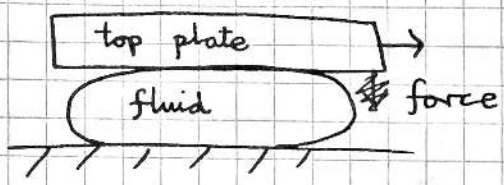
L 1.1

# Fluid Dynamics

- **Description** (Mathematical models)
  - ↳ Practical problems
  - ↳ Assumptions
  - ↳ Simplified models
  - ↳ Solve, verify

## §0 Introduction

What is a fluid?




A fluid cannot support a shear stress while at rest, e.g. gases, liquids

Mathematically,

$$\rho\left[\frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\underline{\nabla}p + \underline{F} \quad , \quad \underline{\nabla}\cdot\underline{u} = 0$$

with some BCs, for inviscid flod, or

$$\rho\left[\frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\underline{\nabla}p + \mu\nabla^2\underline{u} + \underline{F} \quad , \quad \underline{\nabla}\cdot\underline{u} = 0$$

## §1 ~~Introduction~~ Kinematics

### §1.1 Continuum model, Lagrangian and Eulerian descriptions

$10^{25}$ molecules in $1\,m^3$

size of molecule $\lesssim 10^{-9}$ m

spacing $\lesssim 10^{-8}$ m

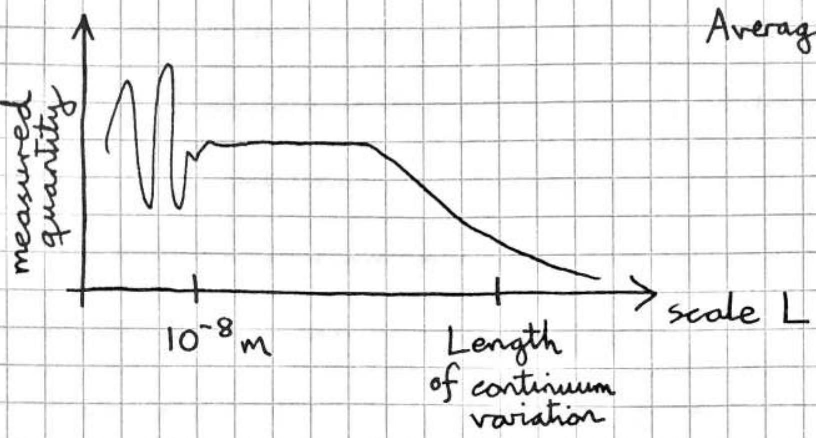
mean free path between collisions $\lesssim 10^{-7}$ m

rms velocity $\sim 10^{2}$ ms$^{-1}$

Define the continuous functions such as density, velocity, pressure as continuous functions of position $\underline{x}$ at any moment of time


$$\rho(\underline{x}, t) = \frac{\text{mass in box centred at } \underline{x}}{\text{volume of box } (\sim L^3)}$$

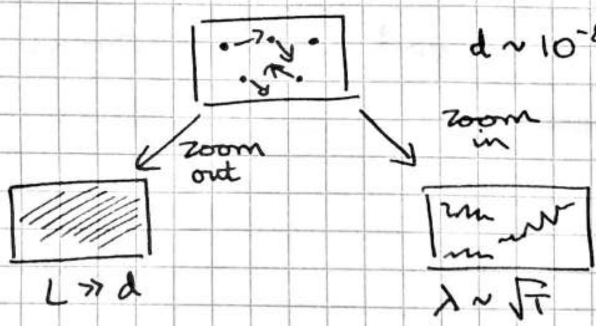
Average over small linear dimension.




Classical and quantum fluids ⊛

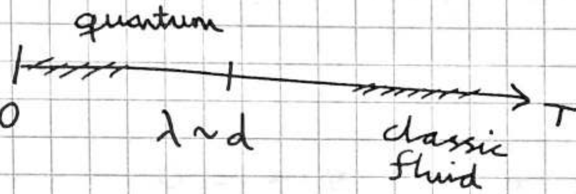
$d \sim 10^{-8}$ m

Bose-Einstein condensates



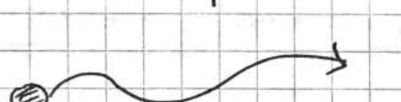

If $\lambda \sim d$, synchronization

Giant matter waves




## Lagrangian vs Eulerian description

Lagrangian description follows individual particles or parcels of fluid.



$\rho$, $\underline{u}$, $p$ etc are functions of initial position $\underline{x}_0$ and $t$.

The position $\underline{x} = \underline{X}(\underline{x}_0, t)$ has to be determined as part of the solution.

Eulerian description: $\rho(\underline{x},t)$, $\underline{u}(\underline{x},t)$, $p(\underline{x},t)$ where $\underline{x}$ is fixed and $t$ varies.

$\underline{u}(\underline{x},t) \rightarrow$ succession of particles / parcels

## §1.2 Flow visualisations: pathlines & streamlines

1) Pathline – trajectory of a single fluid particle/parcel starting at some point ~ long exposure photograph

2) Streamline – curve everywhere tangent to instantaneous flow direction ~ short exposure photograph

Example 2D flow $\underline{u} = (yt, 1)$

1. Pathline

$$\underline{x} = \underline{X}(\underline{x}_0, t)$$

$$\frac{\partial \underline{X}}{\partial t} = \underline{u}(\underline{X}(t), t) \quad \text{with} \quad \underline{X} = \underline{x}_0 = (x_0, y_0) \quad \text{at } t = 0$$

$$\frac{\partial X}{\partial t} = Yt, \quad \frac{\partial Y}{\partial t} = 1$$

$$\Rightarrow Y = y_0 + t$$

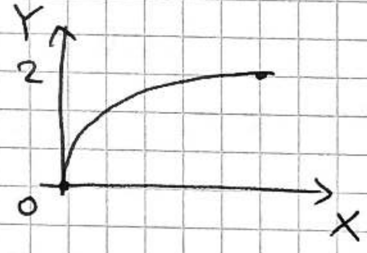
$$\therefore \frac{\partial X}{\partial t} = t(y_0 + t) \Rightarrow X = x_0 + \tfrac{1}{2} y_0 t^2 + \tfrac{1}{3} t^3$$

Eliminate $t$:

$$X = x_0 + \tfrac{1}{2} y_0 (Y - y_0)^2 + \tfrac{1}{3}(Y - y_0)^3$$

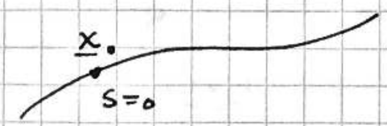
$(x_0, y_0) = (0, 0)$ and $0 \leq t \leq 2$ give

$$X = \tfrac{1}{3} Y^3 \text{ from } Y = 0 \text{ to } 2$$



L 2.1

## Streamlines $\underline{u} = (yt, 1)$




$$\underline{x} = \underline{X}(s, \underline{x}_0, t) \; \left[ \underline{X}(s) \text{ for brevity} \right]$$

$$\frac{\partial \underline{X}}{\partial s} = \underline{u}(\underline{X}(s), t)$$

with $\underline{X} = \underline{x}_0$ at $s = 0$

$\underline{X} = (X, Y)$

$\partial X / \partial s = Yt \quad , \quad \partial Y / \partial s = 1$

$\Rightarrow Y = y_0 + s$

$\Rightarrow X = x_0 + y_0 t s + \frac{1}{2} t s^2$

Eliminate $s = Y - y_0$ and get

$$X = x_0 + y_0 t (Y - y_0) + \tfrac{1}{2} t (Y - y_0)^2$$

Streamline for $t = 2$, $\underline{x}_0 = \underline{0}$ is $X = Y^2$

If the flow is steady, the streamlines and pathlines coincide.

e.g. replace $(yt, 1)$ with $(2y, 1)$

## §1.3 Material derivative

Consider a field $\chi(\underline{x}, t)$ (e.g. $\rho, p, T, u, \dots$)

An observer moving with the flow may see changes of $\chi$ even if at each location $\chi$ is independent of time.

Let the position of the observer at time $t$ be $\underline{x} = \underline{X}(t)$.

They are moving with fluid, so $\frac{d\underline{X}}{dt} = \underline{u}(\underline{X}, t)$.

The rate of change of $\chi$ with time seen by such an observer is

(by chain rule) $$\frac{d}{dt}\chi(\underline{x}, t) = \frac{d\underline{X}}{dt} \cdot \underline{\nabla} \chi \Big|_{\underline{x} = \underline{X}} + \frac{\partial \phi}{\partial t}$$

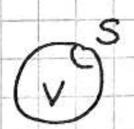
Define material derivative $\frac{D}{Dt}$ as

$$\underbrace{\frac{D\chi}{Dt}}_{\substack{\text{material} \\ \text{derivative}}} = \underbrace{\frac{\partial \chi}{\partial t}}_{\substack{\text{Eulerian} \\ \text{derivative}}} + \underbrace{\underline{u} \cdot \underline{\nabla} \chi}_{\substack{\text{convective} \\ \text{term}}} \quad (\text{at } \underline{x} = \underline{X})$$

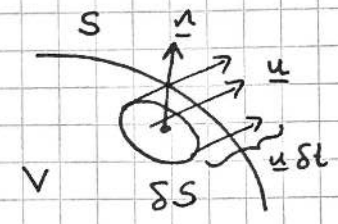
(Lagrangian, total, etc.)

## §1.4 Conservation of mass

Assume that fluid is neither created or destroyed.




Fluid mass conserved.




In time $\delta t$, $\rho\, \underline{n} \cdot \underline{u}\, \delta t$ mass of fluid crosses $\delta S$.

Over the whole surface, we get

$$\left( \int_S \rho \underline{u} \cdot d\underline{S} \right) \delta t$$

Total mass in $V$ is $\int_V \rho \, dV$.

Therefore $\frac{d}{dt} \int_V \rho \, dV = - \int_S \rho \underline{u} \cdot d\underline{S}$

By div thm, $\int_V \dot{\rho} \, dV = - \int_S \rho \underline{u} \cdot d\underline{S}$ — Integral form of mass conservation

becomes $\dot{\rho} = - \underline{\nabla} \cdot (\underbrace{\rho \underline{u}}_{\text{flux (amount of mass crossing unit sfc area)}})$. — Diff form of mass conservation

Note $\underline{\nabla} \cdot (\rho \underline{u}) = \rho\, \underline{\nabla} \cdot \underline{u} + \underline{\nabla} \rho \cdot \underline{u}$

$$\frac{D\rho}{Dt} = - \rho (\underline{\nabla} \cdot \underline{u}) \quad \#\#$$

Lagrangian form of mass conservation

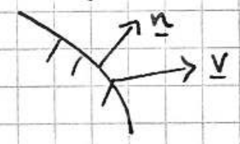
## §1.5 Incompressibility

$\rho = \text{const} \Rightarrow \underline{\nabla} \cdot \underline{u} = 0$

## §1.6 Boundary conditions at an incompremeable boundary


No fluid across if $\underline{u} \cdot \underline{n} = 0$ (stationary boundary)

Moving boundary



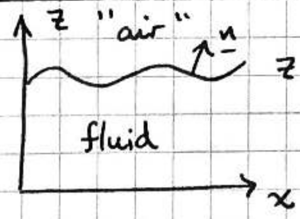

Relative to moving boundary the velocity $\underline{u} - \underline{v}$, so $(\underline{u} - \underline{v}) \cdot \underline{n} = 0$.

Note we require $\underline{u} = \underline{0}$ on solid wall only if fluid is viscous.

For inviscid fluids (most of course) only $\underline{u} \cdot \underline{n} = 0$ is required.

L2.3

## Application to free surface




$z = \zeta(x, y, t)$

$\varphi = z - \zeta(x, y, t)$

zero on sfc

$\underline{n} = \underline{\nabla}\varphi / |\underline{\nabla}\varphi|$

Apply $\underline{U} \cdot \underline{n} = \underline{u} \cdot \underline{n}$ $\qquad \underline{u} = (u, v, w)$

Express $\underline{n}$ and $\underline{U}$ in terms of $\zeta(x, y, t)$

$$\nabla\varphi = \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right)$$

$$\underline{U} = \left(0, 0, \frac{\partial \zeta}{\partial t}\right)$$

$$\therefore \ \underline{u} \cdot \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) = \left(0, 0, \frac{\partial \zeta}{\partial t}\right) \cdot \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right)$$

$$\therefore \ \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} - w = 0$$

$$\frac{D\zeta(x, y, t)}{Dt} - \frac{Dz}{Dt} = 0$$

$$\frac{D}{Dt}\left(\zeta(x, y, t) - z\right) = 0$$

L3.1

## §1.7 Streamfunction

2D fluids, $\underline{u} = (u, v)$

$\underline{\nabla} \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$ (by conservation of mass*)

So $\exists\, \psi(x,y)$ s.t. $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Note $\underline{\nabla} \cdot \underline{u} = 0$ is satisfied for such $u, v$.

$(dx, dy) \to (dy, -dx)$

$\psi$ is called a <u>streamfunction</u>

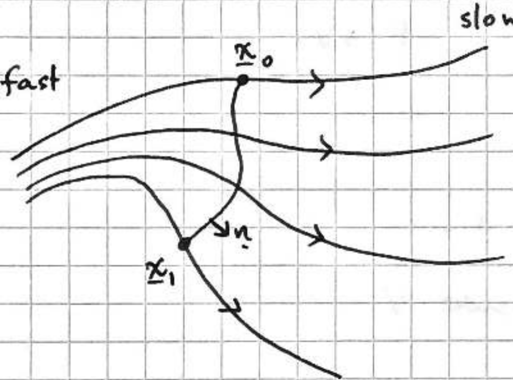
<u>Properties</u>

1) Curves of constant $\psi$ are tangent to $\underline{u}$ at each $(x,y)$.

i.e. they are streamlines

Indeed, $\underline{\nabla}\psi \cdot \underline{u} = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \cdot \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) = 0$.

2) $|\underline{u}| = |\underline{\nabla}\psi| \Rightarrow$ fluid moves faster where streamlines are dense




3) $\psi(\underline{x}_1) - \psi(\underline{x}_0)$

$$= \int_C d\psi = \int_C \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= \int_C -v\, dx + u\, dy$$

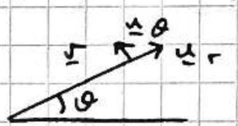
$$= \int_C \underline{u} \cdot \underline{n}\, ds$$

$\psi$ gives volume flux across curves $C$ between $\underline{x}_0, \underline{x}_1$

NB orientation matters

<u>Note</u> $\underline{u} = -\underline{k} \times \underline{\nabla}\psi$, $\underline{k} = (0,0,1)$ uh... sign error? YES

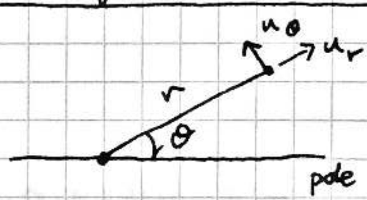
<u>2D polars</u>




$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r \underbrace{u_r}_{\frac{\partial \psi}{\partial \theta}}) + \frac{1}{r}\frac{\partial}{\partial \theta} \underbrace{u_\theta}_{-\frac{\partial \psi}{\partial r}} = 0$$

Streamfunction verifies $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$.

<u>3D axisymmetric case</u> $(r, \theta, \varphi)$ ← indep of $\varphi$ though




$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, u_\theta) = 0$$

$$= \frac{\partial}{\partial r}(r^2 \sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, u_\theta) = 0$$

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \cancel{r} \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

L 3.2

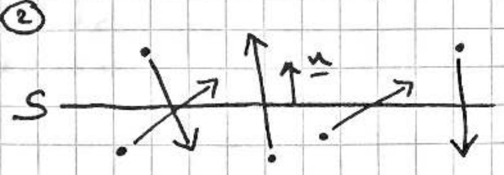
## §2 Dynamics

### §2.1 Surface vs body forces, and the concept of pressure

$\underline{F} \propto$ surface area or volume

(pressure, friction) (gravity, EM)

↑ neglect




A force in $\underline{n}$ direction = "pressure force"

Pressure force is isotropic

$p(\underline{x}, t)$ (pressure)

Force is $\underline{n}\, p\, \delta S$ for element $\delta S$

$\underline{n}\, p\, \delta S$ is the force exerted on the fluid into which $\underline{n}$ points

Volume forces $\underline{F}$ – force per unit mass

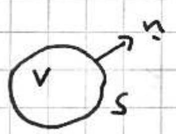
Ex $\underline{F} = \rho\, \underline{g}\, \delta V$ where $\underline{g}$ is gravitational acceleration

Force per unit volume $\rho \underline{F}$, force on $\delta V$ $\rho \underline{F}\, \delta V$

Write $\underline{g} = -\underline{\nabla}\varphi$ ($\underline{F}$ is conservative)

### §2.2 Momentum equation for inviscid flow

Arbitrary $V$




Momentum inside $V$

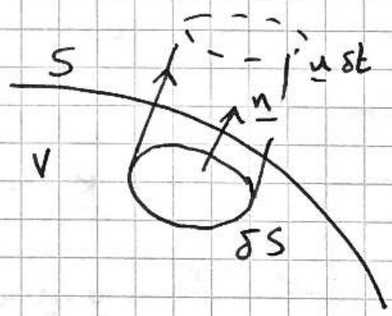
$$\int_V \rho \underline{u}\, dV$$

How does momentum change?

1) volume forces

2) surface forces

3) fluid leaving or entering $V$




$$\delta \underline{p} = \rho \underline{u}\, (\underline{u}\,\delta t)\cdot(\underline{n}\,\delta S) \Rightarrow \int_S \rho \underline{u}\, (\underline{u}\,\delta t)\cdot \underline{n}\, dS$$

$$\therefore \frac{d}{dt}\int_V \rho \underline{u}\, dV = \int_V \rho \underline{F}\, dV - \int_S p\, \underline{n}\, dS - \int_S \rho \underline{u}\, (\underline{u}\cdot\underline{n})\, dS$$

~ Integral form of conservation of momentum

$i^{th}$ component

$$\frac{d}{dt}\int_V \rho u_i\, dV = \int_V \rho F_i\, dV - \int_S p\, n_i\, dS - \int_S \rho u_i u_j n_j\, dS$$

L 3.3

Use generalised divergence theorem

- $$\frac{d}{dt}\int_V \rho u_i \, dV = \int_V \left( \rho F_i - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(\rho u_i u_j) \right) dV$$

$$\therefore \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho F_i$$

$$\therefore \rho \underbrace{\left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right)}_{\frac{Du_i}{Dt}} + u_i \underbrace{\left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right\}}_{\text{zero by mass cons.}} = -\frac{\partial p}{\partial x_i} + \rho F_i$$

$$\therefore \rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F} \qquad \text{(Newton's second law)}$$

- Euler equation + conservation of mass + BCs on surface fully determines the motion of inviscid flow

L 4.1

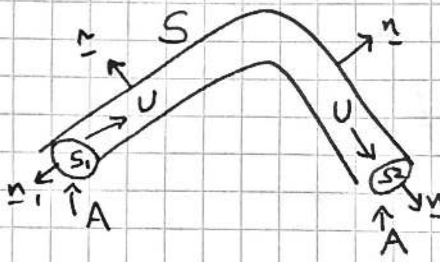
<u>Recall</u> $\quad \frac{d}{dt}\int_V \rho \underline{u}\, dv = \int_V \rho \underline{F}\, dV - \int_S p\underline{n}\, dS - \int_S \rho \underline{u}(\underline{u}\cdot\underline{n})\, dS$

- $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$ $\qquad \underline{F} = -\underline{\nabla}\phi$ gravity acceleration

  $\underline{\nabla}\cdot\underline{u} = 0$ or $\int_S (\underline{u}\cdot\underline{n})\, dS = 0$ for $\rho = \text{const.}$

## §2.3 Applications of momentum integral to inviscid flows

1. Pressure force on a curved section of pipe




Assumption: given $p$, steady flow $U$, incompressible, inviscid flow, velocity parallel to pipe

- Find force on curved section of pipe.

Take the whole bend as our volume $V$.

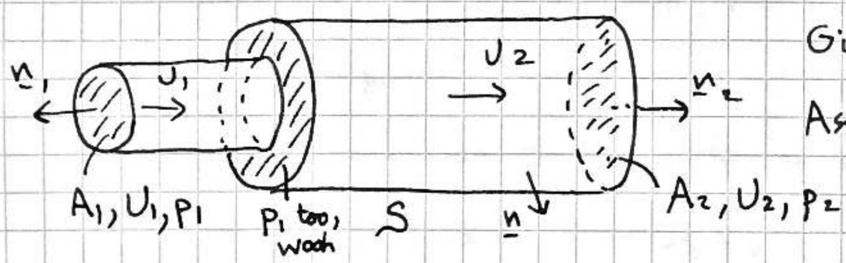
$$\underbrace{\frac{d}{dt}\int_V \rho\underline{u}\, dV}_{\text{zero}} = \underbrace{\int_V \rho\underline{g}\, dV}_{\text{weight of fluid}} - \int_{S\cup S_1 \cup S_2} p\underline{n}\, dS - \int_{S\cup S_1\cup S_2} \rho\underline{u}(\underline{u}\cdot\underline{n})\, dS$$

$$\underline{F} = \int_S p\underline{n}\, dS = \text{weight} - \int_{S_1\cup S_2} p\underline{n}\, dS - \int_{S_1\cup S_2}\rho\underline{u}(\underline{u}\cdot\underline{n})\, dS$$

$$= \text{weight} - Ap(\underline{n}_1 + \underline{n}_2) - (-\rho U A)(-U\underline{n}_1) - (\rho U A)U\underline{n}_2$$

$$= \text{weight} - A(p + \underbrace{\rho U^2}_{\text{beeg}})(\underline{n}_1 + \underline{n}_2)$$

2. Pressure change at an abrupt change in pipe diameter




Given $U_1, A_1, A_2$, find pressure $\Delta$

Assumptions : steady flow

$$0 = \hat{\underline{x}}\cdot \text{RHS} = A_1(-U_1)(-U_1)\rho + p_1 A_1 + p_1(A_2 - A_1)$$
$$- A_2(U_2)(U_2)\rho \cancel{+} A_2 p_2$$

- $\therefore A_1\rho U_1^2 + A_2 p_{\cancel{2}1} = A_2\rho U_2^2 + A_2 p_2$

by conservation of mass, $p_1 - p_2 = \rho U_1^2\left(\frac{A_1^2}{A_2^2} - \frac{A_1}{A_2}\right) < 0$ if $A_2 > A_1$

L4.2

## § 2.4 Bernoulli's streamline theorem for steady flows with potential forces

Assumptions: steady $\frac{\partial \underline{u}}{\partial t} = \underline{0}$, $\underline{F} = -\underline{\nabla}\phi$, $\rho = \text{const}$, inviscid

Vector identity $(\underline{u}\cdot\underline{\nabla})\underline{u} = (\underline{\nabla}\times\underline{u})\times\underline{u} + \nabla(\frac{1}{2}|\underline{u}|^2)$

Note $\underline{\omega} = \underline{\nabla}\times\underline{u}$ vorticity

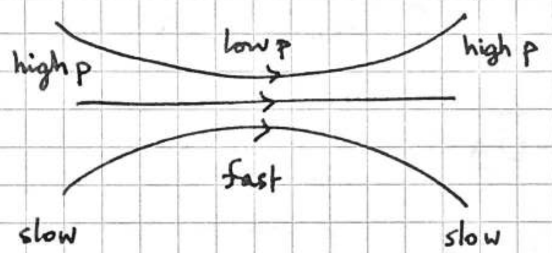
Euler equation $$\underbrace{\frac{\partial \underline{u}}{\partial t}}_{\text{zero}} + \underbrace{(\underline{u}\cdot\underline{\nabla})\underline{u}}_{\text{vector identity}} = -\frac{1}{\rho}\underline{\nabla}p + \underbrace{\underline{F}}_{-\underline{\nabla}\phi}$$

$$\underline{\omega}\times\underline{u} + \underline{\nabla}\underbrace{\left(\frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \phi\right)}_{H} = \underline{0}$$

Dotting with $\underline{u}, \underline{\omega}$ gives $\boxed{\underline{u}\cdot\underline{\nabla}H = 0}$, $\underline{\omega}\cdot\underline{\nabla}H = 0$

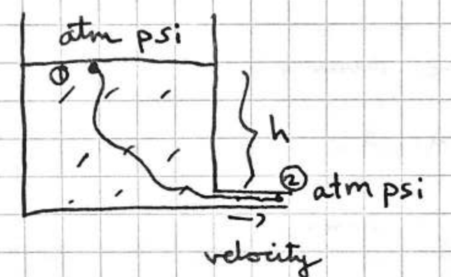
$\Rightarrow$ H constant along streamlines (*Bernoulli's thm*)

Also, H constant along vortex lines




## § 2.5 Applications of Bernoullis thm

1. Flow of water from a tank through a small hole on side




① & ② on same streamline

$U_1 \approx 0$, $p_1 = 1$, $\phi = gh$

$U_2 = ?$, $p_2 = 1$, $\phi = 0$ (lowkey BS)

$\therefore \frac{1}{2}U_2^2 = gh$

$\therefore U_2 = \sqrt{2gh}$

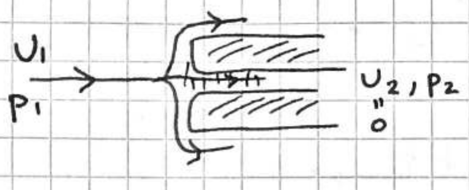
L5.1 Still Ch 2 Dynamics

Recall Euler eq$^n$ $\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \underline{F}$

Assumptions steady, incompressible, inviscid flow, potential forces $\underline{F} = -\underline{\nabla}\varphi$.

Bernoulli's thm : $H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \varphi$ constant along streamlines

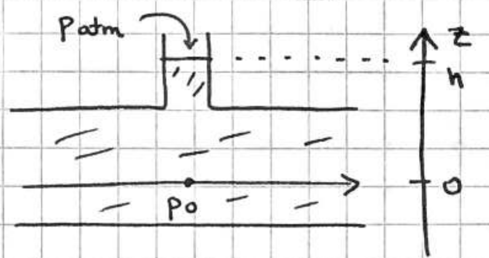
Pitot tube to measure air velocity




$$\frac{1}{2}U_1^2 + \frac{p_1}{\rho} = \frac{1}{2}U_2^2 + \frac{p_2}{\rho}$$ ~stagnate

$$\therefore U_1 = \sqrt{2(p_2 - p_1)/\rho}$$

Hydrostatic pressure



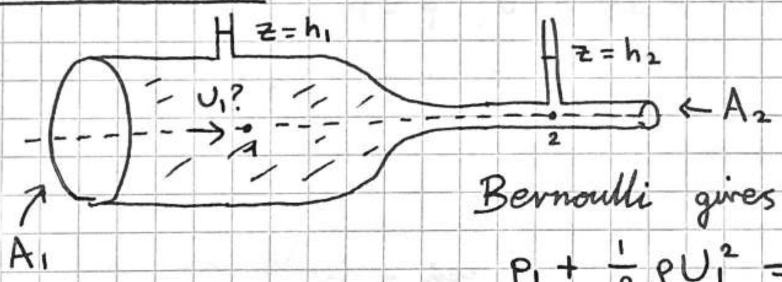

$\underline{u} = \underline{0}$ when fluid at rest ($u_z = 0$ ✓)

z-component of Euler eq$^n$

$$0 = -\frac{1}{\rho}\frac{dp}{dz} - g$$

Integrate between 0, h to get $p_0 = p_{atm} + \rho g h$

③ Venturi meter for measuring flow velocity




Given $A_1, A_2, g, h_1, h_2$

Find $U_1$.

Bernoulli gives

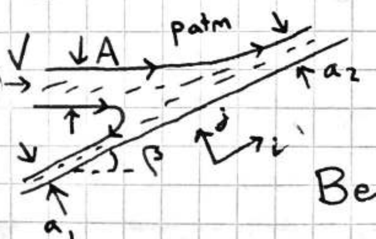
$$p_1 + \frac{1}{2}\rho U_1^2 = p_2 + \frac{1}{2}\rho U_2^2$$

Conservation of mass $A_1 U_1 = A_2 U_2$

$$\therefore p_1 - p_2 = \frac{1}{2}\rho U_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)$$

$$\therefore \rho g (h_1 - h_2) = \frac{1}{2}\rho U_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)$$

④ 2D water jet incident to oblique plate




Fast jet ⇒ neglect gravity

Find $a_1, a_2$ and force on plate

Bernoulli ⇒ on free sfc streamline, constant V

Mass conservation $\Rightarrow$ $VA = Va_1 + Va_2$ i.e. $A = a_1 + a_2$

Momentum

$$\hat{\imath} \cdot \int_S \left( \rho \underline{u} (\underline{u} \cdot \underline{n}) + p \underline{n} \right) dS = 0$$

$$\rho a V^2 \cos\beta = \rho a_2 V^2 - \rho a_1 V^2 \qquad \text{(only pressure at plate)}$$

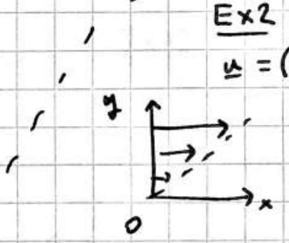
$$\Rightarrow a_2 = \tfrac{1}{2}(1+\cos\beta)\, a, \quad a_1 = \tfrac{1}{2}(1-\cos\beta)\, a$$

Now in $\hat{\jmath}$ direction,

$$\hat{\jmath} \cdot \int_S \left( \rho \underline{u} (\underline{u} \cdot \underline{n}) + \underbrace{p \underline{n}}_{\text{force}} \right) dS = 0$$

Ex2 Shear flow

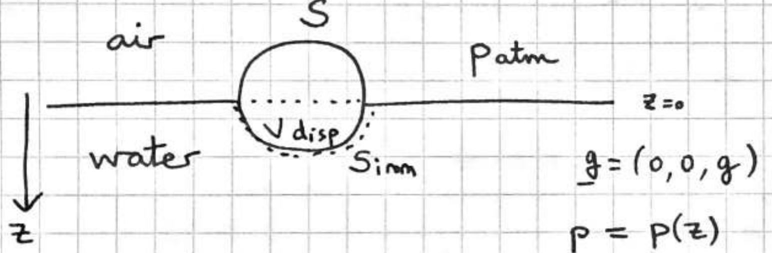
$\underline{u} = (\alpha y, 0, 0)$




$\underline{\omega} = (0, 0, -\alpha)$

$$\text{Force} = \rho a V^2 \sin\beta$$

## 5) Archimedes' principle




$\underline{g} = (0, 0, g)$

$p = p(z)$

$$F_{body} = -\int_S p \underline{n}\, dS = -\int_S (p - p_{atm}) \underline{n}\, dS$$

If $z > 0$, $p = p_{atm} + \rho g z$

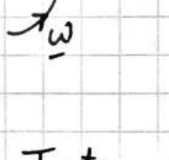
If $z < 0$, $p = p_{atm}$

$$F_{body} = -\int_{S_{imm}} \rho g z \underline{n}\, dS$$

$$= -\int_{V_{disp}} \rho g \hat{z}\, dV = -\cancel{\rho} \underline{g} \text{ (mass of displaced water)}$$

## §2.6 Vorticity

Def$^n$ $\underline{\omega} = \underline{\nabla} \times \underline{u}$ A vortex line at some time $t$ is a curve with tangent vector $\underline{\omega}$ at each point.

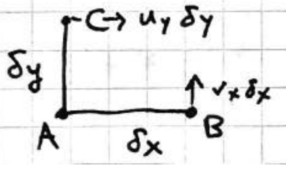



Ex ① Rigid body rotation $\underline{u} = \underline{\Omega} \times \underline{x}$

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = (2\Omega_1, 2\Omega_2, 2\Omega_3) = 2\underline{\Omega} \quad (\ddot{o})$$

Interpretation of vorticity in 2D flows $\underline{u} = (u, v, 0)$

$$\underline{\omega} = (0, 0, v_x - u_y)$$




$v(x+\delta x, y, t) - v(x, y, t) \approx v_x \delta x$

$\frac{1}{2}(\omega) = \frac{1}{2}(v_x - u_y)$ = avg angular velocity

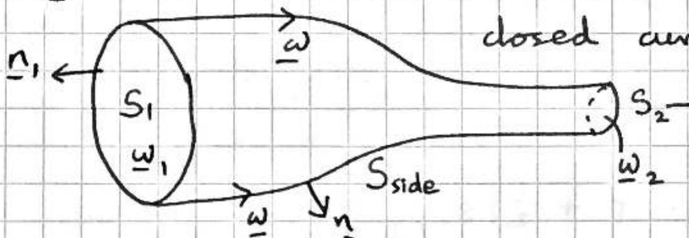
Ex ③ <u>Line vortex</u> $\underline{u} = \frac{K}{r}\underline{e}_\theta$ in cylindrical polars $(r, \theta, z)$

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u_r & ru_\theta & u_z \end{vmatrix} = 0$$


except at the origin, where neither $\underline{u}$ or $\underline{\omega}$ is defined.

| velocity | vorticity |
|---|---|
| streamlines $\int_u$ | vortex lines $\int_\omega$ |
| $\underline{\nabla}\cdot\underline{u} = 0$ | $\underline{\nabla}\cdot\underline{\omega} = 0$ |
| $\int_S \underline{u}\cdot\underline{n}\, dS = 0$ | $\int_S \underline{\omega}\cdot\underline{n}\, dS = 0$ |
| conservation of mass | conservation of vorticity |
| | conservation of ang momentum |

Vortex tube is formed from vortex lines passing through a simple closed curve.




$$\int \underline{\omega}\cdot\underline{n}\, dS = 0$$

$$\therefore \int_{S_1} \underline{\omega}\cdot\underline{n}\, dS = -\int_{S_2} \underline{\omega}\cdot\underline{n}\, dS$$

$$|\omega_1|\, S_1 = |\omega_2|\, S_2$$

"stretching amplifies vorticity" ballerina effect

<u>Recall</u> 1) $(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\omega}\times\underline{u} + \underline{\nabla}\left(\frac{1}{2}|\underline{u}|^2\right)$

2) $\underline{\nabla}\times(\underline{\omega}\times\underline{u}) = \underline{\omega}\underbrace{(\underline{\nabla}\cdot\underline{u})}_{zero} - (\underline{\omega}\cdot\underline{\nabla})\underline{u} - \underline{u}\underbrace{(\underline{\nabla}\cdot\underline{\omega})}_{zero} + (\underline{u}\cdot\underline{\nabla})\underline{\omega}$

3) $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{u}) = 0$

4) $\underline{\nabla}\times\underline{\nabla}\varphi = \underline{0}$

Take curl of Euler equation:

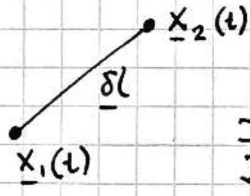
$$\underline{\nabla}\times\left[\frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = \underline{\nabla}\times\left[-\underline{\nabla}\left(\frac{p}{\rho}\right) - \underline{\nabla}\varphi\right]$$

$$\therefore \frac{\partial\underline{\omega}}{\partial t} + \underline{\nabla}\times(\underline{\omega}\times\underline{u}) = \underline{0}$$

$$\therefore \frac{\partial\underline{\omega}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{\omega} = (\underline{\omega}\cdot\underline{\nabla})\underline{u} = \frac{D\underline{\omega}}{Dt}$$

Vorticity equation for incompressible, inviscid with potential forces

<u>Line element</u>




$$\underline{\delta l} = \underline{X}_2 - \underline{X}_1$$

$$\frac{D\underline{\delta l}}{Dt} = \frac{d\underline{X}_2(t)}{dt} - \frac{d\underline{X}_1(t)}{dt} = \underline{u}(\underline{x}_1 + \underline{\delta l}) - \underline{u}(\underline{x}_1)$$

$$\simeq (\underline{\delta l}\cdot\underline{\nabla})\underline{u}\Big|_{\underline{x}_1}$$

L 6.2

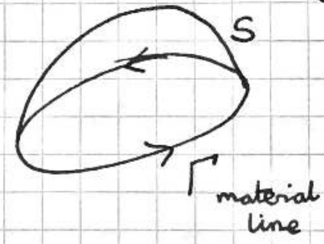
In components $\dfrac{D\delta l_i}{Dt} = \delta l_j \dfrac{\partial u_i}{\partial x_j}$

Line elements (vortex lines too) are stretched and rotated by flow.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{symmetric} \\ \sim \text{pure strain}}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{antisymmetric} \\ \sim \frac{1}{2}\varepsilon_{ijk}\omega_k}}$$

## §2.7 Kelvin circulation theorem

Circulation $C = \oint_\Gamma \underline{u}\cdot d\underline{l} = \int_S \underline{\omega}\cdot\underline{n}\, dS$




$$\frac{dC}{dt} = \oint_\Gamma \frac{D\underline{u}}{Dt}\cdot\underline{\delta l} + \underline{u}\cdot\frac{D\underline{\delta l}}{Dt}$$

$$= \oint_\Gamma \left[-\frac{\underline{\nabla} p}{\rho} - \underline{\nabla}\varphi\right]\cdot\underline{\delta l} + \underline{u}\cdot(\underline{\delta l}\cdot\underline{\nabla})\underline{u}$$

$$= \oint_\Gamma d\underline{l}\cdot\underline{\nabla}\left[-\frac{p}{\rho} - \varphi + \tfrac{1}{2}|\underline{u}|^2\right] = 0$$

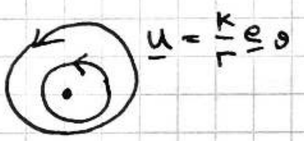
for $\Gamma$ a closed material curve. Assume inviscid, incompressible, potential forces

Define Irrotational flow $\Leftrightarrow$ $\underline{\omega} = \underline{0}$ everywhere.

Irrotational flow stays irrotational.

Proof Kelvin, $C = \int_S \underline{\omega}\cdot\underline{n}\, dS$

## §2.8 The circulation of line vortex




Choose $\Gamma$ = circle radius $R$

$$C = \oint_\Gamma \underline{u}\cdot d\underline{l} = \int_0^{2\pi} \frac{k}{\cancel{r}}\cancel{r}\, d\theta = 2\pi k$$

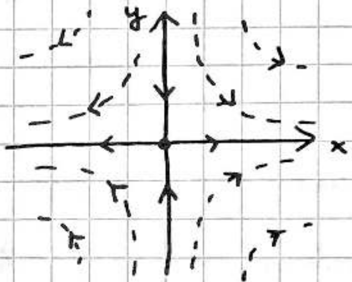
$\underline{u} = \dfrac{\kappa}{2\pi r}\underline{e}_\theta$, $\underline{e}_\theta = -\sin\theta\,\hat{\underline{i}} + \cos\theta\,\hat{\underline{j}}$, $\varkappa$

$$\underline{u} = \frac{\kappa}{2\pi}\left[-\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}\right]$$

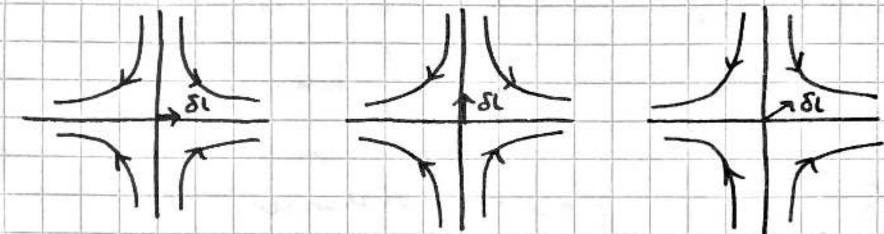
## §2.9 Applications of vorticity equation and material line element eq$^n$

<u>Ex</u> $\underline{u} = (\alpha x, -\alpha y, 0)$, $\alpha > 0$ $\qquad \oint_{\text{Renai}} \underline{u}\cdot d\underline{l}$

$\psi = \alpha x y$



Three line elements



$\underline{\omega} = \underline{0}$

$$\frac{D\delta\underline{l}}{Dt} = (\delta\underline{l}\cdot\underline{\nabla})\,\underline{u}$$

1) $\delta\underline{l} = (\delta l, 0, 0) \Rightarrow \text{RHS} = \delta l \cdot \alpha\, \underline{e}_x$

so $\delta l$ grows exponentially

3) $\delta\underline{l} = (\delta l_1, \delta l_2) \Rightarrow \text{RHS} = (\alpha\delta l_1, -\alpha\delta l_2)$ $\quad \ddot{\circ}$

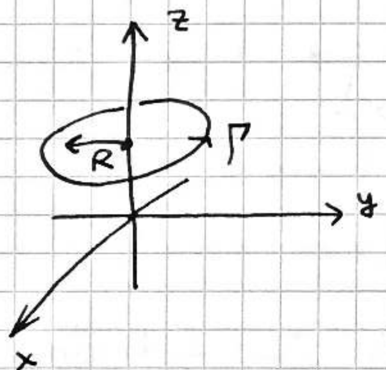
so $\delta\underline{l} \propto (\delta l_1 e^{\alpha t}, \delta l_2 e^{-\alpha t})$ stretches and rotates

<u>Ex 2</u> Pure stretching of vortex lines in axissymmetric flow

$\underline{u}_{\text{strain}} = (-\beta x, -\beta y, 2\beta z)$, $\beta > 0$

$\underline{u} = \underline{u}_{\text{strain}} + \underline{u}_{\text{vort}}$

$\underline{u}_{\text{vort}} = \Omega(t)\,(-y, x, 0)$




$\underline{\omega} = (0, 0, 2\Omega(t))$

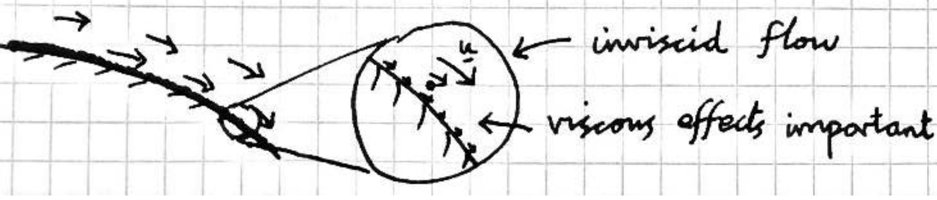
Circulation of $\underline{u}$ around $x^2+y^2 = R^2$

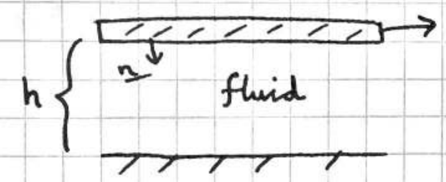
$$C = \oint_{\Gamma} \underline{u}\cdot d\underline{l} = \int_S \underline{\omega}\cdot\underline{n}\, dS = \pi R^2 \cdot 2\Omega(t)$$

As $\Omega(t)$ changes, so will $R$ of material $\Gamma$, by Kelvin Circulation theorem.

## §3 Elementary Viscous Flow

Viscous effects are important in a <u>thin layer</u> of boundary next to the fluid where $\underline{u} = \underline{0}$.






apply force $\tau$ to move at velocity $U$

$$\tau = \mu \frac{U}{h}$$

viscosity

seems reasonable, confirmed by experiment

This suggests that the tangential shear stress exerted <u>by fluid</u> on a boundary surface with $\underline{n}$ pointing <u>into</u> the ~~boundary~~ fluid is

$$\tau = \mu \frac{\partial \underline{u}_{tang}}{\partial \underline{n}} = \mu \underline{n} \cdot \underline{\nabla} \underline{u}_{tang}$$

$\mu$ – absolute viscosity
dynamic

$\nu = \frac{\mu}{\rho}$ kinematic viscosity

$$[\tau] = [Pa] = \left[\frac{kg}{m s^2}\right]$$
$$[\mu] = [Pa \cdot s]$$
$$[\nu] = [m^s s^{-1}]$$

$\mu_{air} = 0.01 \mu_{water}$

$\nu_{air} = 10 \nu_{water}$

really makes you think

## §3.1 Navier-Stokes equations for incompressible fluid

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho} \underline{\nabla} p + \underbrace{\nu \nabla^2 \underline{u}}_{\text{NEW}} + \underline{F} \quad , \quad \underline{\nabla} \cdot \underline{u} = 0$$

+ BCs $\underline{u} = 0$ on boundary

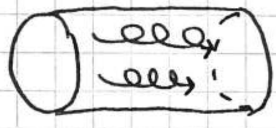
Inviscid flow

vortices generated by viscosity!

Viscous flow

almost the same

## § 3.2 <u>Reynolds number</u>



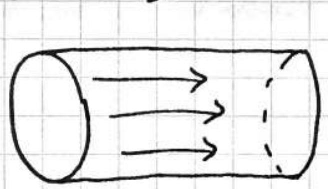
Turbulent flow
Re > 4000

$U$ – typical flow sweep

$L$ – characteristic length

$\nu$ – kinematic viscosity

$$Re = \frac{UL}{\nu}$$ is dimensionless &



Laminar flow
Re < 2000

L 8.1

Recall Navier - Stokes eq$^n$, $\nu = \mu/\rho$ - kinematic viscosity.

Reynold's number $Re = UL/\nu$.

What is the relative importance of the terms in N-S eq$^n$.

Ignore $\underline{F}$. $\underline{x} = L\underset{\uparrow}{\underline{x}'}$, $\underline{u} = U\underset{\uparrow}{\underline{u}'}$, $p = P\underset{\uparrow}{p'}$, $t = \frac{L}{U}\underset{\uparrow}{t'}$

dim$^{less}$

N-S becomes

$$\frac{V^2}{L}\frac{\partial \underline{u}'}{\partial t'} + \frac{U^2}{L}(\underline{u}'\cdot\underline{\nabla}')\underline{u}' = -\frac{P}{\rho L}\underline{\nabla}'p' + \frac{\nu U}{L^2}\underline{\nabla}'^2\underline{u}'$$

$$\therefore \frac{\partial \underline{u}'}{\partial t'} + \underbrace{(\underline{u}'\cdot\underline{\nabla}')\underline{u}'}_{\text{inertia term}} = -\frac{P}{\rho U^2}\underline{\nabla}'p' + \underbrace{\frac{1}{Re}\underline{\nabla}^{2'}\underline{u}'}_{\text{viscous term}}$$

$$\begin{aligned}|(\underline{u}\cdot\underline{\nabla})\underline{u}| &= O\left(\frac{U^2}{L}\right) \\ |\nu\nabla^2\underline{u}| &= O\left(\frac{\nu U}{L^2}\right)\end{aligned} \Rightarrow \frac{|\text{inertia term}|}{|\text{viscous term}|} = O(Re)$$

So Re gives a rough indication of the magnitude of key terms in N-S.

## § 3.2.1 High Re flows

Insignificant viscosity effects.

Ex flow over aerofoil at small angle of attack

Later on: the thickness of the boundary layer $\delta$ goes as $O\left(\frac{L}{\sqrt{Re}}\right)$

## § 3.2.2 Low Re flows

Note $\frac{\partial \underline{u}}{\partial t}$ scales as the inertia term.

This, and the inertia term, are negligible.

something something ?

NS eq becomes Stokes equation $0 = -\underline{\nabla}p + \mu\nabla^2\underline{u} + \underline{F}$, $\underline{\nabla}\cdot\underline{u} = 0$

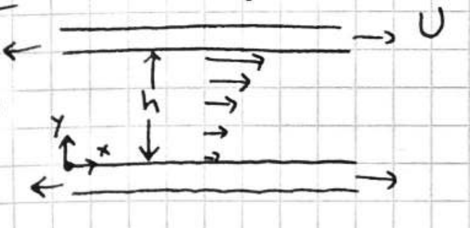
## § 3.3 Some simple viscous flows

### § 3.3.1 Plane parallel shear flow

Usually $\underline{u} = (u(y,t), 0, 0)$. $\underline{\nabla}\cdot\underline{u}$ satisfied

<u>Ex</u> Couette flow

Assume steady flow.




x-component of NS-eq$^n$

$$\underbrace{\cancel{\frac{\partial \underline{u}}{\partial t}}}_{\text{zero}} + \underbrace{(\underline{u}\cdot\cancel{\underline{\nabla}})\underline{u}}_{\text{zero}} = -\frac{1}{\rho}\overbrace{\underline{\nabla}p}^{\text{no x-comp}} + \nu\nabla^2\underline{u} + \underline{F}$$

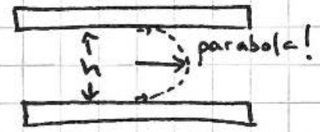
so $u''(y) = 0$, and given bc.s

$$u = \frac{U}{h}y.$$

L8.2

## Ex 2 Poiseuille flow

Steady flow driven by constant pressure gradient




$x$-comp is $\quad 0 = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial y^2}$

$y$-comp is $\quad 0 = -\dfrac{\partial p}{\partial y} - \rho g$

$p = -\rho g \cancel{x}\, y + f(x)$ and $\mu u''(y) = f'(x) \equiv -G$ (sep$^n$ argument)

Solution becomes $u = \dfrac{G}{2\mu} y (h-y)$ ¨

### § 3.3.2 Properties

1. Volume flux $q$ – volume of fluid passing a cross-section / time

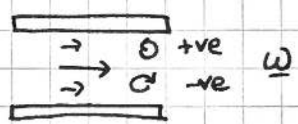
$$q = \int_0^{\cancel{*}h} u \, dy = \frac{1}{2} U h \quad \text{for Couette flow}$$

$$q = \frac{Gh^3}{12\mu} \quad \text{for Poiseuille flow}$$

2. Vorticity $\underline{\omega} = \underline{\nabla} \times \underline{u}$

$$\underline{\omega} = \left(0, 0, -\frac{U}{h}\right) \quad \text{for Couette flow}$$

$$\underline{\omega} = \left(0, 0, \frac{G}{\mu}\left(y - \frac{h}{2}\right)\right) \quad \text{for Poiseuille flow}$$



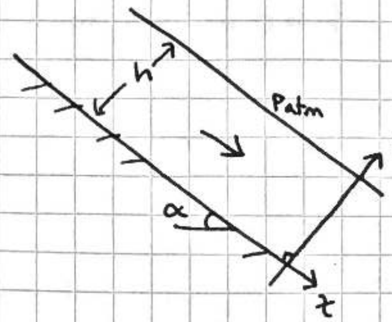

3. Surface stress $\tau$ exerted by fluid

$$\tau = \mu \frac{\partial u_{tang}}{\partial n} \quad \text{so} \quad \tau|_{y=0} = \mu \frac{U}{h} \quad \text{for Couette flow}$$

$$\tau|_{y=h} = -\mu \frac{U}{h}$$

and $\tau = \dfrac{Gh}{2}$ at both boundaries for Poiseuille flow

## §3.3.3 Steady flow under gravity down an inclined plane




$\downarrow g$ assume $\underline{u} = (u(y), v(y), 0)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow v \text{ is const}$$

bc $\Rightarrow$ $v$ identically $0$

Take $x, y$ components of Navier–Stokes

$$x: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu u''(y) + g \sin\alpha \qquad (*)$$

$$y: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g\cos\alpha \times (-1)$$

$$\therefore p = -\rho g y \cos\alpha + f(x)$$

now, for $y = h$, $p = p_{atm}$

$$\therefore p = p_{atm} + \rho g (h - y)\cos\alpha$$

Substitute into $(*)$: $\nu u''(y) = -g \sin\alpha$

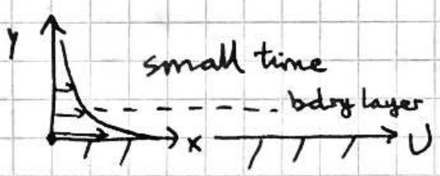
$u(0) = 0$, $u'(h) = 0$ ← since free surface is a streamline? NO WAT something about shear stress

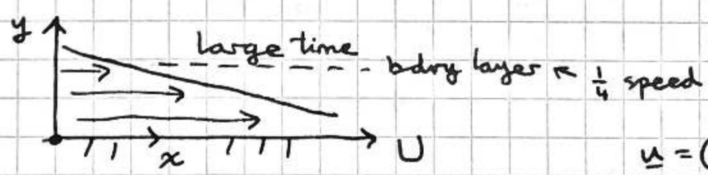
$$\therefore u = \frac{g}{2\nu} y(2h - y) \sin\alpha \quad \text{parabolic velocity profile}$$

The volume flux down the plane per unit length in $z$-dir$^n$

$$q = \int_0^h u\, dy = \frac{gh^3}{3\nu}\sin\alpha \qquad \nu_{air} \approx 15 \nu_{water}$$

## §3.3.4 The flow due to an impulsively moved boundary




$\underline{u} = (u(y,t), 0, 0)$

at time $t=0$, plane set in motion, constant velocity $U$

$x$-component of NS is

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \quad \text{heat eq}^n$$

IC: $u(y,0) = 0$ for $y \geq 0$

BC: $u(0,t) = U$, $u(\infty, t) = 0$ for $t > 0$

Find similarity sol$^n$: eq$^n$ unchanged under $y \rightsquigarrow \alpha y$, $t \rightsquigarrow \alpha^2 t$

So $\exists$ sol$^n$ depending only on $\frac{y}{\sqrt{t}}$, might as well $\eta = \frac{y}{\sqrt{\nu t}}$ dim$^{less}$

Substitute $u(y) = U f(\eta)$ to obtain $f'' + \frac{\eta}{2} f' = 0$

$f(0) = 1$, $f \to 0$ as $\eta \to \infty$

$\cancel{f = A + Be^{\frac{1}{2}\eta}}$ LOL

$$f' = Be^{-\eta^2/4}$$
$$f = A + B\int_0^\eta e^{-s^2/4}\,ds$$
$$\therefore\ u = U\left(1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-s^2/4}\,ds\right)$$

Comments

1) $\nu_{air} \approx 15\,\nu_{water}$

disturbance propagates faster in air

2) tangential stress exerted by fluid on $y=0$ is $\tau = \mu \frac{\partial u}{\partial n}$
$$= -\mu \frac{U}{\sqrt{\pi \nu t}}$$

## § 3.4 Stagnation-point flow

Ignore gravity (fast jet)

$\exists$ solution of NS such that

$$\underline{u} \longrightarrow (Ex, -Ey, 0)$$

as $y \to \infty$

Note $\underline{u} = (Ex, -Ey, 0)$ satisfies $\nabla \cdot \underline{u} = 0$

$$\psi = Exy$$

But does not satisfy $\underline{u} = 0$ at $y = 0$

Try $\psi = Exf(y)$ with

$$\underline{u} = (Exf'(y), -Ef(y), 0) \quad ⊛$$

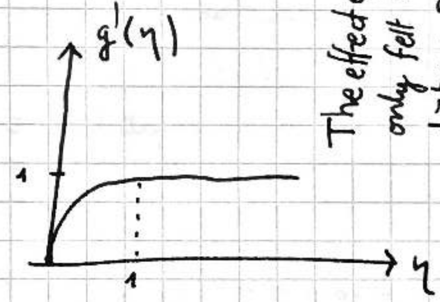
into NS equation

$$(\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$

$x$-comp: $E^2 x(f'^2 - ff'') = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu E x f'''$ (a)

$y$-comp: $E^2 f' f = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu E f''$ (b)

The effect of no-slip only felt within a distance $\delta \sim O\left(\sqrt{\frac{\nu}{E}}\right) \sim O\left(\frac{1}{\sqrt{Re}}\right)$



Diff (b) wrt $x$ $\Rightarrow$ $p_{xy} = 0$

Diff (a) wrt $y$ $\Rightarrow$ $(f'^2 - ff'')' = \frac{\nu}{E} f^{(4)}$

no slip $\underline{u} = 0$ on $y = 0$ $\Rightarrow$ $f(0) = f'(0) = 0$

also $f(y) \to y$ as $y \to \infty$, so $f' \to 1$ as $y \to \infty$

$\frac{\nu}{E} f''' = f'^2 - ff'' - 1$ ⚠ wow not obvious but ok

Rescale $f = \left(\frac{\nu}{E}\right)^{1/2} g(\eta)$, $\eta = \sqrt{\frac{E}{\nu}}\, y$

$g''' - g'^2 = -gg'' - 1$ with $g(0) = 0$, $g'(0) = 0$, $g' \to 1$ as $\eta \to \infty$

L10.1 High af

## §3.5 Vorticity equations in viscous flow

$$\underline{\omega} = \nabla \times \underline{u} \quad , \quad \frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \varphi + \nu \nabla^2 \underline{u}$$

Take curl of NS, use $(\underline{u}\cdot\nabla)\underline{u} = \nabla\left(\frac{1}{2}u^2\right) + \underline{\omega}\times\underline{u}$

$$\therefore \frac{\partial \underline{\omega}}{\partial t} + \nabla \times (\underline{\omega}\times\underline{u}) = \nu \nabla^2 \underline{\omega}$$

Now, $\nabla \times (\underline{\omega}\times\underline{u}) = \underline{\omega}(\nabla\cdot\underline{u}) - (\underline{\omega}\cdot\nabla)\underline{u} - \underline{u}(\nabla\cdot\underline{\omega}) + (\underline{u}\cdot\nabla)\underline{\omega}$

$\nabla\cdot\underline{u} = 0$ and div $\circ$ curl $= 0$

$$\therefore \frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}$$

## §4 Inviscid Flow now irrotational

A velocity field $\underline{u}(\underline{x},t)$ can be written $\underline{u} = \nabla\varphi(\underline{x},t)$ iff $\underline{u}$ is irrotational, i.e. $\underline{\omega} = \nabla\times\underline{u} = \underline{0}$ cf Vector Calculus.

This works so long as well work in simply connected domains.

Indeed, define $\phi(\underline{x},t) = \int_{\underline{x}_0}^{\underline{x}} \underline{u}(\underline{x}',t)\cdot d\underline{l}'$ for a fixed $\underline{x}_0$ and some path joining $\underline{x}_0$ and $\underline{x}$.

This is well defined by Stokes.

In non-simply connected domains the def$^n$ for $\phi(\underline{x},t)$ may be multivalued, which is cool haha.

This occurs when there is circulation round hole.

$\oint \underline{u}\cdot d\underline{l} \neq 0$

For irrotational flow to be a good approximation need in

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}$$

last term to be negligible, go figure

Some basic properties

$\nabla\cdot\underline{u} = 0 \Rightarrow \nabla^2\phi = 0$ so that $\phi$ is harmonic ⍤

Boundary condition for impermeable boundaries,

$$\underline{U}\cdot\underline{n} = \underline{u}\cdot\underline{n} = \underline{n}\cdot\nabla\phi = \frac{\partial\phi}{\partial n}$$

i.e. a Neumann boundary condition

L10.2

## §4.2 A Spherical geometry, axisymmetric flow

General axisymm. solution to Laplace's equation

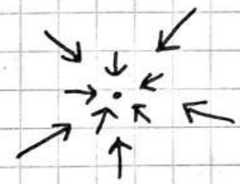

$$\phi = \sum_{n=0}^{\infty} \left\{ A_n r^n + B_n r^{-(n+1)} \right\} P_n(\cos\theta)$$

↑ Legender poly

$P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$

(i) all $A_n, B_n$ zero except for $B_0$

$$\phi = \frac{B_0}{r} \Rightarrow \underline{u} = -\frac{B_0 \underline{e}_r}{r^2} \quad \sim \text{source/sink flow}$$

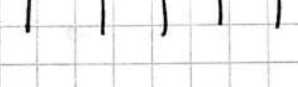


Check that incompressibility, mass conservation satisfied

e.g. flux out of a ball radius $r$ always same

(ii) all $A_n, B_n$ zero except for $A_1$

$$\phi = A_1 r\cos\theta = A_1 z \Rightarrow \underline{u} = A_1 \underline{e}_z \quad \sim \text{uniform}$$

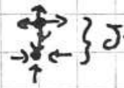


(iii) all $A_n, B_n$ zero except for $B_1$

$$\phi = \frac{B_1 \cos\theta}{r^2} = \frac{B_1 z}{r^3} \Rightarrow \underline{u} = B_1\left(\frac{\underline{e}_z}{r^3} - \frac{3z\underline{e}_r}{r^4}\right)$$

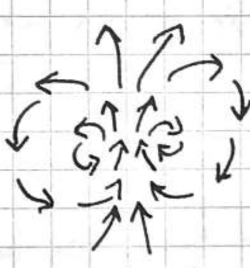
We note this is $-\partial/\partial z$ of the case (i) above

Limiting case of a superposition:




$$\phi = \frac{B_1 z}{r^3} \text{ is supperposition } \frac{-B_1/\delta z}{\{x^2+y^2+(z+\frac{1}{2}\delta z)^2\}^{1/2}}$$

$$\frac{+B_1/\delta z}{\{x^2+y^2+(z-\frac{1}{2}\delta z)^2\}^{1/2}}$$



«dipole»

L 11.1

## A. Spherical geometry, axisymmetry

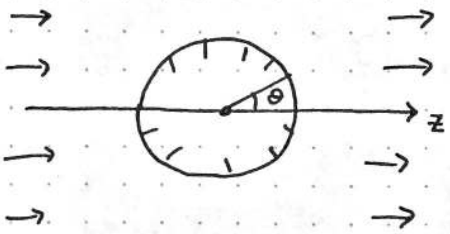
$$\phi = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta).$$

$\phi = \dfrac{B_0}{r}$ source ($B_0 < 0$) or sink ($B_0 > 0$)

$\phi = A_1 r\cos\theta = A_1 z$ uniform flow with velocity $A_1$ in z-dir$^n$

$\phi = \dfrac{B_1 \cos\theta}{r^2}$ dipole (doublet)

(iv) Uniform flow past a sphere of radius a




$\nabla^2\phi = 0$ for $r > a$,

$\partial\phi/\partial r = 0$ at $r = a$,

$\phi \to Ur\cos\theta$ at $r \to \infty$ (unif flow)

Try $\phi = \left(A_1 r + \dfrac{B_1}{r^2}\right)\cos\theta$.

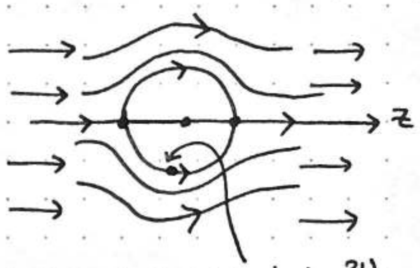
Now, $\phi \to Ur\cos\theta$, Let $A_1 = U$.

$\partial\phi/\partial r = (A_1 - 2B_1 r^{-3})\cos\theta = 0$ at $r = a$

$\therefore B_1 = a^3 U/2$

$\therefore \phi = U(r + a^3/2r^2)\cos\theta$

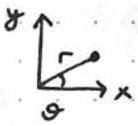
$$\therefore \underline{u} = \underline{\nabla}\phi = \left(\partial\phi/\partial r, \tfrac{1}{r}\partial\phi/\partial\theta, 0\right)$$
$$= \left( U(1 - a^3/r^3)\cos\theta, \; -U(1 + a^3/2r^3)\sin\theta, \; 0 \right)$$




$\max|\underline{u}| = \dfrac{3U}{2}$

## B. Cylindrical geometry, in 2D

$$\phi = A_0 \log r + B_0\theta + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta + \alpha_n) + B_n r^{-n}\cos(n\theta + \beta_n)$$




(i) $A_0 \neq 0$, $\phi = A_0 \log r$

$\underline{u} = \underline{\nabla}\phi = \dfrac{A_0 \underline{e}_r}{r}$



(ii) $B_0 \neq 0$, $\phi = B_0\theta$

$\underline{u} = \dfrac{B_0 \underline{e}_\theta}{r}$

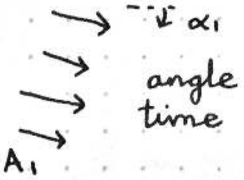

vortex circulation

$\kappa = 2\pi r(B_0/r)$

$\therefore \underline{u} = \dfrac{\kappa}{2\pi r}\underline{e}_\theta$

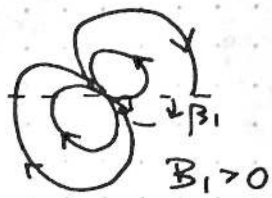
(iii) $A_1 \neq 0$, $\phi = A_1 r \cos(\theta + \alpha_1) = A_1(x\cos\alpha_1 - y\sin\alpha_1)$

$\therefore\ \underline{u} = (A_1\cos\alpha_1, -A_1\sin\alpha_1)$

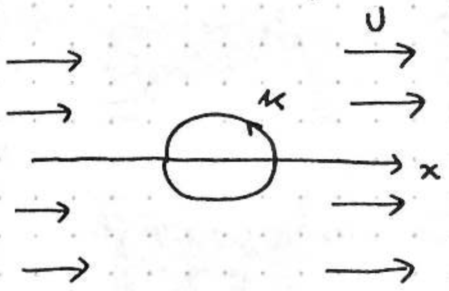



(iv) $B_1 \neq 0$, $\phi = B_1 \dfrac{\cos(\theta+\beta_1)}{r}$

2D dipole pointing in $\theta = -\beta_1$




(v) Uniform flow past a cylinder with circulation!!!




$\nabla^2\phi = 0$ for $r > a$

$\phi \to Ur\cos\theta + \dfrac{\kappa\theta}{2\pi}$ as $r \to \infty$

$\partial\phi/\partial r = 0$ at $r = a$

Solution $\phi = U\cos\theta\left(r + \dfrac{a^2}{r}\right) + \dfrac{\kappa\theta}{2\pi}$ coolio

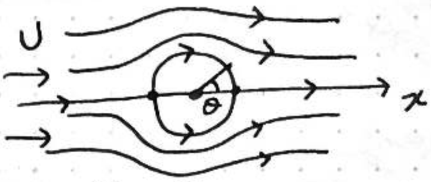
$\underline{u} = \left(\dfrac{\partial\phi}{\partial r}, \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right) = \left(U\cos\theta\left(1 - \dfrac{a^2}{r^2}\right), -U\sin\theta\left(1 + \dfrac{a^2}{r^2}\right) + \dfrac{\kappa}{2\pi r}\right)$

Stagnation points:

(a) $r = a$ and $\sin\theta = \dfrac{\kappa}{4\pi a U}$ (only possible when $\left|\dfrac{\kappa}{4\pi a U}\right| \leq 1$)

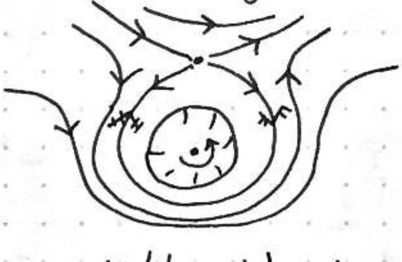
(b) $\cos\theta = 0$ and $r$ s.t. $1 + \dfrac{a^2}{r^2} \mp \dfrac{\kappa}{2\pi U a}\cdot\dfrac{a}{r} = 0$

one real root in $r \geq a$ if $\left|\dfrac{\kappa}{4\pi a U}\right| \geq 1$

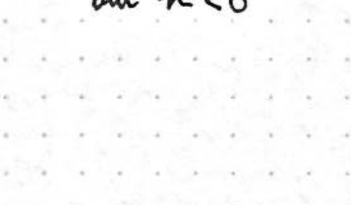



$\kappa = 0$

max velocity at $r = a$, $\theta = \pm\frac{\pi}{2}$, $2U$



$\kappa/4\pi a U > 1$




$0 < |\kappa/4\pi a U| < 1$

but $\kappa < 0$

L12.1

## §4.3 Time-dependent Bernoulli theorem

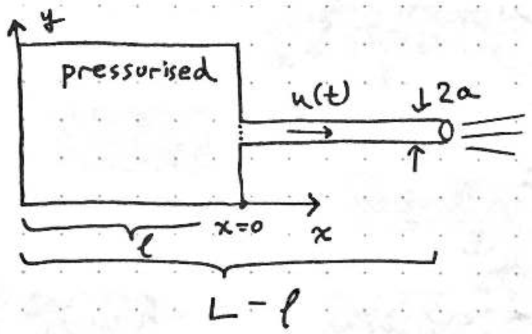
$$\frac{\partial}{\partial t}\underline{\nabla}\varphi + \tfrac{1}{2}\underline{\nabla}|\underline{u}|^2 + \frac{1}{\rho}\underline{\nabla}p + \underline{\nabla}\Phi = 0$$ ← via Euler, $(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\nabla}(\tfrac{1}{2}|\underline{u}|^2)$ for irrotational

$$\Rightarrow \frac{\partial}{\partial t}\varphi + \tfrac{1}{2}|\underline{u}|^2 + \frac{1}{\rho}p + \Phi = \tilde{H}(t) \quad \text{is independent of } \underline{x}$$

## §4.4 Applications of irrotational Bernoulli

### §4.4.1 Fast jet generator, version 1

Neglect gravity, flow starts from rest at $t=0$




What is velocity in tube?

$p|_{x=0} = p_{atm} + p_0(t)$, $p_0 = 0$ if $t \le 0$,

$p_0 > 0$ if $t > 0$.

$t=0$ at rest $\Rightarrow$ irrotational at $t=0$

$\Rightarrow$ irrotational at $t>0$

$\Rightarrow \underline{u} = \underline{\nabla}\varphi$

$$\varphi = u(t)x + \chi(t)$$
$$\dot{\varphi} = \dot{u}(t)x + \dot{\chi}(t)$$

Apply Bernoulli, $\dot{u}x + \dot{\chi} + \tfrac{1}{2}u^2 + \dfrac{p}{\rho} = \text{const. in } x$

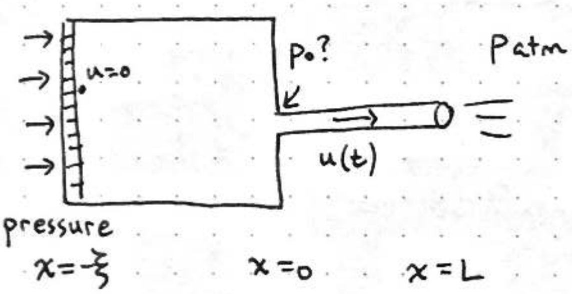
Compare at $x = 0, L$

$$\cancel{\dot{\chi}} + \cancel{\tfrac{1}{2}u^2} + \frac{\cancel{p_{atm}}+p_0}{\rho} = \dot{u}L + \cancel{\dot{\chi}} + \cancel{\tfrac{1}{2}u^2} + \cancel{\frac{p_{atm}}{\rho}}$$

$$\therefore \dot{u} = \frac{p_0}{\rho L} \quad \text{cool}, \quad u = \frac{p_0 t}{\rho L}$$ since $u(0) = 0$ if $p_0$ constant

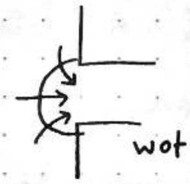
$$u = \frac{1}{\rho L}\int_0^t p_0(t')\,dt'$$

### §4.4.2 Fast jet generator, version 2




$$\varphi = u(t)x + \cancel{\chi(t)}$$ ← small

inside tube (*)




Apply Bernoulli for $x = L, -\xi$

$$\dot{u}L + \tfrac{1}{2}u^2 + \cancel{\frac{p_{atm}}{\rho}} = 0 + \frac{\cancel{p_{atm}}+p_0}{\rho}$$

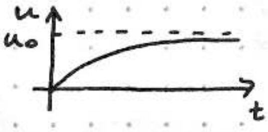
$$\therefore \dot{u}L + \tfrac{1}{2}u^2 = \frac{p_0(t)}{\rho} \quad (**)$$ non-linear

L12.2

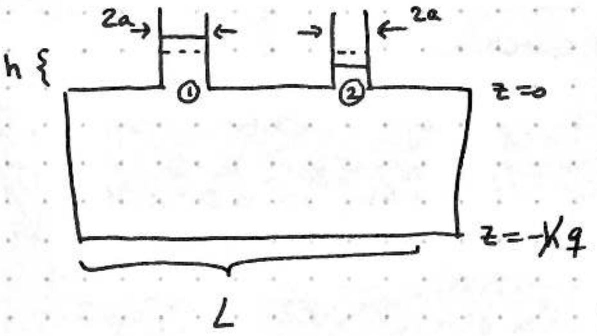
Denote $u_0 = \sqrt{\frac{2p_0}{\rho}}$, $T = \frac{u_0}{2L} t$, $U = \frac{u(t)}{u_0}$

$$\frac{dU}{dT} = 1 - U^2, \quad U(0) = 0 \quad \Rightarrow \quad U = \tanh T$$

$$u = u_0 \tanh\left(\frac{u_0}{2L} t\right)$$




## §4.43 Inviscid manometer oscillations MANOLOMETER




$L \gg a$, $\cancel{l}\, q \gg a$, $h \gg a$

goes to $h+\zeta_1$, $h + \cancel{\zeta_2}\,\zeta_2 = h - \zeta_1$

Tube ①, $\underline{u} = (0, 0, \dot\zeta_1)$

Tube ②, $\underline{u} = (0, 0, -\dot\zeta_{\cancel{2}1})$

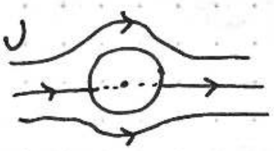
① $\varphi = \dot\zeta_1 z$, ② $\varphi = -\dot\zeta_{\cancel{2}1} z$

$\tilde{H} = \frac{\partial\varphi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + gz$ has the same values at $z = h \pm \zeta_1$

$$\left(\ddot\zeta_1 z + \tfrac{1}{2}\dot\zeta_1^2 + p_{atm}/\rho + gz\right)\Big|_{z=h+\zeta_1} = \left(-\ddot\zeta_1 z + \tfrac{1}{2}\dot\zeta_1^2 + \frac{p_{atm}}{\rho} + gz\right)\Big|_{\text{else}}$$

$\therefore\ \ddot\zeta_1 h + g\zeta_1 = 0$ period $2\pi\sqrt{\frac{h}{g}} \sim \cancel{25}\ 1$ s for $h = 25$ cm

## §4.5 Translating sphere




$$\varphi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$

$$\underline{u} = \underline{\nabla}\varphi = \left(U\cos\theta\left(1 - \frac{a^3}{r^3}\right), -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), 0\right)$$

Time-dep Bernoulli $\underbrace{\frac{\partial\varphi}{\partial t}}_{\text{zero}} + \frac{1}{2}u^2 + \frac{1}{\rho}p + \underbrace{\Phi}_{\text{zero}} = \tilde{H}(t) = \frac{1}{2}U^2 + \frac{p_\infty}{\rho}$

$$p(r,\theta) = p_\infty - \tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho U^2$$

$$p(a,\theta) = p_\infty - \tfrac{1}{2}\rho u^2\big|_{r=a} + \tfrac{1}{2}\rho U^2$$

$$= p_\infty - \tfrac{1}{2}\rho\,\tfrac{9}{4}U^2\sin^2\theta + \tfrac{1}{2}\rho U^2$$

$\underline{u} = (0, -U\sin\theta\cdot\frac{3}{2}, 0)$ for $r = a$

$$F_{\parallel} = \left(\int p\,\underline{n}\,dS\right)_{\parallel} = \int_0^\pi p(a,\theta)\,2\pi a^2 \sin\theta\,d\theta = 0$$

($\cos\theta$)

Similarly, $F_\perp = 0 \Rightarrow$ zero drag on the sphere

~ D'Alembert's paradox

L13.1

Recall $\phi$, $\underline{u}$ for flow past cylinder with circulation.

At $r=a$, $|\underline{u}| = \left| -2U\sin\theta + \dfrac{\kappa}{2\pi a} \right|$.

Apply Bernoulli eq$^n$, compare at $\infty$,

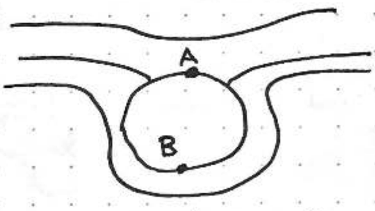
$$\frac{1}{2}\rho\left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)^2 + p(a,\theta) = \frac{1}{2}\rho U^2 + p_\infty$$

$$\therefore \underline{F} = -\int_{\text{cylinder}} p\,\underline{n}\,dS = -\int_0^{2\pi}\left\{\underbrace{\tfrac{1}{2}\rho U^2 + p_\infty}_{\text{integral zero}} - \tfrac{1}{2}\rho\left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)^2\right\} \times (\cos\theta, \sin\theta)\, a\, d\theta$$

$$= \frac{1}{2}\rho a \int_0^{2\pi}\left(4U^2\sin^2\theta - \frac{2\kappa U\sin\theta}{\pi a}\right)(\cos\theta, \sin\theta)\,d\theta$$

$$= \frac{1}{2}\rho a \cdot \left(-\frac{2\kappa U}{\pi a}\right)\int_0^{2\pi}(0, \sin^2\theta)\,d\theta$$

$$= (0, -\rho\kappa U) \quad \text{per unit length}$$

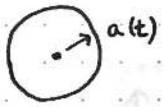



$v_A < v_B \Rightarrow p_A > p_B$

"Planes fly lol"

§4.7 Bubbles and cavities

Consider a spherical bubble of radius $a(t)$.




Bubble oscillates at rate $\dot{a}(t)$.

$\phi \propto \frac{1}{r}$ – source or sink

Boundary moves with $\underline{U} = \dot{a}(t)\underline{e}_r$.

BC $\underline{U}\cdot\underline{n} = \underline{u}\cdot\underline{n} \Rightarrow \phi = -\dfrac{a^2\dot{a}}{r}$, $\underline{u} = \dfrac{a^2\dot{a}}{r^2}\underline{e}_r$

$\tilde{H} = \dfrac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{1}{\rho}p$, neglect gravity (small times)

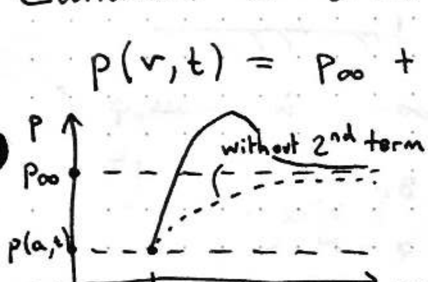
$$\frac{\partial}{\partial t}\left(-\frac{a^2\dot{a}}{r}\right) + \frac{a^4\dot{a}^2}{2r^4} + \frac{p(r,t)}{\rho} = \frac{p_\infty}{\rho}$$

$$\frac{a^2\ddot{a}}{r} + \left(\frac{4a}{r} - \frac{a^4}{r^4}\right)\frac{\dot{a}^2}{2} = \frac{1}{\rho}\left(p(r,t) - p_\infty\right) \qquad (\cdot)$$

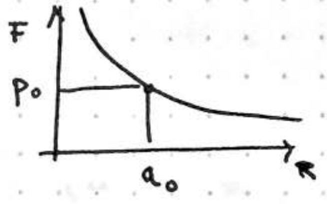
as $r \to a+$, $a\ddot{a} + \frac{3}{2}\dot{a}^2 = \frac{1}{\rho}\left(p(a,t) - p_\infty\right) \qquad (\cdot\cdot)$

Eliminate $\ddot{a}$ from $(\cdot)$ using $(\cdot\cdot)$

$$p(r,t) = p_\infty + \left[p(a,t) - p_\infty\right]\frac{a}{r} + \frac{1}{2}\rho\dot{a}^2\left(\frac{a}{r} - \frac{a^4}{r^4}\right)$$



We know $p(a,t) = F(a)$.




$$F(a) = p_0 \left(\frac{a_0}{a}\right)^{3\gamma} \qquad \gamma \sim \frac{7}{5} \text{ c.f. Thermodynamics}$$

<u>Frequency of small oscillations</u>

$$a = a_0 + \delta a, \quad \delta a \ll a_0$$

$$p_\infty = \text{const.}, \text{ assume}$$

In equilibrium, $p(a) = p_0 = p_\infty$.

Pressure oscillations are $\delta p$ at $r = a$.

$$\delta p = p(a,t) - p_0 \approx -\frac{3\gamma p_0}{a_0}\delta a \quad \text{by linearising}$$

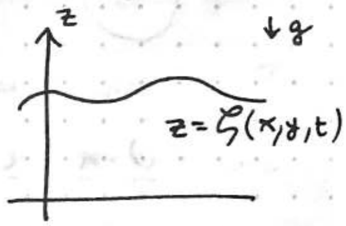
Linearise eqⁿ $(\cdot\cdot)$

$$a_0 \delta\ddot{a} = \delta p/\rho = -\frac{3\gamma p_0}{\rho a_0}\delta a$$

$\therefore$ Frequency of oscillations is $\omega = \sqrt{\dfrac{3\gamma p_0}{\rho a_0^2}}$

## § 5 Waves and geophysical flows

### § 5.1 Free surface waves




$\Phi = gz$, $p = p_{atm}$ on surface, $\underline{u} = \underline{\nabla}\phi$, $\nabla^2\phi = 0$

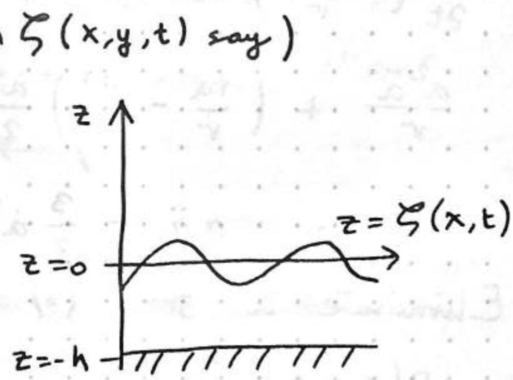
In Ch1 §1.6, kinematic BC

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} - w = 0 \quad \text{on } z = \zeta(x,y,t)$$

Pressure field is via time-dependent Bernoulli

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{1}{\rho}\underset{\substack{\uparrow \\ p_{atm}}}{p} + g\underset{\substack{\uparrow \\ \zeta}}{z} = \tilde{H}(t) \quad (\text{on } \zeta(x,y,t) \text{ say})$$

### § 5.2 Small amplitude water waves




Full problem: $\nabla^2\phi = 0$ in $-h \le z \le \zeta(x,t)$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = \zeta(x,t)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + g\zeta = f(t) \quad \text{on } z = \zeta$$

Linearise in small $\phi, \zeta$, kill $\frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}$, $\frac{1}{2}|\underline{\nabla}\phi|^2$, BCs $z = \zeta$ become $z = 0$

INORIN WILL NEVER LECTURE FLUIDS ☹

Seek solutions

$$\phi = \mathrm{Re}\left( \hat{\phi}(z)\, e^{ikx - i\sigma t} \right)$$
$$\zeta = \mathrm{Re}\left( \hat{\zeta}\, e^{ikx - i\sigma t} \right)$$

Now, $\dfrac{\partial \phi}{\partial t} + g\zeta = f(t) \Rightarrow f(t) \equiv 0.$

$$\left.\begin{array}{llll} \nabla^2\phi = 0 & \Rightarrow & \dfrac{d^2\hat{\phi}}{dz^2} - k^2\hat{\phi} = 0 & \text{in } -h \le z \le 0 \\ \dfrac{\partial \phi}{\partial z} = 0 & \Rightarrow & \dfrac{\partial \hat{\phi}}{\partial z} = 0 & \text{on } z = -h \end{array}\right\} \Rightarrow \hat{\phi} = A\cosh k(z+h)$$

$$\frac{\partial \phi}{\partial z} = \cdots \Rightarrow i\sigma\hat{\zeta} + \frac{d\hat{\phi}}{dz} = 0 \quad \text{on } z = 0$$

Bernoulli $\Rightarrow -i\sigma\hat{\phi} + g\hat{\zeta} = 0$ on $z = 0$

Hence $i\sigma\hat{\zeta} + kA\sinh kh = 0,$

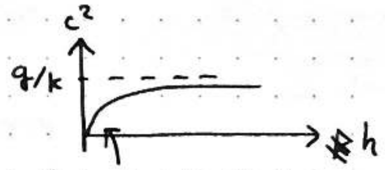
$$g\hat{\zeta} - i\sigma A\cosh kh = 0.$$

Non-trivial sol$^n$ in $\hat{\zeta}$, $A$ iff determinant zero, i.e.

$$\boxed{\sigma^2 = gk\tanh kh}$$ dispersion relation


$\lambda = \dfrac{2\pi}{k}$ crests travel with phase velocity

$$c = \frac{\sigma}{k} \Rightarrow c^2 = \frac{g}{k}\tanh kh$$




$c^2 \approx gh - \frac{1}{3}gk^2h^3$

<u>Special cases</u>

1) Deep water case, $\lambda \ll h \Rightarrow \sigma^2 = gk$

ocean $h \approx 5\,\text{km}$, period $\sim 15$ sec

$$\sigma = \frac{2\pi}{T} \sim 0.4\,\text{s}^{-1}, \quad k^{-1} = g/\sigma^2 \simeq 63\,\text{m}$$
$$\lambda = \frac{2\pi}{k} \sim 400\,\text{m}, \quad c = 25\,\text{ms}^{-1}$$

2) Shallow water, $kh \ll 1 \Rightarrow \sigma^2 = gk^2h$, $c^2 = gh$

Moreover, $c$ maximal for smaller $k$ ☺

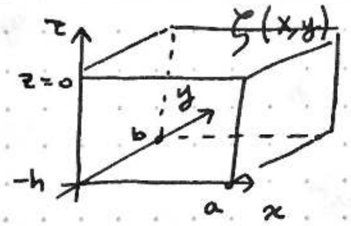
<u>Ex</u> flood in river, $h = 2\,\text{m}$, $c \simeq 4.5\,\text{ms}^{-1}$

Tidal tsunamis in shallow seas $h \approx 50\,\text{m}$, $c \approx 22\,\text{ms}^{-1}$

Transpacific tsunami $h = 5\,\text{km}$, $c \approx 224\,\text{ms}^{-1}$

L14.2

## § 5.3 Free surface modes in a container




Linearise eq$^n$s,

$$\nabla^2\phi = 0 \text{ for } 0 \le x \le a,\ 0 \le y \le b,\ -h \le z \le 0$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ at } z=-h \qquad \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} = 0 \text{ on relevant bdry}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} \text{ at } z=0$$

$$\frac{\partial \phi}{\partial t} + g\zeta = 0 \text{ on } z=0 \quad (!)$$

Seek sol$^n$ $\phi = \mathrm{Re}(\hat{\phi}(x,y,z)e^{-i\sigma t})$

$$\zeta = \mathrm{Re}(\hat{\zeta}(x,y)e^{-i\sigma t})$$

$$i\sigma\hat{\zeta} + \left.\frac{\partial \hat{\phi}}{\partial z}\right|_{z=0} = 0, \quad -i\sigma\hat{\phi}\big|_{z=0} + g\hat{\zeta} = 0$$

Eliminate $\hat{\zeta} \Rightarrow \left.\frac{\partial \hat{\phi}}{\partial z}\right|_{z=0} = \frac{\sigma^2}{g}\hat{\phi}\big|_{z=0}$

BC on $x,y \Rightarrow$ modes $\hat{\phi}_{mn}(z)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

This solves $\nabla^2\phi = 0$ if

$$\frac{d^2\hat{\phi}_{mn}}{dz^2} - K_{mn}^2\hat{\phi}_{mn} = 0 \quad \text{where } K_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

BC in z so $\frac{d\hat{\phi}_{mn}}{dz} = 0$ on $z=-h$

$$\frac{d\hat{\phi}_{mn}}{dz} = \frac{\sigma^2}{g}\hat{\phi}_{mn}(0) \text{ on } z=0 \qquad (*)$$

$\therefore \hat{\phi}_{mn} \propto \cosh(K_{mn}(z+h))$

$(*) \Rightarrow \sigma_{mn}^2 = gK_{mn}\tanh(K_{mn}h)$ dispersion relation yay

## § 5.4 Rayleigh-Taylor instability

Free surface is below, $\sigma_{mn}^2 = -gK_{mn}\tanh(K_{mn}h)$

$\therefore \sigma_{mn}$ imaginary, exponential growth

L14.3

## §5.5 Fluid dynamics in a rotating frame

- Fluid on a spinning Earth.

Work in rotating frame of reference.

$$\text{D\&R} \Rightarrow \frac{D\underline{u}}{Dt} \rightarrow \frac{D\underline{u}}{Dt} + \underbrace{2\underline{\Omega} \times \underline{u}}_{\text{Coriolis}} + \underbrace{\underline{\Omega} \times (\underline{\Omega} \times \underline{x})}_{\text{Centrifugal}}$$

Euler equation becomes

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} + 2\underline{\Omega}\times\underline{u}\right) = -\underline{\nabla} p - \rho\underline{\Omega}\times(\underline{\Omega}\times\underline{x}) + \rho\underline{g}$$

For Earth, $\Omega = 2\pi \times 10^{-5}\,s^{-1}$, largest $|\underline{x}| \approx 10^{7}$m.

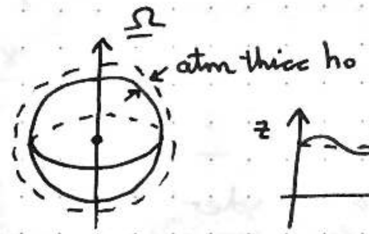
So neglect centrifugal term

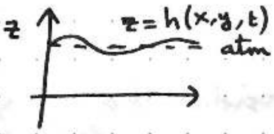
- Consider motion for which $|\underbrace{(\underline{u}\cdot\underline{\nabla})\underline{u}}_{\underline{\omega}\times\underline{u} + \frac{1}{2}\underline{\nabla}(u^2)}| \ll |\underline{\Omega}\times\underline{u}|$

i.e. $|\underline{\omega}| \ll |\underline{\Omega}|$ (ok?)

For atmosphere, $u \approx 10\,ms^{-1}$, $L \approx 10^{6}$m, $\frac{|\omega|}{|\Omega|} \approx \frac{1}{2\pi}$ neglect LOL

L 15.1




Use local cartesian coordinates,

$\hat{x}$ = East, $\hat{y}$ = North = $-\underline{e}_\theta$, $\hat{z}$ = UP

$\underline{\Omega} = (0,\ \Omega\cos\varphi,\ \Omega\sin\varphi)$  latitude

$\underline{u} = (u, v, w)$

$\underline{\Omega} \times \underline{u} = (\Omega w\cos\varphi - \Omega v\sin\varphi,\ \Omega u \sin\varphi,\ -\Omega u\cos\varphi)$

$w \ll u, v$   drop   drop, gravity dominates

$2\underline{\Omega}\times\underline{u} = (-2\Omega v\sin\varphi,\ 2\Omega u\sin\varphi, 0)$

$= 2\Omega\sin\varphi\,(-v, u, 0)$

$= f\hat{z}\times\underline{u}$ where $f = 2\Omega\sin\varphi$ - Coriolis parameter

$f > 0$ in Northern hemisphere, $f < 0$ in Southern hemisphere

In components, the Euler equation becomes

$$\rho\frac{\partial u}{\partial t} - \rho f v = -\frac{\partial p}{\partial x}$$
$$\rho\frac{\partial v}{\partial t} + \rho f u = -\frac{\partial p}{\partial y}$$
$$0 = -\frac{\partial p}{\partial z} - \rho g \implies p = p_0 + \rho g\left[h(x,y) - z\right]$$

neglect $\partial w/\partial t$

Plug back in to get

$$\rho\frac{\partial u}{\partial t} - \rho f v = -\rho g\frac{\partial h}{\partial x}$$
$$\rho\frac{\partial v}{\partial t} + \rho f u = -\rho g\frac{\partial h}{\partial y} \qquad (**)$$

Independent of $z$.

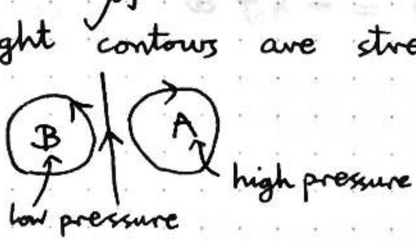
Steady state (centrifugal balance)

$$u = \frac{\partial}{\partial y}\left(-\frac{gh}{f}\right), \quad v = -\frac{\partial}{\partial x}\left(-\frac{gh}{f}\right)$$
$$= \frac{\partial}{\partial y}\left(-\frac{p}{\rho f}\right) \qquad = -\frac{\partial}{\partial x}\left(-\frac{p}{\rho f}\right)$$

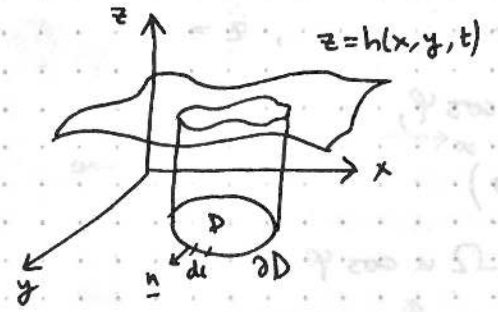
Streamfunction $\psi = -\frac{gh}{f} = -\frac{p}{\rho f}$

Therefore, pressure / height contours are streamlines

In Northern hemisphere



Mass conservation   Consider a cylinder




$\underline{u}_H = (u, v)$

change of mass inside the cylinder

$$\frac{d}{dt}\int_D \rho h \, dS = \int_{\partial D} -h\rho \underline{u}_H \cdot \underline{n} \, dl$$

$$\therefore \frac{\partial h}{\partial t} + \underline{\nabla}_H (h \underline{u}_H) = 0 \qquad \underline{\nabla}_H \cdot \underline{u}_H \neq 0$$

In components, $\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \quad (***)$

Linearised eq$^n$s

$h = h_0 + \eta(x, y, t)$, $\eta \ll h_0$ and $u$, $v$ are small

(**) & (***) drop quadratic terms

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}, \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \qquad (1)$$

$$\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (2)$$

Eliminate $\eta$ from (1):

$$\frac{\partial \zeta}{\partial t} + f(\underline{\nabla}_H \cdot \underline{u}_H) = 0 \quad \text{where } \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \text{ is relative vorticity}$$

From (2), substituting for $\underline{\nabla}_H \cdot \underline{u}_H$,

$$-\cancel{\frac{1}{h_0}} \frac{\partial}{\partial t}\left(\zeta - \frac{\eta}{h_0} f\right) = 0$$

Define "potential vorticity" $\Phi_v = \zeta - \frac{\eta}{h_0} f$ is constant in time

So $\Phi_v = \Phi_r(t=0) = \Phi_0(x, y)$.

Now, do $\frac{\partial}{\partial x}(1a) + \frac{\partial}{\partial y}(1b)$ to obtain

$$\frac{\partial}{\partial t}(\underline{\nabla}_H \cdot \underline{u}_H) - f\zeta = -g\nabla^2\eta$$

$$\therefore -\frac{1}{h_0}\frac{\partial^2 \eta}{\partial t^2} - f\zeta = -g\nabla^2\eta$$

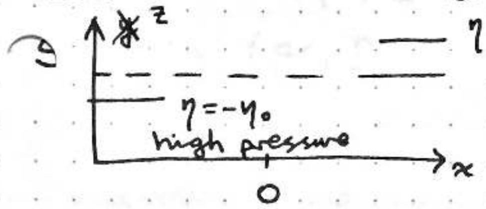
Conservation of potential vorticity yields

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\nabla^2\eta + f^2\eta = -h_0 f \Phi_0 = \text{const.}$$

L15.3



**Ex**




$$\Phi_0 = \zeta - \frac{\eta f}{h_0} = \pm \frac{\eta_0}{h_0} f$$

$$\frac{\partial^2 \eta}{\partial t^2} - g h_0 \nabla^2 \eta + f^2 \eta = \pm f^2 \eta_0 \quad \text{for } x \neq 0$$

Steady flow $\eta = \eta(x)$ has

$$\eta'' - \frac{1}{R^2}\eta = \mp \frac{1}{R^2}\eta_0 \quad \text{where } R = \frac{\sqrt{g h_0}}{f}$$

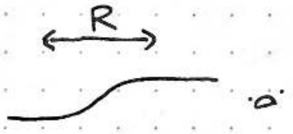
Rossby radius

BC: $\eta, \eta'$ continuous at $x = 0$

$\eta \to \pm \eta_0$ as $x \to \pm\infty$

$\eta = \eta_0 (1 - e^{-x/R})$, $x > 0$

$\eta = -\eta_0 (1 - e^{x/R})$, $x < 0$




| BC inviscid | viscous | free surface |
|---|---|---|
| $\underline{u}\cdot\underline{n} = 0$ | $\underline{u} = 0$ | $D(\zeta - z) = 0$ |

**Mass conservation** $\dfrac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u}) = 0$ $\xrightarrow{\rho \text{ const}}$ $\nabla\cdot\underline{u} = 0$, $\exists \psi$ in 2D $\xrightarrow{\nabla\times\underline{u} = 0}$ $\underline{u} = \underline{\nabla}\phi$, $\nabla^2\phi = 0$

$\boxed{\rho \dfrac{D\underline{u}}{Dt} = -\underline{\nabla} p + \underline{F}}$ Euler, inviscid - - - - - $\boxed{\rho \dfrac{D\underline{u}}{Dt} = -\underline{\nabla} p + \nu \nabla^2 \underline{u} + \underline{F}}$ viscous

$\rho$ = const.
F = conservative
$= -\rho \underline{\nabla} \varphi$

steady → Bernoulli $(\underline{u}\cdot\underline{\nabla})H = 0$

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\underline{\nabla})\underline{u} + \nu \nabla^2 \underline{\omega}$$

vorticity eq$^n$

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\underline{\nabla})\underline{u}$$

Kelvin circulation thm

irrotational: time dep Bernoulli

strong rotation

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \underline{g}$$