\partial \underline{x}}{\partial \theta}\right|$.

Then
$$\underline{e_r} = \frac{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

and
$$\underline{e_\theta} = \frac{\begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Again see that $\{\underline{e_r}, \underline{e_\theta}\}$ are orthonormal like $\{\underline{e_x}, \underline{e_y}\}$.

$$\underline{dx} = \frac{\partial \underline{x}}{\partial r} dr + \frac{\partial \underline{x}}{\partial \theta} d\theta = \underline{e_r}\, dr + r\underline{e_\theta}\, d\theta$$

i.e. If $\theta \to \theta + \delta\theta$, $\underline{x} \longrightarrow \underline{x} + r\delta\theta\, \underline{e_\theta}$

§ 2.2.1 Orthogonal curvilinear coordinates

(u, v, w) are orthogonal curvilinear coords for $\mathbb{R}^3$ if

$$\underline{e_u} = \frac{\partial \underline{x}/\partial u}{|\partial \underline{x}/\partial u|}, \quad \underline{e_v} = \cdots, \quad \underline{e_w} = \cdots$$

are orthonormal, right-handed basis vecs for each (u, v, w).

Call $h_u = |\partial \underline{x}/\partial u|$, $h_v = |\partial \underline{x}/\partial v|$, $h_w = |\partial \underline{x}/\partial w|$

scale factors. Note

$$\begin{aligned} d\underline{x} &= \frac{\partial \underline{x}}{\partial u} du + \frac{\partial \underline{x}}{\partial v} dv + \frac{\partial \underline{x}}{\partial w} dw \\ &= h_u\, du\, \underline{e_u} + h_v\, dv\, \underline{e_v} + h_w\, dw\, \underline{e_w} \end{aligned}$$

i.e. small change $u \to u + \delta u$ results in $\underline{x} + h_u \delta u\, \underline{e_u} \leftarrow \underline{x}$

§ 2.2.2 Cylindrical Polars

(ρ, ϕ, z) defined via

$$\underline{x} = \underline{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} \quad \text{where} \quad \begin{aligned} 0 \le \rho &< \infty, \\ 0 \le \phi &< 2\pi, \\ -\infty < z &< \infty. \end{aligned}$$

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$$\underline{e}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad \underline{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad \underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Scale factors 1, ρ, 1 to give

$$d\underline{x} = \underline{e}_\rho \, d\rho + \underline{e}_\phi \, \rho \, d\phi + \underline{e}_z \, dz$$

§ 2.2.3 Spherical Polars

(r, θ, φ) defined via

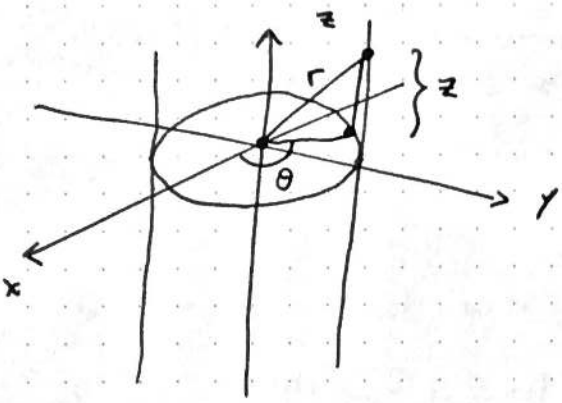
$$\underline{x} = \underline{x}(r, \theta, \varphi) = \begin{pmatrix} r\cos\varphi \sin\theta \\ r\sin\varphi \sin\theta \\ r\cos\theta \end{pmatrix}$$

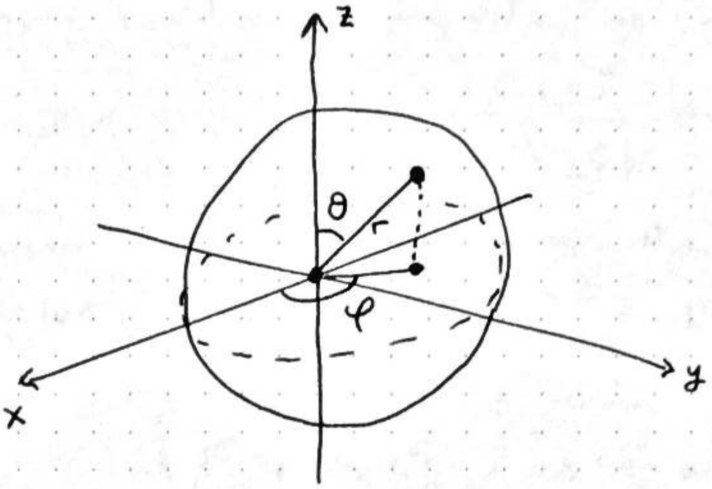
where $0 \leq r < \cancel{0}\infty$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$ and

$$\underline{e}_r = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} \quad \underline{e}_\theta = \begin{pmatrix} \cos\varphi \cos\theta \\ \sin\varphi \cos\theta \\ -\sin\theta \end{pmatrix} \quad \underline{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

with scale factors $h_r, h_\theta, h_\varphi = 1, r, r\sin\theta$ so that

$$d\underline{x} = \underline{e}_r \, dr + r \, \underline{e}_\theta \, d\theta + r\sin\theta \, \underline{e}_\varphi \, d\varphi.$$





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L4.1

§2.3: Gradient operator

For a function $f: \mathbb{R}^3 \to \mathbb{R}$ define gradient, denoted ∇f, by

$$f(\underline{x}+\underline{h}) = f(\underline{x}) + \underline{h}\cdot\nabla f(\underline{x}) + o(\underline{h}) \qquad (\dagger)$$

Define directional derivative in direction $\underline{v}$

$$D_{\underline{v}} f(\underline{x}) = \lim_{t\to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t}$$

or equivalently $f(\underline{x}+t\underline{v}) = f(\underline{x}) + t\, D_{\underline{v}} f(\underline{x}) + o(t) \qquad (\dagger\dagger)$

Setting $\underline{h} = t\underline{v}$ in $(\dagger)$, compare to $(\dagger\dagger)$

$$\Rightarrow D_{\underline{v}} f(\underline{x}) = \underline{v}\cdot\nabla f(\underline{x})$$

Recall from Cauchy-Schwarz, $\underline{a}\cdot\underline{b}$ maximised when $\underline{a}$ points in the same direction as $\underline{b}$.

$$\Rightarrow \begin{cases} \nabla f \text{ points in direction of greatest increase of } f \\ f \text{ does not change in direction } \perp \text{ to } \nabla f \end{cases}$$

[$-\nabla f$ points in direction of greatest decrease of f]

Example $f(\underline{x}) = \frac{1}{2}|\underline{x}|^2$

$$f(\underline{x}+\underline{h}) = \tfrac{1}{2}(\underline{x}+\underline{h})\cdot(\underline{x}+\underline{h}) = \tfrac{1}{2}|\underline{x}|^2 + \underline{h}\cdot\underline{x} + \tfrac{1}{2}|\underline{h}|^2$$
$$= f(\underline{x}) + \underline{h}\cdot\underline{x} + o(\underline{h})$$

$$\Rightarrow \nabla f(\underline{x}) = \underline{x}$$

For curve $t \to \underline{x}(t)$, consider

$$F(t) = f(\underline{x}(t))$$

Set $\delta\underline{x} = \underline{x}(t+\delta t) - \underline{x}(t)$.

$$F(t+\delta t) = f(\underline{x}(t) + \delta\underline{x})$$
$$= f(\underline{x}(t)) + \delta\underline{x}\cdot\nabla f(\underline{x}(t)) + o(\delta\underline{x})$$

Note $\delta\underline{x} = \underline{x}(t+\delta t) - \underline{x}(t) = \delta t\, \underline{x}'(t) + o(\delta t)$

$$\Rightarrow F(t+\delta t) = F(t) + \delta t\, \underline{x}'(t)\cdot\nabla f(\underline{x}(t)) + o(\delta t)$$

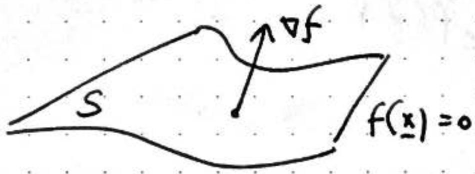
Hence $\dfrac{dF}{dt} = \dfrac{d}{dt} f(\underline{x}(t)) = \underline{x}'(t)\cdot\nabla f(\underline{x}(t))$

Finally, consider surface $S = \{ \underline{x} : f(\underline{x}) = 0 \}$.

For $t \to \underline{x}(t)$ a curve in S, $f(\underline{x}(t)) \equiv 0$.

$$\Rightarrow \quad 0 = \frac{d}{dt} f(\underline{x}(t)) = \underline{x}'(t) \cdot \nabla f(\underline{x}(t))$$

So ∇f orthogonal to tangent vector. Since curve arbitrary, ∇f points in direction of normal to surface.



§2.4: Computing the gradient

If we use occ (u, v, w), how to "wobble" in such a way to change $\underline{x}(u, v, w)$ to $\underline{x} + \underline{h}$?

Easy in Cartesians:

$$f(\underline{x} + \delta \underline{x}) = f(x + \delta x, y + \delta y, z + \delta z)$$
$$= f(x, y, z) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z} + o(\delta \underline{x})$$
$$= f(\underline{x}) + \delta \underline{x} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) + o(\delta \underline{x})$$

So IN CARTESIANS,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x_i} \underline{e_i} \quad (\Sigma \text{ convention})$$

i.e. $[\nabla f]_i = \dfrac{\partial f}{\partial x_i}$

Think of ∇ as vector differential operator.

In Cartesians $\quad \nabla = \underline{e_i} \, \partial / \partial x_i$.

<u>Example</u> Let $f = \frac{1}{2}(x^2 + y^2 + z^2) = \frac{1}{2} |\underline{x}|^2$.

$$[\nabla f]_i = \partial / \partial x_i \left(\tfrac{1}{2} x_j x_j \right) = x_j \delta_{ij} = x_i$$

$$\Rightarrow \quad \nabla f = x_i \underline{e_i} = \underline{x} \quad \text{as before}$$

L4.3 Recall in Cartesians

$$d\underline{x} = dx\,\underline{e_x} + dy\,\underline{e_y} + dz\,\underline{e_z} \quad \text{i.e. } \{h_x, h_y, h_z\} = \{1\}$$
$$= dx_i\,\underline{e_i}$$

Also, if $f(\underline{x}) = f(x_1, x_2, x_3)$,

$$df = \frac{\partial f}{\partial x_i}\,dx_i$$

So
$$\nabla f \cdot d\underline{x} = \underline{e_i}\frac{\partial f}{\partial x_i} \cdot \underline{e_j}\,dx_j = (\underline{e_i}\cdot\underline{e_j})\frac{\partial f}{\partial x_i}\,dx_j$$
$$= \delta_{ij}\frac{\partial f}{\partial x_i}\,dx_j = \frac{\partial f}{\partial x_i}\,dx_i = df$$

Note both sides have coord. free expression, so get

$$\boxed{df = \nabla f \cdot d\underline{x}}$$

<u>Proposition</u> For (u, v, w) an orthogonal curvilinear coord system and $f = f(u,v,w)$, then

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\underline{e_u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\underline{e_v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\underline{e_w}$$

where $h_u = |\partial \underline{x}/\partial u|$, ...

<u>Proof</u> Use $df = \nabla f \cdot d\underline{x}$. Set $\nabla f = [\nabla f]_u\,\underline{e_u} + [\nabla f]_v\,\underline{e_v} + [\nabla f]_w\,\underline{e_w}$.

$$\left(\frac{\partial f}{\partial u}du + \cdots + \frac{\partial f}{\partial w}dw\right) = \left([\nabla f]_u\underline{e_u} + \cdots + [\nabla f]_w\underline{e_w}\right)\cdot\left(h_u\,du\,\underline{e_u} + \cdots + h_w\,dw\,\underline{e_w}\right)$$
$$= h_u[\nabla f]_u\,du + \cdots + h_w[\nabla f]_w\,dw$$

By linear independence of du, dv, dw, get

$$[\nabla f]_u = \frac{1}{h_u}\frac{\partial f}{\partial u}, \ldots, [\nabla f]_w = \frac{1}{h_w}\frac{\partial f}{\partial w}. \qquad \square$$

Cylindrical coords:

$$(\rho, \phi, z), \quad \underline{x} = (\rho\cos\phi, \rho\sin\phi, z) = \rho\underline{e_\rho} + z\underline{e_z}$$

$$\nabla f = \frac{\partial f}{\partial \rho}\underline{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\underline{e_\phi} + \frac{\partial f}{\partial z}\underline{e_z}$$

L4.4 Spherical:

(r, ϑ, φ), $\underline{x} = (r\cos\phi \sin\theta, r\sin\phi\sin\theta, r\cos\theta) = r\underline{e}_r$

$$\nabla f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\vartheta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\underline{e}_\varphi$$

Example

$$f = \begin{cases} \frac{1}{2}(x^2+y^2+z^2) & \text{Cartesian} \\ \frac{1}{2}(\rho^2+z^2) & \text{Cylindrical} \\ \frac{1}{2}(r^2) & \text{Spherical} \end{cases} \qquad \left[f(\underline{x}) = \tfrac{1}{2}|\underline{x}|^2 \right]$$

$$\nabla f = \begin{cases} x\underline{e}_x + y\underline{e}_y + z\underline{e}_z \\ \rho\underline{e}_\rho + z\underline{e}_z \\ r\underline{e}_r \end{cases} \qquad \text{so it works}$$

$f(\underline{x}) = f(\underline{x}(u,v,w)) = F(u,v,w)$ ← don't bother distinguishing position ⟷ coordinates

so in reality $\nabla f = \frac{1}{h_u}\frac{\partial F}{\partial u}\underline{e}_u + \cdots$

$f(\underline{x}) = G(r, \vartheta, \phi)$

$f(\underline{x}) = H(r, \phi, z)$

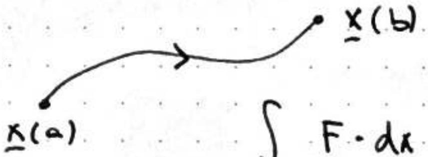
LS.1

§3: Integration over lines, surfaces, and volumes

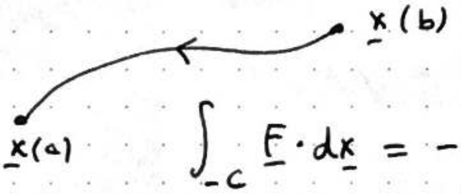
§3.1: Line integrals

- For $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ and curve $C: [a,b] \ni t \longrightarrow \underline{x}(t)$, define

$$\int_C \underline{F} \cdot d\underline{x} = \int_a^b \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t)\, dt$$

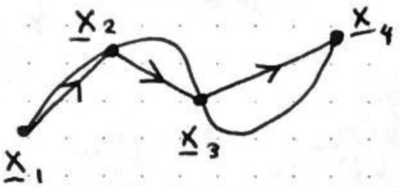


$$\int_C \underline{F} \cdot d\underline{x}$$



$$\int_{-C} \underline{F} \cdot d\underline{x} = -\int_C \underline{F} \cdot d\underline{x}$$

Can interpret as work done on particle moving along C in presence of a force $\underline{F}(\underline{x})$.

- 

$$\lim_{|\Delta \underline{x}| \to 0} \sum_i \underline{F}(\underline{x}_i) \cdot \Delta \underline{x}_i = \int_C \underline{F} \cdot d\underline{x}$$

$$\Delta \underline{x}_i = \underline{x}_{i+1} - \underline{x}_i, \quad |\Delta \underline{x}| = \max_i |\Delta \underline{x}_i|$$

Example

$$\underline{F}(\underline{x}) = \begin{pmatrix} x^2 y \\ yz \\ 2zx \end{pmatrix} \qquad C_1: \underline{x}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix} \quad t \in [0,1]$$

$$C_2: \underline{x}(t) = \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix} \quad t \in [0,1]$$

- $$\int_{C_1} \underline{F} \cdot d\underline{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^2 \\ 2t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt = \frac{1}{4} + \frac{1}{3} + \frac{2}{3} = \frac{5}{4}$$

$$\int_{C_2} \underline{F} \cdot d\underline{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^3 \\ 2t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2t \end{pmatrix} dt = \frac{1}{4} + \frac{1}{4} + \frac{4}{5} = \frac{13}{10}$$

Note they are different.

In general line integral depends on path taken between end points.

- If $\underline{x}(a) = \underline{x}(b)$, say C is closed. Write

$$\oint_C \underline{F} \cdot d\underline{x}$$

Called circulation about C by $\underline{F}$.

<u>Example</u>

$$C: \underline{x}(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ t \end{pmatrix}, \quad t \in [0, 2\pi]$$

$\underline{F} = \rho z \underline{e}_\phi$ Cylindrical polars (ρ, ϕ, z)

Recall in cylindrical polars

$$d\underline{x} = d\rho\, \underline{e}_\rho + \rho\, d\phi\, \underline{e}_\phi + dz\, \underline{e}_z$$

So $\underline{F} \cdot d\underline{x} = \rho^2 z\, d\phi$. On C

$$(\rho, \phi, z) = (a, t, t).$$

$$\left[\underline{x} = \underline{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} \right]$$

$$(d\rho, d\phi, dz) = (0, dt, dt)$$

Hence

$$\int_C \underline{F} \cdot d\underline{x} = \int_0^{2\pi} a^2 t\, dt = 2\pi^2 a^2$$

§3.2 : Conservative forces + Exact differentials

Call objects like $\underline{F} \cdot d\underline{x}$ differential forms.

Say $\underline{F} \cdot d\underline{x}$ is exact if $\underline{F} \cdot d\underline{x} = df$ for some scalar function f.

Recall that

$$df = \nabla f \cdot d\underline{x}$$

So $\underline{F} \cdot d\underline{x}$ is exact iff $\underline{F} = \nabla f$ for some f. Vector fields such as this are called conservative i.e.

$\underline{F} \cdot d\underline{x}$ is exact <=> $\underline{F}$ being conservative

<u>Proposition</u> If $\underline{F} \cdot d\underline{x}$ is exact then

$$\oint_C \underline{F} \cdot d\underline{x} = 0$$

for any closed curve C.

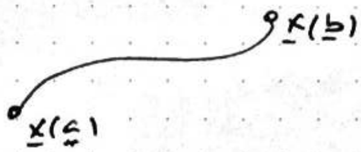
<u>Proof</u> If $\underline{F} = \nabla f$, $C : [a, b] \ni t \to \underline{x}(t)$

$$\oint_C \underline{F} \cdot d\underline{x} = \int_C \nabla f \cdot d\underline{x} = \int_a^b \nabla f(\underline{x}(t)) \cdot \underline{x}'(t)\, dt = \int_a^b \frac{d}{dt}\left(f(\underline{x}(t)) \right) dt$$

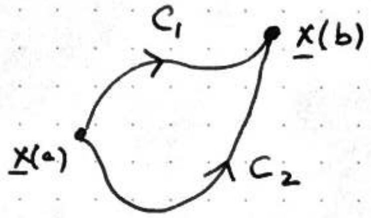
$$= f(\underline{x}(b)) - f(\underline{x}(a)) = 0. \qquad \square$$

L5.3 If $\underline{F}\cdot d\underline{x}$ is exact

$$\int_C \underline{F}\cdot d\underline{x}$$



doesn't depend on the path from $\underline{x}(a)$ to $\underline{x}(b)$.



If $C = C_1 - C_2$, then

$$0 = \int_C \underline{F}\cdot d\underline{x} = \int_{C_1} \underline{F}\cdot d\underline{x} - \int_{C_2} \underline{F}\cdot d\underline{x}.$$

Consider differential in coordinates (u_1, u_2, u_3).

$$\underline{F}\cdot d\underline{x} = \theta_1(u_1,u_2,u_3)\,du_1 + \cdots + \theta_3(u_1,u_2,u_3)\,du_3$$
$$= \theta_i\,du_i$$

If $\underline{F}\cdot d\underline{x}$ exact, must have

$$\theta_i = \frac{\partial f}{\partial u_i} \text{ for some } f.$$

Hence $\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial^2 f}{\partial u_j \partial u_i} = \dfrac{\partial \theta_j}{\partial u_i}$ (by symmetry of mixed partials)

Say $\underline{F}\cdot d\underline{x} = \theta_i\,du_i$ is <u>closed</u> if $\dfrac{\partial\theta_i}{\partial u_j} = \dfrac{\partial\theta_j}{\partial u_i}$.

So $\underline{F}\cdot d\underline{x}$ exact $\Rightarrow$ $\underline{F}\cdot d\underline{x}$ closed

If domain of $\underline{F}\cdot d\underline{x}$ is simply connected, reverse implication true.

<u>Example</u>

(a) Is $\theta = y\,dx - x\,dy$ exact?

$\dfrac{\partial}{\partial x}(-x) \overset{?}{=} \dfrac{\partial}{\partial y}(y)$ No, so <u>not</u> exact

(b) Compute

$$\oint_C 3x^2y\,dx + x^3dy$$

$C: \underline{x}(t) = \begin{bmatrix} \cos\left[\mathrm{Re}\left[\zeta\left(\frac{1}{2}+it\right)\right]\right] \\ \sin\left[\mathrm{Im}\left[\zeta\left(\frac{1}{2}+it\right)\right]\right] \\ \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \end{bmatrix}$ where $t \in [t_1, t_2]$

$\zeta\left(\frac{1}{2}+it_2\right) = \zeta\left(\frac{1}{2}+it_1\right) = 0$

L5.4

Note $d(x^3 y) = 3x^2 y\,dx + x^3 dy \Rightarrow \oint_C \underline{F}\cdot d\underline{x} = 0$

<u>Example</u> Particle mass m, use $m\underline{\ddot{x}} = \underline{F}$,

$$\int_C \underline{F}\cdot d\underline{x} = m\int_a^b \underline{\ddot{x}}(t)\cdot\underline{\dot{x}}(t)\,dt$$

$$= \frac{1}{2}m\int_a^b \frac{d}{dt}\left(\underline{\dot{x}}(t)^2\right)dt$$

$$= \frac{1}{2}m\left(\underline{\dot{x}}(b)^2 - \underline{\dot{x}}(a)^2\right)$$

If $\underline{F} = -\nabla V$, conservative

$$\int_C \underline{F}\cdot d\underline{x} = -\int_C \nabla V\cdot d\underline{x} = -\left(V(b) - V(a)\right)$$

Compare two equations : $\frac{1}{2}m\dot{x}(a)^2 + V(\underline{x}(a)) = \frac{1}{2}m\dot{x}(b)^2 + V(\underline{x}(b))$

So energy conserved.

L6.1 ## §3.3 Integration in $\mathbb{R}^2$

Define integral over $D \subseteq \mathbb{R}^2$ as follows:

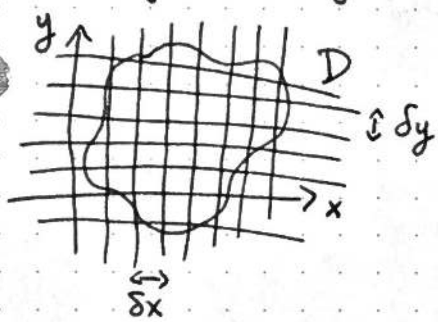
partition D into disjoint sets A_{ij}, with area δA_{ij}, each contained inside disc of radius $\varepsilon > 0$. Pick (x_i, y_j) inside each A_{ij} and set

$$\int_D f \, dA = \lim_{\varepsilon \to 0} \sum_{i,j} f(x_i, y_j)\, \delta A_{ij}.$$

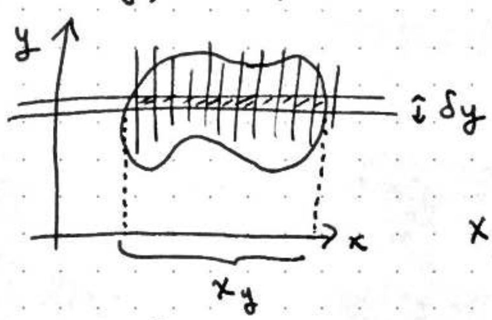
Integral exists if independent of choice of sets A_{ij} and (x_i, y_j).

Obvious choice: use rectangles, so

$$\delta A_{ij} = \delta x \, \delta y$$



Fix y, compute horizontal slice, $\delta x \to 0$



$$\delta y \int_{x_y} f(x,y)\, dx$$

$$x_y = \{x : (x,y) \in D\}$$

Add together horizontal slices, $\delta y \to 0$

$$\int_y \left(\int_{x_y} f(x,y)\, dx \right) dy.$$

Doing vertical slices

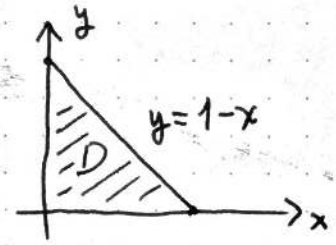
$$\int_x \left(\int_{y_x} f(x,y)\, dy \right) dx \qquad \text{where } y_x = \{y : (x,y) \in D\}.$$

It is a <u>theorem</u> that

$$\int_D f \, dA = \int_x \left(\int_{y_x} f(x,y)\, dy \right) dx = \int_y \left(\int_{x_y} f(x,y)\, dx \right) dy$$

called <u>Fubini's</u> theorem.

<u>Ex:</u>



$$f(x,y) = xy^2$$

$$\int_0^1 \int_0^{1-y} xy^2 \, dx\, dy \qquad \text{over horizontal slices}$$

as $x_y = [0, 1-y]$.

So is $\int_0^1 y^2 \cdot \frac{1}{2}(1-y)^2 \, dy = \frac{1}{60}$.

With vertical slices $\int_0^1 \int_0^{1-x} xy^2 \, dy \, dx = \int_0^1 x \cdot \frac{1}{3}(1-x)^3 \, dx = \frac{1}{60}$.

If $f(x,y) = g(x)h(y)$ and $D = [0,a] \times [0,b]$ then

$$\int_D f \, dA = \left(\int_0^a g(x) \, dx \right) \left(\int_0^b h(y) \, dy \right).$$

Recall if $x = x(u)$

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(x(u)) \frac{dx}{du} \, du \qquad \text{where } x(\alpha) = a, \; x(\beta) = b.$$

If $I = \overset{x}{\cancel{\;}}(I')$ where $I = [a,b]$, then

$$\int_I f(x) \, dx = \int_{I'} f(x(u)) \left| \frac{dx}{du} \right| du$$

Similar in 2D

<u>Proposition</u>

Let $x = x(u,v)$, $y = y(u,v)$ be a differentiable bijection with differentiable inverse mapping D' in the (u,v) plane to D in the (x,y) plane. (in a one-to-one fashion)

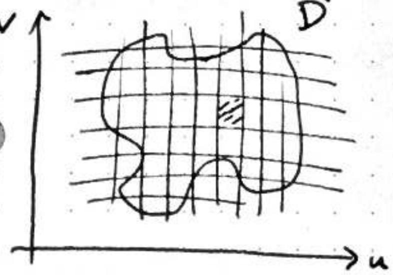
Then $\int_D f(x,y) \, dx dy = \int_{D'} f(x(u,v), y(u,v)) \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| du \, dv$ absolute value of det J.

where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = J$ is the Jacobian.

$$= \left(\frac{\partial \underline{x}}{\partial u} \;\middle|\; \frac{\partial \underline{x}}{\partial v} \right) \quad \text{where } \underline{x} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}.$$

In short, $dA = dx \, dy = \|J\| \, du \, dv$.

If $(x,y) \to (u,v)$ is truly smooth bijection with smooth inverse, $|J| \neq 0$. See Analysis II.



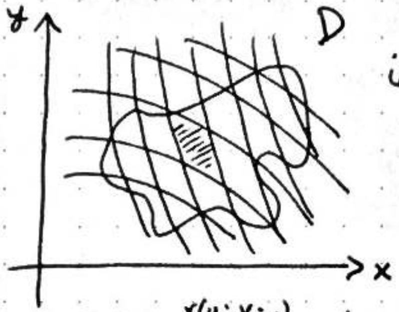
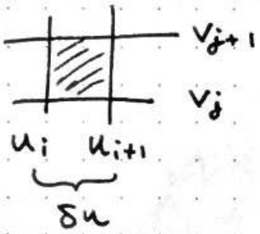
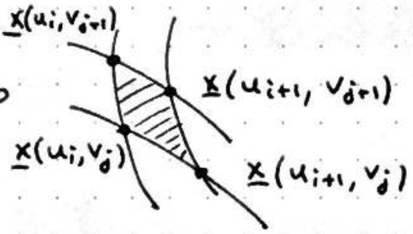
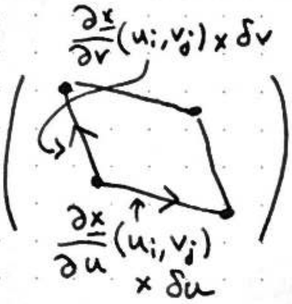


image of lines in u,v plane

Note 

goes to 

.

Area$(A_{ij}) \simeq$ area ()

as $\underline{x}(u_i + \delta u, v_j) - \underline{x}(u_i, v_j) \simeq \frac{\partial \underline{x}}{\partial u}(u_i, v_j)\, \delta u$, etc.

$$= \left| \frac{\partial \underline{x}}{\partial u}(u_i, v_j)\, \delta u \times \frac{\partial \underline{x}}{\partial v}(u_i, v_j)\, \delta v \right|$$

$$= \| J(u_i, v_j) \| \delta u\, \delta v.$$

To leading order $\delta A_{ij} = | J(u_i, v_j) |\, \delta u\, \delta v$.

$$\int_D f\, dA = \lim_{\substack{\delta u \to 0 \\ \delta v \to 0}} \sum_{i,j} f(\cancel{\,}\, x(u_i, v_j), y(u_i, v_j))\, \delta A_{ij}$$

$$= \int_{D'} f(x(u,v), y(u,v))\, |J|\, du\, dv$$

$$= \int_D f(x, y)\, dx\, dy$$

<u>Ex</u>: $x = \rho \cos\phi$, $y = \rho \sin\phi$

$$|J| = \left| \det \begin{pmatrix} \cos\phi & -\rho \sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix} \right| = |\rho| = \rho$$

$$\int_D f(x,y)\, dx\, dy = \int_{D'} f(\rho\cos\phi, \rho\sin\phi)\, \rho\, d\rho\, d\phi$$

L6.4 $\quad D = \{ x \geq 0, y \geq 0, x^2 + y^2 \leq R^2 \}$

$$D' = \{ 0 \leq \rho \leq R, \ 0 \leq \phi \leq \tfrac{\pi}{2} \}$$

Taking $R \to \infty$,

$$\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = \int_0^\infty \int_0^{\pi/2} f(\rho\cos\phi, \rho\sin\phi)\rho\, d\rho\, d\phi$$

$$= \int_0^\infty d\rho \int_0^{\pi/2} d\phi\, \rho f(\rho\cos\phi, \rho\sin\phi)$$

Consider $I = \int_0^\infty e^{-x^2} dx$.

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right)\left(\int_0^\infty e^{-y^2} dy \right)$$

$$= \int_0^\infty dx \int_0^\infty dy \quad e^{-x^2+y^2}$$

$$= \int_0^\infty d\rho \int_0^{\pi/2} d\phi \quad \rho e^{-\rho^2}$$

$$= \left[\tfrac{\pi}{2}\right]\left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^\infty = \frac{\pi}{4}.$$

Hence $I = \dfrac{\sqrt{\pi}}{2}$.

L7.1 $\S 3.4$: Integration in $\mathbb{R}^3$

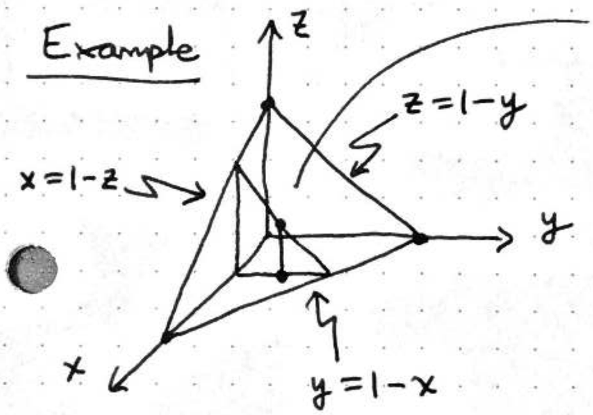
Define integral over volume $V \subset \mathbb{R}^3$, similar to $\S 3.3$

$$\int_V f\, dV = \lim_{\varepsilon \to 0} \sum_{i,j,k} f(x_i, y_j, z_k)\, \delta V_{ijk}$$

Each cell V_{ijk} of partition of V is contained inside ball of radius $\varepsilon > 0$, with $(x_i, y_j, z_k) \in V_{ijk}$.

Find $dV = dx\,dy\,dz$ in Cartesians. Can still perform in any order.

Example



$$\int_V dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1} \tfrac{1}{2}(1-x)^2\, dx = 1/6$$

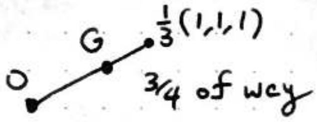
Compute CoM

$$\text{CoM} = \frac{1}{M} \int_V \rho \underline{x}\, dV = \frac{1}{V} \int_V \underline{x}\, dV \quad (\rho = 1). \text{ i.e. uniform}$$

By symmetry CoM $\propto (1,1,1)$.

$$\frac{1}{V} \int_V x\, dV = \frac{1}{V} \int_{x=0}^{1} \tfrac{1}{2} x (1-x)^2\, dx = \frac{\cancel{3\times}\ 1/24}{1/6} = 1/4$$

$\Rightarrow$ CoM $= (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.



Proposition Let $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$. Denote a differentiable bijection that maps the volume D' in the (u,v,w)-space to the volume D in the (x,y,z)-space (in a one-to-one manner).

$$\int_V f(x,y,z)\, dx\, dy\, dz = \int_{V'} f(x(u,v,w), \dots, z(u,v,w))\, |J|\, du\, dv\, dw$$

where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\left(\dfrac{\partial \underline{x}}{\partial u} \middle| \dfrac{\partial \underline{x}}{\partial v} \middle| \dfrac{\partial \underline{x}}{\partial w} \right)$.

L7.2

Jacobian term comes from fact that volume of parallelepiped generated by $\frac{\partial \underline{x}}{\partial u}du$, $\frac{\partial \underline{x}}{\partial v}dv$, $\frac{\partial \underline{x}}{\partial w}dw$ is

$$\left| \frac{\partial \underline{x}}{\partial u}du \cdot \frac{\partial \underline{x}}{\partial v}dv \times \frac{\partial \underline{x}}{\partial w}dw \right| = \left| \frac{\partial \underline{x}}{\partial u} \cdot \frac{\partial \underline{x}}{\partial v} \times \frac{\partial \underline{x}}{\partial w} \right| du\,dv\,dw$$

$$= \left| \det\left(\frac{\partial \underline{x}}{\partial u} \Big| \frac{\partial \underline{x}}{\partial v} \Big| \frac{\partial \underline{x}}{\partial w} \right) \right| du\,dv\,dw$$

<u>Example</u> Cylindrical polars (ρ, ϕ, z).

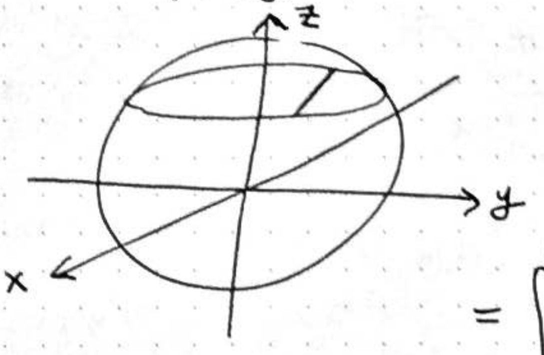
$|J| = \rho$, i.e. $dV = \rho\, d\rho\, d\phi\, dz$

Spherical polars (r, θ, ϕ).

$|J| = r^2 \sin\theta$ i.e. $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

<u>Example</u> Volume of ball radius $R > 0$

$V = \{(x,y,z) : 0 \le x^2+y^2+z^2 \le R^2\}$



$$\int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}} dx\,dy\,dz$$

$$= \int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} 2\sqrt{R^2-y^2-z^2}\, dy\,dz$$

$$= \int_{-R}^{R} \left[y\sqrt{R^2-y^2-z^2} + (R^2-z^2) \arctan\left(\frac{y}{\sqrt{R^2-y^2-z^2}} \right) \right]_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} dz$$

$$= \pi \int_{-R}^{R} (R^2-z^2)\, dz = \frac{4\pi}{3} R^3$$ ☺

<u>OR</u> in spherical polars $\int_0^R dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; r^2 \sin\theta$

$$= \left(\tfrac{1}{3}R^3\right)(2)(2\pi) = \frac{4\pi}{3} R^3$$ ☺

L7.3

Example

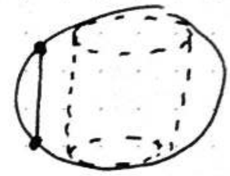
Compute volume of

$V = \{ (x,y,z) : x^2+y^2+z^2 \leq b^2 , \ x^2+y^2 \geq a^2 \}$ $(b > a)$

In cylindrical polars

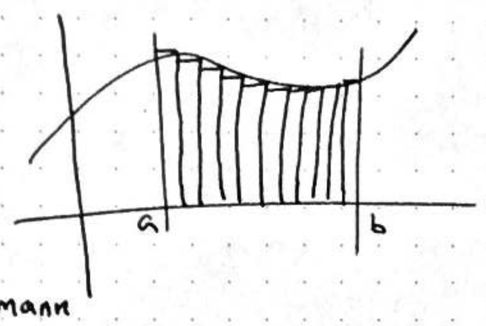
$V = \{ (\rho, \phi, z) : a \leq \rho \leq b , \ \rho^2 + z^2 \leq b^2 , \ 0 \leq \phi \leq 2\pi \}$

$$\int_V dV = \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \int_{z=-\sqrt{b^2-\rho^2}}^{\sqrt{b^2-\rho^2}} \rho \, d\rho \, d\phi \, dz$$



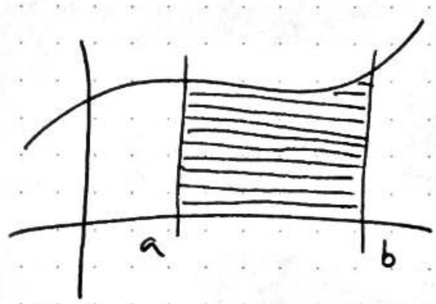
$$= 2\pi \int_a^b 2\sqrt{b^2-\rho^2} \, \rho \, d\rho = \frac{4\pi}{3} (b^2-a^2)^{3/2}$$

Ö



Riemann

$\int_a^b f \, dx$



Lebesgue

$$f(x) = \begin{cases} 1 : x \in \mathbb{Q} \\ 0 : x \notin \mathbb{Q} \end{cases} \qquad \int_0^1 f(x) \, dx = ?$$

$q_1, q_2, \ldots$ . . .

cover w/ $\varepsilon/2$

$\varepsilon/4$

$\varepsilon/8$

L8.1

§3.5 Integration on Surfaces

Can define surface implicitly using a

$$f: \mathbb{R}^3 \to \mathbb{R},$$

$$S = \{ \underline{x} : f(\underline{x}) = 0 \}.$$

Know that $\nabla f(\underline{x})$ is normal to S at $\underline{x}$. Say S is <u>regular</u> if $\nabla f(\underline{x}) \neq 0$ for $\underline{x} \in S$.

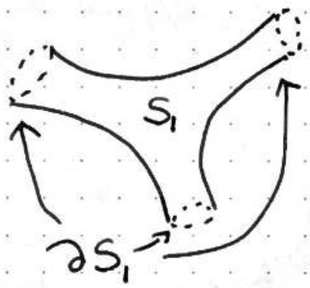
<u>Example</u> $f(\underline{x}) = |\underline{x}|^2 - 1$. Have, in Cartesians,

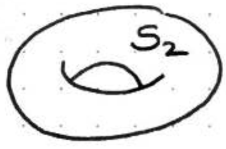
$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\underline{x} \neq 0 \text{ on } S.$$

Sometimes surfaces have boundaries,

e.g. hemisphere $S = \{ (x,y,z) : x^2+y^2+z^2 = 1, z \geq 0 \}$

$$\partial S = \{ (x,y,z) : x^2+y^2 = 1, z = 0 \}$$





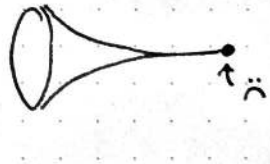
$\partial S_2 = \phi$

Two types of surfaces in this course: closed surfaces ($\partial S = \phi$), and surfaces whose boundary is a ~~st~~ collection of piece-wise smooth curves.

Useful to define surfaces using "local" coordinates (u,v).

$$S = \{ \underline{x} = \underline{x}(u,v) : (u,v) \in D \subseteq \mathbb{R}^2 \}.$$

Say S is <u>regular</u> if $\dfrac{\partial \underline{x}}{\partial u} \times \dfrac{\partial \underline{x}}{\partial v} \neq 0$.

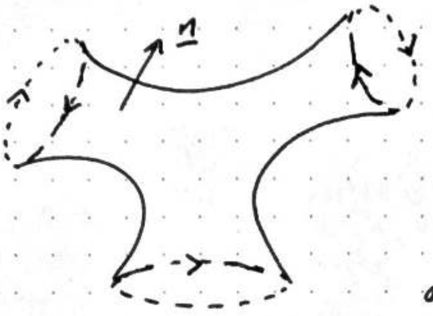


For regular S, define

$$\underline{n} = \frac{\partial \underline{x}/\partial u \times \partial \underline{x}/\partial v}{|\partial \underline{x}/\partial u \times \partial \underline{x}/\partial v|}.$$

L8.2

Using $\underline{n}$, can fix orientation of boundary ∂S. We traverse boundary of S in such a way that the normal is to your <u>left</u>.

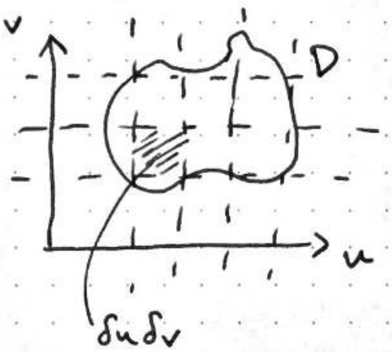


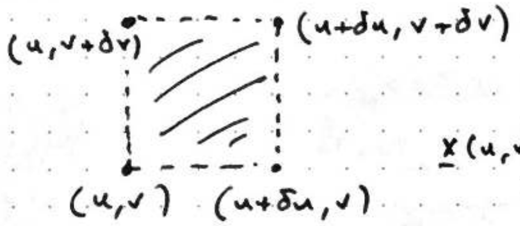
To compute area, might think

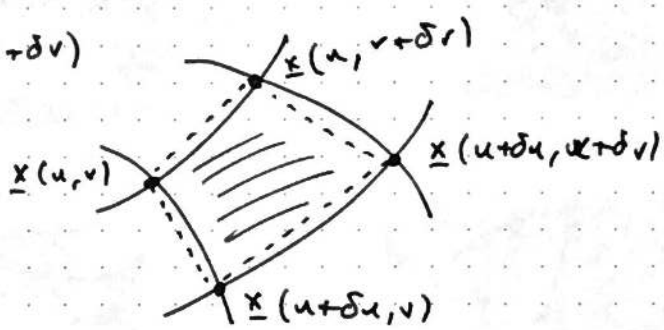
$$S = \{ \underline{x} = \underline{x}(u,v) : (u,v) \in D \}$$

$$\text{area}(S) \stackrel{?}{=} \iint_D du\,dv \quad \text{NO, WTF}$$

area of $\delta u \delta v$ in D does <u>not</u> map to area of same size on S.







To first order, area (⫽) on S is same as area of parallelogram generated by

$$\underline{x}(u+\delta u, v) - \underline{x}(u,v) = \frac{\partial \underline{x}}{\partial u}(u,v)\,\delta u + o(\delta u)$$

$$\underline{x}(u, v+\delta v) - \underline{x}(u,v) = \frac{\partial \underline{x}}{\partial v}(u,v)\,\delta v + o(\delta v)$$

which is $\left| \frac{\partial \underline{x}}{\partial u}\delta u \times \frac{\partial \underline{x}}{\partial v}\delta v \right| = \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| \delta u \delta v$.

Define <u>scalar</u> and <u>vector</u> area elements

$$dS = \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| du\,dv$$

$$d\underline{S} = \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\,dv = \underline{n}\,dS$$

Area of S is then

$$\text{area}(S) = \int_S dS = \iint_D \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| du\, dv$$

<u>Example</u>

Hemisphere radius R, using "spherical polars".

$$S = \left\{ \underline{x} = \begin{pmatrix} R\cos\phi \sin\theta \\ R\sin\phi \sin\theta \\ R\cos\theta \end{pmatrix} : 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le 2\pi \right\}$$

Have $\underline{x} = \underline{x}(\theta, \phi)$.

$$d\underline{S} = \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \phi} d\theta\, d\phi = (R\underline{e}_\theta) \times (R\sin\theta\, \underline{e}_\phi)\, d\theta\, d\phi$$
$$= R^2 \sin\theta\, \underline{e}_r\, d\theta\, d\phi.$$

$$\text{Find area} = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\; R^2 \sin\theta = 2\pi R^2 \int_0^{\pi/2} \sin\theta = 2\pi R^2.$$

Let $\underline{u} = \underline{u}(\underline{x})$ be velocity of fluid. Amount of fluid that crosses patch of area ~~dS~~ δS in time δt would be

$$(\underline{u} \cdot \underline{n}\; \delta S)\, \delta t = (\underline{u} \cdot \delta \underline{S})\, \delta t.$$

So total amount of fluid passing through S in time δt is

$\delta t \int_S \underline{u} \cdot d\underline{S}$, so $\int_S \underline{u} \cdot d\underline{S}$ measures rate at which fluid crosses surface. Called "flux integrals".

Suppose have two different parametrisations

$$S = \{ \underline{x} = \underline{x}(u, v),\; (u, v) \in D \}$$
$$= \{ \underline{x} = \tilde{\underline{x}}(\tilde{u}, \tilde{v}),\; (\tilde{u}, \tilde{v}) \in \tilde{D} \}.$$

Must have that $\underline{x}(u, v) = \tilde{\underline{x}}(\tilde{u}(u, v), \hat{v}(u, v))$ for some $\tilde{u}(u, v), \tilde{v}(u, v)$.

$$\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} = \left(\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial u} \right) \times \left(\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}} \frac{\partial \hat{v}}{\partial v} \right)$$
$$= \left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial u} \right) \left[\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}} \right]$$

L8.4 which is $\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} \frac{\partial\tilde{x}}{\partial\tilde{u}} \times \frac{\partial\tilde{x}}{\partial\tilde{v}}$.

$$\iint_{D'} \left|\frac{\partial\tilde{x}}{\partial\tilde{u}} \times \frac{\partial\tilde{x}}{\partial\tilde{v}}\right| d\tilde{u}\, d\tilde{v} = \iint_D \left|\frac{\partial\tilde{x}}{\partial\tilde{u}} \times \frac{\partial\tilde{x}}{\partial\tilde{v}}\right| \left|\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\right| du\, dv$$

$$= \iint_D \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right| du\, dv .$$

So $\int_S dS$ independent of parametrisation.

$$\int_I f' = f\Big|_{\partial I} = \int_{\partial I} f$$

$$\boxed{\int_M d\eta = \int_{\partial M} \eta}$$ Stoke's Theorem

True Chad

L9.1

§4 Divergence, Curl and Laplacians

§4.1 Definitions

Recall for $f: \mathbb{R}^3 \to \mathbb{R}$, have gradient in Cartesian coords

$$[\underline{\nabla} f]_i = \partial f / \partial x_i \;, \text{ i.e. } \quad \underline{\nabla} = \underline{e}_i \, \partial / \partial x_i$$

For vector field $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ define <u>divergence</u> by

$$\operatorname{div}(\underline{F}) := \underline{\nabla} \cdot \underline{F}$$

So in Cartesians

$$\underline{\nabla} \cdot \underline{F} = (\underline{e}_i \, \partial / \partial x_i) \cdot (F_j \underline{e}_j) = (\underline{e}_i \cdot \underline{e}_j) \, \partial F_j / \partial x_i$$

$$= \delta_{ij} \, \partial F_j / \partial x_i = \partial F_i / \partial x_i .$$

So in Cartesians

$$\boxed{\underline{\nabla} \cdot \underline{F} = \partial F_i / \partial x_i}$$

For $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ define <u>curl</u> by

$$\operatorname{curl}(\underline{F}) := \underline{\nabla} \times \underline{F}.$$

In Cartesians

$$\underline{\nabla} \times \underline{F} = (\underline{e}_j \, \partial / \partial x_j) \times (\underline{e}_k F_k) = (\underline{e}_j \times \underline{e}_k) \, \partial F_k / \partial x_j$$

$$= \varepsilon_{ijk} \, \underline{e}_i \, \partial F_k / \partial x_j \qquad [\, \underline{e}_j \times \underline{e}_k = \varepsilon_{ijk} \, \underline{e}_i \,]$$

i.e. $\boxed{[\underline{\nabla} \times \underline{F}]_i = \varepsilon_{ijk} \, \partial F_k / \partial x_j}$

or, in terms of formal determinant

$$\underline{\nabla} \times \underline{F} = \det \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

For scalar function $f: \mathbb{R}^3 \to \mathbb{R}$, define Laplacian of f

$$[\Delta f =] \nabla^2 f := \underline{\nabla} \cdot \underline{\nabla} f \quad (= \operatorname{div}(\underline{\nabla} f))$$

In Cartesians, using $[\underline{\nabla} f]_i = \partial f / \partial x_i$ and $\underline{\nabla} \cdot \underline{F} = \partial F_i / \partial x_i$, get

$$\boxed{\nabla^2 f = \partial^2 f / \partial x_i \partial x_i}$$

Call $\nabla^2 f = 0$ Laplace's equation and its solutions <u>harmonic functions</u>. :)

Example

$$\underline{F}(\underline{x}) = \underline{x}$$

$$\underline{\nabla}\cdot\underline{F} = \frac{\partial(\underline{x})_i}{\partial x_i} = \delta_{ii} = 3$$

$$[\underline{\nabla}\times\underline{F}]_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} = \varepsilon_{ijk}\delta_{jk} = 0 \qquad i\text{-}i$$

Proposition

For f, g scalar fields, $\underline{F}$, $\underline{G}$ vector fields

$$\underline{\nabla}(fg) = (\underline{\nabla}f)g + f(\underline{\nabla}g)$$

$$\underline{\nabla}\cdot(f\underline{F}) = (\underline{\nabla}f)\cdot\underline{F} + f(\underline{\nabla}\cdot\underline{F})$$

$$\underline{\nabla}\times(f\underline{F}) = (\underline{\nabla}f)\times\underline{F} + f(\underline{\nabla}\times\underline{F})$$

$$\underline{\nabla}(\underline{F}\cdot\underline{G}) = \underline{F}\cdot(\underline{\nabla}\times\underline{G}) + \underline{G}\cdot(\underline{\nabla}\times\underline{F}) + (\underline{F}\cdot\underline{\nabla})\underline{G} + (\underline{G}\cdot\underline{\nabla})\underline{F}$$

$$\underline{\nabla}\times(\underline{F}\times\underline{G}) = \underline{F}(\underline{\nabla}\cdot\underline{G}) - \underline{G}(\underline{\nabla}\cdot\underline{F}) + (\underline{G}\cdot\underline{\nabla})\underline{F} - (\underline{F}\cdot\underline{\nabla})\underline{G}$$

Proof Go to Cartesians, use suffix notation.

E.g. Last one $[\underline{\nabla}\times(\underline{F}\times\underline{G})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}[\underline{F}\times\underline{G}]_k$

$$= \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_j}F_l G_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(G_m\frac{\partial F_l}{\partial x_j} + F_l\frac{\partial G_m}{\partial x_j}\right)$$

$$= G_j\frac{\partial F_i}{\partial x_j} + F_i\frac{\partial G_m}{\partial x_m} - G_i\frac{\partial F_j}{\partial x_j} + F_j\frac{\partial G_i}{\partial x_j}$$

$$= (G_j\,\partial/\partial x_j)F_i + F_i\left(\frac{\partial G_j}{\partial x_j}\right) - G_i\left(\frac{\partial F_j}{\partial x_j}\right) + (F_j\,\partial/\partial x_j)G_i$$

$$= \left\{(\underline{G}\cdot\underline{\nabla})\underline{F} + \underline{F}(\underline{\nabla}\cdot\underline{G}) - \underline{G}(\underline{\nabla}\cdot\underline{F}) + (\underline{F}\cdot\underline{\nabla})\underline{G}\right\}_i$$

Others similar. $\square$

L9.3

In general coordinate systems (u, v, w)

$$\underline{\nabla} = \underline{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \underline{e}_v \frac{1}{h_v}\frac{\partial}{\partial v} + \underline{e}_w \frac{1}{h_w}\frac{\partial}{\partial w}$$

so could compute

$$\underline{\nabla} \cdot \underline{F} = \left(\underline{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \cdots \right) \cdot \left(F_u \underline{e}_u + \cdots \right).$$

However, $\underline{e}_u, \underline{e}_v, \underline{e}_w$ are <u>not</u>, in general, independent of coordinates u, v, w, i.e. $\left(\underline{e}_u \frac{1}{h_u}\frac{\partial}{\partial u}\right) \cdot \left(F_u \underline{e}_u\right) = \left(\underline{e}_u \frac{1}{h_u}\right) \cdot \left(\frac{\partial F_u}{\partial u}\underline{e}_u + F_u \frac{\partial \underline{e}_u}{\partial u}\right)$

This is long-winded $\therefore$ Give formulae

$$\underline{\nabla} \cdot \underline{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w) \right]$$

$$\underline{\nabla} \times \underline{F} = \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v) \right] \underline{e}_u + \text{cyc. perms}$$

Using $\nabla^2 f = \underline{\nabla} \cdot \underline{\nabla} f$ get

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \cdots \right]$$

SEE HANDOUT

Might try to define $\nabla^2 \underline{F}$ by $\underline{\nabla} \cdot \underline{\nabla}\, \underline{F}$ but have not defined $\underline{\nabla}\, \underline{F}$!

In Cartesians, since the $\underline{e}_i$ are constants, reasonable to define

$$(\nabla^2 \underline{F})_i = (\nabla^2 F_i).$$

Can show, using Cartesian coordinates and suffix notation that this gives $\boxed{\nabla^2 \underline{F} = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{F})}$ Dixon ☺

So use this as definition since coordinate independent.

§4.2 Relations between div, grad and curl

<u>Proposition</u> For scalar field f, vector field $\underline{F}$,

$$\underline{\nabla} \times \underline{\nabla} f = 0 \, , \quad \underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = 0$$

i.e. curl ∘ grad = 0, div ∘ curl = 0

<u>Proof</u> In Cartesians

$$[\underline{\nabla} \times \underline{\nabla} f]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_k} \right) = \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = 0 \quad \text{by symmetry}$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = \partial/\partial x_i \, \varepsilon_{ijk} \, \partial/\partial x_j (F_k) = \varepsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} = 0 \quad \text{"}$$

□

Recall that if $\underline{F} = \underline{\nabla} f$, called $\underline{F}$ <u>conservative</u>.

If $\underline{\nabla} \times \underline{F} = \underline{0}$, say $\underline{F}$ is <u>irrotational</u>. Hence

Conservative => Irrotational

Reverse implication true if domain of $\underline{F}$, Ω say, is simply connected (1-connected), i.e. any closed loop in Ω can be continuously shrunk to any point in Ω.

$\mathbb{R}^3$ <u>is</u> 1-connected, but $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is <u>not</u>.

If $\underline{F} = \underline{\nabla} \times \underline{A}$, say $\underline{A}$ is <u>vector potential</u> for $\underline{F}$.

If $\underline{\nabla} \cdot \underline{F} = 0$, say $\underline{F}$ is <u>solenoidal</u>. So:

If $\exists$ vector potential for $\underline{F}$ (i.e. $\underline{F} = \underline{\nabla} \times \underline{A}$)

=> then $\underline{F}$ is solenoidal.

Reverse implication is true if domain of $\underline{F}$, Ω say, is 2-connected, i.e. Ω is one-connected and any sphere in Ω (2 dim) can be continuously shrunk to any point in Ω.

$\mathbb{R}^3$ is 2-connected, but $\mathbb{R}^3 \setminus \{0\}$ is not.

See de Rahm Cohomology for more.

§4.3 Topology via Calculus

We know that

$$\left\{\begin{array}{l}\text{If } \underline{F}: \Omega \to \mathbb{R}^3 \text{ is} \\ \text{irrotational AND } \Omega \\ \text{is simply connected}\end{array}\right\} \implies \begin{array}{l}\underline{F} = \underline{\nabla} f \\ \text{i.e. } \underline{F} \text{ conservative}\end{array} \qquad (\dagger)$$

Is $\Omega = \mathbb{R}^3 \setminus \{z\text{-axis}\}$ simply connected?

Suppose it is. Consider

$$\underline{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad \text{good on } \Omega$$

Can show $\underline{\nabla} \times \underline{F} = \underline{0}$ on Ω. By $(\dagger)$, $\underline{F}$ is conservative, i.e. $\underline{F} = \underline{\nabla} f$. So for closed curve C in Ω

$$\oint_C \underline{F} \cdot d\underline{x} = \oint_C \underline{\nabla} f \cdot d\underline{x} = 0$$

Choose $C = \{x^2+y^2=1, z=0\}$

$= \{(\cos t, \sin t, 0) : t \in [0, 2\pi]\}$.

$$\oint_C \underline{F} \cdot d\underline{x} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = 2\pi$$

so Ω is not simply connected.

§5 Integral Theorems

All these are essentially generalisations of FTC $\int_I f' = \int_{\partial I} f$

§5.1 Green's Theorem: Statement & Examples

Proposition: If $P(x,y)$, $Q(x,y)$ are continuously differentiable on some $A \subset \mathbb{R}^2$, and ∂A is a collection of piece-wise smooth curves, then

$$\oint_{\partial A} P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Orientation: as traverse boundary, A on left.

L10.3

Same convention as with boundaries of surfaces in $\mathbb{R}^3$ if normal to A points <u>out</u> of paper.

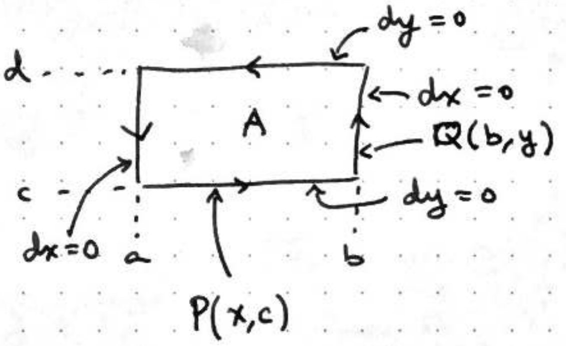
Note it works on $A = [a,b] \times [c,d]$:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_c^d dy \int_a^b dx \left(\frac{\partial Q}{\partial x} \right)$$

$$- \int_a^b dx \int_c^d dy \left(\frac{\partial P}{\partial y} \right)$$

$$= \int_c^d [Q(b,y) - Q(a,y)]\, dy$$

$$- \int_a^b [P(x,d) - P(x,c)]\, dx \qquad = \oint_{\partial A} P\,dx + Q\,dy$$



SO CHECKS OUT ☺

Compare to FTC.

<u>Example</u>

Take $\cancel{Q = \frac{1}{2}x^2}$ $\frac{\partial Q}{\partial x} = \frac{1}{2}$, $\frac{\partial P}{\partial y} = -\frac{1}{2}$

i.e. $Q = \frac{1}{2}x$, $P = -\frac{1}{2}y$. Then

$$\text{Area } A = \iint_A dx\, dy = \iint_A \left(\tfrac{1}{2} - (-\tfrac{1}{2}) \right) dx\, dy$$

$$= \int_{\partial A} \tfrac{1}{2} x\, dy - \tfrac{1}{2} y\, dx \qquad \circlearrowleft$$

Take ellipse $A = \{ x^2/a^2 + y^2/b^2 \leq 1 \}$.

Parametrise $\partial A : [0, 2\pi] \ni t \to \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}$

$$\frac{1}{2} \int_{\partial A} (x\,dy - y\,dx) = \frac{1}{2} \int_0^{2\pi} \big(a\cos t \cdot b\cos t - b\sin t(-a\sin t) \big) dt$$

$$= \pi ab \ ☺$$

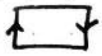
EPIC

§5.2: Stoke's Theorem: Statement + Examples

Proposition Let $\underline{F} = \underline{F}(\underline{x})$ be continuously diff'ble vector field, surface S be orientable, piece-wise regular with piece-wise smooth boundary ∂S. Then

$$\int_S \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \oint_{\partial S} \underline{F} \cdot d\underline{x} \, .$$

Here orientable means S has consistent choice of normal defined on it. Can think of orientable surfaces as those that have 2 sides.

E.g. Möbius strip NOT orientable 

Example $\underline{F} = \begin{pmatrix} -x^2 y \\ 0 \\ 0 \end{pmatrix}$, $\underline{\nabla} \times \underline{F} = \begin{pmatrix} 0 \\ 0 \\ x^2 \end{pmatrix}$

$$S = \left\{ \underline{x}(\theta, \varphi) = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} : 0 \leq \theta \leq \alpha , \ 0 \leq \varphi \leq 2\pi \right\}$$

$$d\underline{S} = \left(\frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \varphi} \right) d\theta \, d\varphi = \underline{e}_\theta \times (\sin\theta \, \underline{e}_\varphi) \, d\theta \, d\varphi$$

$$= \underline{e}_r \sin\theta \, d\theta \, d\varphi$$

$$\int_S \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \int_0^\alpha d\theta \int_0^{2\pi} d\varphi \cdot + (\sin\theta \cos\varphi)^2 \overset{\underline{e}_r \cdot \underline{e}_z}{\cos\theta} \sin\theta$$

$$= \int_0^\alpha d\theta \int_0^{2\pi} d\varphi \ \sin^3\theta \cos\theta \ \cos^2\varphi \, d\varphi$$

$$= \frac{1}{4} \sin^4 \alpha \cdot \pi$$ ☺

$$\partial S = \left\{ \underline{x}(t) = \begin{pmatrix} \sin\alpha \cos t \\ \sin\alpha \sin t \\ \cos\alpha \end{pmatrix} : 0 \leq t \leq 2\pi \right\}$$

$$d\underline{x} = \begin{pmatrix} -\sin\alpha \sin t \\ \sin\alpha \cos t \\ 0 \end{pmatrix} dt$$

$$\oint_{\partial S} \underline{F}\cdot d\underline{x} = \int_0^{2\pi} dt \cdot -(\sin\alpha\cos t)^2(\sin\alpha\sin t)\cdot(-\sin\alpha\sin t)$$

$$= \sin^4\alpha \int_0^{2\pi} \sin^2 t\cos^2 t\; dt = \frac{\pi}{4}\sin^4\alpha \qquad \ddot{\smile}$$

<u>Example</u>

If S is closed $(\partial S = \phi)$ then $\displaystyle\int_S \underline{\nabla}\times\underline{F}\cdot d\underline{x}^S = 0$

<u>Proposition</u> If $\underline{F} = \underline{F}(\underline{x})$ is a continuously diff'ble vector field and $\oint_C \underline{F}\cdot d\underline{x} = 0$ for all closed loops C, then $\underline{\nabla}\times\underline{F} = \underline{0}$.

<u>Proof</u> Assume $\underline{F}$ as above but $\underline{\nabla}\times\underline{F}(\underline{x}_0) \neq \underline{0}$ for some $\underline{x}_0$. So $\exists$ unit vector $\underline{k}$ such that

$$\underline{k}\cdot\underline{\nabla}\times\underline{F}(\underline{x}_0) > 0$$

By continuity, have same for $|\underline{x}-\underline{x}_0| < \delta$ for some $\delta > 0$.
i.e. $\underline{k}\cdot\underline{\nabla}\times\underline{F}(\underline{x}) \geq \varepsilon \overset{?\text{ yes}}{>} 0$ on $|\underline{x}-\underline{x}_0| < \delta$.

Choose surface S inside $|\underline{x}-\underline{x}_0| < \delta$, that lies in plane with normal $\underline{k}$. So

$$0 = \oint_{\partial S} \underline{F}\cdot d\underline{x} = \int_S \underline{\nabla}\times\underline{F}(\underline{x})\cdot d\underline{S}$$

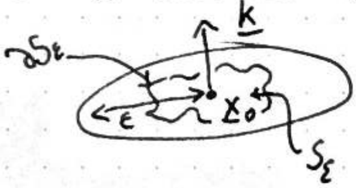
$$= \int_S \underline{\nabla}\times\underline{F}(\underline{x})\cdot\underline{k}\; dS \geq \varepsilon\int_S dS \quad \times\!\!\times$$

Hence $\underline{\nabla}\times\underline{F} = \underline{0}$. $\square$

have shown if
only if requires
1-connected

Find that $\underline{\nabla}\times\underline{F} = \underline{0}$ iff circulation of $\underline{F}$ about any closed C is zero.

<u>Example</u> Let S_ε be a surface contained inside a disc of radius $\varepsilon > 0$ centred at $\underline{x}_0$, with normal $\underline{k}$.



$$\int_{S_\varepsilon} \underline{\nabla}\times\underline{F}\cdot d\underline{S} = \text{area}(S_\varepsilon)\,\underline{\nabla}\times\underline{F}(\underline{x}_0)\cdot\underline{k}$$
$$+ \int_{S_\varepsilon}\left[\underline{\nabla}\times\underline{F}(\underline{x}) - \underline{\nabla}\times\underline{F}(\underline{x}_0)\right]\cdot d\underline{S}$$

L11.3

Note that

- $$\int_{S_\varepsilon} \left[\underline{\nabla} \times \underline{F}(\underline{x}) - \underline{\nabla} \times \underline{F}(\underline{x}_0) \right] \cdot d\underline{S} = o\,(\text{area}(S_\varepsilon))$$

since
$$\left| \int_{S_\varepsilon} \left[\underline{\nabla} \times \underline{F}(\underline{x}) - \underline{\nabla} \times \underline{F}(\underline{x}_0) \right] \cdot d\underline{S} \right|$$
$$\leqslant \underbrace{\sup_{\underline{x} \in S_\varepsilon} \left| \underline{\nabla} \times \underline{F}(\underline{x}) - \underline{\nabla} \times \underline{F}(\underline{x}_0) \right|}_{\substack{\to 0 \text{ as } \varepsilon \to 0 \\ \text{since } \underline{\nabla} \times \underline{F} \text{ continuous}}} \cdot \text{area}(S_\varepsilon)$$

- Hence $\displaystyle\int_{S_\varepsilon} \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \text{area}(S_\varepsilon)\, \underline{\nabla} \times \underline{F}(\underline{x}_0) \cdot \underline{k} + o\,(\text{area}(S_\varepsilon))$.

Divide by area (S_ε), take $\varepsilon \to 0$

$$\underline{k} \cdot \underline{\nabla} \times \underline{F}(\underline{x}_0) = \lim_{\varepsilon \to 0} \frac{1}{\text{area}(S_\varepsilon)} \int_{S_\varepsilon} \underline{\nabla} \times \underline{F} \cdot d\underline{S}$$
$$= \lim_{\varepsilon \to 0} \frac{1}{\text{area}(S_\varepsilon)} \int_{\partial S_\varepsilon} \underline{F} \cdot d\underline{x}$$

So component of $\underline{\nabla} \times \underline{F}(\underline{x}_0)$ in $\underline{k}$ direction is the infinitesimal circulation about the $\underline{k}$ axis per unit area.

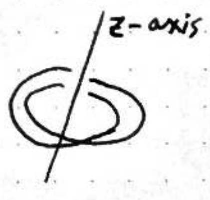
Gives a coordinate free definition of curl.

§5.3 Möbius bands + Stoke's Theorem

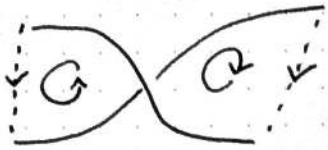
Möbius band NOT orientable.

$$\underline{x}\,(u,v) = \begin{pmatrix} \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u \\ \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \sin u \\ \frac{v}{2} \sin \frac{u}{2} \end{pmatrix} \qquad \begin{array}{l} 0 \leqslant u \leqslant 2\pi \\ -1 \leqslant v \leqslant 1 \end{array}$$

- Note if $\underline{F} = \dfrac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ then $\underline{\nabla} \times \underline{F} = 0$ on S. So if Stokes' theorem works $0 = \displaystyle\oint_{\partial S} \underline{F} \cdot d\underline{x} = 4\pi$.



L11.4



OPЯE

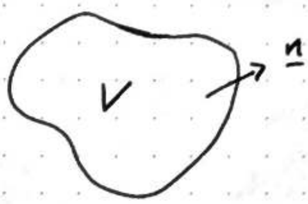
§ 5.4 : Divergence Theorem : statement + examples

<u>Proposition</u> (3D)

If $\underline{F}(\underline{x})$ is a continuously diff'ble vector field, $V \subset \mathbb{R}^3$ is a volume with piece-wise regular boundary ∂V, then

$$\int_V \underline{\nabla} \cdot \underline{F} \, dV = \int_{\partial V} \underline{F} \cdot d\underline{S} \quad \leftarrow \underline{n}\, dS$$

where $\underline{n}$ points <u>out</u> of V.



<u>Proposition</u> (2D)

Let $\underline{F}$, $D \subset \mathbb{R}^2$ be as above. Then

$$\int_D \underline{\nabla} \cdot \underline{F} \, dA = \oint_{\partial D} \underline{F} \cdot \underline{n} \, ds \quad \leftarrow \text{arc-length}$$

where $\underline{n}$ points <u>out</u> of D.

<u>Example</u> $\underline{F}(\underline{x}) = \underline{x}$, $\underline{\nabla} \cdot \underline{F} = 3$

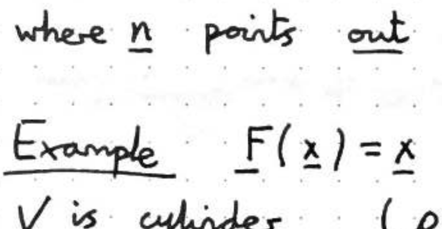
V is cylinder, (ρ, ϕ, z) cylindrical polars

$V = \{(\rho, \phi, z) : 0 \leq \rho \leq R, \ 0 \leq \phi \leq 2\pi, \ -h \leq z \leq h\}$

Have $dV = \rho \, d\rho \, d\phi \, dz$.

$$\int_V \underline{\nabla} \cdot \underline{F} \, dV = \int_0^R d\rho \int_0^{2\pi} d\phi \int_{-h}^{h} dz \cdot 3\rho = 3 \frac{R^2}{2} \cdot 2\pi \cdot 2h$$

$$= \underline{6\pi R^2 h}.$$



Note $\partial V = S_+ \cup S_- \cup S_R$.

$$S_R = \left\{ \underline{x}(\phi, z) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z \end{pmatrix} : 0 \leq \phi \leq 2\pi, -h \leq z \leq h \right\}$$

$$S_\pm = \left\{ \underline{x}(\rho, \phi) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ \pm h \end{pmatrix} : 0 \leq \rho \leq R, 0 \leq \phi \leq 2\pi \right\}$$

We find

$$d\underline{S} = \begin{cases} \underline{e}_\rho R \, d\phi \, dz & \text{on } S_R \\ \pm \underline{e}_z \rho \, d\rho \, d\phi & \text{on } S_\pm \end{cases}$$

L12.2

Use $\underline{x} = \rho \underline{e}_\rho + z \underline{e}_z$,

$$\Rightarrow \underline{x} \cdot d\underline{S} = \begin{cases} R^2 \, d\phi \, dz & \text{on } S_R \\ \underbrace{\pm(\pm h)}_{h \; \ddot{\smile}} \rho \, d\rho \, d\phi & \text{on } S_{\pm} \end{cases}$$

$$\int_{S_R} \underline{F} \cdot d\underline{S} = \int_0^{2\pi} d\phi \int_{-h}^{h} dz \cdot R^2 \cancel{d\phi \, dz} = 2\pi \cdot 2h \cdot R^2 = 4\pi R^2 h$$

$$\int_{S_{\pm}} \underline{F} \cdot d\underline{S} = \int_0^{2\pi} d\phi \int_0^R d\rho \cdot h\rho = 2\pi \cdot \frac{R^2}{2} \cdot h = \pi R^2 h$$

Hence $\displaystyle\int_{\partial V} \underline{F} \cdot d\underline{S} = \left(\int_{S_R} + \int_{S_+} + \int_{S_-} \right) \underline{F} \cdot d\underline{S} = 6\pi R^2 h$ $\ddot{\smile}$

<u>Proposition</u> If $\underline{F}(\underline{x})$ is continuously diff'ble and for every <u>closed</u> surface S

$$\int_S \underline{F} \cdot d\underline{S} = 0 \qquad (+)$$

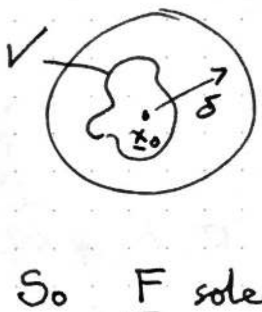
then $\underline{F}$ is solenoidal, i.e. $\underline{\nabla} \cdot \underline{F} = 0$

<u>Proof</u> Suppose not, so $(+)$ holds but $\underline{\nabla} \cdot \underline{F} \neq 0$.

So $\underline{\nabla} \cdot \underline{F}(\underline{x}_0) > 0$ for some $\underline{x}_0$. By continuity, we have

$\underline{\nabla} \cdot \underline{F}(\underline{x}_0) > \varepsilon > 0$ on ball $|\underline{x} - \underline{x}_0| < \delta$.

Choose volume V in $|\underline{x} - \underline{x}_0| < \delta$.



$$\int_{\partial V} \underline{F} \cdot d\underline{S} = \int_V \underline{\nabla} \cdot \underline{F} \, dV > \varepsilon \int_V dV$$

a contradiction since $\int_{\partial V} \underline{F} \cdot d\underline{S} = 0$.

So $\underline{F}$ solenoidal. $\square$

<u>Example</u> Let $V_\varepsilon \subseteq \mathbb{R}^3$ be volume contained inside the ball $\{ \underline{x} : |\underline{x} - \underline{x}_0| < \varepsilon \}$.

$$\int_{V_\varepsilon} \underline{\nabla} \cdot \overset{\text{cont. diff'ble}}{\underline{F}} \, dV = \text{vol}(V_\varepsilon) \cdot \underline{\nabla} \cdot \underline{F}(\underline{x}_0) + \underbrace{\int_{V_\varepsilon} \left(\underline{\nabla} \cdot \overset{\text{at } \underline{x}}{\underline{F}} - \underline{\nabla} \cdot \underline{F}(\underline{x}_0) \right) dV}_{o(\text{vol}(V_\varepsilon))}$$

Since

$$\left| \int_{V_\varepsilon} \left[\underline{\nabla} \cdot \underline{F}(\underline{x}) - \underline{\nabla} \cdot \underline{F}(\underline{x}_0) \right] dV \right| \leq \text{vol}(V_\varepsilon) \cdot \underbrace{\sup_{|\underline{x} - \underline{x}_0| < \varepsilon} \left[| \underline{\nabla} \cdot \underline{F}(\underline{x}) - \underline{\nabla} \cdot \underline{F}(\underline{x}_0) | \right]}_{\substack{\text{goes to } 0 \text{ since} \\ \underline{\nabla} \cdot \underline{F} \text{ continuous}}}$$

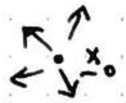
So if we divide by $\text{vol}(V_\varepsilon)$, send $\varepsilon \to 0$, get

$$\underline{\nabla} \cdot \underline{F}(\underline{x}_0) = \lim_{\varepsilon \to 0} \frac{1}{\text{vol}(V_\varepsilon)} \int_{V_\varepsilon} \underline{\nabla} \cdot \underline{F} \, dV$$

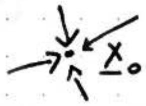
$$= \lim_{\varepsilon \to 0} \frac{1}{\text{vol}(V_\varepsilon)} \int_{\partial V_\varepsilon} \underline{F} \cdot d\underline{S}$$

Divergence measures infinitesimal "flux per unit volume".

$\underline{\nabla} \cdot \underline{F}(\underline{x}_0) > 0$



$\underline{\nabla} \cdot \underline{F}(\underline{x}_0) < 0$



$\underline{\nabla} \cdot \underline{F} = 0$, $\underline{F}$ velocity of incompressible fluid

<u>Example</u> Very common form of eq^n:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$$

$\rho = \rho(\underline{x}, t)$, $\underline{J} = \underline{J}(\underline{x}, t)$

Called conservation laws ; continuity equations

Let $Q(t) = \int_{\mathbb{R}^3} \rho(\underline{x}, t)\, dV$.

Then $$\dot{Q} = \int_{\mathbb{R}^3} \frac{\partial \rho}{\partial t}\, dV = -\int_{\mathbb{R}^3} \underline{\nabla} \cdot \underline{J}\, dV$$

$$= -\lim_{R\to\infty} \int_{|x|<R} \underline{\nabla} \cdot \underline{J}\, dV = -\lim_{R\to\infty} \int_{|x|=R} \underline{J} \cdot d\underline{S} = 0$$

if $|\underline{J}| \to 0$ quickly as $|\underline{x}| \to \infty$ i.e. $Q(t) = \text{const}$.

§5.5 Noether's Theorem

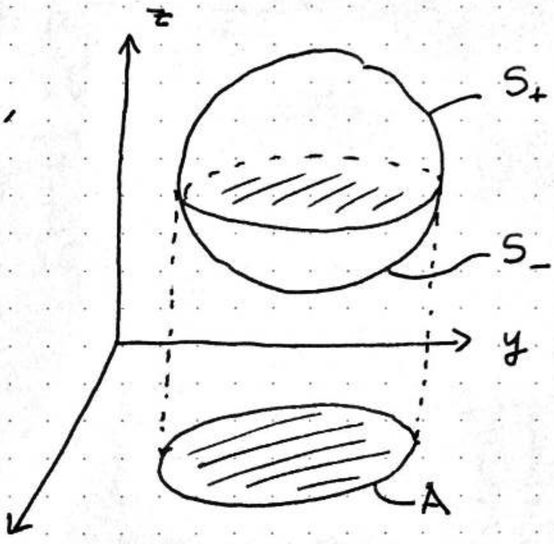
$f = f(t)$
$\tilde{f} = f(t+c)$

§5.6 : Sketch Proofs

Proposition

The divergence theorem is true.

Proof



Suppose $\underline{F} = F_z \underline{e}_z$ with $F_z = F_z(x,y,z)$,

also suppose $\partial V = S_+ \cup S_-$

$$S_{\pm} = \left\{ \underline{x} = \begin{pmatrix} x \\ y \\ g_{\pm}(x,y) \end{pmatrix}, (x,y) \in A \right\}$$

Have

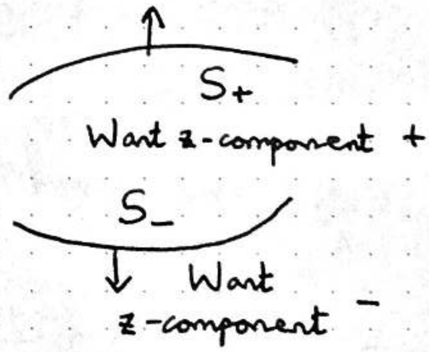
$$\int_V \underline{\nabla} \cdot \underline{F} \, dV = \int_V \frac{\partial F_z}{\partial z} dV$$

$$= \iint_A \left[\int_{z=g_-(x,y)}^{g_+(x,y)} \frac{\partial F_z}{\partial z} dz \right] dx\, dy$$

$$= \iint_A \left[F_z(x,y,g_+(x,y)) - F_z(x,y,g_-(x,y)) \right] dx\, dy$$

This is LHS of div thm. For RHS use

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} \, dx\, dy = \begin{pmatrix} -\partial g_{\pm}/\partial x \\ -\partial g_{\pm}/\partial y \\ 1 \end{pmatrix} dx\, dy$$



on $S_{\pm}$, being careful with sign!

$$d\underline{S}_{\pm} = \pm \begin{pmatrix} -\partial g_{\pm}/\partial x \\ -\partial g_{\pm}/\partial y \\ 1 \end{pmatrix} dx\, dy.$$

So
$$\int_{\partial V} \underline{F} \cdot d\underline{S} = \left[\int_{S_+} + \int_{S_-} \right] \underline{F} \cdot d\underline{S}$$

$$= \iint_A \left[F_z(x,y,g_+(x,y)) - F_z(x,y,g_-(x,y)) \right] dx\, dy = \int_V \underline{\nabla} \cdot \underline{F} \, dV$$

So $\displaystyle\int_V \frac{\partial F_z}{\partial z}\, dV = \int_{\partial V} F_z \, \underline{e}_z \cdot d\underline{S}$.

Similarly $\displaystyle\int_V \frac{\partial F_x}{\partial x}\, dV = \int_{\partial V} F_x \, \underline{e}_x \cdot d\underline{S}$

$$\int_V \frac{\partial F_y}{\partial y}\, dV = \int_{\partial V} F_y \, \underline{e}_y \cdot d\underline{S}.$$

Adding together,

$$\int_V \left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right] dV = \int_{\partial V} \left[F_x \underline{e}_x + F_y \underline{e}_y + F_z \underline{e}_z \right] \cdot d\underline{S}$$

i.e. $\displaystyle\int_V \underline{\nabla} \cdot \underline{F} \, dV = \int_{\partial V} \underline{F} \cdot d\underline{S}$.

$\square$

Can show Div Thm => Green's Thm => Stokes' Thm

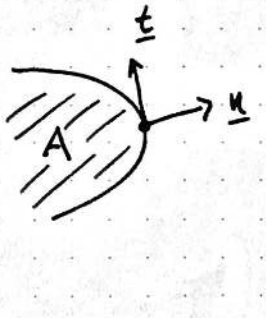
(a) <u>Div Thm => Green's Thm</u>

In 2D

$$\int_A \underline{\nabla} \cdot \underline{F} \, dA = \oint_{\partial A} \underline{F} \cdot \underline{n} \, ds$$

Use $\underline{F} = \begin{pmatrix} Q \\ -P \end{pmatrix}$. So

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \text{LHS}$$



If ∂A parametrised with arc-length,

$\underline{t} = (x'(s), \, y'(s))$

need $\underline{n} = (y'(s), -x'(s))$.

$$\underline{F} \cdot \underline{n} \, ds = Q \frac{dy}{ds} ds + P \frac{dx}{ds} ds = P dx + Q dy$$

Hence $\displaystyle\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_{\partial A} (P dx + Q dy)$

(b) Green's Thm => Stokes' Thm

Suppose

$$S = \{ \underline{x} = \underline{x}(u,v) \ , \ (u,v) \in A \}$$

$$\partial S = \{ \underline{x} = \underline{x}(u,v) \ , \ (u,v) \in \partial A \}$$

We know that

$$\iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du\, dv = \oint_{\partial A} P\,du + Q\,dv$$

We can choose

$$P = \underline{F}(\underline{x}(u,v)) \cdot \frac{\partial \underline{x}}{\partial u} \qquad Q = \underline{F}(\underline{x}(u,v)) \cdot \frac{\partial \underline{x}}{\partial v} .$$

So $P\,du + Q\,dv$

$$= \underline{F}(\underline{x}(u,v)) \cdot \left[\frac{\partial \underline{x}}{\partial u} du + \frac{\partial \underline{x}}{\partial v} dv \right]$$

$$= \underline{F}(\underline{x}(u,v)) \cdot d\underline{x}(u,v).$$

So

$$\oint_{\partial A} P\,du + Q\,dv = \oint_{\partial S} \underline{F} \cdot d\underline{x} .$$

Recall that $d\underline{S} = \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} du\, dv$.

Have

$$\frac{\partial Q}{\partial u} = \frac{\partial}{\partial u}\left[F_j(\underline{x}(u,v)) \cdot \frac{\partial x_j}{\partial v} \right] \qquad \frac{\partial P}{\partial v} = \frac{\partial}{\partial v}\left[F_j(\underline{x}(u,v)) \frac{\partial x_j}{\partial u} \right]$$

$$= \frac{\partial x_i}{\partial u} \frac{\partial F_j}{\partial x_i} + F_j \frac{\partial^2 x_j}{\partial u \partial v} \qquad = \frac{\partial x_i}{\partial v} \frac{\partial F_j}{\partial x_i} \frac{\partial x_j}{\partial u} + F_j \frac{\partial^2 x_j}{\partial v \partial u}$$

$$\frac{\partial x_j}{\partial v}$$

$$\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} = \left[\frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} - \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial u} \right] \frac{\partial F_j}{\partial x_i}$$

$$= \left[\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq} \right] \frac{\partial x_p}{\partial u} \frac{\partial x_q}{\partial v} \frac{\partial F_j}{\partial x_i}$$

$$= \varepsilon_{ijk}\varepsilon_{kpq} \frac{\partial x_p}{\partial u} \frac{\partial x_q}{\partial v} \frac{\partial F_j}{\partial x_i}$$

$$= \left(\varepsilon_{kij} \frac{\partial}{\partial x_i} F_j \right) \left(\varepsilon_{kpq} \frac{\partial x_p}{\partial u} \frac{\partial x_q}{\partial v} \right)$$

L13.4

$$= [\underline{\nabla} \times \underline{F}]_k \left[\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right]_k$$

$$= (\underline{\nabla} \times \underline{F}) \cdot \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right)$$

So $$\iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) \underbrace{}_{du\,dv} = \iint_A \underline{\nabla} \times \underline{F}(\underline{x}(u,v)) \cdot \underbrace{\left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\, dv}_{d\underline{S}}$$

$$= \int_{S \cancel{\partial V}} \underline{\nabla} \times \underline{F} \cdot d\underline{S}$$

Hence $$\int_S \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \oint_{\partial S} \underline{F} \cdot d\underline{x}.$$

§6 : Maxwell's Eq^{n}s

§6.1 : Brief intro to electromagnetism

Have electric field $\underline{E}(\underline{x},t)$ and magnetic field $\underline{B}(\underline{x},t)$. They depend on one another via

(1) $\quad \underline{\nabla}\cdot\underline{E} = \frac{1}{\varepsilon_0}\rho$

(2) $\quad \underline{\nabla}\cdot\underline{B} = 0$

(3) $\quad \underline{\nabla}\times\underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$

(4) $\quad \underline{\nabla}\times\underline{B} - \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t} = \mu_0\underline{J}$

where $\rho(\underline{x},t)$ is the charge density (charge per unit vol), $\underline{J}(\underline{x},t)$ is current density (current per unit area).

ε_0 (μ_0) is the permittivity (permeability) of free space.

$$\frac{1}{\varepsilon_0\mu_0} = c^2 \qquad (c = \text{speed of light})$$

Use $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{B}) = 0$. Take div of (4)

$$-\cancel{\mu_0}\varepsilon_0\frac{\partial}{\partial t}(\underline{\nabla}\cdot\underline{E}) = \cancel{\mu_0}\,\underline{\nabla}\cdot\underline{J}$$

Use (1) to get

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0.$$

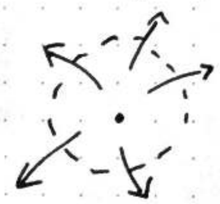
Conservation of charge. [Noether : corresponds to gauge invariance]

§6.2 : Integral formulations

Integrate (1) over volume V, use divergence thm.

$$\int_{\partial V}\underline{E}\cdot d\underline{S} = \frac{1}{\varepsilon_0}\int_V \rho\, dV \equiv \frac{Q}{\varepsilon_0}$$ « Gauss's Law for $\underline{E}$ »

where Q is the total charge in V.



Do same for (2)

$$\int_{\partial V}\underline{B}\cdot d\underline{S} = 0.$$

No magnetic monopoles. (or are there?) (See Dirac, 1931)

L14.2

Integrate (3) over surface S, use Stokes' theorem

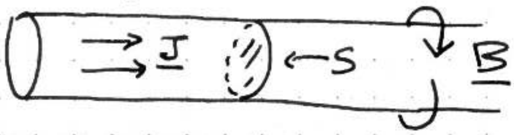
$$\oint_{\partial S} \underline{E} \cdot d\underline{x} = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} = -\frac{d}{dt} \int_S \underline{B} \cdot d\underline{S}$$

i.e. changing magnetic field induces non-zero circulation of $\underline{E}$, i.e. a current (boi wat).

Same for (4)

$$\oint_{\partial S} \underline{B} \cdot d\underline{x} = \mu_0 \int_S \underline{J} \cdot d\underline{S} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \underline{E} \cdot d\underline{S}$$

i.e. steady current induces magnetic field



§ 6.3: Electromagnetic waves

Suppose $\rho = 0$, $\underline{J} = 0$. Then

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{B} - \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = 0$$

Recall that

$$\nabla^2 \underline{E} = \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{E})$$

so in our case,

$$\nabla^2 \underline{E} = -\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = +\underline{\nabla} \times \left[+ \frac{\partial \underline{B}}{\partial t} \right]$$

$$= \frac{\partial}{\partial t} (\underline{\nabla} \times \underline{B})$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \right)$$

$$\Rightarrow \boxed{\nabla^2 \underline{E} - \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} = 0}$$

Waves travel at speed c.

$$\nabla^2 \underline{B} = -\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = -\underline{\nabla} \times \left(\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \right)$$

$$= -\frac{1}{c^2} \frac{\partial}{\partial t} (\underline{\nabla} \times \underline{E}) = \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2}$$

i.e. $$\boxed{\nabla^2 \underline{B} - \frac{1}{c^2}\frac{\partial^2 \underline{B}}{\partial t^2} = 0}$$

§6.4 : Magnetostatics and electrostatics

Assume everything is t-indep. Eq^{n}s decouple

$$\left\{ \underline{\nabla}\cdot\underline{E} = \frac{1}{\cancel{4\pi}\,\varepsilon_0}\rho \;,\quad \underline{\nabla}\times\underline{E} = \underline{0} \right\}$$

$$\left\{ \underline{\nabla}\cdot\underline{B} = 0 \;,\quad \underline{\nabla}\times\underline{B} = \mu_0 \underline{J} \right\}$$

Work on domain $\mathbb{R}^3$ [2 connected]. So

$$\underline{E} = -\underline{\nabla}\phi \;,\quad \underline{B} = \underline{\nabla}\times\underline{A}.$$

So Maxwell's equations reduce to

$$\nabla^2\phi = -\frac{1}{\varepsilon_0}\rho \;,\quad \underline{\nabla}\times(\underline{\nabla}\times\underline{A}) = \mu_0\underline{J}$$

§6.5 Gauge invariance

Work on $\mathbb{R}^3$. Maxwell (2) says

$$\underline{B} = \underline{\nabla}\times\underline{A}$$

Put into (3)

$$\underline{\nabla}\times\left(\underline{E} + \frac{\partial\underline{A}}{\partial t}\right) = 0.$$

So can write

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial\underline{A}}{\partial t}.$$

must subtract $\frac{\partial \chi}{\partial t}$ from ϕ

Note: can always add $\underline{\nabla}\chi$ to $\underline{A}$ without changing $\underline{B}$

Called GAUGE FREEDOM

Recall $\nabla^2\underline{A} = \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \underline{\nabla}\times\underline{\nabla}\times(\underline{A})$.

In (4)

$$\begin{aligned}\underline{\nabla}\times\underline{B} - \mu_0\varepsilon_0\frac{\partial\underline{E}}{\partial t} &= \underline{\nabla}\times(\underline{\nabla}\times\underline{A}) + \mu_0\varepsilon_0\overset{\partial/\partial t}{\left[\underline{\nabla}\phi + \frac{\partial\underline{A}}{\partial t}\right]} \\ &= -\nabla^2 A + \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) + \mu_0\varepsilon_0\frac{\partial}{\partial t}[\cdots] \\ &= -\nabla^2 A + \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t} + \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \\ &= \mu_0\underline{J} \quad [\text{from } (4)]\end{aligned}$$

L14.4

$$-\nabla^2 \underline{A} + \frac{1}{c^2}\frac{\partial \underline{A}}{\partial t} + \underline{\nabla}\left[\underline{\nabla}\cdot\underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right] = \mu_0 \underline{J}$$

Use Gauge freedom to pick $\underline{A}$, ϕ such that

$$\underline{\nabla}\cdot\underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

? $\partial_\mu A^\mu = 0$?

can do by adding $\underline{\nabla}\chi$.

Maxwell's equations reduce to

$$-\nabla^2 \underline{A} + \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J} \qquad (\dagger)$$

? $\Box A^\mu = J^\mu$?

Also using $\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$ in (1)

$$-\nabla^2 \phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\varepsilon_0} \qquad (\dagger\dagger)$$

($\dagger$) and ($\dagger\dagger$) are Maxwell's equations in Lorentz gauge. ⍤

§ 7 : Poisson's Equation

§ 7.1 : The boundary value problem

Many problems in mathematical physics reduce to Poisson's Eq^n:

$$\nabla^2 \varphi = F$$

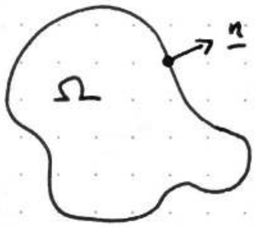
When $F \equiv 0$, it's Laplace's eq^n. Solve on domains $\Omega \subseteq \mathbb{R}^n$ for $n = 2, 3$.
Need some conditions of φ on boundary $\partial\Omega$, or as $|\underline{x}| \to \infty$ if $\Omega = \mathbb{R}^n$.

The Dirichlet problem for Poisson's eq^n

$$\begin{cases} \nabla^2 \varphi = F & \text{on } \Omega, \\ \varphi = f & \text{on } \partial\Omega. \end{cases}$$

The Neumann problem

$$\begin{cases} \nabla^2 \varphi = F & \text{in } \Omega, \\ \partial\varphi/\partial \underline{n} = g & \text{on } \partial\Omega \end{cases}$$



where $\partial\varphi/\partial\underline{n} = \underline{n} \cdot \underline{\nabla} \varphi$.

Note we want φ to approach boundary data continuously as $\underline{x} \to \partial\Omega$.
So we require φ be continuously diff. ble on $\Omega \cup \partial\Omega$.

Ex: Solve, when $r = |\underline{x}|$ $(\underline{x} \in \mathbb{R}^3)$

$$\begin{cases} \nabla^2 \varphi = r & r < a \\ \varphi = 1 & r = a \end{cases} \qquad (\dagger)$$

Guess that $\varphi = \varphi(r)$. Then

$$\nabla^2 \varphi = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\varphi}{dr} \right)$$

So

$$(\dagger) \begin{cases} (r^2 \varphi')' = r^3, \\ \varphi(a) = 1. \end{cases}$$

$$(r^2\varphi')' = r^3 \Rightarrow \varphi' = \frac{r^2}{4} + \frac{A}{r^2} \Rightarrow \varphi = \frac{r^3}{12} - \frac{A}{r} + B$$

A must be zero, else undefined at $r = 0$. $\longleftarrow$ $(*)$

$\varphi(a) = 1$: $1 = \dfrac{a^3}{12} + B$

So $\varphi = 1 + \dfrac{1}{12}\left(r^3 - a^3\right)$.

(*) Note for φ to solve, e.g.

$$\nabla^2\varphi = 0 \qquad |\underline{x}| < R$$

require φ well-defined on $|\underline{x}| < R$.

E.g., NOT the case that

$$\nabla^2\left(\frac{1}{|\underline{x}|}\right) = 0 \quad \forall\, \underline{x} \in \mathbb{R}^3$$

only true for $\underline{x} \neq \underline{0}$ (check).

Consider generic <u>linear</u> problem:

$$\begin{cases} L\varphi = F & \text{in } \Omega \\ B\varphi = f & \text{on } \partial\Omega \end{cases} \qquad (*)$$

where L, B are linear differential operators. If φ_1, φ_2 both solve $(*)$, then by linearity, $\psi := \varphi_1 - \varphi_2$ solves

$$\begin{cases} L\psi = 0 & \text{on } \Omega \\ B\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

If we can show $\psi = 0$, <u>must</u> then have $\varphi_1 = \varphi_2$ so sol^n to $(*)$ unique.

In general, solution to linear problem unique

$\iff$ only solution to homogeneous problem is the zero solution

<u>Proposition</u>: Sol^n to Dirichlet problem is <u>unique</u>.

Sol^n to Neumann problem <u>unique</u> up to a <u>constant</u>.

<u>Proof</u>: Let ψ solve the homogeneous problem

$$\begin{cases} \nabla^2\psi = 0 & \text{in } \Omega \\ B\psi = 0 & \text{on } \partial\Omega \end{cases}$$

where $B\psi = \psi$ or $B\psi = \partial\psi/\partial\underline{n}$ (Dirichlet or Neumann).

Consider

$$I[\psi] = \int_\Omega |\underline{\nabla}\psi|^2\, dV \geq 0.$$

Then $\underset{0}{\underset{\shortparallel}{I[\psi]}} \iff \underline{\nabla}\psi = 0$ on Ω

$\iff \psi = \text{const.}$ on Ω.

Have $I[\psi] = \int_\Omega \underline{\nabla}\psi \cdot \underline{\nabla}\psi \, dV = \int_\Omega [\underline{\nabla} \cdot (\psi \underline{\nabla} \psi) - \underbrace{\psi \nabla^2 \psi}_{\text{zero}}] \, dV$

$$= (\text{by Div. Thm}) = \int_{\partial\Omega} (\psi \underline{\nabla} \psi) \cdot d\underline{S}$$

$$= \int_{\partial\Omega} (\psi \underline{\nabla} \psi) \cdot \underline{n} \, dS = \int_{\partial\Omega} \psi \frac{\partial \psi}{\partial \underline{n}} \, dS$$

This is zero, since $\psi = 0$ or $\dfrac{\partial \psi}{\partial \underline{n}} = 0$ on boundary.

So $\psi =$ const. throughout Ω.

(a) Dirichlet: by continuity, since $\psi = 0$ on $\partial\Omega$ must have $\psi = 0$ throughout Ω. So <u>unique</u> solution.

(b) Neumann: can only say $\psi =$ const. throughout Ω. So any two solutions differ by a constant, as claimed. $\square$

<u>Example</u> (Electrostatics)

Charge density $\rho(\underline{x}) = \begin{cases} 0 & r < a, \\ F(r) & r \geq a. \end{cases}$

Get $\underline{E} = -\underline{\nabla}\phi$ by solving $\nabla^2 \phi = -\frac{1}{\varepsilon_0}\rho$.

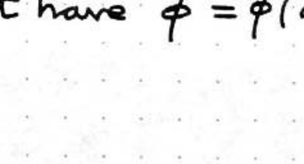
By symmetry $\phi = \phi(r)$, so on $r = a$, must have $\phi = \phi(a)$ = some const. So have

$$\begin{cases} \nabla^2 \phi = 0 & r < a \\ \phi = \text{const.} & r = a \end{cases}$$

So $\phi =$ const. works throughout $r \leq a$, and is solution (by uniqueness).

So $\underline{E} = \underline{0}$ throughout $r \leq a$.

Same argument works for gravitational fields.



§ 7.2 : Gauss' Flux Method

Suppose we want to solve

$$\nabla^2 \varphi = F \qquad (\text{in } \mathbb{R}^3)$$

where F is spherically symmetric, $F = F(r)$, $r = |\underline{x}|$.

Reasonable to assume $\varphi = \varphi(r)$. Write eq^n as

$$\underline{\nabla} \cdot \underline{\nabla} \varphi = F$$

Integrate over $V = \{ \underline{x} : |\underline{x}| \leq R \}$. By div. thm.

$$\int_{\partial V} \nabla \varphi \cdot d\underline{S} = \int_V F \, dV$$

Note on ∂V we have

$$d\underline{S} = \underline{e}_r R^2 \sin\theta \, d\theta \, d\phi \qquad (\text{CHECK})$$

And since $\varphi = \varphi(r)$,

$$\underline{\nabla} \varphi = \varphi'(r) \underline{e}_r.$$

$$\Rightarrow \underline{\nabla} \varphi \cdot d\underline{S} = R^2 \varphi'(R) \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \int_{\partial V} \underline{\nabla} \varphi \cdot d\underline{S} = R^2 \varphi'(R) \int_{\partial V} \sin\theta \, d\theta \, d\phi = 4\pi R^2 \varphi'(R).$$

Hence $\varphi'(R) = \dfrac{1}{4\pi R^2} \displaystyle\int_{|x| \leq r} F \, dV \equiv \dfrac{Q(R)}{4\pi R^2}$.

Where $Q(R)$ is total amount of "stuff" within V.

If we assume $\varphi \to 0$ as $|\underline{x}| \to \infty$,

$$\varphi(r) = -\int_r^\infty \varphi'(R) \, dR = -\int_r^\infty \frac{1}{4\pi R^2} \left[\int_0^R ds \int_0^\pi d\theta \int_0^{2\pi} d\phi \; F(s) \, s^2 \sin\theta \right] dR$$

$$= -\int_r^\infty \frac{1}{R^2} \left[\int_0^R ds \cdot s^2 F(s) \right] dR$$

<u>Example</u> What is the electric field due to (static) charge density

$$\rho(\underline{x}) = \begin{cases} \rho_0 & \text{for } |\underline{x}| \le a, \\ 0 & \text{for } |\underline{x}| > a. \end{cases}$$

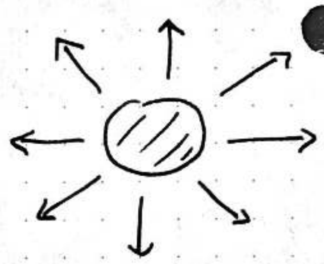
Recall $\underline{E} = -\underline{\nabla}\phi$ where $\nabla^2\phi = -\frac{1}{\varepsilon_0}\rho$.

Know that $\underline{\nabla}\phi = \phi'(r)\,\underline{e}_r$ and

$$\phi'(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}|\le r}\left[-\frac{\rho}{\varepsilon_0}\right] dV = -\frac{Q(r)}{4\pi r^2 \varepsilon_0}$$

Since $\rho = 0$ on $r > a$, $Q(r) = Q(a)$ for $r > a$
$\equiv Q$

$$\left[Q = \frac{4\pi}{3}a^3\rho_0\right] \Rightarrow \underline{E} = \begin{cases} \frac{1}{4\pi\varepsilon_0 r^2} Q(r)\,\underline{e}_r & r \le a \\ \frac{1}{4\pi\varepsilon_0 r^2} Q\,\underline{e}_r & r > a \end{cases}$$



Take $a \to 0$, keeping total charge Q fixed,

$$\underline{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\underline{e}_r}{r^2} = \frac{Q}{4\pi\varepsilon_0}\frac{\underline{x}}{|\underline{x}|^3}$$

Electric field due to point charge Q at $\underline{x} = \underline{0}$. Corresponding potential $\phi = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{|\underline{x}|}$ (CHECK)

Corresponds to $\rho(\underline{x}) = Q\,\delta^{(3)}(\underline{x})$.

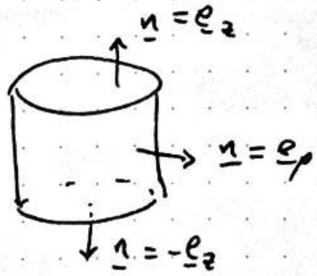
Also use this when there is <u>cylindrical</u> symmetry i.e. $F = F(\rho)$ where $\rho^2 = x^2 + y^2$. Assume that $\varphi = \varphi(\rho)$ in $\nabla^2\varphi = F(\rho)$ i.e. $\underline{\nabla}\cdot\underline{\nabla}\varphi = F(\rho)$.

Integrate over cylinder $V = \{(\rho, \phi, z) : 0 \le \rho \le R,\ 0 \le \phi < 2\pi,\ 0 \le z \le a\}$

By div thm

$$\int_{\partial V} \underline{\nabla}\varphi \cdot d\underline{S} = \int_V F\,dV.$$

Note



Also, since $\varphi = \varphi(\rho)$, $\underline{\nabla}\varphi = \varphi'(\rho)\underline{e}_\rho$.

So $\underline{\nabla}\varphi \cdot d\underline{S} = 0$ on top and bottom of cylinder.

On side of cylinder, $d\underline{S} = \underline{e}_\rho R\, d\phi\, dz$ (CHECK)

So $$\int_{\partial V} \underline{\nabla}\varphi \cdot d\underline{S} = \int_0^a dz \int_0^{2\pi} d\phi \cdot R\varphi'(R) = 2\pi a R\varphi'(R).$$

Also $$\int_V F\,dV = \int_0^R d\rho \int_0^{2\pi} d\phi \int_0^a dz\; F(\rho)\rho\,\cancel{d\rho} = 2\pi a \int_0^R \rho F(\rho)\,d\rho.$$

Hence $$\varphi'(R) = \frac{1}{R}\int_0^R \rho F(\rho)\,d\rho.$$

<u>Example</u>

Electric field from wire of ~~thickness~~ radius a along z-axis, i.e.

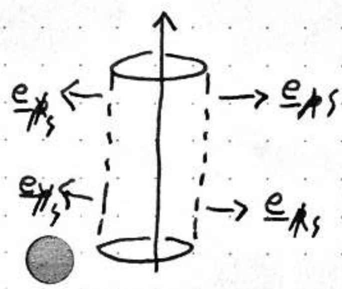
$$\rho(\underline{x}) = \begin{cases} \rho_0 & \text{on } s \leq a \\ 0 & \text{on } s > a \end{cases}$$

where $s^2 = x^2 + y^2$, usual radial coordinate.

$$\nabla^2\phi = -\frac{1}{\varepsilon_0}\rho, \quad \underline{E} = -\underline{\nabla}\phi$$

$$\varphi'(s) = \frac{1}{s}\int_0^s t\left[-\frac{1}{\varepsilon_0}\rho(t)\right]dt = -\frac{1}{s\varepsilon_0}\cdot Q(s), \quad Q(s) = Q(a) \text{ for } s > a$$

Hence $$\underline{E} = \frac{1}{\varepsilon_0 s} Q(s)\,\underline{e}_{\cancel{\rho}s}.$$



Find if $a \to 0$, keep $Q(a)$ constant, get

$$\underline{E} \propto \frac{1}{s}\underline{e}_s$$

Electric field due to <u>line</u> of charge about z-axis.

§7.3: Superposition Principle

Linear problems simple because if

$$L\psi_n = F_n \qquad n=1,2,\dots$$

linear diff. operator

then $L\left(\sum_n \psi_n\right) = \sum_n F_n$.

To solve $L\psi = F$, split F up

$$F = \sum_n F_n$$

then solve $L\psi_n = F_n$ for each n. Then

$$\psi = \sum_n \psi_n.$$

Example

Electric potential due to two point charges Q_a and Q_b situated at $\underline{x} = \underline{a}$ and $\underline{x} = \underline{b}$.

For one point charge

$$-\nabla^2\phi = \frac{1}{\varepsilon_0} Q_a\, \delta^{(3)}(\underline{x}-\underline{a})$$

$$\Rightarrow \phi(\underline{x}) = \frac{Q_a}{4\pi\varepsilon_0}\cdot\frac{1}{|\underline{x}-\underline{a}|}$$

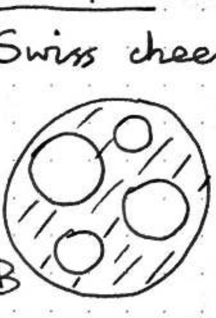
So for two charges

$$-\nabla^2\phi = \frac{Q_a}{\varepsilon_0}\delta^{(3)}(\underline{x}-\underline{a}) + \frac{Q_b}{\varepsilon_0}\delta^{(3)}(\underline{x}-\underline{b})$$

$$\Rightarrow \phi(\underline{x}) = \frac{Q_a}{4\pi\varepsilon_0}\cdot\frac{1}{|\underline{x}-\underline{a}|} + \frac{Q_b}{4\pi\varepsilon_0}\cdot\frac{1}{|\underline{x}-\underline{b}|}$$

Example

Swiss cheese charge density



B

So $\rho(\underline{x}) = \rho_0$ on $|\underline{x}| < R$ but $\rho(\underline{x}) = 0$ on holes:

$$B_i = \{ |\underline{x} - \underline{a}_i| < R_i \}$$

where $|\underline{a}_i| + R_i < R$ for all i

and $|\underline{a}_i - \underline{a}_j| > R_i + R_j$ for all i, j.

What is $\phi(\underline{x})$ on $|\underline{x}| > R$?

If $\mathbb{1}_A$ is indicator function on set A

$$\rho(\underline{x}) = \rho_0 \mathbb{1}_B - \sum_i \rho_0 \mathbb{1}_{B_i}$$

By superposition principle

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{|\underline{x}|} - \sum_i \frac{Q_i}{|\underline{x}-\underline{a}_i|}\right]$$

where $Q = \rho_0 \cdot \frac{4\pi R^3}{3}$ and $Q_i = \rho_0 \cdot \frac{4\pi R_i^3}{3}$.

§ 7.4: Integral Solutions

Have seen potential due to point charge Q at $\underline{x}=\underline{a}$ is proportional to $\frac{Q}{|\underline{x}-\underline{a}|}$.

More generally, collection of point charges Q_i at $\{\underline{a}_i\}$ gives

$$\phi \propto \sum_i \frac{Q_i}{|\underline{x}-\underline{a}_i|}$$

Leads us to consider

$$\phi \propto \int_{\mathbb{R}^3} \frac{F(\underline{y})}{|\underline{x}-\underline{y}|}\, dV(\underline{y})$$

Proposition Suppose $F: \mathbb{R}^3 \to \mathbb{R}$ decays "quickly" as $|\underline{x}| \to \infty$. Then unique solution to $\nabla^2\varphi = F$ on $\mathbb{R}^3$ $\cap$ φ "$\to$" 0 as $|\underline{x}| \to \infty$ is

$$\varphi(\underline{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{F(\underline{y})}{|\underline{x}-\underline{y}|}\, dV(\underline{y})$$

Sketch It would be enough to prove

$$\nabla^2\left(-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}|}\right) = \delta^{(3)}(\underline{x})$$

Since, differentiating under integral (wrt $\underline{x}$)

$$\nabla^2\varphi(\underline{x}) = \int_{\mathbb{R}^3} \nabla^2\left(-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}-\underline{y}|}\right) F(\underline{y})\, dV$$

$$= \int_{\mathbb{R}^3} \delta^{(3)}(\underline{x}-\underline{y}) F(\underline{y})\, dV(\underline{y})$$

$$= F(\underline{x}) \quad ☺$$

Also get $\varphi(\underline{x}) \to 0$ as $|\underline{x}| \to \infty$ (exercise).

For $|\underline{x}| \neq 0$, $\delta^{(3)}(\underline{x}) = 0$. Also, for $|\underline{x}| \neq 0$

$$\nabla^2\left[\frac{1}{|\underline{x}|}\right] = \partial_i\partial_i\left(\frac{1}{r}\right) = \partial_i\left(-\frac{1}{r^3}x_i\right) = \frac{3x_ix_i}{r^5} - \frac{\delta_{ii}}{r^3} = 0 \quad ☺$$

So $\nabla^2\left[-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}|}\right] = \delta^{(3)}(\underline{x})$ for $\underline{x} \neq \underline{0}$.

What about $\underline{x} = \underline{0}$?

IF divergence theorem works with delta functions, for $R > 0$

$$\int_{|\underline{x}|<R} \nabla^2\left[-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}|}\right] dV = -\frac{1}{4\pi}\int_{|\underline{x}|=R} \underline{\nabla}\left(\frac{1}{|\underline{x}|}\right)\cdot d\underline{S}$$

$$= -\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \cdot R^2 \sin\theta \cdot \left(-\frac{\underline{e}_r}{R^2}\right)\cdot \underline{e}_r$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta = 1$$

So $\displaystyle\int_{|\underline{x}|<R} \nabla^2\left[-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}|}\right] dV = \int_{|\underline{x}|<R} \delta^{(3)}(\underline{x})\, dV$

Hence $\displaystyle\nabla^2\left[-\frac{1}{4\pi}\cdot\frac{1}{|\underline{x}|}\right] = \delta^{(3)}(\underline{x})$.

How to solve $\underset{\text{in } \Omega}{\nabla^2\phi = 0} \cap \underset{\text{on } \partial\Omega}{\varphi = f}$?

Try $\displaystyle\varphi(\underline{x}) = \int_{\partial\Omega} \frac{g(\underline{y}, \underline{x})}{|\underline{x}-\underline{y}|} \cancel{dV(\underline{y})}\, dS(\underline{y})$

See that $\displaystyle\nabla^2\left[\frac{1}{|\underline{x}-\underline{y}|}\right] = 0$ if $\underline{x} \in \Omega$ since $\underline{y} \in \partial\Omega$.

Take limit $\underline{x} \longrightarrow \partial\Omega$, get

$$\underbrace{f(\underline{x})}_{\varphi \text{ on boundary}} = \frac{1}{2} g(\underline{x}) + \int_{\partial\Omega} \frac{g(\underline{y})}{|\underline{x}-\underline{y}|} dS \qquad (+)$$

Write $\displaystyle\frac{1}{2}\int_{\partial\Omega} \frac{g(\underline{y})}{|\underline{x}-\underline{y}|} dS = (Kg)(\underline{x})$.

Then $(+)$ equivalent to

$$2f = (1 + K)\, g$$

$$\Rightarrow g = (1+K)^{-1}(2f)$$

For $|\underline{x}| < 1$, $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + \cdots$, so

$$(I + K)^{-1} = \frac{1}{I+K} = I - K + K^2 - \cdots$$

Hence

$$g = (I - K + K^2 - \cdots)(2f).$$

L17.4

QED

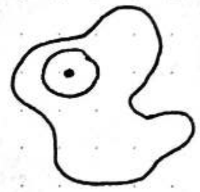
$$S = 1 + \frac{1}{4} + \frac{1}{\sqrt{\quad}} + \cdots = \cdots = 0.124\cdots$$

§7.5: Harmonic Functions

All solutions to Laplace's equation $\nabla^2\varphi = 0$ (are) <u>harmonic</u> functions.
Lots of nice properties. Satisfy mean value property:

<u>Proposition</u> If $\varphi : \Omega \to \mathbb{R}$ $(\Omega \subseteq \mathbb{R}^3)$ is harmonic, then for each $\underline{a} \in \Omega$ we have

$$\varphi(\underline{a}) = \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \varphi(\underline{x})\, dS$$



for $r > 0$ such that $\{|\underline{x}-\underline{a}| < r\} \subseteq \Omega$.

<u>Proof</u> Define

$$F(r) = \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \varphi(\underline{x})\, dS = \frac{1}{4\pi r^2} \int_{|\underline{x}|=r} \varphi(\underline{a}+\underline{x})\, dS.$$

Using spherical polars, get

$$F(r) = \frac{1}{4\pi \cancel{r^2}} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \cdot \cancel{r^2} \sin\theta \cdot \varphi(\underline{a} + r\underline{e}_r) \qquad (\dagger)$$

Now, using $\frac{d}{dt} f(\underline{x}(t)) = \frac{d\underline{x}}{dt} \cdot \underline{\nabla} f(\underline{x}(t))$, get

$$\frac{d}{dr} \varphi(\underline{a} + r\underline{e}_r) = \underline{e}_r \cdot \underline{\nabla}\varphi(\underline{a} + r\underline{e}_r) \quad \text{so}$$

$$F'(r) = \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \cdot \sin\theta \left(\underline{\nabla}\varphi(\underline{a}+r\underline{e}_r) \cdot \underline{e}_r\right)$$

$$= \frac{1}{4\pi r^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \cdot r^2 \sin\theta \left(\underline{\nabla}\varphi(\underline{a}+r\underline{e}_r) \cdot \underline{e}_r\right)$$

$$= \frac{1}{4\pi r^2} \int_{|\underline{x}|=r} \underline{\nabla}\varphi(\underline{a}+\underline{x}) \cdot d\underline{S}$$

$$= \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \underline{\nabla}\varphi(\underline{x}) \cdot d\underline{S}$$

$$= \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|<r} \nabla^2\varphi(\underline{x})\, dV = 0 \quad \text{since } \varphi \text{ harmonic} \quad //$$

So F is constant. From (†), see

$$\lim_{r\to 0} F(r) = \varphi(a)$$

$$\Rightarrow \varphi(a) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r} \varphi(\underline{x})\,dS \qquad \square$$

Tweak the above to get intuitive idea of what Laplacian measures

Proposition If $\varphi: \mathbb{R}^3 \to \mathbb{R}$ is smooth then

$$\nabla^2\varphi(\underline{a}) = \lim_{r\to 0}\frac{6}{r^2}\left[\frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r}\varphi(\underline{x})\,dS - \varphi(\underline{a})\right]$$

In particular, if φ satisfies mean value property, $\nabla^2\varphi = 0$.

Proof: Define

$$G(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r}\varphi(\underline{x})\,dS - \varphi(\underline{a})$$

continuity of φ

By previous, $\lim_{r\to 0} G(r) = 0$. Also

$$G'(r) = F'(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|<r}\nabla^2\varphi(\underline{x})\,dV$$

Note that

$$\int_{|\underline{x}-\underline{a}|<r}\nabla^2\varphi(\underline{x})\,dV = \frac{4\pi}{3}r^3\,\nabla^2\varphi(\underline{a}) + \int_{|\underline{x}-\underline{a}|<r}\left(\nabla^2\varphi(\underline{x}) - \nabla^2\varphi(\underline{a})\right)dV$$

$$= \frac{4\pi}{3}r^3\,\nabla^2\varphi(\underline{a}) + o(r^3)$$

since $\left|\int_{|\underline{x}-\underline{a}|<r}\left(\nabla^2\varphi(\underline{x}) - \nabla^2\varphi(\underline{a})\right)dV\right| \leq \frac{4\pi r^3}{3}\cdot \sup_{|\underline{x}-\underline{a}|<r}\left|\nabla^2\varphi(\underline{x}) - \nabla^2\varphi(\underline{a})\right|.$

$\left[\,\left|\int_A f\,dx\right| \leq \sup_A |f(x)|\cdot|A|\,\right]$

Hence

$$\frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|<r}\nabla^2\varphi(\underline{x})\,dV = \frac{r}{3}\nabla^2\varphi(\underline{a}) + o(r).$$

Hence $G'(r) = \frac{r}{3}\nabla^2\varphi(\underline{a}) + o(r)$.

C.F. $G'(r) = G'(0) + r\,G''(0) + o(r)$

$\Rightarrow G'(0) = 0, \quad G''(0) = \frac{1}{3}\nabla^2\varphi(\underline{a})$

Now use $G(r) = \underbrace{G(0)}_{\text{zero}} + r\,\underbrace{G'(0)}_{\text{zero}} + \frac{1}{2}r^2 G''(0) + o(r^2)$

So $G(r) = \frac{r^2}{6}\nabla^2\varphi(\underline{a}) + o(r^2)$.

$\Rightarrow \nabla^2\varphi(\underline{a}) = \frac{6}{r^2}G(r) + o(1)$.

Hence

$$\nabla^2\varphi(\underline{a}) = \lim_{r\to 0}\frac{6}{r^2}G(r) = \lim_{r\to 0}\frac{6}{r^2}\left[\int_{|\underline{x}-\underline{a}|=r}\varphi(\underline{x})\,dS\cdot\frac{1}{4\pi r^2} - \varphi(\underline{a})\right]. \qquad \square$$

Also have maximum principle for <u>harmonic functions</u>.

<u>Proposition</u> If φ is harmonic on $\Omega^* \subseteq \mathbb{R}^3$ then φ cannot have a maximum in Ω unless $\varphi = \text{const.}$ [i.e. if $\exists\, \underline{a}\in\Omega$ s.t. $\varphi(\underline{x}) \le \varphi(\underline{a})$ $\forall\, \underline{x}\in\Omega$ then $\varphi(\underline{x}) = \varphi(\underline{a})$ throughout Ω].

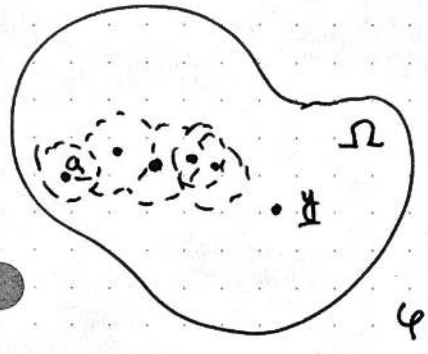
<u>Proof</u> Suppose $\exists\, \underline{a}\in\Omega$ s.t. $\varphi(\underline{a}) \ge \varphi(\underline{x})$ $\forall\, \underline{x}\in\Omega$. Then $\exists\, r>0$ s.t. $\varphi(\underline{a}) - \varphi(\underline{x}) \ge 0$ on $|\underline{x}-\underline{a}| \le r$.

By mean value property

$$\varphi(\underline{a}) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r}\varphi(\underline{x})\,dS$$

$$\Leftrightarrow \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r}\left[\varphi(\underline{a}) - \varphi(\underline{x})\right]dS = 0$$

Hence $\varphi(\underline{a}) = \varphi(\underline{x})$ on $|\underline{x}-\underline{a}| = r$. Applying same argument on any $|\underline{x}-\underline{a}| \le r' < r$ gives that $\varphi(\underline{x}) = \varphi(\underline{a})$ throughout $|\underline{x}-\underline{a}| \le r$.



Take $\underline{y}\in\Omega$ and connect to $\underline{a}$ using overlapping balls such that the centre of the $(n+1)$th ball is contained in the n^{th} ball.

By setup, $\varphi(\underline{x}) = \varphi(\underline{a})$ at centre of 2^{nd} ball, hence $\varphi(\underline{x}) = \varphi(\underline{a})$ throughout second ball by previous. [$\varphi(\underline{a})$ maximal on Ω]

* open, connected

Carry on until you reach $\underline{y}$. Hence $\varphi(\underline{a}) = \varphi(\underline{y})$ where $\underline{y} \in \Omega$ was arbitrary. $\square$
(Need compactness to make rigorous)

~ Consider $U \subset \Omega$ defined as
$U = \{ \underline{x} \in \Omega : \varphi(\underline{x}) = \varphi(\underline{a}) \}$.
By continuity of φ, U is closed in Ω.
But by previous, U is open in Ω.
Hence $U = \Omega$.

#epic

§7.5: Discrete Laplacian

If $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ define

$$\nabla^2 \varphi(\underline{a}) = \frac{\sum_{i=1}^{n} \varphi(\underline{a} + \underline{e}_i) + \sum_{i=1}^{n} \varphi(\underline{a} - \underline{e}_i)}{2n} - \varphi(\underline{a}).$$

If $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ harmonic and bounded, show $\varphi = \text{const}$.

~~$n=1$ is easy~~
~~$n=2$~~

$\boxed{\text{utterly trivial since } \mathbb{Z} \text{ has sup = max}}$

L19.1

§ 8: Cartesian tensors (in $\mathbb{R}^3$)

Throughout this section $\{\underline{e}_i\}$, $\{\underline{e}'_i\}$ will denote set of different orthonormal right-handed basis vectors aligned with some choice of coordinate axes.

§ 8.1: A closer look at vectors

For $\underline{x} \in \mathbb{R}^3$, can use some basis $\{\underline{e}_i\}$ to describe $\underline{x}$ via

$$\underline{x} = x_i \underline{e}_i .$$

Do not identify vector $\underline{x}$ with components $\{x_i\}$ since, with respect to a different basis $\{\underline{e}'_i\}$ we would get

$$\underline{x} = x'_i \underline{e}'_i .$$

Obviously we have

$$x_j \underline{e}_j = x'_j \underline{e}'_j .$$

Have $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, $\underline{e}'_i \cdot \underline{e}'_j = \delta_{ij}$. Hence

$$x'_i = \delta_{ij} x'_j = \underline{e}'_i \cdot \underline{e}'_j x'_j = \underline{e}'_i \cdot x_j \underline{e}_j = (\underline{e}'_i \cdot \underline{e}_j) x_j$$

Denote $R_{ij} = \underline{e}'_i \cdot \underline{e}_j$. Hence

$$\boxed{x'_i = R_{ij} x_j .}$$

Also have

$$x_i = \delta_{ij} x_j = \underline{e}_i \cdot \underline{e}_j x_j = \underline{e}_i \cdot x'_j \underline{e}'_j = (\underbrace{\underline{e}_i \cdot \underline{e}'_j}_{R_{ji}}) x'_j$$

$$x_i = R_{ji} x'_j$$

From previous,

$$x'_i = R_{ij} x_j = R_{ij} R_{kj} x'_k$$

$$\Leftrightarrow 0 = (\delta_{ik} - R_{ij} R_{kj}) x'_k$$

Since x'_i are arbitrary,

$$R_{ij} R_{kj} = \delta_{ik}$$

If R denotes matrix with entries R_{ij}, then

$$R^T R = R R^T = I$$

So R is <u>orthogonal</u> matrix

Note

$$x_i \underline{e}_i = x'_j \underline{e}'_j = R_{jk} x_k \underline{e}'_j = R_{ji} x_i \underline{e}'_j = (R_{ji} \underline{e}'_j) x_i$$

$\Rightarrow \underline{e}_i = R_{ji}\, \underline{e}_j'$

Since $\{\underline{e}_i\}$ right handed, $1 = \underline{e}_1 \cdot \underline{e}_2 \times \underline{e}_3$. Thus

$$1 = R_{j1} R_{k2} R_{l3} \underbrace{\underline{e}_j' \cdot \underline{e}_k' \times \underline{e}_l'}_{\substack{\varepsilon_{jkl} \\ \text{since } \{\underline{e}_i'\} \text{ is} \\ \text{right handed}}} = R_{j1} R_{k2} R_{l3}\, \varepsilon_{jkl} = \det R$$

Hence $\{R_{ij}\}$ are components of special orthogonal matrix. So R is a rotation about some axis.

<u>Moral of story</u>

The components of vector $\underline{v}$ transform according to

$$v_i' = R_{ij} v_j$$

when we go from $\{\underline{e}_i\}$ to $\{\underline{e}_j'\}$.

Call objects whose components transform like this <u>vectors</u>, or rank 1 Cartesian tensors. $R = (R_{ij}) \in SO(3)$

<u>§8.2: A closer look at scalars</u>

Fix $\underline{a}, \underline{b} \in \mathbb{R}^3$, consider

$$\sigma = \underline{a} \cdot \underline{b}$$

Then using $\{\underline{e}_i\}$, $\underline{a} = a_i \underline{e}_i$, etc.

$$\sigma = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j (\underline{e}_i \cdot \underline{e}_j) = a_i b_j \delta_{ij} = a_j b_j$$

If we use $\{\underline{e}_i'\}$, would get

$$\sigma' = a_j' b_j' = a_i' b_i'$$

Use $a_i' = R_{ij} a_j$ etc.

$$\sigma' = R_{ij} a_j R_{ik} b_k = R_{ij} R_{ik} a_j b_k = \delta_{jk} a_j b_k = a_j b_j = \sigma$$

<u>Moral of the story</u>

Objects that transform as

$$\sigma' = \sigma$$

when we change from $\{\underline{e}_i\}$ to $\{\underline{e}_j'\}$ are called <u>scalars</u>, or rank 0 Cartesian tensors.

L18.3

§8.3: A closer look at linear maps

For fixed $\underline{n} \in \mathbb{R}^3$ consider linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T: \underline{x} \mapsto \underline{y} = T(\underline{x}) = \underline{x} - (\underline{x} \cdot \underline{n})\underline{n}.$$

Really is a linear map. (check)

Using $\{\underline{e}_i\}$, $\underline{n} = n_i \underline{e}_i$, etc.

$$y_i \underline{e}_i = x_j \underline{e}_j - x_j n_j n_i \underline{e}_i$$

$$= x_j (\delta_{ij} - n_i n_j) \underline{e}_i$$

$$\Rightarrow y_i = T_{ij} x_j$$

where $T_{ij} = \delta_{ij} - n_i n_j$. These are the components of linear map T with respect to $\underline{e}_i$.

With respect to different basis $\{\underline{e}'_j\}$

$$y'_i = T'_{ij} x'_j$$

where $T'_{ij} = \delta_{ij} - n'_i n'_j$.

But, $n'_i = R_{ip} n_p$, etc.

So:

$$T'_{ij} = \delta_{ij} - R_{ip} R_{jq} n_p n_q$$

$$= R_{ip} R_{jq} (\delta_{pq} - n_p n_q)$$

$$\left[R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij} \right]$$

$$\therefore T'_{ij} = R_{ip} R_{jq} T_{pq}$$

$$\left[T' = R T R^T = R T R^{-1} \right]$$

Moral of story

Objects whose components transform as

$$T'_{ij} = R_{ip} R_{jq} T_{pq}$$

under change $\{\underline{e}_i\}$ to $\{\underline{e}'_j\}$ are called rank 2 Cartesian tensors.

§8.4: Cartesian tensors of rank n

<u>Def^n</u> A tensor of rank n has components $\underbrace{T_{ij\cdots k}}_{n \text{ indices}}$ wrt each Cartesian basis $\{\underline{e}_i\}$ such that

$$T'_{ij\cdots k} = R_{ip} R_{jq} \cdots R_{kr} T_{pq\cdots r}$$

when we go to another Cartesian basis $\{\underline{e}'_j\}$. Here $R_{ij} = \underline{e}'_i \cdot \underline{e}_j$, and are components of $R \in SO(3)$. $R_{ip} R_{jq} \delta_{pq} = \delta_{ij}$.

<u>Example</u> If vectors $\underline{u}, \underline{v}, \ldots, \underline{w}$ have components $u_i, v_j, \ldots, w_k$ wrt $\{\underline{e}_i\}$, define
n in total

$$\underbrace{T_{ij\cdots k}}_{n \text{ indices}} = u_i v_j \cdots w_k$$

$$T'_{ij\cdots k} = u'_i v'_j \cdots w'_k = R_{ip} u_p R_{jq} v_q \cdots R_{kr} w_r$$
$$= R_{ip} R_{jq} \cdots R_{kr} T_{pq\cdots r}$$

So $T_{ij\cdots k}$ are components of n^{th} rank tensor.

<u>Example</u> Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

independently of any basis. So by definition $\delta'_{ij} = \delta_{ij}$. However

$$R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij} = \delta'_{ij}$$

Hence $\delta'_{ij} = R_{ip} R_{jq} \delta_{pq}$ so δ_{ij} define components of rank 2 tensor.

<u>Example</u> The Levi-Civita symbol defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ even perm of } (1,2,3) \\ -1 & \text{if } \quad " \quad \text{odd} \quad " \quad " \\ 0 & \text{otherwise} \end{cases}$$

independently of any basis, so $\varepsilon'_{ijk} = \varepsilon_{ijk}$. But

$$R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} = \det(R)\, \varepsilon_{ijk} \qquad \text{(IA V+M)}$$
$$= \varepsilon_{ijk} = \varepsilon'_{ijk}$$

Hence $\varepsilon'_{ijk} = R_{ip} R_{jq} R_{kr} \varepsilon_{pqr}$ and ε_{ijk} defines the components of a rank 3 tensor.

<u>Example</u> Not every array of numbers defines a tensor, e.g. if

$$A_{ij} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \pi & 1 & 3 \\ 7 & -4 & 10^{10} \end{pmatrix}$$

in some basis $\{\underline{e}_i\}$, but

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in all other Cartesian bases, then A_{ij} are <u>NOT</u> components of rank 2 tensor.

<u>Example</u> Experimental evidence suggests that the current $\underline{J}$ produced inside a material by an electric field $\underline{E}$ depends linearly on it, i.e. $\underline{J} = \sigma \underline{E}$, or $J_i = \sigma_{ij} E_j$ for some array of numbers $\{\sigma_{ij}\}$, the conductivity.

On changing basis, since J_i, E_j are components of vectors,

$$\sigma_{ij}' E_j' = J_i' = R_{ip} J_p = R_{ip} \sigma_{pq} E_q = R_{ip} \sigma_{pq} R_{jq} E_j'$$

since $E_i' = R_{ij} E_j \iff E_i = R_{ji} E_j'$.

This must hold for any $\{E_j'\}$, we conclude

$$\sigma_{ij}' = R_{ip} R_{jq} \sigma_{pq}$$

so σ_{ij} components of <u>rank 2 tensor</u>. cf. Quotient theorem

---

Can add together tensors of same rank, e.g. if $A_{ij\cdots k}$ and $B_{ij\cdots k}$ are components of rank n tensors, define

$$(A+B)_{ij\cdots k} = A_{ij\cdots k} + B_{ij\cdots k}$$

Also for scalar α,

$$(\alpha A)_{ij\cdots k} = \alpha A_{ij\cdots k}$$

CHECK $(A+B)_{ij\cdots k}$ and $(\alpha A)_{ij\cdots k}$ are components of tensors.

If $U_{ij\cdots k}$ and $V_{pq\cdots r}$ are components of tensors of rank m and n respectively, define tensor product

$$(U \otimes V)_{ij\cdots k pq\cdots r} = U_{\underbrace{ij\cdots k}_{m \text{ indices}}} V_{\underbrace{pq\cdots r}_{n \text{ indices}}}$$

L 20.3

So $(U \otimes V)$ defines tensor of rank $n+m$. Indeed,

Mick

$$
\begin{aligned}
(U \otimes V)'_{ij\cdots kpq\cdots r} &= U'_{i\cdots k} V'_{p\cdots r} \\
&= R_{ia} \cdots R_{kc} U_{a\cdots c} R_{pd} \cdots R_{rf} V_{d\cdots f} \\
&= R_{ia} \cdots R_{kc} R_{pd} \cdots R_{rf} (U \otimes V)_{a\cdots cd\cdots f}
\end{aligned}
$$

so tensor transformation law holds.

For tensor of rank n, can define tensor of rank $n-2$ by <u>contraction</u>

E.g. define contraction on indices i, j by

$$\delta_{ij} \underbrace{T_{ijk\cdots l}}_{n \text{ indices}} = T_{ii\underbrace{k\cdots l}_{n-2 \text{ free indices}}}$$

Check RHS defines a tensor of rank $n-2$

$$
\begin{aligned}
T'_{iik\cdots l} &= R_{ip} R_{iq} R_{kr} \cdots R_{ls} T_{pqr\cdots s} \\
&= \delta_{pq} R_{kr} \cdots R_{ls} T_{pqr\cdots s} \\
&= \underbrace{R_{kr} \cdots R_{ls}}_{n-2\ R\text{s}} T_{ppr\cdots s}
\end{aligned}
$$

So $T_{ii\cdots kl}$ define components of rank $n-2$ tensor. Can contract on any two indices, obtain tensor.

L21.1

Say $T_{ijk\cdots l}$ is symmetric in (i,j) if $T_{ij\cdots l} = T_{ji\cdots l}$. Really is a property of tensor, since

$$\begin{aligned} T'_{ij\cdots l} &= R_{ip}R_{jq}\cdots R_{lr}\,T_{pq\cdots r} \\ &= R_{ip}R_{jq}\cdots R_{lr}\,T_{qp\cdots r} \\ &= R_{iq}R_{jp}\cdots R_{lr}\,T_{pq\cdots r} \\ &= R_{jp}R_{iq}\cdots R_{lr}\,T_{pq\cdots r} \\ &= T'_{ji\cdots l} \end{aligned}$$

so symmetry in (i,j) retained under change of basis.

Say $T_{ij\cdots k}$ is antisymmetric in (i,j) if $T_{ij\cdots k} = -T_{ji\cdots k}$.

Say $T_{ij\cdots k}$ is <u>totally</u> symmetric (antisymmetric) if it is symmetric (antisymmetric) in each pair of indices.

<u>Example</u> If $\underline{a}$ is a vector, then $a_i a_j a_k$ is totally symmetric of rank 3.

- Kronecker delta δ_{ij} totally symmetric.
- ε_{ijk} is totally antisymmetric.

In fact, there are no totally antisymmetric tensors on $\mathbb{R}^3$ of rank > 3 [that are non-zero]. Indeed: note if $T_{ij\cdots k}$ is totally antisymmetric, then if any two indices match $T_{ij\cdots k} = 0$.

PHP ① ② ③ pigeonholes

$i, j, k, \ldots, l$  pigeons

So if we have four or more indices, two will match and hence $T_{ij\cdots k}$ will be identically zero.

If rank = 3, T_{ijk} can only have $3! = 6$ non-zero components. If

$$T_{123} = T_{231} = T_{312} = \lambda$$

then by antisymmetry

$$T_{132} = T_{321} = T_{213} = -\lambda.$$

Hence $T_{ijk} = \lambda\,\varepsilon_{ijk}$.

§ 8.5: Tensor Calculus

Recall vector field $\underline{v} = \underline{v}(\underline{x})$ gives you a vector (rank 1 tensor) at each $\underline{x} \in \mathbb{R}^3$. Tensor field of rank n,

$$\underbrace{T_{ij\cdots k}}_{n \text{ indices}}(\underline{x})$$

gives you a n^{th} rank tensor at each $\underline{x} \in \mathbb{R}^3$.

Recall $x'_i = R_{ij} x_j \quad \Leftrightarrow \quad x_j = R_{ij} x'_i$

So: $$\frac{\partial x_k}{\partial x'_j} = \frac{\partial}{\partial x'_j}\left(R_{ik} x'_i \right) = R_{ik}\, \delta_{ij} = R_{jk}$$

just some constants

Hence $$\frac{\partial}{\partial x'_i} = \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = R_{ij} \frac{\partial}{\partial x_j}.$$

Proposition If $T_{ij\cdots k}(\underline{x})$ is a tensor field of rank n, then

$$\underbrace{\left(\frac{\partial}{\partial x_p}\right) \cdots \left(\frac{\partial}{\partial x_q}\right)}_{m \text{ of these}} T_{\underbrace{i\cdots j}_{n \text{ indices}}}(\underline{x}) \qquad (+)$$

is a tensor field of rank m+n.

Proof Denote (+) by $A_{p\cdots q i \cdots j}$.

$$A'_{p\cdots q i\cdots j} = \left(\frac{\partial}{\partial x'_p}\right) \cdots \left(\frac{\partial}{\partial x'_q}\right) T'_{i\cdots j}$$

$$= \left(R_{pa} \frac{\partial}{\partial x_a}\right) \cdots \left(R_{qb} \frac{\partial}{\partial x_b}\right) R_{ic} \cdots R_{jd}\, T_{c\cdots d}$$

$$= \underbrace{R_{pa} \cdots R_{qb} R_{ic} \cdots R_{jd}}_{m+n\ Rs} A_{a\cdots bc\cdots d}$$

$\square$

Examples If $\varphi : \mathbb{R}^3 \to \mathbb{R}$ scalar field, then

$$[\underline{\nabla} \varphi]_i = \partial \varphi / \partial_i$$

is a rank $0+1=1$ tensor field.

If $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ vector field,

$$\underline{\nabla} \cdot \underline{v} = \partial v_i / \partial x_i = \delta_{ij} \underbrace{\partial v_i / \partial x_j}_{\text{rank 2 tensor}}$$

contracting

So $\underline{\nabla}\cdot\underline{v}$ is a tensor field of rank 0.

Look at curl

$$[\underline{\nabla}\times\underline{v}]_i = \varepsilon_{ijk}\,\partial v_k/\partial x_j$$

$$\Big\{ = \underset{\uparrow\ \text{contractions}}{\delta_{aj}\delta_{bk}}\ \underbrace{\varepsilon_{ijk}\,\partial v_b/\partial x_a}_{\text{rank 5 tensor}} \Big\}$$

$\therefore\ [\underline{\nabla}\times\underline{v}]_i$ is a tensor field of rank 1.

<u>or</u> $[\underline{\nabla}\times\underline{v}]'_i = \varepsilon'_{ijk}\,\partial v'_k/\partial x'_j$

$$= R_{ia}R_{jb}R_{kc}\,\varepsilon_{abc}\,R_{jd}R_{ke}\,\partial v_e/\partial x_d$$

$$= R_{ia}\underbrace{R_{jb}R_{jd}}_{\delta_{bd}}\,\underbrace{R_{kc}R_{ke}}_{\delta_{ce}}\,\varepsilon_{abc}\,\partial v_e/\partial x_d$$

$$= R_{ia}\,\varepsilon_{abc}\,\partial v_c/\partial x_b$$

$$= R_{ia}\,[\underline{\nabla}\times\underline{v}]_a \qquad \ddot{\smile}$$

Recall divergence thm

$$\int_V \frac{\partial v_i}{\partial x_i}\,dV = \int_{\partial V} v_i n_i\,dS$$

<u>Proposition</u> If $T_{i\ldots j\ldots k}(\underline{x})$ a tensor field, then

$$\int_V \frac{\partial}{\partial x_j} T_{i\cdots j\cdots k}(\underline{x})\,dV = \int_{\partial V} T_{i\cdots j\cdots k}(\underline{x})\,n_j\,dS$$

So $\left[V, \frac{\partial}{\partial x_j}\right] \longrightarrow [\partial V, n_j]$.

<u>Proof</u> For $a_i,\ldots,b_k$ components of constant vectors, consider vector field $v_j = a_i \cdots b_k\, T_{i\cdots j\cdots k}(\underline{x})$.

$\uparrow$ no j

By div thm,

$$\int_V \frac{\partial v_j}{\partial x_j}\,dV = \int_{\partial V} v_j n_j\,dS$$

$$\Rightarrow a_i\cdots b_k \int_V \frac{\partial}{\partial x_j} T_{i\cdots j\cdots k}\,dV = a_i\cdots b_k \int_{\partial V} T_{i\cdots j\cdots k}\,n_j\,dS$$

True for all choices of $a_i,\ldots,b_k$, so result follows. $\square$

§ 8.6: Rank 2 tensors

For any rank 2 tensor T_{ij}, can write

$$T_{ij} = \underbrace{\tfrac{1}{2}(T_{ij} + T_{ji})}_{S_{ij}} + \underbrace{\tfrac{1}{2}(T_{ij} - T_{ji})}_{A_{ij}}$$

Have S_{ij} symmetric, A_{ij} antisymmetric.

S_{ij} has "6" indep components $(i \geqslant j)$.

A_{ij} has "3" indep components $(i > j)$.

Good, since "9" = "6" + "3".

Maybe write A_{ij} in terms of some vector?

<u>Proposition</u> For any rank 2 tensor T_{ij}, there is a unique decomposition into symmetric and antisymmetric parts, of form

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k$$

for vector $\underline{\omega}$ defined by $\omega_i := \frac{1}{2}\varepsilon_{ijk}T_{jk}$, and S_{ij} symmetric.

<u>Proof</u> As before write

$$T_{ij} = \underbrace{\tfrac{1}{2}(T_{ij} + T_{ji})}_{S_{ij}} + \underbrace{\tfrac{1}{2}(T_{ij} - T_{ji})}_{A_{ij}}$$

Then
$$\begin{aligned} \varepsilon_{ijk}\omega_k &= \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}T_{lm} \\ &= \tfrac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})T_{lm} \\ &= \tfrac{1}{2}(T_{ij} - T_{ji}) \end{aligned}$$

To see uniqueness, suppose

$$S_{ij} + \varepsilon_{ijk}\omega_k = \tilde{S}_{ij} + \varepsilon_{ijk}\tilde{\omega}_k.$$

Taking symmetric parts of each side, get

$$S_{ij} = \tilde{S}_{ij} \quad \text{and} \quad \varepsilon_{ijk}\omega_k = \varepsilon_{ijk}\tilde{\omega}_k \Rightarrow \omega_k = \tilde{\omega}_k.$$ □

<u>Example</u> Suppose a point $\underline{x}$ in a deformable body undergoes a displacement $\underline{u}(\underline{x})$. If two points $\underline{x}, \underline{x}+\delta\underline{x}$ were initially separated by $\delta\underline{x}$, then afterwards they are separated by

$$[\underline{x} + \delta\underline{x} + \underline{u}(\underline{x}+\delta\underline{x})] - [\underline{x} + u(\underline{x})]$$
$$= \delta\underline{x} + \underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$$

<u>Change</u> of displacement is

$$\underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$$

Using Cartesian basis $\{\underline{e}_i\}$

$$u_i(\underline{x}+\delta\underline{x}) - u_i(\underline{x})$$

$$= \frac{\partial u_i}{\partial x_j}\delta x_j + o(\delta\underline{x})$$

From previous

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \varepsilon_{ijk}\omega_k$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ < linear strain tensor >

and $\omega_i = \frac{1}{2}\varepsilon_{ijk}\frac{\partial u_j}{\partial x_k} = -\frac{1}{2}[\underline{\nabla}\times\underline{u}]_i$.

So $u_i(\underline{x}+\delta\underline{x}) - u_i(\underline{x})$

$$= e_{ij}\delta x_j + [\delta\underline{x}\times\underline{\omega}]_i + o(\delta\underline{x})$$

tells us how body stretches or compresses

local rotation

Suppose a body V rotates about origin with angular velocity $\underline{\omega}$.

Then velocity of point $\underline{x}$ in V is $\underline{\omega}\times\underline{x}$.

If density is $\rho(\underline{x})$, angular momentum given by

$$\underline{L} = \int_V \rho(\underline{x})\,\underline{x}\times\underline{v}\,dV$$

$$= \int_V \rho(\underline{x})[\underline{x}\times(\underline{\omega}\times\underline{x})]\,dV$$

Using Cartesian basis $\{\underline{e}_i\}$

$$L_i = \int_V \rho(\underline{x})[x_k x_k \omega_i - x_i x_j \omega_j]\,dV$$

$dx_1 dx_2 dx_3$

$$= I_{ij}\omega_j$$

where we have the <u>inertia tensor</u>

$$I_{ij} = \int_{\mathcal{V}} \rho(\underline{x}) [x_k x_k \delta_{ij} - x_i x_j] \, dV$$

and $\mathcal{V}^F = \{(x_1, x_2, x_3) : \underline{x} = x_i \underline{e}_i \in V\}$.

If we had used a different basis $\{\underline{e}_i'\}$ then

$$I_{ij}' = \int_{\mathcal{V}'} \rho(\underline{x}) [x_k' x_k' \delta_{ij} - x_i' x_j'] \, dV \leftarrow dx_1' dx_2' dx_3'$$

where $\mathcal{V}' = \{(x_1', x_2', x_3') : \underline{x} = x_i' \underline{e}_i' \in V\}$.

Make change of variables

$x_k x_k = x_k' x_k'$

$$x_i' = R_{ij} x_j \quad \Rightarrow \quad |J| = 1$$

Hence

$$I_{ij}' = \int_{\mathcal{V}} \rho(\underline{x}) [x_k x_k \delta_{pq} - x_p x_q] R_{ip} R_{jq} \, dV \leftarrow dx_1 dx_2 dx_3$$

$$= R_{ip} R_{jq} \int_{\mathcal{V}} \rho(\underline{x}) [x_k x_k \delta_{pq} - x_p x_q] \, dV$$

$$= R_{ip} R_{jq} I_{pq}$$ ☺

So I_{ij} is a <u>symmetric</u>, rank 2 tensor.

<u>Example</u>

Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ of uniform density.

Use Cartesian basis aligned with x, y, z axes.

Using scaled spherical polars,

$$x_1 = a r \cos\phi \sin\theta, \quad x_2 = b r \sin\phi \sin\theta, \quad x_3 = c r \cos\theta.$$

Note if $i \neq j$ then

$$I_{ij} = -\int_{\mathcal{V}} \rho x_i x_j \, dV = 0 \quad \text{by symmetry.}$$

$$I_{11} = \int_{\cancel{r}=0}^{1} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \cdot r^2 \sin\theta \, \rho \, (x_1^2 + x_2^2 + x_3^2 - x_1^2) \, abc$$

$$= \rho abc \int_0^1 dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; r^2 (b^2 \sin^2\phi \sin^2\theta + c^2 \cos^2\theta) \, r^2 \sin\theta$$

$$\vdots$$

$$= \frac{M}{5} (b^2 + c^2)$$ ☺ doable!

Similarly $I_{22} = \frac{1}{5} M (a^2 + c^2)$ and $I_{33} = \frac{1}{5} M (b^2 + c^2)$.

L22.4

i.e. $(I_{ij}) = \frac{M}{5}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$

If $a=b=c=R$ get $I_{ij} = \frac{2}{5}MR^2\delta_{ij}$ (sphere).

Proposition For every rank 2 symmetric tensor T_{ij} there exists a choice of basis $\{\underline{e}_i\}$ such that T_{ij} is diagonal.

Proof By IA V+M, real symmetric matrix can be diagonalised using orthogonal transformation R.

Take $\det R = 1$ wlog $\square$

§ 8.7: Isotropic Tensors

Say tensor $T_{ij\cdots k}$ is <u>isotropic</u> if

$$T'_{ij\cdots k} = R_{ip} R_{jq} \cdots R_{kr} T_{pq\cdots r} = T_{ij\cdots k}$$

for all choices of R.

<u>Example</u> i) Scalars are isotropic by definition

ii) Kronecker delta $\delta'_{ij} = R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij}$

iii) Levi-Civita symbol $\varepsilon'_{ijk} = R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} = \varepsilon_{ijk} \left(\cancelto{1}{\det R}\right)$

Can classify all isotropic tensors on $\mathbb{R}^3$.

<u>Proposition</u> Most general isotropic tensor on $\mathbb{R}^3$ is

a) Rank 0: all of them

b) Rank 1: none aside from $\underline{0}$

c) Rank 2: $\alpha \delta_{ij}$ (α a scalar)

d) Rank 3: $\beta \varepsilon_{ijk}$ (β a scalar)

e) Rank 4: $\alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$

f) Rank $n > 4$: linear combinations of products of δ and ε

[e.g. $\delta_{ij} \varepsilon_{pqr}$ is isotropic rank 5 tensor]

<u>Proof</u> a) By def^n.

b) Suppose v_i is isotropic, so

$$v_i = R_{ij} v_j$$

for any choice of R. Try

$$R = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

Then $v_1 = -v_1$, $v_2 = -v_2$.

Doing other rotations get $v_i \equiv 0$.

c) Suppose T_{ij} is isotropic, so

$$T_{ij} = R_{ip} R_{jq} T_{pq}$$

for any choice of R.

L23.2

Take $R = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}$.

Then $T_{13} = R_{12} R_{33} T_{23} = T_{23}$,

$T_{23} = R_{21} R_{33} T_{13} = -T_{13}$.

Hence $T_{13} = T_{23} = 0$. Also get

$T_{11} = R_{12} R_{12} T_{22} = T_{22}$,

$T_{22} = R_{21} R_{21} T_{11} = T_{22}$.

By doing other rotations get $T_{ij} = \alpha \delta_{ij}$.

d) Same idea, more indices. (alright there mate)

e) Proof omitted.

f) " "

Let $V_R = \{\underline{x} : |\underline{x}| \leq R\}$, consider

$$T_{ij\cdots k} = \int_{V_R} f(r)\, x_i x_j \cdots x_k \, dV$$

Note $f(r)$, V_R and dV all invariant wrt rotations.

By def^n

$$T'_{ij\cdots k} = \int_{V_R} f(r)\, x'_i x'_j \cdots x'_k \, dV$$

$$= \int_{V_R} f(r)\, R_{ip} R_{jq} \cdots R_{kr}\, x_i x_j \cdots x_k \, dV$$

Set $y_i = R_{ip} x_p, \ldots$ etc to get (change of variables)

$$= \int_{V_R} f(r)\, y_i \cdots y_k \, dV$$

since the $\{y_i\}$ are dummy variables, this is

$$= \int_{V_R} f(r)\, x_i \cdots x_k \, dV.$$

Hence $T_{ij\cdots k}$ is isotropic ☺.

Take $R \to \infty$ to get $\int_{V_R} \longrightarrow \int_{\mathbb{R}^3}$.

Example

$$T_{ij} = \int_{\mathbb{R}^3} e^{-r^5} x_i x_j \, dV$$

Then T_{ij} isotropic, hence $T_{ij} = \alpha \delta_{ij}$.

Contract on i and j

$$\alpha \delta_{ii} = 3\alpha = \int_{\mathbb{R}^3} e^{-r^5} r^2 \, dV = 4\pi \int_0^\infty r^4 e^{-r^5} \, dr = \frac{4\pi}{5}$$

and hence $T_{ij} = \frac{4}{15}\pi \delta_{ij}$.

Example

Inertia tensor for ball of radius R, density ρ_0.

$$I_{ij} = \int_{|\underline{x}| \leq R} \rho_0 (\underbrace{x_k x_k}_{r^2} \delta_{ij} - x_i x_j) \, dV$$

This is some isotropic tensor, via

$$\rho_0 \delta_{ij} \int_{|\underline{x}| \leq R} r^2 \, dV - \rho_0 \int_{|\underline{x}| \leq R} x_i x_j \, dV.$$

So $I_{ij} = \alpha \delta_{ij}$ for some α. As before,

$$3\alpha = \int_{|\underline{x}| \leq R} \rho_0 (3r^2 - r^2) \, dV = 4\pi\rho_0 \int_0^R 2r^4 \, dr$$

$$\therefore \alpha = \frac{4\pi}{3}\rho_0 R^3 \, \frac{2}{5} R^2 = \frac{2}{5} M R^2$$

Compare to ellipsoid from last lecture.

§ 9.8 Tensors, multi-linear maps & quotient thm

If T_{ij} a tensor, consider bilinear map $t: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $t(\underline{a}, \underline{b}) = T_{ij} a_i b_j$ $\quad (\underline{a}, \underline{b} \in \mathbb{R}^3)$

RHS is scalar, so doesn't depend on choice of basis. So

T_{ij}, rank 2 tensor $\longrightarrow$ bilinear map t

Suppose now $t: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a bilinear map. So using basis $\{\underline{e}_i\}$, can define array of numbers T_{ij} via

$$t(\underline{a}, \underline{b}) = a_i b_j \, t(\underline{e}_i, \underline{e}_j)$$
$$:= a_i b_j T_{ij}$$

If we instead use $\{\underline{e}_i'\}$, then get new array of numbers (if $\underline{e}_i' = R_{ip} e_p$ say)

$$T_{ij}' = t(\underline{e}_i', \underline{e}_j') = R_{ip} R_{jq} \, t(\underline{e}_p, \underline{e}_q) = R_{ip} R_{jq} T_{pq} \quad \ddot\smile$$

So T_{ij} are components of a rank 2 tensor.

bilinear map $t \longrightarrow T_{ij}$, rank 2 tensor

Get one-to-one correspondence between bilinear maps from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and rank 2 tensors. Holds more generally, e.g. if $t: \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n \text{ copies}} \longrightarrow \mathbb{R}$ is multilinear map, define

$$T_{i \ldots j} = t(\underline{e}_i, \ldots, \underline{e}_j)$$ to obtain rank n tensor.

Proposition If $T_{i\ldots j p \ldots q}$ are an array of numbers defined in each Cartesian coordinate system and

$$v_{i\ldots j} := T_{i\ldots j \, p \ldots q} \, u_{p\ldots q}$$

is a tensor for every tensor $u_{p\ldots q}$ then $T_{i\ldots j p\ldots q}$ is a tensor.

Proof Let $u_{p\ldots q} = c_p \cdots d_q$ for some constant vectors $\underline{c}, \ldots, \underline{d}$. Then $v_{i\ldots j} = T_{i\ldots j\, p\ldots q} c_p \cdots d_q$ is a tensor. So for vectors $\{\underline{a}, \ldots, \underline{b}\}$ we know

$$v_{i\ldots j} a_i \cdots b_j = T_{i\ldots j p \ldots q} c_p \cdots d_q a_i \cdots b_j$$

must be a scalar. So have well-defined multilinear map

$$(\underline{a}, \ldots, \underline{b}, \underline{c}, \ldots, \underline{d}) \longmapsto T_{i\ldots j p\ldots q} a_i \cdots b_j c_p \cdots d_q$$

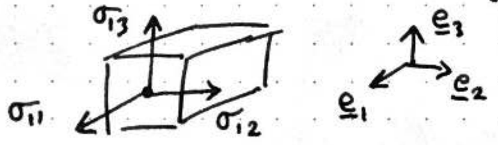
By previous discussion, $T_{i \cdots j p \cdots q}$ are components of a tensor. □

<u>Example</u> Linear strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Experiments suggest that the stresses (internal forces) experienced by a body depend linearly on the strain tensor.

Stress is measured by Cauchy streess tensor σ_{ij}



So $\exists$ array of numbers $(3^4 = 81)$ c_{ijkl} such that $\sigma_{ij} = c_{ijkl}\, e_{kl}$.

Cannot invoke quotient theorem as proven above because e_{kl} is not completely arbitrary (it's symmetric). However if

$$c_{ijkl} = c_{ijlk}$$

then we can (see ES4).

<u>If</u> material isotropic, then

$$c_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \beta\,\delta_{ik}\delta_{jl} + \gamma\,\delta_{il}\delta_{jk}$$

$$\Rightarrow \sigma_{ij} = \lambda\,\delta_{ij}\,e_{kk} + \beta\,e_{ij} + \gamma\,e_{ji}$$

$$= \lambda\,\delta_{ij}\,e_{kk} + 2\mu e_{ij}$$

This is Hooke's law in 3D alright there mate

where $2\mu = \beta + \gamma$.

Contract on (i,j)

$$\sigma_{kk} = (3\lambda + 2\mu)\,e_{kk}$$

$$\Rightarrow 2\mu\, e_{ij} = \sigma_{ij} - \left(\lambda / 3\lambda + 2\mu\right)\delta_{ij}\,\sigma_{kk}$$

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