L1.1

ELECTROMAGNETISM

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Lecture Notes: www. damtp. cam.ac.uk / user / hsr1000 (also Tong)

Example Sheets: www. damtp. cam. ac. uk / user / examples (1's up)

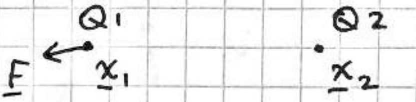
Books: * Zangwill ; Jackson ; Purcell & Morin

1. FUNDAMENTALS

1.1 Electric charge

Expt => all objects possess a property called electric charge Q s.t. ∃ force between 2 bodies at rest, given by Coulomb's law

$$\underset{\substack{\uparrow \\ \text{force on 1} \\ \text{due to 2}}}{\underline{F}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \cdot \frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|^3}$$



(inverse square law)

Q measured in units called Coulombs (C)

$\varepsilon_0 = 8.85 \cdots \times 10^{-12} \, m^{-3} kg^{-1} s^2 C^2$

Q can be either positive or negative

Opposite charges attract, like charges repel

Expt -> charge is conserved, it can be transferred from one body to another but total charge unchanged.

All particles have charges that are integer multiples of

$e = 1.6 \ldots \times 10^{-19} C$ (oh!) $= 1.602176634 \times 10^{-19} C$

proton $+e$ electron $-e$ neutron 0

2 protons : $$\frac{F_{colomb}}{F_{grav}} = \frac{e^2}{4\pi\varepsilon_0 \underset{\substack{\uparrow \\ \text{grav const}}}{G} m^2 \leftarrow \text{proton mass}} \simeq 10^{36}$$

Coulomb force much stronger than gravity

1.2 Charge and current density

On macroscopic scales (large compared to the size of atoms) treat charge as a continuous distribution.

Charge density $\rho(t, \underline{x})$: scalar field such that total charge in volume dV centred on $\underline{x}$ is $\rho(t, \underline{x})\, dV$

Current density $\underline{J}(t, \underline{x})$: vector field such that total charge crossing surface element $d\underline{S}$ in time dt is $(\underline{J} \cdot d\underline{S})\, dt$

Example plasma with N types of charged particle

$n_i(t, \underline{x})\, dV$ particles of type i in dV

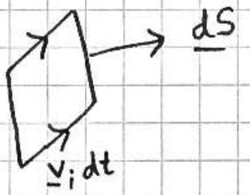
number density

total charge in dV is $\sum_{i=1}^{N} q_i n_i \, dV$

charge of particle i

$\therefore \rho = \sum_i q_i n_i$

assume particles of type i have velocity $\underline{v}_i(t, \underline{x})$



in time dt, all such particles in volume $(\underline{v}_i dt \cdot d\underline{S})$ cross $d\underline{S}$, carry charge $q_i n_i$

$\Rightarrow \underline{J} = \sum_{i=1}^{N} q_i n_i \underline{v}_i$

S: finite surface. Electric current across S $\quad I = \int_S \underline{J} \cdot d\underline{S}$

units : Ampères (A), $1A = 1 C s^{-1}$

V: time-indep volume with boundary S

Charge in V is $Q(t) = \int_V \rho(t, \underline{x})\, dV$

Charge crossing S in time dt is $I\, dt$

Charge conservation $\Rightarrow$ $\underbrace{-\frac{dQ}{dt} dt}_{\text{decrease of } Q \text{ in time } dt} = I\, dt$

$$\therefore \frac{dQ}{dt} = -I = -\int_S \underline{J} \cdot d\underline{S} \qquad (*)$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \underline{\nabla} \cdot \underline{J}\, dV \qquad (\text{div thm})$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} \right) dV = 0$$

V arbitrary $\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0}$ $\quad (\dagger)$

$$\delta^{(3)}(\underline{x}) = \delta(x)\,\delta(y)\,\delta(z)$$

particle of charge q at pos^n $\underline{x}(t)$ gives $\rho(t,\underline{x}) = q\,\delta^{(3)}(\underline{x} - \underline{x}(t))$

$$\underline{J}(t,\underline{x}) = q\,\dot{\underline{x}}(t)\,\delta^{(3)}(\underline{x} - \underline{x}(t))$$

Ex: check $(\dagger)$ is satisfied

N particles: $\rho = \sum_{i=1}^{N} q_i\,\delta^{(3)}(\underline{x} - \underline{x}_i(t))$

$$\underline{J} = \sum_{i=1}^{N} q_i\,\dot{\underline{x}}_i(t)\,\delta^{(3)}(\underline{x} - \underline{x}_i(t))$$

1.3 Lorentz force law

"Test body" object with small size & charge

Expt $\Rightarrow$ a test body of charge q experiences a force given by the Lorentz force law

$$\underline{F} = q\left[\,\underline{E}(t,\underline{x}) + \underset{\substack{\uparrow \\ \text{velocity of body}}}{\underline{v}} \times \underline{B}(t,\underline{x})\,\right]$$

$\underline{E}$: electric field, $\underline{B}$: magnetic field (pseudovector)

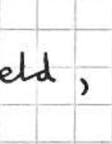
$$\sim B_i' = \det R \; R_{ij} B_j$$

e.g. Coulomb's law $\Rightarrow$ force on static test body due to static body of charge Q, at pos^n $\underline{x}_1$

$$\Rightarrow \underline{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\underline{x} - \underline{x}_1}{|\underline{x} - \underline{x}_1|^3}$$

electric field due to N static bodies $\underline{E} = \sum_{i=1}^{N} \frac{Q_i}{4\pi\varepsilon_0} \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^3}$

$\underline{v} \times \underline{B}$: force $\perp$ direction of motion

constant $\underline{B}$ 

helical motion

E2.1

Electric field lines at time t:

curves $\underline{x}(\lambda)$ with $\frac{dx}{\partial\lambda} = \underline{E}(t,\underline{x})$

View non-test body as a collection of N particles, each viewed as a test body $\Rightarrow$ total force on body given by

$$\underline{F}(t) = \sum_{i=1}^{N} q_i \left[\underline{E}(t,\underline{x}_i) + \dot{\underline{x}}_i(t) \times \underline{B}(t,\underline{x}_i) \right]$$

$$= \sum_{i=1}^{N} \int q_i \, \delta^{(3)}(\underline{x} - \underline{x}_i) \left[\underline{E} + \dot{\underline{x}}_i \times \underline{B} \right] dV$$

$$= \int_{V(t)} \left[\rho(t,\underline{x}) \, \underline{E}(t,\underline{x}) + \underline{J}(t,\underline{x}) \times \underline{B}(t,\underline{x}) \right] dV.$$

$\nwarrow$ volume of body at time t

Similarly find the torque as

$$\underline{\tau}(t) = \int_{V(t)} \underline{x} \times \left[\rho(t,\underline{x}) \, E(t,\underline{x}) + \underline{J}(t,\underline{x}) \times \underline{B}(t,\underline{x}) \right] dV.$$

1.4 Maxwell's equations

$$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0 \quad (M1) \qquad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (M3)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (M2) \qquad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \quad (M4)$$

$\mu_0 = 1.2566\ldots \times 10^{-6}\,\mathrm{NA}^{-2}$ (expt)

$\approx 4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$ (11 s.f.)

will show $c = 2.998 \times 10^{8}\,\mathrm{ms}^{-1}$ = speed of light!

(M1–4) are linear $\Rightarrow$ obey superpos^{n} principle

<u>Show:</u> $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$0 = \underline{\nabla} \cdot \underline{E} - \rho/\varepsilon_0$$

$$= \underline{\nabla} \cdot \frac{\partial}{\partial t} \underline{E} - \frac{\partial}{\partial t} \rho/\varepsilon_0$$

$$= -c^2 \mu_0 \, \underline{\nabla} \cdot \underline{J} - \frac{1}{\varepsilon_0} \frac{\partial}{\partial t} \rho$$

So for consistency with charge conservation, get result.

$$M3 \Rightarrow \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\frac{\partial}{\partial t} \underline{\nabla} \times \underline{B}$$

by VC $\quad \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) + \nabla^2 \underline{E} = -\mu_0 \frac{\partial}{\partial t} \underline{J} - \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} \quad$ by M4

$$-\frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} + \nabla^2 \underline{E} = \frac{1}{\varepsilon_0} \underline{\nabla} \rho + \mu_0 \frac{\partial \underline{J}}{\partial t}$$

L 2.2

Sim^{y} $\underline{\nabla}\times(M4)$, M2, M3 $\Rightarrow -\frac{1}{c^2}\frac{\partial^2 \underline{B}}{\partial t^2} + \nabla^2 \underline{B} = -\mu_0 \underline{\nabla}\times\underline{J}$

$\underline{E}, \underline{B}$ obey inhomogeneous <u>wave eq^{n}s</u>.

In vacuum $(\rho = \underline{J} = 0)$ homogeneous eq^{n} obeyed.

c is speed of waves $\Rightarrow \exists$ sol^{n}s describing of waves speed c

Light is an electromagnetic wave.

1.5 * Averaging *

M1 − 4 are the <u>microscopic</u> Maxwell equations, valid even for small (subatomic) distances.

Inside charged matter we average to obtain smooth ρ, $\underline{J}$, $\underline{E}$ and $\underline{B}$

$$\text{averaging} \longrightarrow \begin{cases} \text{new fields } \underline{D}, \underline{H} \\ \text{macroscopic Maxwell eq}^{n}\text{s} \end{cases}$$

This course valid for (i) micro context

(ii) vacuum (iii) averaging doesn't change much (e.g. air)

1.6 Conductors & Ohm's law

<u>Conductor</u>: a material containing "free charges" that can move in response to $\underline{E}, \underline{B}$ e.g. a solid metal: some electrons can move through lattice of ions.

<u>Insulator</u>: no free charges, i.e. a poor conductor

Conductor at rest in $\underline{E}, \underline{B}$ fields expt $\Rightarrow$ obeys Ohm's law $\boxed{\underline{J} = \overset{\substack{\text{conductivity}\\ \downarrow}}{\sigma} \underline{E}}$

σ is v large for a good conductor (e.g.) metal

v small for an insulator (e.g.) diamond

Consider a conductor occupying region V

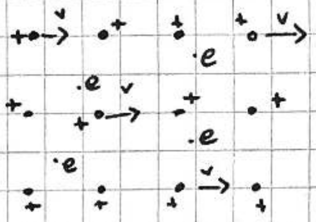
$$0 = \frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} \underset{\text{Ohm}}{=} \frac{\partial \rho}{\partial t} + \sigma\,\underline{\nabla}\cdot\underline{E} = \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon_0}\rho$$

$\Rightarrow \rho(t) = \rho(0)\, e^{-t/t_{decay}}$ with $t_{decay} = \frac{\varepsilon_0}{\sigma}$

$\therefore$ charge decays rapidly inside V, ends up on the surface

L2.3

Consider metal moving with velocity $\underline{v}$



If neutral ($\rho = 0$), Ohm's law becomes

$$\underline{J} = \sigma(\underline{E} + \underline{v} \times \underline{B})$$

2. ELECTROSTATICS

2.1 Time independence

$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0$ (1a) $\quad \underline{\nabla} \times \underline{E} = 0$ (1b) $\leftarrow$ eq^{n}s of electrostatics

$\underline{\nabla} \cdot \underline{B} = 0$ (2a) $\quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$ (2b) $\leftarrow$ eq^{n}s of magnetostatics

Strictly, electrostatics means (a) time independence,

(b) charges at rest, $\underline{J} = \underline{0}$

(c) vanishing $\underline{B}$

L3.1

(1b) $\Rightarrow$ $\underline{E} = -\underline{\nabla}\Phi$, Φ is called the scalar potential (*)

Consider a test body in this field

$$m\ddot{\underline{x}} = \underline{F} = q\underline{E} = -q\underline{\nabla}\Phi$$

$$\therefore \; 0 = m\dot{\underline{x}}\cdot\ddot{\underline{x}} + q\,\dot{\underline{x}}\cdot\underline{\nabla}\Phi = \frac{dE}{dt}$$

where $E = \frac{1}{2}m\dot{\underline{x}}^2 + q\Phi(\underline{x})$ is conserved

$\Rightarrow$ $q\Phi(\underline{x})$ is potential energy of test body

e.g. field of N static charges

$$\underline{E} = \sum_{i=1}^{N} \frac{Q_i(\underline{x}-\underline{x}_i)}{4\pi\varepsilon_0|\underline{x}-\underline{x}_i|^3} \Rightarrow \Phi = \sum_{i=1}^{N}\frac{Q_i}{4\pi\varepsilon_0|\underline{x}-\underline{x}_i|}$$

(*) defines Φ up to a constant , usually demand $\Phi \to 0$ as $\underline{x} \to \infty$, say on the whole of $\mathbb{R}^3$

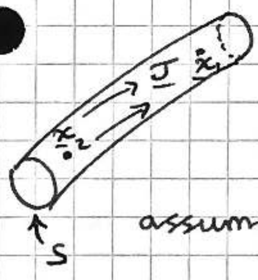
Potential <u>difference</u> unambiguous $\cdot\underline{x}_1 \quad C \quad \cdot\underline{x}_2$

$$\underbrace{\Phi(\underline{x}_2) - \Phi(\underline{x}_1)}_{V} = \int_C \underline{\nabla}\Phi\cdot d\underline{x} = -\int_C \underline{E}\cdot d\underline{x}$$

$W = qV$ is the work needed to move test body from $\underline{x}_1$ to $\underline{x}_2$

<u>Example</u> consider a wire carrying a current $\underline{J} = \sigma\underline{E}$

$\underline{x}_1, \underline{x}_2$ points in the wire s.t. $V > 0$



$$V = \int_{-C}\underline{E}\cdot d\underline{x} = \int_{-C}\frac{1}{\sigma}\underline{J}\cdot d\underline{x}$$

assume (1) length ℓ between $\underline{x}_2, \underline{x}_1$,
constant csa A
(2) constant σ
(3) $\underline{J} \approx$ const and in direction of wire

$\therefore \; V \approx \frac{|\underline{J}|\ell}{\sigma}$ Current in wire $I = \int_S \underline{J}\cdot d\underline{S} = |\underline{J}|\,A$

$\therefore \; V = IR$ where $R = \frac{\ell}{\sigma A}$ is "resistance"

L3.2

(1a) & (*) $\Rightarrow$ $\nabla^2 \Phi = - \rho/\varepsilon_0$ (Poisson eq^n)

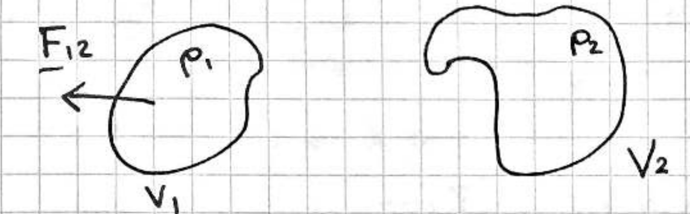
If solving on $\mathbb{R}^3$:

$$\Phi(\underline{x}) = \int \frac{\rho(\underline{x}')}{4\pi\varepsilon_0 |\underline{x}' - \underline{x}|} \, d^3\underline{x}' \qquad \left(\begin{array}{l} \text{assume } \rho \to 0 \text{ at } \infty \\ \text{impose } \Phi \to 0 \text{ at } \infty \end{array} \right)$$

$$\underline{E} = - \underline{\nabla} \Phi = - \int \frac{\rho(\underline{x}')}{4\pi\varepsilon_0} \cdot \underline{\nabla} \left(\frac{1}{|\underline{x}' - \underline{x}|} \right) d^3\underline{x}'$$

$$= \int \frac{\rho(\underline{x}')(\underline{x} - \underline{x}')}{4\pi\varepsilon_0 |\underline{x} - \underline{x}'|^3} \, d^3\underline{x}'$$

<u>Example</u> 2 static charge bodies



$$\underline{E} = \underline{E}_1 + \underline{E}_2$$

$$\underline{F}_{12} = \int_{V_1} d^3\underline{x}_1 \, \rho_1(\underline{x}_1) \, \underline{E}_2(\underline{x}_1)$$

$$= \int_{V_1} d^3\underline{x}_1 \int_{V_2} d^3\underline{x}_2 \, \frac{\rho_1(\underline{x}_1) \rho_2(\underline{x}_2)(\underline{x}_1 - \underline{x}_2)}{4\pi\varepsilon_0 |\underline{x}_1 - \underline{x}_2|^3}$$

need external force $-\underline{F}_{12}$ to maintain equilibrium

the expression for $\underline{F}_{12}$ is antisymmetric in 1, 2 : $\underline{F}_{12} = - \underline{F}_{21}$

Newton III respected

self-force $\rho_1 = \rho_2 = \rho$, $V_1 = V_2 = V$ zero by antisymmetry

2.3 Electrostatic Energy (ee)

The ee of a charge distribution is the work required to create the dist^n by bringing charge from ∞.

e.g. N static particles $\quad E = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 |\underline{x}_i - \underline{x}_j|}$

Proof : add particles one by one. 1st particle ~ no work

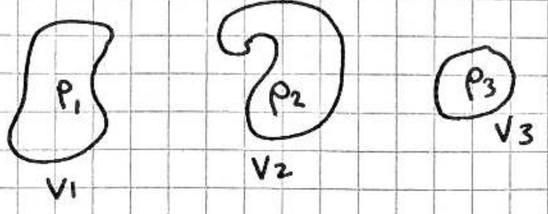
2nd particle sees field $\Phi_1(\underline{x}) = \frac{q_1}{4\pi\varepsilon_0 |\underline{x} - \underline{x}_1|}$

$\therefore$ need work $q_2 \Phi_1(\underline{x}_2) - q_2 \Phi_1(\infty)$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0 |\underline{x}_1 - \underline{x}_2|}$$

General case follows e.g. by induction.

Multiple bodies



$\rho_1 \rightarrow$ field Φ_1 by $(*)$

other bodies Φ_{ext}

View ρ_1 as a superpos^n of N particles $\rho_1(\underline{x}) = \sum_{i=1}^{N} q_i \delta^{(3)}(\underline{x} - \underline{x}_i)$.

Work done against Φ_{ext} to create ρ_1 is

$$W_{ext} = \sum_{i=1}^{N} q_i \Phi_{ext}(\underline{x}_i)$$

$$= \int d^3\underline{x} \ \rho_1(\underline{x}) \Phi_{ext}(\underline{x}).$$

Now calculate ee for arbitrary $\rho(\underline{x})$ producing field $\Phi(\underline{x})$.

Let $\rho_\lambda(\underline{x}) = \lambda \rho(\underline{x})$, $0 \leq \lambda \leq 1$.

Produces field $\Phi_\lambda(\underline{x}) = \lambda \Phi(\underline{x})$.

Increase $\lambda \rightarrow \lambda + \delta\lambda$ by bringing in $\delta\rho_\lambda(\underline{x}) = \delta\lambda \, \rho(\underline{x})$ from ∞.

View $\delta\rho_\lambda$ as a body in external field Φ_λ

$\Rightarrow$ work needed is $\delta E = \int d^3\underline{x} \ \delta\rho_\lambda(\underline{x}) \Phi_\lambda(\underline{x})$

$$= (\lambda \delta\lambda) \int d^3\underline{x} \ \rho(\underline{x}) \Phi(\underline{x})$$

Integrate wrt λ from 0 to 1 to deduce

$$E = \frac{1}{2} \int d^3\underline{x} \ \rho(\underline{x}) \Phi(\underline{x}) = \frac{1}{2} \int d^3\underline{x} \int \frac{\rho(\underline{x}) \rho(\underline{x}')}{4\pi\varepsilon_0 |\underline{x} - \underline{x}'|} d^3\underline{x}'$$

Set $\rho = \varepsilon_0(-\nabla^2\Phi)$ to get

$$E = -\frac{\varepsilon_0}{2} \int d^3\underline{x} \ \Phi(\underline{x}) \nabla^2\Phi(\underline{x})$$

$$= -\frac{\varepsilon_0}{2} \int d^3\underline{x} \left[\underbrace{\underline{\nabla} \cdot (\Phi \underline{\nabla} \Phi)}_{\substack{\text{vanishes at } \infty \\ \text{for nice } \Phi}} - (\underline{\nabla}\Phi)^2 \right]$$

$$= \int d^3\underline{x} \ \left(\frac{1}{2} \varepsilon_0 \underline{E}^2 \right)$$

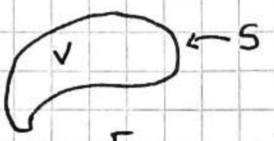
"energy in the electric field"

L4.1

2.4 Gauss' Law

M1 & div thm $\Rightarrow \underbrace{\int_S \underline{E}\cdot d\underline{S}}_{\text{electric flux across } S} = \frac{Q[V]}{\varepsilon_0}$, $\quad Q[V] = \int_V \rho\, dV$

(Gauss' law)



$\underline{E} = -\underline{\nabla}\Phi$, $\quad \int_S (\underline{\nabla}\Phi)\cdot d\underline{S} = -\frac{Q[V]}{\varepsilon_0}$

Ex 1 Spherical polars (r,θ,φ), $\rho = \rho(r)$, $\rho = 0$ for $r > R$

(*) and symmetry tell us $\Phi = \Phi(r)$

$\Rightarrow \underline{E} = E_r(r)\, \underline{e}_r$ ← radial unit vector

S = sphere of radius r

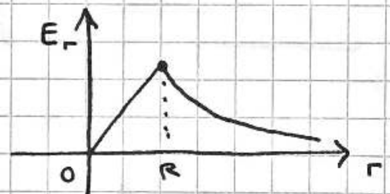
$$4\pi r^2 E_r = \frac{Q(r)}{\varepsilon_0}, \quad Q(r) = \int_0^r dr' \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \rho(r') \sin^{*}\theta\, r'^2$$
$$= 4\pi \int_0^r \rho(r')\, r'^2\, dr'$$

$$\therefore\ E_r = \frac{Q(r)}{4\pi\varepsilon_0 r^2}$$

If $r > R$, $Q(r) = Q \equiv Q(R)$ = total charge

If $r < R$, assume ρ = const. so $Q(r) = \frac{4}{3}\pi r^3 \rho = Q\left(\frac{r}{R}\right)^3$

$$\therefore\ E_r = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}$$



electrostatic energy $\quad \mathcal{E} = \int d^3x\, \frac{\varepsilon_0}{2}\underline{E}^2 = \frac{3Q^2}{20\pi\varepsilon_0 R}$ (const. ρ)

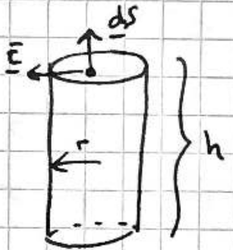
Ex 2 Cylindrical polars (r,φ,z), $\rho = \rho(r)$, $\rho = 0$ for $r > R$

Assume $\Phi = \Phi(r) \Rightarrow \underline{E} = E_r(r)\,\underline{e}_r$

Gauss' law for a cilyinder radius r, height h

$\underline{E}\cdot d\underline{S} = 0$ on the end caps

So LHS $= 2\pi r h E_r = \frac{Q(r,h)}{\varepsilon_0}$

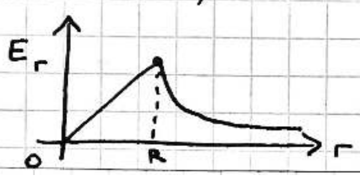


$$Q(r,h) = \int_0^r dr' \int_0^h dz \int_0^{2\pi} d\varphi\, \rho(r')\cdot r' = 2\pi h \int_0^r \rho(r')\, r'\, dr' = h\lambda(r)$$

$r > R$ gives $\lambda(r) = \lambda = \lambda(R)$ = total charge per unit length

$E_r = \frac{\lambda(r)}{2\pi\varepsilon_0 r}$, for $r < R$, assuming ρ = const., $\lambda(r) = \pi r^2 \rho$
$= \lambda\left(\frac{r}{R}\right)^2$

$$\therefore\ E_r = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$



L4.2

$E_r = -\Phi'(r) \Rightarrow \Phi(r) = \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{r}{R}\right) + \Phi_0 \Leftarrow$ const. of integration $(r>R)$

$\Phi \to \infty$ as $r \to \infty$

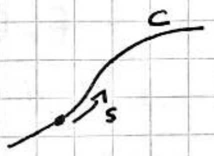
No natural choice for Φ_0

ee $E = \frac{\varepsilon_0}{2}\int dV\, \underline{E}^2 = \infty$ (duh)

ee / unit length ALSO INFINITY

Let $R \to 0$, λ fixed, gives line charge along z axis

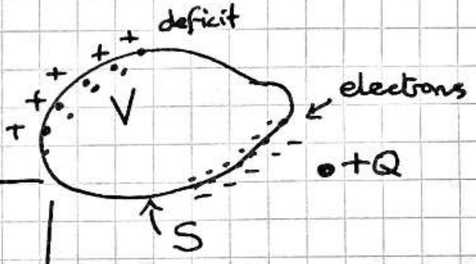
$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad \rho(\underline{x}) = \delta(x)\,\delta(y)\,\lambda$$



More generally, line charge on curve C $\lambda \to \lambda(s)$

charge between $s_1, s_2 = \int_{s_1}^{s_2} \lambda(s)\,ds$

2.5 Conductors & surface charges



Conductor in V, $\underline{J} = \sigma \underline{E}$ (Ohm)

Electrostatics $\Rightarrow \underline{J} = \underline{0}$ so $\boxed{\begin{aligned}&\underline{E} = \underline{0} \text{ in } V\\ &\Phi = \text{const. in } V\end{aligned}}$

$M_1 \Rightarrow \rho = 0$ in V

Charge on the surface S of V

Surface charge density: $\sigma: S \to \mathbb{R}$

s.t. charge on a surface element dS is $\sigma\, dS$

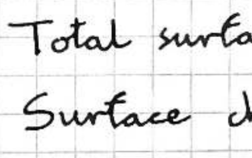
Total surface charge is $Q_{sur} = \int_S \sigma\, dS$

Surface charge $\leftrightarrow$ charge density $\rho \sim \delta$ f^n on S

e.g. $\rho(x,y,z) = \sigma(x,y)\,\delta(z-z_0)$ surface charge in plane $z = z_0$

δ f^n in $\sigma \underset{M1}{\Rightarrow} \underline{E}$ discontinuous at S

We'll determine this discontinuity in general

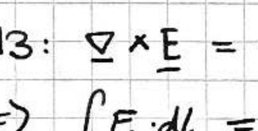


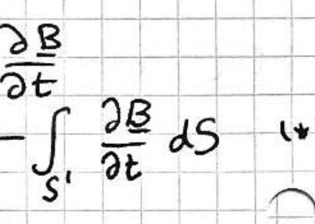
$\underline{E} = \begin{cases} \underline{E}_+ & \text{just above} \\ \underline{E}_- & \text{just below}\end{cases}$

M3: $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

$\underset{\text{Stokes}}{\Rightarrow} \int_C \underline{E} \cdot d\underline{l} = -\int_{S'} \frac{\partial \underline{B}}{\partial t}\, dS$ (*)

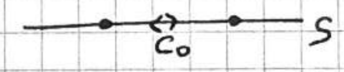
S': rectangle small compared to S





L4.3

Let $h \to 0$. $\quad C_+, C_- \longrightarrow \pm C_0$



RHS of $(*) \longrightarrow 0$ $\quad$ (if $\frac{\partial B}{\partial t}$ bdd)

$$\therefore \int_{C_0} (\underline{E}_+ - \underline{E}_-) \cdot \underline{ds} = 0$$

Take L small enough that integrand ≈ const.

$$\Rightarrow \quad (\underline{E}_+ - \underline{E}_-) \cdot \underset{\uparrow}{\underline{t}} = 0 \qquad (\dagger)$$

tangent to C_0

C_0 arbitrary $\Rightarrow$ $(\dagger)$ holds $\forall\, \underline{t}$ tangent to S

$\therefore$ tangential components of $\underline{E}$ are cts across S

δS = small region of S $\qquad$ Translate by ε up / down to yield a pillbox region δV

top δS_+
bottom δS_-

Apply Gauss' law to yield

boundary $\delta S_+ \cup \delta S_- \cup \Sigma$

$$\int_{\delta S_+ \cup \delta S_- \cup \Sigma} \underline{E} \cdot \underline{dS} = \frac{1}{\varepsilon_0} \int_{\delta S} \sigma \, dS$$

(vol charge negligible in limit $\varepsilon \to 0$)

recall,
$\underline{E}_+$, $\underline{E}$

Let $\varepsilon \to 0$ to obtain

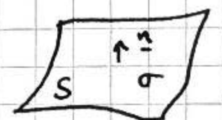
$$\int_{\delta S} (\underline{E}_+ - \underline{E}_-) \cdot \underline{dS} = \frac{1}{\varepsilon_0} \int_{\delta S} \sigma \, dS$$

$\therefore (\underline{E}_+ - \underline{E}_-) \cdot \underline{n} = \frac{\sigma}{\varepsilon_0}$ $\quad$ gives discontinuity in normal component

Hence $\underline{E}_+ - \underline{E}_- = \frac{\sigma}{\varepsilon_0} \underline{n}$ where $\underline{n}$ points from − to +

E5.1 (oops)

Recall discontinuity at a surface charge $\boxed{\underline{E}_+ - \underline{E}_- = \frac{\sigma}{\varepsilon_0}\underline{n}}$

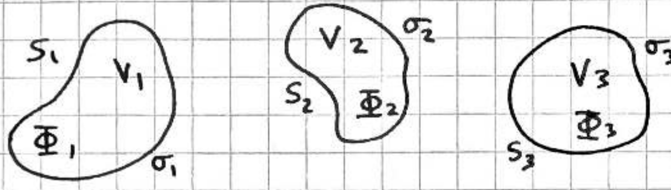


S = surface of conductor

In electrostatics, $\underline{E}_- = \underline{0}$

$\therefore \underline{E}_+ = \frac{\sigma}{\varepsilon_0}\underline{n}$ $\underline{E} = -\underline{\nabla}\Phi \Rightarrow \Phi$ cts across S ($\underline{E}$ has no δ)

$\therefore$ S is a surface of constant Φ (equipotential)



conductors at fixed potentials

Charge Q_i flows to i^{th}, remove batteries giving Φ

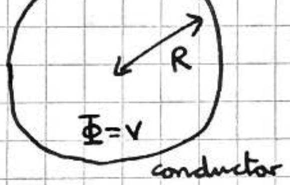
Solve $\nabla^2\Phi = 0$ outside S_i with the bcs $\Phi|_{S_i} = \Phi_i$

(Dirichlet problem for Laplace's eq^n)

Then $\frac{\sigma_i}{\varepsilon_0} = \underline{n}\cdot \underline{E}_+|_{S_i} = \underline{n}\cdot\underline{\nabla}\Phi|_{S_i}$,

and $Q = \int_{S_i} \sigma_i \, dS$.

Ex ①



$r > R$, $\nabla^2\Phi = 0$ assume Φ spherically symm

$\therefore \Phi = A + \frac{B}{r}$ $\Phi \to 0$ at $\infty \Rightarrow A = 0$

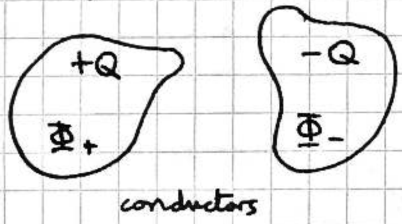
$\Phi|_{r=R} = V \Rightarrow B = VR$

$$\therefore \Phi = \begin{cases} VR/r & : r > R, \\ V & : r < R. \end{cases}$$

$$\underline{E} = -\underline{\nabla}\Phi = \begin{cases} VR/r^2 \, \underline{e}_r & : r > R, \\ 0 & : r < R. \end{cases}$$

$\underline{E}_+ = \frac{V}{R}\underline{e}_r = \frac{V}{R}\underline{n} \Rightarrow \sigma = \frac{\varepsilon_0 V}{R}$, $Q = 4\pi R \varepsilon_0 V$

② Capacitor



$V = \Phi_+ - \Phi_- > 0$ (assumption)

Linearity $\Rightarrow Q = CV$

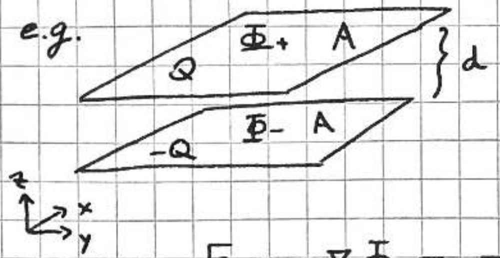
for some constant C – the capacitance depending only on the geometry

If $Q \to Q + \delta Q$, work done is $\delta Q(\Phi_+) + (-\delta Q)\Phi_- = V\delta Q$

So $\delta E = V\delta Q = \frac{1}{C}Q\,\delta Q \Rightarrow E = \frac{1}{2C}Q^2$ integrating

L5.2

e.g.

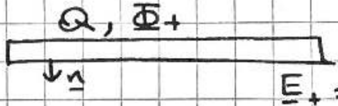


Assume $d \ll \sqrt{A}$, so $\Phi = \Phi(z)$ between plates.

$$\nabla^2 \Phi = 0 \Rightarrow \Phi = A + Bz$$

$$\therefore \quad V = Bd$$

$$\underline{E} = -\underline{\nabla}\Phi = -\frac{V}{d}\hat{k}$$ between plates



upper plate

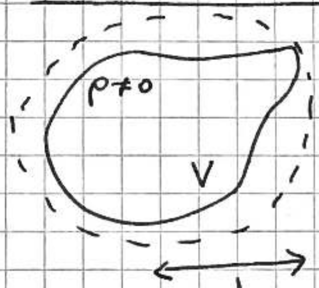
$\underline{E}_+ = -\frac{V}{d}\hat{k} = \frac{V}{d}\underline{n}$

$$\therefore \sigma = \frac{\varepsilon_0 V}{d}, \quad Q = \underbrace{\frac{\varepsilon_0 A}{d}}_{C \ \text{(woo!)}} V$$

$$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)V^2$$

$$= \int \tfrac{1}{2}\varepsilon_0 |\underline{E}|^2 \, dV = Ad \cdot \tfrac{1}{2}\varepsilon_0 |\underline{E}|^2$$

2.6 Electric dipoles & Multipole expansion



$$\Phi(\underline{x}) = \int_V d^3\underline{x}' \, \frac{\rho(\underline{x}')}{4\pi\varepsilon_0 |\underline{x} - \underline{x}'|}$$

Assume $|\underline{x}| \gg d$.

$$|\underline{x} - \underline{x}'|^{-1} = \left((\underline{x} - \underline{x}')^2\right)^{-\frac{1}{2}} = \left(\underline{x}\cdot\underline{x} - 2\underline{x}\cdot\underline{x}' + \underline{x}'\cdot\underline{x}'\right)^{-\frac{1}{2}}$$

$$= \frac{1}{|\underline{x}|}\left(1 - \frac{2\,\underline{x}\cdot\underline{x}'}{|\underline{x}|^2} + \mathcal{O}\left(\frac{d^2}{|\underline{x}|^2}\right)\right)^{-\frac{1}{2}}$$ Let $\hat{\underline{x}} = \frac{\underline{x}}{|\underline{x}|}$.

$$= \frac{1}{|\underline{x}|} + \frac{\hat{\underline{x}}\cdot\underline{x}'}{|\underline{x}|^2} + \mathcal{O}\left(\frac{d^2}{|\underline{x}|^3}\right) \qquad (*)$$

$$\therefore \Phi(\underline{x}) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\underline{x}|} + \frac{\hat{\underline{x}}\cdot\underline{p}}{|\underline{x}|^2} + \mathcal{O}\left(\frac{1}{|\underline{x}|^3}\right)\right)$$

$$Q = \int_V \rho \, dV, \quad \underline{p} = \int_V \underline{x}'\rho(\underline{x}') \, dV$$ dipole moment of charge distⁿ

Next term is $Q_{ij}\dfrac{\hat{x}_i \hat{x}_j}{2|\underline{x}|^3}$ where $Q_{ij} = \displaystyle\int_V \left(3x'_i x'_j - \delta_{ij}|\underline{x}|^2\right)\rho \, dV$

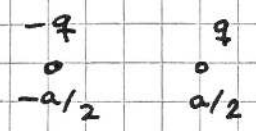
quadrupole moment tensor

$\underline{p}$ gives leading term for neutral matter

In general, $\underline{p}$ depends on the choice of origin (when $Q \neq 0$)

L5.3

Point dipole

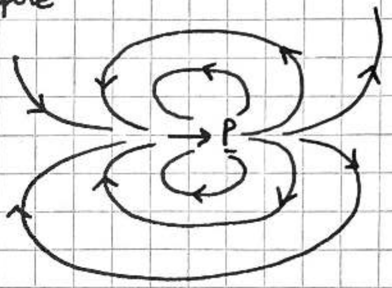


$$\Phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{x}-\underline{x}_{+}|} - \frac{1}{|\underline{x}+\underline{x}_{+}|}\right)$$

Take $a \to 0$ with $\underline{p} = q\,\underline{a}$ fixed.

(*) for $\underline{x}' = \pm \frac{\underline{a}}{2}$ $\Rightarrow$ $\Phi \longrightarrow \frac{\hat{\underline{x}}\cdot\underline{p}}{4\pi\varepsilon_0|\underline{x}|^{\cancel{3}2}} \equiv \Phi_{\text{dipole}}$

$$\underline{E}_{\text{dipole}} = -\,\underline{\nabla}\,\Phi_{\text{dipole}} = \frac{3(\hat{\underline{x}}\cdot\underline{p})\hat{\underline{x}} - \underline{p}}{4\pi\varepsilon_0\,|\underline{x}|^3}$$

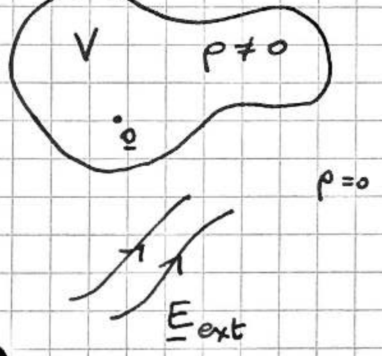


$$\rho(\underline{x}) = q\,\delta^{(3)}(\underline{x} - \tfrac{\underline{a}}{2}) - q\,\delta^{(3)}(\underline{x} + \tfrac{\underline{a}}{2})$$

$$= \cancel{q\,\delta^{(3)}(\underline{x})} - \frac{q\underline{a}}{2}\cdot\underline{\nabla}\,\delta^{(3)}(\underline{x})$$

$$\cancel{-\,q\,\delta^{(3)}(\underline{x})} - \frac{q\underline{a}}{2}\cdot\underline{\nabla}\,\delta^{(3)}(\underline{x}) \quad + \cdots$$

Take $a \to 0$ to get $\quad -\underline{p}\cdot\underline{\nabla}\,\delta^{(3)}(\underline{x}) = \rho_{\text{dipole}}(\underline{x})$.



$$\underline{E} = \underline{E}_{\text{ext}} + \underline{E}_{\text{body}}$$

$\uparrow$ vanishing contribution to force, torque on body

$$\underline{\tau} = \int_V d^3\underline{x}\;\rho(\underline{x})\,\underline{x}\times\underline{E}(\underline{x})$$

$\nwarrow$ ext

assume V small compared to variations in external $\underline{E}$

$$E_i(\underline{x}) = E_i(\underline{0}) + \underline{x}\cdot\underline{\nabla}E_i(\underline{0}) + \cdots$$

$$\Rightarrow \underline{\tau} \cong \int_V d^3\underline{x}\;\rho(\underline{x})\,\underline{x}\times\left(E_i(\underline{0})\,\underline{e}_i\right) = \underline{p}\times\underline{E}(0)$$

zero when $\underline{p}$ parallel with $\underline{E}(0)$

$$F_i = \int_V d^3\underline{x}\;\rho(\underline{x})\,E_i(\underline{x}) \simeq \int_V d^3\underline{x}\;\rho(\underline{x})\left(E_i(\underline{0}) + x_j\,\partial_j\,E_i(\underline{0})\right)$$

$$= Q\,E_i(\underline{0}) + p_j\,\partial_j\,E_i(\underline{0})$$

If $Q = 0$ (neutral matter), $\underline{F} \approx (\underline{p}\cdot\underline{\nabla})\,\underline{E}(\underline{0})$ $\quad$ use $\underline{\nabla}\times\underline{E} = \underline{0}$

$$= \underline{\nabla}(\underline{p}\cdot\underline{E})(\underline{0}) = \underline{F}_{\text{dipole}}$$

$\Rightarrow$ dipole potential energy $\quad V_{\text{dipole}} = -\,\underline{p}\cdot\underline{E}(\underline{x})$

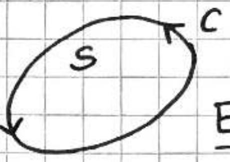
L6.1

III MAGNETOSTATICS

Time indept $\Rightarrow$ $\underline{\nabla}\cdot\underline{J} = 0$ (CC)

$$\underline{\nabla}\cdot\underline{B} = 0 \quad (\text{M2}) \qquad \underline{\nabla}\times\underline{B} = \mu_0\underline{J} \quad (\dagger)$$

Stokes $\Rightarrow$ $\displaystyle\int_C \underline{B}\cdot d\underline{x} = \mu_0\int_S \underline{J}\cdot d\underline{S} \equiv \mu_0 I[\underset{\substack{\uparrow \\ \text{current} \\ \text{through S}}}{S}]$



Ampères Law

Example Cylindrical polars (r, φ, z)

$$\underline{J} = j(r)\,\underline{e}_z \quad , \quad j(r) = 0 \text{ for } r > R$$

Check $\underline{\nabla}\cdot\underline{J} = 0$.

Let $S = D_r$, disc of radius r, centre of z-axis

LHS of Ampère needs $B_\phi \neq 0$.

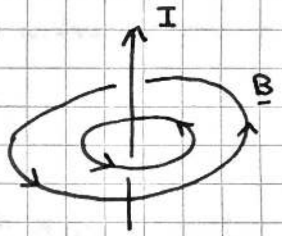
Assume $\underline{B} = B(r)\,\underline{e}_\phi$. Ampère $\Rightarrow$ $2\pi r B(r) = I(r)\mu_0$,

where $I(r) = \int_{D_r} j\, dS = 2\pi\int_0^r j(r')\, r' dr'$.

So
$$\underline{B} = \frac{\mu_0 I(r)}{2\pi r}\,\underline{e}_\phi = \frac{\mu_0 I(r)}{2\pi(x^2+y^2)}(-y, x, 0).$$

This does indeed satisfy (M2) and $(\dagger)$.

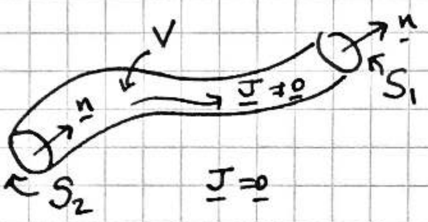
$r > R$: $I(r) = I(R) \equiv I$ $\Rightarrow$ $\underline{B} = \dfrac{\mu_0 I}{2\pi r}\,\underline{e}_\phi$ $(**)$



e.g. thin wire along z-axis carrying current I

Let $R \to 0$ $\Rightarrow$ $(**)$ true for all $r > 0$

(CC) $\Rightarrow$ current same $\forall$ cross-sections of a wire



$$I[S_1] - I[S_2] \underset{\substack{\uparrow \\ \text{zero on} \\ \text{edge}}}{=} \int_V \underline{\nabla}\cdot\underline{J}\, dV = 0$$

3.2 Vector potential

M2 $\Rightarrow$ $\exists\, \underline{A}(\underline{x})$ s.t. $\underline{B} = \underline{\nabla}\times\underline{A}$ $(\dagger\dagger)$ $\underline{A}$ is the vector potential

$\underline{A} + \underline{\nabla}\lambda$ also satisfies $(\dagger\dagger)$ $\underline{A} \mapsto \underline{A} + \underline{\nabla}\lambda$ is a "gauge" transf.

Definition of $\underline{A}$ has "gauge freedom".

Can eliminate via "gauge condition".

<u>Coulomb gauge</u> $\underline{\nabla}\cdot\underline{A} = 0$ — (CG)

$\underline{A}' = \underline{A} + \underline{\nabla}\lambda \Rightarrow \underline{\nabla}\cdot\underline{A}' = \underline{\nabla}\cdot\underline{A} + \nabla^2\lambda$

Choose λ s.t. $\nabla^2\lambda = -\underline{\nabla}\cdot\underline{A}$ (Poisson eq^n) $\Rightarrow \underline{\nabla}\cdot\underline{A}' = 0$.

(†) and (††) $\Rightarrow \mu_0\underline{J} = \underline{\nabla}\times(\underline{\nabla}\times\underline{A}) \underset{\substack{\uparrow \\ \text{vec calc}}}{=} \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \underline{\nabla}^2\underline{A}$

CG $\Rightarrow \nabla^2\underline{A} = -\mu_0\underline{J}$

If solving on $\mathbb{R}^3$,

$$\underline{A}(\underline{x}) = \int \frac{\mu_0\,\underline{J}(\underline{x}')}{4\pi|\underline{x}-\underline{t}'|}\, d\overset{3}{\underline{x}}{}' \quad \text{— } (*)$$

(assumed $\underline{J}\to\underline{0}$, $\underline{A}\to\underline{0}$ at ∞)

Check (CG)

$$\underline{\nabla}\cdot\underline{A}(\underline{x}) = \int \frac{\mu_0}{4\pi}\,\underline{J}(\underline{x}')\cdot\underline{\nabla}\,\frac{1}{|\underline{x}-\underline{x}'|}\,d^3\underline{x}'$$

$$= \frac{\mu_0}{4\pi}\int d^3\underline{x}'\;\underline{J}(\underline{x}')\cdot\Big(-\underset{\substack{\uparrow \\ \text{gradient} \\ \text{wrt } \underline{x}'}}{\underline{\nabla}'}\frac{1}{|\underline{x}-\underline{x}'|}\Big) \qquad \underline{\nabla}f(\underline{x}-\underline{x}') = -\underline{\nabla}'f(\underline{x}-\underline{x}')$$

$$= \frac{\mu_0}{4\pi}\int d^3\underline{x}'\left[-\underline{\nabla}'\cdot\left(\frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\right) + \underbrace{\frac{\underline{\nabla}'\cdot\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}}_{0 \text{ by (cc)}}\right] \cancel{d^3\underline{x}'}$$

= surface term at ∞ (DIV)

= 0 ($J\to 0$ at ∞).

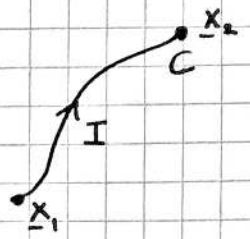
$$B_i = \varepsilon_{ijk}\,\partial_j A_k = \frac{\mu_0}{4\pi}\varepsilon_{ijk}\int J_k(\underline{x}')\,\partial_j\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)d^3\underline{x}'$$

$$= -\frac{\mu_0}{4\pi}\varepsilon_{ijk}\int J_k(\underline{x}')\,\frac{x_j - x_j'}{|\underline{x}-\underline{x}'|^3}\,d^3\underline{x}'$$

$$\therefore\quad \underline{B} = \frac{\mu_0}{4\pi}\int\frac{\underline{J}\times(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}\,d^3\underline{x}'$$

L6.3

3.3 Biot Savart Law

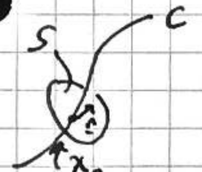


Thin wire carrying current I along curve C

Claim: $\underline{J}(\underline{x}) = I\int_C d\underline{x}'\,\delta^{(3)}(\underline{x}-\underline{x}')$

Check $\underline{J}(\underline{x}) = \underline{0}$ for $\underline{x} \notin C$ ✓

$$\begin{aligned}\underline{\nabla}\cdot\underline{J}(\underline{x}) &= I\int_C dx_i'\,\partial_i\,\delta^{(3)}(\underline{x}-\underline{x}')\\ &= -I\int_C dx_i'\,\partial_i'\,\delta^{(3)}(\underline{x}-\underline{x}')\\ &= -I\int_C d\underline{x}'\cdot\underline{\nabla}'\,\delta^{(3)}(\underline{x}-\underline{x}')\\ &= -I\left[\delta^{(3)}(\underline{x}-\underline{x}_1) - \delta^{(3)}(\underline{x}-\underline{x}_2)\right]\\ &= 0 \quad \text{if } C \text{ closed or if } C \text{ extends to } \infty \ (\underline{x}_1, \underline{x}_2 \text{ big})\end{aligned}$$



$$\int_S \underline{J}\cdot d\underline{S} = I\int_S\int_C d\underline{S}\cdot d\underline{x}'\,\underbrace{\delta^{(3)}(\underset{\substack{\uparrow\\ \text{in } S}}{\underline{x}} - \underset{\substack{\uparrow\\ \text{in } C}}{\underline{x}'})}_{\text{non-zero only at } \underline{x}_0}$$

$$= I\int_{\substack{S'\\ \uparrow}}\int_{\substack{C'\\ \uparrow}} d\underline{S}\cdot d\underline{x}'\,\delta^{(3)}(\underline{x}-\underline{x}')$$

small

$\Rightarrow$ S' flat, C' straight, introduce coords so S' is $\{z=0\}$, C' z-axis

$$\text{RHS} = I\int_{S'}\int_{C'} dx\,dy\,dz\;\delta^{(3)}((x,y,0)-(0,0,z)) = I \quad ✓$$

$$\underline{J}(\underline{x}') = I\int_C d\underline{x}\,\delta^{(3)}(\underline{x}'-\underline{x})$$

Substitute into (*), double prime LA

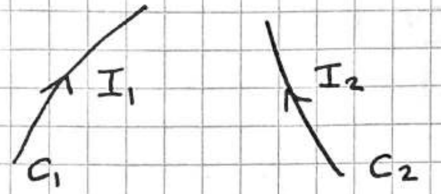
$$A(\underline{x}) = \frac{\mu_0 I}{4\pi}\int d^3\underline{x}''\,\frac{1}{|\underline{x}-\underline{x}''|}\int_C d\underline{x}'\,\delta^{(3)}(\underline{x}''-\underline{x}') = \frac{\mu_0 I}{4\pi}\int_C \frac{d\underline{x}'}{|\underline{x}-\underline{x}'|}$$

Substitute into (**)

$$\Rightarrow \underline{B}(\underline{x}) = \frac{\mu_0 I}{4\pi}\int_C \frac{d\underline{x}'\times(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3} \qquad \text{(Biot-Savart Law)}$$

L7.1

3.4 Force on current dist^n

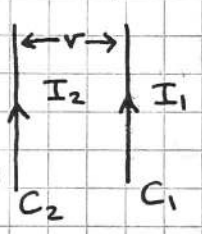


$$\underline{B} = \underline{B}_1 + \underline{B}_2$$

force on 1 due to 2 is

$$\underline{F}_{12} = \int d^3x_1 \; \underline{J}_1(\underline{x}_1) \times \underline{B}_2(\underline{x}_1)$$
$$= \int d^3x_1 \; I_1 \int d\underline{x}' \, \delta^{(3)}(\underline{x}_1 - \underline{x}') \times \underline{B}_2(\underline{x}_1)$$
$$= I_1 \int d\underline{x}' \times \underline{B}_2(\underline{x}') \qquad (***)$$

Example Let C_2 be z-axis, use cyl polars (r, φ, z).



Force on a section of wire 1 of length L.

$$d\underline{x}' = \underline{e}_z \, dz \;, \quad \underline{B}_2 = \frac{\mu_0 I_2}{4\pi r} \cdot 2\underline{e}_\varphi$$

$$\Rightarrow \underline{F}_{12} = I_1 \int_{z_0}^{z_0+L} dz \; \underline{e}_z \times \frac{\mu_0 I_2}{2\pi r} \underline{e}_\varphi = -\frac{\mu_0 I_1 I_2 L}{2\pi r} \underline{e}_r$$

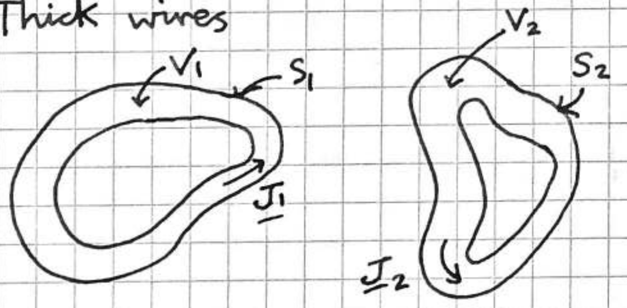
Force attractive if $I_1 I_2 > 0$, repulsive if $I_1 I_2 < 0$.

BS on $(***)$ gives

$$\underline{F}_{12} = \frac{-\mu_0 I_1 I_2}{4\pi} \int_{C_1} \int_{C_2} d\underline{x}_1 \cdot d\underline{x}_2 \; \frac{(\underline{x}_1 - \underline{x}_2)}{|\underline{x}_1 - \underline{x}_2|^3}$$

antisym in $1 \leftrightarrow 2 \Rightarrow \underline{F}_{12} = -\underline{F}_{21}$ Newton III

Thick wires



$$\underline{B} = \underline{B}_1 + \underline{B}_2$$

force on 1 due to 2 is

$$\underline{F}_{12} = \int_{V_1} d^3x_1 \; \underline{J}_1(\underline{x}_1) \times \underline{B}_2(\underline{x}_1)$$

$$\therefore \underline{F}_{12} = \frac{\mu_0}{4\pi} \int_{V_1} \int_{V_2} d^3x_1 \, d^3x_2 \; \frac{\underline{J}_1(\underline{x}_1) \times \left[\underline{J}_2(\underline{x}_2) \times (\underline{x}_1 - \underline{x}_2) \right]}{|\underline{x}_1 - \underline{x}_2|^3}$$

$$= \frac{\mu_0}{4\pi} \int_{V_1} \int_{V_2} d^3x_1 \, d^3x_2 \left[\frac{\underline{J}_1(\underline{x}_1) \cdot (\underline{x}_1 - \underline{x}_2) \, \underline{J}_2(\underline{x}_2)}{|\underline{x}_1 - \underline{x}_2|^3} - \frac{\underline{J}_1(\underline{x}_1) \cdot \underline{J}_2(\underline{x}_2) \overset{(\underline{x}_1 - \underline{x}_2)}{\swarrow}}{|\underline{x}_1 - \underline{x}_2|^3} \right.$$

Now, $$\int_{V_1} d^3x_1 \; \frac{\underline{J}_1(\underline{x}_1) \cdot (\underline{x}_1 - \underline{x}_2)}{|\underline{x}_1 - \underline{x}_2|^3} = \int_{V_1} d^3x_1 \; \underline{J}_1(\underline{x}_1) \cdot \underset{\substack{\uparrow \\ \text{grad wrt } \underline{x}_2}}{\underline{\nabla}} \left(\frac{1}{|\underline{x}_1 - \underline{x}_2|} \right)$$

L7.2

Integrate by parts via divergence theorem

$$= \int_{V_1} d^3x_1 \left[\underbrace{-\underline{\nabla}\cdot\left(\frac{\underline{J}_1(\underline{x}_1)}{|\underline{x}_1-\underline{x}_2|}\right)}_{\text{boundary term vanishes}} + \overbrace{\frac{\underline{\nabla}\cdot\underline{J}_1(\underline{x}_1)}{|\underline{x}_1-\underline{x}_2|}}^{\text{zero by CC}} \right]$$

We deduce that

$$\underline{F}_{12} = -\frac{\mu_0}{4\pi}\int_{V_1}\int_{V_2} d^3x_1\, d^3x_2 \,\frac{\underline{J}_1(\underline{x}_1)\cdot\underline{J}_2(\underline{x}_2)\,(\underline{x}_1-\underline{x}_2)}{|\underline{x}_1-\underline{x}_2|^3} \qquad (*)$$

Antisym in 1, 2 $\Rightarrow$ Newton III, self force = 0 [footnote]

contribution to $\underline{F}_{12}$ from $(\underline{x}_1, \underline{x}_2)$ is attractive if $\underline{J}_1(\underline{x}_1)\cdot\underline{J}_1(\underline{x}_2) > 0$, repulsive if … < 0

Thin wires: replace $V_i \to \mathbb{R}^3$ in $(*)$ $\quad$ ($\underline{J}_i = \underline{0}$ outside V_i)

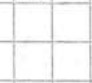
$$\underset{\substack{\uparrow\\(\underline{x}_i)}}{\underline{J}_i} = \int d\underline{x}_i'\, \delta^{(3)}(\underline{x}_i-\underline{x}_i')\, I_i$$

$$\underline{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} d^3x_1\, d^3x_2\, \frac{(\underline{x}_1-\underline{x}_2)}{|\underline{x}_1-\underline{x}_2|^3}\iint_{C_1\, C_2} d\underline{x}_1'\cdot d\underline{x}_2'\, \delta^{(3)}(\underline{x}_1-\underline{x}_1')\,\delta^{(3)}(\underline{x}_2-\underline{x}_2')$$

$$= -\frac{\mu_0 I_1 I_2}{4\pi}\int_{C_1}\int_{C_2} d\underline{x}_1'\cdot d\underline{x}_2'\,\frac{(\underline{x}_1'-\underline{x}_2')}{|\underline{x}_1'-\underline{x}_2'|^3}$$

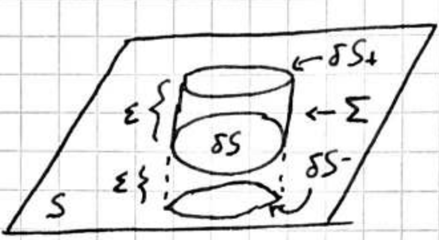
3.5 Surface currents

surface current density: vector field $\underline{K}(t,\underline{x})$ on S

- tangent to S
- charge / unit time crossing inf$^\ell$ curve in S is $\underline{K}\cdot\underline{dl}^\perp$ 

$\underline{B}$ is discontinuous across S

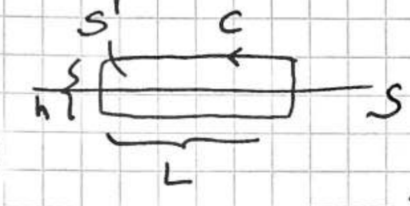
L7.2



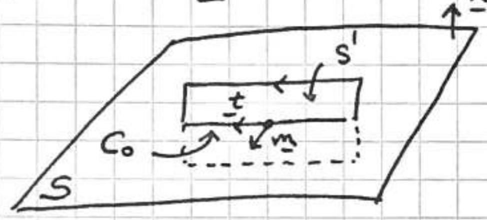
Pill-box argument (M2) $\Rightarrow 0 = \int_{\underset{\uparrow \delta \text{ not } \partial}{\partial V}} \underline{\nabla}\cdot\underline{B}\, dV = \int_{\substack{\delta S^+ \cup\, \delta S^- \\ \cup\, \Sigma}} \underline{B}\cdot d\underline{S}$

Let $\varepsilon \to 0$, deduce $\int_{\delta S} (\underline{B}_+ - \underline{B}_{,})\cdot d\underline{S} = 0$

δS arbitrary $\Rightarrow (\underline{B}_+ - \underline{B}_-)\cdot\underline{n} = 0 \qquad (*)$



$\underset{\text{Stokes}}{\overset{+}{(M4)}} \Rightarrow \int_C \underline{B}\cdot d\underline{x} = \mu_0 I[S'] + \frac{1}{c^2}\int_{S'} \frac{\partial \underline{E}}{\partial t}\cdot d\underline{S}$



$I[S'] = \int_{C_0} \underline{K}\cdot\underline{m}\, dl$ ← normal to C_0 in S

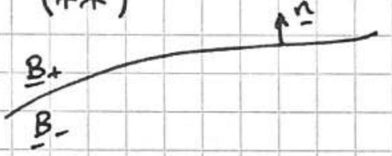
$= \int_{C_0} \underline{K}\cdot(\underline{n}\times\underline{t})\, dl$

Let $h \to 0$, obtain

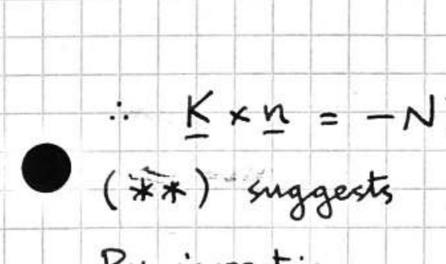
$\int_{C_0} (\underline{B}_+ - \underline{B})\cdot\underline{t}\, dl = \mu_0 \int_{C_0} \underline{K}\cdot(\underline{n}\times\underline{t})\, dl \qquad \left(\frac{\partial \underline{E}}{\partial t} \text{ bdd, so} \to 0\right)$

C_0 arbitrary $\Rightarrow (\underline{B}_+ - \underline{B}_-)\cdot\underline{t} = \mu_0\, \underline{t}\cdot(\underline{K}\times\underline{n}) \quad \forall \underline{t}$

It follows that $\boxed{\underline{B}_+ - \underline{B}_- = \mu_0 \underline{K}\times\underline{n}} \qquad (**)$



<u>Example</u> Infinite solenoid



Assume many turns / unit length $\Rightarrow$ each turn ≈ circular

Treat as surface current $\underline{K}$

cyl polars (r, φ, z)

current crossing dz is $IN\,dz$

$\therefore \underline{K} = NI\,\underline{e}_\varphi \quad , \quad \underline{n} = \underline{e}_r$

$\therefore \underline{K}\times\underline{n} = -NI\,\underline{e}_z$

$(**)$ suggests $\underline{B} \parallel \underline{e}_z$

By inspection, $\underline{B} = \begin{cases} 0 & \text{outside solenoid} \\ \mu_0 NI\,\underline{e}_z & \text{inside solenoid} \end{cases}$

Indeed a solution!

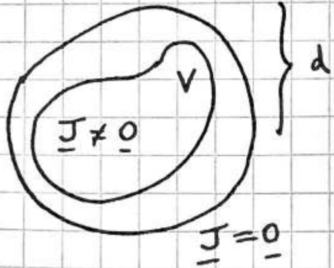
L7.4

At $r = R$, satisfies correct discontinuity.

~ Essentially Maxwell at the boundary

L 8.1

3.6 Magnetic dipoles & field far from a source



$\rho \longleftrightarrow J_i, \quad \Phi \longleftrightarrow A_i, \quad \frac{1}{\varepsilon_0} \longleftrightarrow \mu_0$

Swapping in multipole expansion of Φ,

$$A_i(\underline{x}) \underset{\substack{\uparrow \\ |\underline{x}| \gg d}}{\approx} \frac{\mu_0}{4\pi}\left[\frac{1}{|\underline{x}|}\int_V J_i(\underline{x}')\,d^3x' + \frac{\hat{x}_j}{|\underline{x}|^2}\int_V x_j'\,J_i(\underline{x}')\,d^3x'\right]$$

$$\text{Now,}\quad 0 \underset{\substack{\uparrow \\ \text{div} \\ \text{thm}}}{=} \int_V d^3\underline{x}\,\partial_j\,(x_i J_j(\underline{x}))$$

$$= \int_V d^3x\,\big(J_i(\underline{x}) + x_i \underbrace{\underline{\nabla}\cdot\underline{J}(\underline{x})}_{\text{zero by CC}}\big)$$

So first term zero.

$$\text{Likewise,}\quad 0 = \int_V d^3x\,\partial_k\,(x_i x_j J_k(\underline{x}))$$

$$= \int_V d^3x\,\big(x_j J_i(\underline{x}) + x_i J_j(\underline{x}) + \underbrace{x_i x_j\,\underline{\nabla}\cdot\underline{J}(\underline{x})}_{\text{zero by CC}}\big)$$

ANTISYM

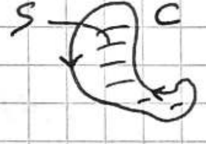
$$\therefore \int_V d^3x\; x_j J_i(\underline{x}) = \frac{1}{2}\int_V d^3x\,\big(x_j J_i(\underline{x}) - x_i J_j(\underline{x})\big)$$

$$= -\varepsilon_{ijk}\, m_k \quad \text{— (4)}$$

where $\underline{m} = \frac{1}{2}\int_V \underline{x} \times \underline{J}(\underline{x})\,d^3x$ is "magnetic dipole moment".

$$\therefore\ \underline{A}(\underline{x}) \approx \frac{\mu_0}{4\pi}\left[\frac{\underline{m}\times\hat{\underline{x}}}{\underline{x}^2} + O\left(\frac{1}{|\underline{x}|^3}\right)\right] \quad \text{— (*)} \quad \text{for } |\underline{x}| \gg d$$

<u>Ex</u> thin wire $\underline{J}(\underline{x}) = I\int_C d\underline{x}'\,\delta^{(3)}(\underline{x}-\underline{x}')$



$$\Rightarrow\ \underline{m} = \frac{I}{2}\int_C \underline{x} \times d\underline{x}$$

$$\underset{\substack{\uparrow \\ \text{const.}}}{\underline{a}}\cdot\underline{m} = \frac{I}{2}\int_C \underline{a}\cdot(\underline{x}\times d\underline{x}) = \frac{I}{2}\int_C d\underline{x}\cdot(\underline{a}\times\underline{x})$$

$$= \frac{I}{2}\int_S \big[\underbrace{\underline{\nabla}\times(\underline{a}\times\underline{x})}_{2\underline{a}}\big]\cdot d\underline{S} = I\,\underline{a}\cdot\int_S d\underline{S}$$

$\therefore\ \underline{m} = I\,\underline{S} \leftarrow$ vector area of S

Planar curve $\underline{S} = \underline{n}\,A$

, $\underline{m} = I\,\underline{n}\,A$

L 8.2

$$\underline{A}_{\text{dipole}}(\underline{x}) = \frac{\mu_0}{4\pi}\,\frac{\underline{m}\times\hat{\underline{x}}}{|\underline{x}|^2}$$ field of a point dipole

$$\underline{B}_{\text{dipole}}(\underline{x}) = \frac{\mu_0\left[\,3(\hat{\underline{x}}\cdot\underline{m})\,\hat{\underline{x}} - \underline{m}\,\right]}{4\pi\,|\underline{x}|^3}$$

Can show $\underline{J}_{\text{dipole}} = \underline{\nabla}\times(\underline{m}\,\delta^{(3)}(\underline{x}))$

$$\underline{B} = \underset{\uparrow}{\underline{B}_{\text{self}}} + \underline{B}_{\text{ext}}$$

gives vanishing contribution to force/torque

torque

$$\underline{\tau} = \int_V \underline{x}\times\left(\underline{J}(\underline{x})\times\underline{B}(\underline{x})\right) d^3x$$
$$= \int_V \left(\underline{x}\cdot\underline{B}(\underline{x})\;\underline{J}(\underline{x}) - \underline{x}\cdot\underline{J}(\underline{x})\;\underline{B}(\underline{x})\right) d^3x$$

Assume V small compared to scale over which $\underline{B}$ varies

$$B_i(\underline{x}) = B_i(\underline{0}) + x_j\partial_j B_i(\underline{0}) + \cdots$$
$$\Rightarrow \tau_i \approx B_j(0)\int_V x_j J_i(\underline{x})\,d^3x - B_i(0)\underbrace{\int_V x_j J_j(\underline{x})\,d^3x}_{=0 \text{ by } (4)}$$
$$\underset{(4)}{=} -\varepsilon_{ijk} B_j(0)\, m_k$$

$\therefore\ \underline{\tau} \approx \underline{m}\times\underline{B}(0)$ vanishes iff $\underline{m}$ parallel to $\underline{B}(0)$

Force $F_i = \int_V d^3x\ \varepsilon_{ijk} J_j(\underline{x})\, B_k(\underline{x})$
$$= \int_V d^3x\ \varepsilon_{ijk} J_j(\underline{x})\left(B_k(0) + \overset{\curvearrowleft}{\partial_l\, x_l}\, B_k(0) + \cdots\right)$$
$$\simeq \varepsilon_{ijk}\,\partial_l B_k(0)\int_V d^3x\ J_j(\underline{x})\, x_l \qquad \delta_{kp}\delta_{il}$$
$$= \varepsilon_{ijk}\,\partial_l B_k(0)\,(-\varepsilon_{jlp}\, m_p)$$
$$= \partial_i B_k(0)\, m_k - \underbrace{\partial_k B_k(0)}_{\text{zero}}\, m_i$$

$\underline{F} \approx \underline{\nabla}(\underline{m}\cdot\underline{B})$

Suggests dipole has energy $-\underline{m}\cdot\underline{B}$.

L8.3

IV TIME DEPENDENCE

4.1 Faraday's law of induction

M3: $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ time varying $\underline{B} \Rightarrow \underline{E} \xrightarrow{\text{conductor}}$ current
"induction"

Stokes $\Rightarrow \int_C \underline{E}\cdot d\underline{x} = -\int_S \frac{\partial \underline{B}}{\partial t}\cdot d\underline{S}$

Assume S independent of time, so $\mathcal{E} = -\frac{d\mathcal{F}}{dt}$ where

$\mathcal{E}(t) = \int_C \underline{E}(t,\underline{x})\cdot d\underline{x}$ electromotive force / emf around C

$\mathcal{F}(t) = \int_S \underline{B}(t,\underline{x})\cdot d\underline{S}$ magnetic flux through S

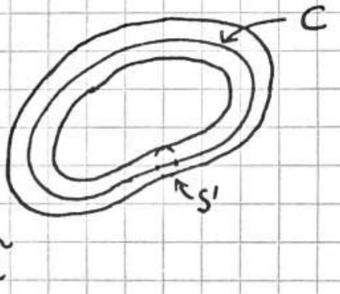
If ∃ thin wire around C, $\underline{F}(t,\underline{x})$ = Lorentz force on charge q

Then $\frac{1}{q}\int_C \underline{F}(t,\underline{x})\cdot d\underline{x} = \int_S (\underline{E} + \underset{\uparrow \text{ tangential to } d\underline{x}}{\dot{\underline{x}}} \times \underline{B})\cdot d\underline{x} = \mathcal{E}(t)$

So when $\mathcal{E}$ is indep of time,

LHS is the work done / unit charge around C.

If C inside (htick) conducting wire, then



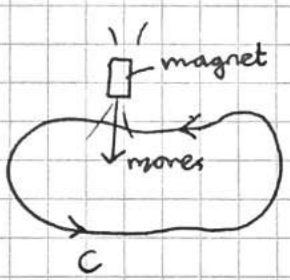
$\mathcal{E} = \int_C \underline{E}\cdot d\underline{x} \underset{\text{Ohm}}{=} \int_C \frac{\underline{J}}{\sigma}\cdot d\underline{x} \approx \frac{|\underline{J}|}{\sigma}\,\underset{\uparrow \text{ length of C}}{\ell}$

$I = \int_{S'} \underline{J}\cdot d\underline{S} \approx |\underline{J}|\,A \leftarrow$ cross-sectional area of wire

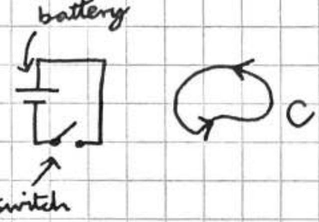
$\therefore\ \mathcal{E} = IR$ where $R = \frac{\ell}{\sigma A}$ - resistance.

$IR = -\frac{d\mathcal{F}}{dt}$

Examples ①



②

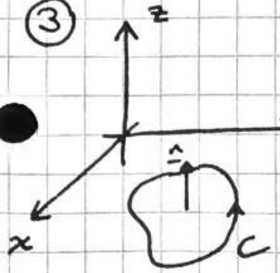


close switch
↳ current in circuit time dependent
↳ time dep $\mathcal{F}$ through C
↳ current in C

Transient Effect

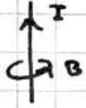
L9.1

③

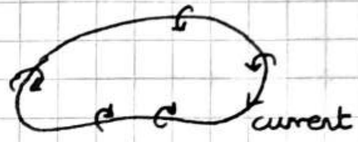


If $\mathcal{F}\uparrow$, then $\mathcal{E} < 0 \Rightarrow I < 0$

induced current clockwise.

Recall  direction of B found by RH rule

Magnetic field produced by induced current contributes negatively to the flux $\mathcal{F}$.



Lenz's Law : induced current produces a magnetic field that opposes the change in $\mathcal{F}$.

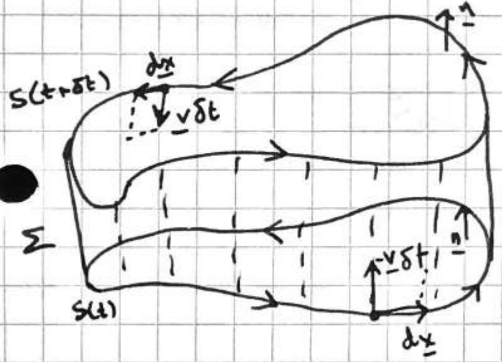
Let $C(t)$ be a closed curve spanned by open surface $S(t)$.

$$\mathcal{F}(t) = \int_{S(t)} \underline{B}(t,\underline{x})\cdot d\underline{S}$$

$$\mathcal{F}(t+\delta t) - \mathcal{F}(t) = \int_{S(t+\delta t)} \underbrace{\underline{B}(t+\delta t,\underline{x})}_{\underline{B}(t,\underline{x})+\delta t\frac{\partial \underline{B}}{\partial t}+\cdots}\cdot d\underline{S} - \int_{S} \underline{B}(t,\underline{x})\cdot d\underline{S}$$

$$= \int_{S(t+\delta t)-S(t)} \underline{B}(t,\underline{x})\cdot d\underline{S} + \delta t \int_{S(t)} \frac{\partial \underline{B}}{\partial t}(t,\underline{x})\cdot d\underline{S} + O(\delta t^2) \qquad (1)$$

replaced $S(t+\delta t)$ with $S(t)$



δV = volume swept by $S(t)$ when $t \to t\delta t$

$$M2 \Rightarrow \int_{\delta V} \underline{\nabla}\cdot\underline{B} = 0 = \int_{S(t+\delta t)-S(t)} \underline{B}\cdot d\underline{S} + \int_{\Sigma} \underline{B}\cdot d\underline{S}$$

On $C(t)$, $\underline{x} = \underline{x}(\lambda, t)$. $\Rightarrow d\underline{x} = \frac{\partial \underline{x}}{\partial \lambda} d\lambda$ (tangent)

parameter round C

$\underline{v} = \frac{\partial \underline{x}}{\partial t}$ (velocity)

Get parallelogram of vector area $d\underline{S} = d\underline{x} \times (\underline{v}\delta t)$

$$\int_{\Sigma} \underline{B}(t,\underline{x})\cdot d\underline{S} = \int_{C(t)} \underline{B}(t,\underline{x})\cdot(d\underline{x}\times\underline{v}\,\delta t) = \delta(t)\int_{C(t)} d\underline{x}\cdot(\underline{v}\times\underline{B}(t,\underline{x}))$$

Combining the above, we get that

$$\mathcal{F}(t+\delta t) - \mathcal{F}(t) = -\delta t \int_{C(t)} d\underline{x}\cdot(\underline{v}\times\underline{B}(t,\underline{x})) + \delta t\int_{S(t)} \frac{\partial \underline{B}}{\partial t}(t,\underline{x})\cdot d\underline{S}$$

L9.2

$$\frac{d\mathcal{F}}{dt} = -\int_{C(t)} d\underline{x}\cdot\left(\underline{v}\times\underline{B}(t,\underline{x})\right) + \int_{S(t)} \underset{\overset{\nwarrow}{= -\underline{\nabla}\times\underline{E}\ \ (M3)}}{\frac{\partial\underline{B}}{\partial t}}(t,\underline{x})\cdot d\underline{S}$$

$$= -\int_{C(t)} d\underline{x}\cdot\left(\underline{E}+\underline{v}\times\underline{B}\right) \qquad \text{(stokes)}$$

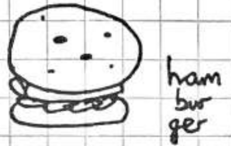
$$\therefore \boxed{\mathcal{E} = -\frac{\partial\mathcal{F}}{dt}} \qquad \text{where } \mathcal{E}(t) = \int_{C(t)} d\underline{x}\cdot\left(\underline{E}(t,\underline{x}) + \underline{v}(t,\underline{x})\times\underline{B}(t,\underline{x})\right)$$

def^n of $\mathcal{E}$ for moving $C(t)$

If thin wire around $C(t)$, free charge $\underline{x}(\lambda(t), t)$

$$\Rightarrow \dot{\underline{x}} = \underset{\overset{\nwarrow}{\frac{\partial x}{\partial\lambda}\dot{\lambda}}}{\underline{v}_r} + \underline{v}$$

$$\frac{1}{q}\int_{C(t)} \underline{F}(t,\underline{x})\cdot d\underline{x}$$



$$= \int_{C(t)} \left(\underline{E}+\underline{v}\times\underline{B} + \underset{\overset{\uparrow}{\text{parallel to } d\underline{x}}}{\underline{v}_r}\times\underline{B}\right)\cdot d\underline{x} = \int_{C(t)} \left(\underline{E}+\underline{v}\times\underline{B}\right)\cdot d\underline{x} = \mathcal{E}$$

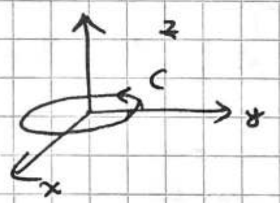
If $C(t)$ inside thick wire with velocity $\underline{v}$, Ohm's $\rightarrow \underline{J} = \sigma(\underline{E}+\underline{v}\times\underline{B})$

$$\Rightarrow \mathcal{E}(t) = \int_{C(t)} \frac{\underline{J}}{\sigma}\cdot d\underline{x}, \quad \mathcal{E}\propto\underline{J},\ \underline{J}\propto I, \quad \Rightarrow \mathcal{E} = IR$$

<u>Example</u> thin wie in xy plane

rotate about x-axis

normal to $S(t)$ $\underline{n} = (0, -\sin t, \cos t)$



$\underline{B} = (0, 0, B)$

$$\mathcal{E} = -\frac{d\mathcal{F}}{dt} = -\frac{d}{dt}\int_{S(t)} \underline{B}\cdot\underline{n}\, dA = AB\sin\theta\frac{d\theta}{dt}$$

$$I = \frac{AB}{R}\sin\theta\frac{d\theta}{dt}$$

e.g. $\theta = \omega t \Rightarrow I = \frac{AB\omega}{R}\sin\omega t$ ~ Alternating Current

$Q(t)$ = total charge crossing point of $C(t)$, $\frac{dQ}{dt} = I$

$\Rightarrow$ charge that flows rotate π is $Q = \frac{AB}{R}\left[-\cos\theta\right]_0^\pi = \frac{2AB}{R}$

L 9.3

4.2 Energy of the electromagnetic field

- Electrostatics $$E = \int_V \frac{\varepsilon_0}{2} \underline{E}^2 \, dV$$

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\underline{E}^2\right) = \frac{\varepsilon_0}{\cancel{2}}\underline{E}\cdot\frac{\partial \underline{E}}{\partial t} \overset{(M4)}{=} \frac{\varepsilon_0 c^2}{\cancel{2}}\underline{E}\cdot\left(\underline{\nabla}\times\underline{B} - \mu_0 \underline{J}\right)$$

$$c^2 = \frac{1}{\mu_0 \varepsilon_0} \qquad = \cancel{\varepsilon}_0 \frac{1}{\mu_0}\left[\underline{B}\cdot\underline{\nabla}\times\underline{E} - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\right] - \underline{E}\cdot\underline{J}$$

$$\underline{\nabla}\cdot(\underline{X}\times\underline{Y}) = -\underline{X}\cdot\underline{\nabla}\times\underline{Y} + \underline{Y}\cdot\underline{\nabla}\times\underline{X} \qquad = \frac{1}{\mu_0}\left[-\underline{B}\cdot\frac{\partial \underline{B}}{\partial t} - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\right] - \underline{E}\cdot\underline{J}$$

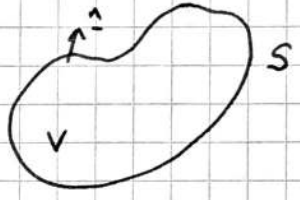
$$w = \frac{\varepsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2$$

$$\therefore \frac{\partial w}{\partial t} + \underline{\nabla}\cdot\underline{S} + \underline{E}\cdot\underline{J} = 0 \qquad (*)$$

$\underline{S} = \frac{1}{\mu_0}\underline{E}\times\underline{B}$ Poynting vector

- Integrate over time-indep V

$$\Rightarrow \frac{d}{dt}\int_V w \, dV = -\int_S \underline{S}\cdot d\underline{A} - \int_V \underline{E}\cdot\underline{J}\, dV$$



w = energy density of em field

$\underline{S}\cdot\underline{n}$ = energy flux across S (i.e. per unit area, unit time)

Assume there are N charged particles in V.

Rate of working of em field on i^{th} particle

- $$\underline{F}\cdot\dot{\underline{x}} = q(\underline{E} + \underline{v}\times\underline{B})\cdot\underline{v} = q\underline{E}\cdot\dot{\underline{x}}$$

$$\Rightarrow \text{total working} = \sum_{i=1}^{N} q_i \underline{E}(t, \underline{x}_i)\cdot\dot{\underline{x}}_i$$

$$= \sum_{i=1}^{N}\int_V q_i \underline{E}(t,\underline{x})\cdot\dot{\underline{x}}_i\, \delta(\underline{x} - \underline{x}_i)\, d^3x$$

$$= \int_V \underline{E}\cdot\underline{J}\, dV \qquad \text{recall } \underline{J} = \sum q_i \dot{\underline{x}}_i \delta(\underline{x} - \underline{x}_i)$$

$\therefore \int_V \underline{E}\cdot\underline{J}\, dV$ is rate of working on charged matter in V

L10.1

Inside a conductor, $\int_V \underline{E}\cdot\underline{J}\,dV = \int_V \frac{1}{\sigma} J^2\,dV > 0.$

Work $\Rightarrow$ kinetic energy of free charges

$\underset{\text{collisions}}{\Rightarrow}$ faster vibrations of metal lattice ↔ higher temp.

∴ work is converted to heat, Joule heating

Thin wire inside V

$$\underline{J} = \int_C d\underline{x}'\, I\, \delta^{(3)}(\underline{x}-\underline{x}')$$

$$\Rightarrow \int_V \underline{E}\cdot\underline{J}\,dV = I\int_C \underline{E}\cdot d\underline{x} = I\mathcal{E} \overset{\mathcal{E}=IR}{=} I^2R \quad \left(\begin{array}{c}\text{rate of}\\ \text{heating}\end{array}\right)$$

e.g. magnetostatics: Energy in $\underline{B}$ field over $\mathbb{R}^3$

$$E_B = \frac{1}{2\mu_0}\int B^2\,d^3x = \frac{1}{2\mu_0}\int \underline{B}\cdot(\underline{\nabla}\times\underline{A})\,d^3x$$

$$= \frac{1}{2\mu_0}\int d^3x\,\Big[\underline{A}\cdot\underbrace{(\underline{\nabla}\times\underline{B})}_{\mu_0\underline{J}} - \underbrace{\underline{\nabla}\cdot(\underline{B}\times\underline{A})}_{\substack{\text{sfc term at } \infty\\ \to 0}}\Big]$$

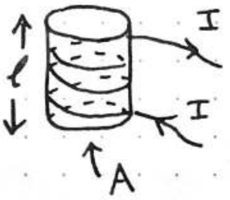
$$= \frac{1}{2\mu_0}\cdot\mu_0\int d^3x\;\underline{J}\cdot\underline{A} \qquad \text{c.f. } E = \tfrac{1}{2}\int d^3x\,\rho\,\Phi$$

$$= \frac{\mu_0}{8\pi}\int d^3x\,d^3x'\,\frac{\underline{J}(\underline{x})\cdot\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|} \qquad \text{c.f. } E = \frac{1}{8\pi\varepsilon_0}\int d^3x\,d^3x'\,\frac{\rho(\underline{x})\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

(energy required to create current dist^n)

Thick wire: $I \propto \underline{J} \Rightarrow E_B = \frac{1}{2}LI^2$ for some constant L, known as the self-inductance of the wire.

Examples ① Finite solenoid



$$\underline{B} \approx \begin{cases} \mu_0 N I\,\underline{e}_z & \text{inside}\\ \underline{0} & \text{outside}\end{cases}$$

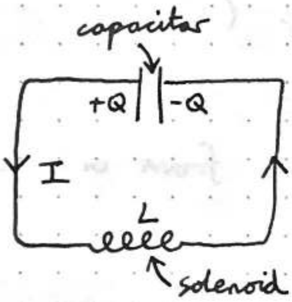
N = # turns / unit Length

n = # turns = $N\ell$

$$E_B = \frac{\mu_0^2 N^2 I^2}{2\mu_0}\cdot \ell A = \frac{\mu_0 n^2 I^2}{2\ell} A$$

$$\Rightarrow L = \frac{\mu_0 n^2 A}{\ell} \quad \text{(not bad)}$$

②



R = resistance of wires

Assume $E = \underbrace{\tfrac{1}{2}CQ^2}_{\substack{\text{energy}\\ \text{in capacitor}\\ \text{(electrostatic)}}} + \underbrace{\tfrac{1}{2}LI^2}_{\substack{\text{energy}\\ \text{in solenoid}\\ \text{(magnetostatic)}}}$

$$\frac{d}{dt}\left(\tfrac{1}{2}CQ^2 + \tfrac{1}{2}LI^2\right) = -I\mathcal{E} = -I^2R \quad \text{(Joule heating)}$$

$\therefore\; CQ\dot{Q} + LI\dot{I} = -I^2R$ now, $I = -\dot{Q}$

$\therefore\; L\ddot{Q} + R\dot{Q} + CQ = 0$

Turn on switch at $t=0$, with $Q = \text{const}$ for $t<0$.

If $R^2 \ll 4LC$, underdampded oscillates at frequency $\sqrt{C/L}$.

§4.3 Electromagnetic Waves

$\underline{\nabla} \cdot \underline{E} = 0$ Monochromatic plane waves (*)

$\underline{\nabla} \cdot \underline{B} = 0$ $\underline{E} = \text{Re}(\underline{E}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}})$ ↙ $\omega, \underline{k}$ real

$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$ $\underline{B} = \text{Re}(\underline{B}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}})$ $\underline{B}_0 \underline{E}_0$ complex

$\underline{\nabla} \times \underline{B} = \frac{1}{c^2}\dot{\underline{E}}$ $(\omega, k, \underline{E}_0, \underline{B}_0)$ same sol^n as $(-\omega, -k, \overline{\underline{E}}_0, \overline{\underline{B}}_0)$

WLOG $\omega > 0$

(Assume $\rho = \underline{J} = 0$) M1 $\Rightarrow \text{Re}(i\underline{k}\cdot\underline{E}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}}) = 0$ iff $\underset{\substack{\uparrow \\ (1)}}{\underline{k}\cdot\underline{E}_0 = 0}$

M2 $\Rightarrow \text{Re}(i\underline{k}\cdot\underline{B}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}}) = 0$ iff $\underline{k}\cdot\underline{B}_0 = 0$ (2)

M3 $\Rightarrow \text{Re}(i\underline{k}\times \overset{E}{\cancel{\underline{B}}}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}}) = \text{Re}(i\omega\underline{B}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}})$

iff $\omega\underline{B}_0 = \underline{k}\times\underline{E}_0$ (3)

M4 $\Rightarrow \text{Re}(i\underline{k}\times\underline{B}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}}) = -\frac{1}{c^2}\text{Re}(i\omega\underline{E}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}})$

iff $\omega\underline{E}_0 = -c^2\underline{k}\times\underline{B}_0$

Now,
$$\omega^2\underline{E}_0 = -c^2\underline{k}\times(\omega\underline{B}_0) = -c^2\underline{k}\times(\underline{k}\times\underline{E}_0)$$
$$= -c^2(\underline{k}\cdot\underline{E}_0\,\underline{k} - k^2\underline{E}_0)$$
$$= c^2k^2\underline{E}_0$$

$\therefore \boxed{\omega = c|\underline{k}|}$ "dispersion relation"

(*) satisfies vacuum Maxwell equations iff

$\omega = c|\underline{k}|$, $\underline{E}_0$ obeys (1), and $\underline{B}_0$ given by (3)

$\underline{E}, \underline{B}$ constant on "surfaces of constant phase" $\underline{k}\cdot\underline{x} = \omega(t-t_0)$

These are planes, normal $\underline{k}$, distance $\frac{\omega}{|\underline{k}|}(t-t_0)$ from origin

$\Rightarrow$ speed $\frac{\omega}{|\underline{k}|} = c$

(1), (2) $\Rightarrow \underline{E}\cdot\underline{k} = \underline{B}\cdot\underline{k} = 0$ "transverse waves"

($\underline{E}, \underline{B}$ oscillate in direction $\perp \underline{k}$) (3) $\Rightarrow \omega\underline{B} = \underline{k}\times\underline{E}$ ~ orthog basis

Choose z-axis such that $\underline{k} = (0,0,k)$, $k>0$, $k = \frac{\omega}{c}$

(1) $\Rightarrow \underline{E}_0 = (\alpha, \beta, 0)$, (3) $\Rightarrow \underline{B}_0 = \frac{1}{c}(-\beta, \alpha, 0)$

L10. 3

We have a two-dimensional family of waves.

$\underline{E}_0, \underline{B}_0$ describe <u>polarization</u> of wave (direction of oscillation)

2 indep polarizations:

Let $\alpha = |\alpha| e^{i\delta_1}$, $\beta = |\beta| e^{i\delta_2}$.

Consider $\beta = 0 \Rightarrow$ $\underline{E} = \underline{e}_x \,\mathrm{Re}(\alpha e^{-i\omega t + i\underline{k}\cdot\underline{x}})$

$$= \underline{e}_x |\alpha| \cos(\omega t - \underline{k}\cdot\underline{x} - \delta_1)$$

$$\underline{B} = \underline{e}_y \,\mathrm{Re}\left(\frac{\alpha}{c} e^{-i\omega t + i\underline{k}\cdot\underline{x}}\right)$$

$$= \underline{e}_y \frac{|\alpha|}{c} \cos(\omega t - \underline{k}\cdot\underline{x} - \delta_1)$$

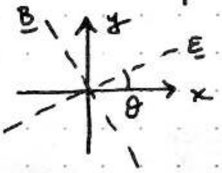
$\underline{E}$ oscillates in x-dir^n, $\underline{B}$ in y-dir^n

Similarly for $\alpha = 0$, $\underline{E}$ oscillates in y-dir^n, $\underline{B}$ in x-dir^n

i.e. $\underline{E}, \underline{B}$ always point in the same direction (up to sign)

This is known as <u>linear polarisation</u>

General such case is for $\delta_1 = \delta_2$



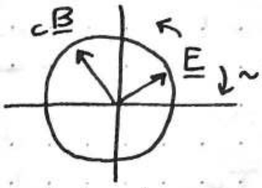
$\theta = \arctan \frac{\beta}{\alpha}$

L12.1 (!)

$\delta_1 \neq \delta_2$ elliptical polarisation

e.g. $\beta = \pm i\alpha \Rightarrow \underline{E} = \underline{e}_x |\alpha| \cos(\omega t - \underline{k}\cdot\underline{x} - \delta_1)$

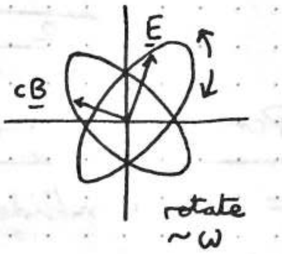
$\pm\, \underline{e}_y |\alpha| \sin(\omega t - \underline{k}\cdot\underline{x} - \delta_1)$



(circular polarisation)

In particular $E_1^2 + E_2^2 = |\alpha|^2 \neq 0$, similarly for $\underline{B}$.

In the general case, have an ellipse for each field.



$$W = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

$$E^2 = \left(\tfrac{1}{2}\left(\underline{E}_0 e^{-i\omega t + i\underline{k}\cdot\underline{x}} + \overline{\underline{E}}_0 e^{i\omega t - i\underline{k}\cdot\underline{x}} \right)\right)^2$$

$$= \frac{1}{2}\mathrm{Re}\left(\underline{E}_0^2 e^{-2i\omega t + 2i\underline{k}\cdot\underline{x}}\right) + \frac{1}{2}|\underline{E}_0|^2$$

$\underline{E}_0^2 = \underline{E}_0 \cdot \underline{E}_0$ cx

$|\underline{E}_0|^2 = \underline{E}_0 \cdot \overline{\underline{E}}_0$ real

Similarly

$$B^2 = \frac{1}{2}\mathrm{Re}\left(\underline{B}_0^2 e^{-2i\omega t + 2i\underline{k}\cdot\underline{x}}\right) + \frac{1}{2}|\underline{B}_0|^2$$

Choose axes as above, then

$$\underline{B}_0^2 = \frac{1}{c^2}(\beta^2 + \alpha^2) = \varepsilon_0\mu_0 \underline{E}_0^2, \quad |\underline{B}_0|^2 = \frac{1}{c^2}(|\beta|^2 + |\alpha|^2)$$

$$\therefore W = \frac{\varepsilon_0}{2}\mathrm{Re}\left(\underline{E}_0^2 * \underset{\substack{\uparrow \\ \text{oscillates} \\ \text{periodically}}}{e^{-2i\omega t + 2i\underline{k}\cdot\underline{x}}}\right) + \frac{\varepsilon_0}{2}|\underline{E}_0|^2 \quad \text{double up!}$$

$$\Rightarrow \langle W \rangle = \mathrm{avg}\, W = \frac{\varepsilon_0}{2}|\underline{E}_0|^2$$

$\underline{S} = \frac{1}{\mu_0}\underline{E}\times\underline{B}$, obtain

$$\langle \underline{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}(\underline{E}_0 \times \overline{\underline{B}}_0)$$

$$= \frac{1}{2\mu_0}\mathrm{Re}(\underline{E}_0 \times (\underline{k}\times\overline{\underline{E}}_0))\cdot\frac{1}{\omega}$$

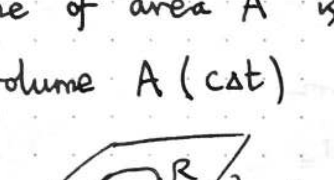
$$= \frac{1}{2\mu_0\omega}\mathrm{Re}\left(|\underline{E}_0|^2\underline{k} - \underbrace{\underline{E}_0\cdot\underline{k}}_{\text{zero}}\,\overline{\underline{E}}_0\right)$$

$$= \frac{|\underline{E}_0|^2}{2\omega\mu_0}\underline{k}$$

$$\therefore \langle \underline{S} \rangle = \frac{c\varepsilon_0}{2}|\underline{E}_0|^2\underline{k}$$

$$= c\langle W \rangle \underline{k}$$

Consider plane $\perp \underline{k}$. In time Δt, energy crossing region R of plane of area A is $\langle \underline{S} \rangle \cdot \hat{\underline{k}} A\Delta t = c\langle W \rangle A \Delta t$, i.e. all energy in a volume $A(c\Delta t)$ crosses R.



Wave transports energy at speed c.

L12.2

4.4 Time-varying fields inside a conductor

Assume all fields are periodic in time with frequency $\omega \neq 0$.

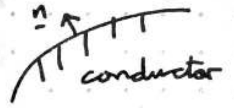
Inside conductor, can show $\underline{E}, \underline{B}$ decay as $e^{-z/\delta}$, where

$z =$ distance from surface

$\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$ "skin depth"

Perfect conductor: $\sigma \to \infty,\ \delta \to 0 \Rightarrow \underline{E}, \underline{B} = 0$ (if $\omega \neq 0$).

$\Rightarrow$ Just outside surface $\underline{E} = \underset{\text{sfc charge!}}{\frac{\sigma}{\epsilon_0}} \underline{n}$, $\underline{B} = \mu_0 \underline{K} \times \underline{n}$



$\Rightarrow$ tangential $\underline{E}$, normal $\underline{B}$ vanish

4.5 Sclar and vector potentials

M2: $\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow \underline{B} = \underline{\nabla} \times \underline{A}$ for some $\underline{A}(t, \underline{x})$

M3: $\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \Rightarrow \underline{\nabla} \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t} \right) = 0$

$\therefore\ \underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla} \Phi$ for some $\Phi(t, \underline{x})$

$(\Phi, \underline{A})$ not unique. If $(\Phi', \underline{A}')$ gives the same $\underline{E}, \underline{B}$ then

$\underline{\nabla} \times (\underline{A} - \underline{A}') = 0 \Rightarrow \underline{A}' = \underline{A} + \underline{\nabla} \lambda$ for some $\lambda(t, \underline{x})$

λ only defined up to freedom to add $f(t)$

$\underline{\nabla}(\Phi' - \Phi) + \frac{\partial}{\partial t}(\underline{A}' - \underline{A}) = 0$

$\therefore\ \Phi' = \Phi - \frac{\partial \lambda}{\partial t} + \text{const.} \leftarrow$ arbitrary

$\therefore\ \underline{A}' = \underline{A} + \underline{\nabla} \lambda,\ \Phi' = \Phi - \frac{\partial \lambda}{\partial t}$

V SPECIAL RELATIVITY

5.1 Galilean transformations

Alice, Bob inertial observers. A coords $(t, \underline{x})$; B pos^n $\underline{x} = \underline{v}t$

B coords $(t', \underline{x}')$. In Newtonian physics, B's coords related to A's coords by a Galilean transformation $t' = t,\ \underline{x}' = \underline{x} - \underline{v}t$

Charged particle in A's coords $m \ddot{\underline{x}}(t) = q \left[\underline{E}(t, \underline{x}(t)) + \dot{\underline{x}}(t) \times \underline{B}(t, \underline{x}(t)) \right]$

Invert via $t = t',\ \underline{x} = \underline{x}' + \underline{v}t$,

L12.3

so in B's coords

$$m\ddot{\underline{x}}'(t) = q\left[\underline{E}(t', \underline{x}'(t)+\underline{v}t') + \begin{matrix}\underline{v}\times\underline{B}(t', \underline{x}'(t)+\underline{v}t') \\ \dot{\underline{x}}'(t')\times\underline{B}(t', \underline{x}'(t)+\underline{v}t')\end{matrix}\right]$$

$\Rightarrow$ B sees fields $\underline{E}(t', \underline{x}') = \underline{E}(t', \underline{x}'(t)+vt') + \underline{v}\times\underline{B}(t', \underline{x}'(t)+\underline{v}t')$

$$\underline{B}(t', \underline{x}') = \underline{B}(t', \underline{x}'(t)+vt').$$

Assume $\rho = \underline{J} = 0$ and Maxwell's eq^{n}s hold for A.

But $\underline{\nabla}\cdot\underline{E}' \neq 0$ (Ex, in general). M1 not valid for Bob.

Maxwell's equations are not invariant under Galilean transformations.

Principle of relativity Laws of physics take the same form in every inertial frame.

We'll show invariance of Maxwell's eq^{n}s $\leftrightarrow$ coords related by Lorentz

5.2 Lorentz transf^{m}s

A : inertial frame S

B : " S' moving in x-dir^n wrt S

Lorentz $t' = \gamma\left(t - \frac{vx}{c^2}\right)$, $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

(B has pos^n $x' = 0 \leftrightarrow x = vt$ so speed v in S)

Event : definite time & pos^n, label by $(t, \underline{x})$

2 events with separation $(\Delta t, \Delta\underline{x})$ in S, $(\Delta t', \Delta\underline{x}')$ in S'

$$\Delta t' = \gamma\left(\Delta t^{*} - v\frac{\Delta x^{*}}{c^2}\right), \quad \Delta x' = \gamma(\Delta x^{*} - vt), \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z$$

$$-c^2(\Delta t')^2 + (\Delta\underline{x}')^2 = -c^2(\Delta t)^2 + (\Delta\underline{x})^2$$

$\Rightarrow$ A, B agree on value of invariant interval $(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta\underline{x})^2$

Events are $\left\{\begin{matrix}\text{timelike} \\ \text{spacelike} \\ \text{null / lightlike}\end{matrix}\right\}$ separated if $(\Delta s)^2 \left\{\begin{matrix}< 0 \\ > 0 \\ = 0\end{matrix}\right\}$

5.3 Minkowski spacetime

Let $x^0 = ct$, $x^i = x_i$.

$$(\Delta S)^2 = -(\Delta x^0)^2 + (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2$$

$$(\Delta S_E)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 \quad \text{(Euclidean space)}$$

Minkowski spacetime : $\mathbb{R}^4$ with invariant interval $(\Delta s)^2$

Indices $\mu, \nu, \ldots$ take values 0, 1, 2, 3

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \quad \text{where} \quad \underset{\swarrow \text{"metric tensor"}}{\eta_{\mu\nu}} = \mathrm{diag}(-1, 1, 1, 1) \quad \leftarrow (*)$$

(Summation convention – appear once as subscript, superscript)

$(\Delta s)^2$ is a quadratic form of signature $(- + + +)$

c.f. Euclidean space $(\Delta s_E)^2 = \delta_{ij} \Delta x_i \Delta x_j$ for Cartesian coords.

2 Cartesian coord systems related by $x_i' = \underset{\substack{\uparrow \\ \text{orthogonal} \\ \text{(rotation/refl)}}}{O_{ij}} x_j + \underset{\substack{\nwarrow \text{constant} \\ \text{(translation)}}}{a_i}$

$$\Rightarrow \delta_{ij} \Delta x_i \Delta x_j \underset{\substack{\uparrow \\ \text{since } O \text{ orthog}}}{=} \delta_{ij} \Delta x_i' \Delta x_j'$$

Isometries of Euclidean space, generate Eucl group

With Mink spacetime,

inertial frame ↔ coords x^μ s.t. (*) holds.

If S, S' inertial, coords x^μ, x'^μ, then

$$x'^\mu = \underset{\substack{\uparrow \\ \text{matrix}}}{\Lambda^\mu{}_\nu} x^\nu + \underset{\nwarrow \text{spacetime translation}}{a^\mu} \quad \leftarrow (**)$$

$$\Delta x'^\mu = \Lambda^\mu{}_\nu \Delta x^\nu$$

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = (\Delta s)^2 = \eta_{\mu\nu} \Delta x'^\mu \Delta x'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \Delta x^\rho \Delta^\sigma$$

$$= \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \Delta x^\mu \Delta x^\nu$$

Δx^μ arbitrary $\Rightarrow$ $\boxed{\eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}} \leftarrow (\dagger)$

i.e. $\Lambda^T \eta \Lambda = \eta \Rightarrow \det \Lambda = \pm 1$

e.g. (1) rotation of spacial coords $x'^0 = x^0$, $x'^i = \underset{\nwarrow \text{orthogonal}}{O_{ij}} x^j$

$\Lambda^0{}_0 = 1$, $\Lambda^i{}_j = O_{ij}$, other components vanish

(2) Lorentz transformation in x-dir^n

$$x'^0 = \gamma(x^0 - \beta x^1), \quad x'^1 = \gamma(x^1 - \beta x^0), \quad x'^2 = x^2, \quad x'^3 = x^3$$

where $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$.

$$\Rightarrow \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{in } y\text{-dir}^n: \begin{pmatrix} \gamma & & -\gamma\beta & \\ & 1 & & \\ -\gamma\beta & & \gamma & \\ & & & 1 \end{pmatrix}$$

$\Lambda^\mu{}_\nu$ satisfying (†) form a group called the <u>Lorentz group</u>

Transformations (**) with Λ obeying (†) form the <u>Poincaré group</u>

5.4 Sclars & vectors

Minkowski – scalar: takes same value wrt all inertial frames; $(\Delta s)^2$

Eucl space – vector $v_i' = O_{ij} v_j$

Minkowski – a <u>4-vector</u> $\underline{V}$ is a map from an inertial frame S with coords x^μ to numbers V^μ, the <u>components</u> of $\underline{V}$, s.t. the components of 2 inertial frames are related by $V'^\mu = \Lambda^\mu{}_\nu V^\nu$

(transforms like Δx^μ)

Euclidean vector ↦ 3-vector (Spatial components of a vector are such)

Euclidean scalar product $\underline{v} \cdot \underline{w} = \delta_{ij} v_i w_j$ (a scalar!)

Minkowski " " $\underline{V} \cdot \underline{W} = \eta_{\mu\nu} V^\mu W^\nu$,

scalar because $\eta_{\mu\nu} V'^\mu W'^\nu = \eta_{\mu\nu} V^\rho W^\sigma \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$

$= \eta_{\rho\sigma} V^\rho W^\sigma$ by (†).

$\underline{V}^2 = \underline{V} \cdot \underline{V}$,

$$\underline{V} \text{ is } \left\{ \begin{matrix} \text{timelike} \\ \text{spacelike} \\ \text{null} \end{matrix} \right\} \text{ if } \underline{V}^2 \text{ is } \left\{ \begin{matrix} <0 \\ >0 \\ =0 \end{matrix} \right\}.$$

5.5 Proper time, 4-velocity & 4-momentum

Curve in Mink spacetime $x^\mu = x^\mu(\lambda)$ in S

Tangent vector $V^\mu = \dfrac{dx^\mu}{d\lambda}$ — a 4-vector

e.g. B's worldline (trajectory) in S, $x = vt$ i.e. $x^1 = \beta x^0$

set $\lambda = x^0 \Rightarrow x^\mu = (\lambda, \beta\lambda, 0, 0) \Rightarrow V^\mu = (1, \beta, 0, 0)$

$$\text{A curve is } \left\{ \begin{matrix} \text{timelike} \\ \text{spacelike} \\ \text{null} \end{matrix} \right\} \text{ iff its tangent is everywhere } \left\{ \begin{matrix} \text{timelike} \\ \text{spacelike} \\ \text{null} \end{matrix} \right\}.$$

e.g. $\eta_{\mu\nu} V^{\mu} V^{\nu} = -1 + \beta^2 < 0 \quad (\beta^2 < 1) \Rightarrow$ worldline of inertial observer is timelike

<u>Rest frame</u> of an inertial observer O: inertial frame s.t. O is at rest at the origin $(x^i = 0)$.

2 events along worldline, $(\Delta s)^2 = -(\Delta x^0)^2 = -c^2(\Delta t)^2$

Δt is time interval between events measured by O.

Call this the proper time τ.

$c^2(\Delta\tau)^2 = -(\Delta s)^2$ RHS a scalar $\Rightarrow \tau$ a scalar

$c^2(\Delta\tau)^2 = -\eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}$ gives proper time between 2 events on O's worldline as measured by O

Can calculate in any frame. ↖ infinitesimal version

<u>Ideal clock</u> unaffected by acc^n

Consider ideal clock attached to a body undergoing non-inertial motion with trajectory $x^{\mu}(\lambda)$.

At any point consider an inertial observer with same tangent vector velocity as body.

Ideal clock measures proper time according to (*) valid for this observer, $\therefore$ $c\dfrac{d\tau}{d\lambda} = \sqrt{-\eta_{\mu\nu}V^{\mu}(\lambda)V^{\nu}(\lambda)}$

So proper time along curve from $x^{\mu}(0)$ is given by integrating wrt λ, with $\tau(0) = 0$.

L14.1

<u>4-velocity</u> of curve $U^\mu = \frac{dx^\mu}{d\tau}$

$\eta_{\mu\nu}U^\mu U^\nu = -c^2$ by def^n of τ

<u>3-velocity</u> $v^i = \frac{dx^i}{dt}$

$$U^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau}, \quad U^i = \frac{dx^i}{d\tau} = v^i\frac{dt}{d\tau}$$

$$\therefore \; -c^2 = \left(\frac{dt}{d\tau}\right)^2(-c^2+v^iv^i) \Rightarrow \frac{dt}{d\tau} = \gamma =: \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\boxed{U^0 = \gamma c, \; U^i = \gamma v^i}$$

Any body has a rest mass, +ve scalar m

4-momentum $\underline{P}$ defined by $P^\mu = mU^\mu \Rightarrow \eta_{\mu\nu}P^\mu P^\nu = -m^2c^2$

$P^0 = \gamma mc$, $P^i = \gamma m v^i$

4-momentum is <u>conserved</u>: respects principle of relativity

Components of $\underline{P}$ in observer O's rest frame give the energy and 3-momentum as observed by O.

$P^0 = E/c$, $P^i = p^i$

If $|\underline{v}| \ll c$, $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \Rightarrow P^0 \approx \frac{1}{c}\Big(\underset{\substack{\uparrow \\ \text{rest mass} \\ \text{energy}}}{mc^2} + \underset{\substack{\uparrow \\ \text{usual} \\ \text{K.E.}}}{\tfrac{1}{2}mv^2}\Big)$, $P^i = \gamma m v^i$

γ

If O has 4-velocity $\underline{U}$ then $-\eta_{\mu\nu}U^\mu P^\nu \underset{\substack{\uparrow \\ \text{evaluate} \\ \text{in rest frame}}}{=} cP^0 = E \leftarrow$ energy of body as seen by O

LHS scalar, can calculate in any frame $\quad U^\mu = (c,0,0,0)$

<u>5.6 Covectors & tensors</u>

$x_j = O^{-1}_{ji}(x'_i - a_i)$

f scalar, Euclidean change of coords $x'_i = O_{ij}x_j + a_i$

$$(\nabla' f)_i = \frac{\partial f}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial f}{\partial x_j} = (O^{-1})_{ji}(\nabla f)_j \underset{\substack{\uparrow \\ O \text{ is ortho}}}{=} O_{ij}(\nabla f)_j$$

$\Rightarrow$ components of $\underline{\nabla} f$ are vector components

$x^\nu = (\Lambda^{-1})^\nu{}_\mu(x'^\mu - a^\mu)$

In Minkowski spacetime, $(\nabla f)_\mu = \frac{\partial f}{\partial x^\mu}$ and

$$(\nabla' f)_\mu = \frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu(\nabla f)_\nu$$

$(\Lambda^{-1})^\nu{}_\mu \neq \Lambda^\mu{}_\nu$ ☹ $\quad \therefore (\nabla f)_\mu$ not a 4-vector

A <u>covector</u> $\underline{Z}$ is a map from an inertial frame to numbers Z_μ for which the components are s.t. $Z'_\nu = (\Lambda^{-1})^\mu{}_\nu Z_\mu$

∇f a covec

Up/down Greek indices distinguish vectors and covectors

A (4-) tensor of type (r,s) is a map $\underline{T}$ from an inertial frame to components $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$ s.t. the components obey

$$T'^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\mu_r}{}_{\rho_r} (\Lambda^{-1})^{\sigma_1}{}_{\nu_1} \cdots (\Lambda^{-1})^{\sigma_s}{}_{\nu_s} T^{\rho_1 \cdots \rho_r}{}_{\sigma_1 \cdots \sigma_s} \quad \overset{(*)}{\swarrow}$$

$(1,0) \longleftrightarrow$ 4-vector , $(0,1) \longleftrightarrow$ covector

e.g. $\underline{V}$ 4-vector , Z covector , define $(1,1)$ tensor $\underline{V} \otimes \underline{Z}$ (outer/tensor product) via $(\underline{V} \otimes \underline{Z})^\mu{}_\nu = V^\mu Z_\nu$ satisfies $(*)$

Tensors of type (r,s) form a vector space of dimension 4^{r+s}

Tensor eq^n: LHS and RHS are tensors of same type

If such eq^n holds in one frame, it holds in all frames

$\Rightarrow$ Satisfies principle of relativity

A tensor is <u>isotropic</u> if its components are the same wrt all frames.

e.g. δ with components $\delta^\mu{}_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \text{else} \end{cases}$ (isotropic by def^n)

$$\delta'^\mu{}_\nu = \delta^\mu{}_\nu = \Lambda^\mu{}_\rho (\Lambda^{-1})^\rho{}_\nu = \Lambda^\mu{}_\rho (\Lambda^{-1})^\sigma{}_\nu \delta^\rho{}_\sigma$$

$\Rightarrow (1,1)$ tensor

$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ also isotropic

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \eta_{\rho\sigma} = (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma \eta_{\alpha\beta}$$

$$= \eta_{\mu\nu} \quad \text{so yes} \quad \Rightarrow (0,2) \text{ tensor}$$

$(0,2)$ tensor , cpts $T_{\mu\nu} \longleftrightarrow 4 \times 4$ matrix

If invertible , then $(T^{-1})^{\mu\nu}$ are cpts of a $(2,0)$ tensor (sheet 3)

$\eta_{\mu\nu} \to (\eta^{-1})^{\mu\nu} \equiv \eta^{\mu\nu}$ "inverse metric"

$\varepsilon_{\mu\nu\rho\sigma}$: antisymmetric , $\varepsilon_{0123} = +1$ defines isotropic $\overset{\text{factor of } \det\Lambda}{\swarrow}$ <u>pseudotensor</u> of type $(0,4)$

L14.3

Contraction of an up index with a down index

e.g. $\underline{T}$ type (2,1), Let $V^\mu = T^{\mu\nu}{}_\nu$

$$V'^\mu = T'^{\mu\nu}{}_\nu = \Lambda^\mu{}_{\rho_1} (\Lambda^{-1})^\sigma{}_\nu \Lambda^\nu{}_{\rho_2} T^{\rho_1\rho_2}{}_\sigma$$
$$= \Lambda^\mu{}_{\rho_1} \delta^\sigma{}_{\rho_2} T^{\rho_1\rho_2}{}_\sigma$$
$$= \Lambda^\mu{}_{\rho_1} T^{\rho_1\sigma}{}_\sigma$$

Contraction of up with up (not a tensor!)

e.g. $\Delta x^\mu \Delta x^\mu = (\Delta x^0)^2 + \dots + (\Delta x^3)^2$ changes from frame to frame

(0,2) tensor symmetric if $T_{\mu\nu} = T_{\nu\mu}$

antisym if $T_{\mu\nu} = -T_{\nu\mu}$

4-vector $\underline{V} \xrightarrow[\text{product}]{\text{out}}$ (1, 2) tensor, cpts $\eta_{\mu\nu} V^\rho \xrightarrow[\text{contract}]{}$ covector $V_\mu = \eta_{\mu\nu} V^\nu$

defines map from vectors to covectors

inverse:

covector $\underline{Z}$, cpts $Z_\mu \rightarrow$ 4-vector $\underline{Z}$, cpts $Z^\mu = \eta^{\mu\nu} Z_\nu$

generalise defn of tensors so up/down in any order, e.g.

$T_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma} T^\rho{}_\sigma$ transforms via $(\Lambda^{-1})^\rho{}_\mu \Lambda^\nu{}_\sigma T_\rho{}^\sigma$

Define new tensors by raising or lowering indices with η, η^{-1}

(e.g.) (1,1) tensor, cpts $T^\mu{}_\nu$ define new tensors $T_{\mu\nu} = \eta_{\mu\rho} T^\rho{}_\nu$

$$T^{\mu\nu} = \eta^{\nu\rho} T^\mu{}_\rho$$
$$T_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma} T^\rho{}_\sigma$$

$\eta_{\mu\nu} = \text{diag}(-1,1,1,1) = \eta^{\mu\nu}$!! $\Rightarrow$ raise or lower by multiplying by ±1

e.g. $T^0{}_i = -T_{0i}$, $T^{ij} = T_{ij}$, $T^{00} = T_{00}$

$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$

L15.1

Tensor field of type (r, s): tensor of type (r, s) at each point of Mink spacetime. It is diff iff its cpts in S are diff functions of x^μ ($\Rightarrow$ also diff in S')

Derivative of a tensor field $\underline{T}$ of type (r, s) is a tensor field $\underline{\nabla}\,\underline{T}$ of type (r, s+1) with components $\partial_\mu T^{\nu_1 \cdots \nu_r}{}_{\rho_1 \cdots \rho_s}$.

e.g. $$\partial'_\mu V'^\nu = \frac{\partial x^\rho}{\partial x'^\mu}\underset{\nwarrow \partial_\rho}{} V'^\nu = (\Lambda^{-1})^\rho{}_\mu\, \partial_\rho (\Lambda^\nu{}_\sigma V^\sigma) = (\Lambda^{-1})^\rho{}_\mu \Lambda^\nu{}_\sigma\, \partial_\rho V^\sigma$$

which indeed is the desired transformation law

$\partial_\mu V^\mu$: divergence of vector field (scalar field)

$\partial_\mu f$: gradient of scalar field (vector field)

$\partial_\mu Z_\nu - \partial_\nu Z_\mu$: curl of covector field ((0,2) tensor)

d'Alembertian operator : $$\Box = \partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$$
$$= -\left(\frac{\partial}{\partial x^0}\right)^2 + \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^3}\right)^2$$
$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

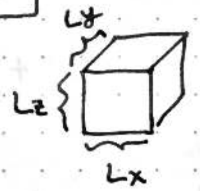
$\therefore$ wave eqⁿ for scalar field Φ is $\Box\Phi = 0$

LHS is scalar $\Rightarrow$ satisfies principle of relativity

(wave equation with speed $\neq c$ does not!)

5.7 Charge and current density

In S



N^3 particles each of charge q at rest

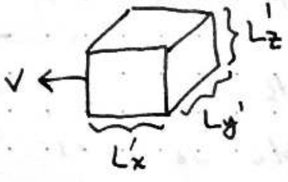
$L_x = L_y = L_z = L$

R = region of spacetime occupied by cube

charge density : $\frac{N^3 q}{L^3} = \rho$ (in R)

current density : $\underline{J} = 0$

In S'



$L'_x = L_x/\gamma = L/\gamma$

$L'_y = L'_z = L$

charge density : $\rho' = \frac{N^3 q}{L^3}\gamma = \gamma\rho$

current density : $\underline{J}' = q\frac{\gamma N^3 v}{L^3} = \gamma\rho v$

$\underline{J}' = (-\gamma\rho v, 0, 0)$ in R

to compute $\underline{J}$:

distance between particles in x'-dirⁿ

L'_x/N

particles crossing plane ⊥ x'-dirⁿ / time

$(vN/L'_x)N^2$

$= \gamma v N^3/L$

Let $j^\mu = (\rho c, \underline{J})$ in any inertial frame.

$S: j^\mu = (\rho c, \underline{0})$, $S': j'^\mu = (\gamma \rho c, -\gamma \rho v, 0, 0)$

$$= \underset{\nwarrow}{\Lambda^\mu{}_\nu} j^\nu$$ Lorentz boost in x-dirⁿ

∴ j^μ transforms as cpts of a 4-vector

the charge-current density 4-vector

$$\partial_t \rho + \underline{\nabla} \cdot \underline{J} = 0 \iff \partial_0(\rho c) + \partial_i J_i = 0 \iff \boxed{\partial_\mu J^\mu = 0}$$

5.8 Maxwell tensor

$\underline{F} = m\underline{a} = \frac{d\underline{p}}{dt}$

In SR, $\underline{F} \to \underset{\substack{\uparrow \\ \text{4-force}}}{f^\mu}$, $\frac{d\underline{p}}{dt} \to \underset{\nwarrow \text{scalar}}{\frac{d}{d\tau}} P^\mu \leftarrow$ 4 vector suggests $f^\mu = \frac{\partial P^\mu}{d\tau}$ (*)

$$\therefore \eta_{\mu\nu} f^\mu P^\nu = \eta_{\mu\nu} \frac{dP^\mu}{d\tau} P^\nu = \frac{1}{2} \frac{d}{d\tau}\Big(\underbrace{\eta_{\mu\nu} P^\mu P^\nu}_{= -m^2c^2}\Big) = 0$$

Try $$f^\mu = q \underset{\nwarrow (1,1)\text{-tensor}}{F^\mu{}_\nu} U^\nu$$

Need $0 = \eta_{\mu\nu} f^\mu P^\nu = m f_\mu U^\nu = qm F_{\mu\nu} U^\mu U^\nu$

Guaranteed if $F_{\mu\nu} = -F_{\nu\mu}$.

$$\therefore f^\mu = q\, \eta^{\mu\nu} \underset{\nwarrow \text{antisym } (0,2)\text{-tensor}}{F_{\nu\rho}} U^\rho$$

(*) $\perp U^\mu \Rightarrow$ only 3 independent components

$$\frac{dP^i}{d\tau} = q\, \eta^{ij} F_{j\rho} U^\rho = q F_{i\rho} U^\rho = q\left(F_{i0} U^0 + F_{ij} U^j\right) \qquad (\dagger)$$
$$= q\gamma\left(-c F_{0i} + F_{ij} v^j\right)$$

Assume $|\underline{v}| \ll c$, so $\gamma \approx 1$, $\tau \approx t$, $P^i = mv^i$ $\Rightarrow m \frac{dv_i}{dt} = q(-cF_{0i} + F_{ij} v^j)$

Non-relativistic $\Rightarrow$ usual Lorentz force law should be valid

∴ $F_{0i} = -E_i/c$, $F_{ij} = \varepsilon_{ijk} B_k$

(Antisymm: $F_{i0} = -F_{0i}$, $F_{00} = 0$)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

So the electromagnetic field is described by a $(0,2)$ tensor $\underline{F}$, the Maxwell tensor.

L15.3

$(\dagger) \Leftrightarrow \cancel{\gamma}\frac{d}{d\tau}(m\gamma v^i) = q\gamma(\underline{E} + \underline{v} \times \underline{B})$ eq^n of motion of relativistic charged ptcle

$E_i = -cF_{0i}\,, \quad B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$

$\rho = j^0/c\,, \quad J_i = j^i$

(M1) $\partial_i(-cF_{0i}) = \frac{1}{c\varepsilon_0}j^0 \Leftrightarrow \partial_i(F^{0i}) = \mu_0 j^0 \qquad \left(\frac{1}{c^2} = \mu_0\varepsilon_0\right)$

$F^{\mu\nu}$ antisym (check) $\Rightarrow F^{00} = 0$

$\therefore \partial_0(F^{00}) + \partial_i(F^{0i}) = \mu_0 j^0$

$\therefore \partial_\nu F^{0\nu} = \mu_0 j^0$ — (1)

(M4) $\varepsilon_{ijk}\partial_j B_k = \frac{1}{c\cancel{2}}\partial_0 E_i + \mu_0 J_i$

$\Leftrightarrow \partial_j F_{ij} + \partial_0 F_{0i} = \mu_0 j^i$

$\Leftrightarrow -\partial_0 F_{i0} + \partial_j F_{ij} = \mu_0 j^i$

$\Leftrightarrow \partial_0 F^{i0} + \partial_j F^{ij} = \mu_0 j^i$

$\Leftrightarrow \partial_\nu F^{i\nu} = \mu_0 j^i$ — (2)

(1) & (2) $\Leftrightarrow \boxed{\partial_\nu F^{\mu\nu} = \mu_0 j^\mu}$ tensor eq^n! equivalent to M1, M4

L16.1

(M2) $0 = \partial_i B_i = \frac{1}{2}\varepsilon_{ijk}\partial_i F_{jk} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}$

$\therefore$ M2 $\Longleftrightarrow$ $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ — (3)

(M3) $\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t}$, $0 = \varepsilon_{ijk}\partial_j E_k + c\partial_0 B_i$

$$= -c\,\varepsilon_{ijk}\partial_j F_{0k} + \frac{c}{2}\partial_0 \varepsilon_{ijk} F_{jk}$$

$(A)\ \partial_\nu F^{\mu\nu} = \mu_0 j^\mu$

multiply by $\frac{1}{c}\varepsilon_{ilm}$

$\Rightarrow$ $\partial_l F_{0m} + \partial_m F_{l0} + \partial_0 F_{ml} = 0$ — (4)

(3), (4) $\Longleftrightarrow$ $\boxed{\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0}$ — (B)

Bianchi identity (M2) & (M3)

Maxwell equations $\Longleftrightarrow$ 2 tensor equations (A, B)

$\therefore$ respect principle of relativity

NB (A) : $\mu_0 \partial_\mu j^\mu = \underbrace{\partial_\mu \partial_\nu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{anti}} = 0$

$\therefore$ j^μ must be conserved

$F'_{\mu\nu} = (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu F_{\rho\sigma}$ i.e. $F' = (\Lambda^{-1})^T F (\Lambda^{-1})$ — (*)

Λ : Lorentz trans^n in x-dir^n, velocity v

Λ^{-1} : " " , " $-$v

$$\Lambda^{-1} \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \beta = \frac{v}{c}$$

Exercise : subst into (*) to determine $F'_{\mu\nu}$

$\Rightarrow$ $E'_1 = E_1$, $B'_1 = B_1$

$E'_2 = \gamma(E_2 - vB_3)$, $B'_2 = \gamma(B_2 + \frac{v}{c^2}E_3)$

$E'_3 = \gamma(E_3 + vB_2)$, $B'_3 = \gamma(B_3 - \frac{v}{c^2}E_2)$

$\underline{v} = (v, 0, 0)$, $\underline{E} = \underset{\substack{\uparrow \\ \text{parallel} \\ \text{to } \underline{v}}}{\underline{E}_\parallel} + \underset{\substack{\nwarrow \\ \text{perp to } \underline{v}}}{\underline{E}_\perp}$, $\underline{B} = \underline{B}_\parallel + \underline{B}_\perp$

then $\underline{E}'_\parallel = \underline{E}_\parallel$, $\underline{B}'_\parallel = \underline{B}_\parallel$

$\underline{E}'_\perp = \gamma(\underline{E}_\perp + \underline{v} \times \underline{B})$, $\underline{B}'_\perp = \gamma(\underline{B}_\perp - \frac{1}{c^2}\underline{v} \times \underline{E})$ — (†)

These hold for a Lorentz trans^n in arbitrary $\underline{v}$

- rotate axes of S, S' so $\underline{v} = (v, 0, 0)$

L16.2

- apply (†)
- rotate back to original axes

(†) are 3-vector eq^{n}s so hold wrt all axes

$$\begin{aligned} F_{\mu\nu} F^{\mu\nu} &= F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} \\ &= 2 F_{0i} F^{0i} + F_{ij} F^{ij} \\ &= -2 \frac{E_i}{c} \cdot \frac{E_i}{c} + \varepsilon_{ijk} B_k \varepsilon_{ijl} B_l \\ &= -2 \left(E^2/c^2 - B^2 \right) \end{aligned}$$

LHS is a scalar $\Rightarrow E^2/c^2 - B^2$ scalar, smalle value in all frames

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} &= \varepsilon_{0ijk} F^{0i} F^{jk} + \varepsilon_{i0jk} F^{i0} F^{jk} \\ &\quad + \varepsilon_{ij0k} F^{ij} F^{0k} + \varepsilon_{ijk0} F^{ij} F^{k0} \\ &= -4 \varepsilon_{ijk} F_{0i} F_{jk} \\ &= -4 \varepsilon_{ijk} (-E_i/c) \varepsilon_{jkl} B_l \\ &= \frac{8}{c} \underline{E} \cdot \underline{B} \end{aligned}$$

$\varepsilon_{0ijk} = \varepsilon_{ijk}$

LHS pseudoscalar $\Rightarrow \underline{E} \cdot \underline{B}$ also a pseudoscalar

e.g. if $\underline{B} = \underline{0}$ in S then $\underline{E} \cdot \underline{B} = 0$ in S

$\Rightarrow \underline{E}' \cdot \underline{B}' = 0$ in S'

$\Rightarrow \underline{E}' \perp \underline{B}'$ in S'

5.9 Electromagnetic Potential

$$F_{ij} = \varepsilon_{ijk} B_k \underset{\substack{\uparrow \\ \underline{B} = \underline{\nabla} \times \underline{A}}}{=} \varepsilon_{ijk} \varepsilon_{klm} \partial_l A_m = \partial_i A_j - \partial_j A_i \quad \text{—— (1)}$$

$$F_{0i} = -E_i/c \underset{\substack{\uparrow \\ \underline{E} = -\underline{\nabla} \Phi - \dot{\underline{A}}}}{=} \frac{1}{c} \partial_i \Phi + \partial_0 A_i = \partial_0 A_i - \partial_i \left(-\Phi/c \right) \quad \text{—— (2)}$$

Let $A_\mu = (-\Phi/c, A_i)$.

(1, 2) $\Longleftrightarrow$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ —— (*)

Gauge freedom : $\tilde{\underline{\Phi}} = \underline{\Phi} - \frac{\partial\lambda}{\partial t}$, $\tilde{\underline{A}} = \underline{A} + \underline{\nabla}\lambda$

$$\Leftrightarrow \quad \tilde{A}_\mu = A_\mu + \partial_\mu \lambda$$

check : $\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \cancel{\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda}$ $\checkmark$

$\therefore$ A_μ , $\tilde{A}_\mu$ give same $F_{\mu\nu}$

Can eliminate gauge freedom with a <u>gauge condition</u>

Lorenz gauge : $\partial_\mu A^\mu = 0$

$$\partial_\mu \tilde{A}^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \lambda = \partial_\mu A^\mu + \Box \lambda$$

Choose λ to obey inhomogeneous wave equation , source $\partial_\mu A^\mu$

$$\Rightarrow \partial_\mu \tilde{A}^\mu = 0$$

LHS of $(*)$ tensor components , suggests A_μ a covector, actually only a "covector up to a gauge transf^{m}"

$$A'_\mu = (\Lambda^{-1})^\nu{}_\mu A_\nu + \partial'_\mu \lambda \qquad \text{for some } \lambda .$$

A_μ : "covector potential"

A^μ : "4-vector potential"

Maxwell eq^{n}s using A_μ

(B) automatically satisfied (Ex)

(A) $\partial_\nu F^{\mu\nu} = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 j^\mu$

$$\partial^\mu (\partial_\nu A^\nu) - \partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu$$

Lorenz gauge : $\Box A^\mu = -\mu_0 j^\mu$ inhomogeneous wave eq^n