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Part IB Analysis and Topology

1. Uniform convergence
2. Uniform continuity
3. Metric spaces
4. Completeness and the CMT
5. Topological spaces
6. Connectedness
7. Compactness
8. Differentiation

Prerequisites: IA Analysis I

Books: Burk & Burk: A Second Course in Math Analysis
Sutherland: Metric and Topological Spaces
Rudin: Principles of Math Analysis

Ex. Sheets: 4

Notes: no notes

1. Uniform Convergence

recall: a $\underset{\substack{\uparrow \\ \text{real/complex}}}{\text{scalar}}$ sequence (x_n) converges to scalar x if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \ n \geqslant N \Rightarrow |x - x_n| < \varepsilon$$

This makes the notion of "getting close to" precise.

<u>Def</u> Let (f_n) be a sequence of scalar-valued functions on a set S. Let f be a " on S.

We say <u>(f_n) converges to f pointwise on S</u> ($f_n \to f$ ptwise on S as $n \to \infty$) if, for every $x \in S$,

$(f_n(x))_n$ converges to $f(x)$

Formally, $\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant$ ~~#~~ N

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

<u>Note</u> N depends on ε and x.

<u>Examples</u> ① $S = [0,1]$, $f_n(x) = x^n$, $x \in S$, $n \in \mathbb{N}$

[recall: if $z \in \mathbb{C}$, $|z| < 1$, then $z^n \to 0$ as $n \to \infty$]

So $f_n \to f$ ptwise on S, where $f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$

<u>Note</u> f_n are cts (even smooth) yet f is not

② $S=[0,\infty)$, $f_n(x)=x^2e^{-nx}$, $x\in S$, $n\in\mathbb{N}$

[recall $\exp(z)=\sum_{n=0}^{\infty} z^n/n!$ $z\in\mathbb{C}$; diff'ble $\exp'=\exp$
$\exp(z+w)=\exp z\cdot\exp w$ has period $2\pi i$]

exp on $\mathbb{R}$ is a diff'ble bijection $\mathbb{R}\to(0,\infty)$ with diff'ble inverse called log

For $a>0$, $z\in\mathbb{C}$, let $a^z=\exp(z\log a)$

Let $e=\exp(1)=\lim_{n\to\infty}(1+\frac{1}{n})^n$, so $e^z=\exp(z)$]

$$0\le x^2e^{-nx}=\frac{x^2}{e^{nx}}=\frac{x^2}{1+(nx)+\frac{(nx)^2}{2}+\dots}\le\frac{x^2}{nx}=\frac{x}{n}\ \nwarrow_{x\neq 0}$$

So $0\le f_n(x)\le x/n$ $\forall x\in S$, $\forall n\in\mathbb{N}$

Fix $x\in S$, $x/n\to 0$ as $n\to\infty$, so $f_n(x)\to 0$ as $n\to\infty$

So $f_n\to 0$ ptwise on S.

__Def__ Let S, (f_n), f be as before. Say __(f_n) converges uniformly to f on S__ ($f_n\to f$ unif on S) if

$\forall\varepsilon>0, \exists N\in\mathbb{N}, \forall x\in S, \forall n\in\mathbb{N}, n\ge N \Rightarrow |f_n(x)-f(x)|<\varepsilon$

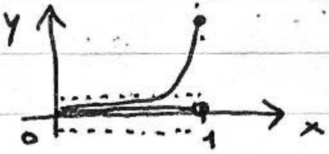
(pt $\forall\varepsilon>0, \forall x\in S, \exists N\in\mathbb{N}, \forall n\in\mathbb{N}, n\ge N\Rightarrow |f_n(x)-f(x)|<\varepsilon$)

__Note__ For uniform convergence N depends on ε but not on x.

__Picture__

__Examples__ ① $S=[0,1]$, $f_n(x)=x^n$, $x\in S$, $n\in\mathbb{N}$

$$f(x)=\begin{cases}0 & \text{for } 0\le x<1\\ 1 & \text{for } x=1\end{cases}$$



Let $\varepsilon=1/2$. Let $n\in\mathbb{N}$ be arbitrary.
Since $f_n(1)=1$ and f_n is continuous, there is $x<1$ s.t. $|f_n(x)-f_n(1)|<1/2$.

So $f_n(x)>1/2$, so $|f_n(x)-f(x)|\not<1/2$.

So $f_n\not\to f$ uniformly.

Strategy: Q: does (f_n) converge uniformly?

__Step 1__: " pointwise?

__Step 2__: If $(f_n)\to f$ ptwise, need only check if $(f_n)\to f$ unif or not.

② $S=[0,\infty)$, $f_n(x)=x^2e^{-nx}$, $x\in S$, $n\in\mathbb{N}$

We know $f_n\to 0$ ptwise.

$$0 \leq x^2 e^{-nx} = \frac{x^2}{1+(nx)+\frac{(nx)^2}{2}+\dots} \leq \frac{x^2}{(nx)^2/2} = \frac{2}{n^2} \quad (x>0)$$

So $0 \leq f_n(x) \leq 2/n^2 \ \forall x \in S, \forall n \in \mathbb{N}$.

Given $\varepsilon > 0$, since $2/n^2 \to 0$ as $n \to \infty$, $\exists N \in \mathbb{N}, \forall n \geq N$, $\Rightarrow 2/n^2 < \varepsilon$.

So $\forall x \in S, \forall n \in \mathbb{N}$ if $n \geq N$ then $|f_n(x)| \leq 2/n^2 < \varepsilon$.

So $f_n \to 0$ unif on S.

<u>Theorem 1</u> Let S be a $\overset{\uparrow}{\text{sub}}$ set $\overset{\uparrow}{\text{of } \mathbb{C}}$, $f_n (n \in \mathbb{N})$, f functions from S to $\mathbb{R}$ or $\mathbb{C}$ such that $f_n \to f$ unif.

If f_n is cts at a $\forall n \in \mathbb{N}$, then f is cts at a.

So if f_n is cts $\underset{\forall n \in \mathbb{N}}{\uparrow}$, then f is cts.

L2.1

Thm 1 Let S be a subset of $\mathbb{R}$ or $\mathbb{C}$. Assume $f_n \to f$ unif. on S. Let ~~x~~ $a \in S$. If $\forall n$, f_n is cts at a, then f is cts at a. So if f_n is continuous on S $\forall n$, then f is continuous on S.

[recall: <u>f cts at a</u> means that $\forall \varepsilon > 0, \exists \delta > 0$, s.t.

$$\forall x \in S,\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

informally: $x \approx a \Rightarrow f(x) \approx f(a)$

<u>f cts on S</u> means f is cts at a $\forall a \in S$

So $\forall a \in S, \forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall x \in S$,

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

Note δ depends on ε and a]

<u>Proof</u> [Idea: choose large n s.t. $f_n(x) \approx f(x)$ $\forall x \in S$

$x \approx a \Rightarrow f_n(x) \approx f_n(a)$

So $f(x) \underset{\substack{\uparrow \\ \forall x \in S}}{\approx} f_n(x) \underset{\substack{\uparrow \\ \forall x \approx a}}{\approx} f_n(a) \approx f(a)$ "3ε-proof"]

Fix $\varepsilon > 0$. Since $f_n \to f$ unif. on S, $\exists N \in \mathbb{N}$ s.t. $\forall x \in S$, $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$. (*)

Fix $n \in \mathbb{N}$ s.t. $n \geq N$, e.g. $n = N$.

f_n is cts at a, so $\exists \delta > 0$ s.t. $\forall x \in S$,

$$|x-a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon \qquad (**)$$

Now for any $x \in S$, if $|x-a| < \delta$ then

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &\leq \underset{\text{by } (*)}{\varepsilon} + \underset{\text{by } (**)}{\varepsilon} + \underset{\text{by } (*)}{\varepsilon} \end{aligned}$$

☐

<u>Remarks</u> 1. This gives new proof that x^n does not converge unif on $[0,1]$ to $f(x) = \begin{cases} 0, & x < 1, \\ 1, & x = 1. \end{cases}$

2. It's <u>not</u> true that if f_n is diff'ble at a $\forall n$ & $f_n \to f$ unif., then f is diff'ble at a. But see Thm 4.

Thm 2 Let $f_n : [a,b] \longrightarrow \mathbb{R}$ be (Riemann) integrable $\forall n \in \mathbb{N}$. If $f_n \to f$ unif on $[a,b]$, then f is integrable and $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$ as $n \longrightarrow \infty$.

[recall: $f : [a,b] \to \mathbb{R}$ is integrable means f is bounded on $[a,b]$ and $\underline{\int_a^b} f = \overline{\int_a^b} f$.

dissection $\mathcal{D} : a = x_0 < x_1 < \dots < x_n = b$

upper sum $\mathcal{U}_\mathcal{D}(f) = \sum (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f$

lower sum $\mathcal{L}_\mathcal{D}(f) = \sum (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f$

lower integral $\underline{\int_a^b} f = \sup_\mathcal{D} \mathcal{L}_\mathcal{D}(f)$

upper integral $\overline{\int_a^b} f = \inf_\mathcal{D} \mathcal{U}_\mathcal{D}(f)$

always $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

Riemann's criterion: f integrable $\iff \forall \varepsilon > 0 \; \exists \mathcal{D}$ s.t.
$\mathcal{U}_\mathcal{D}(f) - \mathcal{L}_\mathcal{D}(f) < \varepsilon$

inequalities: $\left| \int_a^b f \right| \leq \int_a^b |f| \leq (b-a) \sup_{[a,b]} |f|$]

Proof f bounded? Can choose $n \in \mathbb{N}$ s.t.
$|f_n(x) - f(x)| < 1 \quad \forall x \in [a,b]$

f_n is bounded, so $\exists M, \forall x \in [a,b], |f_n(x)| \leq M$

So $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq M + 1 \quad \forall x \in [a,b]$

f integrable? Fix $\varepsilon > 0$. Choose $n \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| \leq \varepsilon$ $\forall x \in [a,b]$. f_n integrable, so $\exists \mathcal{D}$ of $[a,b]$ s.t.
$\mathcal{U}_\mathcal{D}(f_n) - \mathcal{L}_\mathcal{D}(f_n) < \varepsilon$

Say $\mathcal{D}$ is $a = x_0 < x_1 < \dots < x_m = b$.
For each $k = 1, \dots, m$, $\forall x \in [x_{k-1}, x_k]$,

$$f(x) = f(x) - f_n(x) + f_n(x) \leq \varepsilon + \sup_{[x_{k-1}, x_k]} f_n$$

$$\therefore \sup_{[x_{k-1}, x_k]} f(x) \leq \varepsilon + \sup_{[x_{k-1}, x_k]} f_n(x)$$

$$\mathcal{U}_\mathcal{D}(f) = \sum_{k=1}^m (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f(x) \leq \sum_{k=1}^m (x_k - x_{k-1}) \left(\varepsilon + \sup_{[x_{k-1}, x_k]} f_n \right)$$

$$= \varepsilon (b-a) + \mathcal{U}_\mathcal{D}(f_n)$$

L2.3

Similarly, $L_D(f) \geqslant L_D(f_n) - \varepsilon(b-a)$.

So $U_D(f) - L_D(f) \leqslant 2\varepsilon(b-a) + (U_D(f_n) - L_D(f_n))$

$$\leqslant \varepsilon(1 + 2(b-a))$$ ✓

Finally, fix $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall x \in [a,b]$, $\forall n \geqslant N$,
$|f_n(x) - f(x)| < \varepsilon$

So $\forall n \geqslant N$, $\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right|$

$$= \left| \int_a^b (f_x(x) - f_n(x))\,dx \right|$$

$$\leqslant (b-a) \sup_{[a,b]} |f - f_n|$$

$$\leqslant \varepsilon(b-a).$$

So $\int_a^b f_n \longrightarrow \int_a^b f$. □

Remark 1. $\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

2. If $f_n \to f$ unif on S and f_n are bounded on S, then f is bounded on S.

Cor 3 Let $f_n : [a,b] \to \mathbb{R}$ be integrable $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ to some function f. Then f is integrable and

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Note $\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$

Proof Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$, $x \in [a,b]$, $n \in \mathbb{N}$
The assumption says that $F_n \to f$ unif. on $[a,b]$.
We know that F_n is integrable &

$$\int_a^b F_n = \sum_{k=1}^{n} \int_a^b f_k$$

So by Thm 2, f is integrable, and moreover

L2.4

$$\int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty} F_n(x)\,dx$$

$$= \lim_{n\to\infty} \int_a^b F_n(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k(x)\,dx$$

$$= \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx . \qquad \square$$

Thm 4 Let $f_n : [a,b] \longrightarrow \mathbb{R}$ be continuously diff'ble on $[a,b]$. Assume

(i) $\sum f_n'(x)$ converges unif. on $[a,b]$

(ii) $\exists\, c \in [a,b]$ s.t. $\sum f_n(c)$ converges

Then $\sum f_n(x)$ converges unif. to some continuously diff'ble $f : [a,b] \longrightarrow \mathbb{R}$, and moreover $f'(x) = \sum f_n'(x)$.

Note $\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} \frac{df_n}{dx}$

idea

$g(x) = \sum f_n'(x)$

want f s.t. $f' = g$, $f(c) = \sum f_n(c)$

& $\sum f_n$ converges unif to f

Let $f(x) = \sum f_n(c) + \int_c^x g(t)\,dt$

Thm 4 $f_n : [a,b] \to \mathbb{R}$ ctsly diff'ble $\forall n$

(i) $\sum_{n=1}^{\infty} f_n'(x)$ conv unif on $[a,b]$

(ii) $\sum_{n=1}^{\infty} f_n(c)$ conv for some $c \in [a,b]$

Then $\sum_{n=1}^{\infty} f_n(x)$ conv unif on $[a,b]$ to a ctsly diff'ble function f on $[a,b]$ and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad \forall x \in [a,b]$$

Note $\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} f_n'(x)$

Proof Let $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ for $x \in [a,b]$.

By (i) + Thm 1, g is cts so integrable on $[a,b]$.

Define $f : [a,b] \to \mathbb{R}$ by $f(x) = \int_c^x g(t)\,dt + \lambda$,

where $\lambda = \sum_{n=1}^{\infty} f_n(c)$.

[Idea: solve d.e. $f' = g$, $f(c) = \sum_{n=1}^{\infty} f_n(c)$]

By FTC, f is diff'ble & $f'(x) = g(x)$ $\forall x \in [a,b]$.

So f is continuously diff'ble. Also, $f(c) = \lambda$.

Remains to show that $\sum f_n(x)$ conv. unif to $f(x)$.

$$f(x) - \sum_{k=1}^{n} f_k(x) = \int_c^x g(t)\,dt + \lambda - \sum_{k=1}^{n}\left(\int_c^x f_k'(t)\,dt + f_k(c)\right)$$

↑ by FTC

$$= \int_c^x \left(g(t) - \sum_{k=1}^{n} f_k'(t)\right) dt + \left(\lambda - \sum_{k=1}^{n} f_k(c)\right)$$

Fix ε. $\exists N \in \mathbb{N}$ s.t. $\forall \overset{t}{\cancel{x}} \in [a,b]$, $\forall n \in \mathbb{N}$, $n \geq N$

$$\Rightarrow \left| g(t) - \sum_{k=1}^{n} f_k'(t) \right| < \varepsilon$$

$\forall n \in \mathbb{N}, n \geq N$

$$\Rightarrow \left| \lambda - \sum_{k=1}^{N} f_k(c) \right| < \varepsilon$$

So $\forall x \in [a,b]$, $\forall n \in \mathbb{N}$, if $n \geqslant N$ then

$$\left| f(x) - \sum_{k=1}^{n} f_k(x) \right| \leqslant \left| \int_c^x \left(g(t) - \sum_{k=1}^{n} f_k'(x)\right) dt \right| + \left| \lambda - \sum_{k=1}^{n} f_k(c) \right|$$

$$\leqslant (b-a)\varepsilon + \varepsilon = \varepsilon(b-a+1)$$ □

<u>recall</u> A scalar sequence (x_n) is <u>Cauchy</u> if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \geqslant N, |x_m - x_n| < \varepsilon$

convergent $\underset{\text{GPC}}{\iff}$ Cauchy $\implies$ bounded

<u>Def</u> Let (f_n) be a sequence of scalar functions on a set S. We say <u>(f_n) is uniformly Cauchy on S</u> if
$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in S, \forall m,n \in \mathbb{N}, m,n \geqslant N$
$\implies |f_m(x) - f_n(x)| < \varepsilon$

<u>Thm 5</u> (GPUC – general principle of uniform convergence)
(f_n) uniformly Cauchy on S
$\implies$ (f_n) uniformly convergent on S

<u>Pf</u> <u>STEP 1</u> (f_n) converges pointwise
Fix $x \in S$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}, \forall m,n \in \mathbb{N}, \forall t \in S$, $m,n \geqslant N \implies |f_m(t) - f_n(t)| < \varepsilon$.
So in particular, $\forall m,n \in \mathbb{N}$, $m,n \geqslant N$
$\implies |f_m(x) - f_n(x)| < \varepsilon$.
Thus, $(f_n(x))_{n=1}^{\infty}$ is Cauchy $\implies$ convergent.
Let $f(x) = \lim_{n\to\infty} f_n(x)$.
Do this for every $x \in S$ to obtain pointwise limit $f : S \to \mathbb{R}/\mathbb{C}$.
<u>STEP 2</u> $f_n \to f$ unif on S
Fix $\varepsilon > 0$. $\exists N \in \mathbb{N}, \forall x \in S, \forall m,n \in \mathbb{N}$, $m,n \geqslant N \implies |f_n(x) - f_m(x)| < \varepsilon$
Fix $x \in S$, ~~m~~ $n \in \mathbb{N}$ with $n \geqslant N$. Since $f_m(x) \to f(x)$ as $m \to \infty$, can choose $m \in \mathbb{N}$ s.t. $m \geqslant N$ & $|f_m(x) - f(x)| < \varepsilon$ (may depend on x, ε)
Then $|f(x) - f_n(x)| \leqslant |f(x) - f_m(x)| + |f_m(x) - f_n(x)| < 2\varepsilon$ □
<u>OR</u> consider $|f_m(x) - f_n(x)| < \varepsilon$ $\forall x \in S$, $\forall m,n \geqslant N$.
Fix $x \in S$, $n \geqslant N$, take $\lim_{m\to\infty}$: $|f(x) - f_n(x)| \leqslant \varepsilon$

L3.3

Cor 6. (Weierstrass M-test) Let (f_n) be a sequence of scalar functions on a set S. Let $\sum_{n=1}^{\infty} M_n$ be a convergent series with $M_n \geq 0 \ \forall n$.
If $|f_n(x)| \leq M_n \ \forall x \in S, \forall n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.

Proof Let $F_n(x) = \sum_{k=1}^{n} f_n(x)$, $x \in S$, $n \in \mathbb{N}$. Observe

$$|F_m(x) - F_n(x)| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

True $\forall x \in S, \forall m, n \in \mathbb{N}, m \leq n$.
Fix $\varepsilon > 0$. Since $\sum M_k$ converges, it's Cauchy, so $\exists N \in \mathbb{N}$ s.t. $\forall m, n \in \mathbb{N}, m, n \geq N \Rightarrow \sum_{k=m+1}^{n} M_k < \varepsilon$
By above, $|F_m(x) - F_n(x)| < \varepsilon \ \forall x \in S, \forall m, n \geq N$.

So (F_n) is unif Cauchy, so we're done by Thm 5. □

Consider the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$.
$(a_n)_{n=0}^{\infty}$ is complex seq. (fixed), $a \in \mathbb{C}$ (fixed), $z \in \mathbb{C}$ variable
Let R = radius of convergence (r.o.c)
[$|z-a| < R \Rightarrow \sum a_n (z-a)^n$ conv absolutely
$|z-a| > R \Rightarrow \sum a_n (z-a)^n$ diverges]
Define $f : D(a, R) = \{z \in \mathbb{C} : |z-a| < R\} \longrightarrow \mathbb{C}$
by $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

So the power series converges pointwise to f on $D(a; R)$.
Q: is the convergence uniform?
A: not in general
e.g. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $R = 1$

$$\left|\sum_{n=0}^{\infty} z^n\right| \leq N+1 \text{ on } D(0,1)$$

$\frac{1}{1-z}$ not bounded on $D(0,1)$

L3.4

Thm 7 Let $\sum a_n (z-a)^n$, R, f be as above. For any r with $0 < r < R$,

$$\sum a_n (z-a)^n \xrightarrow[\text{unif}]{} f \quad \text{on } D(a,r)$$

Proof Fix $w \in \mathbb{C}$ s.t. $r < |w-a| < R$.

$\sum a_n (w-a)^n$ converges, so $a_n (w-a)^n \to 0$ as $n \to \infty$

So $\exists M \geq 0$ s.t. $\left(|a_n (w-a)|^n \leq M \quad \forall n \right)$ (conv $\Rightarrow$ bdd)

Let $f_n(z) = a_n (z-a)^n$, $z \in \mathbb{C}$, $n \in \mathbb{N}$

If $|z-a| < r$, $n \in \mathbb{N}$, then

$$|f_n(z)| = \left| a_n (w-a)^n \frac{(z-a)^n}{(w-a)^n} \right| \leq M \left| \frac{z-a}{w-a} \right|^n \leq M\rho^n$$

where $\rho = \left| \frac{r^{\cancel{n}}}{(w-a)} \right|$.

Since $\rho < 1$, $\sum_{n=0}^{\infty} M\rho^n$ converges.

So by Cor 6, $\sum f_n(z)$ converges unif on $D(a,r)$. □

Remarks ① It follows that f is continuous on $D(a,R)$

② $\sum_{n=1}^{\infty} a_n \cdot n (z-a)^{n-1}$ has roc R as well

By a complex analogue of Thm 4, can prove that f is cx diff'ble & $\frac{d}{dz}\left(\sum a_n (z-a)^n \right) = \sum_{n=1}^{\infty} a_n n (z-a)^n$.

L4.1

Thm 7 A power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ with r.o.c R converges uniformly on $D(a,r)$ for every r with $0<r<R$.

Remarks ① $f: D(a,R) \to \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ is continuous on $D(a,R)$

② $\sum_{n=1}^{\infty} a_n \cdot n(z-a)^{n-1}$ also has r.o.c. R. A complex version of Thm 4 shows f is cx diff'ble on $D(a,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} a_n \cdot n(z-a)^{n-1}$$

(see IB Complex Analysis)

③ Fix $w \in D(a,R)$. Choose r s.t. $|w-a| < r < R$, choose $\delta > 0$ s.t. $|w-a| + \delta < r$. Then $D(w,\delta) \subset D(a,r)$.
(if $|z-w| < \delta$ then $|z-a| \leq |z-w| + |w-a| < \delta + |w-a|$)
So $\sum_{n=1}^{\infty} a_n z^n$ converges uniformly on $D(w,\delta)$.
We say $\sum_{n=1}^{\infty} a_n(z-a)^n$ converges locally uniformly on $D(a,R)$.

§2. Uniform Continuity

Let $U \subset \mathbb{R}$ or $\mathbb{C}$, f a scalar function on U.
For $x \in U$, say f is cts at x if $\forall \varepsilon > 0, \exists \delta > 0, \forall y \in U$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Say f is cts on U if f is cts at x for every $x \in U$, i.e.
$\forall x \in U, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in U, |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Say f is uniformly cts on U if
$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in U, \forall y \in U, |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. This is cts, not unif cts on $\mathbb{R}$. Indeed, let

$$(x + \delta/2)^2 - x^2 = \delta x + \delta^2/4$$

Given $\delta > 0$, let $x = 1/\delta$, $y = x + \delta/2$. Then
$|f(y) - f(x)| > 1$.

Theorem 1 Let f be a scalar function on a closed, bdd interval $[a,b]$. If f is continuous on $[a,b]$, then f is unif. continuous on $[a,b]$.

Proof Assume not. Then

$\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in \overset{[a,b]}{U}, |x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Fix such a "bad" $\varepsilon > 0$. $\forall n \in \mathbb{N}$, $\exists\, x_n, y_n \in [a,b]$ s.t. $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. (above w/ $1/n$)

By Bolzano-Weierstrass there's a subsequence $(x_{k_n})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ s.t. $x_{k_n} \to x$, say, as $n \to \infty$, and $x \in [a,b]$.

Then $|y_{k_n} - \underbrace{x_{k_n}}_{x}| \leq |y_{k_n} - x_{k_n}| + |x_{k_n} - x| < \frac{1}{k_n} + |x_{k_n} - x|$

$\leq 1/n + |x_{k_n} - x| \to 0$ as $n \to \infty$

So $y_{k_n} \to x$ as $n \to \infty$.

f is cont. at x, so $\exists\, \delta > 0$ s.t. $\forall y \in [a,b]$,

$|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon/2+1$

As $x_{k_n} \to x$ and $y_{k_n} \to x$, $\exists\, n \in \mathbb{N}$ s.t.

$|x_{k_n} - x| < \delta$ and $|y_{k_n} - x| < \delta$, so ⨳ $\square$

$|f(x_{k_n}) - f(y_{k_n})| \leq |f(x_{k_n}) - f(x)| + |f(y_{k_n}) - f(x)| < 2\varepsilon/3$

This gives a new (?) proof that cts $f : [a,b] \to \mathbb{R}$ is integrable.

Given $\varepsilon > 0$, by Thm 1 $\exists\, \delta > 0$ s.t. $\forall x, y \in [a,b]$,

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Choose ~~P~~ $\mathcal{P}$ $a = x_0 < x_1 < \cdots < x_n = b$ s.t. $x_k - x_{k-1} < \delta$

For every k, $x, y \in [x_{k-1}, x_k]$, $|f(y) - f(x)| < \varepsilon$

Hence $\sup_{[x_{k-1}, x_k]} f - \inf_{[x_{k-1}, x_k]} f \leq \varepsilon$.

Apply $\Sigma_k (x_k - x_{k-1})$: $U_{\mathcal{D}} - L_{\mathcal{D}} \leq \varepsilon(b-a)$

§3. Metric Spaces

Let M be a set. A metric on M is a function $d : M \times M \to \mathbb{R}$ such that (i) (positivity) $d(x,y) \geq 0$ $\forall x, y \in M$

and $d(x,y) = 0 \Leftrightarrow x = y$

(ii) (symmetry) $d(x,y) = d(y,x)$ $\forall x, y \in M$

(iii) (Δ ineq) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x, y, z \in M$

A metric space is a pair (M, d) where M is a metric and d is a metric on M.

Examples ① $M = \mathbb{R}$ or $\mathbb{C}$, $d(x,y) = |x - y|$ the usual metric

(always assume unless otherwise stated)

② $M = \mathbb{R}^n$ or $\mathbb{C}^n$, $d_2((x_i)_{i=1}^{n}, (y_i)_{i=1}^{n}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$

X

L4.3

called the Euclidean metric

Triangle inequality uses Cauchy-Schwarz

(assumed unless otherwise stated)

(M, d_2) is denoted ℓ_2^n

OR $d_1((x_i)_{i=1}^n, (y_i)_{i=1}^n) = \sum_{i=1}^n |x_i - y_i|$

From $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$, we get

$$d_1((x_i)_{i=1}^n, (z_i)_{i=1}^n) \le d_1((x_i),(y_i)) + d_1((y_i),(z_i))$$

(M, d_1) is denoted ℓ_1^n

OR $d_\infty((x_i),(y_i)) = \max_{1 \le i \le n} |x_i - y_i|$

(M, d_∞) is ℓ_∞^n

[ℓ_p^n, $1 \le p \le \infty$ See Part II Linear Analysis]

③ Let S be any $\ne \emptyset$ set. Let $\ell_\infty(S)$ = set of all bdd scalar functions on S.

(f is bdd : $\exists\, C \geq 0$ s.t. $|f(x)| \le C \ \forall x \in S$)

E.g. $\ell_\infty^n = \ell_\infty(\{1, \dots, n\})$

The uniform metric D on $\ell_\infty(S)$ is defined by

$$D(f,g) = \sup_{x \in S} |f(x) - g(x)| \qquad (\text{well-defined as } f, g \text{ bounded})$$

$D(f,g) \geq 0$ ✓ if $f = g$ then $D(f,g) = 0$ ✓

if $f \ne g$ then $\exists\, x \in S$ s.t. $f(x) \ne g(x)$

so $D(f,g) \geq |f(x) - g(x)| > 0$

Symmetry ✓

$$|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)|$$
$$\le D(f,g) + D(g,h)$$

So $D(f,h) \le D(f,g) + D(g,h)$.

LAST TIME metric, metric space

examples ① $\mathbb{R}$ or $\mathbb{C}$ with $d(x,y) = |x-y|$ usual metric

② $\mathbb{R}^n$ or $\mathbb{C}^n$ with usual, Euclidean metric $d_2(x,y) = \left(\sum |x_i - y_i|^2\right)^{1/2}$

Other metrics: d_1, d_∞ Notation $\ell_p^n = (\mathbb{R}^n, d_p)$ $p = 1, 2, \infty$

③ $\ell_\infty(S)$ with uniform metric ($S \neq \phi$ set)

④ Let M be any set.

Define $d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$ (Check is metric)

d is called the discrete metric

(M, d) is called a discrete metric space

⑤ Let G be a group generated by a symmetric set $S \subset G$
($e \notin S$, $\forall x \in S$, $x^{-1} \in S$)

Define $d(x,y)$ = least $n \geqslant 0$ s.t. $\exists\, s_1, \dots, s_n \in S$
with $y = x s_1 \cdots s_n$ ($\Leftrightarrow$ $x^{-1} y = s_1 \cdots s_n$)

Check this is a metric

$\rightsquigarrow$ geometric group theory

⑥ G is a graph, G connected
$d(x,y)$ = Length of shortest path x to y

⑦ Riemannian metrics in geometry (see IB Geometry)

⑧ $M = \mathbb{Z}$, fix a prime p, define $d_p(x,y) = 0$ if $x = y$,
& if $x \neq y$, then write $x - y = p^n a$ where $n \geqslant 0$ and
$p \nmid a$ and set $d_p(x,y) = p^{-n}$.

Check positivity ✓
symmetry ✓

triangle inequality: given $x, y, z \in \mathbb{Z}$ need
$d(x,z) \leq d(x,y) + d(y,z)$

WLOG x, y, z pairwise distinct. Assume $d(x,y) = p^{-m}$,
and $d(y,z) = p^{-n}$.

So $x - y = p^{+m} a$, $y - z = p^n b$ where $p \nmid a, b$.

So $x - z = p^m a + p^n b$. Let $s = \min(m, n)$.

$x - z = p^{+s}(p^{m-s} a + p^{n-s} b)$

So $d(x,z) \underset{\leq}{\cancel{=}} p^{-s} = \max(p^{-m}, p^{-n}) = \max(d(x,y), d(y,z))$
$\leq d(x,y) + d(y,z)$.

$$d(x,z) \le \overset{\swarrow\ \text{max}}{} d(x,y),\ d(y,z) \quad \forall x,y,z$$

then d is called an <u>ultrametric</u>

& (M,d) is an <u>ultrametric space</u>

$\mathbb{Z}$ with d_p is called the <u>p-adic integers</u>

ⓐ $M = \mathbb{R}^{\mathbb{N}}$ = set of all functions $\mathbb{N} \to \mathbb{R}$.

For $x = (x_n),\ y = (y_n) \in \mathbb{R}^{\mathbb{N}}$, let

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \min(1, |x_n - y_n|) \qquad \text{(Check a metric)}$$

<u>New objects from old</u>

<u>Subspace</u> Let (M,d) be a metric space & $N \subset M$. Then $d|_{N\times N}$ is a metric on N (often also denoted by d) & N with this metric is called a <u>(metric) subspace</u> of (M,d).

<u>E.g.</u> $C([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ cts} \}$

From Analysis I, $f \in C[0,1] \Rightarrow f \in \ell_\infty([0,1])$.

So $C[0,1]$ with the uniform metric is a subspace of ℓ_∞.

<u>Other metrics on $C[0,1]$</u> L_1-metric

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$

L_2-metric

$$d_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2\, dx \right)^{1/2}$$

<u>Product of metric spaces</u> Let (M,d) & (M',d') be metric spaces. Metrics on $M \times M'$: given $a = (x,x'),\ b = (y,y') \in M \times M'$ define

$$d_1(a,b) = d(x,y) + d'(x',y')$$

$$d_2(a,b) = \left(d(x,y)^2 + d'(x',y')^2 \right)^{1/2}$$

$$d_\infty(a,b) = \max\{ d(x,y), d'(x',y') \}.$$

<u>Note</u> $d_\infty \le d_2 \underset{\substack{\uparrow \\ \text{square}}}{\le} d_1 \le 2 d_\infty$

Notation: $(M \times M', d_p) = M \oplus_p M' \qquad (p = 1, 2, \infty)$

L5.3

Generalises to any finite sum of metric spaces

Eg $\ell_2^n = (\mathbb{R}^n, d_2) = \mathbb{R} \oplus_2 \mathbb{R} \oplus_2 \cdots \oplus_2 \mathbb{R}$ (n times)

Quotients we need to generalise to topological spaces

Convergence Let M be a metric space (d understood)
Given a sequence (x_n) in M and $x \in M$, say (xn) converges to x (write $x_n \to x$ as $n \to \infty$) if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geqslant N \Rightarrow d(x_n, x) < \varepsilon$.

Note $x_n \to x$ in $M \Leftrightarrow d(x_n, x) \to 0$ in $\mathbb{R}$

A sequence (x_n) in M is said to be convergent in M if $\exists$ $x \in M$ s.t. $(x_n) \to x$ as $n \to \infty$.

Lemma 1 Assume $x_n \to x$ and $x_n \to y$ in a metric space M. Then $x = y$.

Proof Assume not & Let $\varepsilon = d(x,y)$.
Since $x \neq y$, we have $\varepsilon > 0$.
Since $x_n \to x$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geqslant N_1 \Rightarrow$
$\Rightarrow d(x_n, x) < \varepsilon/10$,
$x_n \to y$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geqslant N_2 \Rightarrow$
$\Rightarrow d(x_n, y) < \varepsilon/10$.
Fix $n \geqslant \max(N_1, N_2)$. Then
$$\varepsilon = d(x,y) \leq d(x,x_n) + d(x_n,y) < \varepsilon/5. \quad ⨳ \quad \square$$

So if (x_n) is convergent in M, we can let $\lim_{n\to\infty} x_n$ be the unique $x \in M$ s.t. $x_n \to x$, called the limit of (x_n).

Examples ① In $\mathbb{R}$ & $\mathbb{C}$ this is the usual notion of convergence
② In 2-adic integers, $2^n \to 0$ as $n \to \infty$
since $2^n - 0 = 2^n \cdot 1$ so $d(2^n, 0) = 2^{-n} \to 0$ ö
③ If $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geqslant N \Rightarrow x_n = x$
so (x_n) is eventually constant x,
then $x_n \to x$. Converse clearly false.
But true if metric is discrete.
Indeed, if $x_n \to x$, $\exists N \ \forall n \geqslant N$, $d(x_n, x) < 1/2$.
So $\forall n \geqslant N$, $d(x_n, x) = 0$ i.e. $x_n = x$.

Examples

④ Let $S \neq \emptyset$ be a set. In $\ell_\infty(S)$ we have $f_n \to f$ in the uniform metric $\iff$ $f_n \to f$ uniformly on S

For $\varepsilon > 0$, $n \in \mathbb{N}$ we have

$$D(f_n, f) = \sup |f_n(x) - f(x)| \leq \varepsilon$$

$$\iff |f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in S$$

Note e.g. in $\mathbb{R}$, $f_n(x) = x + 1/n$, $f(x) = x$

So $f_n \to f$ unif. on $\mathbb{R}$, but $f_n, f \notin \ell_\infty(\mathbb{R})$

⑤ $\mathbb{R}^{\mathbb{N}}$, $d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min(1, |x_n - y_n|)$

Given $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots) \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$

$x = (x_1, x_2, \dots)$

Then $x^{(n)} \to x$ in $(\mathbb{R}^{\mathbb{N}}, d) \iff \forall i,\ x_i^{(n)} \to x_i$ as $n \to \infty$

(Exercise) ← EPIC

Q Given any set S, ? $\exists$ metric d on $\mathbb{R}^S$ = all f^n's $S \to \mathbb{R}$ s.t. $f_n \to f$ wrt d $\iff$ $f_n \to f$ pointwise on S ?

A No. We need topology.

⑥ $C[0,1]$, $f_n(x) = x^n$, $x \in [0,1]$, $n \in \mathbb{N}$

; $f(x) = 0$ for $0 \leq x < 1$, 1 for $x = 1$

$f_n \in C[0,1]$, $f \notin C[0,1]$, $f_n \to f$ ptwise on $[0,1]$

But $d_1(f_n, 0) = \int_0^1 |f_n(x)|\,dx = \frac{1}{n+1} \to 0$ as $n \to \infty$

So $f_n \to 0$ in $(C[0,1], d_1)$

⑦ $M \oplus_p M'$, $p = 1, 2, \infty$

Given (a_n), a in $M \times M'$, say $a_n = (x_n, x_n')$

$a = (x, x')$

we have $a_n \to a$ in $M \oplus_p M' \iff x_n \to x$ in M,

$x_n' \to x'$ in M'

$$0 \leq d(x_n, x),\ d'(x_n', x') \leq d_p((x_n, x_n'), (x, x'))$$

$$\leq d(x_n, x) + d'(x_n', x')$$

Note d_1, d_2, d_∞ are different.

⑧ Let N be a subspace of a metric space M. Let (x_n) be a sequence in N.

Then (x_n) is convergent in $N \Rightarrow (x_n)$ convergent in M

If $x_n \to x$ in M & $x \in M \setminus N$, ~~*~~ then (x_n) cannot be convergent in N.

Indeed, if $x_n \to y$ in N, then $x_n \to y$ in M.

By Lemma 1, $x = y$, so $y \notin N$ ※

Let $f: M \to M'$ be a f^n between metric spaces.

For $a \in M$, <u>f is continuous at a</u> if $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in M$, $d(x,a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon$.

Say <u>f is continuous</u> if $\forall a \in M$, f is continuous at a, i.e.

$\forall a \in M, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in M, d(x,a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon$

Note δ depends on ε and a (and f)

More generally, for $N \subset M$, <u>f is continuous on N</u> if ~~f(~~ $\forall a \in N$, f is continuous at a

<u>Note</u> f cont. on $N \Rightarrow f|_N : N \to M'$ is cont.

↙① $\not\Leftarrow$ ②

$\forall a \in N, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in M, d(a,x) < \delta \Rightarrow d'(f(a), f(x)) < \varepsilon$

↙

$\forall a \in N, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in N, d(a,x) < \delta \Rightarrow d'(f(a), f(x)) < \varepsilon$

$M = \mathbb{R}$, $f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$

$N = [0, \infty)$

<u>Prop 2</u> For $a \in M$, TFAE

(i) f is cont at a

(ii) if $x_n \to a$ in M, then $f(x_n) \to f(a)$ in M'

So if f is continuous, then for convergent (x_n) in M, $(f(x_n))$ is conv. in M' & $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$

<u>Pf</u> $\Rightarrow$ Given $\varepsilon > 0$, (want $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d'(f(x_n), f(a)) < \varepsilon$)

there is $\delta > 0$ s.t. $\forall x \in M$ $d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$

Since $x_n \to a$, $\exists N \in \mathbb{N}, \forall n \geq N, d(x_n, a) < \delta$, and so $d'(f(x_n), f(a)) < \varepsilon$ ✓

<u>$\Leftarrow$</u> Assume not. $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in M, d(x,a) < \delta$ but $d'(f(x), f(a)) \geq \varepsilon$.

Fix such a "bad" ε. By above, $\forall n \in \mathbb{N}\ \exists x_n \in M$ s.t.

$d(x_n, a) < 1/n$ and $d'(f(x_n), f(a)) \geq \varepsilon$ (take $\delta = \frac{1}{n}$)

Then $x_n \longrightarrow a$, $f(x_n) \not\rightarrow f(a)$ $\#$ ☐

<u>Cor 3</u> Given scalar functions f, g on a metric space M, let $a \in M$. If f, g are continuous at a, so are $f+g$, fg and assuming $g(x) \neq 0 \ \forall x$, so is f/g.

<u>Pf</u> Assume $x_n \to a$ in M. Since f, g are cont at a, by Prop 2, $f(x_n) \to f(a)$, $g(x_n) \longrightarrow g(a)$.

So $f(x_n) + g(x_n) \longrightarrow f(a) + g(a) = (f+g)(a)$.

By Prop 2, $f+g$ is cont at a.

fg, f/g similar. ☐

<u>Prop 4</u> Given functions $f : M \to M'$, $g : M' \longrightarrow M''$ between metric spaces, if f is continuous at $a \in M$, and g is cont. at $f(a)$, then $g \circ f : M \to M''$ is cont. at a.

<u>Proof</u> Assume $x_n \to a$ in M. Since f is cont at a, by Prop 2, $f(x_n) \longrightarrow f(a)$ in M'. Since g is cont at $f(a)$, $g(f(x_n)) \to g(f(a))$ i.e. $(g \circ f)(x_n) \longrightarrow (g \circ f)(a)$. By Prop 2, $g \circ f$ cont at a. ☐

<u>Exercise</u> Prove Cor 3, Prop 4 using $\varepsilon - \delta$ arguments.

<u>Examples</u> ① constant function : $f : M \to M'$ $\quad (c \in M')$
$\qquad x \to c$

identity function : $g : M \to M$ $\quad (\delta = \varepsilon)$
$\qquad x \to x$

By Cor 3, real & complex polynomials are cts

② Given a metric space (M, d), $d : M \times M \longrightarrow \mathbb{R}$ is continuous wrt d_p $(p = 1, 2, \infty)$

$d(x,x') \leq d(x,y) + d(y,y') + d(y',x')$ $\quad (x,x') \ (y,y')$

$d(x,x') - d(y,y') \leq d(x,y) + d(y',x')$ $\quad |d(x,x') - d(y,y')|$ $\qquad d(x,y) + d(x',y')$

$= d_1((x,x'),(y,y'))$

$\leq 2 d_p((x,x'),(y,y'))$

So $d(y,y') - d(x,x') \leq 2 d_p((y,y'),(x,x'))$

So $|d(x,x') - d(y,y')| \leq 2 d_p((x,x'),(y,y'))$

Examples

② (M,d) a metric space

$$d : M \oplus_p M \longrightarrow \mathbb{R}$$

$$|d(x,x') - d(y,y')| \leq d(x,y) + d(x',y') = d_1((x,x'),(y,y'))$$
$$\leq 2\, d_p((x,x'),(y,y'))$$

Def A function $f : M \to M'$ between metric spaces is *Lipschitz* if $\exists\, C \geq 0$ s.t. $\forall\, x,y \in M$

$$d'(f(x), f(y)) \leq C\, d(x,y) \qquad (C\text{-Lipschitz})$$

Say f is *uniformly continuous* if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x, y \in M$, $d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$

Note Lipschitz $\Rightarrow$ uniformly cont. $\Rightarrow$ cont.

Back to examples

③ Let M, M' be metric spaces, fix $y \in M'$ & consider

$$f : M \to M \oplus_p M'$$
$$x \mapsto (x,y)$$

$$d_p(f(x), f(z)) = d_p((x,y),(z,y)) = d(x,z)$$

Def A map $g : M \to M'$ between metric spaces is *isometric* if $d'(g(x), g(y)) = d(x,y)\ \forall\, x,y \in M$

Note isometric $\Rightarrow$ 1-Lipschitz

isometric $\Rightarrow$ injective

Back to examples

④ Projections: Let M, M' be metric spaces. Consider

$$q : M \oplus_p M' \to M \qquad q' : M \oplus_p M' \to M'$$
$$(x,x') \mapsto x \qquad (x,x') \mapsto x'$$

$$d(q(x,x'), q(y,y')) = d(x,y) \leq d_\infty((x,x'),(y,y'))$$
$$\leq d_p((x,x'),(y,y'))$$

So q is 1-Lipschitz (as is q')

e.g. $\mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto x_k$ is 1-Lipshitz

fix $(a_i)_{i=1}^{n} \in \mathbb{R}^n$, $\mathbb{R} \to \mathbb{R}^n$, $x \mapsto (a_1, \dots, a_{k-1}, x, a_{k+1}, \dots, a_n)$

The topology of metric spaces

Two observations:

· In a product metric, convergence does not depend on which of the three metrics d_1, d_2, d_∞ we use

· continuity depends only on convergent sequences

Fix a metric space (M, d).

For $x \in M$, $r \in \mathbb{R}$, $r \geq 0$, the <u>open ball $D_r(x)$ of centre x and radius r</u> is the set $D_r(x) = \{ y \in M : d(x,y) < r\}$

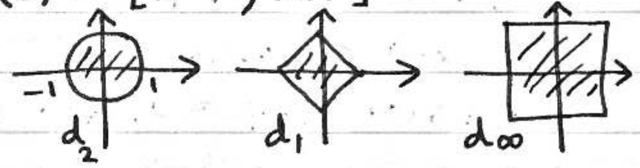
<u>Note</u> $x_n \to x \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N,$
$n \geq N \Rightarrow x_n \in D_\varepsilon(x)$

$f : M \to M'$ is cont. at $a \iff \forall \varepsilon > 0, \exists \delta > 0, \forall x \in M,$
$x \in D_\delta(a) \Rightarrow f(x) \in D_\varepsilon(f(a))$
$\iff \forall \varepsilon > 0, \exists \delta > 0$
$f(D_\delta(a)) \subset D_\varepsilon(f(a))$

The <u>closed ball $B_r(x)$ of centre x and radius r</u> is the set $B_r(x) = \{ y \in M : d(y,x) \leq r\}$.

<u>Examples</u> ① In $\mathbb{R}$, $D_r(x) = (x-r, x+r)$,
$B_r(x) = [x-r, x+r]$

② In $\mathbb{R}^2$, $B_1((0,0))$



③ If (M, d) is a discrete metric space,
$D_1(x) = \{x\}$, $B_1(x) = M$

<u>Note</u> $B_s(x) \subset D_r(x) \subset B_r(x)$ if $s < r$

Given $U \subset M$, say <u>U is a neighbourhood of x (in M)</u> if $\exists \varepsilon > 0$, $D_\varepsilon(x) \subset U$
$\iff \exists \varepsilon > 0$, $B_\varepsilon(x) \subset U$

Given $U \subset M$, say <u>U is open (in M)</u> if $\forall x \in U, \exists r > 0$, $D_r(x) \subset U$.

So U is open $\iff$ U is a nghd of x $\forall x \in U$

<u>Lemma 5</u> Open balls are open

<u>Proof</u> Consider $\overset{D}{\cancel{B}}_r(x)$.

Given $y \in D_r(x)$, seek $s > 0$ s.t. $D_s(y) \subset D_r(x)$.

Try $s = r - d(x,y) > 0$ by defⁿ of $D_r(x)$.

We show $D_s(y) \subset D_r(x)$.

Given $z \in D_s(y)$, $d(x,z) \leq d(x,y) + d(y,z) < s + d(x,y) = r$

So $z \in D_r(x)$. □

Prop 6 In (M, d) TFAE

(i) $x_n \longrightarrow x$

(ii) $\forall$ ngbds U of x, $\exists N \in \mathbb{N}, n \geqslant N \Rightarrow x_n \in U$

(iii) $\forall$ open $U \ni x$, ———— " ————

Proof (i) $\Rightarrow$ (ii) $\exists r > 0$ s.t. $D_r(x) \subset U$

Since $x_n \to x$, $\exists N \in \mathbb{N}, \forall n \in \mathbb{N}$, $n \geqslant N \Rightarrow d(x_n, x) < r$

So if $n \geqslant N$, then $x_n \in D_r(x) \subset U$.

(ii) $\Rightarrow$ (iii) ✓

(iii) $\Rightarrow$ (i) Given $\varepsilon > 0$, by Lemma 5, $D_\varepsilon(x)$ is open.

Since $x \in D_\varepsilon(x)$, by (iii), $\exists N \in \mathbb{N}, \forall n \in \mathbb{N}$, $n \geqslant N$

$\Rightarrow x_n \in D_\varepsilon(x)$ i.e. $d(x_n, x) < \varepsilon$ □

Prop 7 Given $f : M \to M'$ between metric spaces

(A) for $a \in M$, TFAE: (i) f cts at a

(ii) $\forall$ ngbds V of $f(a)$ in M',
$\exists$ ngbd U of a in M s.t. $f(U) \subset V$

(iii) $\forall$ ngbds V of $f(a)$ in M',
$f^{-1}(V)$ is a ngbd of a in M

(B) TFAE: (i) f is cts

(ii) $\forall$ open subsets V of M', $f^{-1}(V)$ is open in M

Proof (A)

(i) $\Rightarrow$ (ii) Given ngbd V of $f(a)$, there's $\varepsilon > 0$ s.t. $D_\varepsilon(f(a)) \subset V$. As f cts at a, there's $\delta > 0$ s.t.

$f(D_\delta(a)) \subset D_\varepsilon(f(a)) \subset V$

So $U = D_\delta(a)$ works.

(ii) $\Rightarrow$ (iii) Given ngbd V of $f(a)$, by (ii) $\exists$ ngbd U of a s.t. $f(U) \subset V$. Then $U \subset f^{-1}(V)$ and so $f^{-1}(V)$ is a ngbd of a.

(iii) $\Rightarrow$ (i) Given $\varepsilon > 0$, $V = D_\varepsilon(f(a))$ is a ngbd of $f(a)$, so by (iii), $f^{-1}(V)$ is a ngbd of a. So there's $\delta > 0$ s.t.

$D_\delta(a) \subset f^{-1}(V)$ Then $f(D_\delta(a)) \subset V = D_\varepsilon(f(a))$. ✓

(B) (i) $\Rightarrow$ (ii) Given open $V \subset M'$, need $f^{-1}(V)$ open in M.
For $x \in f^{-1}(V)$, $f(x) \in V$, so V is a ngbd of $f(x)$.

L7.4

By (i), f is cts at x, so by (A), $f^{-1}(V)$ is a ngbd of x.

<u>(ii) => (i)</u> Given $x \in M$, given open set V

Prop 7 $f : M \to M'$

(A) For $a \in M$, TFAE

(i) f cts at a

(ii) $\forall$ ngbds V of $f(a)$ in M' $\exists$ ngbd U of a in M s.t. $f(U) \subset V$

(iii) $\forall$ ngbds V of $f(a)$ in M', $f^{-1}(V)$ is ngbd of a in M

(B) TFAE

(i) f is cts

(ii) $\forall$ open subsets of V in M', $f^{-1}(V)$ is open in M

Proof of (B)

(i) $\Rightarrow$ (ii) Given open $V \subset M'$, given $x \in f^{-1}(V)$, we have $f(x) \in V$, since V open in M' $\exists$ $\varepsilon > 0$ s.t. $D_\varepsilon(f(x)) \subset V$. Since f is cts, $\exists$ $\delta > 0$ s.t. $f(D_\delta(x)) \subset D_\varepsilon(f(x)) \subset V$. Thus $D_\delta(x) \subset f^{-1}(V)$. So $f^{-1}(V)$ is open in M. ✓

(ii) $\Rightarrow$ (i) Given $x \in M$, given $\varepsilon > 0$, by Lemma 5 $V = D_\varepsilon$ of $f(x)$ is open in M'. So $f^{-1}(V)$ is open in M, and $x \in f^{-1}(V)$. So $\exists$ $\delta > 0$ s.t. $D_\delta(x) \subset f^{-1}(V)$. This means $f(D_\delta(x)) \subset V = D_\varepsilon(f(x))$. □

The topology of a metric space (M, d) is the collection of open subsets of M.

Prop 8 In a metric space M, we have

(i) $\emptyset$, M are open

(ii) if U_i is open in M $\forall i \in I$, then $\bigcup_{i \in I} U_i$ is open

(iii) if U, V open, then $U \cap V$ open

Proof (i) is clear

(ii) Given $x \in \bigcup_{i \in I} U_i$, there exists $j \in I$ s.t. $x \in U_j$. Since U_j is open, $\exists$ $r > 0$ s.t. $D_r(x) \subset U_j \subset \bigcup_{i \in I} U_i$ ✓

(iii) Given $x \in U \cap V$, since U is open, $\exists$ $r_1 > 0$ s.t. $D_{r_1}(x) \subset U$, and since V is open, $\exists$ $r_2 > 0$ s.t. $D_{r_2}(x) \subset V$. Then $r \doteq \min(r_1, r_2)$ verifies $D_r(x) \subset U \cap V$. ✓ □

Def A subset A of a metric space M is closed if whenever a sequence (a_n) in A is convergent in M, $\lim a_n \in A$

Examples ① Closed balls are closed: given (a_n) in $B_r(z)$, assume $a_n \to x$ in M. Then $d(a_n, z) \to d(x, z)$,

(since $d : M \times M \to \mathbb{R}$ is cts & $(a_n, z) \to (x, z)$ in $M \times M$)
Since $d(a_n, z) \leq r \ \forall n$, it follows that $d(x,z) \leq r$, i.e. $x \in B_r(z)$.

② In $\mathbb{R}$, $[a,b]$ is closed (recall (a,b) is open)
$\mathbb{R}$ is both open and closed
$[0,1)$ is neither open nor closed
$\uparrow$ $D_r(0) = (-r, r) \not\subset [0,1)$
$\nwarrow$ $1 - \frac{1}{n} \in [0,1])$ but $1 - \frac{1}{n} \to 1$

<u>Lemma 9</u> A subset A of a metric space M is closed in M iff $M \setminus A$ is open in M.

<u>Pf</u> "$\Rightarrow$" assume not. So $\exists x \in M \setminus A$ s.t. $\forall r > 0$, $D_r(x) \not\subset M \setminus A$. So $\forall r > 0$, $\exists y \in A$ s.t. $y \in D_r(x)$.
In particular, $\forall n \in \mathbb{N}$ $\exists a_n \in A \cap D_{1/n}(x)$.
Then $d(a_n, x) < 1/n \to 0$ so $a_n \to x$ in M.
But $x \notin A$. ⨳

"$\Leftarrow$" assume not. So $\exists$ sequence (a_n) in A s.t. $a_n \to x$ in M, but $x \notin A$. Since $M \setminus A$ is open, $\exists \varepsilon > 0$ s.t. $D_\varepsilon(x) \subset M \setminus A$.
Since $a_n \to x$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $d(a_n, x) < \varepsilon$.
In particular, $a_N \in D_\varepsilon(x) \in M \setminus A$, but $a_N \in A$. ⨳ ☐

<u>Example</u> If (M, d) is a discrete space, then $\{x\} = D_1(x)$ is open in M $\forall x \in M$. By Prop 8, $A = \bigcup_{x \in A} \{x\}$ is open $\forall A \subset M$. So every set is open, and so by Lemma 9, every set is closed.

<u>Def</u> Two metrics d, d' on a set M are <u>equivalent</u> $(d \sim d')$ if (M,d) & (M,d') have the same topology
$\Leftarrow\!\Rightarrow$ ——— " ——— convergent sequences
$\uparrow$ by prop 6, Lemma 9
$\Leftarrow\!\Rightarrow$ ——— " ——— continuous f^ns (both to & from)
$\uparrow$ by prop 7

Note that $d \sim d' \Leftrightarrow id : (M, d) \to \cancel{M'} (M^{\dagger}, d')$ & $id : (M^{\dagger}, d') \to (M, d)$ are cts.

<u>Def^{n}s</u> A map $g : M \to M'$ between metric spaces is a <u>homeomorphism</u> if g is a bijection & both g and g^{-1} are continuous.

L8.3

Say g is an isometry if g is a bijection & g is isometric.

Say M, M' are homeomorphic if $\exists$ hom $M \to M'$

" isometric — isom $M \to M'$

Remarks (1.) bijection + cts $\not\Rightarrow$ homeomorphic

$(\text{id} : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{Eucl}))$

(2.) isometr~~y~~ic + surj $\equiv$ isometry

Examples ① $(0,1)$ & $(0,\infty)$ are homeomorphic

$x \mapsto \frac{1}{x} - 1$

② $\mathbb{R}^2$ & $\mathbb{C}$ are isometric $((x,y) \to x+iy)$

Note $d \sim d' \iff \text{id} : (M,d) \to (M,d')$ is homeomorphism

Say d, d' are uniformly equivalent if $\text{id} : (M,d) \to (M,d')$ & $\text{id} : (M,d') \to (M,d)$ are unif. cts

Say d, d' are Lipschitz equivalent if $\exists\, a>0, b>0$ s.t.

$a \cdot d(x,y) \leq d'(x,y) \leq b \cdot d(x,y) \quad \forall x, y \in M$

Example ① On $M \times M'$, $d_1 \sim_{Lip} d_2 \sim_{Lip} d_\infty$

② On $C[0,1]$, $D \nsim d_1$ (unif, L_1)

if $f_n = x^n$ then $f_n \to 0$ in d_1, but not in D

§4 Completeness

A sequence (x_n) in a metric space (M,d) is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \in \mathbb{N}$ s.t. $m,n \geqslant N$, $d(x_m, x_n) < \varepsilon$

A subset A of M is bounded if $\exists x \in M, r>0$ s.t. $A \subset B_r(x)$

$(\forall w \in M,\ B_r(z) \subset B_R(w)$ for $R = r + |z-w|)$

Lemma convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded

L9.1

Lemma 1 convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded

Proof assume $x_n \to x$ in a metric space M.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geqslant N$, $d(x_n, x) < \varepsilon$,

and so $\forall m, n \geqslant N$, $d(x_m, x_n) \leqslant d(x_m, x) + d(x_n, x) < 2\varepsilon$. ✓

Assume (x_n) is Cauchy in M. Need $\{x_n : n \in \mathbb{N}\} \subset$ ball

$\exists N \in \mathbb{N}$, $\forall m, n \geqslant N$, $d(x_m, x_n) < 1$, so in particular

$x_n \in B_1(x_N)$ $\forall n \geqslant N$

Let $R = \max(1, d(x_1, x_N), \dots, d(x_{N-1}, x_N))$.

Then $x_n \in B_R(x_N)$ $\forall n \in \mathbb{N}$. ✓

Note bounded $\not\Rightarrow$ Cauchy e.g. in $\mathbb{R}$: $0, 1, 0, 1, \dots$

Cauchy $\not\Rightarrow$ convergent e.g. $x_n = \frac{1}{n} \to 0$ in $\mathbb{R}$

so (x_n) is Cauchy in $\mathbb{R}$

so (x_n) is Cauchy in $\mathbb{R} \setminus \{0\}$

but not convergent in $\mathbb{R} \setminus \{0\}$

Def A metric space M is complete if every Cauchy sequence in M converges in M.

Ex $\mathbb{R}$ or $\mathbb{C}$

Prop 2 If M, M' are complete metric spaces, then so is

$M \oplus_p M' = (M \times M', d_p)$ $(p = 1, 2, \infty)$

Proof Let (a_n) be Cauchy in $M \oplus_p M'$.

Say $a_n = (x_n, x_n')$, $n \in \mathbb{N}$.

$\forall m, n \in \mathbb{N}$, $d(x_m, x_n) \leqslant d_p(a_m, a_n)$,

so (x_n) is Cauchy in M.

Similarly, (x_n') is Cauchy in M'.

M, M' are complete, so $\exists x \in M, x' \in M'$ s.t. $x_n \to x$, $x_n' \to x'$

Hence $a_n = (x_n, x_n') \to (x, x')$ in $M \oplus_p M'$. $\square$

Ex $\mathbb{R}^n$, $\mathbb{C}^n$ are complete in the Euclidean metric

Thm 3 Let S be a non-empty set. Then $(\ell_\infty(S))$ is complete in the uniform metric D.

Proof Let (f_n) be Cauchy in $\ell_\infty(S)$.

STEP 1 Consider ptwise convergence. Fix $x \in S$.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m, n \geqslant N$, $D(f_m, f_n) < \varepsilon$.

So $\forall m, n \geqslant N$, $|f_m(x) - f_n(x)| \leqslant D(f_m, f_n) < \varepsilon$.

Since ℝ or ℂ is complete $(f_n(x))_{n\in\mathbb{N}}$ is convergent
Let $f(x) = \lim_{n\to\infty} f_n(x)$. Do this to obtain a scalar function f on S s.t. $f_n \to f$ ptwise on S.

STEP 2 $f \in \ell_\infty(S)$?

Choose N s.t. $\forall n \geqslant N$, $D(f_n, f) < 1$. Fix $n \geqslant N$ (e.g. $n = N$).
f_n is bounded, so $\exists\, C \geqslant 0, \forall x \in S, |f_n(x)| \leqslant C$.
Then $\forall x \in S$, $|f(x)| \leqslant |f_n(x)| + |f(x) - f_n(x)| \leqslant 1 + C$ ✓

STEP 3. $f_n \to f$ in D

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\forall n, m \geqslant N$, $D(f_m, f_n) < \varepsilon$
Fix $n \geqslant N$, $x \in S$. $\forall m \geqslant N$, $|f_m(x) - f_n(x)| < \varepsilon$
Let $m \to \infty$ to get $|f(x) - f_n(x)| \leqslant \varepsilon$
$x \in S$ was arbitrary, so $D(f, f_n) \leqslant \varepsilon$
This holds $\forall n \geqslant N$. □

Prop 4 Let N be a subspace of a metric space M.
(i) If N is complete, then N is closed in M.
(ii) If N is closed in M and M is complete, then N is complete.
So in a complete metric space, a subspace is complete iff it is closed.

Proof (i) Let (x_n) be a sequence in N s.t. $x_n \to x$ in M.
By Lemma 1, (x_n) is Cauchy in M, so in N.
Since N is complete, $\exists\, y \in N$ s.t. $x_n \to y$ in N.
So $x_n \to y$ in M. By Lemma 3.1, $x = y$, so $x \in N$. ✓
(ii) Let (x_n) be Cauchy in N. Then (x_n) is Cauchy in M.
M is complete, so $\exists\, x \in M$ s.t. $x_n \to x$ in M.
~~M~~ N is closed in M, so $x \in N$., and $x_n \to x$ in N. □

Thm 4: Let M be a metric space. Then
$$C_b(M) = \{ f : M \to \mathbb{R} (\text{or } \mathbb{C}) : f \text{ is bounded \& cts} \}$$
is complete in the uniform metric.

Proof We have $C_b(M) \subset \ell_\infty(M)$, so by Thm 3 & Prop 4 (ii), it's enough to check $C_b(M)$ is closed in $\ell_\infty(M)$.
Assume $f_n \to f$ in $\ell_\infty(M)$ where $f_n \in C_b(M)$ $\forall n \in \mathbb{N}$.
Need f is continuous
Fix $x \in M$, $\varepsilon > 0$. Can choose $N \in \mathbb{N}$ s.t. $\forall n \geqslant N$, $D(f_n, f) < \varepsilon$.

Fix $n \geqslant N$ (e.g. $n=N$). Since f_n is cts, $\exists\, \delta>0, \forall y \in M$, $d(x,y)<\delta \Rightarrow |f_n(y)-f_n(x)|<\varepsilon$.
Then if $d(y,x)<\delta$, then

$$\begin{aligned} |f(y)-f(x)| &\leqslant |f(y)-f_n(y)| + |f_n(y)-f_n(x)| + |f_n(x)-f(x)| \\ &\leqslant 2D(f_n,f)+\varepsilon \\ &< 3\varepsilon. \end{aligned}$$

□

Fix a set $S \neq \phi$, a metric space (N,d').
Let $\ell_\infty(S,N) = \{ f: S \to N \mid f \text{ is bounded} \}$.
(Here f bdd means $\operatorname{im} f$ is a bdd subset of N)
Define the uniform metric on $\ell_\infty(S,N)$ by

$$D(f,g) = \underbrace{\sup_{x \in S} d'(f(x),g(x))}_{\substack{\exists z \in \mathbb{R} \text{ s.t.} \\ f(x), g(x) \in B_R(z)\ \forall x \in S \\ \text{so } d'(f(x),g(x)) \leq 2R}}$$

Given a metric space (M,d), let

$$C_b(M,N) = \{ f: M \to N \mid f \text{ cts, bdd} \} \subset \ell_\infty(M,N)$$

Theorem 6 Let S, M, N be as above. Assume N is complete. (i) $\ell_\infty(S,N)$ is complete in the unif metric D
(ii) $C_b(M,N)$ is a closed subspace of $\ell_\infty(M,N)$ and hence complete in the unif metric

Proof Follow proofs of Thms 3, 5 ; use metric d' of N instead of usual metric on $\mathbb{R}$ or $\mathbb{C}$. □

Ex For any closed, bdd interval $[a,b]$ in $\mathbb{R}$,

$$C[a,b] \underset{\substack{\uparrow \\ \text{analysis I}}}{=} C_b[a,b] \text{ is complete in unif metric}$$

Def A map $f : M \to M'$ between metric spaces is a contraction mapping if $\exists \lambda \in \mathbb{R}$ with $0 \leq \lambda < 1$ s.t.

$$d'(f(x), f(y)) \leq \lambda\, d(x,y) \quad \forall x, y \in M.$$

Note that f is λ-Lipshitz, so in particular cts.

Thm 7 (Contraction mapping Theorem, CMT or Banach's fixed point theorem). Let M be a non-empty complete metric space. Let $f : M \to M$ be a contraction mapping. Then f has a unique fixed point, i.e. $\exists!\, x \in M$ s.t. $f(x) = x$

Remarks 1. $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}\setminus\{0\}$, $t \mapsto t/2$ is a contraction mapping ($\lambda = \frac{1}{2}$) but has no fixed point

2. $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto t+1$ is isometric ($\lambda = 1$) but no fixed pt

3. $f : [1, \infty) \to [1, \infty)$, $x \mapsto x + \frac{1}{x}$. Then $\forall x \neq y$, $|f(x) - f(y)| < |x - y|$, but no fixed point.

Proof of Thm 7 Choose $\lambda < 1$ s.t. $d(f(x), f(y)) \leq \lambda\, d(x,y)$ $\forall x, y$.

Fix $x_0 \in M$ ($M \neq \emptyset$). Define (x_n) by $x_n = f(x_{n-1})$.

So $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, ..., $x_n = (\underbrace{f \circ \cdots \circ f}_{n \text{ times}})(x_0)$, etc.

We show that (x_n) is Cauchy

For $n \geq 2$,
$$\begin{aligned} d(x_n, x_{n-1}) &= d(f(x_{n-1}), f(x_{n-2})) \\ &\leq \lambda\, d(x_{n-1}, x_{n-2}) \end{aligned}$$

Inductively, we get $d(x_n, x_{n-1}) \leq \lambda^{n-1} d(x_1, x_0)$.

So for $m > n$,
$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n)\, d(x_1, x_0) \\ &= \frac{\lambda^n - \lambda^m}{1-\lambda}\, d(x_1, x_0) \\ &\leq \frac{\lambda^n}{1-\lambda}\, d(x_1, x_0) \end{aligned}$$

Since $\lambda < 1$, $\frac{\lambda^n}{1-\lambda} d(x_1, x_0) \longrightarrow 0$. So given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\frac{\lambda^n}{1-\lambda} d(x_1, x_0) < \varepsilon$.

Hence $\forall m, n \geq N$, $d(x_n, x_m) < \varepsilon$. ✓

Since M is complete, so $\exists z \in M$ s.t. $x_n \to z$ as $n \to \infty$.

Since f is cts, $f(x_n) \to f(z)$. But $f(x_n) = x_{n+1} \to z$.

By Lemma 3.1, we have $z = f(z)$. ✓

<u>uniqueness</u> assume $f(w) = w$, $f(z) = z$.

Then $d(w,z) = d(f(w), f(z)) \leq \lambda\, d(w,z)$. So $d(w,z) = 0$, $w = z$. ✓□

<u>Note</u> In proof we had $d(x_m, x_n) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0)$.

Let $m \to \infty$, so $d(z, x_n) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0)$.

So $x_n \to z$ exponentially fast.

<u>An application</u> The IVP ($f'(t) = f(t^2)$, $f(0) = y_0$) has a unique solution on $[0, \frac{1}{2}]$

<u>idea</u> assume f is a solution. Then f is diff'ble, so cts, which in turn implies f' is cts.

By FTC, $f(t) - f(0) = \int_0^t f'(s)\,ds$

so $f(t) = y_0 + \int_0^t f(s^2)\,ds$.

Let $M = C[0, \frac{1}{2}]$ which is non-empty and complete in $\overset{\text{unif metric}}{D}$.

Define $T : M \to M$ by $(Tg)(t) = y_0 + \int_0^t g(s^2)\,ds$

<u>T well-defined?</u> i.e. $g \in M \overset{?}{\Rightarrow} Tg \in M$

By FTC, Tg is in fact diff'ble, $(Tg)'(t) = g(t^2)$

So Tg is cts.

<u>f a sol^n of IVP $\Leftrightarrow$ f cts, fixed pt of T</u>

$\Rightarrow$ ✓

$\Leftarrow$ If $f = Tf$, then f is diff'ble, and $f'(t) = (Tf)'(t) = f(t^2)$.

And $f(0) = (Tf)(0) = y_0$ ✓

<u>T has unique fixed pt</u> For $g, h \in M$

~~$\sup_t$~~ $|(Tg)(t) - (Th)(t)| = \left| \int_0^t (g(s^2) - h(s^2))\,ds \right|$

$\leq t \sup_t |g(t) - h(t)| \leq \frac{1}{2} D(g,h)$

Take sup over $t \in [0, \frac{1}{2}]$ to get

$D(Tg, Th) \leq \frac{1}{2} D(g,h)$ ✓ □

L10.3

Notation For $x = (x_i)_{i=1}^{n} \in \mathbb{R}^n$ (or $\mathbb{C}^n$)

let $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ the Euclidean norm (or length) of x

Note that $d_2(x,y) = \|x-y\|$.

Thm 8 (Lindelöf – Picard) Let $a<b$, $R>0$ be in $\mathbb{R}$, $y_0 \in \mathbb{R}^n$ & $\varphi \underset{\text{a cts}}{\wedge}: [a,b] \times B_R(y_0) \longrightarrow \mathbb{R}^n$. Assume $\exists\, k>0$ s.t.

$$\|\varphi(t,x) - \varphi(t,y)\| \leq k\|x-y\| \quad \forall\, t \in [a,b]\ \forall\, x,y \in B_R(y_0)$$

Then $\exists\, \varepsilon>0$ s.t. $\forall\, t_0 \in [a,b]$ the IVP

$$f'(t) = \varphi(t, f(t)) \quad , \quad f(t_0) = y_0$$

has a unique sol^n on $[t_0-\varepsilon, t_0+\varepsilon] \cap [a,b]$.

Remark 1. Think of $t \in [a,b]$ as time, $x \in B_R(y_0)$ as position & $f(t)$ of position of particle at time t.

2. A solution is a function $f: [t_0-\varepsilon, t_0+\varepsilon] \cap [a,b] \to B_R(y_0)$ which is diff'ble and satisfies the IVP.

Proof $\mathbb{R}^n$ is complete (Prop 2) & $B_R(y_0)$ is closed in $\mathbb{R}^n$, so by Prop 4, $B_R(y_0)$ is complete.

Since φ is continuous, it is bounded on the closed bdd set

$[a,b] \times B_R(y_0) \subset \mathbb{R}^{n+1}$.

Let $C = \sup\{\|\varphi(t,x)\| ; t \in [a,b], x \in B_R(y_0)\}$.

Let $\varepsilon = \min\left(\frac{R}{C}, \frac{1}{2K}\right)$. Let $[c,d] = [t_0-\varepsilon, t_0+\varepsilon] \cap [a,b]$.

Let $M = C([c,d], B_R(y_0))$. Then $M \neq \phi$, complete in uniform metric D by Thm 6.

Define $T: M \to M$ by $(Tg)(t) = y_0 + \int_{t_0}^{t} \varphi(s, g(s))\, ds$.

Well-defined $Tg \in M$?

Tg continuous $s \mapsto \varphi(s, g(s))$ is continuous, so by FTC, Tg is even diff'ble & $(Tg)'(t) = \varphi(t, g(t))$.

Tg takes values in $B_R(y_0)$ $\|(Tg)(t) - y_0\| = \left\|\int_{t_0}^{t} \varphi(s,g(s))\, ds\right\|$

$$\leq \left|\int_{t_0}^{t} \|\varphi(s,g(s))\|\, ds\right| \leq \varepsilon C \leq R \quad ✓$$

Thm 8 (Lindelöf - Picard) Let $a<b, R>0$ be in $\mathbb{R}$, $y_0 \in \mathbb{R}^n$, $\varphi : [a,b] \times B_R(y_0) \to \mathbb{R}^n$ be continuous.

Assume $\exists\, k>0$ s.t. $\|\varphi(t,x) - \varphi(t,y)\| \leqslant K\|x-y\|$ $\forall\, t \in [a,b]$, $\forall x,y \in B_R(y_0)$. Then $\exists\, \varepsilon>0$, $\forall t_0 \in [a,b]$ the IVP

$$f'(t) = \varphi(t, f(t)) \quad , \quad f(t_0) = y_0$$

has unique sol^n on $[t_0-\varepsilon, t_0+\varepsilon] \cap [a,b]$.

Pf $C = \sup\{\|\varphi(t,x)\| : t \in [a,b], x \in B_R(y_0)\}$

$$\varepsilon = \min\left(\frac{R}{C}, \frac{1}{2K}\right)$$

$[c,d] = [t_0-\varepsilon, t_0+\varepsilon] \cap [a,b]$ $\quad c<d$, $t_0 \in [c,d]$

$M = C([c,d], B_R(y_0))$ complete in unif metric D.

$T : M \to M$, $(Tg)(t) = y_0 + \int_{t_0}^{t} \varphi(s, g(s))\, ds$

T well-defined ✓

f is a sol^n of the IVP $\Leftrightarrow$ $f \in M$ & $Tf = f$

$\Rightarrow$ f is diff'ble, so f is continuous & takes values in $B_R(y_0)$

So $f \in M$. Since $s \mapsto \varphi(s, f(s))$ is continuous, we have f' is also continuous. So by FTC

$$f(t) - f(t_0) = \int_{t_0}^{t} \varphi(s, f(s))\, ds$$

So $f(t) = y_0 + \int_{t_0}^{t} \varphi(s,f(s))\, ds = (Tf)(t)$.

Hence $Tf = f$.

$\Leftarrow$ If $f \in M$, then $s \mapsto \varphi(s, f(s))$ is continuous, so by FTC, Tf is diff'ble & $(Tf)'(t) = \varphi(t, f(t))$.

Now $f = Tf$, so f is diff'ble and $f'(t) = \varphi(t, f(t))$.

Also, $f(t_0) = (Tf)(t_0) = y_0$ ✓

T is a contraction map on M For $g, h \in M$,

$$\|(Tg)(t) - (Th)(t)\| = \left\| \int_{t_0}^{t} (\varphi(s,g(s)) - \varphi(s,h(s)))\, ds \right\|$$

$$\leqslant \left| \int_{t_0}^{t} \|\varphi(s,g(s)) - \varphi(s,h(s))\|\, ds \right|$$

$$\leqslant \varepsilon \cdot K D(g,h) \leqslant \tfrac{1}{2} D(g,h)$$

since $\|\varphi(s,g(s)) - \varphi(s,h(s))\| \leqslant k\|g(s) - h(s)\| \leqslant k D(g,h)$

L11.2

Taking sup over all $t \in [c,d]$, we get

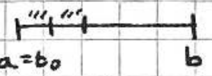
$D(Tg, Th) \leq \frac{1}{2} D(g,h)$ ✓

⊗

Apply Thm 7 (CMT)

□

<u>Remarks</u> ① In general, we cannot guarantee a global solution on $[a,b]$.



⊗ [For the IVP $f'(t) = f(t^2)$, $f(0) = y_0$

we found unique sol^n on $[0, \frac{1}{2}]$

By same argument $\forall \lambda \in (0,1)$ $\exists$ unique sol^n f_λ on $[0,\lambda]$

For $0 < \lambda < \mu < 1$, $f_\mu\big|_{[0,\lambda]} = f_\lambda$ by uniqueness

This gives a sol^n on $[0,1)$.

In fact can extend to $[0,b]$ for some $b > 1$.]

② Suppose we have $\psi : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (similar conditions as before), want to solve the IVP

need more $f^{(j)}(t_0) = z_j$

$$f^n(t) = \psi(t, f(t)), \quad f(t_0) = z_0$$

$f'(t), \dots, f^{(n-1)}(t)$

some $n \in \mathbb{N}$

Consider $\varphi : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x_0, \dots, x_{n-1}$

$$(t, x_0, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, \psi(t,))$$

If g is a sol^n of $\left.\begin{aligned} g'(t) &= \varphi(t, g(t)) \\ g(t_0) &= (z_0, \dots, z_n) \end{aligned}\right\}$ $g(t) = \Big(g_1(t), \dots, g_n(t)\Big)$

Let $f = g_1$. $(g_1'(t), \dots, g_n'(t)) = (g_2(t), g_3(t), \dots, g_n(t), \psi(t, g(t)))$

$$g_2 = f', \; g_3 = g_2' = f'', \dots, g_n = f^{(n-1)}(t), \; f^{(n)} = g_n' = \psi(t, \dots, g_n)$$
$$= \psi(t, f, f^{(2)}, \dots, f^{(n-1)})$$

<u>Addenda</u> ① $f : [a,b] \longrightarrow \mathbb{R}^n$ can be written as

$f(t) = (f_1(t), \dots, f_n(t))$ where $f_j(t) = (f(t))_j$

So $f_j : [a,b] \longrightarrow \mathbb{R}$.

So $f'(t) = (f_1'(t), \dots, f_n'(t))$ assuming f_j diff'ble $\forall j$ ← DEFINITIONS

$\int_a^b f(t)\,dt = \left(\int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt\right) = v$.

e.g. FTC for f follows from FTC for f_j

L 11. 3

$\|v\| = \left(\sum_{j=1}^{n} v_j^2 \right)^{1/2}$

So $\|v\|^2 = \sum_{j=1}^{n} v_j \int_a^b f_j(t)\,dt = \int_a^b \sum_{j=1}^{n} v_j f_j(t)\,dt$

$\leq \int_a^b \left(\sum_{j=1}^{n} v_j^2 \right)^{1/2} \left(\sum_{j=1}^{n} f_j(t)^2 \right)^{1/2} dt$

$\uparrow$
Cauchy Schwarz

$= \|v\| \int_a^b \|f(t)\|\,dt$

$\therefore \|v\| = \left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt$ ✓

② <u>B-W in $\mathbb{R}^n$</u> assume $(x^{(k)})$ is a bdd sequence in $\mathbb{R}^n$

Say $x^{(k)} = (x_j^{(k)})_{j=1}^{n}$. Since $|x_j^{(k)}| \leq \|x^{(k)}\|$ each sequence $(x_j^{(k)})_{k=1}^{\infty}$ is bdd in $\mathbb{R}$.

By Bolzano-Weierstrass in $\mathbb{R}$ - n times - get subsequence $(y^{(k)})$ of $(x^{(k)})$ s.t. $(y_j^{(k)})_{k=1}^{\infty}$ is convergent $\forall j$.

So then $y^{(k)}$ is convergent in $\mathbb{R}^n$.

③ Let $A \subset \mathbb{R}^n$ be closed and bdd, $f: A \to \mathbb{R}$ continuous.

Then f is bdd on A.

If not, $\exists\ (x^{(k)})$ in A s.t. $\|f(x^{(k)})\| > k \quad \forall k$.

$\exists$ convergent subsequence $(y^{(k)}) \longrightarrow z \in A$ by closedness

Since f cts at z, $f(y^{(k)}) \longrightarrow f(z)$.

Hence $(f(y^{(k)}))$ is bdd ※

<u>correction to Thm 3</u> $\ell_\infty(S)$ complete in D

Start with Cauchy seq (f_n) in $\ell_\infty(S)$.

<u>STEP 1</u> $f_n \to f$ pointwise for some scalar f on S

<u>STEP 2</u> ~~$D(f_n, f)$~~ can choose ~~n~~ N s.t. $\forall\, m, n \geq N$, $D(f_n, f_m) \leq 1$

For fixed $x \in S$, $\forall n \geq N$, $|f_n(x) - f_N(x)| \leq 1$

Let $n \to \infty$ so $|f(x) - f_N(x)| \leq 1$.

This holds $\forall x$, so f_N bdd $\Rightarrow$ f bdd.

<u>STEP 3</u> fine ✓

L|2.1

5. Topological Spaces

Def Given a set X, a topology on X is a collection τ of subsets of X (i.e. $\tau \subset \mathcal{P}X$) s.t.

(i) $\phi, X \in \tau$

(ii) if $\forall i \in I$, $U_i \in \tau$, then $\bigcup_{i \in I} U_i \in \tau$

(iii) if $U, V \in \tau$, then $U \cap V \in \tau$

Note (iii) $\Rightarrow$ $\forall n \in \mathbb{N}$, $\forall U_1, \dots, U_n \in \tau$, we have $\bigcap_{i=1}^{n} U_i \in \tau$

Def A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

Members of τ are called open sets. So for $U \subset X$, U is open (or τ-open or open in X) $\Leftrightarrow$ $U \in \tau$

Examples 1. Metric topologies by Prop 3.8

From now on a metric space is always assumed to carry the metric topology (unless otherwise stated)

E.g. $\mathbb{R}$ has the Euclidean topology, same for $\mathbb{R}^n$

Def A topological space X (or the topology of X) is metrizable if $\exists$ metric d on X whose metric topology is the given topology

(In that case any other metric $d' \sim d$ will give this topology)

2. Indiscrete topology on a set X is $\tau = \{\phi, X\}$

Def Given topologies τ_1, τ_2 on X, say τ_1 is coarser than τ_2 (or τ_2 is finer than τ_1) if $\tau_1 \subset \tau_2$.

Note: the indiscrete topology is the coarsest on any set

If $|X| \geqslant 2$, then this is not metrizable. Given metric d on X, fix $x \neq y$ in X. Let $r = d(x,y)$ and $U = D_r(x)$.

Then $x \in U$ $(r > 0)$, $y \notin U$, U is open $\underset{\text{wrt } d}{\wedge}$ (Lemma 3.5)

"So" $U \notin \tau$, as desired.

3. Discrete topology on a set X is $\tau = \mathcal{P}X$.

This is metrizable by the discrete metric.

This is the *finest* topology on X.

4. The *cofinite topology* on a set X is

$$\tau = \{\emptyset\} \cup \{ \mathcal{U} \subset X \mid \mathcal{U} \text{ is cofinite, i.e. } X \setminus U \text{ is finite} \}$$

When X is finite, then τ is the discrete topology.

When X is infinite, then τ is not metrizable.

Fix $x \neq y$ in X. Assume $\mathcal{U}, V$ are open, $x \in U$, $y \in V$. Then U and V are cofinite. So $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite.

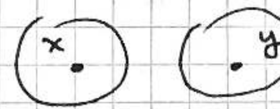
Since X is infinite, $U \cap V \neq \emptyset$.

Def A top. space Y is *Hausdorff* if $\forall\, x \neq y$ in Y $\exists$ open sets $\mathcal{U}, V$ s.t. $\mathcal{U} \ni x$, $V \ni y$ and $U \cap V = \emptyset$.

Note that for infinite X, the cofinite topology is not Hausdorff.

Prop 1 Metric spaces are Hausdorff

Proof Fix $x \neq y$ in a metric space (M, d)



Let $r = \frac{1}{2} d(x,y)$, Let $U = D_r(x)$, $V = D_r(y)$.

$r > 0$, so $x \in \mathcal{U}$, $y \in V$. $\mathcal{U}, V$ are open in M (Lemma 3.5)

If $z \in U \cap V$, then $d(x,y) \leq d(x,z) + d(z,y) < 2r = d(x,y)$. $\divideontimes$

So $\mathcal{U}, V$ are disjoint. $\square$

Def A subset A of a top. space (X, τ) is *closed* (or *τ-closed* or *closed in X*) if $X \setminus A$ is open.

Note In a metric space, this agrees with the previous def^n, by Lemma 3.9

Prop 2 Properties of closed sets in a top. space X

(i) $\emptyset, X$ are closed

(ii) if $\forall i \in I \neq \emptyset$, A_i is closed, then $\bigcap_{i \in I} A_i$ is closed

(iii) if $n \in \mathbb{N}$, $A_1, \dots, A_n$ are closed, then $\bigcup_{k=1}^{n} A_k$ is closed $\square$

Ex In cofinite topology, A is closed $\Leftrightarrow$ $A = X$ or A finite

Def Let X be a topological space, $x \in X$, $\mathcal{U} \subset X$. We say *$\mathcal{U}$ is a ngbd of X* if $\exists$ open set V s.t. $x \in V \subset \mathcal{U}$

Note In a metric space this agrees with the previous def^n. Indeed,

L12.3

Let (M,d) be a metric space, $x \in M$, $\mathcal{U} \subset M$. Assume $\exists r > 0$, $D_r(x) \subset \mathcal{U}$.

Then $V = D_r(x)$ is open & $x \in V \subset \mathcal{U}$. ✓ Conversely, assume $\exists$ open V s.t. $x \in V \subset \mathcal{U}$. Since V is open, $\exists r > 0$, $D_r(x) \subset V \subset \mathcal{U}$. ✓

<u>Prop 3</u> Let $\mathcal{U}$ be a subset of a top. space X. Then $\mathcal{U}$ is open iff $\forall x \in \mathcal{U}$, $\mathcal{U}$ is a ngbd of x.

<u>Pf</u> "$\Rightarrow$" given $x \in \mathcal{U}$, take $V = \mathcal{U}$. So V is open, $x \in V \subset \mathcal{U}$.

$\subsetneq$oof

"$\Leftarrow$" for every $x \in \mathcal{U}$, $\exists$ open $V_x \ni x$ and $V_x \subset \mathcal{U}$

Then $\mathcal{U} = \bigcup_{x \in \mathcal{U}} V_x$ is open, being a union of open sets. $\square$

<u>Def</u> Let (x_n) be a sequence in a top. space X & let $x \in X$.

Say <u>(x_n) converges to x</u> (write $x_n \to x$ as $n \to \infty$) if $\forall$ ngbds $\mathcal{U}$ of x, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $x_n \in \mathcal{U}$.

<u>Note</u> In a metric space, this agrees with the previous def^n (Prop 3.6)

<u>Example</u> In an indiscrete space every sequence converges to every pt.

<u>Prop 4</u> In a Hausdorff space, if $x_n \to x$ and $x_n \to y$, then $x = y$.

<u>Proof</u> Assume not. Then $\exists$ disjoint open $\mathcal{U} \ni x$, $V \ni y$.

$x_n \to x$ so $\exists N_1$ s.t. $\forall n \geq N_1$, $x_n \in \mathcal{U}$

$x_n \to y$ so $\exists N_2$ s.t. $\forall n \geq N_2$, $x_n \in V$

So for $n \geq \max\{N_1, N_2\}$, $x_n \in \mathcal{U} \cap V = \phi$. ⨳ $\square$

L13.1

Remark In a metric space, a set A is closed $\Leftrightarrow$ whenever (x_n) in A converges to x in the whole space, then $x \in A$.

"$\Rightarrow$" holds in any top space, but "$\Leftarrow$" does not in general

Def Let X be a top space, $A \subset X$, $x \in X$. We say x is an accumulation point of A (or limit point or cluster point) if $\forall$ ngbds $\mathcal{U}$ of x, $\exists\ a \in A \cap \mathcal{U}$ with $a \neq x$, i.e. $U \cap (A \setminus \{x\}) \neq \emptyset$

The derived set A' of A is the set of all accumulation points of A.

Ex In $\mathbb{R}$, $A = [0,1) \cup \{2\}$ $\quad A' = [0,1]$

$$\mathbb{Q}' = \mathbb{R}, \quad \mathbb{Z}' = \emptyset$$

Prop 5 Let X be a top space, $A \subset X$. Then A is closed iff $A' \subset A$.

Proof "$\Rightarrow$" If A is closed, then $\mathcal{U} = X \setminus A$ is open.

So for any $x \in X \setminus A$, $\mathcal{U}$ is a ngbd of x and $\mathcal{U} \cap A = \emptyset$.

So $x \notin A'$. Thus, $A' \subset A$. ✓

"$\Leftarrow$" Given $x \in X \setminus A$, we have $x \notin A'$. So $\exists$ ngbd $\mathcal{U}$ of x s.t. $\mathcal{U} \cap (A \setminus \{x\}) = \mathcal{U} \cap A = \emptyset$. Choose open set V s.t. $x \in V \subset \mathcal{U}$. Then $V \cap A = \emptyset$, i.e. $V \subset X \setminus A$. So $X \setminus A$ is a ngbd of all its points & hence open (Prop 3) □

Def Let A be a subset of a top space X. Define the interior of A as the set $\operatorname{int} A = A^\circ = \bigcup \{\mathcal{U} \subset X : \mathcal{U} \text{ open}, \mathcal{U} \subset A\}$.

The closure of A is $Cl(A) = \overline{A} = \bigcap \{F \subset X : F \text{ closed}, A \subset F\}$.

Note A° is open, $A^\circ \subset A$, and if $\mathcal{U} \subset A$, $\mathcal{U}$ open, then $\mathcal{U} \subset A^\circ$

So A is open iff $A^\circ = A$. [X is closed & $X \supset A$] $\overline{A}$ is closed, $A \subset \overline{A}$, and if $F \supset A$, F closed, then $\overline{A} \subset F$. So A is closed iff $A = \overline{A}$.

Prop 6 We have (a) $A^\circ = \{x \in X \mid A \text{ is a ngbd of } x\}$

(b) $\overline{A} = \{x \in X \mid \forall \text{ ngbds } \mathcal{U} \text{ of } x,\ \mathcal{U} \cap A \neq \emptyset\} = A \cup A'$

Pf (a) $x \in A^\circ$ iff $\exists$ open $\mathcal{U}$ s.t. $\mathcal{U} \subset A$, and $x \in \mathcal{U}$. So done ✓

(b) Assume $x \notin \overline{A}$. Then $x \in \mathcal{U} = X \setminus \overline{A}$, which is open. So $\mathcal{U}$ is a ngbd of x. Moreover, $\mathcal{U} \cap A = \phi$. So $x \notin A \cup A'$. étru

Conversely, assume $\exists$ ngbd $\mathcal{U}$ of x s.t. $\mathcal{U} \cap A = \phi$. Choose open set V s.t. $x \in V \subset \mathcal{U}$. We still have $V \cap A = \phi$, i.e. $A \subset X \setminus V$

So $\overline{A} \subset X \setminus V$, since $X \setminus V$ is closed. Hence $x \notin \overline{A}$. $\square$

Ex In $\mathbb{R}$, $A = [0,1) \cup \{2\}$, then $\overline{A} = [0,1] \cup \{2\}$.

And $A^\circ = (0,1)$.

$\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}^\circ = \phi$, $\overline{\mathbb{Z}} = \mathbb{Z}$, $\mathbb{Z}^\circ = \phi$

Remark In a metric space, $x \in \overline{A} \iff \exists (x_n)$ in A s.t. $x_n \to x$

"$\Leftarrow$" true in any top space, "$\Rightarrow$" is not in general   coconut ☺

Convergent sequences determine metric topologies.

False in general!

Def Let X be a top space. Say $A \subset X$ is dense in X if $\overline{A} = X$. Say X is separable if $\exists$ ctble dense set in X.

E.g. $\mathbb{R}$ is separable, by say $\mathbb{Q}$. So is $\mathbb{R}^n$.

An uncountable set with discrete topology is not separable.

Subspaces Let (X, τ) be a top space & $Y \subset X$. The subspace topology on Y is the topology $\{\mathcal{U} \cap Y \mid \mathcal{U} \in \tau\}$. Also called the topology on Y induced by τ. So for $\mathcal{U} \subset Y$, $\mathcal{U}$ is open in Y iff $\exists$ open set V in X s.t. $\mathcal{U} = V \cap Y$.

Eg $X = \mathbb{R}$, $Y = [0,2]$. Then $\mathcal{U} = (1,2]$ is open in Y, since $\mathcal{U} = Y \cap (1,3)$ & $(1,3)$ is open in $\mathbb{R}$.

$\mathcal{U}$ is not open in $\mathbb{R}$.

Remarks 1. $Z \subset Y \subset X$. Then X induces the subspace topology on Y, which induces a subspace top on Y. But also, X induces a subspace top on Z. These two are the same.

2. $N \subset M$, M a metric space. The metric on M induces the metric

topology on M, which induces the subspace top on N.

Also, restricting the metric to N makes N a metric space with its metric top. These two are the same.

Prop 7 Let Y be a subspace of a top space X.

(a) $A \subset Y$ is closed in $Y \iff \exists$ closed set $B \subset X$ s.t. $B \cap Y = A$

(b) For $A \subset Y$, $\overline{A}^Y = \overline{A}^X \cap Y$, where

$\overline{A}^Y =$ closure of A in Y

$\overline{A}^X =$ " X

Note (b) fails for interior E.g. $X = \mathbb{R}$, $Y = \{0\} = A$.

In Y, $A^\circ = A$. In X, $A^\circ = \emptyset$

Proof (a) A is closed in $Y \iff Y \setminus A$ open in Y (def)

$\iff \exists$ open V in X s.t. $Y \setminus A = Y \cap V$ (def)

$\iff$ " $A = Y \cap (X \setminus V)$

$\iff \exists$ closed B in X s.t. $A = Y \cap B$

(b) $\overline{A}^X$ is closed in X, so $\overline{A}^X \cap Y$ is closed in Y, and $A \subset Y \cap \overline{A}^X$.

Hence $\overline{A}^Y \subset Y \cap \overline{A}^X$.

Given $y \in Y \cap \overline{A}^X$, a nghd $\mathcal{U}'$ of y in Y, we can choose open set $\mathcal{U}$ in Y s.t. $y \in \mathcal{U} \subset \mathcal{U}'$.

So $\exists$ open set V in X s.t. $\mathcal{U} = Y \cap V$. So $y \in V$, and hence $V \cap A \neq \emptyset$ since $y \in \overline{A}^X$. Hence $U \cap A = Y \cap V \cap A$

$$= V \cap A \neq \emptyset.$$

Thus $\mathcal{U}' \cap A \neq \emptyset$. So by ~~defn~~ Prop 6 $y \in \overline{A}^Y$. $\square$

The purpose of this page is to spell out in detail the example given in lectures on applying Theorem 4.8 (Lindelöf–Picard) to n^{th}-order ODEs.

Example. Let $a < b$ and $R > 0$ be real numbers, let $z = (z_0, z_1, \dots, z_{n-1}) \in \mathbb{R}^n$ and let

$$\psi\colon [a,b] \times B_R(z) \to \mathbb{R}$$

be a continuous function. Assume that for some $K > 0$ we have

$$|\psi(t,x) - \psi(t,y)| \leqslant K\|x-y\| \qquad \text{for all } t \in [a,b] \text{ and all } x,y \in B_R(z)\ .$$

Then there exists $\varepsilon > 0$ such the for any $t_0 \in [a,b]$ the n^{th}-order IVP (initial value problem)

$$\text{(1)} \qquad \begin{aligned} g^{(n)}(t) &= \psi(t, g(t), g^{(1)}(t), g^{(2)}(t), \dots, g^{(n-1)}(t)) \\ g^{(j)}(t_0) &= z_j \quad \text{for } 0 \leqslant j \leqslant n-1 \end{aligned}$$

has a unique solution on $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$.

Note. This means that there is a unique n-times differentiable function

$$g\colon [c,d] \to \mathbb{R}$$

that satisfies (1) for all $t \in [c,d]$. This implicitly includes the assumption that

$$(g(t), g^{(1)}(t), g^{(2)}(t), \dots, g^{(n-1)}(t)) \in B_R(z)$$

for all $t \in [c,d]$.

Proof. Let us define $\varphi\colon [a,b] \times B_R(z) \to \mathbb{R}^n$ by setting

$$\varphi(t, x_0, x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, \psi(t, x_0, x_1, \dots, x_{n-1}))$$

for $t \in [a,b]$ and $x = (x_0, x_1, \dots, x_{n-1}) \in B_R(z)$. Then φ is continuous and satisfies

$$\|\varphi(t,x) - \varphi(t,y)\| \leqslant (K+1)\|x-y\| \qquad \text{for all } t \in [a,b] \text{ and all } x,y \in B_R(z)\ .$$

By Lindelöf–Picard (Theorem 4.8 in the lectures), there exists $\varepsilon > 0$ such that the IVP

$$\text{(2)} \qquad f'(t) = \varphi(t, f(t))\ , \qquad f(t_0) = z$$

has a unique solution on $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$. Let f be this unique solution. Thus, $f\colon [c,d] \to B_R(z)$ is a differentiable function with $f(t_0) = z$ and $f'(t) = \varphi(t, f(t))$ for all $t \in [c,d]$. Let $f_0, f_1, \dots, f_{n-1}$ be the components of f, *i.e.*, functions $f_j\colon [c,d] \to \mathbb{R}$ such that $f(t) = (f_0(t), f_1(t), \dots, f_{n-1}(t))$ for all $t \in [c,d]$. Since f is a solution of (2), each f_j is differentiable and

$$\text{(3)} \quad \begin{aligned} (f_0'(t), f_1'(t), \dots, f_{n-1}'(t)) &= f'(t) = \varphi(t, f(t)) \\ &= (f_1(t), f_2(t), \dots, f_{n-1}(t), \psi(t, f_0(t), f_1(t), \dots, f_{n-1}(t))) \end{aligned}$$

for all $t \in [c,d]$. Set $g = f_0$. Comparing coordinates in (3) shows that g is an n-times differentiable function $[c,d] \to \mathbb{R}$ with $g^{(j)} = f_j$ for $0 \leqslant j < n$ (induction on j), and moreover

$$\begin{aligned} g^{(n)}(t) = f_{n-1}'(t) &= \psi(t, f_0(t), f_1(t), \dots, f_{n-1}(t)) \\ &= \psi(t, g(t), g^{(1)}(t), \dots, g^{(n-1)}(t)) \end{aligned}$$

for all $t \in [c,d]$. Finally, since $f(t_0) = z$, we have $g^{(j)}(t_0) = f_j(t_0) = z_j$ for $0 \leqslant j \leqslant n-1$. This completes the proof of existence.

To prove uniqueness, assume that $\tilde{g}$ is another solution to (1) on $[c,d]$. Define $\tilde{f}\colon [c,d] \to B_R(z)$ by setting $\tilde{f}(t) = (\tilde{g}(t), \tilde{g}^{(1)}(t), \dots, \tilde{g}^{(n-1)}(t))$. It is straightforward to verify that $\tilde{f}$ is a solution to (2). It follows that $\tilde{f} = f$ and $\tilde{g} = g$. □

L14.1 LAST TIME SUBSPACES

TODAY PRODUCTS AND QUOTIENTS

● BUT FIRST BASE AND CONTINUITY

Def A base for a top space (X, τ) is a family $\mathcal{B} \subset \tau$ s.t. $\forall \mathcal{U} \in \tau, \exists \mathcal{C} \subset \mathcal{B}$ s.t. $\mathcal{U} = \bigcup_{B \in \mathcal{C}} B$, equivalently $\tau = \{ \bigcup_{B \in \mathcal{C}} B \mid \mathcal{C} \subset \mathcal{B} \}$. So a base always determines the topology.

Examples $\{(a,b) \mid a<b, a,b \in \mathbb{R}\}$ is a base for $\mathbb{R}$

In general, the collection of all open balls in a metric space is a base for the metric topology

● Lemma 8 Let X be a set and $\mathcal{B} \subset \mathcal{P}X$. Assume

(i) $X = \bigcup_{B \in \mathcal{B}} B$

(2) $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B}$ s.t.

$$x \in B \subset B_1 \cap B_2$$

Then there's a unique topology on X for which $\mathcal{B}$ is a base.

Note Often, we'll have $X \in \mathcal{B}$ ($\Rightarrow$ (i)), $\forall B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{B}$ ($\Rightarrow$ (ii))

Proof Define $\tau = \{ \bigcup_{B \in \mathcal{C}} B \mid \mathcal{C} \subset \mathcal{B} \}$

● $\emptyset \in \tau$ ($\mathcal{C} = \emptyset$), $X \in \tau$ ($\mathcal{C} = \mathcal{B}$ by (i))

Given $\mathcal{U}_i \in \tau$ for all i in some set I. So $\forall i \in I, \exists \mathcal{C}_i \subset \mathcal{B}$ s.t. $\mathcal{U}_i = \bigcup_{B \in \mathcal{C}_i} B$. Then $\bigcup_{i \in I} \mathcal{U}_i = \bigcup_{i \in I} \bigcup_{B \in \mathcal{C}_i} B = \bigcup_{B \in \bigcup_i \mathcal{C}_i} B$ ✓

Finally, assume $\mathcal{U}_1, \mathcal{U}_2 \in \tau$. Then

$\exists \mathcal{C}_j \subset \mathcal{B}$ s.t. $\mathcal{U}_j = \bigcup_{B \in \mathcal{C}_j} B$ for $j = 1, 2$.

Given $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, $\exists B_1 \in \mathcal{C}_1, B_2 \in \mathcal{C}_2$ s.t. $x \in B_1 \cap B_2$

By (ii) $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset B_1 \cap B_2 \subset \mathcal{U}_1 \cap \mathcal{U}_2$.

So $\mathcal{U}_1 \cap \mathcal{U}_2 = \bigcup_{x \in \mathcal{U}_1 \cap \mathcal{U}_2} B_x \in \tau$.

● So τ is a topology, and by def^n $\mathcal{B}$ is a base for τ. □

Def A top space is 2nd countable if it has a countable base.

Eg $\{ (a,b) \mid a<b, a,b \in \mathbb{Q} \}$ is a countable base for $\mathbb{R}$

$\{ D_r(x) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}, r>0 \}$ $\mathbb{R}^n$

Def A map $f: X \to Y$ between top spaces is continuous if $\forall$ open sets V in Y, $f^{-1}(V)$ is open in X

Note Between metric spaces, this agrees with previous def^n by 3.7

Prop 9 Let $f: X \to Y$ be a map between top spaces. Then

(a) f is cts $\Leftrightarrow$ $\forall$ closed V in Y, $f^{-1}(V)$ is closed in X

(b) If $\mathcal{B}$ is a base for Y, then f is cts $\Leftrightarrow$

$\forall B \in \mathcal{B}$, $f^{-1}(B)$ is open in X

(c) If f is cts, and $g: Y \to Z$ is cts, where Z is top space, then $g \circ f$ is continuous.

Proof (a) $\forall$ closed V in Y, $f^{-1}(V)$ is closed in X

$\Leftrightarrow$ " , $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ is open in X

$\Leftrightarrow$ $\forall$ open V in Y, $f^{-1}(V)$ open in X

(b) $\underline{\Rightarrow}$ true since each $B \in \mathcal{B}$ is open in Y

$\underline{\Leftarrow}$ Given open $V \subset Y$, then $V = \bigcup_{B \in \mathcal{C}} B$, and

$f^{-1}(V) = f^{-1}(\cup B) = \cup f^{-1}(B)$ — open in X ✓

(c) W open in $Z \Rightarrow g^{-1}(W)$ open in $Y \Rightarrow f^{-1}(g^{-1}(W))$ open in X

$\Rightarrow (g \circ f)^{-1}(W)$ open in X □

Examples 1. Constant functions $f: X \to Y$, $f(x) = y_0 \ \forall x \in X$

For $V \subset Y$, $f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \emptyset & \text{else} \end{cases}$

2. Identity $id: X \to X$, $(id)^{-1}(V) = V$

3. If $Y \subset X$, the inclusion map $i: Y \to X$, $i(y) = y \ \forall y \in Y$ is continuous, since $i^{-1}(V) = V \cap Y$.

So if $f: X \to Z$ is cts, so is $f \circ i = f|_Y : Y \to Z$.

L14.4

f is continuous $\Leftrightarrow$ $\pi_X \circ f$, $\pi_Y \circ f$ are continuous

<u>Note</u> Let $g = \pi_X \circ f$, $h = \pi_Y \circ f$

$$Z \to X \qquad Z \to Y$$

$$f(z) = (g(z), h(z))$$

(ii) says f cts iff g, h cts

L1~~6~~5.1

Prop 10 Let X, Y be top spaces

(a) $\pi_X : X \times Y \to X$, $(x,y) \mapsto x$
$\pi_Y : X \times Y \to Y$, $(x,y) \mapsto y$ are cts

(b) Let Z be a top space & $f : Z \to X \times Y$ be a function.

Then f is cts $\Leftrightarrow$ $\pi_X \circ f$, $\pi_Y \circ f$ are cts

Pf (a) Given an open set U in X, $\pi_X^{-1}(U) = U \times Y$ which is open in $X \times Y$. So π_X is cts. Similarly, π_Y is cts.

(b) "$\Rightarrow$" Follows from (a) and Prop 9 (c)

"$\Leftarrow$" Given open sets U in X, V in Y,

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(U \times Y \cap X \times V) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}(\pi_X^{-1}(U)) \cap f^{-1}(\pi_Y^{-1}(V)) \\ \overset{\text{in } Z}{} &= (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V) \end{aligned}$$

which is open by assumption. By Prop 9 (b) it follows that f is cts. $\square$

All of the above extend to finite products. Given $n \in \mathbb{N}$, top spaces $X_1, \dots, X_n$, the family

$$\mathcal{B} = \{ U_1 \times U_2 \times \cdots \times U_n \mid U_i \text{ open in } X_i,\ 1 \leq i \leq n \}$$

is a base for a (unique) topology on $X_1 \times \cdots \times X_n$, the product topology.

For $W \subset X_1 \times \cdots \times X_n$, W is open $\Leftrightarrow$ $\forall x \in W$, $\exists$ open U_i in X_i s.t. $x \in U_1 \times \cdots \times U_n \subset W$

If each X_i is a metric space, with metrics d_i, $1 \leq i \leq n$, then the product topology on $X_1 \times \cdots \times X_n$ is given by e.g. the metric

$$D_1(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i) \qquad \text{for } x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n)$$

Taking D_∞ also works.

Ex The product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ is the usual Euclidean topology.

L16.2

Arbitrary products - non-examinable

Quotient Spaces

Let X be a top. space & let R be an equivalence rel^n on X.

[So $R \subset X \times X$ is s.t. writing $x \sim y$ for $(x,y) \in R$,

$\forall x \in X,\ x \sim x$

$\forall x, y \in X,\ x \sim y \Rightarrow y \sim x$ $\qquad \forall x,y,z \in X,\ \begin{matrix} x \sim y, \\ y \sim z \end{matrix} \Rightarrow x \sim z.$]

Let X/R be the set of all equivalence classes, the quotient set.

Let $q : X \to X/R$ be the quotient map, so for $x \in X$,

$q(x) = \{ y \in X : y \sim x \}$ is the equiv class of x.

The <u>quotient topology</u> on X/R is $\{ V \subset X/R : q^{-1}(V) \text{ open in } X \}$

[sketch : ϕ is open since $q^{-1}(\phi) = \phi$

$q^{-1}(X/R) = X$ so X/R open

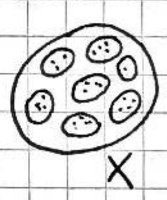
$q^{-1}\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} q^{-1}(V_i)$ so open

$q^{-1}(U_1 \cap U_2) = q^{-1}(U_1) \cap q^{-1}(U_2)$ so open]

<u>Note</u> 1. q is surjective & continuous

2. For $x \in X$, $t \in X/R$, we have $x \in t \Leftrightarrow q(x) = t$

For $V \subset X/R$,
$$\begin{aligned} q^{-1}(V) &= \{ x \in X : q(x) \in V \} \\ &= \{ x \in X : \exists\, t \in V \text{ s.t. } q(x) = t \} \\ &= \{ x \in X : \exists\, t \in V \text{ s.t. } x \in t \} \\ &= \bigcup_{t \in V} t \end{aligned}$$
WOT



<u>Ex</u> $\mathbb{R}$ with usual top

$\mathbb{R}$ is an abelian group under $+$

$\mathbb{Q} \leqslant \mathbb{R}$, $\mathbb{R}/\mathbb{Q}$ = set of cosets of $\mathbb{Q}$ in $\mathbb{R}$

comes from the eq. rel^n $x \sim y$ iff $x - y \in \mathbb{Q}$

What is quotient topology on $\mathbb{R}/\mathbb{Q}$?

Assume $V \subset \mathbb{R}/\mathbb{Q}$ is open and non-empty.

So $\exists\ a < b$ in $\mathbb{R}$ s.t. $(a,b) \subset q^{-1}(V)$

Given $x \in \mathbb{R}$, $\exists\ r \in \mathbb{Q} \cap (a - x, b - x)$.

So $r+x \in (a,b)$. So $q(x) = q(r+x) \in q(q^{-1}(V)) \underset{\substack{\uparrow \\ q \text{ surjective}}}{=} V$

So $V = \mathbb{R}/\mathbb{Q}$.

So quotient topology is indiscrete, so not metrizable.

Recall Let X be a set, R an eq. rel. on X, $q : X \to X/R$ be the quotient map. Assume $f : X \to Y$ respects R : $\forall x, y \in X$, $x \sim y \Rightarrow f(x) = f(y)$. Then $\exists!\ \tilde{f} : X/R \to Y$ s.t. $f = \tilde{f} \circ q$

Picture

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & Y \\ & X/R \arrow[u, dotted, "\exists!\,\tilde{f}\text{ s.t. diagram commutes}"'] \end{tikzcd}$$

In addition, 1. $\operatorname{im}\tilde{f} = \operatorname{im} f$ (as q surjective)

2. If f fully respects R : $\forall x, y \in X$, $x \sim y \Leftrightarrow f(x) = f(y)$, then $\tilde{f}$ is injective ($\tilde{f}(q(x)) = \tilde{f}(q(y)) \Rightarrow f(x) = f(y)$
$\Rightarrow x \sim y \Rightarrow q(x) = q(y)$)

Prop 11 Let X, Y be top spaces, R an equiv rel^n on X, $q : X \to X/R$ be the quotient map, $f : X \to Y$ some function that respects R. Let $\tilde{f} : X/R \to Y$ be the unique map s.t. $f = \tilde{f} \circ q$. Then

(a) f cts $\Rightarrow \tilde{f}$ cts

(b) f open map $\Rightarrow \tilde{f}$ open map

So if f fully respects R and is surjective, cts & open, then $\tilde{f}$ is a homeomorphism.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & Y \\ & X/R \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

Proof (a) Let V be open in Y

Is $\tilde{f}^{-1}(V)$ open in X/R?

$q^{-1}(\tilde{f}^{-1}(V)) = (\tilde{f} \circ q)^{-1}(V) = f^{-1}(V)$ is open in X ✓

(b) Given open set V in X/R, $U = q^{-1}(V)$ is open in X, and $V = q(q^{-1}(V)) = q(U)$. So $\tilde{f}(V) = \tilde{f}(q(U)) = f(U)$ is open in Y. ✓

The rest follows. □

Remark Work "upstairs"

E.g. $\mathbb{R}/\mathbb{Z}$ ($x \sim y \Leftrightarrow x - y \in \mathbb{Z}$) is homeomorphic to the circle $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

L16.4

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "q"'] & S^1 \\ \mathbb{R}/\mathbb{Z} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$f(t) = e^{2\pi i t}$

fully respects relⁿ

surjective onto S^1

cts because exp is

$\tilde{f} : x + \mathbb{Z} \mapsto e^{2\pi i x}$ is a cts bijection. Is $\tilde{f}$ open?

LAST TIME $f : \mathbb{R} \to S^1$, $f(x) = e^{2\pi i x}$

Is f open? ($\mathbb{R}/\mathbb{Z}$ homeomorphic to S^1)

Sketch assume $\mathcal{U} \subset \mathbb{R}$ open, $f(\mathcal{U})$ not open

$\exists z_n \in S^1 \setminus f(\mathcal{U})$ s.t. $z_n \to z \in f(\mathcal{U})$

Choose $x \in \mathcal{U}$ s.t. $z = f(x)$.

Choose $x_n \in [x - \frac{1}{2}, x + \frac{1}{2}]$ s.t. $f(x_n) = z_n$. Note $x_n \in \mathbb{R} \setminus \mathcal{U}$.

By Bolzano-Weierstrass, WLOG $x_n \to y \in [x - \frac{1}{2}, x + \frac{1}{2}]$

Since f is cts, $z_n = f(x_n) \to f(y) = z = f(x)$. So $y - x \in \mathbb{Z}$.

Thus, $y = x$. Since $\mathbb{R} \setminus \mathcal{U}$ is closed, $y = x \in \mathbb{R} \setminus \mathcal{U}$ ⁂

6 Connectedness

Let $I \subset \mathbb{R}$ be an interval: $\forall x, y, z \in \mathbb{R}$ if $x < y < z$ and $x, z \in I$, then $y \in I$. (So I is of the form (a,b), $[a,b]$, $(a,b]$, $[a,b)$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, $(-\infty,\infty)$)

Recall from IA the IVT: if $f : I \to \mathbb{R}$ is cts, then $\forall x, y \in I$, $\forall c \in \mathbb{R}$, if $x < y$ and $f(x) < c < f(y)$ (OR around), then $\exists z \in I$ s.t. $x < z < y$ and $f(z) = c$. In other words, $f(I)$ is an interval.

Ex $f : (0,1) \cup (1,2) \to \mathbb{R}$,

$$x \mapsto \begin{cases} 0 & \text{if } x \in (0,1), \\ 1 & \text{if } x \in (1,2). \end{cases}$$

Then f is cts but $\operatorname{im} f$ is not an interval.

Def A topological space X is disconnected if $\exists$ subsets $\mathcal{U}, V$ of X s.t. $\mathcal{U}, V$ open, nonempty, disjoint, and $\mathcal{U} \cup V = X$.

Say $\mathcal{U}, V$ disconnect X.

X is connected if it's not disconnected.

Thm 1 For a topological space X TFAE

(i) X is connected

(ii) if $X \to \mathbb{R}$ is cts then $f(X)$ is an interval

(iii) every continuous $f : X \to \mathbb{Z}$ is constant

__Pf (i) ⇒ (ii)__ Assume not. So $\exists\, x < y < z$ in $\mathbb{R}$ s.t. $x, z \in f(X)$ but $y \notin f(X)$. Let $\mathcal{U} = f^{-1}((-\infty, y))$, $V = f^{-1}((y, \infty))$.

Since f is cts, $\mathcal{U}, V$ are open. Since $x \in (-\infty, y) \cap f(X)$, $\mathcal{U}$ is non-empty. Similarly for V.

$\mathcal{U} \cap V = f^{-1}((-\infty, y) \cap (y, \infty)) = f^{-1}(\phi) = \phi$

$\mathcal{U} \cup V = f^{-1}(\mathbb{R} \setminus \{y\}) = X$ since $y \notin f(X)$

So $\mathcal{U}, V$ disconnect X. ※

__(ii) ⇒ (iii)__ immediate

__(iii) ⇒ (i)__ Suppose $\mathcal{U}, V$ disconnect X. Define $f : X \to \mathbb{Z}$ by $f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{U}, \\ 1 & \text{if } x \in V. \end{cases}$ Well-defined since $\mathcal{U} \cap V = \phi$, $\mathcal{U} \cup V = X$.

$$f^{-1}(A) = \begin{cases} \phi & \text{if } 0, 1 \notin A \\ U & \text{if } 0 \in A, 1 \notin A \\ V & \text{if } 1 \in A, 0 \notin A \\ X & \text{if } 0, 1 \in A \end{cases} \quad \text{all open}$$

So f is cts. Since $\mathcal{U}, V$ are non-empty, f is not constant. ※ ☐

__Cor 2__ Let $X \subset \mathbb{R}$. Then X is connected ⇔ X is an interval

__Pf__ ⇒ Since $f : X \to \mathbb{R}$, $f(x) = x$ is cts, by Thm 1 we have $f(X) = X$ is an interval ✓

__⇐__ __easy__ by IVT, (ii) in Thm 1 holds, so X is connected.

__directly__ assume X is not connected. Let $\mathcal{U}, V \subset X$ disconnect X.

Fix $x \in \mathcal{U}$, $y \in V$. WLOG $x < y$.

Then $[x, y] \subset X$ as X is an interval.

Let $z = \sup(\mathcal{U} \cap [x, y])$. Note that $x \leq z \leq y$.

__Either__ $z \in \mathcal{U}$, so $z < y$ and $\exists\, N \in \mathbb{N}$ s.t. $z + \frac{1}{n} \in (z, y]$ $\forall\, n \geq N$. Since $(z, y] \subset V$ & $z + \frac{1}{n} \to z$, we have $z \in V$ since $V = X \setminus \mathcal{U}$ is closed. So $z \in \mathcal{U} \cap V$ ※

__Or__ $z \in V$. Then $\forall\, n \in \mathbb{N}$, since $z - \frac{1}{n} < z$, $\exists\, x_n \in \mathcal{U} \cap [x, y]$ s.t. $x_n > z - \frac{1}{n}$. So $x_n \to z$ and hence $z \in \mathcal{U} = X \setminus V$ (closed in X) ※ ☐

L16.3

More examples 1. $\emptyset$, $\{x\}$ are connected

2. Any indiscrete topological space is connected

3. The cofinite topology on an infinite set is connected

4. The discrete topology on a set of size $\geqslant 2$ is disconnected

Lemma 3 A subspace Y of a topological space X is disconnected iff $\exists$ open sets U, V in X s.t. $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$,

$$U \cap V \cap Y = \emptyset, \quad U \cup V \supset Y.$$

Pf "$\Rightarrow$" Let U', V' disconnect Y. Then $\exists$ open U, V in X s.t. $U' = Y \cap U$ and $V' = Y \cap V$. These work.

"$\Leftarrow$" Given U, V as in the statement, taking $U' = U \cap Y$, $V' = Y \cap V$, these disconnect Y. □

Prop 4 Let Y be a connected subspace of a topological space X. Then $\overline{Y}$ is connected.

Pf Assume not. So by Lemma 3, $\exists$ open sets U, V in X s.t.

$U \cap \overline{Y}$, $V \cap \overline{Y}$ non-empty $\qquad U \cup V \supset \overline{Y}$

$U \cap V \cap \overline{Y} = \emptyset$

Hence, $U \cap V \cap Y = \emptyset$ and $U \cup V \supset Y$. So we cannot have $U \cap Y$, $V \cap Y$ both non-empty, else Y is disconnected.

WLOG $U \cap Y = \emptyset$. Then $Y \subset X \setminus U$, which is closed.

Hence $\overline{Y} \subset X \setminus U$, so $U \cap \overline{Y} = \emptyset$. ※ □

Remarks 1. Could use Thm 1 (iii) to prove Prop 4.

2. Any Z s.t. $Y \subset Z \subset \overline{Y}$ is connected

In the space Z, the closure of Y is Z.

Thm 5 Let $f: X \to Y$ be a cts map between top spaces

If X is connected then so is $f(X)$

Proof WLOG $f(X) = Y$. Indeed, f viewed as a function $X \to f(X)$ is continuous: if $\mathcal{U}$ open in $f(X)$ then $\mathcal{U} = V \cap f(X)$ of some open V in Y, so $f^{-1}(\mathcal{U}) = f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(V)$ is open.

Assume $g: Y \to \mathbb{Z}$ is continuous. Then $g \circ f: X \to \mathbb{Z}$ is cts.

Since X is connected, $g \circ f$ is constant: $\exists\, n \in \mathbb{Z}$ s.t. $\forall\, x \in X$,

$g(f(x)) = n$. Since $f(X) = Y$, we have $g(y) = n \quad \forall\, y \in Y$.

So g is constant and hence Y is connected. ☐

Remark 1. Connectedness is a topological property: if X, Y are homeomorphic, then X connected $\Leftrightarrow$ Y connected.

2. If $f: X \to Y$ is cts, & $A \subset X$, A is connected, then $f(A)$ is connected. Apply Thm 5 to $f|_A : A \to Y$.

Corollary 6 If X is connected and R is an equiv rel^n on X, then X/R is connected.

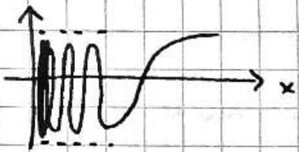
Pf The quotient map $q: X \to X/R$ is continuous & surjective. ☐

Example Let $Y = \{ (x, \sin(\frac{1}{x})) : x > 0 \} \subset \mathbb{R}^2$

Then $Y = \operatorname{im} f$, $f: (0, \infty) \to \mathbb{R}^2$, $x \mapsto (x, \sin(\frac{1}{x}))$.

f is continuous (look at components) & $(0,\infty)$ is connected (Thm 2)

By Thm 5, Y is connected.



By Prop 4, $\overline{Y}$ is also connected.

Note that $\overline{Y} = Y \cup \{0\} \times [-1, 1]$.

If $(x,y) \in \overline{Y}$, then $\exists\, (x_n, y_n) \in Y$ s.t. $x_n \to x$, $y_n \to y$

Also, $x_n > 0$, $y_n = \sin 1/x_n \; \forall n$.

So $-1 \leq y_n \leq 1 \; \forall n$, so $-1 \leq y \leq 1$.

If $x = 0$, we happy, $(x,y) \in \{0\} \times [-1,1]$.

If $x > 0$, by continuity of f at x, $y_n \to \sin 1/x$.

So $(x,y) \in Y$.

Given $-1 \leq y \leq 1$, we show $(0,y) \in \overline{Y}$.

$\forall n \in \mathbb{N}$, $x \mapsto \frac{1}{x}$ maps $(0, \frac{1}{n})$ to (n, ∞), & hence

$x \mapsto \sin(\frac{1}{x})$ maps $(0, \frac{1}{n})$ to $[-1,1]$.

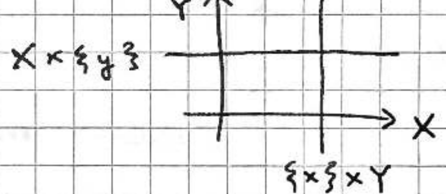
So $\exists x_n \in (0, \frac{1}{n})$ s.t. $\sin(1/x_n) = y$.

Now $(x_n, \sin(1/x_n)) = (x_n, y) \to (0, y)$. ✓

Lemma 7 Let $\mathcal{A}$ be a family of connected subsets of a topological space X. If $\forall A, B \in \mathcal{A}$ we have $A \cap B \neq \emptyset$, then $\bigcup_{A \in \mathcal{A}} A$ is connected.

Pf Assume $f : \bigcup_{A \in \mathcal{A}} A \to \mathbb{Z}$ is continuous. Then by Thm 1, for any $A \in \mathcal{A}$, $f|_A$ is constant for each $A \in \mathcal{A}$: so $\exists n_A \in \mathbb{Z}$ s.t. $f(x) = n_A$ $\forall x \in A$. For $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$ so $n_A = n_B$.
So f is constant. So by Thm 1, $\bigcup_{A \in \mathcal{A}} A$ is connected. □

Thm 8 If X, Y are connected topological spaces, then $X \times Y$ is connected in the product topology.



Pf WLOG $X \neq \emptyset$, $Y \neq \emptyset$

Fix $x \in X$. We show $\{x\} \times Y$ is connected.

Define $f : Y \to X \times Y$, $y \mapsto (x, y)$.

Note $\pi_X \circ f(y) = x$, $\pi_Y \circ f(y) = y$ are continuous. So is f.

By Thm 5, $\operatorname{im} f = \{x\} \times Y$ is connected.

Similarly, $X \times \{y\}$ is connected $\forall y \in Y$.

Fix $x_0 \in X$. For $y \in Y$, let $A_y = \{x_0\} \times Y \cup X \times \{y\}$.

Since $\{x_0\} \times Y \cap X \times \{y\} \ni (x_0, y)$ is non empty, by Lemma 7, A_y is connected $\forall y \in Y$.

Next, $\forall y, z \in Y$, $A_y \cap A_z \supset \{x_0\} \times Y$ is non-empty.

So by Lemma 7, $\bigcup_{y \in Y} A_y$ is connected. But this is $X \times Y$. □

Eg. $\mathbb{R}^n$ is connected

<u>Components</u> Let X be a top space. For $x, y \in X$, let

$$x \sim y \iff \exists \text{ connected subset } A \text{ of } X \text{ s.t. } x, y \in A$$

This is an equivalence relation: $x \sim x$ as $\{x\}$ is connected; $x \sim y \Rightarrow y \sim x$ is clear; if $x \sim y$, $y \sim z$, $\exists$ connected subsets A, B of X s.t. $x, y \in A$, $y, z \in B$. Then $A \cap B \neq \emptyset$, so by Lemma 7, $A \cup B$ is connected & $x, z \in A \cup B$, so $x \sim z$.

Let C_x be the equivalence class of $x \in X$.

An equivalence class is called a <u>connected component</u> of X.

<u>Prop 9</u> Connected components are non-empty, maximal wrt inclusion connected subsets, they're closed, and they partition X.

<u>Pf</u> Let C be a connected component.

So $C = C_x$ for some $x \in X$. Since $x \sim x$, $x \in C_x$ so $C \neq \emptyset$.

Given $y \in C$, $y \sim x$, so $\exists$ connected set A_y s.t. $x, y \in A_y$.

$\forall z \in A_y$, we have $z, x \in A_y$, so $z \sim x$ which means $z \in C$.

So $A_y \subset C$. By Lemma 7, $\bigcup_{y \in C} A_y = C$ is connected.

If $C \subset D$, D connected, then $\forall y \in D$, $x, y \in D$, so $y \sim x$, so $y \in C$. So $C = D$. Thus C is maximal.

By Prop 4, $\overline{C}$ is connected, $C \subset \overline{C}$ so by maximality, $C = \overline{C}$, i.e. C is closed. Last part follows from IA. □

<u>Def^n</u> A top space X is <u>path connected</u> if $\forall x, y \in X$, $\exists$ cts $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.

<u>Thm 10</u> Any path connected space is connected.

L18.1

<u>Def</u> X is path connected if $\forall x, y \in X$ $\exists$ cts $\gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

<u>Thm 10</u> path connected $\Rightarrow$ connected

<u>Proof</u> Assume X is path connected but not connected.

Let $\mathcal{U}, V$ disconnect X. Fix ~~x~~ $x \in \mathcal{U}$, $y \in V$.

$\exists$ cts $\gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

Then $\gamma^{-1}(\mathcal{U}), \gamma^{-1}(V)$ are open in $[0,1]$ as γ is continuous.

$0 \in \gamma^{-1}(\mathcal{U})$, $1 \in \gamma^{-1}(V)$, so $\gamma^{-1}(\mathcal{U}), \gamma^{-1}(V)$ non-empty

$\gamma^{-1}(\mathcal{U}) \cap \gamma^{-1}(V) = \gamma^{-1}(\mathcal{U} \cap V) = \phi$

$\gamma^{-1}(\mathcal{U}) \cup \gamma^{-1}(V) = \gamma^{-1}(\mathcal{U} \cup V) = [0,1]$

So $\gamma^{-1}(\mathcal{U}), \gamma^{-1}(V)$ disconnect $[0,1]$ ⋇ (Thm 2) $\square$

<u>Example</u> $Y = \{ (x, \sin\frac{1}{x}) : x > 0 \}$

$X = \overline{Y} = Y \cup \{0\} \times [-1,1]$

X is connected

We show X is NOT path connected

Let $x = (0,0)$, $y = (1, \sin(1))$. Assume $\gamma : [0,1] \longrightarrow X$ is continuous, $\gamma(0) = x$, $\gamma(1) = y$. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{R}$ be the components of γ: $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $t \in [0,1]$

These are continuous.

If $\gamma_1(t) > 0$, then $\gamma_1([0,t]) \supset [0, \gamma_1(t)]$ by IVT, so $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{2\pi n}, \frac{1}{2\pi n + \frac{\pi}{2}} \in (0, \gamma_1(t)) \subset \gamma_1([0,t])$

So $\exists\, a, b \in (0,t)$ s.t. $\gamma_1(a) = \frac{1}{2\pi n}$, $\gamma_1(b) = \frac{1}{2\pi n + \frac{\pi}{2}}$

Hence $\gamma_2(a) = 0$, $\gamma_2(b) = 1$.

Starting with 1, we inductively construct a sequence

$1 > t_1 > t_2 > t_3 > \cdots$ s.t. $\gamma_2(t_n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

Now $t_n \rightarrow s$, say, so $\gamma_2(t_n) \longrightarrow \gamma_2(s)$ ⋇

<u>Lemma 11</u> Let $f : X \rightarrow Y$ be a function between topological spaces. Assume $X = A \cup B$, A, B closed sets & $f|_A : A \rightarrow Y$,

$f|_B : B \to Y$ are continuous. Then f is continuous.

Proof Given a closed set V in Y,

$$f^{-1}(V) = (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B)$$

$$= \underbrace{f|_A^{-1}(V)}_{\text{closed in A}} \cup \underbrace{f|_B^{-1}(V)}_{\text{closed in B}}$$

$$= (W_A \cap A) \cup (W_B \cap B)$$

for some closed sets W_A, W_B in X. Since A, B, W_A, W_B are closed in X, $f^{-1}(V)$ is closed in X ✓ ☐

Corollary 12 Let X be a top space & for $x, y \in X$ define $x \sim y$ $\Leftrightarrow$ $\exists$ cts $\gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$. Then this is an equivalence relation.

Pf For $x \in X$, we have $x \sim x$ by taking $\gamma(t) = x$ $\forall t$.

If $x \sim y$, so $\gamma : [0,1] \longrightarrow X$ is cts with $\gamma(0) = x$, $\gamma(1) = y$, then let $(-\gamma) : [0,1] \longrightarrow X$ be given by $(-\gamma)(t) = \gamma(1-t)$. This is cts, $(-\gamma)(0) = \gamma(1) = y$, $(-\gamma)(1) = \gamma(0) = x$, so $y \sim x$.

Assume $x \sim y$, $y \sim z$ and $\gamma, \delta : [0,1] \longrightarrow X$ are cts with $\gamma(0) = x$, $\gamma(1) = \delta(0) = y$, $\delta(1) = z$.

Define $\eta : [0,1] \to X$ by $\eta(t) = \begin{cases} \gamma(2t) & t \in [0, \tfrac{1}{2}], \\ \delta(2t-1) & t \in [\tfrac{1}{2}, 1]. \end{cases}$

At $t = \frac{1}{2}$, $\eta(\frac{1}{2}) = \gamma(1) = \delta(0) = y$, so η well-defined.

By Lemma 11, η is cts & $\eta(0) = x$, $\eta(1) = z$ so $x \sim z$. ☐

Theorem 13 Let $\mathcal{U} \subset \mathbb{R}^n$ be open. Then $\mathcal{U}$ is connected iff $\mathcal{U}$ is path-connected.

Pf $\Leftarrow$ Thm 10

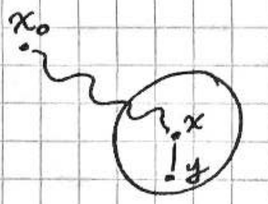
$\Rightarrow$ WLOG $\mathcal{U} \neq \emptyset$. Fix $x_0 \in \mathcal{U}$. Let $V = \{x \in \mathcal{U} \mid x \sim x_0\}$.

We show that $V, \mathcal{U} \setminus V$ are open. Since $\mathcal{U}$ is connected, one of $V, \mathcal{U} \setminus V$ is $\emptyset$, but $x_0 \in V$ so $\mathcal{U} \setminus V = \emptyset$ i.e. $\mathcal{U} = V$.

This will show that $\mathcal{U}$ is path connected.

V open Take $x \in V$. Since $\mathcal{U}$ is open $\exists r > 0$, $D_r(x) \subset \mathcal{U}$

L18.3



Given $y \in D_r(x)$, $\gamma(t) = (1-t)x + ty$,

$t \in [0,1]$ is continuous and connects x to y.

So $y \sim x$ & hence $y \sim x_0$, i.e. $y \in V$.

Thus $D_r(x) \subset V$, and so V is open.

$\mathcal{U} \setminus V$ open Fix $x \in \mathcal{U} \setminus V$. $\exists r > 0$ s.t. $D_r(x) \subset \mathcal{U}$.

For $y \in D_r(x)$, $y \sim x$ as before, so $y \not\sim x_0$, else $x \sim x_0$ ⨳ (Cor 12)

Thus, $D_r(x) \subset \mathcal{U} \setminus V$. ✓

□

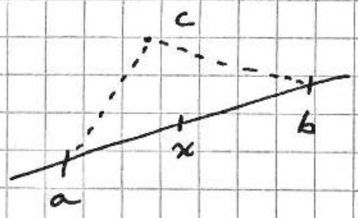
Example For $n \geq 2$, $\mathbb{R}^n$ not homeomorphic to $\mathbb{R}$

Assume $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a homeomorphism.

Fix $x \in \mathbb{R}^n$ & let $y = f(x)$. Then $f|_{\mathbb{R}^n \setminus \{x\}} : \mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R} \setminus \{y\}$ is a homeomorphism & $(f|_{\mathbb{R}^n \setminus \{x\}})^{-1} = f^{-1}|_{\mathbb{R} \setminus \{y\}}$.

$\mathbb{R} \setminus \{y\} = (-\infty, y) \cup (y, \infty)$ is disconnected,

$\mathbb{R}^n \setminus \{x\}$ is path connected, so connected ⨳



7. Compactness

Recall that a real valued, cts function on a closed, bdd theorem is bdd & attains its bounds

Q For which topological spaces X is every if ~~every~~ bdd cts $f : X \to \mathbb{R}$

Some examples 1. X finite

2. If $\forall$ cts $f : X \to \mathbb{R}$, $\exists n \in \mathbb{N}$, $\exists A_1, \dots, A_n \subset X$ s.t. $X = \bigcup_{j=1}^{n} A_j$ & f bdd on each A_j, then ✓

Note given continuous $f : X \to \mathbb{R}$, for $x \in X$,

$\mathcal{U}_x = f^{-1}((f(x) - 1, f(x) + 1))$ is open, $x \in \mathcal{U}_x$ & $\forall y \in \mathcal{U}_x$

$|f(y)| \leq |f(x)| + 1$, so f is bdd on $\mathcal{U}_x$ & $X = \bigcup_{x \in X} \mathcal{U}_x$.

If $\exists$ *finite* $F \subset X$ s.t. $X = \bigcup_{x \in F} \mathcal{U}_x$, then f is bdd on X.

Def An *open cover* is a family $\mathcal{U}$ of open sets in X s.t.

$$X = \bigcup \{ U : U \in \mathcal{U} \}.$$

A *subcover* of $\mathcal{U}$ is a subset $\mathcal{V} \subset \mathcal{U}$ s.t. $X = \bigcup_{U \in \mathcal{V}} U$, it is a *finite subcover* if $\mathcal{V}$ is a finite set.

L18.4

X is compact if every open cover for X has a finite subcover

Thm 1 If X is compact & $f: X \to \mathbb{R}$ is continuous, then f is bounded & attains its bounds.

Thm 1 X is $\overset{\neq\emptyset}{\swarrow}$ compact & $f : X \to \mathbb{R}$ is continuous, then f is continuous, then f is bounded & attains its bounds.

Pf f bounded For $x \in X$, $\mathcal{U}_x = f^{-1}((f(x)-1, f(x)+1))$ is open and contains x & f is bdd on $\mathcal{U}_x$ by $|f(x)| + 1$.

Since $X = \bigcup_{x \in X} \mathcal{U}_x$ & X is compact, $\exists$ finite $F \subset X$ s.t. $\bigcup_{x \in F} \mathcal{U}_x = X$. So f is bdd on X by $\max\{|f(x)|+1 : x \in F\}$.

f attains its bounds Let $m = \inf\{f(x) : x \in X\}$ (exists, since $X \neq \emptyset$ & f bdd) Assume $\nexists\, x$ s.t. $f(x) = m$.

For $x \in X$, $f(x) > m$, so $\exists\, m_x$ s.t. $f(x) > m_x > m$.

Let $\mathcal{U}_x = f^{-1}((m_x, \infty))$ — open, contains x, $\inf_{\mathcal{U}_x} f \geqslant m_x > m$.

X is compact & $X = \bigcup_{x \in X} \mathcal{U}_x$, so $\exists$ finite $F \subset X$ s.t. $X = \bigcup_{x \in F} \mathcal{U}_x$.

Then $\forall y \in X$, $f(y) \geqslant \min\{m_x : x \in F\} > m$ ⨳

Sup is similar. □

Note For $Y \subset X$, Y is compact $\Leftrightarrow$ whenever $\mathcal{U}$ is a family of open sets in X s.t. $Y \subset \bigcup_{U \in \mathcal{U}} U$, then $\exists$ finite $\mathcal{V} \subset \mathcal{U}$ s.t. $Y \subset \bigcup_{U \in \mathcal{V}} U$.

Thm 2 $[0,1]$ is compact

Pf Let $\mathcal{U}$ be a family of open sets in $\mathbb{R}$ s.t. $[0,1] \subset \bigcup_{U \in \mathcal{U}} U$.

Assume $\nexists$ finite $\mathcal{V} \in \mathcal{U}$ s.t. $[0,1] \subset \bigcup_{U \in \mathcal{V}} U$.

In general, if $0 \leqslant a < b \leqslant 1$ & $[a,b]$ cannot be covered by finitely many $U \in \mathcal{U}$, then let $c = \frac{1}{2}(a,b)$ & since $[a,b]$ is $[a,c] \cup [c,b]$, one of $[a,c], [c,b]$ cannot be covered by finitely many $U \in \mathcal{U}$. Note that $c - a = b - c = \frac{1}{2}(b-a)$.

Inductively, find intervals $[0,1] \supset I_1 \supset I_2 \supset \cdots$ s.t. I_n cannot be covered by finitely many $U \in \mathcal{U}$ & if $I_n = [a_n, b_n]$ then $b_n - a_n = 2^{-n}$. From IA Analysis, $a_n \to x$ say $x \in [0,1]$ & $b_n = a_n + (b_n - a_n) \longrightarrow x$. Now $\exists\, U \in \mathcal{U}$ s.t. $x \in U$.

U is open, so $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U$.

Since $a_n, b_n \to x$, $\exists n$ s.t. $a_n, b_n \in (x_n - \varepsilon, x_n + \varepsilon)$.

So $I_n \subset \mathcal{U}$. $\divideontimes$ $\square$

Prop 3 Let X be a top. space & $Y \subset X$.

(a) X compact, Y closed in X $\Rightarrow$ Y compact

(b) X Hausdorff, Y compact $\Rightarrow$ Y closed in X

Pf (a) Let $\mathcal{U}$ be a family of open sets in X s.t. $Y \subset \bigcup_{U \in \mathcal{U}} U$.

Then $\mathcal{U} \cup \{X \setminus Y\}$ is an open cover for X.

X compact, so $\exists$ finite $\mathcal{V} \subset \mathcal{U}$ s.t. $\bigcup_{U \in \mathcal{V}} U \cup (X \setminus Y) = X$.

Hence $Y \subset \bigcup_{U \in \mathcal{V}} U$ ✓

(b) Fix $x \in X \setminus Y$. $\forall y \in Y$ $\exists$ disjoint open sets U_y, V_y s.t. $x \in U_y$, $y \in V_y$.

Since $Y \subset \bigcup_{y \in Y} V_y$ & Y compact, $\exists$ finite $F \subset Y$ s.t. $Y \subset \bigcup_{y \in F} V_y$.

Then $U = \bigcap_{y \in F} U_y$ is open, $x \in U$, $\forall z \in Y$ $\exists y \in F$ s.t. $z \in V_y$ and so $z \notin U_y$, and hence $z \notin U$.

Thus $U \subset X \setminus Y$. So $X \setminus Y$ is a ngbd of all of its points, & hence open. $\square$

Prop 4 If X is compact & $f : X \to Y$ is continuous, then $f(X)$ is compact.

Pf Let $\mathcal{U}$ be a family of open sets in Y s.t. $f(X) \subset \bigcup_{U \in \mathcal{U}} U$.

Then $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover for X. Since X is compact, $\exists$ finite $\mathcal{V} \subset \mathcal{U}$ s.t. $X = \bigcup_{U \in \mathcal{V}} f^{-1}(U)$.

So $f(X) \subset \bigcup_{U \in \mathcal{V}} U$. $\square$

Remarks 1. Compactness is a topological property

2. If $f : X \to Y$ is cts, $A \subset X$, A is compact, then $f(A)$ is compact.

$(f|_A : A \to Y$ is cts$)$

EX For $a \leq b$ in $\mathbb{R}$, $[a,b] = f([0,1])$ where $f(x) = (b-a)x + a$.

So $[a,b]$ is compact.

Cor 5 If X is compact & R is an equiv relⁿ on X, then X/R is compact. $\square$

Thm 6 (Topological Inverse Function Theorem, TIFT)

If $f: X \to Y$ is a continuous bijection, X is compact, Y is Hausdorff, then f is a homeomorphism.

Pf Need f to be an open map. Since f bijection, this is the same as f being a closed map (image of closed set is closed).

If $V \subset X$ is closed, then V is compact (Prop 3).

f cts, so $f(V)$ is compact (Prop ~~3~~ 4)

Y Hausdorff, so $f(V)$ is closed in Y (Prop 3). $\square$

Ex $f: \mathbb{R} \to S^1$, $f(t) = e^{2\pi i t}$, induces a cts bijection $\tilde{f}: \mathbb{R}/\mathbb{Z} \to S^1$

$\mathbb{R}/\mathbb{Z} = q(\mathbb{R}) = q([0,1])$ ↑quotient map So $\mathbb{R}/\mathbb{Z}$ is compact by Thm 1, Prop 4.

S^1 is Hausdorff since its' a metric space.

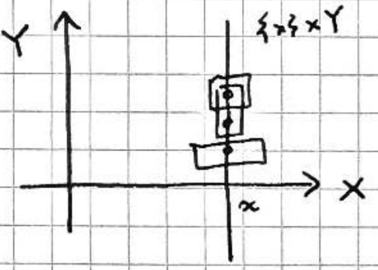
So $\mathbb{R}/\mathbb{Z} \cong S^1$ ☺

Thm 7 (Tychonov) The product of compact spaces is compact

Pf (For two, & hence finitely many, spaces)
See notes on arbitrary products

Assume X, Y are compact & $\mathcal{U}$ is an open cover for $X \times Y$.

Stage 1 Fix $x \in X$.



$\forall y \in Y \ \exists W_y \in \mathcal{U}$ s.t. $(x,y) \in W_y$.

$\exists$ open sets U_y in X, V_y in Y s.t.

$(x,y) \in U_y \times V_y \subset W_y$

Then $\{V_y : y \in Y\}$ is an open cover for Y.

Y compact, so $\exists$ finite $F \subset Y$ s.t. $\bigcup_{y \in F} V_y = Y$.

Let $U = \bigcap_{y \in F} U_y$ Then U is open, $x \in U$, and

$$U \times Y = \bigcup_{y \in F} (U \times V_y) \subset \bigcup_{y \in F} (U_y \times V_y) \subset \bigcup_{y \in F} W_y$$

Conclusion: $\forall x \in X \ \exists$ open ngbd T_x of X in x s.t. $T_x \times Y$ is finitely covered by $\mathcal{U}$.

L19.4

Stage 2 $\{T_x : x \in X\}$ is an open cover for X. X is compact, so $\exists$ finite $E \subset X$ s.t. $\bigcup_{x \in E} T_x = X$. So

$$X \times Y = \bigcup_{x \in E} T_x \times Y$$ # is finitely covered by $\mathcal{U}$. $\square$

Remark Tychonov extends to any finite product: if $n \in \mathbb{N}$ & $X_1, \dots, X_n$ are compact, then so is $X_1 \times \cdots \times X_n$.

Thm 8 (Heine-Borel) A subset K of $\mathbb{R}^n$ is compact if and only if K is closed and bounded.

Proof $\underline{\Rightarrow}$ $f: K \to \mathbb{R}$, $f(x) = \|x\|$, is cts, & thus bdd by Thm 1, i.e. $\exists M \geqslant 0$ s.t. $\|x\| \leqslant M$ for all $x \in K$. So K is bdd.

Prop 3 implies K is closed.

$\underline{\Leftarrow}$ Choose M s.t. $\|x\| \leqslant M$ $\forall x \in K$.

So $K \subset [-M, M]^n$.

By Thm 2 & Prop 4, $[-M, M]$ is compact.

By Tychonov (Thm 7), $[-M, M]^n$ is compact.

Since K is closed, it is compact by Prop 3. □

Def Given an open set $\mathcal{U} \subset \mathbb{R}^n$, a sequence of functions $f_k : \mathcal{U} \to \mathbb{R}$ converges locally uniformly on $\mathcal{U}$ to some function $f : \mathcal{U} \to \mathbb{R}$ if $\forall x \in \mathcal{U}$, $\exists r > 0$ s.t. $D_r(x) \subset \mathcal{U}$ & $f_k \to f$ uniformly on $D_r(x)$. This happens $\Leftrightarrow$ $f_k \to f$ uniformly on any compact subset of $\mathcal{U}$.

Def A top. space X is sequentially compact if every sequence in X has a convergent subsequence.

E.g. any closed, bdd subset of $\mathbb{R}^n$ (by Bolzano-Weierstrass)

Def Fix a metric space (M, d). For $\varepsilon > 0$, $F \subset M$, say F is an ε-net for M if $\forall x \in M$, $\exists y \in F$ s.t. $d(x,y) \leqslant \varepsilon$. (i.e. M is covered by $B_\varepsilon(y)$ for $y \in F$. Say F is a finite ε-net if, in addition, F is a finite set. Say M is totally bounded if $\forall \varepsilon > 0$, $\exists$ finite ε-net for M.

Thm 9 TFAE: (i) M is compact

(ii) M is sequentially compact

(iii) M is totally bounded & complete

L20.2

<u>Pf</u> <u>(i) ⇒ (ii)</u> Let (x_n) be a sequence in M.

For $n \in \mathbb{N}$, let $A_n = \{x_k : k > n\}$. We show that $\bigcap_{n\in\mathbb{N}} \overline{A_n} \neq \emptyset$.

Assume not. Then $\bigcup_{n\in\mathbb{N}} (\underbrace{M \setminus \overline{A}_n}) = M$. Since M is compact, $\exists N$ s.t. $\bigcup_{n=1}^{N} (M\setminus \overline{A}_n) = M$. $\overset{\text{open}}{\text{Since}}$ $A_m \supset A_n$ $\forall\, m \leq n$, we have $M \setminus \overline{A}_N = M$, i.e. $\overline{A}_N = \emptyset$. $\divideontimes$

Fix $x \in \bigcap_{n\in\mathbb{N}} \overline{A}_n$. We'll construct a subseq. of (x_n) converging to x. Now, $x \in \overline{A}_1$, so $D_1(x) \cap A_1 \neq \emptyset$, so $\exists\, k_1 > 1$ s.t. $x_{k_1} \in D_1(x)$.

$x \in \overline{A}_{k_1}$, so $D_{\frac{1}{2}}(x) \cap A_{k_1} \neq \emptyset$, so $\exists\, k_2 > k_1$ s.t. $x_{k_2} \in D_{\frac{1}{2}}(x)$.

Continue inductively to yield $k_1 < k_2 < \cdots$ s.t. $x_{k_m} \in D_{\frac{1}{m}}(x)$ $\forall m$.

So $x_{k_m} \longrightarrow x$ as $m \longrightarrow \infty$.

<u>(ii) ⇒ (iii)</u> M is complete since a Cauchy sequence with a convergent subsequence is convergent. Assume M is not totally bounded.

So $\exists\, \varepsilon > 0$ s.t. M has no finite ε-net. Pick $x_1 \in M$.

Having picked $x_1, x_2, \ldots, x_n$ pick $x_{n+1} \in M \setminus \bigcup_{j=1}^{n} B_\varepsilon(x_j)$.

The sequence (x_n) satisfies $d(x_m, x_n) \geq \varepsilon$ $\forall\, m \neq n$.

So (x_n) has no Cauchy subsequence. $\divideontimes$

<u>(iii) ⇒ (i)</u> Assume M is not compact. So $\exists$ open cover $\mathcal{U}$ for M with no finite subcover. Call $A \subset M$ "bad" if $\nexists$ finite $\mathcal{V} \subset \mathcal{U}$ with $A \subset \bigcup_{U\in\mathcal{V}} U$.

<u>E.g.</u> M is bad

Note if $A = \bigcup_{i=1}^{n} B_i$ & A is bad then $\exists\, i$ s.t. B_i is bad.

We next show that if A is bad & $\varepsilon > 0$, then $\exists\, B \subset A$ with B bad and $\operatorname{diam} B = \sup\{d(x,y) : x, y \in B\} < \varepsilon$.

Since M is totally bounded, $\exists$ finite $F \subset M$ s.t. $M = \bigcup_{x\in F} B_{\varepsilon/2}(x)$

Then $A = \bigcup_{x\in F} (A \cap B_{\varepsilon/2}(x))$, so $\exists\, x \in F$ s.t. $B = B_{\varepsilon/2}(x)$ is bad and $\operatorname{diam} B < \varepsilon$.

Using this construct a sequence $M \supset A_1 \supset A_2 \supset \cdots$ s.t. $\forall n$, A_n is bad & $\operatorname{diam} A < \frac{1}{n}$.

L 20.3

Pick $x_n \in A_n$ ($A_n \neq \phi$ since A_n is bad)

$\forall m, n \geqslant N$, $d(x_m, x_n) \leqslant \frac{1}{N}$ since $x_n, x_m \in A_N$

So (x_n) is Cauchy, & hence $x_n \longrightarrow x$ for some $x \in M$, since M is complete.

Now $\exists\, U \in \mathcal{U}$ s.t. $x \in U$. Since U is open, $\exists\, r > 0$ s.t. $D_r(x) \subset U$. Can choose n s.t. $d(x, x_n) < \frac{r}{2}$ & $\frac{1}{n} < \frac{r}{3}$.

$$\begin{aligned}\forall y \in A_n, \quad d(y,x) &\leqslant d(y, x_n) + d(x_n, x) \leqslant \operatorname{diam} A_n + d(x_n, x) \\ &< \frac{1}{n} + d(x_n, x) < r.\end{aligned}$$

So $A_n \subset D_r(x) \subset U$. $\divideontimes$ ☐

<u>Remarks</u> 1. New proof of B-W in $\mathbb{R}^n$.

2. New proof of Tychonov for metric spaces.

3. Thm 9 fails in general top spaces.

8. Differentiation

<u>Preliminaries</u> Fix $m, n \in \mathbb{N}$. $L(\mathbb{R}^m, \mathbb{R}^n) = \{ T : \mathbb{R}^m \to \mathbb{R}^n \mid T \text{ linear} \}$

$L(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^{mn}$. Let e_i, $1 \leq i \leq m$ be the SB for $\mathbb{R}^m$,
& e_j', $1 \leq j \leq n$ " $\mathbb{R}^n$.

We can identify $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ with the $n \times m$ matrix (T_{ji}) where $T_{ji} = \langle Te_i, e_j' \rangle$ $1 \leq i \leq m$, $1 \leq j \leq n$.

⌈ $\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$ for $x = (x_j)$, $y = (y_j) \in \mathbb{R}^n$;
$|\langle x, y \rangle| \leq \|x\| \|y\|$ ⌋

So $L(\mathbb{R}^m, \mathbb{R}^n)$ becomes a Euclidean space

$$\|T\| = \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |T_{ji}|^2 \Big)^{1/2} = \Big(\sum_{i=1}^{m} \|Te_i\|^2 \Big)^{1/2}$$

For $S, T \in L(\mathbb{R}^m, \mathbb{R}^n)$, $d(S,T) = \|S - T\|$.

<u>Lemma 1</u> (a) Given $T \in L(\mathbb{R}^m, \mathbb{R}^n)$, $\forall x \in \mathbb{R}^m$, $\|Tx\| \leq \|T\| \|x\|$
So T is Lipschitz & hence continuous.

(b) For $S \in L(\mathbb{R}^m, \mathbb{R}^n)$, $T \in L(\mathbb{R}^n, \mathbb{R}^p)$, $\|TS\| \leq \|T\| \|S\|$.

<u>Proof</u> (a) If $x = \sum_{i=1}^{m} x_i e_i$, then $\|Tx\| = \| \sum_{i=1}^{m} x_i Te_i \| \leq$

L 20.4

$$\leqslant \sum_{i=1}^{m} |x_i| \|Te_i\| \leqslant \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{m} \|Te_i\|^2 \right)^{1/2} = \|x\| \, \|T\|. \quad \checkmark$$

So for $x, y \in \mathbb{R}^m$, $d(Tx, Ty) = \|Tx - Ty\| = \|T(x-y)\|$

$$\leqslant \|T\| \|x-y\| = \|T\| \, d(x,y) \quad \checkmark$$

(b) $\|TS\| = \left(\sum_{i=1}^{m} \|TSe_i\|^2 \right)^{1/2} \leqslant \left(\sum_{i=1}^{m} \|T\|^2 \|Se_i\|^2 \right)^{1/2} = \|T\| \|S\|$

□

L21.1

<u>recall</u> $f : \mathbb{R} \to \mathbb{R}$ is diff at a if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

The limit is denoted $f'(a)$.

Let $\varepsilon(h) = \frac{f(a+h) - f(a)}{h} - f'(a)$.

Then $f(a+h) = \underbrace{f(a)}_{\text{const.}} + \underbrace{hf'(a)}_{\text{linear}} + \underbrace{h\,\varepsilon(h)}_{\text{error}}$.

$\varepsilon(h) \to 0$ as $h \to 0$ (same as $\varepsilon(0) = 0$ & ε cts at 0).

So the error is $o(h)$.

Def^n We are given $m, n \in \mathbb{N}$, an open set $\mathcal{U} \in \mathbb{R}^m$, a function $f : \mathcal{U} \to \mathbb{R}^n$, and $a \in \mathcal{U}$. Say f is diff'ble at a if $\exists\ T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and a function ε s.t.

$$f(a+h) = f(a) + T(h) + \varepsilon(h)\|h\| \quad ⊛$$

where $\varepsilon(h) \to 0$ as $h \to 0$ (i.e. $\varepsilon(0) = 0$ & ε cts at 0).

Remarks
$$\varepsilon(h) = \begin{cases} 0 & \text{if } h = 0, \\ \frac{f(a+h) - f(a) - T(h)}{\|h\|} & \text{if } h \neq 0,\ a+h \in \mathcal{U}. \end{cases}$$

domain of ε is $\{h \in \mathbb{R}^m : a+h \in \mathcal{U}\}$ on which ⊛ holds.

Since $\mathcal{U}$ is open $\exists\ r > 0$ s.t. $D_r(a) \subset \mathcal{U}$, so $D_r(0) \subset$ domain of ε.

Crucial: $\varepsilon(h) \to 0$ as $h \to 0$ $\iff$ $\varepsilon(0) = 0$ & cts at 0.

$\iff$ the error $\varepsilon(h)\|h\|$ is $o(\|h\|)$

Next, note that T is unique. Indeed, assume $S, T \in L(\mathbb{R}^m, \mathbb{R}^n)$, and $f(a+h) = f(a) + T(h) + o(\|h\|)$

$f(a+h) = f(a) + S(h) + o(\|h\|)$.

Then $\frac{S(h) - T(h)}{\|h\|} \to 0$ as $h \to 0$.

For $x \neq 0$ in $\mathbb{R}^m$,
$$\frac{S(x) - T(x)}{\|x\|} = \frac{S(\frac{1}{n}x) - T(\frac{1}{n}x)}{\|\frac{1}{n}x\|} \to 0 \text{ as } n \to \infty.$$

So $S = T$.

If f is diff at ~~x~~; then the unique $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ is the <u>derivative of f at a</u>, denoted by $f'(a)$.

So $f(a+h) = f(a) + f'(a)(h) + o(\|h\|)$.

Other notations for $f'(a)$: $Df(a)$ OR $Df|_a$.

Say f is differentiable on $\mathcal{U}$ if f is diff'ble at a $\forall a \in \mathcal{U}$, and the derivative of f on $\mathcal{U}$ is $f' : \mathcal{U} \to L(\mathbb{R}^m, \mathbb{R}^n)$, $a \mapsto f'(a)$.

Remark Case $m=1$ For $T \in L(\mathbb{R}, \mathbb{R}^m)$, setting $v = T(1)$, we have $\forall x \in \mathbb{R}$, $T(x) = T(x1) = xT(1) = xv$.

So $L(\mathbb{R}, \mathbb{R}^n) \cong \mathbb{R}^n$, $T \mapsto T(1)$.

So for open $\mathcal{U} \subset \mathbb{R}$, $f : \mathcal{U} \to \mathbb{R}^n$, $a \in \mathcal{U}$, we have f diff'ble at a iff $\exists v \in \mathbb{R}^n$ s.t. $f(a+h) = f(a) + hv + o(h)$.

Then $v = f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

Examples 1. Constant functions: $f : \mathbb{R}^m \to \mathbb{R}^n$, $f(x) = c \ \forall x$

$$f(a+h) = c = f(a) + 0(h) + 0$$

So f is diff'ble at a & $f'(a) = 0 \in L(\mathbb{R}^m, \mathbb{R}^n)$.

So $f' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$ is constant 0.

2. Linear maps $f : \mathbb{R}^m \to \mathbb{R}^n$ linear

$$f(a+h) = f(a) + f(h) + 0$$

So f diff'ble at a & $f'(a) = f$.

So $f' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$ maps $a \mapsto f$ $\forall a$.

3. Bilinear maps $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ bilinear

$$f((a,b) + (h,k)) = f(a+h, b+k)$$

$$= \underbrace{f(a,b)} + \underbrace{f(a,k) + f(h,b)}_{\text{linear in } (h,k)} + \underbrace{f(h,k)}_{\text{error } = o(\|(h,k)\|)?}$$

$$\|f(h,k)\| = \|f(\sum_{i=1}^{m} h_i e_i, \sum_{j=1}^{n} k_j e_j')\| = \|\sum_{i,j} h_i k_j f(e_i, e_j')\|$$

$$\leq \sum_{i,j} |h_i||k_j| \, \|f(e_i, e_j')\| \leq \|(h,k)\|^2 \left(\sum_{i,j} \|f(e_i, e_j')\| \right)$$

The error is $O(\|(h,k)\|^2)$, so $o(\|(h,k)\|)$.

So $f'(a,b)(h,k) = f(a,k) + f(h,b)$,

$f' : \mathbb{R}^m \times \mathbb{R}^n \to L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^p)$, is linear

4. $f : \mathbb{R}^m \to \mathbb{R}$, $f(x) = \|x\|^2$

$$f(a+h) = \|a+h\|^2 = \underbrace{\|a\|^2}_{f(a)} + \underbrace{2\langle a,h\rangle}_{\text{linear}} + \underbrace{\|h\|^2}_{O(\|h\|^2)}$$

So $f'(a)(h) = 2\langle a,h\rangle$.

5. Let M_n = space of $n\times n$ real matrices
$\cong L(\mathbb{R}^n, \mathbb{R}^n)$

$f : M_n \to M_n$, $f(A) = A^2$

$$f(A+H) = (A+H)^2 = \underbrace{A^2}_{f(A)} + \underbrace{AH + HA}_{\text{linear in } H} + \underbrace{H^2}_{\text{error}}$$

By Lemma 1(b), $\|H^2\| \leq \|H\|^2$, so error is $o(\|H\|)$.

So $f'(A)(H) = AH + HA$.

Prop 2 f diff at $a \Rightarrow f$ cts at a

Pf $f(a+h) = f(a) + f'(a)(h) + \varepsilon(h)\|h\|$, where $\varepsilon(0)=0$, ε cts at 0.
$f'(a)$ is cts by Lemma 1(a), $\|\cdot\|$ is cts, ε is cts at 0.
So the RHS is cts at 0, so $h \mapsto f(a+h)$ is cts at 0. ☐

Prop 3 (Chain Rule) We have open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, functions $f : U \to \mathbb{R}^n$, $g : V \to \mathbb{R}^p$, $f(U) \subset V$, $a \in U$, $b = f(a)$.
If f is diff'ble at a, g is diff'ble at b, then $g\circ f$ is diff'ble at a, and $(g\circ f)'(a) = g'(f(a)) \circ f'(a)$.

Pf Let $S = f'(a)$, $T = g'(b)$. Then

$$f(a+h) = f(a) + S(h) + \varepsilon(h)\|h\|$$
$$g(b+k) = g(b) + T(k) + \delta(k)\|k\|$$

where ε, δ are 0 at 0, and cts at 0.

$$(g\circ f)(\underset{+h}{a}) = g(\underbrace{f(a)}_{b} + \underbrace{S(h) + \varepsilon(h)\|h\|}_{k=k(h)})$$
$$= g(b) + T(k) + \delta(k)\|k\|$$
$$= (g\circ f)(a) + T\circ S(h) + \|h\|\,T(\varepsilon(h)) +$$
$$+ \delta(k(h))\,\|S(h) + \varepsilon(h)\|h\|\,\|$$

$T\circ\varepsilon$ is cts at 0, so $\|h\|\,T(\varepsilon(h)) = o(\|h\|)$.
& $T\circ\varepsilon(0) = 0$

L21.4

$k(0) = 0$, so $\delta(k(h)) = 0$ & $\delta \circ k$ is cts at 0

$$\frac{\| S(h) + \varepsilon(h) \|h\| \|}{\|h\|} \leqslant \frac{\|S(h)\|}{\|h\|} + \|\varepsilon(h)\| \underset{\substack{\uparrow \\ \text{Lemma 1}}}{\leqslant} \|S\| + \|\varepsilon(h)\|$$

So $$\|\delta(k(h))\| \frac{\| S(h) + \varepsilon(h)\|h\| \|}{\|h\|} \leqslant \underbrace{\|\delta(k(h))\| \cdot \left(\|S\| + \|\varepsilon(h)\|\right)}_{\to 0 \text{ as } h \to 0.}$$

So $(g\circ f)(a+h) = (g \circ f)(a) + g'(f(a)) \circ f'(a)(h) + o(\|h\|)$. □

L22.1

$\mathcal{U} \subset \mathbb{R}^m$ open, $f : \mathcal{U} \to \mathbb{R}^{\cancel{m} n}$, $a \in \mathcal{U}$

Prop 4 f is diff'ble at a $\Leftrightarrow$ each component f_j is diff at a

Then $f'(a)(h) = \sum_{j=1}^{n} f_j'(a)(h) e_j'$.

Pf Have $\pi_j : \mathbb{R}^{\cancel{m} n} \to \mathbb{R}$, $\pi_j(x) = \langle x, e_j' \rangle$

& $f_j = \pi_j \circ f$, so $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

"$\Rightarrow$" Chain rule, π_j linear so diff'ble everywhere

"$\Leftarrow$" $f_j(a+h) = f_j(a) + f_j'(a)(h) + \varepsilon_j(h)\|h\|$, $\varepsilon_j(0) = 0$, cts at 0

$$f(a+h) = \sum_{j=1}^{n} f_j(a+h) e_j' = \sum_{j=1}^{n} \left(f_j(a) + f_j'(a)(h) + \varepsilon_j(h)\|h\| \right) e_j'$$

$$= f(a) + \underbrace{\sum_{j=1}^{n} f_j'(a)(h) e_j'}_{\text{linear in } h} + \Big(\underbrace{\sum_{j=1}^{n} \varepsilon_j(h) e_j'}_{\substack{\text{zero at zero,} \\ \text{cts at zero}}}\Big) \|h\|$$

□

Prop 5 If $f, g : \mathcal{U} \to \mathbb{R}^n$ & $\varphi : \mathcal{U} \to \mathbb{R}$ are diff'ble at a, then so are $f+g$ & φf $(x \mapsto \varphi(x) f(x))$ &

$$(f+g)'(a) = f'(a) + g'(a)$$

$$(\varphi f)'(a)(h) = \varphi'(a)(h) f(a) + \varphi(a) \cdot f'(a)(h)$$

Pf We have

$$\left.\begin{aligned} f(a+h) &= f(a) + f'(a)(h) + \varepsilon(h)\|h\| \\ g(a+h) &= g(a) + g'(a)(h) + \delta(h)\|h\| \\ \varphi(a+h) &= \varphi(a) + \varphi'(a)(h) + \eta(h)\|h\| \end{aligned}\right\}$$

where $\varepsilon, \delta, \eta$ are zero at zero and cts at zero

Hence, $(f+g)(a+h) = (f+g)(a) + (\underbrace{f'(a) + g'(a)}_{\substack{\text{linear map} \\ \mathbb{R}^m \to \mathbb{R}^n}})(h) + (\underbrace{\varepsilon(h) + \delta(h)}_{\substack{0 \text{ at } 0, \\ \text{cts at } 0}}) \|h\|$.

✓

Can expand $\varphi(a+h) f(a+h)$ & get the product rule that way.

OR Define $F : \mathcal{U} \to \mathbb{R} \times \mathbb{R}^{n\cancel{m}}$, $x \mapsto (\varphi(x), f(x))$,

$G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(\lambda, y) \mapsto \lambda y$.

Note $\varphi f = G \circ F$, F is diff'ble at a by Prop 4, &

$F'(a)(h) = (\varphi'(a)(h), f'(a)(h))$.

G is diff'ble being bilinear:

$G'(\mu, b)(\lambda, h) = \mu h + \lambda b$

By the chain rule, $\varphi \cancel{\circ} f$ is diff'ble at a and

$$\begin{aligned}(\varphi f)'(a)(h) &= G'(F(a)) \circ F'(a)(h) \\ &= G'(\varphi(a), f(a))\,(\varphi'(a)(h),\ f'(a)(h)) \\ &= \varphi(a)\, f'(a)(h) + \varphi'(a)(h)\, f(a) \ \checkmark\end{aligned}$$

□

Partial derivatives $U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$, $a \in U$

Fix a direction $u \in \mathbb{R}^m$, i.e. $u \neq 0$.

If it exists, the limit $\lim_{t \to 0} \frac{f(a+tu)-f(a)}{t}$ is called the directional derivative of f at a in direction u & is denoted by $D_u f(a)$

Note 1. $f(a+tu) = f(a) + t D_u f(a) + o(t)$, $D_u f(a) \in \mathbb{R}^n$

2. Let $\gamma(t) = a + tu$. Then $(f \circ \gamma)'(0) = D_u f(a)$.

Special case: if $u = e_i$, $1 \leq i \leq m$, write $D_i f(a)$ for $D_{e_i} f(a)$ and call it the i^{th} partial derivative of f at a.

Prop 6 If f is diff'ble at a, then $\forall u \neq 0$, $D_u f(a)$ exists, and $D_u f(a) = f'(a)(u)$. So for $h = \sum_{i=1}^m h_i e_i$, then

$$f'(a)(h) = \sum_{i=1}^m h_i D_i f(a).$$

Pf We have $f(a+h) = f(a) + f'(a)(h) + \varepsilon(h)\|h\|$ where $\varepsilon = 0$ at 0, cts at 0.

Put $h = tu$: $f(a+tu) = f(a) + t \underset{\substack{\uparrow \\ \text{linear}}}{f'(a)(u)} + \varepsilon(tu)\|tu\|$

$$\frac{f(a+tu) - f(a)}{t} = f'(a)(u) + \varepsilon(tu)\frac{|t|}{t}\|u\|$$
$$\to f'(a)(u) \quad \text{as } t \to 0$$

□

Remark 1. Assume f is diff'ble at a. Q: Matrix of $f'(a)$?

$f'(a)(e_i) = D_i f(a)$ is i^{th} column

(i, j) entry

$$\therefore\ \pi_j(f'(a)(e_i)) \underset{\substack{\uparrow \\ \pi_j \text{ linear} \\ + \text{ chain rule}}}{=} (\pi_j \circ f)'(a)(e_i) = f_j'(a)(e_i) = D_i f_j(a)$$

Other notation $D_i f_j(a) = \frac{\partial f_j}{\partial x_i}(a)$.

Matrix of $f'(a)$ is $\left(\frac{\partial f_j}{\partial x_i}(a)\right)_{1 \leq j \leq n,\ 1 \leq i \leq m}$, the Jacobian of f at a $Jf(a)$.

2. If $D_u f(a)$ exists, then so does $D_u f_j(a)$ $\forall j$ &

$$\frac{f_j(a+tu) - f_j(a)}{t} = \pi_j\left(\frac{f(a+tu)-f(a)}{t}\right) \longrightarrow \pi_j(D_u f(a)).$$

So $D_u f_j(a) = \pi_j(D_u f(a))$. So $D_u \pi_j = \pi_j D_u$.

3. Converse of Prop 6 fails massively.

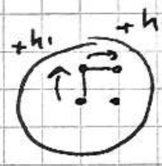
Theorem 7 $\mathcal{U} \subset \mathbb{R}^m$ open, $f : \mathcal{U} \to \mathbb{R}^n$, $a \in \mathcal{U}$

Assume $\exists\, r > 0$ s.t. $D_r(a) \subset \mathcal{U}$ & $D_i f$ exists on $D_r(a)$ for all $i = 1, \dots, m$ and are continuous at a. Then f is diff at a.

Proof WLOG $n = 1$ (Prop 4, Remark 2 above)

We do the proof for $m = 2$ (general case is similar)

Let $a = (a_1, a_2)$ & consider $h = (h_1, h_2) \in D_r(0)$.



$$f(a_1+h_1, a_2+h_2) - f(a_1, a_2) - h_1 D_1 f(a_1, a_2) - h_2 D_2 f(a_1, a_2)$$

$$= f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2) - h_2 D_2 f(a_1, a_2) \qquad \textcircled{\scriptsize I}$$

$$+ f(a_1+h_1, a_2) - f(a_1, a_2) - h_1 D_1 f(a_1, a_2) \qquad \textcircled{\scriptsize II}$$

$\overset{?}{=} o(\|(h_1, h_2)\|)$ if so, done

By def^n, $\textcircled{\scriptsize II}$ is $o(h_1)$ & so $o(\|(h_1, h_2)\|)$.

Let $\varphi(t) := f(a_1+h_1, a_2+t)$, $t \in [-|h_2|, |h_2|]$

φ is cts & diff'ble on $(-|h_2|, |h_2|)$

& $\varphi'(t) = D_2 f(a_1+h_1, a_2+t)$

By MVT, $\exists\, \theta = \vartheta(h_1, h_2) \in (0, 1)$ s.t.

$$\varphi(h_2) - \varphi(0) = \varphi'(\vartheta h_2)\, h_2$$

$$\textcircled{\scriptsize I} = \varphi(h_2) - \varphi(0) - h_2 D_2 f(a_1, a_2)$$

$$= h_2 \left[D_2 f(a_1+h_1, a_2 + \vartheta h_2) - D_2 f(a_1, a_2) \right]$$

is $o(h_2)$ and hence $o(\|(h_1, h_2)\|)$, since $D_2 f$ is cts at (a_1, a_2)

$\square$

Thm 8 (Mean Value Inequality) We have an open set $\mathcal{U} \subset \mathbb{R}^m$, a function $f : \mathcal{U} \to \mathbb{R}^n$. Assume f is diff. on $\mathcal{U}$. Assume we are given $a, b \in \mathcal{U}$ s.t.

$$[a,b] = \{(1-t)a + tb : t \in [0,1]\} \subset \mathcal{U} \quad \& \quad \exists M \geqslant 0 \text{ s.t.}$$

$$\|f'(z)\| \leqslant M \quad \forall z \in [a,b]. \text{ Then}$$

$$\|f(b) - f(a)\| \leqslant M \cdot \|b-a\|.$$

Proof Let $v = f(b) - f(a)$. Consider

$$\varphi : [0,1] \longrightarrow \mathbb{R}, \quad \varphi(t) = \langle f((1-t)a + tb), v \rangle.$$

$$\varphi(1) - \varphi(0) = \langle f(b) - f(a), v \rangle = \|f(b) - f(a)\|^2$$

Note $\varphi = \psi \circ f \circ \gamma$ where $\gamma(t) = (1-t)a + tb$, $\psi(x) = \langle x, v \rangle$.

Hence φ is cts on $[0,1]$ & diff on $(0,1)$ with 1.1

$$\varphi'(t) = \psi'(f \circ \gamma(t)) \circ f'(\gamma(t)) \circ \underbrace{\gamma'(t)}_{b-a} = \langle f'(\gamma(t))(b-a), v \rangle$$

By MVT, $\exists\, \theta \in (0,1)$ s.t.

$$\varphi(1) - \varphi(0) = \varphi'(\theta).$$

$$\text{So } \|f(b) - f(a)\|^2 = \langle f'(\gamma(\theta))(b-a), v \rangle \underset{\uparrow \text{Cauchy Schwartz}}{\leqslant} \|f'(\gamma(\theta))(b-a)\| \|v\|$$

$$\leqslant \|f'(\gamma(\theta))\| \cdot \|b-a\| \cdot \|v\|$$

$$\leqslant M \|b-a\| \cdot \|f(b) - f(a)\|.$$

Result follows. ☐

Cor 9 Let $\mathcal{U} \subset \mathbb{R}^m$ be open & connected, $f : \mathcal{U} \longrightarrow \mathbb{R}^n$ diff on $\mathcal{U}$, $f'(x) = 0 \ \forall x \in \mathcal{U}$. Then f is constant.

Proof For $a \in \mathcal{U}$, $\exists\, r > 0$ s.t. $D_r(a) \subset \mathcal{U}$.

$\forall b \in D_r(a)$, $[a,b] \subset D_r(a) \subset \mathcal{U}$

& $f'(z) = 0 \ \forall z \in [a,b]$

So by Thm 8, $\|f(b) - f(a)\| = 0$ i.e. $f(b) = f(a)$.

So f is constant on $D_r(a)$.

Hence $\forall y \in \mathbb{R}^n$, $\mathcal{U}_y = \{x \in \mathcal{U} : f(x) = y\}$ is open.

Also, $\mathcal{U}_y = f^{-1}(\{y\})$ is closed in $\mathcal{U}$.

So $\mathcal{U}_y$, $\mathcal{U} \setminus \mathcal{U}_y$ would disconnect $\mathcal{U}$ unless one were ϕ.

So for $y \in \operatorname{im} f$, $\mathcal{U}_y \neq \phi$ so $\mathcal{U}_y = \mathcal{U}$. ☐

Remarks Suppose $\mathcal{U} \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are open, $f : \mathcal{U} \to V$ a bijection, f is diff at $a \in \mathcal{U}$ & $f^{-1} : V \longrightarrow \mathcal{U}$ is diff. at $f(a)$. Let $S = f'(a)$, $T = (f^{-1})'(f(a))$. Then

Then

$$ST = f'(a) \circ (f^{-1})'(f(a)) \overset{\text{Chain Rule}}{=} (f \circ f^{-1})'(f(a)) = I_n$$

$$TS = (f^{-1})'(f(a)) \circ f'(a) = (f^{-1} \circ f)'(a) = I_m$$

So $n = rk(ST) = rk(TS) = m$.

<u>Thm 10 (Inverse Function Theorem)</u> We have an open set $\mathcal{U} \subset \mathbb{R}^n$, a C^1 function (continuously diff) $f : \mathcal{U} \to \mathbb{R}^n$, a point $a \in \mathcal{U}$ with $f'(a)$ invertible. Then $\exists$ open sets $V \subset \mathcal{U}$, $W \subset \mathbb{R}^n$ with $a \in V$, $f(a) \in W$ s.t.

$f|_V : V \to W$ is a bijection with a C^1 inverse

$g : W \to V$ & $g'(y) = [f'(g(y))]^{-1} \ \forall y \in W$

<u>Proof</u> <u>step 1</u> WLOG $a = f(a) = 0$ & $f'(a) = I$

<u>Pf</u> Consider $h : \mathcal{U} - a = \{x - a : x \in \mathcal{U}\} \to \mathbb{R}^n$

$$h(x) = T^{-1}(f(x+a) - f(a))$$

where $T = f'(a)$. By Chain Rule h is diff ~~at~~ & $h'(x) = T^{-1} \circ f'(x+a)$. So h is C^1, $h(0) = 0$ & $h'(0) = T^{-1} \circ f'(a) = I$. Assuming the result for h, we get it for f since $f(x) = T(h(x-a)) + f(a)$. □

Now assume $f(0) = 0$, $f'(0) = I$. We can fix $r > 0$ s.t. $D_r(0) \subset \mathcal{U}$ ($\mathcal{U}$ is open) & $\|f'(x) - I\| \leq \frac{1}{2} \ \forall x \in D_r(0)$ (f' is continuous on $\mathcal{U}$).

<u>step 2</u> $\forall x, y \in D_r(0)$, $\|f(x) - f(y)\| \geq \frac{1}{2}\|x - y\|$. In particular, f is injective on $D_r(0)$.

<u>Pf</u> Consider $h(x) = x - f(x)$. Then $h'(x) = I - f'(x)$. So $\|h'(x)\| \leq \frac{1}{2} \ \forall x \in D_r(0)$.

By Thm 8, $\|h(x) - h(y)\| = \|x - f(x) - (y - f(y)\| \leq \frac{1}{2}\|x - y\|$ for all $x, y \in D_r(0)$. Hence

$$\frac{1}{2}\|x - y\| \geq \|x - y\| - \|f(x) - f(y)\| \quad (\Delta\text{-ineq})$$

So $\|f(x) - f(y)\| \geq \frac{1}{2}\|x - y\|$. □

<u>step 3</u> For $0 < s < \frac{r}{2}$, $D_s(0) \subset f(B_{2s}(0)) \subset f(\mathcal{U})$

<u>Pf</u> Fix $y \in D_s(0)$. Consider $h : B_{2s}(0) \to \mathbb{R}^n$, given by $h(x) = y - f(x) + x$.

We have $h'(x) = -f'(x) + I$, so $\|h'(x)\| \leq \frac{1}{2} \ \forall x \in B_{2s}(0)$.

So by Thm 8, $\|h(x) - h(z)\| \leq \frac{1}{2}\|x - z\| \ \forall x, z \in B_{2s}(0)$.

For $x \in B_{2s}(0)$, $\|h(x)\| = \| h(x) - h(0) + y \|$
$\leq \|h(x) - h(0)\| + \|y\|$
$\leq \frac{1}{2}\|x\| + \|y\|$
$\leq \frac{1}{2} \cdot 2s + s$
$= 2s.$

So $h : B_{2s}(0) \longrightarrow B_{2s}(0)$, $B_{2s}(0)$ is $\neq \phi$, complete, and h is a contraction mapping. So $\exists\, x \in B_{2s}(0)$ with $h(x) = x$, i.e. $y = f(x)$ ✓ □

<u>step 4</u> Fix s, $0 < s < r/2$. Let $W = D_s(0)$ and
$V = f^{-1}(D_s(0)) \cap D_r(0)$.

W is open, V is open, $f(V) \subset W$,
$f(V) = W$ by step 3, $f|_V$ is injective by step 2
So $f|_V : V \to W$ is a bijection. Let $g : W \to V$ be the inverse.
Given $a, b \in W$, let $x = g(a)$, $y = g(b)$. Then $x, y \in V \subset D_r(0)$ so by step 2, $\|f(x) - f(y)\| \geq \frac{1}{2}\|x - y\|$, so $\|a - b\| \geq \frac{1}{2}\| g(a) - g(b)\|$. So g is Lipschitz, so continuous. □

g diff (non-examinable)
By chain rule, $I = (f \circ g)'(y) = f'(g(y)) \circ g'(y)$
so $g'(y) = [f'(g(y))]^{-1}$. □

second derivatives

We have open set $U \subset \mathbb{R}^m$, a function $f: U \to \mathbb{R}^n$, $a \in U$.

f is twice differentiable at a if f is diff. on some open set V with $a \in V \subset U$ & the derivative

$$f' : V \to L(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^{mn}$$ is diff at a. Let

$$f''(a) = (f')'(a)$$ – called the second derivative of f at a.

We have $f''(a) \in L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n))$ & there's a function ε s.t. $f'(a+h) = f'(a) + f''(a)(h) + \varepsilon(h)\|h\|$

where $\varepsilon(h) \to 0$ as $h \to 0$. So $\varepsilon(h) \in L(\mathbb{R}^m, \mathbb{R}^n)$ &

$$\|\varepsilon(h)\| = \left(\sum_{i=1}^m \|\varepsilon(h)(e_i)\|^2\right)^{1/2} \to 0 \text{ as } h \to 0$$

$\Leftrightarrow$ $\varepsilon(h)(e_i) \to 0$ as $h \to 0$ for each i

$\Leftrightarrow$ $\varepsilon(h)(k) = \sum_{i=1}^m k_i \varepsilon(h)(e_i) \to 0$ as $h \to 0$ for each fixed $k = \sum_{i=1}^m k_i e_i$.

So $$f'(a+h)(k) = f'(a)(k) + f''(a)(h)(k) + \underbrace{\varepsilon(h)(k) \cdot \|h\|}_{o(\|h\|)}$$
for each fixed $k \in \mathbb{R}^m$.

Note $$L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n)) \cong Bil(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$$
$$T \longleftrightarrow B$$
via $B(h,k) = T(h)(k)$, works in both direction.

We'll think of $f''(a) \in Bil(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$.

In summary, if f is diff. on open V, $a \in V \subset U$, then f is twice diff. at a $\Leftrightarrow$ $\exists\, B \in Bil(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ s.t. for each $k \in \mathbb{R}^m$, we have

$$f'(a+h)(k) = f'(a)(k) + B(h,k) + o(\|h\|) \qquad (k \text{ fixed})$$

Then $B = f''(a)$.

EX $f: M_n \to M_n$, $f(A) = A^3$

$$f(A+H) = (A+H)^3 = \underset{f(A)}{\boxed{A^3}} + \underbrace{(A^2H + AHA + HA^2)}_{\text{linear in } H} + \underbrace{H^2A + HAH + AH^2 + H^3}_{o(\|H\|)}$$

L 24.2

So $f'(A)(H) = HA^2 + AHA + A^2H$.

Then $f'(A+H)(K) = K(A+H)^2 + (A+H)K(A+H) + (A+H)^2K$

$$= \underbrace{[KA^2 + AKA + A^2K]}_{f'(A)(K)} + \underbrace{[KAH + KHA + AKH + HKA + AHK + HAK]}_{\text{bilinear in } (H,K)} + O(\|H\|^2)$$

So $f''(A)(H,K) = KAH + KHA + AHK + HKA + KAH + KHA$

Assume $f''(a)$ exists. Then

$$f'(a+h)(k) = f'(a)(k) + f''(a)(h,k) + o(\|h,\cancel{k}\|)$$

Fix $u, v \in \mathbb{R}^m \setminus \{0\}$. Put $k = v$.

$$D_v f(a+h) = D_v f(a) + f''(a)(h,v) + o(\|h\|)$$

So $D_v f$ is diff at a & $(D_v f)'(a)(h) = f''(a)(h,v)$.

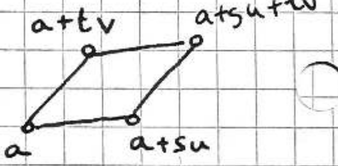
So $D_u D_v f(a) = (D_v f)'(a)(u) = f''(a)(u,v)$.

So $D_i D_j f(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a) = f''(a)(e_i, e_j)$.

<u>Theorem 11</u> Let $\mathcal{U} \subset \mathbb{R}^m$ be open, $f: \mathcal{U} \to \mathbb{R}^n$ be twice diff on $\mathcal{U}$ with f'' cts at a. Then $\forall\, u, v \in \mathbb{R}^m \setminus \{0\}$,

$D_u D_v f(a) = D_v D_u f(a)$, i.e. $f''(a)$ is symmetric.

<u>Proof</u> WLOG $n = 1$ $\left((f_j)'' = (f'')_j \right)$



Define $\varphi(s,t) = f(a+su+tv) - f(a+tv)$

$$- (f(a+su) - f(a))$$

$$= f(a+su+tv) - f(a+su)$$

$$- (f(a+tv) - f(a)).$$

Fix s,t. Consider $\psi(x) = f(a+xu+tv) - f(a+xu)$.

$$\varphi(s,t) = \psi(s) - \psi(0) \underset{MVT}{=} s\,\psi'(\alpha s) \qquad \alpha = \alpha(s,t) \in (0,1)$$

So $\varphi(s,t) = (D_u f(a + \alpha su + tv) - D_u f(a + \alpha su))\, s$.

Consider $\psi(y) = D_u f(a + \alpha su + yv)$.

$$\varphi(s,t) = s(\psi(t) - \psi(0)) \underset{MVT}{=} st\,\psi'(\beta t) \qquad \beta = \beta(s,t) \in (0,1)$$

$$= st\, D_v D_u f(a + \alpha su + \underset{\beta t}{\cancel{y}}\, v)$$

$$\frac{\varphi(s,t)}{st} = D_v D_u f(a+\alpha s u + \beta t v) = f''(a+\alpha s u+\beta t v)(v,u)$$
$$\longrightarrow f''(a)(v,u) \quad \text{as} \quad (s,t) \to (0,0)$$

Repeat in the other order to get

$$\frac{\varphi(s,t)}{st} = D_u D_v f(a+\tilde{\alpha} s u + \tilde{\beta} t v) = f''(a+\tilde{\alpha} s u+\tilde{\beta} t v)(u,v)$$
$$\longrightarrow f''(a)(u,v) \quad \text{as} \quad (s,t) \to 0.$$

☐

$U \subset \mathbb{R}^m$, open, $f: U \to \mathbb{R}$, $a \in U$

f has a <u>local maximum at a</u> if $\exists\, r>0$ s.t. $D_r(a) \subset U$,
$\forall x \in D_r(a)$, $f(x) \leq f(a)$.

Similarly define <u>local minimum at a</u>.

f has a <u>stationary point at a</u> if f is diff at a & $f'(a) = 0$.

Easy to see that if f is diff. at a & f has local max / min at a, then $f'(a) = 0$.

<u>Theorem 12</u> Let $U \subset \mathbb{R}^m$ be open, $f: U \to \mathbb{R}$ be twice diff on U, f'' continuous at a & $f'(a) = 0$. Then if the symmetric bilinear form $f''(a)$ is <u>positive definite</u> then f has a local min at a, & if $f''(a)$ is negative definite, then a local max at a.

<u>Proof</u> FACT: $f(a) + f'(a)(h) + \frac{1}{2} f''(a)(h,h) + \varepsilon(h)\|h\|^2$ where $\varepsilon(h) \to 0$ as $h \to 0$.

Recall $f''(a)$ is diagonalisable: $\exists$ basis $u_1, \dots, u_n$ of $\mathbb{R}^m$ s.t.

$$f''(a)(u_i, u_j) = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j. \end{cases}$$

Assume $f''(a)$ is positive def: $f''(a)(h,h) \geq 0 \ \forall h$
& "=" iff $h = 0$

So $\lambda_i = f''(a)(u_i, u_i) > 0 \ \forall i$. So $\mu = \min\{\lambda_i\} > 0$.

For $h \in \mathbb{R}^m$,
$$f''(a)(h,h) = \sum_{i,j} h_i h_j f''(a)(u_i, u_j)$$
$$= \sum_i h_i^2 \lambda_i \geq \mu \|h\|^2 \leftarrow \text{whh}$$

BACT: $f(a+h) - f(a) \geq \left(\frac{1}{2}\mu + \varepsilon(h)\right)\|h\|^2$

L24.4

$\exists r > 0$ s.t. $D_r(a) \subset \mathcal{U}$, $|\varepsilon(h)| \leq \frac{1}{4}\mu \quad \forall h \in D_r(0)$

For such h, $f(a+h) - f(a) \geqslant \left(\frac{1}{4}\mu\right) \|h\|^2 \geqslant 0$.

So f has a local minimum at a. $\square$