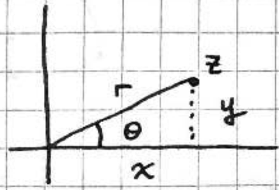
Complex Methods

by Dr Robert Hunt

Chapter 1 (Analytic Functions)

1.1 The Complex Plane and the Riemann Sphere



Any $z \in \mathbb{C}$ can be written in the form

$x + iy$ (where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $x, y \in \mathbb{R}$)

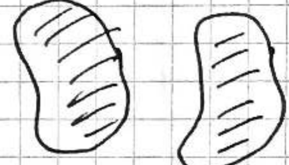
or $re^{i\theta}$ where the <u>modulus</u> is $|z| = r = \sqrt{x^2+y^2}$

and the <u>argument</u> $\arg z = \theta$ satisfies $y = x \tan\theta$. The argument is only defined only up to multiples of 2π; the <u>principal value of the argument</u> is the value of θ in the range $(-\pi, \pi]$. Note that the formula $\tan^{-1}(y/x)$ gives the correct value for the principal value of θ only if $x > 0$; if $x \leq 0$ then it might be out by $\pm\pi$ (consider $z = +1+i$ and $-1-i$).

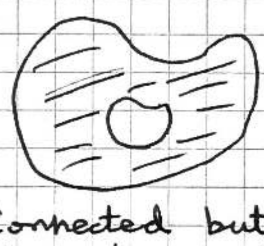
A~~n~~ <u>open set</u> D is a subset of $\mathbb{C}$ which does not include its boundary. (More technically, $D \subseteq \mathbb{C}$ is open if $\forall\, z_0 \in D$, $\exists\, \delta > 0$ s.t. the disc $\{|z - z_0| < \delta\}$ is contained in D)

A <u>neighbourhood</u> of a point z is an open set containing z.

A <u>domain</u> is an open set that is connected (i.e. cannot be split into two disjoint open sets). A <u>simply-connected domain</u> in $\mathbb{C}$ is one with no holes (i.e. any curve lying in the domain can be shrunk continuously to a point without ever leaving the domain)



Not connected



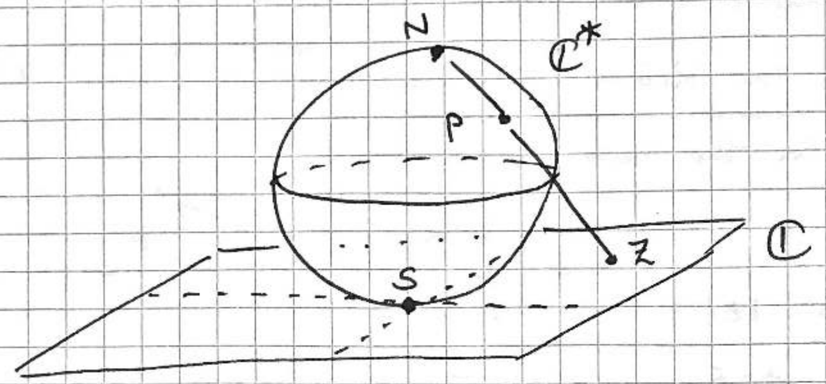
Connected but not simply-connected

Simply-connected domain

The <u>extended complex plane</u> is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

We can reach the "point at ∞" by going off in any direction in the plane, and all are equivalent. Conceptually, we may use the <u>Riemann sphere</u>, which is a sphere resting on the complex plane with its

"South Pole" S at $z = 0$.



For any point z in $\mathbb{C}$, drawing a line through the "North Pole" N of the sphere to z, and noting where this line intersects the sphere, specifies an equivalent point P on the sphere. Then ∞ is equivalent to the "North Pole" itself.

To investigate properties of ∞ we use the substitution $\zeta = \frac{1}{z}$. A function $f(z)$ is said to have a particular property <u>at ∞</u> if $f(1/\zeta)$ has that same property at $\zeta = 0$.

1.2 Complex Differentiation

Recall the definition of differentiation for a real function $f(x)$:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

It is implicit that the limit must be the same whichever dir^n we approach x from. Consider $|x|$ at $x = 0$ for example; if we approach from the right ($\delta x \to 0^+$), the limit is $+1$, whereas from the left it is -1.

Because these limits are different we say that $|x|$ is not diff'ble at $x = 0$.

Now extend the definition to complex functions $f(z)$: f is <u>differentiable</u> at z if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists (and is therefore independent of the direction of approach — but now there eis an infinity of possible directions.

We say that f is <u>analytic</u> at a point z if there exists a neighbourhood of z throughout which f' exist. The terms <u>regular</u> and <u>holomorphic</u> are also used.

A function which is analytic throughout $\mathbb{C}$ is called <u>entire</u>.

A <u>singularity</u> of f is a point at which it is not analytic, or not even defined.

The property of analyticity is in fact a surprisingly strong one! For example, two consequences include (see §4.3):

(i) If a function is analytic then it is infinitely diff'ble (cf the existence of real functions which can be differentiated N times but no more, for any given N).

(ii) A bounded entire function is constant. (cf tanh on $\mathbb{R}$)

<u>The Cauchy-Riemann Equations</u>

Separate f and z into real and imaginary parts:

$$f(z) = u(x,y) + i\,v(x,y)$$

where $z = x+iy$ and u, v are real functions. Suppose that f is diff'ble at z. We may take δz in any direction; first take it to be real, $\delta z = \delta x$. Then

$$f'(z) = \lim_{\delta x \to 0} \frac{f(z+\delta x) - f(z)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now take δz to be pure imaginary, $\delta z = i\delta y$. Then

$$f'(z) = \frac{f(z+i\delta y) - f(z)}{i\delta y} \quad \lim_{\delta y \to 0}(\cdot)$$

$$= \lim_{\delta y \to 0} \frac{u(x, y+\delta y) + iv(x, y+\delta y) - u(x,y) - iv(x,y)}{i\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} .$$

Equate real and imaginary parts to deduce

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

– the Cauchy-Riemann equations.

L2.1

The converse of C-R doesn't really hold. One needs further conditions

A For example, that u_x, u_y, v_x, v_y are continuous.

<u>Examples</u> (i) $f(z) = z$ is entire. Here $f'(z) = 1$.

Here $u = x$, $v = y$ and the C-R equations are satisfied.

(ii) $f(z) = e^z = e^x(\cos y + i \sin y)$ is entire since

$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x$$

The derivative is

$$f'(z) = u_x + i v_x = e^x(\cos y + i \sin y) = e^z,$$

B as expected.

(iii) If $f(z)$ is analytic and $f(z_0) \neq 0$, then $\frac{1}{f(z)}$ is analytic at z_0.

So $\frac{1}{z}$ is analytic except at the origin.

(iv) Any rational function — i.e. $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials — is analytic except at points where $Q(z) = 0$.

For instance, $\frac{z}{z^2+1}$ is analytic except at $\pm i$.

(v) Many standard real functions can be extended naturally to complex functions, and obey the usual rules for derivatives. For example:

- z^n has derivative $n z^{n-1}$
- $\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i}$ has derivative $\cos z \equiv \frac{e^{iz} + e^{-iz}}{2}$
- Similarly for $\cos z$, $\sinh z$, $\cosh z$, etc.
- $\log z \equiv \operatorname{Log}|z| + i \arg z$ has derivative $\frac{1}{z}$

Note: it can sometimes be useful to write

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y,$$

etc.

(vi) The product and composition of analytic functions is also analytic. The product, quotient and chain rule can be used just as for real functions.

Examples (i) $f(z) = \text{Re}\, z$ has $u = x$, $v = 0$;

but $u_x = 1 \neq 0 = v_y$ so $\text{Re}\, z$ is nowhere diff'ble

(ii) $f(z) = |z|$ has $u = \sqrt{x^2+y^2}$, $v = 0$, is nowhere analytic

(iii) $f(z) = \bar{z} = x - iy$ (complex conjugate, also denoted z^*)

has $u = x$, $v = -y$ and is nowhere analytic

(iv) For $f(z) = |z|^2 = x^2 + y^2$ the C-R equations

$(2x = 0, 2y = 0)$ are satisfied at the origin $x = y = 0$.

So f is diff at the origin, but nowhere else.

So it is nowhere analytic, not even at 0.

* Analytic Continuation   \# SOY

If we are given the values of an analytic function in some restricted region — which could be rather small, e.g. a curve in the complex plane — then there is a unique extension to the rest of $\mathbb{C}$ that is still analytic. (No proof) This extension might have some singularities, and it might be multivalued.

This fact can be useful in extending the domain of definition of a function. We shall see an example in §5.3. *

1.3 Harmonic functions

If $f(z) = u + iv$ is analytic, then by C-R

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = -(u_y)_y$$

$$\therefore\ u_{xx} + u_{yy} = 0$$

So u satisfies Laplace's equation in 2 dimensions.

Similarly for v. A function satisfying Laplace's equation in some 2D domain is said to be harmonic there.

Functions u, v satisfying the C-R equations are called harmonic conjugates. If we know one then we can find the other, up to a constant.

L2.3

For example, consider $u(x,y) = x^2-y^2$, which is harmonic.

A Its harmonic conjugate satisfies

$$v_y = u_x = 2x \Rightarrow v = 2xy + g(x)$$

for some function $g(x)$. So

$$-2y = u_y = -v_x = -2y - g'(x)$$

$$\Rightarrow g(x) = \text{const.} = \alpha \text{ say}$$

The corresponding analytic function whose real part is u is therefore

$$f(z) = x^2-y^2 + 2ixy + i\alpha = z^2 + i\alpha$$

If the domain in which the functions are harmonic is not simply B connected, then this method might give a solution that is multivalued. For example, if $u = \frac{1}{2}\log(x^2+y^2)$, which is harmonic in the domain $\mathbb{C}\setminus\{0\}$. The corresponding $f(z)$ is $\log z$, which is multivalued (see §1.4).

Contours of harmonic conjugate functions are $\perp$ to each other.

<u>Proof</u>: $\nabla u = (u_x, u_y)^T$ is perpendicular to contours of u, by IA Vector Calculus? Similarly for ∇v and contours of v.

But
$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$= \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= 0. \qquad \square$$

§1.4 Multivalued functions

For $z = re^{i\theta}$, define $\log z = \log r + i\theta$.

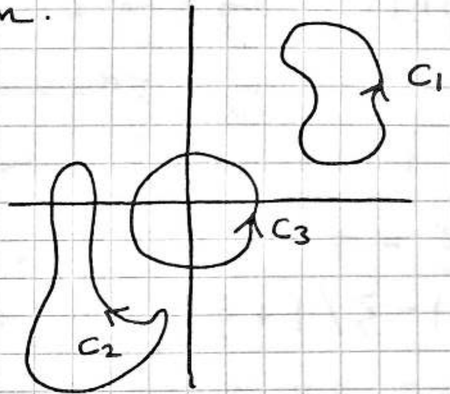
There are thus an infinity of values of $\log z$, for θ may take an infinity of values. For example

$$\log i = \frac{\pi i}{2} \text{ or } \frac{5\pi i}{2} \text{ or } -\frac{3\pi i}{2} \text{ or} \cdots,$$

depending on which choice of θ we make.

Branch points

Consider the three curves shown in the diagram.



On C_1 we could choose θ to be always in the range $(0, \frac{\pi}{2})$, and then $\log z$ would be continuous and single valued ("CSV") going round C_1. On C_2, we could choose $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\log z$ would again be CSV. But for C_3, which encircles the origin, there is no such choice; whatever we do, $\log z$ cannot be made CSV around C_3 (it must either "jump" somewhere or be multivalued).

A <u>branch point</u> of a multi-valued function – here, the origin – is a point which is impossible to encircle with an arbitrarily small curve upon which the function is both continuous and single-valued. The function is said to have a <u>branch point singularity</u> at that point.

Examples: (i) $\log(z-a)$ has a branch point at $z=a$

(ii) $\log\left(\frac{z-1}{z+1}\right) = \log(z-1) - \log(z+1)$ has two branch points, at ± 1.

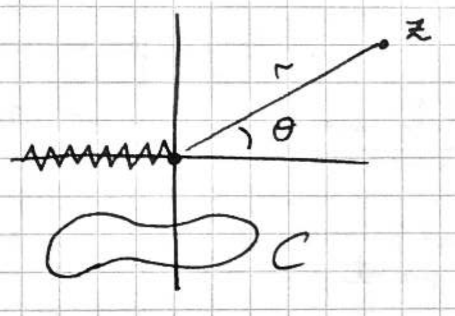
(iii) $z^\alpha = r^\alpha e^{i\alpha\theta}$ has a branch point at the origin for $\alpha \notin \mathbb{Z}$. Consider a circle of radius r_0 centred at O, and suppose WLOG that we start at $\theta = 0$ and go once round anticlockwise. θ must vary continuously to ensure continuity of $e^{i\alpha\theta}$ (!), so as we get back we almost reach 2π. But then there will be a jump in θ back to 0 (to satisfy the single valued requirement) and hence a jump in z^α from $r_0^\alpha e^{2\pi i\alpha}$ to r_0^α. We cannot, therefore, make z^α both continuous <u>and</u> single-valued on the circle. (Note that if $\alpha \in \mathbb{Z}$ then there is <u>no</u> jump, since $e^{2\pi i\alpha} = 1$ for $\alpha \in \mathbb{Z}$)

(iv) $\log z$ also has a branch point at ∞, because if $\zeta = \frac{1}{z}$ (see §1.1), $\log z = -\log \zeta$ which has a branch point at $\zeta = 0$.
Similarly, z^α has a branch point at ∞ when $\alpha \notin \mathbb{Z}$.

(v) $\log\left(\frac{z-1}{z+1}\right)$ does not have a branch point at ∞, because if $\zeta = \frac{1}{z}$, then $\log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{1-\zeta}{1+\zeta}\right)$. For ζ close to zero, $1-\zeta/1+\zeta$ remains close to 1, and therefore well away from the branch point of log at the origin. So we can encircle $\zeta = 0$ without $\log\left(\frac{1-\zeta}{1+\zeta}\right)$ being discontinuous.

Branch cuts

If we wish to ensure that $\log z$ is CSV on any curve, we must stop curves from encircling the origin. We do this by introducing a branch cut from $-\infty$ on the real axis to the origin. No curve is allowed to cross this cut.



We can then decide for values of θ lying in the range $(-\pi, \pi]$ only, and we have defined a branch of $\log z$ which is CSV on any curve C that doesn't cross the cut.

This branch is analytic everywhere expect on the -ve real axis, where it is not even continuous, and the branch points themselves.

The cut described above is the canonical (i.e. standard) branch cut for $\log z$, and the branch of $\log z$ is called the principal value of the logarithm. It is usual to include $+\pi$ in the range of θ and not $-\pi$; but in fact this is of no consequence since we shall never evaluate the function precisely on the branch cut.

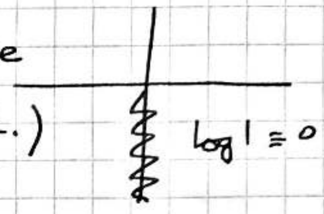
What are the values of $\log z$ just above and below the branch cut? Consider a point on the negative axis $\{z : x < 0\}$. Just above the cut, at $z = x + i0^+$, $\theta = \pi$, so $\log z = \log|x| + i\pi$. Just below it, at

L3.3

$z = x + i0^-$, $\log z = \log|x| - i\pi$.

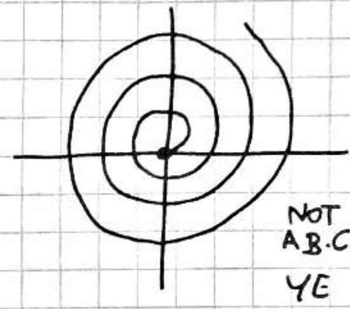
This is not the only possible branch of $\log z$:

(a) We could place the cut along the -ve Im axis and choose $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. (Note θ is still measured from the horizontal even though the branch cut is now vertical.)



(b) With a branch cut along the negative real axis we could choose $\theta \in (\pi, 3\pi)$.

(c) With the branch cut illustrated it is more difficult to write down the exact choice of θ, but this is equally valid. Any branch cut that stops curves wrapping round the branch point will do.



NOT A.B.C.? YE

Exactly the same considerations (and possible branch cuts) apply for $z^\alpha = r^\alpha e^{i\alpha\theta}$, $\alpha \notin \mathbb{Z}$. One way of seeing this is to note that $z^\alpha = e^{\alpha \log z}$ (ish)

Whenever a problem requires the use of a branch, it is important to specify it clearly. This can be done in two ways:

- Define the function and parameter explicitly, e.g.

$$\log z = \log|z| + i\arg z, \quad \arg z \in (-\pi, \pi]$$

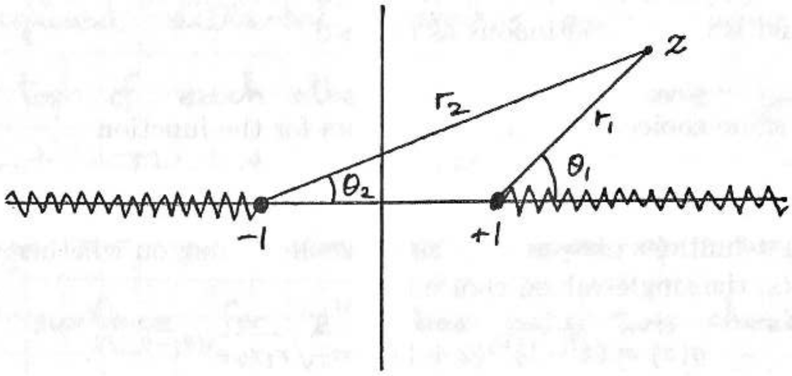
- Specify the location of the branch cut and give the value of the required branch at a single point not on the cut. The values everywhere else (except at the cut) are then defined uniquely by continuity. For example, $\log z$ with a branch cut along $\mathbb{R}^-$ and $\log 1 = 0$.

Worked Example
Branch Cuts for Multiple Branch Points

What branch cuts would we require for the function

$$f(z) = \log \frac{z-1}{z+1}\ ?$$

It is clear that there are branch points at ± 1, but we have a non-trivial choice of branch cuts. Define $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, as shown in the following diagram.

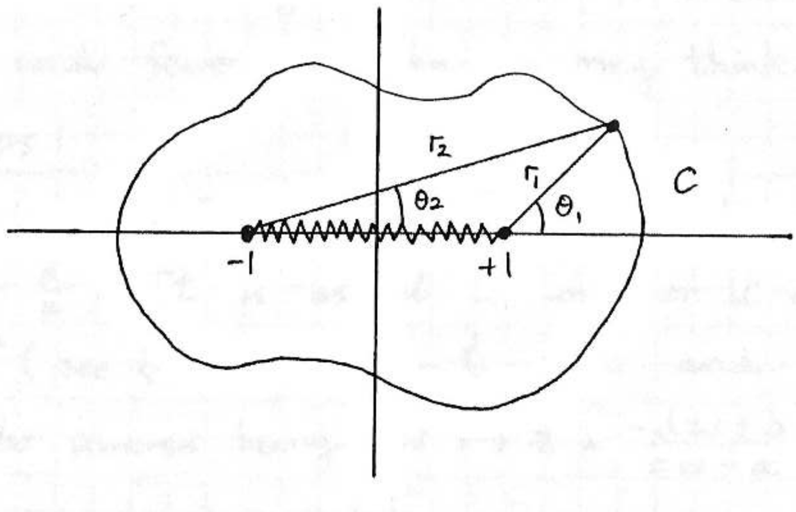


The most straightforward choice is to take two branch cuts, one emanating from each branch point to infinity. In the case shown, we choose $0 \leqslant \theta_1 < 2\pi$ and $-\pi < \theta_2 \leqslant \pi$, and the consequent single-valued definition of $f(z)$ is

$$\begin{aligned} f(z) &= \log(z-1) - \log(z+1) \\ &= (\log r_1 + i\theta_1) - (\log r_2 + i\theta_2) \\ &= \log(r_1/r_2) + i(\theta_1 - \theta_2). \end{aligned}$$

The two cuts make it impossible for z to "wind around" either of the two branch points, so we have obtained a single-valued function that is analytic except along the branch cuts.

Note that this choice of branch cuts also stops z "winding around ∞" (e.g., moving around a circle of arbitrarily large radius). This restriction is not in fact necessary, because ∞ is not a branch point of $f(z)$. So an alternative choice is to take only *one* branch cut, between -1 and 1, as shown below. This time, we choose both $0 \leqslant \theta_1 < 2\pi$ and $0 \leqslant \theta_2 < 2\pi$. (This definition for θ_1 initially seems at odds with the location of the branch cut, but the reason will become clear.) The definition of $f(z)$ is as before, but with these different ranges for θ_1 and θ_2.



If z were to wind around just *one* branch point then it would cross the branch cut, from above to below say. While θ_1 would be unchanged (at π), θ_2 would "jump" from 0 to 2π, causing a discontinuity in $f(z)$. This is, of course, not allowed, as we may not cross branch cuts.

But it *is* now possible for z to wind around *both* of the branch points together. Consider a curve C which does so. Starting from the point on C lying on the positive real axis (where $\theta_1 = \theta_2 = 0$) and moving anti-clockwise, both θ_1 and θ_2 increase. When we have made one complete revolution and returned to the positive real axis, having encircled both branch points exactly once, θ_1 and θ_2 both suddenly "jump" from 2π back to 0. But this jump does *not* result in a jump in the value of $\theta_1 - \theta_2$; so $f(z)$ is not affected, and is indeed continuous as claimed.

Exactly the same choice of branch cuts occurs for the function

$$g(z) = (z^2 - 1)^{1/2}.$$

With the same definitions of θ_1 and θ_2, as above depending on whether we have one or two branch cuts, the single-valued choice is

$$g(z) = (z-1)^{1/2}(z+1)^{1/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}.$$

This time the single branch cut works because, when both θ_1 and θ_2 jump by 2π, $\frac{1}{2}(\theta_1 + \theta_2)$ jumps by 2π also; and $e^{2\pi i} = 1$. The cut prevents either θ_1 or θ_2 jumping on its own.

This idea can be extended to higher numbers of branch points in the right circumstances.

L4.1

Note that a branch cut alone does not specify a branch (compare the principal branch with (b) above, which is a different branch even though it uses the same branch cut), nor is a single value of the function sufficient by itself (compare the principal branch with (c) above).

* Riemann Surfaces

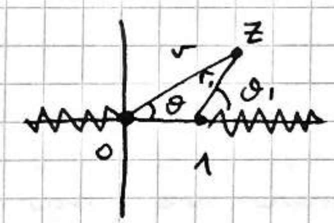
Riemann imagined different branches as separate copies of $\mathbb{C}$, all stacked on top of each other but each one joined to the next at the branch cut. This structure is a Riemann surface.

See Priestly for the Riemann surface of $\log z$.

The Riemann surface for $z^{1/2}$ has only two sheets, because for a given branch cut there are only two possible branches. *

Multiple Branch Points

When there is more than one branch point we may need more than one branch cut. For $f(z) = \{z(z-1)\}^{1/3}$ there are branch points at $0, 1$ and ∞, so we need several branch cut; a possibility is shown below. Then no curve can encircle $0, 1$ or ∞.



For any z write $z = re^{i\theta}$ and $z - 1 = r_1 e^{i\theta_1}$ with $\theta \in (-\pi, \pi]$, $\theta_1 \in [0, 2\pi)$, and define

$$f(z) = \sqrt[3]{r r_1}\, e^{i(\theta + \theta_1)/3}$$

This is continuous, so long as we don't cross either cut.

Sometimes we need fewer cuts than we may think.

1.5 Möbius maps

The Möbius map $z \mapsto w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ is analytic except at $z = -\frac{d}{c}$. It is useful to consider it as a map from $\mathbb{C}^*$ to $\mathbb{C}^*$ (see §1.1) with $-\frac{d}{c} \to \infty$ and $\infty \to \frac{a}{c}$. It is then bijective, the inverse being $w \mapsto z = \dfrac{-dw+b}{cw-a}$, another Möbius map.

L4.2

A <u>circline</u> is either a circle or a line. Geometrically it is clear that choosing three distinct points $\alpha, \beta, \gamma \in \mathbb{C}^*$ uniquely specifies a circline through those points. (If one of the points is ∞ then we have the line through the other points)

Usefully, we can find a Möbius map taking any given circline to any other.

Lemma 1: Möbius maps take circlines to circlines

* Proof: any circle can be expressed as a circle of Apollonius,

$|z - z_1| = \lambda |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$, $\lambda \in \mathbb{R}^+$ (recall V+M: the case $\lambda = 1$ corresponds to a line, $\lambda \neq 1$ to a circle). We then have

$$\left| \frac{-dw + b}{cw - a} - z_1 \right| = \lambda \left| \frac{-dw+b}{cw - a} - z_2 \right|$$

$$\Leftrightarrow \quad |(cz_1 + d)w - (az_1 + b)| = \lambda |(cz_2 + d)w - (az_2 + b)|$$

$$\Leftrightarrow \quad |w - w_1| = \lambda \left| \frac{cz_2 + d}{cz_1 + d} \right| |w - w_2|$$

where $w_1 = \dfrac{az_1 + b}{cz_1 + d}$, $w_2 = \dfrac{az_2 + b}{cz_2 + d}$. * □

Lemma 2: Given $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}^*$ we can find a (unique) Möbius map sending $\alpha \mapsto \alpha'$, $\beta \mapsto \beta'$, $\gamma \mapsto \gamma'$.

* Proof: the Möbius map $f_1(z) = \left(\frac{\beta - \gamma}{\beta - \alpha}\right) \frac{z - \alpha}{z - \gamma}$ sends $\alpha \to 0$, $\beta \to 1$, $\gamma \to \infty$. (If one the points is already ∞ then we change the map, e.g. if $\alpha = \infty$ then $f_1(z) = (\beta - \gamma)\frac{1}{z - \gamma}$.) Let f_2 be the map sending $\alpha' \to 0$, $\beta' \to 1$, $\gamma' \to \infty$.

Then $f_2^{-1} \circ f_1$ is the required map. It's a Möbius map (!). * □

Worked Example
Conformal Mapping from a Half-Disc to a Disc

How could we map the half-disc $|z| < 1$, $\operatorname{Im} z > 0$ conformally to the full disc $|z| < 1$? It is tempting simply to apply $z \mapsto z^2$, but this is incorrect because the image would *exclude* the non-negative real axis $z = x \in [0, 1)$. Instead we build up a map in stages.

It is necessary to have a plan of attack. We shall first find a map f_1 from the half-disc to a quadrant; then a map f_2 from the quadrant to a half-plane; then a map f_3 (which we already know) from the half-plane to the disc. So we use the following three maps:

1. Since the required map involves quadrants of the unit circle and the complex plane, we shall try using $f_1(z) = (z-1)/(z+1)$. We can find the image of the half-disc by considering the images of specific points on the circlines making up its boundary. We see that the interval $[-1, 1]$ on the real axis is mapped to the negative real axis (since $1 \mapsto 0$, $0 \mapsto -1$ and $-1 \mapsto \infty$). The semicircle is mapped to the positive imaginary axis (since $1 \mapsto 0$, $i \mapsto i$ and $-1 \mapsto \infty$). Hence the half-disc maps *either* to the second quadrant of the plane, *or* to quadrants one, three and four. To determine which, we consider an interior point: $\frac{1}{2}i \mapsto \frac{1}{5}(4i - 3)$, which is in the second quadrant.

   (Alternatively, we could note that angles are preserved at $z = 1$; since the interior of the region subtends an angle $\pi/2$ there, we must obtain just a single quadrant.)

   Therefore, f_1 takes the half-disc to the second quadrant of the complex plane.

2. We would now like the left-hand half-plane, because we already know how to map that to the whole disc; so we apply $f_2(z) = -iz^2$ to take the second quadrant to the left-hand half-plane.

3. Finally, we apply $f_3(z) = (1+z)/(1-z)$ to obtain the disc.

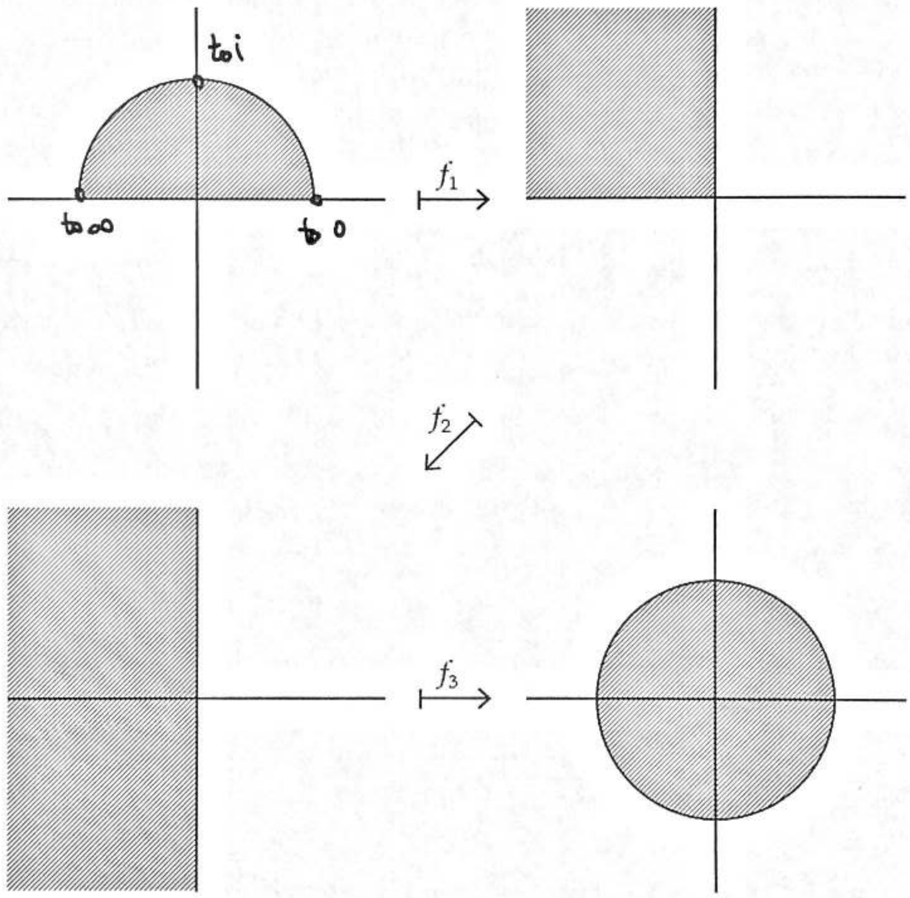
   Our composite map taking the half-disc directly to the full disc is therefore $f = f_3 \circ f_2 \circ f_1$. A diagram showing the sequence of mappings is shown overleaf.

We may leave our answer in this form, although if required we can also simplify the expression:

$$f(z) = \frac{1 - i\left(\frac{z-1}{z+1}\right)^2}{1 + i\left(\frac{z-1}{z+1}\right)^2} = \frac{(1-i)z^2 + 2(1+i)z + 1 - i}{(1+i)z^2 + 2(1-i)z + 1 + i} = -i\,\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}.$$

The factor of $-i$ at the front of this expression is in fact redundant since it simply rotates the whole disc, so a suitable map is

$$z \mapsto \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}.$$



to i
to ∞
to 0
f_1
f_2
f_3

1.6 Conformal maps

A conformal map $f: \mathcal{U} \to V$, where $\mathcal{U}, V$ are open subsets of $\mathbb{C}$, i.e. one which is analytic with non-zero derivative in $\mathcal{U}$.

Although not part of the definition, it is usual (and helpful) to require that f be a bijection between $\mathcal{U}$ and V.

An alternative definition is that a conformal map is one that preserves the angle (in both magnitude and orientation) between intersecting curves. We shall show that our definition implies this; the converse is also true (proof omitted), so the two definitions are equivalent.

Suppose that $z_1(t)$ is a curve in $\mathbb{C}$, parametrised by $t \in \mathbb{R}$, which passes through a point z_0 when $t = t_1$. Suppose further that its tangent there, $z_1'(t_1)$ has a well-defined direction; then $z_1'(t_1) \neq 0$ and the curve makes an angle $\phi = \arg z_1'(t_1)$ to the Re-axis at z_0. Consider the image of the curve,

$$Z_1(t) = f(z_1(t)).$$

Its tangent direction at $t = t_1$ is

$$Z_1'(t_1) = z_1'(t_1) f'(z_1(t_1)) = z_1'(t_1) f'(z_0)$$

and therefore makes an angle with the Re-axis of

$$\arg(z_1'(t_1) f'(z_1(t_1))) = \phi + \arg f'(z_0)$$

(noting that $\arg f'(z_0)$ exists, since f is conformal so $f'(z_0) \neq 0$).

In other words, the tangent direction is rotated by $\arg f'(z_0)$.

Now if $z_2(t)$ is another curve passing through z_0, then its tangent direction will also be rotated by $\arg f'(z_0)$.

The result follows.

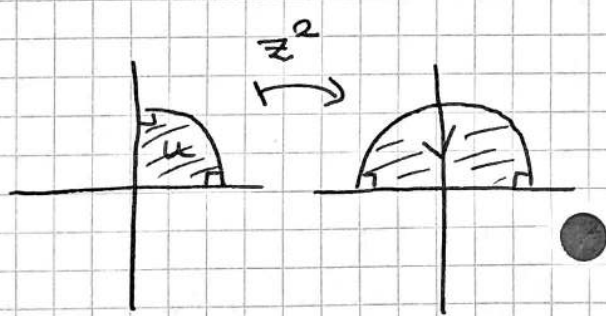
Sometimes we do not know what V, the image set of f on $\mathcal{U}$, is in advance. Often, the easiest way to find it is first to find the image of the boundary $\partial\mathcal{U}$, which will form the boundary ∂V of V;

but, since this does not reveal upon which side of ∂V V lies, to then find the image of a single point of our choice within $\mathcal{U}$, which will be within V.

Examples of conformal maps

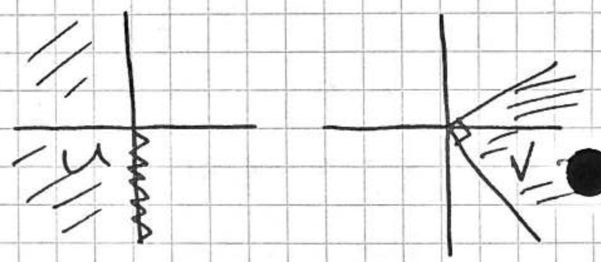
(i) The map $z \mapsto az + b$, $a, b \in \mathbb{C}$, $a \neq 0$, rotates by $\arg a$, enlarges by $|a|$ and translates by b, and is conformal everywhee.

(ii) $f(z) = z^2$ is a conformal map from $\mathcal{U} = \{ z : 0 < |z| < 1, 0 < \arg z < \frac{\pi}{2} \}$ to $V = \{ z : 0 < |z| < 1, 0 < \arg z < \pi \}$



Note that the right angle between the two boundary curves at $z = 1$ is preserved, because f is conformal there; similarly at $z = i$. But the right angle at $z = 0$ is not presered, because f is not conformal there ($f'(0) = 0$). Fortunately, this doesn't matter since $\mathcal{U}$ is an open set, does not include 0.

(iii) How could we conformally map the left half plane $\mathcal{U} = \{ z : \operatorname{Re} z < 0 \}$ to the wedge $V = \{ z : |\arg z| < \frac{\pi}{4} \}$.



We could try to halve the angle, so try using $z^{1/2}$, for which we need to choose a branch. The branch cut must not lie in $\mathcal{U}$ (since $z^{1/2}$ is not analytic on the branch cut) so choose for example a cut along the negative imaginary axis:

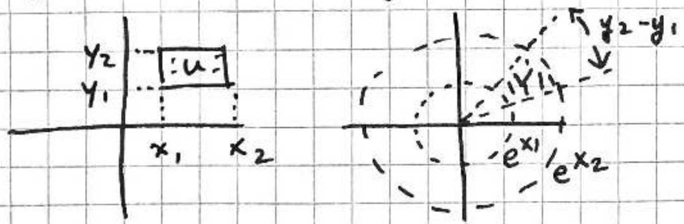
$$re^{i\theta} \mapsto \sqrt{r}\, e^{i\theta/2} \text{ where } \theta \in \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right].$$

We now apply $z^{1/2}$ to $\mathcal{U}$ to produce $\{ z' : \frac{\pi}{4} < \arg z' < \frac{3\pi}{4} \}$.

Just rotate, multiply by $-i$.

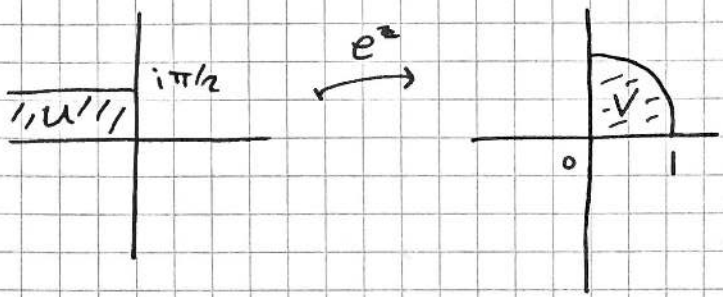
The final map is $f(z) = -i z^{1/2}$

(iv) e^z takes rectangles conformally to sectors of annuli



With an appropriate choice of branch, $\log z$ does the reverse.

Semi-infinite or doubly infinite strips can be treated similarly (by effectively taking $x_1 = -\infty$ and/or $x_2 + \infty$); for example, e^z maps the sime-infinite strip shown below to a quarter disc.



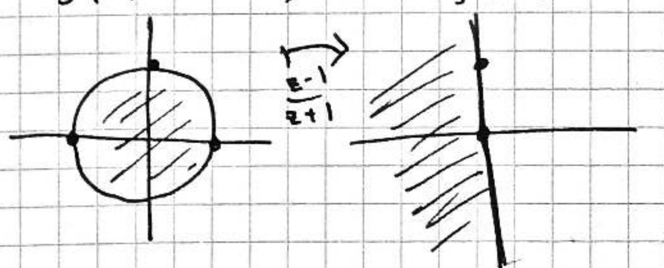
(v) Möbius maps are conformal everywhere, except at $-\frac{d}{c}$.

They are extremely useful in taking circlines to circlines.

Consider $f(z) = \dfrac{z-1}{z+1}$ acting on the unit disc $U = \{z : |z| < 1\}$.

The boundary of U is a circle; the three points $-1, 1, i$ get sent to $\infty, 0, \frac{i-1}{i+1} = i$.

Therefore (see §1.5) the image of ∂U is the imaginary axis. since $f(0) = -1$, we see $f(U)$ is the left half-plane.



The inverse map, which is $z \longmapsto \dfrac{1+z}{1-z}$, maps V to U conformally.

L6.1

$*$ Alternative derivation: $w = \frac{z-1}{z+1} \iff z = \frac{1+w}{1-w}$. So

$|z| < 1 \iff |w+1| < |w-1|$, i.e. w is closer to -1 than it is to $+1$, which describes precisely the left-hand half plane $*$

In fact this particular map can usefully be deployed more generally on quadrants of the unit disc or of the complex plane.

(vi) $f(z) = \frac{1}{z}$ is a simple Möbius map useful for acting on vertical or horizontal lines, which map to circles passing through the origin with centres on one of the axes, or for mapping sectors within the unit disc to sectors outside it, or vice versa.

In practice, complicated conformal maps are usually assembled as the composition of simpler conformal maps (note the composition of conformal maps is conformal, via chain rule).

1.7 Solving Laplace's eq^n using conformal maps

The following algorithm can be used to solve Laplace's equation

$$\nabla^2 \phi(x,y) = 0$$

on a tricky domain $\mathcal{U} \subseteq \mathbb{R}^2$ with given Dirichlet boundary conditions on $\partial\mathcal{U}$. We identify subsets of $\mathbb{R}^2$ with subsets of $\mathbb{C}$ in the obvious manner.

1. Find a conformal map $f: \mathcal{U} \to V$, where $\mathcal{U}$ is now considered a subset of $\mathbb{C}$ and V is a "nice" domain of our choice. Our aim is to find a harmonic function Φ in V that satisfies the same boundary conditions as ϕ.
2. Map the boundary conditions on $\partial\mathcal{U}$ directly to the equivalent points ~~a~~ on ∂V.
3. Now solve $\nabla^2 \Phi = 0$ in V with the new boundary conditions.
4. The required harmonic function ϕ in $\mathcal{U}$ is then given by

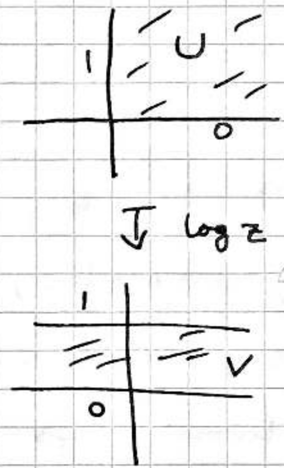
$$\phi(x,y) = \Phi(\operatorname{Re} f(x+iy), \operatorname{Im} f(x+iy)).$$

L6.2

This works because, since Φ is harmonic, it is the real part of some complex analytic function $F(z) = \Phi(x,y) + i\Psi(x,y)$ where $z = x+iy$ (see §1.3). Now $F(f(z))$ is analytic as it is a composition of analytic functions; so its real part, $\Phi(\operatorname{Re} f, \operatorname{Im} f)$, is harmonic.

<u>Example</u> $\nabla^2\phi = 0$ in the first quadrant of $\mathbb{R}^2$ subject to $\phi(x,0)=0$, $\phi(0,y)=1$, and with ϕ bounded near the origin and at ∞.

We choose $f(z) = \log z$ (principal branch) which maps U to the strip $0 < \operatorname{Im} z < \frac{\pi}{2}$. It takes the +ve real axis to $\operatorname{Im} z = 0$ and the +ve imaginary axis to $\operatorname{Im} z = \frac{\pi}{2}$.



Therefore, we need to solve

$\nabla^2\Phi = 0$ in V subject to $\Phi(x,0) = 1$,

$\Phi(x,\frac{\pi}{2}) = 1 \quad \forall x \in \mathbb{R}$.

$$Dg = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \bar{z}}d\bar{z}$$

Need Φ bdd as $x \to \pm\infty$.

By inspection the solution is $\Phi(x,y) = \frac{2}{\pi}y$.

Hence $\phi = \frac{2}{\pi}\arg z$ is sol^n on U.

* Alternative notations: some prefer to use a variable name other than z for the image of the conformal map. Common choices include

$\zeta = f(z)$ with $\zeta = \xi + i\eta$; or

$w = f(z)$ with $w = u + iv$.

This enables them to draw diagrams of U in the z-plane directly alongside V in the ζ-plane or w-plane.

The transformed version of Laplace's equation is now

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} = 0 \quad \text{or} \quad \frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0$$

and the solution is $\phi(x,y) = \Phi(\xi,\eta)$ or $\Phi(u,v)$. *

L6.3

Chapter 2. Contour integration and Cauchy's Theorem

2.1 Contours and integrals

A curve $\gamma(t)$ is a (continuous) map $\gamma : [0,1] \to \mathbb{C}$.

A closed curve is one where $\gamma(0) = \gamma(1)$.

A simple curve is one which is injective (except at 0, 1 say)

A contour is a piecewise smooth curve.

Given two contours γ_1 and γ_2 with matching endpoints, i.e. $\gamma_1(1) = \gamma_2(0)$, then $\gamma_1 + \gamma_2$ denotes two contours joined end-to-end.

The contour $-\gamma$ is the contour γ traversed in the opposite dir^n.

- The contour integral $\int_\gamma f(z)\,dz$ is defined to be

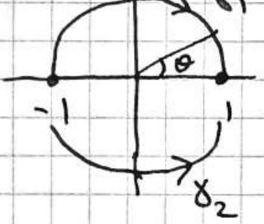
$$\int_0^1 f(\gamma(t))\,\gamma'(t)\,dt.$$

Alternatively, and equivalently, for a simple contour we dissect the contour at $z_0, z_1, \dots, z_N$, where $z_0 = \gamma(0)$, $z_N = \gamma(1)$, and let $\delta z_n = z_{n+1} - z_n$ $(n = 0, \dots, N-1)$. Then

$$\int_\gamma f(z)\,dz = \lim_{\Delta \to 0} \sum_{n=0}^{N-1} f(z_n)\,\delta z_n$$

where $\Delta = \max\limits_{n=0,\dots,N-1} |\delta z_n|$ and, as $\Delta \to 0$, $N \to \infty$.

- The result of a contour integral between two points in $\mathbb{C}$ may depend on the choice of contour. For example, consider



$$I_1 = \int_{\gamma_1} \frac{dz}{z} \quad \text{and} \quad I_2 = \int_{\gamma_2} \frac{dz}{z}$$

where in both cases we integrate from $z = -1$ to $+1$ around a unit semicircle: γ_1 above, γ_2 below the real axis.

Substitute $e^{i\theta}$, $dz = ie^{i\theta}d\theta$:

$$I_1 = \int_\pi^0 \frac{ie^{i\theta}d\theta}{e^{i\theta}} = -i\pi \quad , \text{ but } \quad I_2 = \int_{-\pi}^0 i\,d\theta = +i\pi .$$

Elementary properties of the integral

(i) $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$.

(ii) $\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$.

(iii) If γ is a contour from a to b in $\mathbb{C}$ then

$$\int_\gamma f'(z)\,dz = f(b) - f(a)$$

so long as f is diff'ble at every point on γ (so for example we must not cross a branch cut of f).

(iv) Integration by substitution and by parts work exactly as they do for integrals on the real line.

(v) If γ has length L and $|f(z)| \leqslant M$ on γ then

$$\left| \int_\gamma f(z)\,dz \right| \leqslant LM ,$$

since $\left| \int_\gamma f(z)\,dz \right| \leqslant \int_\gamma |f(z)||dz| \leqslant M\int_\gamma |dz| = LM$.

Integrals on closed contours

The notation $\oint f(z)\,dz$ denotes an integral round a closed contour. It doesn't matter where we start from so long as we go all the way round.

The usual direction is anticlockwise (the "positive sense") ; if we traverse γ in a negative sense (clockwise) then we get negative the previous result.

2.2 Cauchy's Theorem

If $f(z)$ is analytic in a simply-connected domain D, then for every simple closed contour γ in D,

$$\oint_\gamma f(z)\,dz = 0 .$$

* The proof (non-examinable) of this remarkable theorem is

simple and follows from the Cauchy - Riemann equations and Green's theorem. Let u, v be the real and imaginary parts of f. Then

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u+iv)(dx+i\,dy)$$
$$= \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (v\,dx + u\,dy)$$
$$= \iint_S (-v_x - u_y)\,dx\,dy + i\iint_S (u_x - v_y)\,dx\,dy$$

where S is the region enclosed by γ, by applying Green's theorem in the plane, $\oint_{\partial S} (P\,dx + Q\,dy) = \iint_S (Q_x - P_y)\,dx\,dy$.

But $u_x = v_y$, $v_x = -u_y$ so both integrands vanish. *

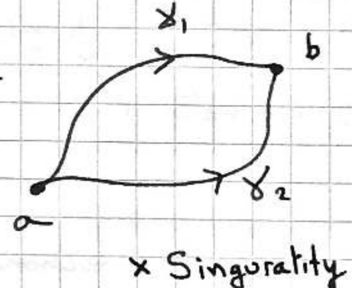
** In fact this proof requires u and v to have continuous partial derivatives in S (for otherwise Green's theorem does not apply).

We shall see later that f is in fact diff'ble infinitely many times so u, v do have continuous partial derivatives ; but our proof of that will utilise Cauchy's theorem! ** GANG

2.3 The Deformation Theorem

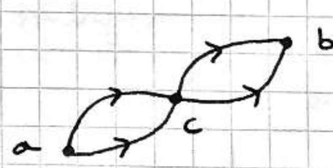
Suppose that γ_1, γ_2 are contours from a to b, and that f is analytic on and between them. Then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$



Proof : suppose γ_1 and γ_2 do not cross. Then $\gamma_1 - \gamma_2$ is a simple closed contour, so $\oint_{\gamma_1 - \gamma_2} f(z)\,dz = 0$ by Cauchy's theorem.

The result follows.



If γ_1, γ_2 do cross then dissect them at each crossing point (e.g. c in the diagram) and apply the above to each section.

So, if f has no singularities at all, $\int_a^b f(z)\,dz$ does not depend on the chosen contour at all.

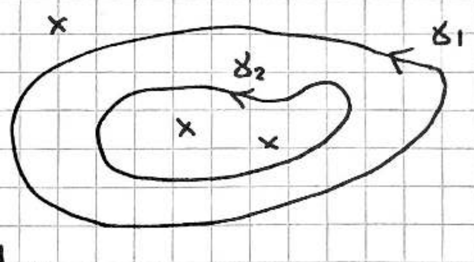
* Another way to think about path independence, and indeed Cauchy's theorem itself, is to consider $\int f(z)\,dz$ as a path integral in $\mathbb{R}^2$.

Then $f(z)dz = (u+iv)(dx+idy) = (u+iv)\,dx + (-v+iu)\,dy$

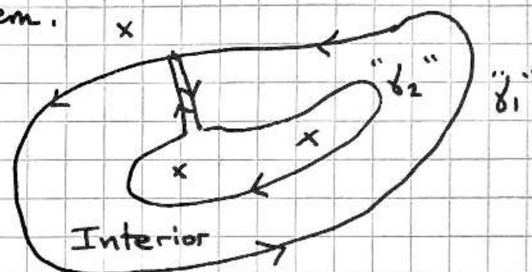
is an exact differential: $\frac{\partial}{\partial y}(u+iv) = \frac{\partial}{\partial x}(-v+iu)$

from the C-R equations. $*$

The deformation theorem also applies to closed contours.



Suppose that γ_1 is a closed contour that can be continuously deformed into another, γ_2, inside it; and that there are no singularities in the region between them.



Consider the contour γ shown;

$$\oint_\gamma f(z)\,dz = 0.$$

Now let the distance between the two "cross-cuts" tend to zero: those contributions cancel out and, in the limit, we have

$$\oint_{\gamma_1} f(z)\,dz = +\oint_{\gamma_2} f(z)\,dz = 0.$$

Chapter 3. Laurent Series & Singularities

3.1 Taylor series

If f is analytic at z_0 then it has a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

valid in some ngbd of z_0. See § 3.3 for information about the radius of convergence.

All the standard Taylor series from real analysis apply in $\mathbb{C}$ as well; for example

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (\text{convergent } \forall z)$$

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n \qquad (\text{convergent for } |z|<1)$$

3.2 Zeros

The zeros of an analytic function f are the points z_0 where $f(z_0)=0$.

A zero is of order N, in its Taylor Series $\sum a_n (z-z_0)^n$, the first non-zero coeff is a_N.

Alternatively, it is of order N if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) \text{ but } f^{(N)}(z_0) \neq 0.$$

A zero of order one (or two, three, etc) is also called a <u>simple zero</u> (or double zero, triple zero, etc)

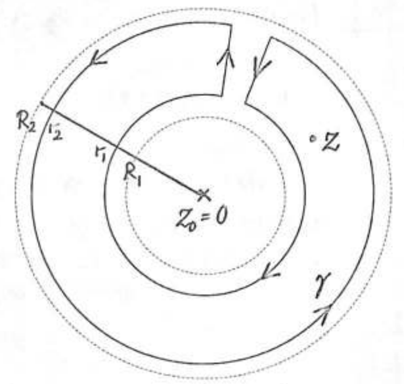
Examples: (i) $z^3 + iz^2 + z + i = (z-i)(z+i)^2$ has a simple zero at $z=i$ and a zero of order 2 at $z=-i$.

Non-Examinable Material
Proof of the Existence of Laurent Series

If $f(z)$ is analytic in some annulus $R_1 < |z - z_0| < R_2$ then it has a Laurent Series about z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

convergent within the annulus.



Proof: without loss of generality, $z_0 = 0$. Given some z in the annulus, choose r_1, r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$ and let γ_1, γ_2 be the contours $|z| = r_1$, $|z| = r_2$ traversed anti-clockwise. Choosing γ to be the contour shown, let

$$I(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

We now calculate $I(z)$ in two different ways.

Firstly, by letting the distance between the cross-cuts tend to zero, so that they cancel, we obtain

$$I(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(w)}{w - z} \, \mathrm{d}w - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Consider

$$\oint_{\gamma_1} \frac{f(w)}{w - z} \, \mathrm{d}w = -\frac{1}{z} \oint_{\gamma_1} \frac{f(w)}{1 - w/z} \, \mathrm{d}w$$

$$= -\frac{1}{z} \oint_{\gamma_1} f(w) \sum_{m=0}^{\infty} \left(\frac{w}{z}\right)^m \mathrm{d}w$$

(using a Taylor Series for $(1 - w/z)^{-1}$, valid since $|w| = r_1 < |z|$ on γ_1)

$$= -\sum_{m=0}^{\infty} z^{-m-1} \oint_{\gamma_1} f(w) w^m \, \mathrm{d}w$$

($\oint \sum = \sum \oint$ by uniform convergence on γ_1), so that

$$-\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(w)}{w - z} \, \mathrm{d}w = \sum_{n=-\infty}^{-1} a_n z^n$$

where $a_n = \frac{1}{2\pi i} \oint_{\gamma_1} f(w) w^{-n-1} \, \mathrm{d}w$, $n < 0$.

Similarly

$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{w-z}\,\mathrm{d}w = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n = \frac{1}{2\pi i}\oint_{\gamma_2} f(w)w^{-n-1}\,\mathrm{d}w$, $n \geqslant 0$, by expanding

$$\frac{1}{w-z} = \frac{1}{w}\,\frac{1}{1-z/w} = \sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}$$

(valid since $|w| = r_2 > |z|$ on γ_2).

Combining these results, $I(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

Alternatively, we may calculate $I(z)$ using the deformation theorem. Since there are no singularities within γ except at $w = z$ itself, we may change γ to a circle of radius ε centred at $w = z$. Letting $w = z + \varepsilon e^{i\theta}$,

$$\begin{aligned} I(z) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\big(f(z) + O(\varepsilon)\big)\,\mathrm{d}\theta \\ &\to f(z) \text{ as } \varepsilon \to 0. \end{aligned}$$

Since $I(z)$ does not in fact depend on ε, we conclude that $I(z) = f(z)$.

The result follows after translating the origin by z_0.

L8.1

(ii) $\sinh z$ has zeroes where

$$\frac{e^{z}-e^{-z}}{2} = 0 \iff e^{2z} = 1 \iff z = n\pi i, \ n \in \mathbb{Z}.$$

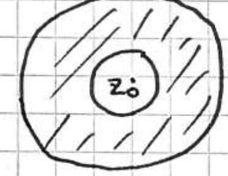
The zeros are all simple (since its derivative is $\cosh n\pi i = \cos n\pi$)

(iii) Since $\sinh z$ has a simple zero at $z = \pi i$, $\sinh^3 z$ has a zero of order 3 there. If needed, we can find its Taylor series about πi by writing $\zeta = z - \pi i$:

$$\begin{aligned}
\sinh^3 z &= [\sinh(\zeta + \pi i)]^3 \\
&= [-\sinh \zeta]^3 \\
&= -\left(\zeta + \tfrac{1}{3!}\zeta + \dots\right)^3 \qquad \text{(using Taylor series for sinh)} \\
&= -\zeta^3 - \tfrac{1}{2}\zeta^5 - \dots \\
&= -(z-\pi i)^3 - \tfrac{1}{2}(z-\pi i)^5 - \dots
\end{aligned}$$

3.3 Laurent Series

If f might have a singularity at z_0 then we cannot a priori expect it to have a Taylor Series there. Instead, if we know that f is analytic in an annulus $R_1 < |z - z_0| < R_2$ then it has a Laurent Series about z_0.



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

convergent within the annulus.

* Proof (non-examinable) : see the separate sheet *

It can be shown that the Laurent series for f about a particular z_0 is unique within any given annulus. Note that Taylor Series are just a special case of Laurent Series (with $a_n = 0 \ \forall n < 0$ and $R_1 = 0$).

Examples: (i) $\frac{e^z}{z^3}$ has Laurent Series about $z_0 = 0$ given by

$$\frac{e^z}{z^3} = \sum_{n=-\infty}^{\infty} \frac{z^{m-3}}{m!} = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}.$$

L 8.2

(ii) $e^{1/z}$ about $z_0 = 0$ has

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

so $a_n = \frac{1}{(-n)!}$ for $n \leq 0$.

(iii) $f(z) = \frac{e^z}{z^2-1}$ has a singularity at $z_0 = 1$ but is analytic in an annulus $0 < |z - z_0| < 2$. (since -1, the other singularity, is at a distance of 2). Write everything in terms of $\zeta = z - z_0$, so

$$f(z) = \frac{e^{\zeta} e^{z_0}}{\zeta(\zeta+2)} = \frac{e^{z_0}}{2\zeta} e^{\zeta}\left(1 + \tfrac{1}{2}\zeta\right)^{-1}$$

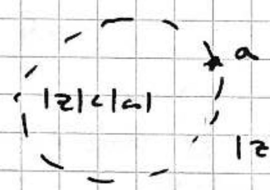
$$= \frac{e}{2\zeta}\left(1 + \zeta + \frac{1}{2!}\zeta^2 + \cdots\right)\left(1 - \frac{1}{2}\zeta + \frac{1}{4}\zeta^2 + \cdots\right)$$

$$= \frac{e}{2\zeta}\left(1 + \frac{1}{2}\zeta + \frac{1}{4}\zeta^2 + \cdots\right)$$

$$= \frac{1}{2}e\left(\frac{1}{z - z_0} + \frac{1}{2} + \frac{z - z_0}{4} + \cdots\right).$$

Hence $a_{-1} = \frac{1}{2}e$, $a_0 = \frac{1}{4}e$, etc. This series is valid in the whole annulus (our expression for $(1 + \frac{1}{2}\zeta)^{-1}$ was valid for $|\frac{1}{2}\zeta| < 1$, i.e. $|z - z_0| < 2$).

(iv) If $f(z) = \frac{1}{z-a}$ where $a \in \mathbb{C}$ then it is analytic in $|z| < |a|$ so it has a Taylor series about $z_0 = 0$ given by



$$\frac{1}{z-a} = -\frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1} = -\sum_{n=0}^{\infty} a^{-n-1} z^n,$$

the binomial expansion being valid since $\left|\frac{z}{a}\right| < 1$.

In $|z| > |a|$, it has a Laurent Series (in the "annulus" $|a| < |z| < \infty$) given by

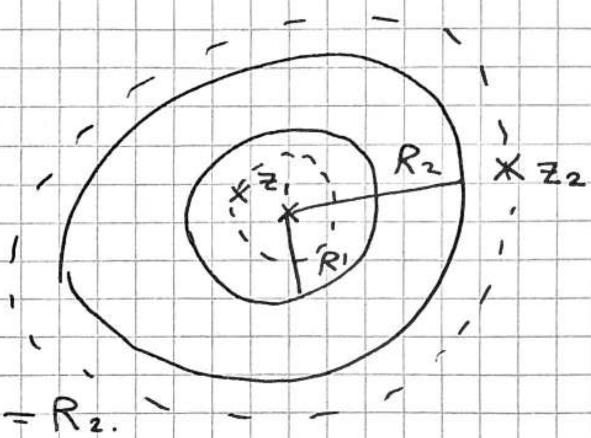
$$\frac{1}{z-a} = \frac{1}{z}\left(1 - \frac{a}{z}\right)^{-1} = \sum_{m=0}^{\infty} \frac{a^m}{z^{m+1}} = \sum_{n=-\infty}^{-1} a^{-n-1} z^n,$$

the binomial expansion is valid since $\left|\frac{a}{z}\right| < 1$.

(v) This doesn't seem to work for $f(z) = z^{-1/2}$: we cannot find a Laurent series about $z_0 = 0$. The reason is that the required branch cut (see §1.4) would pass through any annulus about the origin, so we cannot find such an annulus in which f is analytic.

L 8.3

(Of course $z^{-1/2}$ has Taylor Series about other points $z_0 \neq 0$, except those on the branch cut.)



Radius of convergence

Spose we have a Laurent series that we know to be valid in some annulus $R_1 < |z - z_0| < R_2$ but there are no singularities on $|z - z_0| = R_2$.

Then the outer radius of convergence can actually be pushed outwards, until the circle touches the singularity, at z_2 say.

Similarly if there are no singularities on $|z - z_0| = R_1$, then the inner radius can be pulled inwards until that circle touches a singularity, at z_1 say.

Then our Laurent Series in fact converges in the enlarged annulus

$$|z_1 - z_0| = R_1' < |z - z_0| < R_2' = |z_2 - z_0|.$$

In other words, the annulus of convergence of a Laurent series is always maximally large, with a singularity on each of the bounding circles, even if we originally derived it in a smaller region.

* This is because we could have started with R_1' and R_2' in the first place instead of R_1 and R_2 in the proof that Laurent Series exist; and because Laurent Series are unique, this new one must be the same as the old one, they agree in a region. *

A Taylor series is just a special case of a Laurent Series, resulting in the following statement: the radius of convergence of a Taylor Series is always the distance to the nearest singularity.

Example: cosec z has Laurent Series $(z - z^3/3! + \cdots)^{-1}$

$$= z^{-1}(1 - z^2/3! + \cdots)^{-1} = z^{-1} + \frac{z}{6} + \cdots$$

for z small enough.

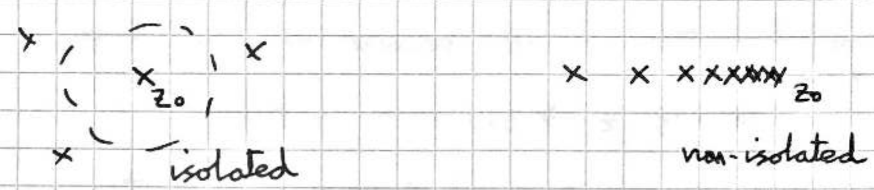
L 8.4

but it is not immediately obvious where the binomial expansion will be valid. The singularities of cosec z closest to the origin are at $z = \pm\pi$ so the annulus of convergence is in fact $0 < |z| < \pi$.

L 9.1

3.4 Classification of singularities

Suppose that f has a singularity at $z = z_0$. If there is a neighbourhood of z_0 within which f is analytic, except at z_0 itself, then f has an <u>isolated singularity</u> at z_0. If there is no such neighbourhood, then f has a <u>non-isolated singularity</u> at z_0. (Some authors call this an "essential singularity" but that creates confusion with another type described below.)



Examples: (i) $\operatorname{cosech} z$ has isolated singularities at $z = n\pi i$, $n \in \mathbb{Z}$

(ii) $\operatorname{cosech}\left(\frac{1}{z}\right)$ has isolated singularities at $z = \frac{1}{n\pi i}$, $n \in \mathbb{Z} \setminus \{0\}$, and a non-isolated singularity at $z = 0$. (since there are other arbitrarily close singularities)

(iii) $\operatorname{cosech} z$ has a non-isolated singularity at ∞ (see §1.1)

(iv) $z^{-1/2}$ has a branch point singularity at $z = 0$ (see §1.4); this is a type of non-isolated singularity, because $z^{-1/2}$ is not analytic at any point on the branch cut, but is usually treated as a separate case.

If f has an isolated singularity at z_0, we can find an annulus $0 < |z - z_0| < r_0$ say within which f is analytic, and it therefore has a Laurent Series. This gives us a way to classify singularities.

(a) Check for a branch point singularity

(b) Check for a non-isolated singularity

(c) Otherwise, consider the coeffs of the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

1. If $a_n = 0 \ \forall\, n < 0$, then f has a <u>removable singularity</u> at z_0

2. If $\exists N > 0$ s.t. $a_n = 0 \ \forall\, n < -N$ but $a_{-N} = 0$ then f has a <u>pole of order N</u> at z_0. (If $N = 1, 2, \ldots$ this is also called a <u>simple pole</u>,

double pole, etc.)

3. If there does not exist such an N then f has an essential isolated singularity (juicy!).

The behaviour of near z_0 is as follows:

1. At a removable singularity, where $f = a_0 + a_1(z-z_0) + \cdots$ for $0<|z-z_0|<r_0$, $f(z) \to a_0$ as $z \to z_0$; so an easy way to tell that a singularity is removable is that f has a finite limit. We can "remove the singularity" by redefining $f(z_0) = a_0 = \lim_{z \to z_0} f$; then f will become analytic at z_0.

2. At a pole, $|f(z)| \to \infty$ as $z \to z_0$.

3. At an essential isolated singularity, f does not tend to any finite or infinite limit. * In fact, it can be show than f all possible complex values (bar at most one) in any nghd of z_0 however small * For example, $e^{1/z}$ (see EX (vi) below) takes all values except 0.

Examples (i) $\frac{\cos z}{z}$ has Laurent Series $z^{-1} - \frac{1}{2}z + \frac{1}{24}z^3 - \cdots$ about the origin so has a simple pole at $z=0$.

(ii) $\frac{1}{z-i}$ has a simple pole at $z=i$ (since it is its own Laurent series)

(iii) $\frac{z^2}{(z-1)^2(z-i)^3}$ has a double pole at $z=1$ and a triple pole at $z=i$

To show formally that, for instance, there is a double pole at $z=1$, first note that $\frac{z^2}{(z-i)^3}$ is analytic there, so has a Taylor Series

$b_0 + b_1(z-1) + b_2(z-1)^2 + \cdots$ (note $b_0 \neq 0$)

for some b_n. Hence $\frac{z^2}{(z-1)^2(z-i)^3} = \frac{b_0}{(z-1)^2} + \frac{b_1}{z-1} + b_2 + \cdots$

(iv) If $g(z)$ has a zero of order N at $z=z_0$, then $g(z)^{-1}$ has a pole of order N there. To prove this, write $g(z) = (z-z_0)^N G(z)$ for some G with $G(z_0) \neq 0$, and note that $\frac{1}{G(z)}$ has a Taylor Series about z_0.

Hence $\cot z$ has a simple pole at the origin, because $\tan z$ has a simple zero there.

(v) z^2 has a double pole at ∞ (see §1.1)

(vi) $e^{1/z}$ has an essential isolated singularity at $z=0$, because all the a_ns are non-zero for $n \le 0$ (see § 3.3 (ii))

(vii) $\sin \frac{1}{z}$ also has an essential isolated singularity at $z=0$ because (using the standard Taylor series for sin) there are non-zero a_n for infinitely many negative n.

(viii) $f(z) = \dfrac{e^z - 1}{z}$ has a removable singularity at $z=0$ because

$$f(z) = 1 + \frac{1}{2!} z + \frac{1}{3!} z^2 + \dots$$

By defining $f(0)=1$ we would remove the singularity and we would obtain an entire function.

(ix) $f(z) = \dfrac{\sin z}{z}$ is not defined at $z=0$ but has a removable singularity there; remove it by setting $f(0)=1$.

(x) A rational function $f(z) = \dfrac{P(z)}{Q(z)}$ with P, Q polynomials. Has a singularity at any point where Q has a zero; but if P also has a zero there then it is removable, redefining

$$f(z_0) = \frac{P'(z_0)}{Q'(z_0)} \quad \text{(assuming } Q \text{ has a simple zero)}.$$

3.5 Closed Contour Integrals of Laurent Series

Suppose that f is analytic within some annulus, so has a Laurent Series $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ there, and that γ is an anticlockwise, simple closed contour lying within the annulus. What is $\oint_\gamma f(z)\,dz$?

$2\pi i a_{-1} \times$ winding number

Choose a circular contour γ_r lying inside γ but still within the annulus.

E10.1

By deformation thm of §2.3,

$$\oint_\gamma f(z)\,dz = \oint_{\gamma^r} f(z)\,dz = \sum a_n \oint_{\gamma^r} (z-z_0)^n\,dz$$

(because in fact we have uniform convergence on γ^r – proof omitted).

But $\oint_{\gamma^r} (z-z_0)^n dz = \int_0^{2\pi} r^n e^{in\theta} \cdot ire^{i\theta} d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= \begin{cases} 2\pi i r^{n+1} & n=-1, \\ 0 & n \neq -1. \end{cases}$$

Hence $\oint_\gamma f(z)\,dz = 2\pi i a_{-1}$.

Chap 4 The Calculus of Residues

4.1 Residues

If f has an isolated singularity at z_0 then it has a Laurent series expansion about that point (see §3.4).

The residue of f at z_0 is the coeff a_{-1} of its Laurent series. (We have already seen in §3.5 that this coeff is important for evaluating integrals). There is no standard notation but we shall denote the residue by $\operatorname*{res}_{z=z_0} f(z)$.

At a simple pole, the residue is given by

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z \to z_0} \{(z-z_0) f(z)\}$$

since the RHS is equal to

$$\lim_{z \to z_0} \left\{ (z-z_0)\left(\frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots\right)\right\}$$

($a_n = 0 \ \forall n < -1$ at a simple pole). More generally, at a pole of order N, the residue is given by

$$\lim_{z \to z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left((z-z_0)^N f(z)\right) \right\}$$

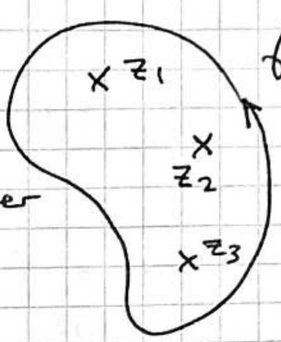
which can be proved in a similar manner.

In practice, a variety of techniques can be used to evaluate residues: no single technique is optimal for all situations.

See the worked examples.

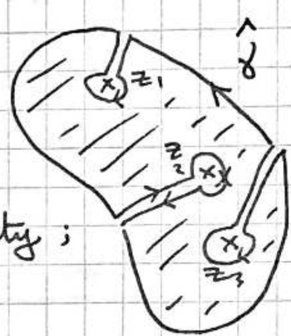
L10.2

4.2 The Residue Theorem



Suppose that f is analytic in a simply connected domain, except at a finite number of isolated singularities $z_1, z_2, \ldots, z_n$; and that a simple closed contour γ encircles the singularities anticlockwise. Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{res}_{z=z_k} f(z)$$



Proof: consider the curve $\hat{\gamma}$ shown, consisting of small clockwise circles around each singularity; cross-cuts, which cancel in the limit as they approach each other in pairs, and the large outer curve (which is the same as γ in the limit. $\hat{\gamma}$ encircles no singularities, so $\oint_{\hat{\gamma}} f(z)\,dz = 0$ by Cauchy's theorem.

Hence in the limit where the cross-cuts cancel, we have

$$\oint_\gamma f(z)\,dz + \sum_{k=1} \oint_{\gamma_k} f(z)\,dz = 0.$$

But about each isolated singularity z_k there is a Laurent series valid locally in some annulus, so by §3.5 and §4.1 we have

$$\oint_{\gamma_k} f(z)\,dz = -2\pi i \operatorname*{res}_{z=z_k} f(z)$$

(minus because γ_k is clockwise). The result follows.

4.3 Cauchy's integral formula

Suppose f is analytic in a simply connected domain D and that $z \in D$. Then for any simple closed contour γ in D encircling z anticlockwise,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z}\,dw$$

(Note that if z lies <u>outside</u> γ then RHS $=0$ by Cauchy's theorem)

<u>Proof</u> The only singularity of the integrand is at $w=z$, and is a simple pole (since f is analytic there). Hence, using the residue theorem and the formula in §4.1, the integral is equal to

↳10.3

$$2\pi i \operatorname*{res}_{w=z} \frac{f(w)}{w-z} = 2\pi i \lim_{w\to z} f(w) = 2\pi i f(z). \qquad \square$$

* So, if we know f on γ, then we know f at all points within γ. Another way of looking at this is to write $f=u+iv$, where u, v are harmonic, and are both specified on γ. Then we have Laplace's equation for u & v with Dirichlet BCs, which has a unique solution inside γ using a result from IA Vector Calculus. CIF gives us a way of calculating that solution explicitly *

** Normally, Cauchy's integral formula is proved <u>before</u> the residue theorem, because it is needed to prove the existence of Laurent Series. Our proof in §3.3 actually contained a hidden proof of CIF without the residue theorem! **

Worked Example
Calculating Residues

Example: e^z/z^3 at $z=0$

By expanding e^z as a Taylor series, we see that $f(z)=e^z/z^3$ has a Laurent expansion about $z=0$ given by

$$z^{-3}+z^{-2}+\tfrac{1}{2}z^{-1}+\tfrac{1}{3!}+\cdots.$$

Hence the residue is $\frac{1}{2}$ (the coefficient of z^{-1}).

Alternatively, we note that f has a pole of order 3 at $z=0$, so we can use the general formula for the residue at a pole:

$$\operatorname*{res}_{z=0} f(z)=\lim_{z\to 0}\left\{\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(z^3 f(z)\right)\right\}=\tfrac{1}{2}\lim_{z\to 0}\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}e^z\right\}=\tfrac{1}{2}.$$

Example: $e^z/(z^2-1)$ at $z=1$

We have already calculated the Laurent expansion of $g(z)=e^z/(z^2-1)$ at $z=1$:

$$\frac{e^z}{z^2-1}=\frac{e}{2}\left(\frac{1}{z-1}+\frac{1}{2}+\cdots\right),$$

so the residue is $\frac{1}{2}e$.

Alternatively, we use the formula for the residue at a simple pole:

$$\operatorname*{res}_{z=1} g(z)=\lim_{z\to 1}\frac{(z-1)e^z}{z^2-1}=\lim_{z\to 1}\frac{e^z}{z+1}=\tfrac{1}{2}e.$$

Example: $1/(z^8-w^8)$ at $z=w$ $\neq 0$

For any complex constant w, $h(z)=(z^8-w^8)^{-1}$ has 8 simple poles, at $z=we^{n\pi i/4}$ $(n=0,1,\ldots,7)$. The residue at $z=w$ can be evaluated by factorizing z^8-w^8 into its eight linear factors:

$$\lim_{z\to w}\frac{z-w}{z^8-w^8} = \lim_{z\to w} \frac{1}{z^7+z^6 w+\cdots}$$

$$=\lim_{z\to w}\frac{1}{(z-we^{\pi i/4})(z-iw)(z-we^{3\pi i/4})(z+w)(z+we^{\pi i/4})(z+iw)(z+we^{3\pi i/4})}$$

$$=\frac{1}{\left(w^2-(we^{\pi i/4})^2\right)\left(2w^2\right)\left(w^2-(we^{3\pi i/4})^2\right)\left(2w\right)}$$

$$=\frac{1}{(1-i)(2)(1+i)(2)w^7}=1/8w^7.$$

Alternatively, it is *much* easier to use L'Hôpital's Rule:

$$\operatorname*{res}_{z=w} h(z)=\lim_{z\to w}\frac{z-w}{z^8-w^8}=\lim_{z\to w}\frac{1}{8z^7}=1/8w^7.$$

We can differentiate Cauchy's formula

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^2}\, dw .$$

(Differentiation under integral sign is valid because integrand, both before and after, is a continuous function in z, w) Repeating,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw .$$

Hence at any point where f is analytic, all its derivatives exist, so it is differentiable infinititely many times as advertised in §1.2.

* An application of Cauchy's integral formula is Liouville's theorem (non-examinable): any bounded entire function is constant.

Suppose $|f| \leq M$, and consider a circle of radius R centred at a given z. Then

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2}\, dw$$

$$\Rightarrow |f'(z)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^2} \quad (\text{from § 2.1 (v)})$$

$$\to 0 \quad \text{as } R \to \infty$$

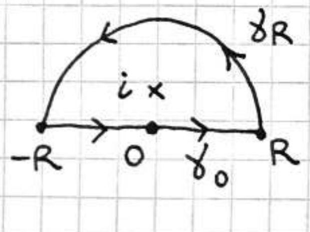
Hence $f'(z) = 0 \ \forall z \in \mathbb{C}$ and f is constant. *

* Another application is the maximum modulus principle (non-examinable): if f is analytic within a bounded domain and on its boundary, then $|f|$ attains its maximum on the boundary. *

4.4 Applications of the Residue Theorem

To illustrate the technique known as "the calculus of residues" we shall evaluate $I = \int_0^\infty \frac{dx}{1+x^2}$ (which we can already do by trigonometry).

Consider $\oint_\gamma \frac{dz}{1+z^2}$ where γ is the contour shown: from $-R$ to R along the real axis (γ_0) then returning to $-R$ via a semicircle of radius R in the upper half-plane. (γ_R)



L11.2

This is known as "closing in the upper half-plane" or "closing above".

Now $\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$ so the only singularity enclosed by γ is a simple pole at i, where the residue is $\lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$. Hence

$$\oint_\gamma \frac{dz}{1+z^2} = \oint_{\gamma_0} \frac{dz}{1+z^2} + \oint_{\gamma_R} \frac{dz}{1+z^2} = 2\pi i \operatorname{res}_i \frac{1}{1+z^2} = \pi .$$

But $\int_{\gamma_0} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} \longrightarrow 2I$ as $R \to \infty$.

Also $\int_{\gamma_R} \frac{dz}{1+z^2} \longrightarrow 0$ as $R\to\infty$ (see below), so we obtain in the limit $2I + 0 = \pi$, i.e. $I = \frac{\pi}{2}$.

To justify $\int_{\gamma_R} \frac{dz}{1+z^2} \longrightarrow 0$ as $R\to\infty$, we can use a formal or an informal argument.

Formal: $|1+z^2| \geqslant |z|^2 - 1 \geqslant R^2 - 1$ on γ_R

$$\text{so } \left| \int_{\gamma_R} \frac{dz}{1+z^2} \right| \leqslant \pi R \cdot \frac{1}{R^2-1} \longrightarrow 0 \quad \text{as } R \to \infty$$

Informal:

$$\text{for } z \in \gamma_R, \ \left| \frac{1}{1+z^2} \right| = O(R^{-2})$$

$$\text{so } \left| \int_{\gamma_R} \frac{dz}{1+z^2} \right| \leqslant \pi R\, O(R^{-2}) = O(R^{-1}) \longrightarrow 0$$

This example in itself is not impressive, but the method adapts easily to more complicated integrals.

Further examples: (i) To find $I = \int_0^\infty \frac{dx}{(x^2+a^2)^2}$, where $a > 0$ is a real constant, consider

$\oint_\gamma \frac{dz}{(z^2+a^2)^2}$ where γ is the real axis closed in the upper half plane.

The only singularities is a pole of order 2 at ia, within γ.

The residue is

$$\lim_{z\to ia} \frac{d}{dz} \frac{1}{(z+ia)^2} = \lim_{z\to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{-8ia^3} = -\tfrac{1}{4} i a^{-3}.$$

The integral round γ_R still vanishes as $R\to\infty$ since now

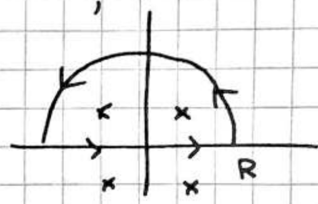
$$\left| \gamma_R \frac{dz}{(z^2+a^2)^2} \right| \leqslant \pi R\, O(R^{-4}) = O(R^{-3})$$

Therefore $2I = 2\pi i\left(-\tfrac{1}{4} i a^{-3}\right) \Rightarrow I = \frac{\pi}{4a^3}$.

L11.3

For $I = \int_0^\infty \frac{dx}{1+x^4}$ we can use the same contour.

There are simple poles at $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{-\pi i/4}$, $e^{-3\pi i/4}$, but only the first two poles are enclosed, with residues $-\frac{1}{4}e^{\pi i/4}$ and $\frac{1}{4}e^{-\pi i/4}$ respectively.

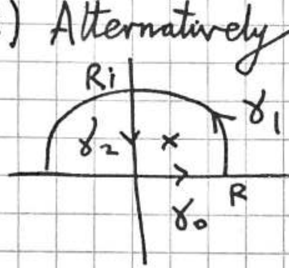


The integral around γ_R is $\pi R O(R^{-4}) \to 0$.

Hence $2I = 2\pi i \cdot \frac{1}{4}\left(e^{-\pi i/4} - e^{\pi i/4}\right)$

$$\Rightarrow I = \frac{\pi}{2\sqrt{2}}.$$

(iii) Alternatively for $\int_0^\infty \frac{dx}{1+x^4}$ we could use a quarter circle as shown.



Let γ consist of real axis from 0 to R (γ_0), the arc of circle from R to iR (γ_1) and the imaginary axis from iR to 0 (γ_2).

Now, $\int_{\gamma_0} \frac{dz}{1+z^4} \longrightarrow I$ as $R \to \infty$; along γ_2 we have

$$\int_{\gamma_2} \frac{dz}{1+z^4} = \int_R^0 \frac{i\,dy}{1+y^4} \longrightarrow -iI \quad \text{as } R \to \infty.$$

$\int_{\gamma_1} \to 0$ as before.

We enclose only one pole, $e^{i\pi/4}$ which yields

$$I(1-i) = 2\pi i\left(-\tfrac{1}{4}e^{\pi i/4}\right) \quad \Rightarrow I = \frac{\pi}{2\sqrt{2}}$$

as before.

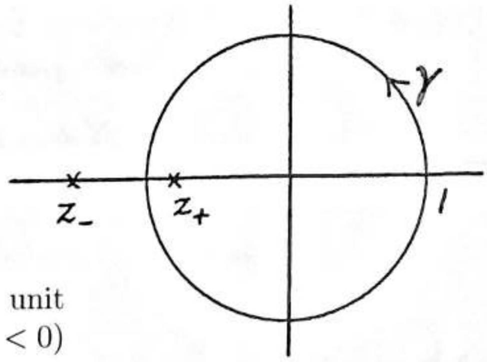
(iv) For trigonometric integrals of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$ substitute $z = e^{i\theta}$, $dz = iz\,d\theta$, $\cos\theta = \frac{1}{2}(z + z^{-1})$, $\sin\theta = \frac{1}{2i}(z - z^{-1})$, to obtain a closed contour integral. See the worked example.

Worked Example
Integrals of Trigonometric Functions

We wish to evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$$

where $a > 1$ (so that the integrand is always finite). Substitute $z = e^{i\theta}$, so that $dz = iz\,d\theta$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$. As θ increases from 0 to 2π, z moves round the circle γ of radius 1 in the complex plane. Hence



$$I = \oint_\gamma \frac{(iz)^{-1}\,dz}{a + \frac{1}{2}(z + z^{-1})} = -2i \oint_\gamma \frac{dz}{z^2 + 2az + 1}.$$

The integrand has poles at

$$z_\pm = -a \pm \sqrt{a^2 - 1},$$

both on the real axis. Note that z_+ is inside the unit circle (check that $a - 1 < \sqrt{a^2-1} < a$, so $-1 < z_+ < 0$) whereas z_- is outside it. The integrand is equal to

$$\frac{1}{(z - z_+)(z - z_-)}$$

so the residue at $z = z_+$ is $1/(z_+ - z_-) = 1/2\sqrt{a^2-1}$. Hence

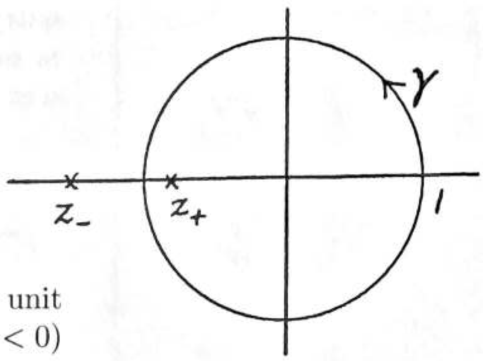
$$I = -2i\left(\frac{2\pi i}{2\sqrt{a^2-1}}\right) = \frac{2\pi}{\sqrt{a^2-1}}.$$

Worked Example
Integrals of Trigonometric Functions

We wish to evaluate

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

where $a > 1$ (so that the integrand is always finite). Substitute $z = e^{i\theta}$, so that $\mathrm{d}z = iz\,\mathrm{d}\theta$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$. As θ increases from 0 to 2π, z moves round the circle γ of radius 1 in the complex plane. Hence



$$I = \oint_\gamma \frac{(iz)^{-1}\,\mathrm{d}z}{a + \frac{1}{2}(z + z^{-1})} = -2i \oint_\gamma \frac{\mathrm{d}z}{z^2 + 2az + 1}.$$

The integrand has poles at

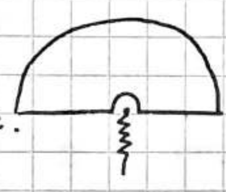
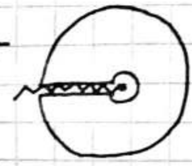
$$z_\pm = -a \pm \sqrt{a^2 - 1},$$

both on the real axis. Note that z_+ is inside the unit circle (check that $a - 1 < \sqrt{a^2 - 1} < a$, so $-1 < z_+ < 0$) whereas z_- is outside it. The integrand is equal to

$$\frac{1}{(z - z_+)(z - z_-)}$$

so the residue at $z = z_+$ is $1/(z_+ - z_-) = 1/2\sqrt{a^2 - 1}$. Hence

$$I = -2i\left(\frac{2\pi i}{2\sqrt{a^2 - 1}}\right) = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

(v) For integrals involving a function with a branch cut, we can use an indented contour  or a keyhole contour .
See the worked example.

4.5 Using rectangular contours

Lemma: if $z = x+iy$, then

$$|\sinh y| \leqslant |\sin z|, |\cos z| \leqslant \cosh y,$$

$$|\sinh x| \leqslant |\sinh z|, |\cosh z| \leqslant \cosh x.$$

Hence as $y \to \pm\infty$, $|\sin z|$ and $|\cos z| \sim \frac{1}{2}e^{|y|}$,

as $x \to \pm\infty$, $|\sinh z|$ and $|\cosh z| \sim \frac{1}{2}e^{|x|}$.

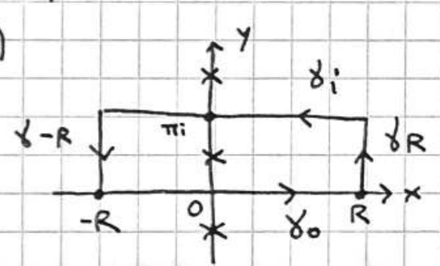
Proof: for $\sin z$, first note that $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y}$.
Now apply the triangle inequalities

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$$

to $z_1 = e^{iz}$, $z_2 = e^{-iz}$.
Similarly for $\cos z$, $\sinh z$, $\cosh z$.

Examples of integrals using rectangular contours:

(i)



To evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh x}\, dx$$

where $\alpha \in (-1,1)$ is real, use a rectangular contour as shown. We see that

$$\int_{\gamma_0} \frac{e^{\alpha z}}{\cosh z}\, dz \to I \quad \text{as } R \to \infty$$

$$\int_{\gamma_1} \frac{e^{\alpha z}}{\cosh z}\, dz = \int_R^{-R} \frac{e^{\alpha(x+\pi i)}}{\cosh(x+\pi i)}\, dx = e^{\alpha\pi i}\int_R^{-R} \frac{e^{\alpha x}}{-\cosh x}\, dx$$

$$\to e^{\alpha \pi i} I \quad \text{as } R \to \infty$$

On γ_R we have $z = R+iy$ so $|\cosh z| \sim \frac{1}{2}e^R$ from Lemma, and $|e^{\alpha z}| = e^{\alpha R}$. Hence $\left|\frac{e^{\alpha z}}{\cosh z}\right| = O(e^{(\alpha-1)R}) \to 0$ as $R \to \infty$.
So $\int_{\gamma_R} \to 0$. Similarly for $\int_{\gamma_{-R}}$.

Worked Example
Integration Around a Branch Cut

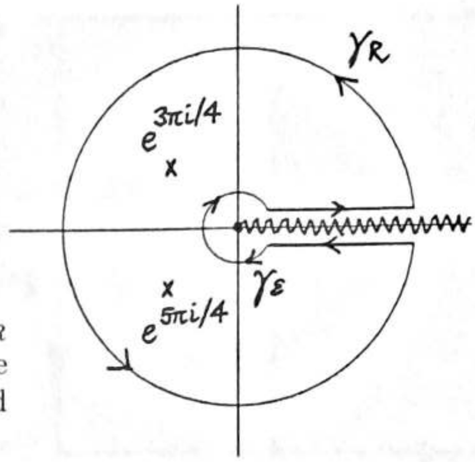
We wish to evaluate

$$I = \int_0^\infty \frac{x^\alpha}{1+\sqrt{2}\,x+x^2}\,\mathrm{d}x$$

where $-1 < \alpha < 1$ so that the integral converges. We will need a branch cut for z^α; we take this along the positive real axis and define

$$z^\alpha = r^\alpha e^{i\alpha\theta}$$

where $z = re^{i\theta}$ and $0 \leqslant \theta < 2\pi$.



Consider

$$\oint_\gamma \frac{z^\alpha}{1+\sqrt{2}\,z+z^2}\,\mathrm{d}z$$

where the *keyhole contour* γ consists of a large circle γ_R of radius R, a small circle γ_ε of radius ε (to avoid the singularity of z^α at $z=0$) and two lines just above and below the branch cut, as shown.

The contribution from γ_R is at most $2\pi R \times O(R^{\alpha-2}) = O(R^{\alpha-1}) \to 0$ as $R \to \infty$, since $\alpha < 1$.

The contribution from γ_ε is at most $2\pi\varepsilon \times O(\varepsilon^\alpha) = O(\varepsilon^{\alpha+1}) \to 0$ as $\varepsilon \to 0$, since $\alpha > -1$.

The contribution from just above the branch cut is

$$\int_\varepsilon^R \frac{x^\alpha}{1+\sqrt{2}\,x+x^2}\,\mathrm{d}x \to I$$

as $\varepsilon \to 0$ and $R \to \infty$, and the contribution from just below the branch cut is

$$\int_R^\varepsilon \frac{x^\alpha e^{2\alpha\pi i}}{1+\sqrt{2}\,x+x^2}\,\mathrm{d}x \to -e^{2\alpha\pi i} I$$

as $\varepsilon \to 0$ and $R \to \infty$.

Hence

$$\oint_\gamma \frac{z^\alpha}{1+\sqrt{2}\,z+z^2}\,\mathrm{d}z \to (1-e^{2\alpha\pi i})I$$

as $\varepsilon \to 0$ and $R \to \infty$. But the integrand is equal to

$$\frac{z^\alpha}{(z-e^{3\pi i/4})(z-e^{5\pi i/4})}$$

(by finding the roots of the quadratic), so the poles inside γ are at $e^{3\pi i/4}$ with residue $e^{3\alpha\pi i/4}/(\sqrt{2}\,i)$ and at $e^{5\pi i/4}$ with residue $e^{5\alpha\pi i/4}/(-\sqrt{2}\,i)$. (Note that it would not be wise to write the location of the second pole as $e^{-3\pi i/4}$, since that would be inconsistent with our chosen branch for z^α.)

The only singularity inside the contour is at $\frac{\pi i}{2}$, where the residue is $\frac{e^{\alpha\pi i/2}}{\sinh\frac{\pi i}{2}} = -i e^{\alpha\pi i/2}$. Hence

$$I(1+e^{\alpha\pi i}) = 2\pi i(-ie^{\alpha\pi i/2})$$

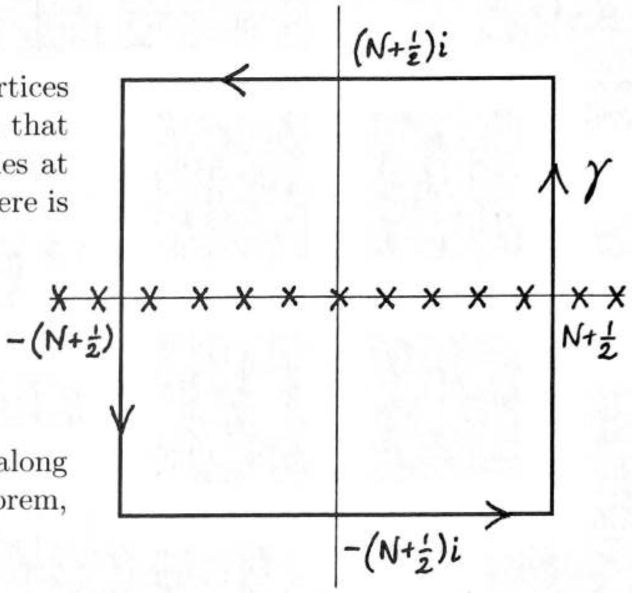
$$\therefore \underline{I = \pi \sec\frac{\alpha\pi}{2}}$$

(ii) It is possible to use contour integration to sum infinite series. This is traditionally done using $\cot z$ and a large square contour that has been chosen carefully to avoid all of the singularities. See the worked example.

Worked Example
Summation of Series using a Square Contour

We shall evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ by integrating a suitable function around a square contour. Consider

$$\oint_\gamma \frac{\cot \pi z}{z^2}\,\mathrm{d}z$$



where γ is the contour shown in the diagram with vertices at $(N+\frac{1}{2})(\pm 1 \pm i)$. Here N is a large integer, so that γ avoids all the singularities. There are simple poles at $z = n \in \mathbb{Z}$, $n \neq 0$, with residues $1/n^2\pi$. At $z = 0$ there is a triple pole:

$$\begin{aligned}\frac{\cot \pi z}{z^2} &= \frac{1}{z^2(\pi z + \frac{1}{3}\pi^3 z^3 + \cdots)} \\ &= \pi^{-1} z^{-3}(1 - \tfrac{1}{3}\pi^2 z^2 + \cdots),\end{aligned}$$

so the residue is $-\frac{1}{3}\pi$. But as $N \to \infty$, the integrals along the sides vanish (see below), so from the residue theorem,

$$2\pi i\left(2\sum_{n=1}^{N}\frac{1}{n^2\pi} - \tfrac{1}{3}\pi\right) \to 0,$$

i.e., $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

To demonstrate the behaviour of the integrals along the sides, we shall show that on γ, $\cot \pi z$ is bounded by a constant M that does not depend on N. Then

$$\left|\oint_\gamma \frac{\cot \pi z}{z^2}\,\mathrm{d}z\right| \leqslant 4(2N+1)\frac{M}{(N+\frac{1}{2})^2} \to 0 \text{ as } N \to \infty.$$

First consider the right-hand side, on which $z = N + \frac{1}{2} + iy$:

$$|\cot \pi z| = \frac{\left|\cos\left((N+\frac{1}{2})\pi + i\pi y\right)\right|}{\left|\sin\left((N+\frac{1}{2})\pi + i\pi y\right)\right|} = |-\tan i\pi y| = |\tanh \pi y| \leqslant 1.$$

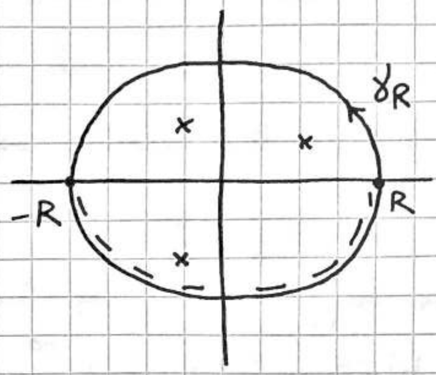
Similarly for the left-hand side. Now consider the top side, on which $z = x + (N+\frac{1}{2})i$:

$$|\cot \pi z| = \frac{|\cos \pi z|}{|\sin \pi z|} \leqslant \frac{\cosh(N+\frac{1}{2})\pi}{|\sinh(N+\frac{1}{2})\pi|} = \coth(N+\tfrac{1}{2})\pi,$$

where we have used the general results $|\sin z| \geqslant |\sinh \operatorname{Im} z|$, $|\cos z| \leqslant \cosh \operatorname{Im} z$. Now $\coth\theta \to 1$ as $\theta \to \infty$, so in particular it is bounded (by 2 say) for sufficiently large θ; hence $\cot \pi z$ is bounded by 2 for sufficiently large N. Similarly for the bottom side.

Hence we may choose $M = 2$ for sufficiently large N.

4.6 Jordan's Lemma



- Suppose f is an analytic function, except for a finite number of singularities, and that $f(z) \to 0$ as $|z| \to \infty$.

Then for any real constant $\lambda > 0$,

$$\int_{\gamma_R} f(z)\, e^{i\lambda z}\, dz \longrightarrow 0$$

as $R \to \infty$, where γ_R is the semicircle of radius R in $\mathbb{H}$.

For $\lambda < 0$, the same conclusion holds for the semicircle contour γ_R' in the lower half-plane.

- Such integrals arise frequently in applications, particularly with Fourier Transforms as we shall see in Chapter 5.

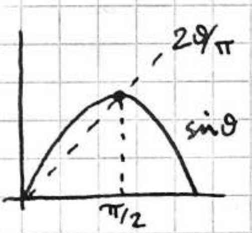
The result is easy to show in "straightforward cases", when $f(z)$ decays <u>more rapidly</u> than $\frac{1}{z}$ as $|z| \to \infty$. Then $|f(z)| = o\left(\frac{1}{R}\right)$ on γ_R, and since $|e^{i\lambda z}| = e^{-\lambda \operatorname{Im} z} \leq 1$ on γ_R, we have

$$\left| \int_{\gamma_R} f(z)\, e^{i\lambda z}\, dz \right| \leq \pi R \cdot o\left(\frac{1}{R}\right) \longrightarrow 0$$

as $R \to \infty$, as required.

But in "sensitive cases", when f decays <u>less rapidly</u> than $\frac{1}{z}$, the

- proof relies on the fact that for $\theta \in [0, \frac{\pi}{2}]$, $\sin\theta \geq \frac{2\theta}{\pi}$ (see pic)



Let $M_R = \sup\limits_{z \in \gamma_R} |f(z)|$, then $M_R \to 0$ as $R \to \infty$.

Now
$$\left| \int_{\gamma_R} f(z)\, e^{i\lambda z}\, dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})\, e^{i\lambda Re^{i\theta}}\, iRe^{i\theta}\, d\theta \right|$$

$$\leq R \int_0^{\pi} |f(Re^{i\theta})|\, |e^{i\lambda Re^{i\theta}}|\, d\theta$$

$$\leq 2RM_R \int_0^{\pi/2} e^{-\lambda R \sin\theta}\, d\theta$$

$$\leq 2RM_R \int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\, d\theta$$

$$= \frac{\pi}{\lambda}\left(1 - e^{-\lambda R}\right) \underset{\searrow 0}{M_R} \longrightarrow 0 \text{ as } R \to \infty.$$

Similarly for γ_R' when $\lambda < 0$.

Examples (i) To calculate $I = \int_0^\infty \frac{\cos \alpha x}{1+x^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos \alpha x}{1+x^2}\, dx$,
where $\alpha > 0$ is real. We might try the following approaches

(a) Close the contour above, and try to show that

$$\int_{\gamma_R} \frac{\cos \alpha z}{1+z^2}\, dz \to 0 \quad \text{as } R \to \infty.$$

This approach fails because in fact $\cos \alpha z$ is exponentially large (at ∞)

(b) Write $I = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\, dx + \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+x^2}\, dx$.

We can close the contour above for the first integral and use Jordan's lemma on γ_R with $\lambda = \alpha > 0$; for the second integral we close below using γ_R' with $\lambda = -\alpha < 0$.

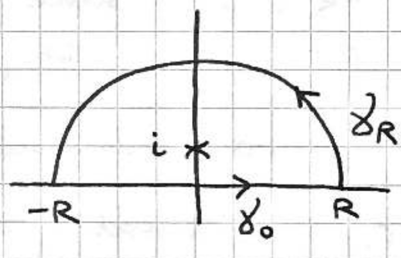
(c) A quicker method is to note that

$$I = \frac{1}{2}\,\mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\, dx,$$

so we only need one contour, closing above.

(d) Even quicker is to note that $\int_{-\infty}^{\infty} \frac{\sin \alpha x}{1+x^2}\, dx = 0$, so in fact
$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\, dx$.

Consider $\int_{\gamma_0+\gamma_R} \frac{e^{i\alpha z}}{1+z^2}\, dz$.



Along γ_0 obtain $2I$ as $R \to \infty$.

On γ_R obtain 0, by Jordan's lemma, $\alpha > 0$.

(This is a "straightforward" case, since $\frac{1}{1+z^2} = o(\frac{1}{z})$ as $|z| \to \infty$)

So $I = \frac{1}{2}\, 2\pi i \operatorname*{res}_{z=i} \frac{e^{i\alpha z}}{1+z^2} = \pi i \cdot \frac{e^{-\alpha}}{2i} = \frac{1}{2}\pi e^{-\alpha}$.

(ii) To find $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$ we can also use Jordan's lemma.
(this is a "sensitive" case because the decay at ∞ is $O(\frac{1}{z})$)
We have the complication of a singularity of e^{iz}/z on the real axis.

Chapter 5 Transform Theory

5.1 Fourier Transforms

The Fourier transform of a function $f(x)$ that decays sufficiently rapidly at $|\infty| \to z$ is $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$.

Worked Example
Integration Around a Singular Point on the Real Axis

We wish to evaluate
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

This integrand is well-behaved at the origin, so the integral is non-singular. But we will need to use Jordan's Lemma, splitting $\sin z$ into exponentials and then splitting the integral in two (closing $\int (e^{iz}/z)\,\mathrm{d}z$ in the *upper* half-plane and $\int (e^{-iz}/z)\,\mathrm{d}z$ in the *lower* half-plane). This will cause a problem because e^{iz}/z and e^{-iz}/z are singular at $z = 0$, so the contour will pass directly *through* a singularity in both integrals.

To avoid this, we note that because $(\sin x)/x$ is bounded near the origin we can exclude a small interval $(-\varepsilon, \varepsilon)$ from the integral and later let $\varepsilon \to 0$, without affecting the value of the integral. In other words,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{\sin x}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\sin x}{x}\,\mathrm{d}x \right) \\
&= \frac{1}{2i} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{-R}^{-\varepsilon} \frac{e^{-iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{\varepsilon}^{R} \frac{e^{-iz}}{z}\,\mathrm{d}z \right) \\
&= \frac{1}{2i} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z \right) \\
&\qquad - \frac{1}{2i} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{e^{-iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{-iz}}{z}\,\mathrm{d}z \right).
\end{aligned}$$

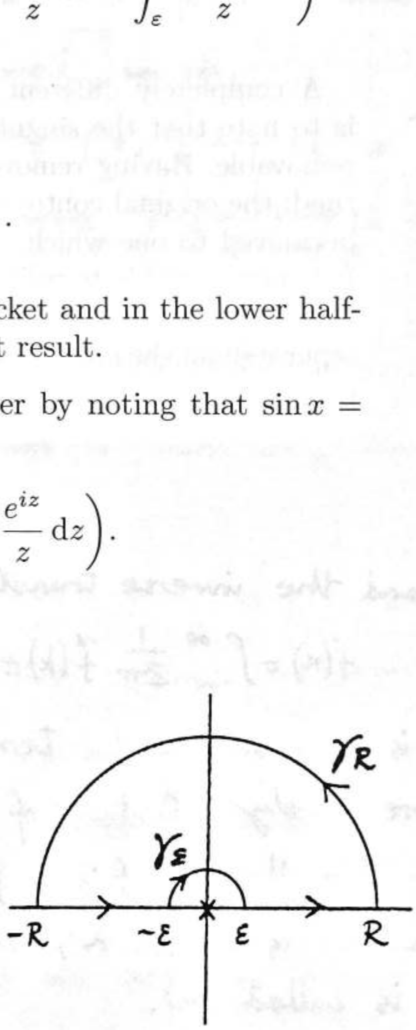
We could now close in the upper half-plane for the first bracket and in the lower half-plane for the second bracket, which would obtain the correct result.

Alternatively, we can simplify the calculation even further by noting that $\sin x = \operatorname{Im}(e^{ix})$ and that, therefore,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z \right).$$

We are now ready to proceed.

Let γ be the contour from $-R$ to $-\varepsilon$ along the real axis, then round a semicircle γ_ε of radius ε, then from ε to R, returning via a semicircle γ_R of radius R. Then γ encloses no poles of e^{iz}/z, so



$$\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z = -\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{\gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z.$$

Jordan's Lemma tells us that the integral round γ_R vanishes as $R \to \infty$. On γ_ε, $z = \varepsilon e^{i\theta}$ and $e^{iz} = 1 + O(\varepsilon)$; so

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_\pi^0 \frac{1+O(\varepsilon)}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta = -i\pi + O(\varepsilon). \qquad (*)$$

Hence, taking the limit as $\varepsilon \to 0$ and $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \mathrm{Im}(i\pi) = \pi.$$

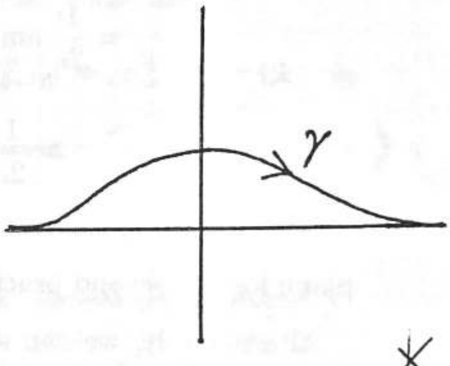
Equation $(*)$ is an example of the *indentation lemma*, which states that if γ_ε is an anticlockwise arc of radius ε and angle α centred around a simple pole z_0 of a function $f(z)$, then

$$\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z \to i\alpha \operatorname*{res}_{z=z_0} f(z) \qquad \text{as } \varepsilon \to 0$$

(i.e., a fraction $\alpha/2\pi$ of the value of the integral around an entire circle). This is easily proved using the Laurent Series. In the current case the residue of e^{iz}/z is 1, $\alpha = \pi$ and the contour goes *clockwise* around $z = 0$, so we obtain $-i\pi$.

The indentation lemma does *not* work for poles of higher order.

A completely different approach to this entire method is to note that the singularity of $(\sin z)/z$ at the origin is removable. Having removed it, we have an analytic integrand; the original contour along the real axis can therefore be moved to one which does not pass through the origin. It is now possible to write $\sin z = (e^{iz} - e^{-iz})/2i$, split the integrand in two and apply Jordan's Lemma to each part separately in the normal way, because our contour no longer passes through a singularity.



and the inverse transform is

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \tilde{f}(k) e^{ikx}\,dk.$$

It is common for the terms e^{-ikx} and e^{ikx} to be swapped around. More rarely, factors of 2π or $\sqrt{2\pi}$ are rearranged. Traditionally, if f is a function of position x, then the transform variable is called k; while if f is a function of time t then it is called ω.

* In fact a more precise version of the inverse transform is

$$\frac{1}{2}\left(f(x_+) + f(x_-)\right) = \frac{1}{2\pi}\, \text{p.v.} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, dk .$$

The LHS indicated that at a discontinuity, the inverse transform gives the average value. The RHS shows that only the principal value of the integral (denoted $PV\int$ or $\mathcal{P}\int$ or $⨍$), i.e. $\lim_{R\to\infty} \int_{-R}^{R}$ rather than $\lim_{\substack{R\to\infty \\ S\to\infty}} \int_{-R}^{S}$. This is convenient for us in light of the semi-circle method of §4.4 and §4.6.

(Several functions have p.v. integrals but not normal ones:

e.g. $\text{p.v.} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx = 0$ but $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$ does not exist, since each of the limits $R \to \infty$ and $S \to \infty$ diverges individually so the value depends on the way in which R and S do.

For instance, $\int_{-R}^{2R} \frac{x}{1+x^2}\, dx \to \ln 2$ as $R \to \infty$, not zero.) *

Properties of the Fourier transform

(i) linearity: $\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$.

The Fourier transform can be denoted by $\tilde{f} = \mathcal{F}(f)$ or $\tilde{f}(k) = \mathcal{F}(f)(k)$

In an abuse of notation, we often write $\tilde{f}(k) = \mathcal{F}(f(x))$.

(ii) translation: $\mathcal{F}(f(x-x_0)) = e^{-ikx_0} \tilde{f}(k)$

(iii) scaling: $\mathcal{F}(f(\lambda x)) = \frac{1}{|\lambda|} \tilde{f}\left(\frac{k}{\lambda}\right)$

(iv) shifting: $\mathcal{F}(e^{ik_0 x} f(x)) = \tilde{f}(k-k_0)$

(v) transform of a derivative: $\mathcal{F}(f'(x)) = ik\tilde{f}(k)$.

More generally, $\mathcal{F}(f^{(n)}(x)) = (ik)^n \tilde{f}(k)$.

(vi) derivative of a transform: $\tilde{f}'(k) = -i\, \mathcal{F}(x f(x))$.

More generally, $\tilde{f}^{(n)}(k) = (-i)^n \mathcal{F}(x^n f(x))$.

(vii) Parseval's identity: $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk$.

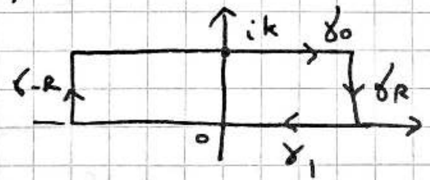
(viii) Convolution: if $h = f * g$, that is $h(x) = \int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx'$, then $\tilde{h}(k) = \tilde{f}(k)\, \tilde{g}(k)$.

§5.2 Calculating and inverting Fourier transforms using the calculus of residues

Examples: (i) When inverting Fourier transforms, we generally use a semicircle contour (in the upper half-plane if $x>0$, lower if $x<0$) and apply Jordan's Lemma: see the worked example.

(ii) If $f(x) = e^{-x^2/2}$ then

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ikx}\,dx$$
$$= \int_{-\infty}^{\infty} e^{-(x+ik)^2/2}\, e^{-k^2/2}\,dx$$
$$= e^{-k^2/2} \int_{\gamma_0} e^{-z^2/2}\,dz \qquad (z = x+ik)$$



where γ_0 is the contour shown, running along the line $\operatorname{Im} z = k$, in the limit $R \to \infty$.

Now $\int_{R} \to 0$ and $\int_{-R} \to 0$ (both are bdd by $ke^{-(R^2-k^2)/2}$ which is $O(e^{-R^2/2})$ for fixed k), and there are no singularities, so $\int_{\gamma_0} = -\int_{\gamma_1}$ and $\tilde{f}(k) = e^{-k^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}\,e^{-k^2/2}$ using a standard result.

5.3 Laplace Transforms

The Fourier transform is a powerful tool for solving differential equations and investigating physical systems, but it has two key restrictions: firstly, many functions of interest grow exponentially and so do not have Fourier transforms; and secondly, there is no way of incorporating initial or boundary conditions on the variable being transformed. (When used to solve an ODE, the Fourier transform merely gives a *particular* solution: there are no arbitrary constants produced by the method.)

To get around these restrictions we introduce the *Laplace transform*, but we have to pay the price ~~of~~ with a different restriction: it is only defined for functions $f(t)$ that vanish for $t<0$.

Worked Example
Contour Integration: Inverse Fourier Transforms

Consider

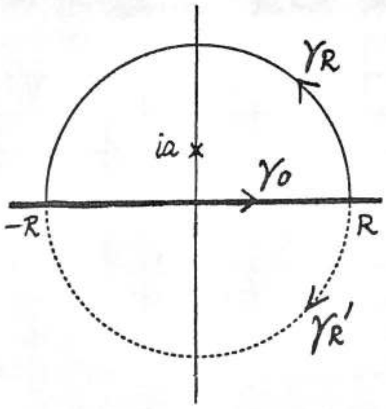
$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

where $a > 0$ is a real constant. The Fourier Transform of $f(x)$ is

$$\begin{aligned} \widetilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x \\ &= \int_{0}^{\infty} e^{-ax-ikx}\,\mathrm{d}x \\ &= -\frac{1}{a+ik}\left[e^{-ax-ikx}\right]_0^\infty \\ &= \frac{1}{a+ik}. \end{aligned}$$

We shall verify the Inverse Fourier Transform by evaluating

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k.$$



In the complex k-plane, let γ_0 be the contour from $-R$ to R on the real axis, γ_R be the semicircle of radius R in the upper half-plane and γ_R' be the semicircle of radius R in the lower half-plane. Let γ be γ_0 followed by γ_R and let γ' be γ_0 followed by γ_R'.

Now $\widetilde{f}(k)$ has only one pole, at $k = ia$, which is simple, so

$$\oint_\gamma \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = 2\pi i \operatorname*{res}_{k=ia} \frac{e^{ikx}}{i(k-ia)} = 2\pi e^{-ax},$$

whereas

$$\oint_{\gamma'} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = 0.$$

(Note that γ' is traversed in a negative sense, so if there had been any poles within γ' we would have had to introduce a minus sign.)

Now, if $x > 0$, we can apply Jordan's Lemma (with $\lambda = x$) to γ_R to show that $\int_{\gamma_R} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k \to 0$ as $R \to \infty$, since $\widetilde{f}(k) = O(1/k)$ as $|k| \to \infty$. Hence for $x > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = \frac{1}{2\pi}\lim_{R\to\infty}\oint_\gamma \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = e^{-ax}.$$

For $x < 0$ we close in the lower half-plane instead and use γ':

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\gamma'} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = 0.$$

Hence, combining the above results, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{f}(k)e^{ikx}\,\mathrm{d}k = \begin{cases} 0 & x<0 \\ e^{-ax} & x>0 \end{cases}$$

as expected.

Note that by taking real and imaginary parts of this equality we can deduce the values of particular real integrals. The imaginary part gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a\sin kx - k\cos kx}{a^2+k^2}\,\mathrm{d}k = 0,$$

which is obvious anyway as the integrand is an odd function of k. But the real part gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a\cos kx + k\sin kx}{a^2+k^2}\,\mathrm{d}k = \begin{cases} 0 & x<0 \\ e^{-ax} & x>0 \end{cases}$$

and in particular

$$\int_{-\infty}^{\infty}\frac{a\cos\theta + \theta\sin\theta}{a^2+\theta^2}\,\mathrm{d}\theta = 2\pi e^{-a}.$$

From now on we shall make this assumption, so that if we refer to the function $f(t) = e^{-t}$ for instance, we really mean $f(t) = e^{-t} H(t)$ where H is the Heaviside function.

The Laplace transform of such a function is defined by

$$\hat{f}(s) = \int_0^\infty f(t) e^{-st}\, dt.$$

It exists for functions that grow no more than exponentially fast as $t \to \infty$. Many functions (e.g. t and e^t) which do not have Fourier transforms do have Laplace transforms.

The notation $\hat{f} = \mathcal{L}(f)$ or $\hat{f}(s) = \mathcal{L}(f(t))$ is also used, and the symbol p is sometimes used instead of ~~f~~ s. The notation $\hat{f}$ is not standard; in fact there is no standard.

* Note that $\hat{f}(s) = \tilde{f}(-is)$ provided that both transforms exist. *

Examples: (i) $\mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$

(ii) $\mathcal{L}(t) = \frac{1}{s^2}$ by integration by parts

(iii) $\mathcal{L}(e^{\lambda t}) = \int_0^\infty e^{(\lambda - s)t}\, dt = \frac{1}{s-\lambda}$

(iv) $\mathcal{L}(\sin t) = \mathcal{L}\left(\frac{1}{2i}(e^{it} - e^{-it})\right)$

$$= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) \quad \text{by (iii)}$$

$$= \frac{1}{s^2+1}.$$

* Note that the integral only converges if Re s is sufficiently large; for instance in (iii) we require Re $s >$ Re λ. In fact we often implicitly assume s is real as well when calculating $\hat{f}(s)$. However, once we have calculated $\hat{f}$ for this range of s we can consider it to exist everywhere in the complex plane, except at singularities (such as at $s = \lambda$ in this example), using analytic continuation, as described in § 1.2. *

It is useful to build up a "library" of Laplace transforms:

$f(t)$	$\hat{f}(s)$	$f(t)$	$\hat{f}(s)$
1	$1/s$	$\sin \omega t$	$\omega / s^2+\omega^2$
$e^{\lambda t}$	$1/s-\lambda$	$\cos \omega t$	$s / s^2+\omega^2$
t^n	$n!/s^{n+1}$	$\sinh \lambda t$	$\lambda / s^2-\lambda^2$
$t^n e^{\lambda t}$	$n!/(s-\lambda)^{n+1}$	$\cosh \lambda t$	$s / s^2-\lambda^2$
$\delta(t-t_0)$	$e^{-t_0 s}$		

5.4 Properties of the Laplace transform

The first 4 properties are easily proved by substitution.

(i) Linearity: $\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$

(ii) Translation: $\mathcal{L}(f(t-t_0) H(t-t_0)) = e^{-t_0 s} \hat{f}(s)$

(iii) Scaling: $\mathcal{L}(f(\lambda t)) = \frac{1}{\lambda} \hat{f}(\frac{s}{\lambda})$ for $\lambda > 0$ (so $f(\lambda t) = 0$ for $t < 0$)

(iv) Shifting: $\mathcal{L}(e^{s_0 t} f(t)) = \hat{f}(s - s_0)$

(v) Transform of a derivative: $\mathcal{L}(f'(t)) = s\hat{f}(s) - f(0)$.

<u>Proof</u>: $\int_0^\infty f'(t) e^{-st}\, dt = [f(t) e^{-st}]_0^\infty + s\int_0^\infty f(t) e^{-st}\, dt$

$$= s\hat{f}(s) - f(0)$$

Repeating the process,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s^2 \hat{f}(s) - sf(0) - f'(0)$$

and so on.

This is the key fact needed for solving ODEs using Laplace transforms.

(vi) Derivative of a transform $\hat{f}'(s) = \mathcal{L}(-tf(t))$

<u>Proof</u>: $\hat{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$

$$\Rightarrow \hat{f}'(s) = -\int_0^\infty t f(t) e^{-st}\,dt = \mathcal{L}(-tf(t)).$$

More generally, $\hat{f}^{(n)}(s) = \mathcal{L}\left((-t)^n f(t)\right)$.

So, for example,

$$\mathcal{L}(t\sin t) = -\frac{d}{ds}\frac{1}{s^2+1} \quad (\text{from } \S 5.3\,(\text{iv}))$$
$$= \frac{2s}{(s^2+1)^2}$$

(vii) Asymptotic limits: as $s \to \infty$, $s\hat{f}(s) \to f(0)$.

Similarly, as $s \to 0$, $s\hat{f}(s) \to f(\infty)$, so long as $f(\infty)$ exists and is finite (and this is not a mere technicality).

<u>Proof</u>: from (v) above, $s\hat{f}(s) = f(0) + \int_0^\infty f'(t)e^{-st}\,dt$

As $s \to \infty$, $e^{-st} \to 0\ \forall t$, so $s\hat{f}(s) \to f(0)$ (since f' grows no more than exponentially fast.) Similarly, as $s \to 0$, $e^{-st} \to 1$ so

$$s\hat{f}(s) \to f(0) + \int_0^\infty f'(t)\,dt = f(\infty).$$

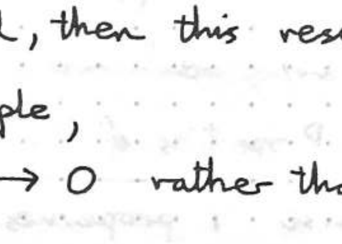
Note that if $f(\infty)$ is not finite or is undefined, then this result may give positively misleading answers. For example,

$$\mathcal{L}(e^t) = \frac{1}{s-1} \text{ but as } s \to 0,\ \frac{s}{s-1} \to 0 \text{ rather than } \infty.$$

§5.5 The Inverse Laplace Transform

Given $\hat{f}(s)$ we can calculate $f(t)$ using the <u>Bromwich inversion formula</u>

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)e^{st}\,ds$$



(proof at the end of this section).

Here c is a real constant and the Bromwich inversion contour runs along the vertical line $\operatorname{Re} s = c$.

Γ <u>must</u> lie to the right of all the singularities of $\hat{f}(s)$.

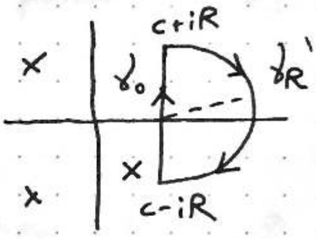
L15.3

In the case that $\hat{f}(s)$ has only a finite number of singularities s_k, $k=1,\dots,n$, and $\hat{f}(s)\to 0$ as $|s|\to\infty$,

$$f(t) = \sum_{k=1}^{n} \operatorname*{res}_{s=s_k} (\hat{f}(s)e^{st})$$

for $t>0$, and $f(t)$ vanishes for $t<0$.

Note that this result <u>does not hold</u> if $\hat{f}(s) \not\to 0$ at ∞ (see example (iii) below.



<u>Proof</u>: (a) When $t<0$, consider the contour $\gamma' = \gamma_0 + \gamma_R'$ shown, which encloses no singularities. If $\hat{f}(s) = o(\frac{1}{s})$ as $|s|\to\infty$, then $\left|\int_{\gamma_R'} \hat{f}(s)e^{st}\,ds\right| \leq \pi R e^{ct} \sup_{s\in\gamma_R'} |\hat{f}(s)|$

$$\to 0 \text{ as } R\to\infty.$$

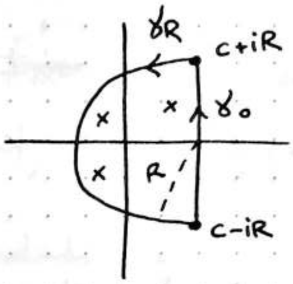
(Here we have used $|e^{st}| \leq e^{ct}$ which arises from $\mathrm{Re}(st)\leq ct$ on γ_R', noting that $t<0$). If $\hat{f}(s)$ decays less rapidly at ∞, but still tends to zero there, then the same result still holds by a slight modification of Jordan's Lemma. So in either case, $\int_{\gamma_R'} \to 0$.

Also, $\int_{\gamma_0} \to \int_{\Gamma}$. By Cauchy's theorem, $f(t)=0$ for $t<0$.

(As it must do for any function with a Laplace transform, this explains why Γ must lie to the right of the singularities.

L16.1

(b) When $t > 0$, we close the contour to the left instead, and let $\gamma = \gamma_0 + \gamma_R$ as shown.



Once again, we can show that

$$\int_{\gamma_R} \to 0 \text{ as } R \to \infty.$$

Hence by the residue theorem,

$$\int_{\Gamma} \hat{f}(s) e^{st}\, ds = \lim_{R\to\infty} \int_{\gamma_0} (\cdot)\, ds$$
$$= \lim_{R\to\infty} \int_{\gamma} (\cdot)\, ds$$
$$= 2\pi i \hat{\sum_{k=1}} \operatorname*{res}_{s=s_k} (\hat{f}(s) e^{st})$$

The result follows from the Bromwich inversion formula.

Examples: (i) $\hat{f}(s) = \frac{1}{s-1}$ has a pole at $s=1$, so we must use $c > 1$. We have $\hat{f}(s) \to 0$ as $|s| \to \infty$, so Jordan's lemma applies as above. Hence $f(t) = 0$ for $t < 0$ and, for $t > 0$,

$$f(t) = \operatorname*{res}_{s=1}\left(\frac{e^{st}}{s-1}\right) = e^t$$

(ii) $\hat{f}(s) = s^{-n}$ has a pole of order n at $s=0$, so $c > 0$, and $\hat{f}(s) \to 0$ as $|s| \to \infty$. Hence for $t > 0$,

$$f(t) = \operatorname*{res}_{s=0}\left(\frac{e^{st}}{s^n}\right) = \lim_{s\to 0}\left\{\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}} e^{st}\right\} = \frac{t^{n-1}}{(n-1)!}$$

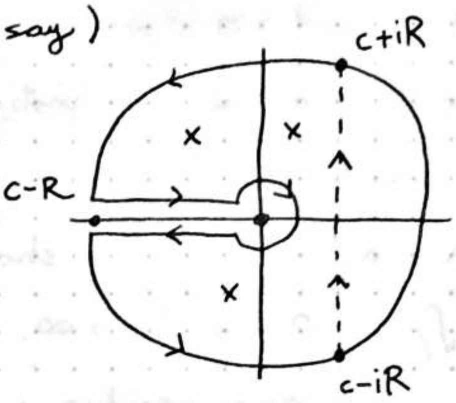
(iii) In the case $\hat{f}(s) = e^{-s}s^{-n}$, we cannot use the standard result above since $\hat{f}(s) \not\to 0$ as $s \to |\infty|$ in the left-hand half-plane. But we can write

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma} \frac{e^{-s}}{s} e^{st}\, ds \quad \text{for } n = 1,$$
$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{s} e^{st'}\, ds \quad \text{where } t' = t-1.$$

Now we can close to the right when $t' < 0$ and to the left when $t' > 0$, picking up the residue from the pole at $s=0$. Hence

$$f(t) = \begin{cases} 0 & t' < 0 \\ 1 & t' > 0 \end{cases} = \begin{cases} 0 & t < 1 \\ 1 & t > 1 \end{cases} = H(t-1).$$

(iv) If $\hat{f}(s)$ has a branch point (at $s=0$ say) then we must use a Bromwich keyhole contour as shown.



* Derivation of inverse Laplace transform *

Since f has a Laplace transform, it grows no more than exponentially fast; hence $\exists\, c \in \mathbb{R}$ such that $g(t) = f(t)e^{-ct}$ decays at ∞ (zero for $t<0$).

So g has a Fourier transform

$$\begin{aligned}\tilde{g}(\omega) &= \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt \\ &= \int_{-\infty}^{\infty} f(t)\, e^{-ct} e^{-i\omega t}\, dt \\ &= \hat{f}(c+i\omega).\end{aligned}$$

Then $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(c+i\omega)\, e^{i\omega t}\, d\omega$

$$\Rightarrow f(t)\, e^{-ct} = \frac{1}{2\pi i}\int_{c-i\omega}^{c+i\omega} \hat{f}(s)\, e^{(s-c)t}\, ds \qquad (\text{subs } s = c+i\omega).$$

The result follows.

5.6 The Convolution Theorem for Laplace transforms

The convolution of two functions f and g,

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-t')\, g(t')\, dt'$$

simplifies when f, g vanish for negative time to

$$(f*g)(t) = \int_0^t f(t-t')\, g(t')\, dt'.$$

The convolution theorem states that $\widehat{f*g}(s) = \hat{f}(s)\,\hat{g}(s)$, just as for Fourier transforms.

Proof:

$$\begin{aligned}\widehat{f*g}(s) &= \int_0^\infty \int_0^t f(t')\, g(t-t')\, dt'\, e^{-st}\, dt \\ &= \int_0^\infty \left\{ \int_0^t f(t-t')\, g(t')\, e^{-st}\, dt' \right\} dt \\ &= \int_0^\infty \left\{ \int_{t'}^\infty f(t-t')\, g(t')\, e^{-st}\, dt \right\} dt' \\ &= \int_0^\infty \left\{ \int_0^\infty f(u)\, g(t')\, e^{-su} e^{-st'}\, du \right\} dt' \qquad (u = t-t') \\ &= \int_0^\infty \left\{ \int_0^\infty f(u)\, e^{-su}\, du \right\} g(t')\, e^{-st'}\, dt' \\ &= \hat{f}(s)\, \hat{g}(s), \quad \text{as desired.}\end{aligned}$$

Example from §5.3, $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$. So if $f(t) = t$, then

$\hat{f}(s) = \frac{1}{s^2}$ and $\widehat{f*f}(s) = \frac{1}{s^4}$, whence $(f*f)(t) = \frac{1}{6}t^3$.

This is easily verified by direct calculation:

$$(f*f)(t) = \int_0^t (t-t')t'\,dt'$$
$$= \frac{1}{6}t^3$$

5.7 Solution of Differential Equations using Laplace transforms

Examples: (i) The Laplace transform converts constant coeff ODEs to algebraic equations (and PDEs to ODEs): see the worked example.

(ii) Laguerre's equation of order n,

$$t\ddot{y} + (1-t)\dot{y} + ny = 0$$

has a solution that is well-behaved at $t=0$ (the other has a logarithmic singularity). We seek the finite solution with $y(0)=1$.

Now
$$\mathcal{L}(t\dot{y}) = -\frac{d}{ds}\mathcal{L}(\dot{y}) = -\frac{d}{ds}(s\hat{y} - y(0))$$
$$= -s\hat{y}' - \hat{y}$$

Similarly $\mathcal{L}(t\ddot{y}) = -s^2\hat{y}' - 2s\hat{y} + y(0)$.

Hence on taking the Laplace transform we obtain (after simplif.)

$$s(s-1)\hat{y}' = (n+1-s)\hat{y},$$

which is a first-order ODE for $\hat{y}(s)$ solved as

$$\hat{y} = A\frac{(s-1)^n}{(s^{n+1})}$$ where A is arbitrary constant.

But $y(0) = \lim_{s\to\infty} s\hat{y}(s) = A$, so $A=1$. Hence

$$\hat{y} = \frac{(s-1)^n}{s^{n+1}} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k s^{-(k+1)}$$

$$\Rightarrow y = \sum_{k=0}^{n}\binom{n}{k}(-t)^k/k!$$ a cool Polynomial

Worked Example
Solving Differential Equations using the Laplace Transform and its Inverse

We shall solve

$$\ddot{x} + x = 2\sin t$$

for $x(t)$, with initial conditions $x(0) = 0$, $\dot{x}(0) = 2$. Taking the Laplace transform with respect to time,

$$\left(s^2\widehat{x}(s) - sx(0) - \dot{x}(0)\right) + \widehat{x}(s) = \frac{2}{s^2+1}.$$

Using the initial conditions, we obtain

$$s^2\widehat{x} - 2 + \widehat{x} = \frac{2}{s^2+1}$$

from which we deduce that

$$\widehat{x} = \frac{2s^2+4}{(s^2+1)^2}.$$

To invert this we write down the Bromwich inversion formula

$$x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2s^2+4}{(s^2+1)^2}\, e^{st}\,\mathrm{d}s.$$

The integrand has poles of order two at $s = \pm i$, so we must have $c > 0$ in order that the integration contour lies to the right of the singularities.

We next check whether $\widehat{x}(s) \to 0$ as $|s| \to \infty$. Clearly it does: $\widehat{x}(s) = O(|s|^{-2})$. That being the case, we may close the integration contour to the left when $t > 0$ and use the standard result involving the sum of the residues at the poles.

At $s = i$, the residue is

$$\begin{aligned}\lim_{s\to i}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2s^2+4}{(s+i)^2}\,e^{st}\right) &= \lim_{s\to i}\left(\frac{(s+i)\left(4s+(2s^2+4)t\right)-2(2s^2+4)}{(s+i)^3}\,e^{st}\right)\\ &= -\tfrac{1}{2}(t+3i)e^{it}.\end{aligned}$$

Similarly, at $s = -i$ the residue is $-\frac{1}{2}(t-3i)e^{-it}$.

Combining these results we obtain

$$\begin{aligned}x(t) &= -\tfrac{1}{2}(t+3i)e^{it} - \tfrac{1}{2}(t-3i)e^{-it}\\ &= -\tfrac{1}{2}\left(2t\cos t + 3i(2i\sin t)\right)\\ &= 3\sin t - t\cos t.\end{aligned}$$