L1.1 Complex Analysis ~ Neshan Wickramasekera

Lecture notes Webpage of T. Scholl

Many books A classic is Ahlfors, Complex Analysis

The course will build on notions from real analysis (Analysis I, Analysis & Topology) e.g. continuity, diff'bility, uniform convergence, and topological notions, such as compactness, connectedness etc.

§1 Basic notions

Notation: $\mathbb{C}$ is the complex plane

$\bar{z}$ denotes the complex conjugate of z in $\mathbb{C}$

$|z|$ is the modulus (absolute value) of $z \in \mathbb{C}$, $\sqrt{z\bar{z}}$

$D(a,r)$ is the open ball centred at a of radius r

$= \{z \in \mathbb{C} : |z-a| < r\}$

$U \subseteq \mathbb{C}$ is open if for every $a \in U$, $\exists\, r > 0$ s.t.

$D(a,r) \subset U$

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ (via $x+iy \mapsto (x,y)$), and give $\mathbb{R}^2$ the topology induced by the Euclidean metric, then $U \subseteq \mathbb{C}$ is open iff U as a subset of $\mathbb{R}^2$ is open.

This course is about complex valued functions defined on some open subset (possibly $\mathbb{C}$) of $\mathbb{C}$, $f : U \to \mathbb{C}$.

Def Let $U \subseteq \mathbb{C}$ be open, $w \in U$, $f : U \to \mathbb{C}$.

(i) We say $\lim_{z \to w} f(z) = A$ if $\forall\, \varepsilon > 0$, then $\exists\, \delta > 0$ s.t. $0 < |z-w| < \delta \Rightarrow |f(z) - A| < \varepsilon$.

(ii) f is continuous at w if $\lim_{z \to w} f(z) = f(w)$. f is continuous on U if f is continuous at each point $w \in U$.

We can write $f(x+iy) = u(x,y) + iv(x,y)$ with $u, v : U \to \mathbb{R}$ i.e. $u = \operatorname{Re} f$, $v = \operatorname{Im} f$.

With the identification of $\mathbb{C}$ with $\mathbb{R}^2$ given the Euclidean metric, continuity of f at $w \in \mathbb{C}$ is the same as continuity of u and v at w.

Complex differentiation

Def Extend the deff in the real case.

(i) Let $f : U \to \mathbb{C}$, $U \subseteq \mathbb{C}$ open, and let $w \in U$. Say f is diff'ble at w if the limit $f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$ exists.

$f'(w)$ is the derivative of f at w

(ii) f is holomorphic at $w \in U$ if f is diff'ble at every point in a neighbourhood of w (say a disc $D(w, \varepsilon)$)

f is holomorphic on U if f is holomorphic at every $w \in U$, i.e. f is diff'ble at every $w \in U$.

Somtimes the word "analytic" is used for holomorphic

Complex differentiation enjoys the same properties as real differentiation for sums, products, quotients, and the chain rule holds for composition of functions. Proofs of all these are identical to the real case.

Ex: polynomials $p(z) = \sum_{i=0}^{n} a_i z^i$, $a_0, \dots, a_n \in \mathbb{C}$ are diff'ble on $\mathbb{C}$.

If p, q are polynomials with $q \not\equiv 0$, then p/q is holomorphic on $\mathbb{C} \setminus \{z : q(z) = 0\}$.

Write $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$

Q: Is diff'bility of f at $w \in U$ the same as diff'bility of $u, v : U \to \mathbb{R}$ at the same pt.

A: No

Theorem 1.1 (Cauchy-Riemann equations) $f : U \to \mathbb{C}$ is diff'ble at $w = c + id \in U$ iff the functions u, v are diff'ble at (c, d) and satisfy the Cauchy Riemann equations at (c, d)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } (c, d)$$

Recall $u : \mathcal{U} \to \mathbb{R}$ is diff at (c,d) if $\exists$ linear map $L : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - L(x-c, y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

In this case L is uniquely defined by $L(x,y) = \left.\frac{\partial u}{\partial x}\right|_{(c,d)} x + \left.\frac{\partial u}{\partial y}\right|_{(c,d)} y$

Say $L = Du(c,d)$, the derivative.

Proof By the definition (!), f is diff'ble at w with derivative $f'(w) = p + iq$ iff

$$\lim_{z\to w} \frac{f(z) - f(w) - f'(w)(z-w)}{|z-w|} = 0,$$

equivalently separating real and imaginary parts,

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - (p(x - \overset{c}{\cancel{y}}) - q(y-d))}{\|(x-c, y-d)\|}$$

and $$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - (q(x-\overset{c}{\cancel{y}}) + p(y-d))}{\|(x-c, y-d)\|}.$$

So f is diff'ble at $w = c + id$ with derivative $f'(w) = p + iq$ iff u and v are diff'ble at (c,d) with derivatives

$$Du(w) = (p, -q)$$
$$Dv(w) = (q, p),$$

or equivalently u, v are diff'ble at (c,d) with C-R. $\square$

L 2.1

<u>Last time</u>: $f = u + iv$ is diff at a point $w = c + id$ iff u and v are diff at (c, d), and satisfy CR equations;

i.e. $u_x = v_y$, $u_y = -v_x$, at (c, d)

<u>Warning</u>: Just the fact that u and v have partials at (c, d) and satisfy CR eq^{n}s does not guarantee diff'bility of f at w. (important that u, v are diff, see ES1)

<u>Deeper question</u> (beyond this course) If we knew that CR eq^{n}s hold in an open set, then does that guarantee diff. of f? No! With certain additional conditions, e.g. continuity of partials, the answer is yes.

<u>Remark</u>: From the proof of the above thm, we obtain the expression $f'(w) = u_x(c, d) + i v_x(c, d)$ (& similar variants)

<u>Remark</u>: If we just want to show that f diff at w $\Rightarrow$ partials of u & v exist at (c, d) and satisfy CR, then we can proceed more simply as follows:

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{h \to 0} \frac{f(w+h) - f(w)}{h}$$

Approaching 0 from real, imaginary axes gives

$f'(w) = u_x + i v_x$ and $f'(w) = v_y - i u_y$.

Something something limit of a complex f^n via real, imaginary parts.

<u>Ex</u>: $f(z) = \bar{z}$, $u(x, y) = x$, $v(x, y) = -y$

Then $u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$.

So CR are not satisfied anywhere, and f is not diff'ble.

<u>Remark</u> Complex diff'bility is much more restrictive than just real differentiability of real & imaginary parts. We do indeed have some surprising conclusions:

(a) If $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded then it is constant.

(b) If $f: \mathcal{U} \to \mathbb{C}$ is holomorphic, then $f': \mathcal{U} \to \mathbb{C}$ is holomorphic.

(c) If $f_n: \mathcal{U} \to \mathbb{C}$ is a sequence of holomorphic functions and if $f_n \to f$ uniformly, then f is holomorphic.

We'll prove these later on. V)

In view of ⓑ, we have that partial derivatives of all orders of $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$ exist and are continuous.

So if $f = u + iv : \mathcal{U} \to \mathbb{C}$ is holomorphic, we may differentiate CR equations

$$u_x = v_y \Rightarrow u_{xx} = v_{yx}$$

$$u_y = -v_x \Rightarrow u_{yy} = -v_{xy} \overset{\text{continuous}}{=} -v_{yx}$$

Hence $\Delta u = 0$ on $\mathcal{U}$.

So the real and imaginary parts of a holomorphic functions are <u>harmonic</u> functions.

<u>Corollary 1.2</u>: Let $f = u + iv : \mathcal{U} \to \mathbb{C}$. Suppose that u, v have continuous partial derivatives in $\mathcal{U}$, and u, v satisfy the CR equations. Then f is holomorphic on $\mathcal{U}$.

<u>Proof</u> From IB Anal Topology and Theorem 1.1 □

<u>Remark</u> Once we know ⓑ, the converse of Cor 1.2 is true.

<u>Def^{n}s</u> • A <u>curve</u> is a continuous map from a closed interval, i.e. cts $\gamma : [a, b] \to \mathbb{C}$. A curve is <u>continuously diff</u>, i.e. C^1, if γ' exists and is continuous on $[a, b]$ (at the end points have one-sided derivatives)

• An open subset $\mathcal{U} \subseteq \mathbb{C}$ is <u>path-connected</u> if for any pair of points $z_1, z_2 \in \mathcal{U}$, there is a curve $\gamma : [a, b] \to \mathcal{U}$ s.t. $\gamma(a) = z_1$, $\gamma(b) = z_2$.

• A <u>domain</u> is a non-empty, open, path-connected subset of $\mathbb{C}$.

<u>Corollary 1.3</u>: If $\mathcal{U} \subseteq \mathbb{C}$ is a domain, $f : \mathcal{U} \to \mathbb{C}$ is holomorphic and $f'(z) = 0 \ \forall z \in \mathcal{U}$, then f is constant.

<u>Proof</u>: Use components u, v and IB Analyst's Topology. □

Power Series

Recall (from IA Analysis I):

Thm 1.4 (Radius of convergence) Let (c_n) be a sequence of complex numbers. Then $\exists R \in [0, \infty]$ s.t. the power series

$\sum_{n=0}^{\infty} c_n (z-a)^n$ converges absolutely if $|z-a| < R$ and diverges if $|z-a| > R$. If $R > 0$ and if $0 < r < R$, then the convergence of the series is uniform on $\overline{D_r(a)}$.

R = radius of convergence of the series

A formula for R is $R = \frac{1}{\lambda}$, where $\lambda = \limsup_{n\to\infty} (|c_n|)^{1/n}$

Thm 1.5 Let $\sum_{n=0}^{\infty} c_n (z-a)^n$ be a p.s. with r.o.c. $R > 0$.

Define f by $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for $z \in D(a, R)$. Then:

(i) f is holomorphic in $D(a,R)$,

(ii) the derived series $\sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$ has r.o.c. R as well, and

(iii) $f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1} \; \forall z \in D(a,R)$,

(iv) f has derivatives of all orders on $D(a,R)$ and $c_n = \frac{f^{(n)}(a)}{n!}$,

(v) if f vanishes in some open disc around a, then $f \equiv 0$.

Proof (iv) follows from (ii)/(iii) by differentiating repeatedly.

(v) follows from the formula $c_n = \frac{f^{n}(a)}{n!}$ and the fact that $f^{(n)}(a) = 0 \; \forall n$ if $f \equiv 0$ in some ball $B(a, r)$.

(i)-(iii) WLOG $a = 0$, shifting our argument $g(z) = f(a+z)$.

So we have $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with r.o.c equal to $R > 0$.

Let $|z| < R$, and choose ρ s.t. $|z| < \rho < R$.

$$\lim_{n\to\infty} \frac{n|c_n||z|^{n-1}}{|c_n|\rho^n} = \lim_{n\to\infty} \frac{n}{\rho}\left(\frac{|z|}{\rho}\right)^{n-1} = 0,$$

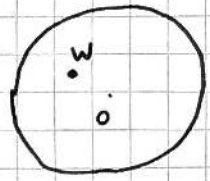
so, by the comparison test, since $\sum |c_n|\rho^n$ converges, we have that the desired series converges.

So the r.o.c of the desired series $R_1 \geq R$.

Also $n|c_n| \geq |c_n|$ implies $R_1 \leq R \Rightarrow R_1 = R$

L 3.2

To show that f is holomorphic with f' given by the derived series, fix $w \in D(a,R)$. Note that f is diff. at w with derivative $f'(w) = \sum n c_n w^{n-1}$ iff the function



$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{for } z \neq w \\ \sum_{n=1}^{\infty} n c_n w^{n-1} & \text{for } z = w \end{cases}$$

is continuous at w.

But $g(z) = \sum_{n=0}^{\infty} h_n(z)$ where $h_n(z) = \begin{cases} c_n \dfrac{z^n - w^n}{z-w} & \text{if } z \neq w, \\ n c_n w^{n-1} & \text{if } z = w. \end{cases}$

N.B. for $n=0$, $h_n \equiv 0$

Continuity of g at w will follow if we can show that the convergence in the above is uniform in some ngbd of w.

For any $z \in D(0,r)$, where $|w| < r < R$, we have for $z \neq w$,

$$|h_n(z)| = \left| \frac{c_n(z^n - w^n)}{z-w} \right| = |c_n| \left| z^{n-1} + w z^{n-2} + \cdots + w^{n-1} \right|$$
$$\leq n |c_n| r^{n-1}.$$

The same bound holds for $z = w$, so we are done by the Weierstrass - M test.
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□

<u>Def^n</u> The <u>complex exponential</u> function is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

This has r.o.c. ∞, so is <u>entire</u> i.e. diff on $\mathbb{C}$

<u>Prop 1.6</u> (i) e^z is an entire function with derivative e^z,

(ii) $e^{z+w} = e^z e^w \ \forall z, w \in \mathbb{C}$, and $e^z \neq 0 \ \forall z \in \mathbb{C}$,

(iii) $e^{x+iy} = e^x(\cos y + i \sin y)$,

(iv) $e^z = 1$ iff $z = 2n\pi i$ for $n \in \mathbb{Z}$,

(v) If $w \in \mathbb{C} \setminus \{0\}$ then $\exists$ ~~!~~ $z \in \mathbb{C}$ s.t. $e^z = w$.

<u>Proof</u> (i) r.o.c. is ∞

(ii) Fix $w \in \mathbb{C}$. Let $F(z) = e^{z+w} e^{-z}$. This is holomorphic on $\mathbb{C}$ with zero derivative, so is constant $= 1 \cdot e^w$.

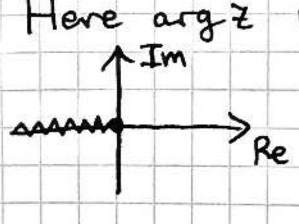
Take $w = 0$ to get $e^z \cdot e^{-z} = 1$ and plug back in.

L4.1

<u>Def^n</u> Given $z \in \mathbb{C}$, we say $w \in \mathbb{C}$ is a <u>logarithm</u> of z if
$$e^w = z.$$
From Prop 1.6 (v), z has a logarithm iff $z \neq 0$. By Prop 1.6(v), if $z \neq 0$, then z has infinitely many logarithms, differing from one another by integer multiples of $2\pi i$. (There is no preferred log; both $i\pi$ and $-i\pi$ are equally valid logs of -1; there is no mathematical reason to choose one over the other)

<u>Def^n</u> Let $\mathcal{U} \subseteq \mathbb{C} \setminus \{0\} = \mathbb{C}^*$ be open. A <u>branch of logarithm on $\mathcal{U}$</u> is a continuous function $\lambda : \mathcal{U} \to \mathbb{C}$ such that
$$\forall z \in \mathcal{U}, \ e^{\lambda(z)} = z.$$
<u>Remarks</u> ① Any branch of log on an open set $\mathcal{U} \subseteq \mathbb{C}^*$ is automatically holomorphic. To check this, note first that since $e^{\lambda(z)} = z \ \forall z \in \mathcal{U}$, $\lambda(z) \neq \lambda(w)$ if $z \neq w$.
$$\lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{z - w} = \lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{e^{\lambda(z)} - e^{\lambda(w)}}$$
$$= \lim_{z \to w} \left(\frac{e^{\lambda(z)} - e^{\lambda(w)}}{\lambda(z) - \lambda(w)} \right)^{-1} \underset{\substack{\text{continuity of } \lambda \text{ means} \\ \lambda(z) \to \lambda(w) \\ \text{as } z \to w}}{=} \frac{1}{e^{\lambda(w)}} = \frac{1}{w}$$
So moreover, $\lambda'(w) = \frac{1}{w}$.

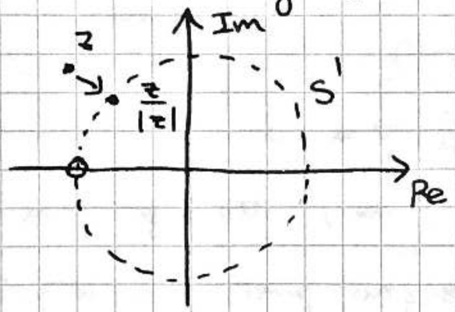
② From the definition it follows that
$$|z| = e^{\mathrm{Re}\,\lambda(z)} \Rightarrow \mathrm{Re}\,\lambda(z) = \log|z|$$
<u>Def^n</u> The <u>principal branch of logarithm</u> is the function
$$\mathrm{Log} : \mathcal{U} = \mathbb{C} \setminus \{x \in \mathbb{R}, x \leq 0\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \log|z| + i \arg z$$
Here $\arg z$ is the unique argument of z in $(-\pi, \pi)$.



To check that Log is a branch of logarithm, we need continuity and it being an inverse.

L4.2

To see continuity of $\arg(z)$, consider the map $z \mapsto \frac{z}{|z|}$ from



$\mathbb{C}^*$ onto $S^1 \setminus \{(-1,0)\}$, continuous.

Then $t \mapsto e^{it}$ gives a homeo$^{\underline{m}}$ from $(-\pi, \pi)$ to $S^1 \setminus \{(-1,0)\}$.

Also, $e^{\log|z| + i\arg z} = |z|(\cos\arg z + i\sin\arg z)$

KINDA CRINGE $\qquad = z$ by properties of $\cancel{z}$ cos, sin.

Prop 1.7 (i) Log is holomorphic with $\mathrm{Log}'(z) = \frac{1}{z}$.

(ii) For $|z| < 1$, $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(1)^{n-1} z^n}{n}$.

Proof (i) follows from above

(ii) First note that roc of the given series is 1.

We may differentiate

$$\frac{d}{dz}\left(\mathrm{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}\right) = \frac{1}{1+z} - \sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1} = 0$$

So $\mathrm{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$ differ by a constant.

Plug $z = 0$ in to get the constant to be zero. $\square$

Remark There is no way to extend Log to a holomorphic function (not even to a continuous function) on $\mathbb{C}^*$. Indeed,

$$\lim_{\theta \to \pi^-} \mathrm{Log}(e^{i\theta}) = i\pi, \quad \lim_{\theta \to -\pi^+} \mathrm{Log}(e^{i\theta}) = -i\pi.$$

We will later prove there is no branch of logarithm on $\mathbb{C}^*$.

Using exp and log, we may extend the def^n of familiar functions from real analysis to the complex domain.

e.g. For $\alpha \in \mathbb{C}$ and $z \in U_1$, $z^\alpha = e^{\alpha \mathrm{Log} z}$

Then $\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}$.

e.g. $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $\quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$\cosh z = \frac{e^z + e^{-z}}{2}$, $\quad \sinh z = \frac{e^z - e^{-z}}{2}$

These functions are holomorphic everywhere, with the expected derivatives from real analysis.

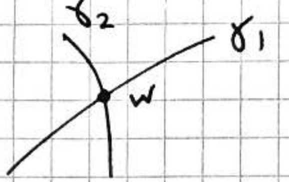
L4.3

<u>Conformality</u> Let $f: \mathcal{U} \to \mathbb{C}$ be holomorphic.

Let $w \in \mathcal{U}$ s.t. $f'(w) \neq 0$.

Such points are "nice" in two ways: geometrically and analytically.

<u>Geometric part</u> f preserves angles at w in the following sense



Let $\gamma_1, \gamma_2 : [-1, 1] \to \mathbb{C}$ be C^1 curves s.t. $\gamma_1(0) = \gamma_2(0) = w$, $\gamma_1'(0), \gamma_2'(0)$ are non-zero

Then we have $(f \circ \gamma_i)'(0) = f'(w)\, \gamma_i'(0) \neq 0$.

In particular, $\dfrac{(f\circ\gamma_1)'(0)}{(f\circ\gamma_2)'(0)} = \dfrac{\gamma_1'(0)}{\gamma_2'(0)}$.

Taking arguments,

$$\arg (f\circ\gamma_1)'(0) - \arg (f\circ\gamma_2)'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0),$$

i.e. the angle between γ_1, γ_2 at w is the same as the angle between $f\circ\gamma_1, f\circ\gamma_2$ at $f(w)$.

<u>Analytic part</u> f is locally invertible near w: viewing f as a map $(x, y) \mapsto (u(x,y), v(x,y))$, $\mathcal{U} \to \mathbb{R}^2$,

$$Df(w) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}, \quad \det(Df(w)) = |f'(w)|^2 > 0$$

The result follows by the inverse function theorem from Analysis & Topology, provided we know f is C^1 (which it is)

Moreover, the inverse map has derivative $(Df)^{-1}$.

L5.1

If $f'(w) \neq 0$, f holom. $U \to \mathbb{C}$, then via real inverse function theorem, f has a C^1 inverse f^{-1}, locally near w.

Moreover, $Df^{-1}\big|_{f(\tilde{w})} = \left(Df\big|_{\tilde{w}}\right)^{-1}$ for all $\tilde{w}$ near w.

$$= \frac{1}{|f'(\tilde{w})|^2}\begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}\Bigg|_{\tilde{w}}$$

From this, can check that f^{-1} satisfies CR equations. Hence f^{-1} is <u>holomorphic near $f(w)$</u>.

<u>Def^{n}s</u> ① A holomorphic f^n $f: U \to \mathbb{C}$ is <u>conformal at $w \in U$</u> if $f'(w) \neq 0$.

② If $f: D \to \tilde{D}$ is a bijective holomorphic f^n on a domain D, with $f'(z) \neq 0$ $\forall z \in D$, we say that f is a <u>conformal equivalence</u> between D and $\tilde{D}^{\ddagger}$.

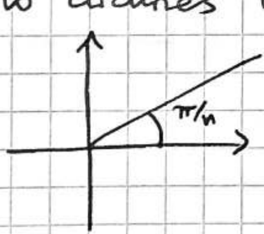
(Automatically get that $\tilde{D}$ is a domain; also f^{-1} conformal)

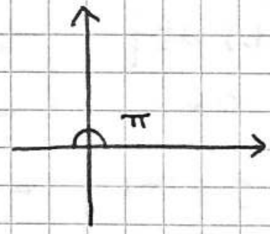
<u>Ex</u> ① Any Möbius map $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$

$$z \mapsto \frac{az+b}{cz+d}$$

with $ad-bc \neq 0$ is a conformal equivalence from $\hat{\mathbb{C}}$ to itself.

Maps circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$ ($\Leftrightarrow$ circlines in $\mathbb{C}$ map to circlines in $\mathbb{C}$).

② 

$z \mapsto z^n$, $n \geq 1$ integer


conformal with inverse $z \mapsto z^{1/n}$, principal branch

③ $\exp: \{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) < \pi\} \to \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$

Inverse is Log (principal branch).

④ $z \in \mathbb{H} = \{\operatorname{Im}(z) > 0\} \Leftrightarrow z$ is closer to i than to $-i$

$$\Leftrightarrow |z-i| < |z+i|$$

$$\Leftrightarrow \left|\frac{z-i}{z+i}\right| < 1$$

So $g(z) = \frac{z-i}{z+i}$ maps $\mathbb{H}$ into $D(0,1)$.

In fact, $g : \mathbb{H} \to D(0,1)$ is a conformal equivalence.

A very important, juicy theorem about conformal mappings is

<u>Riemann mapping thm</u> Let D be a domain bounded by a simple closed curve (i.e. an injective continuous map $S^1 \to \mathbb{C}$). Then D is conformally equivalent to the open unit disc $D(0,1)$.

In fact, much more generally, any simply connected domain that is <u>not</u> the whole of $\mathbb{C}$ is conformally equivalent to $D(0,1)$.

§2 Complex Integration I

We aim to extend real (Riemann) integration to integration of $\mathbb{C}$ complex valued f^ns of complex variable along curves.

Start with complex functions of a real variable.

<u>Def^n</u> If $f : [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ is continuous, then we define

$$\int_a^b f(t)\,dt = \int_a^b \operatorname{Re} f(t)\,dt + i \int_a^b \operatorname{Im} f(t)\,dt,$$

both understood as Riemann integrals.

Easy to check (by direct calculation), that

$$w \int_a^b f(t)\,dt = \int_a^b w f(t)\,dt \quad , \forall w \in \mathbb{C}.$$

<u>Prop 2.1</u> (Basic estimate)

$$\left| \int_a^b f(t)\,dt \right| \leq (b-a) \sup_{t\in[a,b]} |f(t)|.$$

Equality iff f is constant.

<u>Proof</u> If $\int_a^b f(t)\,dt = 0$, there is nothing to prove.

Assume $\int_a^b f(t)\,dt = re^{i\theta} \neq 0$. Then

$$\left| \int_a^b f(t)\,dt \right| = r = e^{-i\theta} \int_a^b f(t)\,dt = \int_a^b e^{-i\theta} f(t)\,dt$$

$$= \int_a^b \operatorname{Re}(e^{-i\theta} f(t))\,dt + i \underbrace{\int_a^b \operatorname{Im}(e^{-i\theta} f(t))\,dt}_{\text{zero}}$$

So

$$\left| \int_a^b f(t)\,dt \right| = \int_a^b \operatorname{Re}(e^{-i\theta} f(t))\,dt$$

$$\leq \int_a^b \left| \operatorname{Re}(e^{-i\theta} f(t)) \right| dt$$

$$\leq \int_a^b \left| e^{-i\theta} f(t) \right| dt$$

$$\leq (b-a) \sup_{t\in[a,b]} |f(t)|$$

L 5.3

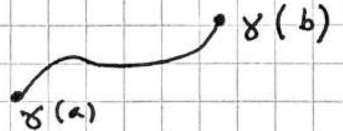
We have equality iff $|f(t)| = M \ \forall t$,

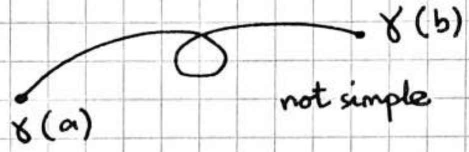
$$|\mathrm{Re}(e^{-i\theta} f(t))| = M \ \forall t$$

So $f = Me^{i\theta} \ \forall t$ is constant. $\square$

<u>Def^n</u> Let $\gamma : [a,b] \to \mathbb{C}$ be a C^1 curve. The <u>arc length</u> of γ is defined as $\text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

We say γ is <u>simple</u> if $\gamma(t_1) \neq \gamma(t_2)$ unless $t_1 \neq t_2$ or $\{t_1, t_2\} = \{a, b\}$.





~ simplicity depends on parametrisation

<u>Def^n</u> Let $f : \mathcal{U} \to \mathbb{C}$ be continuous, and let $\gamma : [a,b] \to \mathbb{C}$ be a C^1 curve. Then the <u>integral of f along γ</u> is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt \qquad \text{(see previous def}^n\text{)}$$

Some <u>basic properties</u>

① <u>linearity</u>: $\int_\gamma (c_1 f_1(z) + c_2 f_2(z))\, dz$

$$= c_1 \int_\gamma f_1(z)\, dz + c_2 \int_\gamma f_2(z)\, dz \qquad \forall c_1, c_2 \in \mathbb{C}$$

② <u>additivity</u>: If $\gamma : [a,b] \to \mathcal{U}$ is a C^1 curve, and $a < a' < b$, then $\int_\gamma f(z)\, dz = \int_{\gamma|_{[a,a']}} f(z)\, dz + \int_{\gamma|_{[a',b]}} f(z)\, dz$

③ <u>inverse path</u>: If $(-\gamma) : [-b, -a] \to \mathbb{C}$ is the curve $(-\gamma)(t) = \gamma(-t)$, then $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$,

④ <u>invariance under reparametrisation</u>

If $\varphi : [a', b'] \to [a,b]$ is injective C^1 function with $\varphi(a') = a$, $\varphi(b') = b$, then if $\delta = \gamma \circ \varphi : [a', b'] \to \mathcal{U}$, we have $\int_\delta f(z)\, dz = \int_\gamma f(z)\, dz$. (Proof: use real change of variables)

Def^n A piecewise C^1 curve is a continuous curve $\gamma : [a,b] \to \mathbb{C}$ s.t. $\exists$ $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ with the property that $\gamma_i = \gamma|_{[a_{i-1}, a_i]} : [a_{i-1}, a_i] \to \mathbb{C}$ is a C^1 curve for each $i = 1, \dots, n$.

If γ is piecewise C^1, we define

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$$

$$\circledast \quad \text{Length}(\gamma) = \sum_{i=1}^{n} \text{length}(\gamma_i) = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \gamma'(t)\,dt$$

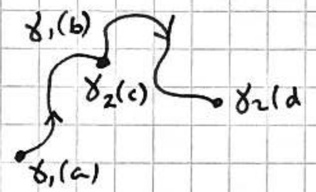
By the additivity property (2) (last time), the def^{n}s are independent of the choice of partition.

From now on, all <u>curves</u> will be assumed to be piecewise C^1.

It will be useful also to combine two curves as follows:

given $\gamma_1 : [a,b] \to \mathbb{C}$, $\gamma_2 : [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$, then define

$$(\gamma_1 + \gamma_2) : [a, b+d-c] \to \mathbb{C}$$



$$t \longmapsto \begin{cases} \gamma_1(t) & \text{for } t \in [a,b], \\ \gamma_2(c+t-b) & \text{for } t \in [b, b+d-c]. \end{cases}$$

Prop 2.2 For any continuous $f : U \to \mathbb{C}$ and any curve $\gamma : [a,b] \to U$, we have that

$$\left| \int_\gamma f(z)\,dz \right| \leq \text{Length}(\gamma) \cdot \sup_\gamma |f(\gamma(t))|$$

Proof If γ is C^1, then

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right|$$

$$\leq \int_a^b |f(\gamma(t))\,\gamma'(t)|\,dt$$

did it in A&T L11

$$\leq \sup_\gamma |f(\gamma(t))| \int_a^b |\gamma'(t)|\,dt$$

L 6.2

If γ is piecewise C^1, use $\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$ and the Δ inequality. $\square$

<u>Thm 2.3</u> (Fundamental Thm of Calculus)

Suppose $f: \mathcal{U} \to \mathbb{C}$ is continuous.

If there is a holomorphic $F: \mathcal{U} \to \mathbb{C}$ s.t. $F' = f$, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) \quad \text{for any curve } \gamma: [a,b] \to \mathcal{U}.$$

In particular, if γ is closed then $\int_\gamma f(z)\,dz = 0$.

<u>Proof</u> $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$

$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt$$

$$= F(\gamma(b)) - F(\gamma(a)). \quad \square$$

We call such an F an antiderivative of f.

<u>Remark</u> We will drop the assumption that f is continuous once we show $F' = f$ implies f holomorphic.

<u>Ex</u> $\int_\gamma z^n\,dz = \,?$, $n \in \mathbb{Z}$ for $\gamma: [0,1] \to \mathbb{C}$,
$t \mapsto Re^{2\pi i t}$

For $n \neq -1$, $\frac{1}{n+1} z^{n+1}$ is an antiderivative of z^n in $\mathbb{C}^*$.

So $\int_\gamma z^n\,dz = 0$ for $n \neq -1$.

For $n \neq -1$ NOT, from def^n

$$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{Re^{2\pi i t}} \cdot R \cdot 2\pi i\, e^{2\pi i t}\,dt = 2\pi i.$$

Since this is non-zero, this shows $\frac{1}{z}$ has no antiderivative in any open set containing γ.

In particular, Log has no branch on $\mathbb{C}^*$.

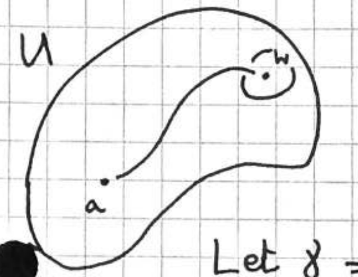
<u>Thm 2.4</u> (Converse to FTC) Let $\mathcal{U} \subseteq \mathbb{C}$ be a domain.

If $f: \mathcal{U} \to \mathbb{C}$ is continuous and if $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in $\mathcal{U}$, then there is a holomorphic $F: \mathcal{U} \to \mathbb{C}$ s.t. $F' = f$.

L 6.3

In fact, given any $a \in \mathcal{U}$, then $F(w) = \int_{\gamma_w} f(z)\,dz$ is an antiderivative for f, where γ_w is a curve from a to w.

<u>Proof</u> Fix $a \in \mathcal{U}$, let $F(w) = \int_{\gamma_w} f(z)\,dz$, where $\gamma_w : [0,1] \to \mathcal{U}$ is a piecewise C^1 curve s.t. $\gamma(0) = a$, $\gamma(1) = w$. (A continuous curve exists by connectedness of $\mathcal{U}$. We can then construct a piecewise C^1 curve, even a polygonal one, from a to w)



Since $\mathcal{U}$ is open, $\exists\, r > 0$ s.t. $D(w, r) \subset \mathcal{U}$. For any $h \in \mathbb{C}$ with $|h| < r$, let δ_h be the straight line segment $t \mapsto w + th$ for $0 \le t \le 1$.

Let $\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$, a closed curve.

So $\int_\gamma f(z)\,dz = 0$ by assumption. So

$$F(w+h) = F(w) + \int_{\delta_h} f(z)\,dz = F(w) + h f(w) + \int_{\delta_h} (f(z) - f(w))\,dz$$

So

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| = \left| \frac{1}{h} \int_{\delta_h} (f(z) - f(w))\,dz \right|$$

$$\le \underbrace{\frac{\text{length}(\delta_h)}{|h|}}_{1} \cdot \underbrace{\sup_{z \in \delta_h} |f(z) - f(w)|}_{\to 0 \text{ by cts}}$$

$\square$

<u>Def^n</u> A domain $\mathcal{U}$ is <u>star-shaped</u> ☆ if $\exists\, a \in \mathcal{U}$ s.t. $\forall\, w \in \mathcal{U}$, the straight line segment $[a, w] \subseteq \mathcal{U}$.

$\mathcal{U}$ is disc $\Rightarrow$ $\mathcal{U}$ is convex $\Rightarrow$ $\mathcal{U}$ star-shaped $\Rightarrow$ (path) connected

<u>Cor 2.5</u> If $\mathcal{U}$ is star-shaped, $f : \mathcal{U} \to \mathbb{C}$ is cts, and if $\int_\gamma f(z)\,dz = 0$ for any simple closed curve parametrising the boundary of a triangle $T \subseteq \mathcal{U}$, then f has an antiderivative.

<u>Proof</u> Pick a as in the above proof as the point wrt which $\mathcal{U}$ is star-shaped. Also take γ_w to be $[a, w] \subseteq \mathcal{U}$.

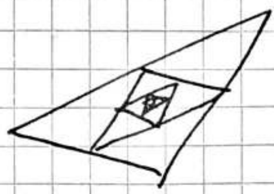
The note $\gamma_w + \delta_h + (-\gamma_{w+h})$ is essentially a triangle. $\square$

L 6.4

Next goal: Cauchy's Theorem

Will say that $\int_\gamma f(z)\,dz = 0$ when f is holo., γ closed and under certain other assumptions.

Simplest case: γ is a triangle

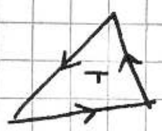


L7.1

<u>Notation</u> A triangle T is the convex hull of 3 points in $\mathbb{C}$.

$$T = \{ a z_1 + b z_2 + c z_3 \; : \; a,b,c \in \mathbb{R},\ 0 \le a,b,c \le 1,\ a+b+c = 1 \}$$

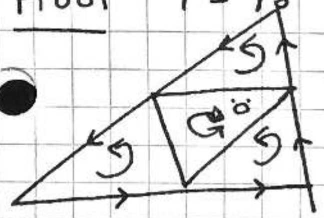
We'll denote by ∂T the simple, closed curve whose image is the boundary of T. Then ∂T consists of 3 line segments which we will direct so that T lies to the left.



<u>Thm 2.6</u> (Cauchy's theorem for triangles)

Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic.
If T is a triangle in U, then $\int_{\partial T} f(z)\, dz = 0$.

<u>Proof</u> $T = T_0$ Subdivide T into 4 smaller triangles by connecting the midpoints of its sides.



Call them $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$.

$$\int_{\partial T} f(z)\, dz = \sum_{j=1}^{4} \int_{\partial T^{(j)}} f(z)\, dz \qquad \text{(cancel)}$$

So if we let $\eta(T) = \int_{\partial T} f(z)\, dz$, then $\eta(T) = \sum_{j=1}^{4} \eta(T^{(j)})$.

So for some $j \in \{1, \dots, 4\}$,

$$|\eta(T^{(j)})| \ge \frac{1}{4} |\eta(T)|.$$

Let this $T^{(j)}$ be T_1. So $|\eta(T_1)| \ge \frac{1}{4} |\eta(T_0)|$,
and $\text{Length}(\partial T_1) = \frac{1}{2} \text{length}(\partial T_0)$.

Repeat this process indefinitely.

Get a sequence of triangles $T_0 \supset T_1 \supset T_2 \supset \cdots$ s.t.

$$|\eta(T_n)| \ge \frac{1}{4^n} |\eta(T_0)|, \quad \text{Length}(\partial T_n) = \frac{1}{2^n} \text{length}(\partial T_0)$$

There exists $z_0 \in U$ s.t. $\bigcap_{j=1}^{\infty} T_j = \{z_0\}$. (IB Analysis & Topology)

By diff'bility of f at z_0, $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\forall |z - z_0| < \delta, \quad |f(z) - (f(z_0) + f'(z_0)(z - z_0))| < \varepsilon |z - z_0|.$$

For n large enough, $T_n \subseteq D(z_0, \delta)$.

$$\eta(T_n) = \int_{\partial T_n} f(z)\, dz = \int_{\partial T_n} \big(f(z) - (f(z_0) + f'(z_0)(z - z_0)) \big)\, dz$$

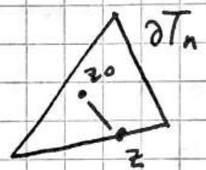
since poly has antideriv

$$\therefore |\eta(T_n)| \le \sup_{\partial T_n} |f(z) - (f(z_0) + (z - z_0) f'(z_0))| \cdot \text{Length}(\partial T_n)$$

$$\le \varepsilon \sup_{\partial T_n} |z - z_0| \cdot \text{Length}(\partial T_n)$$

Get the bound

$$|\eta(T_n)| \leq \varepsilon\,(\text{Length}(\partial T_n))^2.$$



Combining with the lower bound above,

$$\frac{|\eta(T)|}{\cancel{4^n}} \leq \varepsilon \cdot \frac{(\text{Length}(T))^2}{\cancel{4^n}}$$

Let $\varepsilon \to 0$ to get the result. $\square$

For later purposes, we need to extend this to allow a finite set where f may (a priori) not be differentiable.

<u>Theorem 2.7</u> Let $f: \mathcal{U} \to \mathbb{C}$ be continuous. If $S \subseteq \mathcal{U}$ is a finite set and if f is holomorphic in $\mathcal{U} \setminus S$, then

$$\int_{\partial T} f(z)\,dz = 0$$

for any triangle T in $\mathcal{U}$.

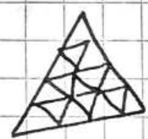
<u>Proof</u> Subdivide as before into $N = 4^n$ subtriangles $T_1, \dots, T_N$ as before (by connecting midpoints). Then $\int_{\partial T} f(z)\,dz = \sum_{i=1}^{N} \int_{\partial T_i} f(z)\,dz$.

By previous theorem,

$$\int_{\partial T} f(z)\,dz = \sum_{j \in I} \int_{\partial T_j} f(z)\,dz$$

where $I = \{ j : T_j \cap S \neq \emptyset \}$.

Each point can be contained in at most 6 subtriangles.



Hence $|I| \leq 6|S|$. So

$$\left| \sum_{j \in I} \int_{\partial T_j} f(z)\,dz \right| \leq \sum_{j \in I} \sup_{z \in T} |f(z)| \cdot \text{length}(\partial T_j)$$

$$\leq 6|S| \sup_{z \in T} |f(z)| \cdot \frac{\text{length}(\partial T)}{2^n}.$$

Let $n \to \infty$. $\square$

<u>Corollary</u> ("Convex Cauchy") Let $\mathcal{U} \subseteq \mathbb{C}$ be a star domain and $f: \mathcal{U} \to \mathbb{C}$ be cts and holomorphic in $\mathcal{U} \setminus S$ where S is a finite set. Then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in $\mathcal{U}$.

L7.3

<u>Proof</u> By thm 2.7 and corollary 2.5, f has an antiderivative.
By FTC, the claim follows. $\square$

<u>Thm 2.9</u> (Cauchy integral formula for a disc)

Let $D = D(a,r)$ and $f : D \to \mathbb{C}$ be holomorphic.

Then for any $0 < \rho < r$ and $w \in D(a,\rho)$, we have that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \, dz$$

Here $\partial D(a,\rho)$ is the curve $t \mapsto a + \rho e^{2\pi i t}$ for $t \in [0,1]$.

In particular, taking $w = a$,

$$f(a) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-a} \, dz = \int_0^1 f(a + \rho e^{2\pi i t}) \, dt$$

(Mean value property)

<u>Proof</u> Let

$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{for } z \neq w, \\ f'(w) & \text{for } z = w. \end{cases}$$

Then g is holomorphic on $D \setminus \{w\}$, and is continuous.

By the previous theorem,

$$\int_{\partial D(a,\rho)} \frac{f(z)-f(w)}{z-w} \, dz = 0.$$

Now $$\int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \, dz = f(w) \int_{\partial D(a,\rho)} \frac{1}{z-w} \, dz.$$

Use $$\frac{1}{z-w} = \frac{1}{z-a} \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^n}$$ via geometric series.

Convergence is uniform for $z \in \partial D(a,\rho)$.

So $$\int_{\partial D(a,\rho)} \frac{1}{z-w} \, dz = \underbrace{\int_{\partial D(a,\rho)} \frac{1}{z-a} \, dz}_{= 2\pi i \text{ (direct)}} + \sum_{n=1}^{\infty} \underbrace{\int \left(\frac{(a-w)^n}{(z-a)^{n+1}} \right) dz}_{\text{zero by FTC}}$$

$\square$

In the proof of the CIF last time, we used the following

<u>fact</u>: If $f_n \to f$ uniformly on a curve γ, then

$$\int_\gamma f_n(z)\,dz \longrightarrow \int_\gamma f(z)\,dz$$

<u>Proof</u>: $$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| \leq \Big(\underbrace{\sup_\gamma |f_n(z) - f(z)|}_{\downarrow\, 0} \Big)\, L(\gamma)$$

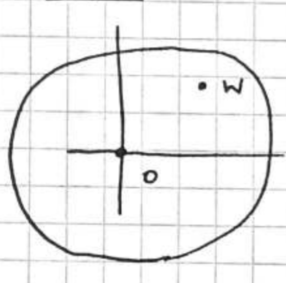
$\square$

<u>Consequences of the CIF</u>

<u>Thm 2.10</u> (Liouville's Thm) If $f: \mathbb{C} \to \mathbb{C}$ is an entire function (i.e. f is holomorphic everywhere), and if f is bounded ($|f(z)| \leq K$ for some constant K), then f is constant.

In fact, it suffices to assume f has sub-linear growth in place of boundedness, i.e. $|f(z)| \leq C(1 + |z|^\alpha)$ for some constants $C \geq 0$, $\alpha \in (0,1)$.

<u>Proof</u> Let $w \in \mathbb{C}$. By CIF, for any $\rho > |w|$,



$$f(w) = \frac{1}{2\pi i} \int_{\partial D_\rho(0)} \frac{f(z)}{z-w}\,dz, \quad f(0) = \frac{1}{2\pi i} \int_{\partial D_\rho(0)} \frac{f(z)}{z}\,dz$$

Now estimate

$$|f(w) - f(\overset{0}{\cancel{z}})| \leq \frac{1}{2\pi} \left| \int_{\partial D_\rho(0)} f(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dz \right|$$

$$= \frac{|w|}{2\pi} \left| \int_{\partial D_\rho(0)} \frac{f(z)}{z(z-w)} \right|$$

$$\leq \frac{|w|}{2\pi}\, 2\pi\rho \cdot \sup_{\partial D_\rho(0)} \frac{|f(z)|}{\underbrace{|z|}_{\rho}\,|z-w|}$$

$$\leq \sup_{\partial D_\rho(0)} \frac{|w|\, C(1+\rho^\alpha)}{|z-w|} \leq \frac{|w|\, C(1+\rho^\alpha)}{\rho - |w|}.$$

Let $\rho \to \infty$. Get $f(w) = f(0)\ \forall w \in \mathbb{C}$. $\square$

L8.2

Thm 2.11 (Fundamental Theorem of Algebra) Every non-constant polynomial with complex coefficients has a complex root.

Proof Let $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $a_n \neq 0$, $n \geqslant 1$.

If $\forall z$, $p_n(z) \neq 0$, then $f(z) = \frac{1}{p_n(z)}$ is entire.

Since $|p_n(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|$ for $z \neq 0$

$$\longrightarrow \infty \quad \text{as } |z| \to \infty,$$

we have that $|p_n(z)| \geqslant 1$ for $z \in \mathbb{C} \setminus \overline{D(0,R)}$ for some $R > 0$.

So $|f_n(z)| \leqslant 1$ for $z \in \mathbb{C} \setminus \overline{D(0,R)}$.

Since $\overline{D(0,R)}$ is compact, and f is fixed, we also have $|f|$ is bounded in $\overline{D(0,R)}$.

So f is a bounded, entire function, hence constant $\times\!\!\times$ $\square$

Thm 2.12 (local maximum modulus principle) Let $f: D(a,r) \to \mathbb{C}$ be holomorphic. If $|f(z)| \leqslant |f(a)|$ for all $z \in D(a,r)$, then f is constant.

Proof By the mean value property (Thm 2.9 CIF for $w = a$),

$$f(a) = \int_0^1 f(a + \rho e^{2\pi i t})\, dt \quad \text{for } \rho < r.$$

$$|f(a)| \overset{\circledast}{\leqslant} \sup_{0 \leqslant t \leqslant 1} |f(a + \rho e^{2\pi i t})| \leqslant |f(a)|.$$

Hence equality must hold, $\circledast$ implies $f(a + \rho e^{2\pi i t})$ is constant in t, so must be $f(a)$. $\square$

Taylor Series

Thm 2.13 Let $f: D(a,r) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation on $D(a,r)$ given by

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n \quad \text{where} \quad c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

<u>Proof</u> For any $w \in D(a,r)$, and any ρ with $|w-a| < \rho < R$,
by CIF
$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w}\, dz.$$

Expand
$$\frac{1}{z-w} = \frac{1}{z-a} \cdot \frac{1}{1-\frac{w-a}{z-a}} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$
with convergence uniform in $\partial D(a,\rho)$.

So
$$f(w) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, dz \right) (w-a)^n$$
□

<u>Cor 2.14</u> If $f: \mathcal{U} \to \mathbb{C}$ is holomorphic on an open set $\mathcal{U} \subseteq \mathbb{C}$, then f has derivatives of all orders, and all of its derivatives are holomorphic on $\mathcal{U}$.

<u>Proof</u> Thm 2.13 + Thm 1.4 (power-series infinitely diff) □

<u>Remark</u> We say that a function (real or complex) is <u>analytic</u> in an open set if it has a convergent Taylor series in a ball about every point. For complex functions, analytic <=> holomorphic.
<=> smooth

For real functions, implications do not reverse, e.g. $e^{-\frac{1}{x^2}}$

From now on, we'll use analytic & holomorphic interchangeably.

<u>Remark</u> We now also have that $f = u + iv$ is holomorphic in some open set $\mathcal{U}$ iff u, v have continuous partial derivatives and ~~satisfy~~ Cauchy-Riemann.

<u>Thm 2.15</u> (Morera's Thm) Let $\mathcal{U} \subseteq \mathbb{C}$ be open, $f: \mathcal{U} \to \mathbb{C}$ be continuous. If $\int_\gamma f(z)\, dz = 0$ for any closed curve in $\mathcal{U}$, then f is holomorphic.

<u>Proof</u> Antiderivative exists, $F' = f$. But F holomorphic, so $F' = f$ is holomorphic. #epic □

L9.1

<u>Thm 2.16</u> (CIF for the derivative) Let f be holomorphic on a disc $D(a,r)$. For any $w \in D$ and any ρ s.t. $|w-a| < \rho < r$, we have that

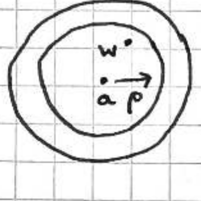
$$f'(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^2}\, dz .$$

$a + \rho e^{2\pi i t}$

More generally, if $f^{(n)}$ is the n^{th} derivative of f, then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{n+1}}\, dz .$$

<u>Remark</u> We have already seen this (in the derivation of Taylor series) in the special case $w = a$.



<u>Proof</u> Let $g(z) = \dfrac{f(z)}{z-w}$, holomorphic in $D \setminus \{w\}$, with $g'(z) = \dfrac{f'(z)}{z-w} - \dfrac{f(z)}{(z-w)^2}$.

By the FTC, $\int_{\partial D(a,\rho)} g'(z)\, dz = 0$.

So $\int_{\partial D(a,\rho)} \dfrac{f'(z)}{z-w}\, dz = \int_{\partial D(a,\rho)} \dfrac{f(z)}{(z-w)^2}\, dz$.

By the CIF applied to f', $f'(w) = \dfrac{1}{2\pi i} \int_{\partial D(a,\rho)} \dfrac{f'(z)}{(z-w)}\, dz$.

So the claim follows.

To prove the formula for $n \geq 2$, induct on n.

(Apply inductive hypothesis i.e. $f^{(n)}(w) = \dfrac{n!}{2\pi i} \int_{\partial D(a,\rho)} \dfrac{f(z)}{(z-w)^{n+1}}\, dz$ with f' in place of f, and then repeat the previous argument with $g(z) = \dfrac{f(z)}{(z-w)^{n+1}}$). $\square$

Uniform limits of holomorphic functions.

<u>Def^n</u> Let $U \subseteq \mathbb{C}$ be open. Let (f_n) be a sequence of functions. We say (f_n) <u>converges locally uniformly</u> on U if for any $a \in U$, there exists a disc $D(a,r)$, $r > 0$, s.t. (f_n) converges uniformly on $D(a,r)$.

<u>Prop 2.17</u> (f_n) converges locally uniformly on $U \iff (f_n)$ converges locally uniformly on every compact subset of U.

<u>Proof</u> Easy exercise. $\square$

<u>Thm 2.18</u> Let $f_n : \mathcal{U} \to \mathbb{C}$ be holomorphic, and suppose that $f_n \to f$ locally uniformly on $\mathcal{U}$ (where $f : \mathcal{U} \to \mathbb{C}$ is some function defined by the pointwise limit of the f_n). Then f is holomorphic.

<u>Remark</u> Such a theorem does not hold for real analytic functions, cf the Stone–Weierstrass theorem on uniform limits of polynomials. See II Linear Analysis.

<u>Proof</u> First (from IB Analysis & Topology) we know f is continuous. Let $a \in \mathcal{U}$, with $\overline{D(a,r)} \subset \mathcal{U}$. Then by Cauchy's thm (for ★), we have that $\int_\gamma f_n(z)\,dz = 0$ for any γ with $\text{image}(\gamma) \subset \overline{D(a,r)}$. careful!

But by local uniform convergence, (Prop 2.17), we have $f_n \to f$ uniformly on $\overline{D(a,r)}$ so $\int_\gamma f(z)\,dz = 0$.

Hence by Morera's theorem, f is analytic in $D(a,r)$. $\square$

[Moreover, $f_n' \to f'$ locally uniformly on $\mathcal{U}$]

<u>Proof</u> (cont.) Use CIF for derivatives. For any $w \in D(a, \frac{r}{2})$,

$$f_n'(w) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f_n(z)}{(z-w)^2}\,dz, \text{ and}$$

$$f'(w) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(z)}{(z-w)^2}\,dz.$$

$$\text{So } |f_n'(w) - f'(w)| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \sup_{\partial D(a,r)} |f_n - f| \cdot \underbrace{\sup_{\partial D(a,r)} \frac{1}{(z-w)^2}}_{\leq 4/r^2} \to 0$$

$\square$

<u>Next</u>: Zeros of a holomorphic function (non-zero) are isolated.

To show this, let $f : D(a,r) \to \mathbb{C}$ be holomorphic, note that by the Taylor series we have $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for $z \in D(a,r)$.

If $f \not\equiv 0$ in $D(a,r)$, then $c_n \neq 0$ for some n. Let m be the smallest such n. Then $f(z) = (z-a)^m g(z)$, $g(z) = \sum_{n=m}^{\infty} c_n (z-a)^{n-m}$.

Now g is holomorphic in $D(a,r)$, with $g(a) = c_m \neq 0$.

If $m \geq 1$, we say f has a zero of order m at a.

<u>Thm 2.19</u> (Principle of isolated zeros) If $f: D(a,r) \to \mathbb{C}$ is holomorphic, then there exists $r'>0$ s.t. $f(z) \neq 0$ for all z with $0 < |z-a| < r'$.

CHANGE MY MIND

<u>Proof</u> If $f(a) \neq 0$, done by continuity.

If $f(a) = 0$, we have $f(z) = (z-a)^m g(z)$ for some $m \geq 1$, and g with $g(a) \neq 0$. Then by continuity of g, $\exists\, r'>0$ s.t. $g(z) \neq 0$ for all $z \in D(a,r')$, i.e. $f(z) \neq 0$ for all $z \in D(a,r') \setminus \{a\}$. $\square$

<u>Analytic continuation</u>

By Taylor Series, we know that a holomorphic function f on a disc $D(a,r)$ is completely determined by the values of f and all its derivatives at a.

Q: Does this generalise to arbitrary domain? A: Yes! (next time)

Thm 2.20 (Unique continuation of analytic functions)

Let $D' \subset D$ be domains, and let $f : D' \to \mathbb{C}$ be analytic. Then there is at most one analytic $g : D \to \mathbb{C}$ s.t. $g(z) = f(z)\ \forall z \in D'$.

[Such a function g, if it exists, is called an analytic continuation of f to D.]

Proof Let $g_1, g_2 : D \to \mathbb{C}$ with $g_1 = g_2 = f$ in D'.

Let $h = g_1 - g_2$, so $h = 0$ in D'.

Let $D_0 = \{ z \in D : h \equiv 0 \text{ in some disc } D(z,r) \subset D, r > 0 \}$.

Then D_0 is an open subset of D.

Let $D_1 = \{ z \in D : h^{(n)}(z) \neq 0 \text{ for some } n \} = D \setminus D_0$.

Indeed, if $z \in D$ is s.t. $h \not\equiv 0$ in a disc $D(z,r)$, then by Taylor series $z \notin D_0$. So $D = D_0 \sqcup D_1$.

Also, D_1 is open. So by connectedness of D, one of D_0, D_1 is empty. Since $D' \subset D_0$, it follows that $D = D_0$. $\square$

Remark The above proof only depended on having a convergent power series about every point. So the thm is valid for real analytic f$^{\text{n}}$s.

Remark Analytic continuations to larger domains need not exist.

$f(z) = \sum_{n=0}^{\infty} z^{n!}$, has roc $= 1$, hence defines an analytic function on $D(0,1)$. But this has no analytic continuation to a larger domain containing $D(0,1)$ as a strict subset. (Ex Sheet 2)

$\partial D(0,1)$ is called the "natural boundary" of f.

Rmk about the principle of isolated zeros (Thm 2.19)

Zeros of a holomorphic function can have a limit point at the boundary of the domain (Thm only rules out limit points within the interior of the domain for f$^{\text{n}}$s not identically zero)

$f(z) = \sin \frac{1}{z}$, $U = \mathbb{C} \setminus \{0\}$, $z_n = \frac{1}{2n\pi} \to 0$, $f(z_n) = 0$

<u>Corollary 2.21</u> (Identity principle) Let $\mathcal{U} \subseteq \mathbb{C}$ be a <u>domain</u>, and let $f, g : \mathcal{U} \to \mathbb{C}$ be holomorphic. If the set $S = \{f(z) = g(z)\}$ has a non-isolated point, then $f \equiv g$.

<u>Proof</u> Let $h = f - g$. By hypothesis, h has a non-isolated zero z in $\mathcal{U}$. So by thm 2.19, $\exists r > 0$ s.t. $h \equiv 0$ in $D(z, r)$. Then by Thm 2.20, $h \equiv 0$ in $\mathcal{U}$. $\square$

<u>Corollary 2.22</u> (Global maximum principle) Let $\mathcal{U} \subseteq \mathbb{C}$ be a bounded domain, and let $f : \overline{\mathcal{U}} \to \mathbb{C}$ be continuous and holomorphic in $\mathcal{U}$. Then $\sup_{\overline{\mathcal{U}}} |f|$ is attained at a point in ∂U.

<u>Proof</u> Let $z_0 \in \overline{\mathcal{U}}$ be a point at which $|f|$ attains its sup.

If $z_0 \in \mathcal{U}$, then we can apply the local maximum principle in a suitable disc $D(z_0, r)$ to conclude $f \equiv c$, a constant in $D(z_0, r)$.

Then we can apply Identity principle with $g = c$, to conclude that $f \equiv c$ on $\overline{\mathcal{U}}$. The conclusion follows.

If $z_0 \in \partial U$, then also have the conclusion. $\square$

<u>Remark</u> There is a very close connection between some of the key theorems we've proved so far and the theory of harmonic functions (or more generally to elliptic PDE's, of which harmonic functions are a special case).

E.g. Unique continuation holds for harmonic f^ns (in any dimension)

Regularity holds for harmonic f^ns (i.e. once diff $\Rightarrow$ ∞ diff)

Maximum principles hold (both local & global) for harmonic f^ns

Mean value property also holds (only for harmonic f^ns, not for general elliptic eq^ns)

The proofs of these are via different techniques, and are valid in arbitrary dimensions.

§3 Complex Integration II

Key goals (1) Characterise domains for which Cauchy's theorem is valid.

(2) Generalise CIF for certain closed curves which are more general than circles

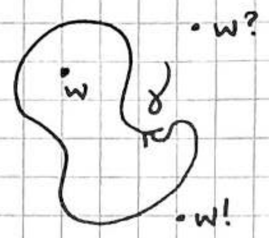
We need the notion of winding number of a closed curve.

Informally, the no. of times a closed curve winds around a given point.

Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve. Fix $w \in \mathbb{C} \setminus \text{image}(\gamma)$.

Write for each $t \in [a,b]$,

$$\gamma(t) = w + r(t) e^{i\theta(t)}.$$



It's clear that $r(t)$ is uniquely determined

$$r(t) = |\gamma(t) - w|.$$

If γ is continuous, then $r(t)$ is cts. If γ is piecewise C^1, then $r(t)$ is also piecewise C^1.

If we can find a continuous $\theta(t)$ such that the above holds, then define winding number of γ around w (or index of γ around w) by $I(\gamma; w) = \dfrac{\theta(b) - \theta(a)}{2\pi}$.

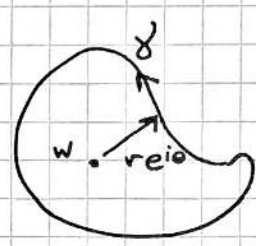
Clear that $I(\gamma; w)$ is an integer, since $\gamma(b) = \gamma(a)$, from which $r(b) = r(a)$ and $e^{i\theta(a)} = e^{i\theta(b)}$.

If θ_1 is another continuous choice, then $\theta - \theta_1$ takes values in $2\pi\mathbb{Z}$, so is constant by connectedness of $[a,b]$.

So $I(\gamma; w)$ is independent of the choice of θ.

Winding number $\gamma(t) = w + r(t)\, e^{i\theta(t)}$, θ cts

$$I(\gamma; w) = \frac{1}{2\pi}\left(\theta(b) - \theta(a)\right)$$



Thm 3.1 sike! Lemma

Let $w \in \mathbb{C}$. If $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ is piecewise C^1, then $\exists$ piecewise C^1 $\theta : [a,b] \to \mathbb{R}$ s.t.

$$\gamma(t) = w + r(t)\, e^{i\theta(t)} \quad \forall\ t \in [a,b]$$

where $r(t) = |\gamma(t) - w|$ is piecewise C^1.

Math Lab If γ is C^1, and if the above holds, then

$$\gamma'(t) = r'(t)\, e^{i\theta(t)} + \underbrace{r(t)\, e^{i\theta(t)}} * i\theta'(t)$$

$$= \left(\gamma(t) - w\right)\left(\frac{r'(t)}{r(t)} + i\theta'(t)\right).$$

So $\theta' = \mathrm{Im}\left(\dfrac{\gamma'(t)}{\gamma(t) - w}\right)$, and $\theta(t) - \theta(a) = \overset{\mathrm{Im}}{\int_a^t} \dfrac{\gamma'(s)}{\gamma(s) - w}\, ds$.

Proof Let $h(t) = \displaystyle\int_a^t \frac{\gamma'(s)\, ds}{\gamma(s) - w}$.

Then h is continuous on $[a,b]$, and

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - w} \quad \text{whenever } \gamma'(t) \text{ is continuous.}$$

So h is piecewise C^1.

$$\frac{d}{dt}\left(\left(\gamma(t) - w\right) e^{-h(t)}\right) = \gamma'(t)\, e^{-h(t)} - \left(\gamma(t) - w\right) e^{-h(t)}\, h'(t)$$
$$= 0$$

whenever h is diff'ble (i.e. except for a finite number of points).

Since $(\gamma(t) - w)\, e^{-h(t)}$ is continuous, this means that it is constant.

So $(\gamma(t) - w)\, e^{-h(t)} = \gamma(a) - w$

$$\Rightarrow \gamma(t) - w = (\gamma(a) - w)\, e^{h(t)}$$
$$= (\gamma(a) - w)\, e^{\mathrm{Re}\, h(t)}\, e^{i\, \mathrm{Im}\, h(t)}$$
$$= |\gamma(a) - w|\, e^{\mathrm{Re}\, h(t)}\, e^{i[\arg(\gamma(a) - w) + \mathrm{Im}\, h(t)]}$$

where $e^{i\alpha} = \dfrac{\gamma(a) - w}{|\gamma(a) - w|}$. Set $\theta(t) = \alpha + \mathrm{Im}\, h(t)$. $\square$

<u>Corollary</u> (of the proof) If additionally γ is a closed curve, then

$$\boxed{I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w}} \qquad \forall\, w \notin \text{Image}(\gamma)$$

<u>Proof</u> Let $\theta(t) = \alpha + \operatorname{Im} h(t)$ as in the proof of the Lemma.

By def^n, $I(\gamma; w) = \dfrac{\theta(b) - \theta(a)}{2\pi} = \dfrac{\operatorname{Im} h(b) - \operatorname{Im} h(a)}{2\pi}$

$$= \frac{\operatorname{Im} h(b)}{2\pi}.$$

Also we have $(\gamma(t) - w) = (\gamma(a) - w)\, e^{h(t)} \implies e^{h(b)} = 1$. (closed)

So $\operatorname{Re} h(b) = 0$.

So $\operatorname{Im} h(b) = \frac{1}{i} h(b)$.

So
$$I(\gamma; w) = \frac{1}{2\pi i} h(b) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)\, ds}{\gamma(s) - w} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w}. \qquad \square$$

<u>Remark</u> In fact Lemma 3.1 holds for continuous γ, with the assertion that $r(t)$ and $\theta(t)$ are continuous. We won't need this, but the proof is elementary, based on uniform continuity.

<u>Prop 3.2</u> If $\gamma : [a,b] \longrightarrow D(z_0, R)$ is closed, and $w \notin D(z_0, R)$, then $I(\gamma; w) = 0$.

<u>Proof</u> The function $z \mapsto \frac{1}{z-w}$ is holomorphic in $D(z_0, R)$.

So $\int_\gamma \frac{dz}{z-w} = 0$, the result follows. (Convex Cauchy) $\square$

<u>Prop 3.3</u> Let γ be a closed curve. The function $w \mapsto I(\gamma; w)$ is continuous on $\mathbb{C} \setminus \text{Image}(\gamma)$. Hence, since $I(\gamma; w) \in \mathbb{Z}$, this function is locally constant.

<u>Proof</u> Trivial $\square$

<u>Def^{n}s</u> Let $U \subseteq \mathbb{C}$ be open.

(i) a closed cure γ in U is <u>homologous to zero in U</u> if
$I(\gamma; w) = 0$ whenever $w \in \mathbb{C} \setminus U$.

(ii) U is <u>simply connected</u> if every closed curve in U is homologous to zero in U.

A disc is simply connected (Lemma).

A nnulus is not simply connected. (CIF)

<u>Theorem 3.4</u> (CIF, general version)

Let $U \subseteq \mathbb{C}$ be open, and let $\gamma : [a,b] \to U$ be a closed curve homologous to zero in U. Then for any holomorphic $f : U \to \mathbb{C}$, we have

(a) $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z-w} = I(\gamma; w) f(w)$ for $w \in U \setminus \mathrm{Image}\,\gamma$

(b) $\displaystyle \int_\gamma f(z)\,dz = 0.$

Immediate corollary is the following

<u>Cor 3.5</u> (General version of Cauchy's theorem)

If U is simply connected, then for any closed curve γ in U and any holomorphic $f : U \to \mathbb{C}$, we have that $\int_\gamma f(z)\,dz = 0$.

<u>Remark</u> Cauchy's thm says that if $\int_\gamma f(z)\,dz$ has zero for all functions of the form $\frac{1}{z-w}$, $w \in \mathbb{C} \setminus U$, then it holds for every holomorphic f.

<u>Proof</u> (b) follows from (a), because given holomorphic f, pick any $w \in U \setminus \mathrm{image}(\gamma)$ and apply (a) to the function $F(z) = (z-w) f(z)$. Then F is holomorphic on U, $F(w) = 0$, so by (a) $\displaystyle \int_\gamma f(z) \overset{dz}{=} \int_\gamma \frac{F(z)}{z-w}\,dz = 0.$

To prove (a), we proceed in the following way:

<u>STEP 1</u> Define $g : U \times U \to \mathbb{C}$

$$(z,w) \mapsto \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{for } z \neq w, \\ f'(w) & \text{for } z = w. \end{cases}$$

and $h(w) = \int_\gamma g(z,w)\,dz$ for $w \in U$

$$h_1(w) = \int_\gamma \frac{f(z)}{z-w}\,dz \quad \text{for } w \in \mathbb{C} \setminus \mathrm{image}(\gamma) \text{ with } I(\gamma, w) = 0$$

Thm 3.3(a) CIF, general version

$\mathcal{U} \subseteq \mathbb{C}$ open, $\gamma : [a,b] \to \mathcal{U}$ closed curve homologous to zero in $\mathcal{U}$, i.e. $I(\gamma; w) = 0 \ \forall \ w \in \mathbb{C} \setminus \mathcal{U}$. Then for holomorphic $f : \mathcal{U} \to \mathbb{C}$ we have $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \, dz = I(\gamma; w) f(w)$ for $w \in \mathcal{U} \setminus \text{image}(\gamma)$.

Proof

$$\text{CIF} \iff \frac{1}{2\pi i} \int_\gamma \left(\frac{f(z) - f(w)}{z-w} \right) dz = 0 \quad \forall \ w \in \mathcal{U} \setminus \text{image}(\gamma).$$

Let $g : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$

$$(z,w) \mapsto \begin{cases} \dfrac{f(z) - f(w)}{z-w} & \text{for } z \neq w, \\ f'(w) & \text{for } z = w. \end{cases}$$

Claim g is continuous. Clear at points (z,w) with $z \neq w$.

For a point $(a,a) \in \mathcal{U} \times \mathcal{U}$, given $\varepsilon > 0$, we can choose $\delta > 0$ s.t. $\xi \in D(a,\delta) \Rightarrow |f'(\xi) - f'(a)| < \varepsilon$.

Now

$$f(z) - f(w) = \int_0^1 \frac{d}{dt} f(tz + (1-t)w) \, dt$$
$$= \int_0^1 f'(tz + (1-t)w)(z-w) \, dt$$

$$\therefore \left| \frac{f(z) - f(w)}{z-w} - f'(a) \right| = \left| \int_0^1 \left(f'(tz + (1-t)w) - f'(a) \right) dt \right| \leq \varepsilon$$

for all $z, w \in D(a,\delta)$, since $tz + (1-t)w \in D(a,\delta)$ as well.

So D is continuous at (a,a), hence everywhere.

For a fixed z, the function $w \mapsto g(z,w)$ is holomorphic.

This is clear at points $w \in \mathcal{U} \setminus \{z\}$. To check holomorphy at $w = z$ choose a ball $\overline{D(z,\rho)} \subset \mathcal{U}$, and apply Convex Cauchy + Morera, for closed curves in $\overline{D(z,\rho)}$.

allows a priori points were not diff'ble, but cts

Now let

$$h(w) = \int_\gamma g(z,w) \, dz.$$

$\text{CIF} \iff$ for $w \notin \text{image}(\gamma)$, $h = 0$.

Claim h is continuous. To see this, pick $w \in \mathcal{U}$, choose disc $\overline{D(w,\rho)} \subset \mathcal{U}$. Note that g is uniformly continuous on $\text{image}(\gamma) \times \overline{D(w,\rho)}$.

So for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. $w' \in D(w,\delta)$, $z \in \text{image}(\gamma)$

$$\Rightarrow |g(z,w') - g(z,w)| < \varepsilon.$$

Then for $w' \in D(w,\delta)$, $|h(w') - h(w)| \leq \varepsilon \, \text{length}(\gamma)$. So h is cts.

Claim h is holomorphic. Fix $w \in \mathcal{U}$, and $\overline{D(w,\rho)} \subset \mathcal{U}$.

For any closed curve $\tilde{\gamma} : [c,d] \to D(w,\rho)$, we have

$$\begin{aligned}
\int_{\tilde{\gamma}} h(w)\,dw &= \int_{\tilde{\gamma}} \int_{\gamma} g(z,w)\,dz\,dw \\
&= \int_c^d \int_a^b g(\gamma(s), \tilde{\gamma}(t))\, \gamma'(s)\,ds\, \tilde{\gamma}'(t)\,dt \\
&\overset{*}{=} \int_a^b \int_c^d g(\gamma(s), \tilde{\gamma}(t))\, \tilde{\gamma}'(t)\,dt\, \gamma'(s)\,ds \\
&= \int_{\gamma} \underbrace{\int_{\tilde{\gamma}} g(z,w)\,dw}_{\text{zero by Convex Cauchy}}\,dz = 0.
\end{aligned}$$

Hence by Morera, h is holomorphic.

Next let $h_1(w) = \int_{\gamma} \frac{f(z)}{z-w}\,dz$ for $w \in \mathbb{C} \setminus \text{image}(\gamma)$ s.t. $I(\gamma; w) = 0$

Let $\mathcal{U}_1$ be the domain, open by continuity.

Then h_1 is continuous on $\mathcal{U}_1$ (check), and also holomorphic on $\mathcal{U}_1$.
(Use the same Convex Cauchy + Morera argument)

Finally, by assumption $I(\gamma; w) = 0$ for any $w \notin \mathcal{U}$.

Hence $\mathbb{C} = \mathcal{U} \cup \mathcal{U}_1$. And on $\mathcal{U} \cap \mathcal{U}_1$, h and h_1 agree.

Indeed, for ~~$h \in$~~ $w \in \mathcal{U} \cap \mathcal{U}_1$, $h(w) = \int_{\gamma} \frac{f(z) - f(w)}{z-w}\,dz \underset{\uparrow\, I(\gamma;w)=0}{=} \int_{\gamma} \frac{f(z)}{z-w}\,dz$

So $\varphi : \mathbb{C} \to \mathbb{C}$

$$z \mapsto \begin{cases} h(z) & \text{for } z \in \mathcal{U}, \\ h_1(z) & \text{for } z \in \mathcal{U}_1 \end{cases}$$

is an entire function.

Choose $R>0$ s.t. image$(\gamma) \subset D(0,R)$.

Then for $w \notin D(0,R)$, $I(\gamma; w) = 0$.

Therefore, for such w, $\varphi(w) = h_1(w)$ and so

$$|\varphi(w)| = \left| \int_\gamma \frac{f(z)}{z-w}\, dz \right| \leq (\sup_\gamma |f|)\ \text{length}(\gamma)\ \frac{1}{|w|-R} \longrightarrow 0$$

as $w \to \infty$.

By Liouville, φ is constant and equal to zero.

Hence for $w \in U \setminus$ image(γ), $h(w) = 0$ as desired. $\square$

<u>Remark</u> It is pretty cool that we use a global theorem (Liouville) to prove this local result, namely CIF.

Next we want to connect our notion of simply connected to the more standard definition in algebraic topology.

<u>Def^n</u> Let $U \subseteq \mathbb{C}$ be open. Let $\gamma_0, \gamma_1 : [0,1] \to U$ be two closed (piecewise C^1) curves. Then γ_0 is <u>homotopic to γ_1 in U</u> if there exists a continuous map $H : [0,1] \times [0,1] \longrightarrow U$ s.t.

$$H(0,t) = \gamma_0(t) \quad \text{for } t \in [0,1]$$

$$H(1,t) = \gamma_1(t) \quad \text{for } t \in [0,1]$$

$$H(s,0) = H(s,1) \quad \forall\, s \in [0,1]$$

<u>Thm 3.4</u> If $\gamma_0, \gamma_1 : [0,1] \to U$ are closed curves, and if γ_0 is homotopic to γ_1 in U, then for $w \in \mathbb{C} \setminus U$,

$$I(\gamma_0; w) = I(\gamma_1; w)$$

The proof relies on the following:

<u>Lemma 3.5</u> If $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$, and $w \in \mathbb{C}$, and $\cancel{\gamma}$

$$|\gamma_0(t) - \gamma_1(t)| < |w - \gamma_0(t)| \quad \text{for each } t \in [0,1],$$

then $I(\gamma_0; w) = I(\gamma_1; w)$.

<u>Proof</u> Ex Sheet.

<u>Thm 3.6</u> If $\gamma_0, \gamma_1 : [0,1] \to \mathcal{U}$ are closed curves with γ_0 homotopic to γ_1 in $\mathcal{U}$, then $I(\gamma_0; w) = I(\gamma_1; w)$ for any $w \notin \mathcal{U}$.

<u>Proof</u> We have cts $H : [0,1]^2 \to \mathcal{U}$ s.t. $H(0,t) = \gamma_0(t)$, and $H(1,t) = \gamma_1(t)$ $\forall\, t \in [0,1]$ and $H(s,0) = H(s,1)$ $\forall\, s \in [0,1]$.

$H([0,1]^2) \subseteq \mathcal{U}$ is compact, and $w \notin \mathcal{U}$.

So $\exists\ \varepsilon > 0$ s.t. $|w - H(s,t)| > 2\varepsilon$ $\forall\ (s,t) \in [0,1]^2$.

H is uniformly continuous, so we can find $n \in \mathbb{N}$ s.t.

$|s-s'| + |t-t'| \leq \frac{1}{n} \Rightarrow |H(s,t) - H(s',t')| < \varepsilon$.

Let $\gamma_s(t) = H(s,t)$.

For $k = 1, 2, \dots, n$, $|\gamma_{\frac{k-1}{n}}(t) - \gamma_{\frac{k}{n}}(t)| < \varepsilon$ $\forall\ t \in [0,1]$.

Also, for $k = 0, 1, \dots, n$, $|w - \gamma_{\frac{k}{n}}(t)| > 2\varepsilon$.

NOICE

So $|\gamma_{\frac{k-1}{n}}(t) - \gamma_{\frac{k}{n}}(t)| < |w - \gamma_{\frac{k}{n}}(t)|$.

If γ_s is piecewise C^1 for $s = 0, \frac{1}{n}, \dots, 1$ we can apply Lemma 3.7.

But Lemma holds for continuous curves as well, so eh. □

<u>Remarks</u> ① Thm 3.6 implies that if a closed curve $\gamma : [0,1] \to \mathcal{U}$ is null-homotopic (i.e. homotopic to a constant curve $\tilde{\gamma} : [0,1] \to \mathcal{U}$, $t \mapsto p$), then γ is homologous to zero in $\mathcal{U}$.

(The converse is not true. Think about a counterexample)

② The "usual" definition of simply connected spaces is that any closed curve in the space is null-homotopic. We have shown

$\mathcal{U}$ is simply connected in the sense of alg top

$\Rightarrow$ $\mathcal{U}$ is simply connected in our definition via winding no.

$\Rightarrow$ $\int_\gamma f(z)\,dz$ for holomorphic $f : \mathcal{U} \to \mathbb{C}$, cure γ in $\mathcal{U}$

The reverse implications also hold, (iii) $\Rightarrow$ (ii) easy, (ii) $\Rightarrow$ (i) hard.

The fact that (i) $\Leftrightarrow$ (iii), or even (i) $\Rightarrow$ (iii) is surprising, because (i) is a topological statement and (iii) is an analytic one.

§4 Laurent series, isolated singularities, the residue theorem

Thm 4.1 Suppose f is holomorphic in the annulus

$$A = \{ z \in \mathbb{C} : r < |z-a| < R \}$$

where $0 \le r < R \le \infty$. Then

(i) there exists a unique expansion of f in A of the form

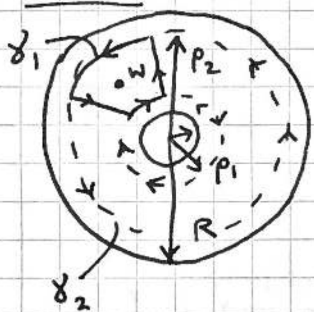
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \left(= \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n} \right),$$

(ii) the coefficients are given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad \text{for any } r < \rho < R,$$

(iii) for any $r < \rho' \le \rho < R$, the series converges uniformly on $\{ z \in \mathbb{C} : \rho' \le |z-a| \le \rho \}$.

Proof Let $w \in A$. Choose ρ_1, ρ_2 s.t. $r < \rho_1 < |w-a| < \rho_2 < R$.



$$\int_{\gamma_2} \frac{f(z)}{z-w}\, dz = 0 \qquad \text{since } z \mapsto \frac{f(z)}{z-w} \text{ holo on } A \setminus \{w\}$$
and γ_2 null-homotopic in $A \setminus \{w\}$

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(z)}{z-w}\, dz = I(\gamma_1 ; w)\, f(w) \qquad \text{(CIF)}$$

$I(\gamma_1 ; w) = 1$ (by def^n or the fact γ_1 homotopic to a small circle around w)

So
$$f(w) = \frac{1}{2\pi i} \int_{\gamma_1 + \gamma_2} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)}{z-w}\, dz$$
$$= f_1(w) + f_2(w)$$

where
$$f_1(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)}{z-w}\, dz, \quad f_2(w) = \frac{-1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)}{z-w}\, dz.$$

For f_1, expand the integral exactly as in Taylor series, get

$$f_1(w) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{with} \quad c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

For f_2, write
$$\frac{1}{z-w} = -\frac{1}{(w-a)\left(1 - \frac{z-a}{w-a}\right)} = -\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

with convergence uniform on $\partial D(a, \rho_1)$.

So $f_2(w) = \sum_{m=1}^{\infty} d_m (w-a)^{-m}$ with $d_m = \frac{1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)}{(z-a)^{-m+1}} dz$.

This gives the existence of Laurent series.

For (ii), (iii) and uniqueness; suppose we have an expansion of the form above. Then the power series part converges for all $w \in A$ so has r.o.c $\geq R$, and uniform convergence in $\{|z-a| \leq \rho\}$.

Letting $\frac{1}{z-a} = u$, we also have $\sum_{n=1}^{\infty} c_{-n} u^n$ with r.o.c. $\geq \frac{1}{r}$.

So convergence is uniform in $\{|z-a| \geq \rho'\}$.

So in ⊛, uniform convergence holds on $\{\rho' \leq |z-a| \leq \rho\}$.

Now we can integrate around $\partial D(a,\rho)$ for any $r < \rho < R$ to get

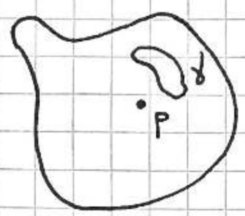
$$\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{m+1}} dz = \sum_{n=-\infty}^{\infty} c_n \cdot \int_{\partial D(a,\rho)} \frac{(z-a)^n}{(z-a)^{m+1}} = 2\pi i c_m.$$

□

L14.1

<u>Rmk</u> about simply connected domains:

- Any star domain is simply connected.



Given $\gamma:[0,1]\to \mathcal{U}$ a closed curve, if p is the centre of $\mathcal{U}$, take $H:[0,1]^2 \to \mathcal{U}$

$$(s,t) \mapsto sp + (1-s)\gamma(t).$$

<u>Last time</u>: If $f: A \to \mathbb{C}$ is holomorphic, $A = \{ r < |z-a| < R \}$, $0 \leq r < R \leq \infty$, then $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \ \forall z \in A$, with

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \text{for any } \rho \in (r, R).$$

- This is the Laurent series of f in A.

<u>Rmk</u> It follows from above that $f(z) = f_1(z) + f_2(z)$ for $z \in A$, where f_1 is holomorphic in $D(0,R)$, f_2 is holomorphic in $\mathbb{C} \setminus \overline{D(a,r)}$
$= \{|z-a| > r\}$

In fact, $f_1(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$,

$f_2(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n$.

Next we discuss isolated singularities, and then classify them using Laurent series.

<u>Def^{n}s</u> A cx-valued function f has an <u>isolated singularity</u> at $a \in \mathbb{C}$

- if f is defined and holomorphic in a punctured disc $D(a,R) \setminus \{a\}$ for $R>0$, but not in $D(a,R)$ (i.e. either not defined at a, or not holomorphic in $D(a,R)$)

<u>Ex</u> $a=0$, $f(z) = \frac{1}{z}$ or $f(z) = \frac{e^z - 1}{z}$ or $f(z) = e^{1/z}$

<u>Def^n</u> The isolated singularity a is a <u>removable singularity</u> if $\exists$ holomorphic $g: D(a,R) \to \mathbb{C}$ that agrees with f in $D(a,R) \setminus \{a\}$.

<u>Prop 4.2</u> f has a removable singularity at $a \in \mathbb{C}$ iff

$$\lim_{z \to a} (z-a) f(z) = 0$$

- <u>Proof</u> "$\Rightarrow$" $(z-a) f(z) = (z-a) \underbrace{g(z)}_{\text{bdd at } a} \to 0$ as $z \to a$

"$\Leftarrow$" Let
$$h(z) = \begin{cases} (z-a)^2 f(z) & z \neq a, \\ 0 & z = a. \end{cases}$$

Then h is holomorphic in some $D(a,R)\setminus\{a\}$.

$$\frac{h(z)-h(a)}{z-a} = (z-a)f(z) \longrightarrow 0 \quad \text{as } z \to a$$

So h is holomorphic in $D(a,R)$ with $h'(a)=0$, $h(a)=0$.

So h has a zero of order ≥ 2, hence $h(z)=(z-a)^2 g(z)$ for some holomorphic g on $D(a,R)$. $\square$

<u>Corollary</u> If f has an isolated singularity at $z=a$, and if f is bdd near a, then a is removable.

<u>Pf</u> Immediate from Prop 4.2. In fact f has a limit at a. $\square$

<u>Def^n</u> Let a be an isolated singularity of f. Then a is a <u>pole</u> if

$$\lim_{z\to a} |f(z)| = \infty.$$

The point a is an <u>essential singularity</u> if a is neither removable nor a pole.

<u>Prop 4.3</u> Let $\mathcal{U}$ be a domain, $a\in\mathcal{U}$ and let $f:\mathcal{U}\setminus\{a\}\longrightarrow\mathbb{C}$ be holomorphic. Then TFAE

(i) the point a is a pole of f,

(ii) there exists $\varepsilon>0$ s.t. $h: D(a,\varepsilon)\to\mathbb{C}$

$$z \longmapsto \begin{cases} 0 & \text{for } z=a, \\ 1/f(z) & \text{for } z\neq a, \end{cases}$$

is holomorphic and non-zero for $z\neq a$,

(iii) there is holomorphic $g:\mathcal{U}\to\mathbb{C}$ such that $f(z)=(z-a)^{-k}g(z)$, for some $k\geq 1$ and with $g(a)\neq 0$. Such g and k are uniquely determined by f.

<u>Proof</u> "(i) $\Rightarrow$ (ii)" $|f(z)|\longrightarrow\infty$ as $z\to a$. So choose $R>0$ s.t. $|f(z)|>1$ for $0<|z-a|<R$. Then $\frac{1}{f}$ is holomorphic in $D(a,R)\setminus\{a\}$ and is bounded. So the singularity at a is removable (Prop 4.2) i.e. $f(z)^{-1}=h(z)$ $\forall z\in D(a,R)\setminus\{a\}$ where $h:D(a,R)\to\mathbb{C}$ is holo. Moreover, $h(0)=0$ since $\lim_{z\to a}\frac{1}{|f(z)|}=0$.

"(ii) $\Rightarrow$ (iii)" Let h be as in (ii). Since $h(a)=0$, the order of a as a zero of h is some $k \geq 1$. Write $h(z) = (z-a)^k h_1(z)$ where $h_1 : D(a,R) \to \mathbb{C}$ is holomorphic and $h_1(a) \neq 0$.

Choosing $g(z) = \frac{1}{h_1(z)}$ so $f(z) = (z-a)^{-k} g(z)$ gives (iii).

Well, g is holomorphic on $D(a,R)$.

Now $g(z) = (z-a)^k f(z)$ on $D(a,R) \setminus \{a\}$.

By identity principle, $(z-a)^k f(z)$ extends g to a holomorphic function on $\mathcal{U}$.

Uniqueness follows from $g(a) \neq 0$:

For uniqueness of k, g, suppose $f(z) = (z-a)^{-\tilde{k}} \tilde{g}(z)$ on $\mathcal{U} \setminus \{a\}$.

If $\tilde{k} > k$, then $\tilde{g}(z) = (z-a)^{\tilde{k}-k} g(z) \to 0$ as $z \to a$, contradicting $\tilde{g}(a) \neq 0$. So $\tilde{k} \leq k$. Similarly $\tilde{k} \geq k$.

It follows that $g = \tilde{g}$.

(iii) $\Rightarrow$ (i) e.z. $\square$

<u>Corollary 4.4</u> If $a \in \mathbb{C}$ is an essential singularity of f, then $\lim_{z \to a} |f(z)|$ does not exist (in $\mathbb{R} \cup \{\infty\}$). Converse true too.

<u>Proof</u> Props 4.2 & 4.3 $\square$

<u>Ex</u> $f(z) = e^{1/z}$ $\quad f\left(\frac{1}{2n\pi i}\right) \to 1$, $\quad f\left(\frac{1}{n}\right) \to \infty$

no limit

<u>Terminology</u> ① If f has a pole at $z=a$, then the unique integer k given by Prop 4.3 (iii), (s.t. near a, $f(z) = (z-a)^{-k} g(z)$ with g holomorphic, $g(a) \neq 0$) is called the <u>order</u> of the pole a.

If $k=1$ then the point a is a simple pole.

② Let $\mathcal{U}$ be a domain, and $S \subseteq \mathcal{U}$ be a set of isolated points in $\mathcal{U}$. If $f : \mathcal{U} \setminus S \to \mathbb{C}$ is holomorphic, each point of S, say a, is either removable or a pole, then we say f is a <u>meromorphic</u> function on $\mathcal{U}$.

(<u>Rmk</u>: As a map into $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ = Riemann sphere, a meromorphic function behaves perfectly nicely. The point ∞ on $\hat{\mathbb{C}}$ is just like any other point. More on this in the Riemann Surfaces course)

<u>Rmk</u> By Corollary 4.4, we know that at an (isolated) essential singularity a of f, the function "oscillates a lot", i.e. the limit of $|f|$ does not exist in $\mathbb{R} \cup \{\infty\}$. Can we be more precise? Yes!

One result is the <u>Casorati - Weierstrass</u> thm (Ex Sheet 2), which says that for (small enough) $\varepsilon > 0$, the image $f(D(a,\varepsilon) \setminus \{a\})$ is dense in $\mathbb{C}$.

A much deeper theorem is the Picard thm, which says that $\exists w \in \mathbb{C}$ s.t. $\mathbb{C} \setminus \{w\} \subseteq f(D(a,\varepsilon) \setminus \{a\})$ for all $\varepsilon > 0$, i.e. every complex number, except possibly one, is attained by f in arbitrarily small ngbds of a.

<u>Connection to Laurent expansion</u>

Let f be holo in $D(a,R) \setminus \{a\}$. Then we have the series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \forall\, z \in D(a,R) \setminus \{a\}$$

(by Laurent series thm taken with $r=0$).

(i) ~~If~~ $c_n = 0 \ \forall n < 0 \iff f(z) = g(z) \quad \forall z \in D(a,R)\setminus\{a\}$

where $g(z) = \sum_{n\geq 0} c_n (z-a)^n$ is holo in $D(a,R)$

$\iff$ a is a removable singularity

(ii) ~~If~~ $c_k \neq 0$ for $k<0$, $c_n = 0 \ \forall n<k \iff$

$$f(z) = \frac{c_k}{(z-a)^{-k}} + \cdots + \frac{c_{-1}}{(z-a)} + \sum_{n=0}^{\infty} c_n (z-a)^n = g(z)/(z-a)^{-k}$$

where g is holo in $D(a,R)$ with $g(a) = c_k \neq 0$

$\iff$ a is a pole of order k.

(iii) ~~If~~ $c_n \neq 0$ for infinitely many negative n

$\iff$ a is an essential singularity

Let the series expansion be as above. Then by unif convergence

$$\int_{\partial D(a,\rho)} f(z)\,dz = \sum_{n=-\infty}^{\infty} \int_{\partial D(a,\rho)} c_n (z-a)^n\,dz = 2\pi i c_{-1}$$

So c_{-1} is special, it is called the residue of f at a,

$$\boxed{\mathrm{Res}_f(a) = c_{-1}}$$

The negative part $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the principal part of f at a.

Thm 4.5 (Residue Theorem) Let $\mathcal{U}$ be a domain, $S = \{a_1, \dots, a_k\} \subseteq \mathcal{U}$, $f: \mathcal{U}\setminus S \to \mathbb{C}$ holomorphic. (So f has isolated singularities at the a_i, only at them). Then if γ is a closed curve in $\mathcal{U}$ homologous to zero in $\mathcal{U}$, and s.t. $S \cap \text{image}(\gamma) = \emptyset$, we have that

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{k} I(\gamma; a_j)\,\mathrm{Res}_f(a_j) \times (2\pi i)$$

Proof Let $g_i(z) = \sum_{n=-\infty}^{-1} c_n^{(i)} (z-a_i)^n$ be the principal part of f at a_i. Note that g_i is holomorphic on $\mathbb{C}\setminus\{a_i\}$.

In particular we have $\sum g_i$ holomorphic on $\mathbb{C}\setminus S$.

So $f - \sum g_i$ is holomorphic on $\mathcal{U}\setminus S$, with removable singularities ~~of~~ at the a_i.

So by Cauchy's theorem, since γ is homologous to zero in $\mathcal{U}$,

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{k} \int_\gamma g_j(z)\,dz,$$

$$= \sum_{j=1}^{k} c_{-1}^{(j)}\, 2\pi i\, I(\gamma; a_j)$$

where in the last step we have used uniform convergence of the series for g_j and CIF for $\int \frac{c_{-1}^{(j)}}{z-a_j}\,dz$. $\square$

Rmk This generalises both the general Cauchy's theorem and CIF.

Useful facts for computing integrals.

① If f has a simple pole at $z=a$, then $\mathrm{Res}_f(a) = \lim_{z\to a}(z-a)f(z)$

② If f has a pole of order k at a, so $f(z) = \frac{g(z)}{(z-a)^k}$ near a, where g is holo, $g(a) \neq 0$, then $\mathrm{Res}_f(a) = \frac{g^{(k-1)}(a)}{(k-1)!}$.

③ If $f = g/h$ with g, h holomorphic at a, $g(a) \neq 0$, h has a simple zero at a, then $\mathrm{Res}_f(a) = g(a)/h'(a)$.

Indeed, $\mathrm{Res}_f(a) = \lim_{z\to a} \frac{z-a}{h(z)} g(z) = \frac{g(a)}{h'(a)}$ using ①.

More straight up fax.

① Jordan's Lemma: If f is holomorphic in $\{|z|>r\}$ for some r, and if $zf(z)$ is bounded for large $|z|$, then

$$\int_{\gamma_R} f(z)e^{i\alpha z}\,dz \to 0 \quad \text{for any } \alpha>0, \text{ as } R\to\infty$$

where $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$.

Proof Ex sheet 3. Need $\frac{\sin t}{t} \geq \frac{2}{\pi}$ for $t \in (0, \frac{\pi}{2}]$.

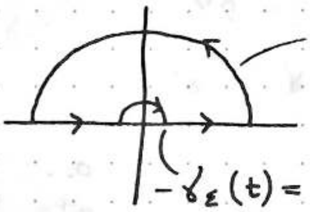
② Let f be holomorphic in $D(a,R)\setminus\{a\}$ with a simple pole at $z=a$. If $\gamma_\varepsilon : [\alpha,\beta] \to \mathbb{C}$ is $\gamma_\varepsilon(t) = a+\varepsilon e^{it}$, then

$$\lim_{\varepsilon\to 0^+} \int_{\gamma_\varepsilon} f(z)\,dz = (\beta-\alpha)\, i\, \mathrm{Res}_f(a)$$

Proof: Use $f(z) = \frac{c_{-1}}{z-a} + g(z)$ near a. $\square$

L16.1

Example Consider $f(z) = \frac{e^{it}}{t}$



$\int_{\gamma_R} f(z)\,dz \to 0$ by Jordan

$\int_{-\gamma_\varepsilon} f(z)\,dz \to \overset{-}{\pi} i \operatorname{Res}_f(0) = \overset{-}{\pi} i$

f is holomorphic in a star domain $\mathcal{U}$, in which γ is homologous to zero.

So $0 = \int_\gamma f(z)\,dz = \left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R\right) e^{ix}/x\,dx + \left(\int_{-\gamma_\varepsilon} + \int_{\gamma_R}\right) f(z)\,dz$

Hence as $\varepsilon \to 0, R \to \infty$, get

$$2i \int_0^\infty \sin x / x\,dx = \pi i \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

§5 Argument principle, local degree and Rouché's theorem

We'll work in domains "bounded by close curves".

Def^n Let D be a domain. We say a closed curve γ bounds D if $I(\gamma; w) = 1 \ \forall w \in D$, and $I(\gamma; w) = 0 \ \forall w \notin D \cup \text{image}(\gamma)$.

$\gamma = a + Re^{it}, 0 \leq t \leq 2\pi$ bounds the dis $D(a, R)$

The orientation of γ matters, if $I(\gamma; w) = -1 \ \forall w \in D$, and $I(\gamma; w) = 0 \ \forall w \notin D \cup \text{image}(\gamma)$, then $(-\gamma)$ bounds D.

note D is bdd, as imγ is bdd

Thm 5.1 (Argument principle)

Let γ be a closed curve bounding a domain D.

Suppose f is meromorphic in an open set $\mathcal{U}$ containing $\overline{D} \cup \text{img}\,\gamma$.

Suppose that imgγ contains no zeros or poles of f, and that f has precisely N zeros and P poles in D, counted with multiplicity.

Then $N - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz = I(\Gamma, 0)$ where $\Gamma = f \circ \gamma$.

Proof $f \not\equiv 0$ in D since γ contains no zeros of f. ← hmm...

So N and P are bounded (since D is finite and zeros and poles are isolated), I guess.

Note that $0 \notin \text{Img}(\Gamma)$ since f has no zeros on img(γ).

Then Γ is a closed curve, and $I(\Gamma; 0)$ is well-defined.

We have $I(\Gamma; 0) = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,dt$

$$= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

If $z=a$ is not a pole or a zero of f, then $\frac{f'}{f}$ is holo. at a.

<u>Claim</u> If f has a zero (pole) of order k at $z=a$, then $\frac{f'}{f}$ has a simple pole at $z=a$, with $\operatorname{Res}_{\frac{f'}{f}}(a) = k$ ($-k$ for pole).

<u>Check</u> If a is a zero, locally $\exists$ $f(z) = (z-a)^k g(z)$ for $g(a) \neq 0$

$\therefore f'(z)/f(z) = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$ and the claim follows.

The same idea works at a pole.

Now apply the residue theorem to $\frac{f'}{f}$, noting that $I(\gamma; w) = 1$ for all $w \in D$, and that γ is homologous to zero in the open set U. (Since $I(\gamma; w) = 0$ for $w \notin U$) $\square$

This has an important application as follows:

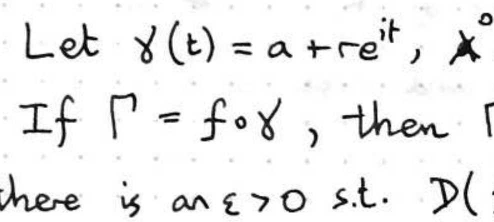
<u>Def^n</u> Let $f: D(a,R) \to \mathbb{C}$ be non-constant holomorphic. Then the <u>local degree of f at a</u>, denoted $\deg_f(a)$ is the order of the zero of $f(z) - f(a)$. This is a positive integer.

<u>Thm 5.2</u> (Local degree theorem) Let $f: D(a,R) \to \mathbb{C}$ be holomorphic, non-constant. Suppose $\deg_f(a) = d > 0$. Then for every sufficiently small $r > 0$, there is an $\varepsilon > 0$ s.t. for any w s.t. $0 < |w - f(a)| < \varepsilon$, the equation $f(z) = w$ has precisely d roots in $D(a,r) \setminus \{a\}$, and they are distinct.

<u>Proof</u> Since f is non-constant, by the principle of isolated zeroes choose $r > 0$ s.t. $f(z) - f(a) \neq 0$ and $f'(z) \neq 0$ $\forall z \in \overline{D(a,r)} \setminus \{a\}$.

Let $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$. Then $f(\gamma(t)) \neq f(a)$ $\forall t \in [0, 2\pi]$.

If $\Gamma = f \circ \gamma$, then Γ misses $f(a)$. Since $\mathbb{C} \setminus \text{image}(\Gamma)$ is open, there is an $\varepsilon > 0$ s.t. $D(f(a), \varepsilon) \subset \mathbb{C} \setminus \text{image}(\Gamma)$.



Then for $w \in D(f(a), \varepsilon)$, by the argument principle, # zeros (w/ multiplicity)

in $D(a,r)$ is $\cancel{N}(\Gamma; w) \underset{\uparrow}{=} I(\Gamma; f(a)) = d$.
I

of $f(z)-w$

I locally constant

If $w \in D(f(a), \varepsilon) \setminus \{f(a)\}$, then these zeros are all in $D(a,r)\setminus\{a\}$ and are distinct because $f'(z) \neq 0$ in $D(a,r)\setminus\{a\}$. $\square$

<u>Corollary 5.3</u> Any holomorphic, non-constant function on a domain is an open map, i.e. maps open sets to open sets.

<u>Proof</u> If $V \subseteq D$ is open, and $W = f(V)$, then for any $b \in W$, find $a \in V$ s.t. $f(a) = b$, and $r > 0$ s.t. $D(a,r) \subseteq V$.

By the previous result, for small enough r we can find $\varepsilon > 0$ s.t. $D(b, \varepsilon) \subseteq W$, as desired. $\square$

Lastly,

<u>Thm 5.4 (Rouché's Thm)</u> Let γ be a closed curve bounding a domain D. Let f, g be holomorphic on an open set $U \supseteq \overline{D} \cup \operatorname{img}\gamma$.

If $|f| > |g|$ on $\operatorname{Img}(\gamma)$, then f and $f+g$ have the same number of zeros in D.

<u>Proof</u> $|f| > |g| \Rightarrow f$ and $f+g$ non-zero on γ

Let $h = \frac{f+g}{f} = 1 + \frac{g}{f}$. Then $|h \circ \gamma - 1| < 1$.

So $\Gamma = h \circ \gamma$ maps into $D(1,1)$ and $I(\cancel{\gamma}\,\Gamma, 0) = 0$.

By the argument principle,

$$\begin{array}{ccc} \#\text{zeros of } f+g & = & \#\text{zeros of } f \\ \| & & \| \\ \#\text{zeros of } h & & \#\text{poles of } h \end{array}$$

after cancelling any pesky common zeros. $\square$

~ The End ~