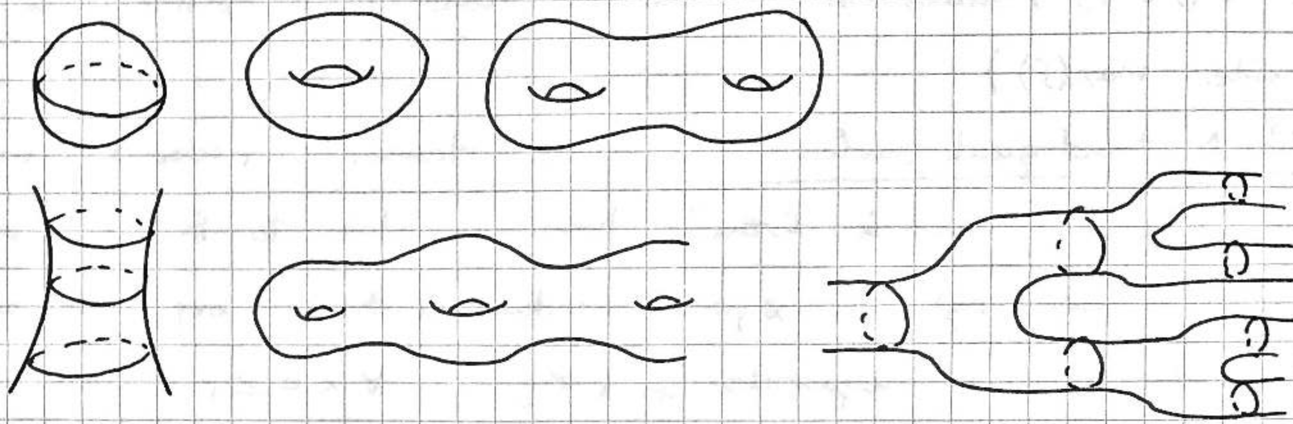
L 1.1

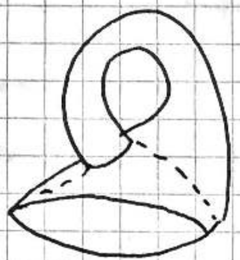
Geometry

Topological Surfaces

A surface is basically what you think it is.



$$Z = \{(0,0)\} \cup \{(\tfrac{1}{n}, 0) : n \in \mathbb{N}\} \subseteq \mathbb{R}^2, \quad \Sigma = \mathbb{R}^2 \setminus Z$$



Klein bottle

Recall: A topological space is a set X equipped with a preferred collection of open subsets, s.t.

(i) $\emptyset$, X are open

(ii) If $\{U_i\}_{i \in I}$ are open, so is $\bigcup_{i \in I} U_i$

If $\{V_j\}_{1 \le j \le n}$ are open, so is $\bigcap_{j=1}^{n} V_j$

Recall if X, Y are spaces, a function $f: X \to Y$ is continuous if $f^{-1}(U)$ is open in X for all U open in Y.

We say X and Y are homeomorphic if $\exists f: X \to Y$ continuous bijection with continuous inverse.

Recall: We say X is Hausdorff if $\forall$ distinct $x, y \in X$, $\exists$ disjoint open $U \ni x$ & $V \ni y$.

Definition: We say spaces X and Y are locally homeomorphic if each $x \in X$ belongs to an open set U (depending on x) s.t. U is homeomorphic to an open subset of Y.

L1.2

<u>Remark</u>: The circle $S^1 \subseteq \mathbb{R}^2$ (Euclidean topology) and the real line $\mathbb{R}$ are locally homeomorphic. But there is no map $f: S^1 \to \mathbb{R}$ which $\forall \theta \in S^1$, takes small open sets near ~~x~~ θ to open sets in $\mathbb{R}$. (Consider $\max(f)$)

<u>Def^n</u> A <u>topological surface</u> Σ is a Hausdorff (second countable) topological space which is locally homeomorphic to $\mathbb{R}^2$, (i.e. $\forall x \in \Sigma$ there is an open ngbd of x, $\mathcal{U}$, which is homeomorphic to an open set in $\mathbb{R}^2$, or equivalently (why ?!) $\forall x \in \Sigma$, there is an open ngbd $\mathcal{U}$ of x homeo^c to an open <u>disc</u> in $\mathbb{R}^2$.)

<u>Remarks</u> (1) $\mathbb{R}^2$ means $\mathbb{R}^2$ with its Euclidean topology

(2) A space X is <u>second countable</u> if it has a countable base for its topology, i.e. $\exists$ a countable collection of open sets $\{\mathcal{U}_i\}_{i \in \mathbb{N}}$ s.t. every open set of X contains one of these [e.g. if $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$, take rational radius open balls centred at points of $\mathbb{Q}^n \cap X$]

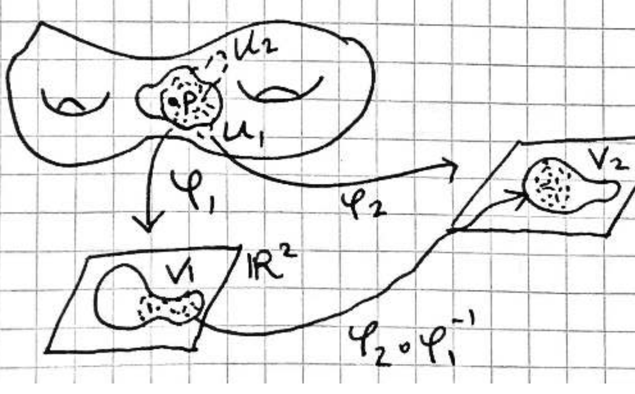
UH OH...

<u>Remark</u> "Recall" subspaces of Hausdorff or second countable spaces inherit these properties.

Let Σ be a topological surface. A pair $(\mathcal{U}, \varphi)$ where $\mathcal{U} \subseteq \Sigma$ is open and $\varphi: \mathcal{U} \to V \underset{\text{open}}{\subseteq} \mathbb{R}^2$ is a homeo$^{\underline{m}}$ is called a <u>chart</u> for Σ. If $p \in \mathcal{U}$, we will talk about a chart for Σ at p.

We usually consider V to be an open disc, e.g. $B(0,1) = \{ z \in \mathbb{C}, |z| < 1 \}$

The inverse $\varphi^{-1}: V \to \mathcal{U} \subseteq \Sigma$ is called a <u>local parametrisation</u> of Σ (V open in $\mathbb{R}^2$) (sometimes called a <u>patch</u> of surface).



Given two charts $(\mathcal{U}_1, \varphi_1)$ & $(\mathcal{U}_2, \varphi_2)$ containing p, we obtain a homeomorphism

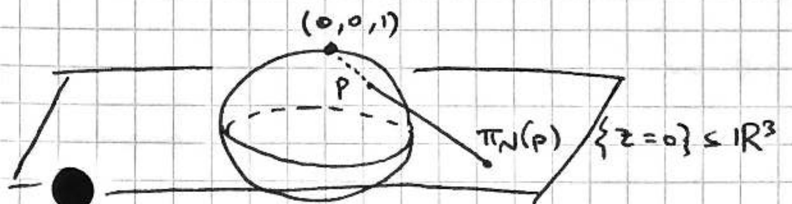
$$\varphi_2 \circ \varphi_1^{-1}: \underset{\displaystyle V_1}{\underset{\cap}{\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)}} \to \underset{\displaystyle V_2}{\underset{\cap}{\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)}}$$

The map $\varphi_2 \circ \varphi_1^{-1}$ is called the <u>transition function</u> for the two charts

<u>Def^n</u> An <u>atlas</u> of charts for Σ is a collection $\{(U_i, \varphi_i)\}$ of charts for which $\bigcup_{i \in I} U_i = \Sigma$. (Such an atlas has associated transition functions)

<u>Example</u> (0) Any open subset of $\mathbb{R}^2$ is a topological surface, with an atlas consisting of a single chart.

(1) Consider $S^2 \subseteq \mathbb{R}^3$, the unit sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$



Stereographic projection from $N=(0,0,1) \in S^2$ by definition sends $p \in S^2 \setminus N$ to $\pi_N(p)$, the intersection of $\{z=0\}$ with the line Np.

If $p=(x,y,z)$, then $\pi_N(p) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \in \mathbb{R}^2 = \{z=0\}$ (since it's of the form $(\alpha, \beta, 0)$ & $(x,y,z) + \lambda(x,y,z-1)$)

This defines a homeo^m from $S^2 \setminus N$ to $\mathbb{R}^2$. ~ check!

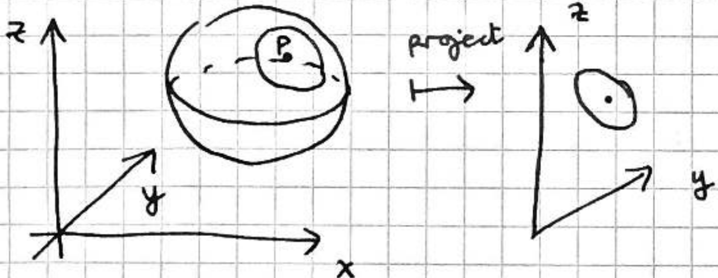
Similarly, stereographic projection from the south pole sends

$$S^2 \setminus (0,0,-1) \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

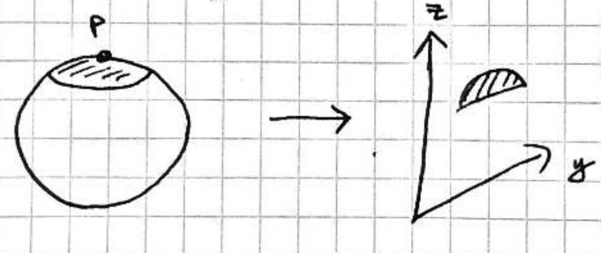
The transition function for the atlas is the map defined on $\mathbb{R}^2 \setminus (0,0)$ (i.e. on the image of $\pi_N|_{S^2 \setminus (0,0,-1)}$).

Ex Sheet 1: The transition map sends $(u,v) \mapsto \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$.

<u>Remark</u> There are many other possible charts!



e.g. projection to a coordinate plane



the projection to the yz plane is <u>not</u> a chart at p: the image of open sets near p are not open in $\mathbb{R}^2$.

L2.1

Examples of Surfaces

<u>Recall</u>: A <u>topological surface</u> is a Hausdorff, second countable space locally homeo^c to $\mathbb{R}^2$; i.e. if Σ is a surface, then $\forall x \in \Sigma$, there's an open $x \in \mathcal{U} \subseteq \Sigma$ s.t. $\mathcal{U} \cong (V \text{ open in } \mathbb{R}^2)$

<u>NB</u> X is locally homeo^c to Y if $\forall x \in X$, $\exists$ open $\mathcal{U} \ni x$ s.t.

$$\mathcal{U} \underset{\text{homeo}^c}{\cong} (\text{open subset of } Y)$$

A nice way to construct surfaces is as <u>identification spaces</u>, a quotient space where you identify sides of a planar polygon in pairs.

<u>Recall</u>: If $X \twoheadrightarrow Y$ is a surjective map of <u>sets</u>, & X is a space, then we define the <u>quotient topology</u> on Y to be the topology s.t.

$$V \subseteq Y \text{ is open} \iff q^{-1}(V) \subseteq X \text{ is open}.$$

This has the feature that if Z is any other space, then $f: Y \to Z$ is continuous iff $f \circ q: X \to Z$ is continuous.

<u>Example</u> The <u>torus</u> T^2 is the quotient (or identification) space

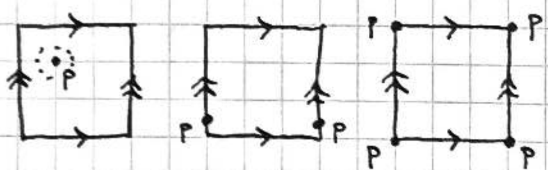
, i.e. we take the space $[0,1] \times [0,1] \subseteq \mathbb{R}^2$, & we impose an equivalence relation via

$(0,t) \sim (1,t) \quad \forall \; 0 \le t \le 1$,

$(s,0) \sim (s,1) \quad \forall \; 0 \le s \le 1$.

Then T^2 is the set of equivalence classes, with the quotient topology from $[0,1]^2$.

<u>CLAIM</u>: T^2 is a topological surface



Pick $\delta > 0$ small s.t. $B(p,\delta) \subseteq (0,1)^2$.

Then $B(p,\delta) \to \underset{\mathbb{R}^2}{\underset{\cap}{B(0,\delta)}}$, $x \mapsto x - p$ is a homeomorphism.

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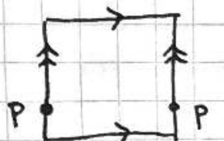


Consider the case $p = (0, y) \sim (1, y) \in T^2 = [0,1]^2/\sim$

An open ngbd of p in T^2 is made up of two small half-discs, i.e.

$$\left.\begin{aligned} U_- &= B((0,y),\delta) \cap [0,1]^2 \\ U_+ &= B((1,y),\delta) \cap [0,1]^2 \end{aligned}\right\} \text{ viewed as subsets of } T^2$$

We define a map $U_- \cup U_+ \longrightarrow B(0,\delta) \underset{\text{open}}{\subseteq} \mathbb{R}^2$

$$(u,v) \longmapsto (u, v-y) \quad \text{if } (u,v) \in U_-$$

$$(u,v) \longmapsto (-1+u, v-y) \quad \text{if } (u,v) \in U_+$$

<u>CLAIM</u>: This is a homeo$^{\underline{m}}$ from $U_- \cup U_+$ to $B(0,\delta) \underset{\text{open}}{\subseteq} \mathbb{R}^2$

One way to check this is to recall

(a) If A is a space, & $A = A' \cup A''$ union of closed subspaces

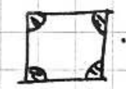
$f' : A' \to Z$ and $f'' : A'' \to Z$ are continuous & f', f'' agree on $A' \cap A''$, then they define a continuous function $f : A \to Z$

[Use: f is cts if $f^{-1}(\text{closed})$ is closed]

(b) If X is compact & Y is Hausdorff, a continuous bijection $f : X \to Y$ is a homeo$^{\underline{m}}$

In our setting, the same maps define $\overline{U_-} \cup \overline{U_+} \longrightarrow \overline{B(0,\delta)}$

(a) & (b) say this map is a homeo$^{\underline{m}}$ from a closed ngbd of $(0,y)$ to $\overline{B(0,\delta)}$. Now restrict to interiors.

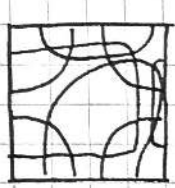
Finally, one can play the same game at the vertex of $[0,1]^2/\sim$, taking a ngbd .

<u>Finally</u>, we should check that T^2 is Hausdorff & second countable. These are both clear in this case, by considering our particular open sets in the first case.

<u>Remark</u>: We've shown T^2 is a topological surface

One can also see the transition functions for an atlas of charts of our shape. These are locally <u>translations</u>.

L2.3



"3 discs",
I. Smith (2020)

An atlas of T^2 with 3 charts.

<u>More examples</u>

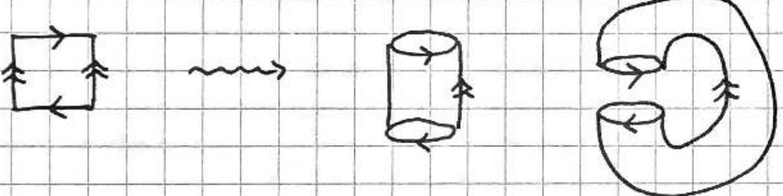
 $\cong S^2$ 

 $\cong$  a cone

$\|?$ flatten

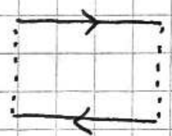
 disc in $\mathbb{R}^2$

<u>Example</u> : Klein bottle, defined to be



glues together abstractly

<u>Def^n</u> A Möbius band is any topological surface homeomorphic to $((0,1) \times [0,1])/\sim$ where $(t, 0) \sim (1-t, 1)$



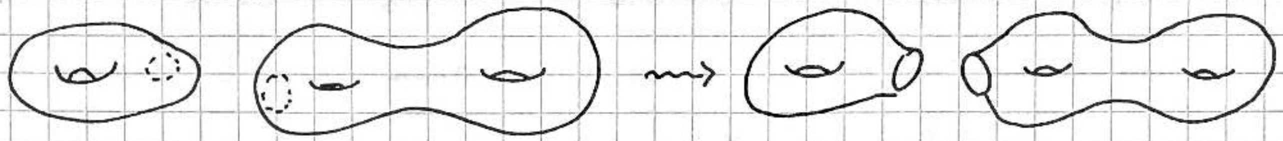
<u>Example</u> The <u>real projective plane</u> is the space 

Exercise: this is the same (as sets) as $S^2/\pm 1$

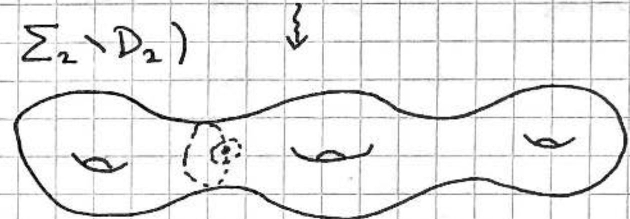
The space of lines

- Recall we were talking about identification spaces.

<u>Def^n</u> If Σ_1 & Σ_2 are topological surfaces, say connected, then a <u>connect sum</u> $\Sigma_1 \# \Sigma_2$ of the Σ_i is the space obtained from removing an open disc D_i from Σ_i & then gluing the boundary circles $\partial D_i = \overline{D_i} \setminus D_i = S^1$ together by a homeo$^{\underline{m}}$ of S^1.

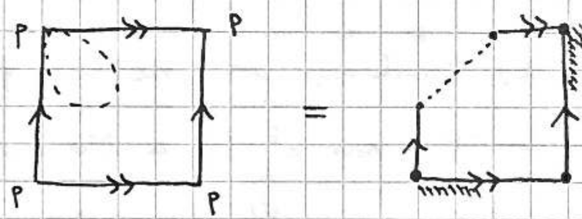


(This is a quotient space of $\Sigma_1 \setminus D_1 \cup \Sigma_2 \setminus D_2$)

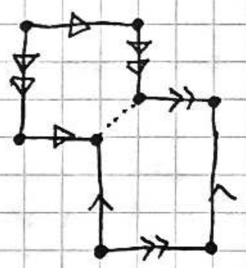


- Then $\Sigma_1 \# \Sigma_2$ is a topological surface, by building charts as for our identification spaces last time.

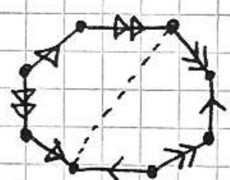
<u>Example</u>



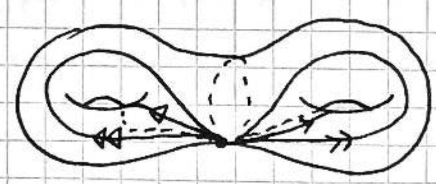
So $T^2 \# T^2$ is the identification space



which is an octagon



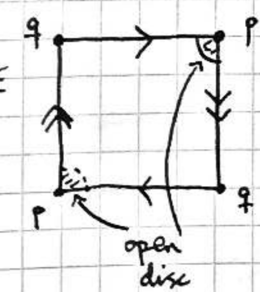
$=$



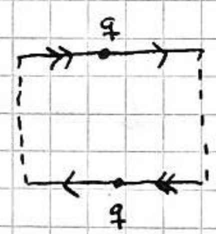
~ sth off maybe
~ no matter

<u>Example</u> If P is the projective plane, then $P \setminus$ (open disc)

$\cong$



$\simeq$

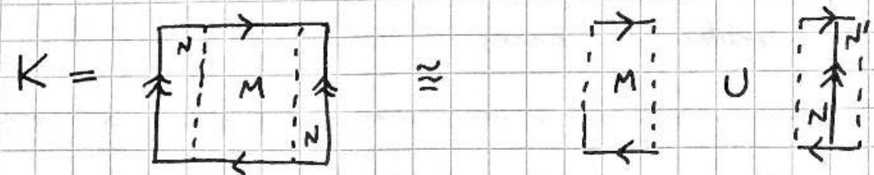


a Möbius band (with its boundary circle)

So $P \# P$ is obtained by gluing two Möbius bands along their boundary.

L3.2

But also the Klein bottle

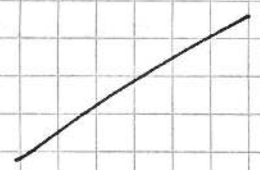


is also obtained by gluing two Möbius bands along their boundary.

So $P \# P \cong K$ are homeo^c.

<u>Remark</u> The operation of $\#$ depends on choices of discs D_i & homeo^m of boundaries, in fact the resulting surface doesn't depend on choices. We will not prove this.

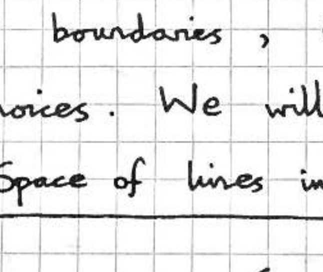
<u>Space of lines in $\mathbb{R}^2$</u>:



We intuitively understand what it means to move a line in $\mathbb{R}^2$ continuously.

<u>Proposition</u>: There is a (natural) topology on the set of straight lines in $\mathbb{R}^2$ with respect to which it's homeo^c to a Möbius band.

<u>Proof</u>: Let $X_h = \{$ non-vertical lines in $\mathbb{R}^2 \}$ (h for horizontal)



We can specify uniquely a point of X_h by giving

(i) a slope $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ of the line to the horizontal

& (ii) the <u>signed</u> orthogonal distance from the line to O, where the distance is positive if the line hits the y-axis at $(0, y)$ with $y > 0$.

Then $(\theta, \ell) : X_h \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$ which is a bijection.

We also have a subset $X_v = \{$ non-horizontal lines in $\mathbb{R}^2 \}$, again specified by an angle to the horizontal & a signed distance to O, where the lign! hits the x-axis at $(\overset{\curvearrowright}{0, x})$ & sign(x) = sign of dist.

We get a new bijection $\varphi_v : X_v \longrightarrow (0, \pi) \times \mathbb{R}$.

If $X_{hv} = X_h \cap X_v$ is the set of lines neither horizontal <u>nor</u>

vertical, then we get a transition map (of sets)

$$\varphi_v \circ \varphi_h^{-1} : \varphi_h(X_{hv}) \longrightarrow \varphi_v(X_{hv})$$

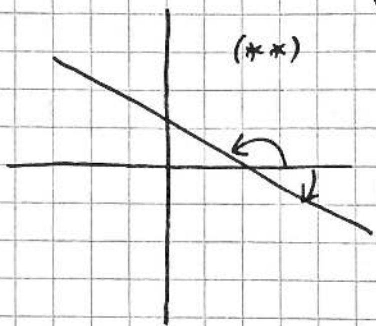
$$\varphi_h(X_{hv}) = \left(-\tfrac{\pi}{2}, 0\right) \times \mathbb{R} \cup \left(0, \tfrac{\pi}{2}\right) \times \mathbb{R}$$

$$\varphi_v(X_{hv}) = \left(0, \tfrac{\pi}{2}\right) \times \mathbb{R} \cup \left(\tfrac{\pi}{2}, \pi\right) \times \mathbb{R}$$

(*) $(\theta, v) \mapsto (\theta, -v)$ ⚠ no!

(**) $(\theta, v) \mapsto (\theta + \pi, v)$

(**)



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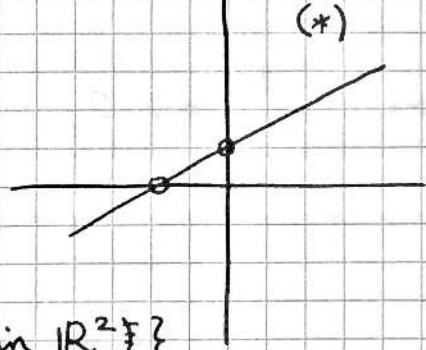


Define a topology on $X = \{\text{set of all lines in } \mathbb{R}^2\}$ by saying that U is open $\iff$ $\varphi_h(U \cap X_h)$ and $\varphi_v(U \cap X_v)$ are both open.

Since the transition map is a homeo$^{\underline{m}}$ of open sets in $\mathbb{R}^2$, X is then locally homeo$^{\underline{c}}$ to $\mathbb{R}^2$.

Indeed, X is the quotient space $\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \mathbb{R}\right) / \sim$, $\left(-\frac{\pi}{2}, v\right) \sim \left(\frac{\pi}{2}, \overset{\text{minus?}}{v}\right)$, a Möbius band! $\square$

Remark (1) The group of rigid motions of $\mathbb{R}^2$ (Euclidean isometries) generated by translations and elements of $O(2)$ acts on X. This action is by continuous maps (& hence homeo$^{\underline{m}}$s). This is what we mean by saying that the topology on X is "natural".

Proposition: There is no metric (in the sense of metric spaces) $d: X \times X \to \mathbb{R}$ which induces the natural topology & is invariant under the group of rigid motions of $\mathbb{R}^2$.

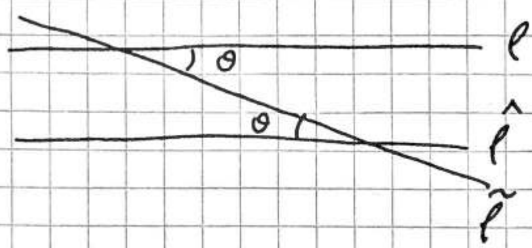
Proof: Given two lines, if such a d existed, & assuming the lines ℓ_1, ℓ_2 are not parallel, we can translate them to meet at O, and so if they meet at an angle θ, then $d(\ell_1, \ell_2) = f(\theta)$ depends only on $\theta \in S^1$.

We also know that $f(\theta)$ is continuous in θ (in order for d to

L 3.4

induce the natural topology) & $f(\theta) \to 0$ as $\theta \to 0$ (as $d(\ell,\ell)=0$).

Let $d(\ell, \hat{\ell}) = d > 0$ for $\ell \neq \hat{\ell}$ parallel.



~~Let~~ Δ-inequality says

$$d \leq d(\ell, \tilde{\ell}) + d(\tilde{\ell} + \hat{d})$$
$$= 2f(\theta)$$
$$\to 0$$

as $\theta(\hat{\ell}) \to 0$. ※

L4.1

Smooth surfaces

The following abstracts the way we put a topology on the space of lines in $\mathbb{R}^2$.

Proposition Let Σ be a set which is given as a countable union

$$\Sigma = \bigcup_{\alpha \in A} \mathcal{U}_\alpha ,$$

where A is countable & for each $\alpha \in A$ we have a bijection

$$\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow V_\alpha ,$$

where $V_\alpha \subseteq \mathbb{R}^2$ is open in $\mathbb{R}^2$.

Declare $\mathcal{U} \subseteq \Sigma$ to be open $\Leftrightarrow$ $\forall \alpha$, $\varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha) \subseteq V_\alpha$ open

Then (a) this does define a topology on Σ, and

(b) Σ is then a topological surface, provided

(i) the "transition maps" $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are homeo^{m}s, $\forall \alpha, \beta \in A$

DOUBT, CONFUSE $\rightarrow$ (ii) $\{ (x,x) : x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \} \subseteq \mathcal{U}_\alpha \times \mathcal{U}_\beta$ is closed $\forall \alpha, \beta \in A$

Sketch of proof

(a) Direct from the definitions, e.g. if $\mathcal{U}_i \subseteq \Sigma$ is open $\forall i \in I$,

$$\varphi_\alpha\Big(\mathcal{U}_\alpha \cap \big(\bigcup_{i \in I} \mathcal{U}_i\big)\Big) = \varphi_\alpha\Big(\bigcup_{i \in I} (\mathcal{U}_\alpha \cap \mathcal{U}_i)\Big) = \bigcup_{i \in I} \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_i)$$

is a union of open sets in $\mathbb{R}^2$, so open.

So arbitrary unions of open sets in Σ are open.

(b) If $p \in \Sigma$, say $p \in \mathcal{U}_\alpha$, then I have $\hat{p} \in V_\alpha$ & I can find an open ball $\hat{B} \ni \hat{p}$ in V_α,
$= \varphi_\alpha(p)$

& consider $B = \varphi_\alpha^{-1}(\hat{B})$, a ngbd of p. (i) says that if $p \in \mathcal{U}_\beta$ & B is small enough to be in $\mathcal{U}_\alpha, \mathcal{U}_\beta$, then $\varphi_\beta(B)$ is open in V_β i.e. (i) $\Rightarrow$ Σ is locally homeo^{c} to $\mathbb{R}^2$

(ii) implies the topology is Hausdorff (!), using from Analysis & Topology that for any topological space Y, Y is Hausdorff iff

L4.2

$\Delta_Y = \{(y,y) : y \in Y\} \subseteq Y \times Y$ is closed.

Finally, countability of A yields second countability of Σ. $\square$

<u>Definition</u> If $U, V \subseteq \mathbb{R}^n$ are open, a countinuous map $f: U \to V$ is <u>smooth</u> if it is infinitely differentiable. Equivalently, f has partial derivatives of all orders. (Also defined for $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$)

A (<u>smooth</u>) <u>diffeo^m</u> $f: U \to V$ is a smooth bijection with smooth inverse.

<u>Definition</u> A <u>smooth structure</u> on a surface Σ (or just a "smooth surface") is a topological surface Σ with an atlas of charts $\{(U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha \underset{\text{open}}{\subseteq} \mathbb{R}^2)\}$ for which the transition functions $\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are smooth <u>diffeo^{m}s</u>

<u>Example</u> (1) $S^2 \subseteq \mathbb{R}^3$ had an atlas $(U_\alpha, \varphi_\alpha)$, (U_s, φ_s) from stereo^c projection from the North & South poles. The transition map is $(u,v) \mapsto \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. This is smooth, as is the inverse (itself).

(2) The space of lines in $\mathbb{R}^2$ had an atlas of charts for which the transition functions were $(\theta, \ell) \mapsto (\theta + \pi, \ell)$ and $(\theta, \ell) \mapsto (\theta, -\ell)$. These are clearly smooth.

<u>Remark</u> (Philosophy, open with care)

Being a topological surface is a <u>property</u>. Being a smooth surface is <u>data</u>.

Observe: it makes no sense to ask if a given topological surface Σ is "locally diffeo^c to $\mathbb{R}^2$"

The choice of smooth structure on Σ enables me/us to make sense of smooth functions on Σ.

We would like an efficient way to recognise smooth surfaces, e.g. s living in $\mathbb{R}^3$.

L4.3

<u>Lemma</u> A subspace $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface if $\forall p \in \Sigma$ there is an open ngbd $p \in W \subseteq \mathbb{R}^3$ and a smooth function $f: W \to \mathbb{R}$ s.t. (i) $\Sigma \cap W = f^{-1}(0)$

& (ii) $\forall x \in \Sigma \cap W$, $Df|_x \neq 0$

(Df is the "matrix" of partial derivatives of f)

Slogan: a smooth surface in $\mathbb{R}^3$ is <u>locally</u> cut out by a single smooth function.

The proof uses

<u>Implicit Function Theorem</u> (special case)

Let $W \subseteq \mathbb{R}^3$ be open & $f: W \to \mathbb{R}$ be smooth, s.t. at $p = (x_0, y_0, z_0) \in W$, $Df|_p \neq 0$. Permuting coordinates if necessary, suppose $\frac{\partial f}{\partial z}\Big|_p \neq 0$. Suppose $f(p) = 0$.

Then $\exists$ an open ngbd $(x_0, y_0) \in V \underset{\text{open}}{\subseteq} \mathbb{R}^2$, an open $W' \subseteq W$, and a smooth $g: V \to \mathbb{R}$, s.t.

$$\begin{cases} f(x, y, g(x,y)) = 0 \text{ if } (x,y) \in V, \ \& \ g(x_0, y_0) = z_0 \\ \text{If } (x,y,z) \in f^{-1}(0) \cap W', \text{ then } z = g(x,y) \text{ for some } (x,y) \text{ (in } V) \end{cases}$$

<u>Proof of Lemma</u>

Let $\mathcal{U} = \Sigma \cap W'$, for some $W' \subseteq W$ as obtained from Implicit f^n theorem, & where we fixed $p \in W$ as in the Lemma.

Define a chart by $(\mathcal{U}, \varphi)$, where $\varphi: \mathcal{U} \to V \underset{\text{open}}{\subseteq} \mathbb{R}^2$

$(x,y,z) \mapsto (x,y)$

The IFT says φ is a homeo$^{\underline{m}}$ with inverse $(x,y) \mapsto (x, y, g(x,y$

The set of such pairs $(\mathcal{U}, \varphi)$ defines an atlas for Σ, where we define φ to be a coordinate plane projection s.t. the corresponding "other" partial derivative of f is non-vanishing.

Suppose we have two charts of the form above, with domains $\mathcal{U} \subseteq \Sigma$ and $\mathcal{U}' \subseteq \Sigma$.

L4.4

The transition map

$$\varphi' \circ \varphi^{-1} : (x,y) \mapsto \begin{cases} (y, g(x,y)) \\ (x, g(x,y)) \\ (x,y) \end{cases}$$

depending on which coordinate plane I project to under φ'.

So these are smooth maps, & therefore the atlas is smooth. $\square$

<u>Corollary</u> If a subspace $\Sigma \subseteq \mathbb{R}^3$ satisfies the conditions of the above Lemma, then <u>locally</u> Σ is the graph of a smooth function over one of the coordinate planes.

Slogan: smooth surfaces in $\mathbb{R}^3$ are locally graphs of smooth f^ns

L5.1 <u>The implict function theorem</u>

<u>Recall</u> Last time we considered a subset $\Sigma \subseteq \mathbb{R}^3$ s.t. $\forall p \in \Sigma$, $\exists$ open ngbd $p \in W \underset{\text{open}}{\subseteq} \mathbb{R}^3$ & $f : W \to \mathbb{R}$ smooth s.t.

$\Sigma \cap W = f^{-1}(0)$ &

$\forall x \in \Sigma \cap W, \ Df|_x \neq 0$

<u>Def^n</u> If Σ is a smooth surface in $\mathbb{R}^3$ in the above sense, we say a chart for Σ is <u>allowable</u> if the chart $(\mathcal{U}, \varphi)$, $\varphi : \mathcal{U} \xrightarrow[\text{homeo}]{\cong} V \subseteq \mathbb{R}^2$ is s.t. (a) the inverse $\varphi^{-1} = \sigma : V \to \mathcal{U} \subseteq \mathbb{R}^3$ is smooth as a map to $\mathbb{R}^3$, & (b) the map $D\sigma|_x$ has rank 2 for all $x \in V$.

We say that σ is an <u>allowable parametrisation</u>.

<u>Remark</u> The charts we constructed last time to show Σ <u>was</u> an abstract smooth surface <u>were</u> allowable. Indeed, our charts had inverses of the form $(x,y) \underset{\sigma}{\mapsto} (x, y, g(x,y))$ for $g : V \to \mathbb{R}$ smooth & $V \subseteq \mathbb{R}^2$ open, so $D\sigma$ has form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{pmatrix}$ which has rank.

<u>Lemma</u> If $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, the set of all allowable charts is a smooth atlas.

<u>Remark</u> This atlas does not depend on any choice except for the embedding of Σ in $\mathbb{R}^3$, i.e. it does not depend on a choice of coordinates in $\mathbb{R}^3$. (<u>N</u>B An embedding $X \xrightarrow[f]{} Y$ is a continuous bijection which is a homeo$^{\underline{m}}$ to $f(X) \subseteq Y$)

<u>Proof of Lemma</u> If $\sigma : V \to \mathbb{R}^3$ & $\sigma' : V' \to \mathbb{R}^3$ are both allowable parametrisations of an open ngbd of $p \in \Sigma$, then as $D\sigma|_x$ has rank 2 $\forall x \in V$ there is a choice of coordinate plane projection $P : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $P \circ \sigma$ is locally invertible by IFT. ~ ngbd of $\sigma^{-1}(p)$

Then $\sigma^{-1} \circ \sigma' = (P \circ \sigma)^{-1} \circ (P \circ \sigma')$ is then a composition of smooth functions, noting this expression does make sense in some perhaps smaller $V'' \subseteq V'$. ~ smaller both b/c of IFT, $\sigma(V) \cap \sigma'(V')$ business

So the transition maps for any pair of allowable charts are smooth so {all allowable charts} is an (intrinsic to the embedding) smooth atlas. $\square$

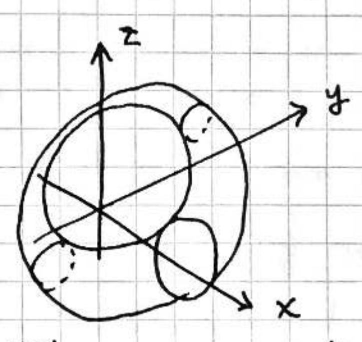
Concretely, to show $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, we can show it's cut out by a smooth function with $Df \neq 0$, a graph $(x, y, g(x,y))$ for smooth g, <u>or</u> covered by allowable parametrisations.

<u>Examples</u> (Surfaces of revolution)

Let $\gamma : (a,b) \longrightarrow \mathbb{R}^3$ ~~is~~ be a smooth arc in the xz-plane,

$$\gamma(u) = (f(u), 0, g(u))$$

We'll assume $f > 0$, and that γ is an embedding (often, $g(u) = u$)



Spin γ around the z-axis, i.e. we consider the surface defined by a local parametrisation

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

for $a < u < b$ & $0 < v < 2\pi$.

Then we compute: $\sigma_u = (f_u \cos v, f_u \sin v, g_u)$

$$\sigma_v = (-f \sin v, f \cos v, 0).$$

Then $\sigma_u \times \sigma_v = (-f g_u \cos v, -f g_u \sin v, f f_u)$ (check!) and hence $\|\sigma_u \times \sigma_v\|^2 = f^2((f_u)^2 + (g_u)^2)$.

So if we parametrise γ by arc-length, i.e. $f_u^2 + g_u^2 = 1$ everywhere, then $\sigma_u \times \sigma_v$ is pointwise non-zero, <u>so</u> σ is an allowable parametrisation and our surface of revolution is smooth.

<u>Recall</u> <u>Inverse Function Theorem</u> Let U, V be open in $\mathbb{R}^n$, & $f : U \to V$ be ctsly diff'ble. Suppose $F(p) = q$ & $Df|_p$ is invertible ($Df|_p = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$). Then $\exists$ open $q \in V' \subseteq V$ & a ctsly diff'ble $\begin{array}{l} G : V' \to U \\ \quad q \mapsto p \end{array}$ s.t. the image of G is an open ngbd of p, & $F \circ G$ is the identity on V'. If F was smooth, then so is G.

<u>Remark</u> Given this, then $D(F \circ G)|_q = DF|_p \circ DG|_q = Id$

$$\underline{\text{so}} \quad DG|_q = (DF|_p)^{-1}.$$

L5.3

Implicit Function Theorem

Let $n = k+m$ & identify $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$.

Let $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$, & $F: \mathcal{U} \to \mathbb{R}^m$ be ctsly diff'ble.

Say $p = (a,b) \in \mathbb{R}^n$, & $F(p) = c$. $\sim p \in \mathcal{U}$

Define $f: \mathbb{R}^m \to \mathbb{R}^m$ by $f(y) = F(a,y)$ (so $f(b) = c$).

Suppose $Df|_b$ is invertible. Then there's an open ngbd

$$(a,c) \in A \times C \subseteq \mathbb{R}^k \times \mathbb{R}^m$$

& a ctsly diff'ble $\varphi: A \times C \to \mathbb{R}^m$ s.t.

(i) $\varphi(a,c) = b$ & image$(\varphi) \subseteq \mathcal{U}$

(ii) $F(x, \varphi(x,z)) = z \quad \forall (x,z) \in A \times C$

(iii) for some $(a,b) \in W \underset{\text{open}}{\subseteq} \mathcal{U}$, if $(x,y) \in W$, $z \in C$, &

$F(x,y) = z$, then $y = \varphi(x,z)$

Remark If F is smooth, then so is φ

Remark Consider $F: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x^2 - 2y$.

If I want to solve $F(x,y) = c$, I could notice that

$$F(x, g(x)) = c, \quad g(x) = \frac{x^2-c}{2}.$$

I could factor out y, & note $D_y F = \frac{\partial F}{\partial y}$ was nowhere zero.

Implicit F^n Theorem

Given $F: \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$, & DF "never vanishes" in the y-variables ($y \in \mathbb{R}^m$), then I can $\underbrace{\text{factor out the } y\text{-variables}}_{\text{i.e. replace } \varphi(x,z)}$ when understanding $F^{-1}(z)$.

Proof of Implicit F^n Theorem

Define $\hat{f}: \mathcal{U} \to \mathbb{R}^n$, $(x,y) \xmapsto{\hat{f}} (x, F(x,y))$ $\quad (x,y) \in \mathbb{R}^k \times \mathbb{R}^m$

Then $D\hat{f}|_{(a,b)} = \begin{pmatrix} 1 & 0 \\ * & Df|_b \end{pmatrix}$ so $D\hat{f}|_{(a,b)}$ is invertible.

IFT $\Rightarrow \exists$ ngbd of $\hat{f}(a,b) = (a,c)$, WLOG an open set of form $A \times C$, and a ctsly diff (or smooth) inverse $\hat{g}: A \times C \to \mathcal{U}$

s.t. $\hat{f} \circ \hat{g} = \text{id}$.

L 5.4

Write $\hat{g}(x,\overset{z}{\cancel{y}}) = (\theta(x,\overset{z}{\cancel{y}}), \varphi(x,\overset{z}{\cancel{y}})) \in \mathbb{R}^k \times \mathbb{R}^m$.

So $\hat{f} \circ \hat{g} = id \Rightarrow \theta(x,z) = x$ & $F(x, \varphi(x,z)) = z$. the image of $\hat{g}$

This constructs the φ. Moreover, $\overset{\hat{f}}{\cancel{\varphi}}$ is injective in a ngbd of (a,b).

So ~~it~~ in this ngbd, there is at most one solution of $F(x,y) = z$. So any such solution necessarily has $y = \varphi(x,z)$.

That φ is ctsly diff / smooth follows from corresponding conclusion in Inverse Function Theorem. $\square$

L 6.1 **Orientability**

Begin with a digression: smoothness of the inverse function.

Suppose we have the continuous inverse G, want that G is diff'ble (or smooth).

Recall: if $A : U \to \mathbb{R}^n$ is continuously diff'ble, then the Mean Value Theorem implies $\| A(x+h) - A(x) \| \leq K \|h\|$,

where $K = \sup_{0 \leq \theta \leq 1} \| DA|_{x+\theta h} \|$.

WLOG, in setting of IFT, assume $F(0) = 0$, $DF|_0 = Id$, & on U we have $\| Df|_x - I \| < 1/2$.

Suppose $G(y) = x$, $G(y+k) = x+h$. Note $F(x+h) - F(x) = k$.

Write $(F(x+h) - (x+h)) - (F(x) - x) = k - h$.

$$\| DF|_{x+\theta h} - I \| < \tfrac{1}{2} \underset{\text{MVT for } A = F - id}{\Rightarrow} \| k - h \| \leq \tfrac{1}{2} \|h\| \quad (*)$$

CLAIM $\dfrac{\| G(y+k) - G(y) - DF|_x^{-1}(k) \|}{\|k\|} \to 0$ as $k \to 0$.

Write
$$\begin{aligned}(+) &= G(y+k) - G(y) - DF|_x^{-1}(k) \\ &= -DF|_x^{-1}\big(F(x+h) - F(x) - DF|_x(\underset{G(y+k)-G(y)}{\underset{\uparrow}{h}})\big)\end{aligned}$$

Apply MVT inequality to the function

$$t \overset{A}{\mapsto} F(x+th) - t\,DF|_x(h) \quad \text{on } 0 \leq t \leq 1.$$

On RHS of $(+)$, we see $-DF|_x^{-1}(A(1) - A(0))$ so MVT says

$$\|(+)\| \leq \| DF|_x^{-1} \| \, \|h\| \, R(h)$$

where $R(h) = \sup_{0 \leq t \leq 1} \| DF|_{x+th} - DF|_x \|$.

Since DF is continuous, $R(h) \to 0$ as $h \to 0$.

Moreover, $(*)$ implies $\|k\| \geq \frac{1}{2}\|h\|$, so as $k \to 0$, $h \to 0$ too.

Combining these, we get the claim, & hence G is ctsly diff with $DG|_y = DF|_x^{-1}$,

i.e. $DG = (\text{inverse}) \circ DF \circ G$ where inverse $GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$.

cts? (and so does $F+1$)

This equation says if G has m^{th} order partial derivatives so does DG, so G being m times diff $\Rightarrow$ DG is $(m+1)$ times diff.

If F is smooth, G is smooth. ← legit, th

NB. $M \mapsto M^{-1}$ is smooth. $\square$

We've seen 3 flavours of surface: topological, smooth, in $\mathbb{R}^3$.

Let $f: \overline{D}^2 \longrightarrow \overline{D}^2$, $(\overline{D}^2 = \{z \in \mathbb{C} : |z| \leq 1\})$ be a homeo$^{\underline{m}}$.

Fact: f sends $S^1 = \{|z| = 1\} \subseteq \overline{D}^2$ to itself

(This needs e.g. II Algebraic Topology)

We say a homeo$^{\underline{m}}$ of S^1 preserves orientation if it takes

the anticlockwise loop $t \mapsto e^{it}$ to

another anticlockwise loop $t \mapsto e^{i\alpha(t)}$ where α increasing.

Using "Fact", we say a homeo$^{\underline{m}}$ of $\mathbb{R}^2$ is orientation preserving if $f|_{\overline{D}^2}: \overline{D}^2 \longrightarrow f(\overline{D}^2)$ is o preserving. (u wot m8)

Def^n If U, V are open in $\mathbb{R}^2$, & $F: U \to V$ is continuously diff, we say F is orientation preserving if F is a diffeo$^{\underline{m}}$ & $\forall p \in U$, $\det(DF|_p) > 0$.

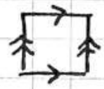
Remark If I take a (f. dim$^{\underline{l}}$) v. space $\mathbb{R}^n$ & consider ordered bases, then $\exists$ an equiv rel^n $(e_1, \dots, e_n) \sim (f_1, \dots, f_n)$ if the change of basis matrix from $\underline{e}$ to $\underline{f}$ has positive determinant.

If $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is smooth & invertible, then F preserves orientation if $\underline{e}$ and $DF|_p(\underline{e})$ lie in the same equivalence class. $(\forall p)$

Note: {considering $F: \mathbb{R}^2 \to \mathbb{R}^2$ smooth}, $F(0) = 0$, as locally approximated by $DF|_0$, a linear map, that if $\det(DF|_0) > 0$ then F is o preserving as a homeo$^{\underline{m}}$ too.

Definition A (topological or smooth) surface is orientable if it admits an atlas of (smooth) charts for which the transition maps are orientation preserving as homeo$^{\underline{m}}$ / diffeo$^{\underline{m}}$ of open subsets of $\mathbb{R}^2$.

<u>NB</u> This does not mean <u>every</u> atlas has this property (Why not?)

<u>Example</u> $S^2 \subseteq \mathbb{R}^3$, $T^2 =$  are orientable surfaces.

For surfaces in $\mathbb{R}^3$, we have another way to approach orientability.

$\Sigma \subseteq \mathbb{R}^3$

Let $p \in \Sigma \subseteq \mathbb{R}^3$, Σ a smooth sfc in $\mathbb{R}^3$

<u>Def^n</u> The tangent plane $T_p\Sigma \subseteq \mathbb{R}^3$ is the affine 2-dim^l subspace spanned by tangent vectors $\gamma'(t) \in \mathbb{R}^2$ to diff'ble curves $\gamma : (-1,1) \longrightarrow \mathbb{R}^3$ s.t. $\gamma(0) = p$, image$(\gamma) \subseteq \Sigma$.

<u>Note</u> If Σ is a smooth sfc in $\mathbb{R}^3$, we can choose an allowable parametrisation $\sigma : V \to \Sigma \subset \mathbb{R}^3$ near p, $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$ locally, in domain V, $\gamma = \sigma(u(t), v(t))$, for some $t \mapsto (u(t), v(t))$ a curve in V, & $\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t) \in \langle \sigma_u, \sigma_v \rangle$.

As $\langle \sigma_u, \sigma_v \rangle$ is 2-dim^l (def^n of allowable), we see $T_p\Sigma$ <u>is</u> 2-dimensional. Moreover, if $\tilde{\sigma} : \tilde{V} \to \Sigma$ is another allowable parametrisation near p, then

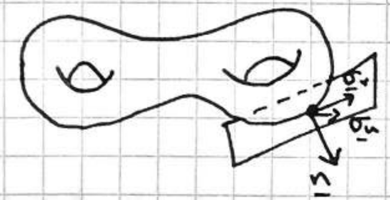
$$D\tilde{\sigma}|_q = D\sigma\,(\underbrace{D\sigma^{-1} \circ D\tilde{\sigma}|_q}_{\cancel{\text{smooth}},\ \text{inv'ble linear map}}),$$

$(\tilde{\sigma}(q) = p)$ because $\sigma^{-1} \circ \tilde{\sigma}$ diffeo^m

so image $D\tilde{\sigma}|_q$ agrees with image $D\sigma|_{\tilde{q}}$ $(\sigma(\tilde{q}) = p)$.

So $T_p\Sigma = \langle \sigma_u, \sigma_v \rangle$ for <u>any</u> choice of allowable σ.

<u>Def^n</u> If $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, a <u>normal vector</u> to Σ at p is a (non-zero, usually <u>unit</u> length) vector orthogonal to $T_p\Sigma$.



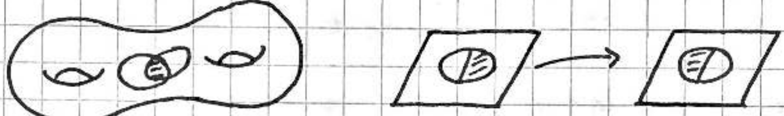
<u>Def^n</u> We say Σ is <u>two sided</u> (for $\Sigma \subseteq \mathbb{R}^3$ smooth) if we can continuously choose one of the two unit normal vectors to Σ globally over Σ.

Orientation & Euler characteristic

"Don't listen to what I say, listen to what I mean" (R. Feynman)

Last time $f : \bar{D}^2 \to \bar{D}^2$ homeo^m preserves orientation if it preserves sense of the boundary.

But def^n given for a homeo^m of an open subset of $\mathbb{R}^2$ to be o. preserving was spinach.



Def^n A topological surface is orientable iff it contains no open subspace homeo^c to a Möbius band.

Proposition (a) A smooth surface is orientable iff it has an atlas of charts s.t. transition maps (smooth!) have $\det Df|_p > 0$ everywhere

(b) A smooth surface in $\mathbb{R}^3$ is orientable iff it's 2-sided (there's a continuous choice of unit normal)

We won't prove (a).

If Σ is a smooth surface in $\mathbb{R}^3$, then we claim:

Lemma Σ is 2-sided $\iff$ it has an atlas with +ve Jacobian transition functions.

Proof Let $\sigma : V \to \Sigma$ be an allowable param^n for a subset of Σ.

Define (given σ) the positive unit normal for points in $\sigma(V)$ by asking that $\langle \sigma_u, \sigma_v, n \rangle$ form a +ve (right-handed) basis of $\mathbb{R}^3$.

If we have another allowable param^n $\tilde{\sigma}$, then $\sigma = \tilde{\sigma} \circ f$ for a transition map f. Write $Df|_p = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Then
$$\left.\begin{aligned} \sigma_u &= \tilde{\sigma}_u \alpha + \tilde{\sigma}_v \gamma \\ \sigma_v &= \tilde{\sigma}_u \beta + \tilde{\sigma}_v \delta \end{aligned}\right\} \text{ at } p.$$

The +ve normal is $\dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ but $\sigma_u \times \sigma_v = \det(Df|_p)\, \tilde{\sigma}_u \times \tilde{\sigma}_v$.

So if the transition maps have +ve Jacobian, the +ve normal is well-defined & Σ is 2-sided.

On the other hand, if Σ is 2-sided, only <u>allow</u> param$^{\underline{n}}$s $\sigma : V \to \Sigma$ s.t. $\langle \sigma_u, \sigma_v, n \rangle$ is a RH basis, where $\underline{n}$ is a chosen unit normal on Σ.

The same computation shows these charts have +ve Jacobian transition functions. $\square$

<u>Corollary</u> $\mathbb{RP}^2$, Klein bottle non-orientable

<u>Theorem</u> A <u>compact</u> non-orientable surface cannot be embedded in $\mathbb{R}^3$. (Compact surfaces in $\mathbb{R}^3$ are 2-sided)

Proof follows easily from results in Pt II, though probably no-one will point that out.

The key other topological invariant of surfaces is their Euler characteristic.

<u>Definition</u> A (polygonal) <u>subdivision</u> of a compact topological surface S contains (i) a finite subset $V \subseteq S$ of <u>vertices</u>

(ii) a finite collection of continuous embeddings $\{ \rho_i : [0,1] \to S \}$ s.t. $\rho_i^{-1}(V) = \{0, 1\}$ & $\text{image}(\rho_i) \cap \text{image}(\rho_j) \subseteq \{ \rho_i(0), \rho_i(1), \rho_j(0), \rho_j(1) \} \subseteq V$ for each $i \neq j$

s.t. (iii) the connected components of $S \setminus (V \cup \bigcup_i \rho_i)$ are homeo$^{\underline{c}}$ to open discs.

The images of ρ_i are <u>edges</u>, & the components in (iii) are <u>faces</u>.

<u>Examples of Subdivisions</u>

(a) 

1 vertex, 0 edges, 1 face, $S = S^2$

(b)  1 vertex, 1 edge, 2 faces (on S^2)

(c)  or  or  are subdivisions of S^2

(d)  e $= T^2$ yields  1 vertex, 2 edges, 1 face

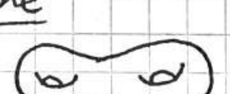
<u>Def^n</u> The Euler characteristic of a subdivision is $\#V - \#E + \#F$

<u>Theorem</u> (a) Every compact topological surface admits a subdivision

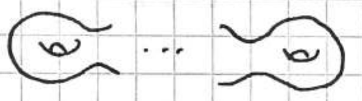
(b) The Euler characteristic is independent of the choice of subdivision, so gives a homeo$^{\underline{m}}$ invt of compact surfaces : $e(S)$ or $\chi(S)$

(c) The homeo$^{\underline{m}}$ class of a compact (connected ?) surface is determined by its orientability & its Euler characteristic.

<u>Example</u>

$\chi\Big($  $\Big) = \chi\Big($  $\Big) \rightsquigarrow$ (c.f. discussion of # sum) 1 vertex, 4 edges, 1 face $\chi = -2$

If I have more holes, g say, this comes from identifying

 sides of a $4g$-gon and labels

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$

1 vertex, $2g$ edges, 1 face & $\chi(S) = 2-2g$.

The <u>genus</u> of a compact orientable surface is defined by $\chi(S) = 2-2g$

It counts the "number of holes".

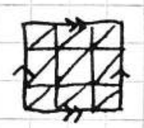
<u>Def^n</u> A subdivision is a <u>triangulation</u> if

(i) the boundary of each face, i.e. its closure in S minus itself, comprises exactly 3 vertices & 3 edges,

(ii) 2 such "triangular" faces are disjoint, or meet in exactly one vertex, or exactly one edge

(iii) each <u>closed</u> face is embedded in S

<u>Example</u>  ✓  <u>not</u> a triangulation of T^2

 triangulation of T^2

All compact surfaces admit triangulations.

<u>Proposition</u> If Σ is a compact orientable surface, then $\chi(\Sigma)$ is even.

<u>Proof</u> (Sketch) Fix a triangulation of Σ. By counting arguments (cf Q sheet 1) one checks (if V, E, F are the numbers of vertices edges, faces),

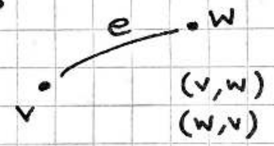
$$2E = 3F \quad (\text{so } F \text{ is even})$$

$$\sum_{v \in V} \text{valence}(v) = 2E$$

$$\Rightarrow \underbrace{|V_{odd}|}_{\text{odd valence vertices}} \equiv 0 \pmod 2$$

<u>Want</u>: $V \equiv E \pmod 2$, so STP $E \equiv |V_{\cancel{odd}\ even}| \pmod 2$

Let $D \subseteq V \times V$ be the set of <u>directed</u> edges



Consider permutations of D

(i) $\eta\ \cancel{\chi} : D \to D$ reverses arrows

(ii) <u>Since</u> Σ is oriented we can  coherently orient all triangles clockwise.

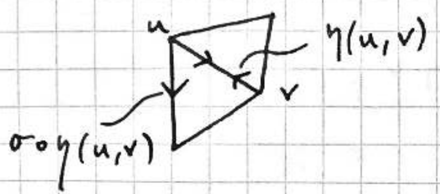
We have $\sigma : D \to D$ s.t. for $(u,v) \in D$, $\sigma(u,v) = (v,w)$ where (u,v,w) is a face of the triangulation & then $\sigma(v,w) = (w,u)$, $\sigma(w,u) = (u,v)$ & $\sigma(v,u) = (u,x)$ where x is the ! elt s.t. (u,v,x) is also a face.

Now consider the <u>sign</u> of the permutation $\sigma \circ \eta$.

$\eta\ \cancel{\chi}$ is a product of E transpositions on D, sign $(-1)^E$

σ is a product of 3-cycles, & there are F of them, so sign $+1$

But composition rotates each edge (u,v) at its vertex, so composition has sign $(-1)^{|V_{even}|}$. Put this together, get parity result. $\square$



$\sigma\eta(u,v)$ moves us clockwise one place

L 8.1

Length

Let $\gamma : (a,b) \to \mathbb{R}^3$ be smooth (or continuously differentiable).

For each $t \in (a,b)$, we have the tangent vector $\gamma'(t)$.

We define the <u>length</u> $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

This is independent of parametrisation in the following sense. If

$s : [A,B] \to [a,b]$ is monotone increasing & C^1 & $\tau(t) = \gamma(s(t))$

then $L(\tau) = \int_A^B |\tau'(t)|\, dt = \int_A^B |\gamma'(s(t))| \underbrace{|s'(t)|}_{s'(t)} dt = \int_a^b |\gamma'(s)|\, ds = L(\gamma)$.

We will call γ a smooth curve if $\gamma'(t)$ is nowhere zero.

<u>Lemma</u> Any smooth (or diff.) curve $\gamma : (a,b) \to \mathbb{R}^3$ can be parametrised by <u>arc-length</u>, i.e. s.t. $\|\gamma'(t)\| = 1 \ \forall t$.

<u>Pf</u> Ex Sheet 2 $\square$

Now let $\Sigma \subseteq \mathbb{R}^3$ be a smooth surface in $\mathbb{R}^3$. Let $\gamma : [0,1] \to \mathbb{R}^3$ have image lying in the image of an allowable parametrisation

$$\sigma : \underset{\substack{\text{open} \\ \mathbb{R}^2}}{V \cap} \to \Sigma \subseteq \mathbb{R}^3$$

NEED u, v be diff.?

We can write $\gamma(t) = \sigma(u(t), v(t))$ for functions u, v.

So $\gamma'(t) = \sigma_u \dot{u} + \sigma_v \dot{v}$

& $\|\gamma'(t)\|^2 = \langle \sigma_u, \sigma_u \rangle \dot{u}^2 + 2\langle \sigma_u, \sigma_v \rangle \dot{u}\dot{v} + \langle \sigma_v, \sigma_v \rangle \dot{v}^2$.

<u>Define</u> $E = \langle \sigma_u, \sigma_u \rangle$, $F = \langle \sigma_u, \sigma_v \rangle$, $G = \langle \sigma_v, \sigma_v \rangle$, smooth functions on V.

The expression $E\, du^2 + 2F\, du\, dv + G\, dv^2$ is the <u>first fundamental form</u> of Σ (wrt the parametrisation σ).

<u>Example</u> (0) If we take $\Sigma = \mathbb{R}^2 = \{z = 0\} \subseteq \mathbb{R}^3$, we have parametrisations $\sigma(u,v) = (u, v, 0)$, so $\sigma_u = (1,0,0)$ and $\sigma_v = (0,1,0) \Rightarrow$ FFF is $du^2 + dv^2$.

If we take Σ and work with polar co-ordinates,

$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, 0)$. Then $\sigma_r = (\cos\theta, \sin\theta)$ and

$\sigma_\theta = (-r\sin\theta, r\cos\theta, 0) \Rightarrow$ FFF is $dr^2 + r^2 d\theta^2$.

<u>Remark</u> What <u>is</u> the 1st fundamental form?

At $p \in \Sigma$, we have $T_p\Sigma = \langle \sigma_u, \sigma_v \rangle$. The FFF is just a symmetric bilinear form on this space, represented by the matrix $\begin{pmatrix} E_p & F_p \\ F_p & G_p \end{pmatrix}$. It gives a way of understanding lengths of curves in $\mathbb{R}^3$ which happen to be in Σ in terms of parametrisations of Σ.

The notation is a mnemonic for remembering that if $\gamma(t) = \sigma(u(t), v(t))$ then $L(\gamma) = \int_\gamma (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\, dt$.

<u>Remark</u> Claim that lengths of curves in a patch of surface $\Sigma \subseteq \mathbb{R}^3$ determine the FFF. Define

$$\gamma_\varepsilon(t) = \sigma(t, 0) \text{ defined on } [0, \varepsilon],$$

where $\sigma : B(0,1) \longrightarrow \Sigma$ is allowable.

$$\begin{array}{ccc} B(0,1) & \longrightarrow & \Sigma \\ \cup\!\!\! & & \cup\!\!\! \\ 0 & \longmapsto & p \end{array}$$

Then $\frac{d}{d\varepsilon} L(\gamma_\varepsilon) = \frac{d}{d\varepsilon}\int_0^\varepsilon E(\sigma(t,0))^{1/2}\, dt = E(\sigma(\varepsilon, 0))^{1/2}$,

so $E|_p^{1/2}$ is then $\lim_{\varepsilon\to 0}\left(\frac{d}{d\varepsilon} L(\gamma_\varepsilon)\right)$.

Similarly $G|_p^{1/2} = \lim_{\varepsilon\to 0}\left(\frac{d}{d\varepsilon} L(\tau_\varepsilon)\right)$ for $\tau_\varepsilon(s) = \overset{\sigma}{(0,s)}$ on $[0, \varepsilon]$.

And if I study $\lim_{\varepsilon\to 0}\left(\frac{d}{d\varepsilon} L(\eta_\varepsilon)\right)$ for $\eta_\varepsilon(t) = \sigma(t,t)$, I get $(E|_p + 2F|_p + G|_p)^{1/2}$.

<u>Def^n</u> Let Σ & Σ' be smooth surfaces in $\mathbb{R}^3$.

Then Σ and Σ' are <u>isometric</u> if there is a smooth $\underset{1,1.}{\text{diffeo}^m}$ $f : \Sigma \to \Sigma'$ s.t. $L(f \circ \gamma) = L(\gamma)$ $\forall$ curves γ on Σ.

(f need <u>not</u> be defined on all of $\mathbb{R}^3$)

<u>Corollary</u> If $p \in \Sigma$ and $q \in \Sigma'$, then Σ and Σ' are locally isometric near p, q (i.e. have isometric open ngbds) if they admit allowable parametrisations having the same first fundamental form.

L8.3

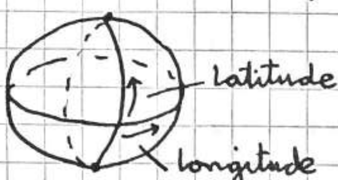
Pf From previous discussion. $\square$

Notation The FFF is also called the <u>induced metric</u> or <u>induced Riemannian metric</u> on Σ from $\mathbb{R}^3$, and written

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad \text{or} \quad g_\Sigma = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

<u>Examples of First Fundamental Form</u>

(1) The sphere $\{x^2+y^2+z^2=a^2\} \subseteq \mathbb{R}^3$ has an open set with parametrisation $\sigma(u,v) = (a\cos u\cos v,\ a\cos u\sin v,\ a\sin u)$



v = longitude $\in (0, 2\pi)$

u = latitude $\in (-\frac{\pi}{2}, \frac{\pi}{2})$

& param$^{\underline{n}}$ covers the complement of a half great circle.

Then $\sigma_u = (-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u)$

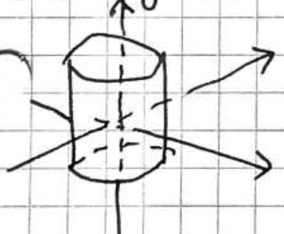
$\sigma_v = (-a\cos u\sin v,\ a\cos u\cos v,\ 0)$

$E = \langle \sigma_u, \sigma_u \rangle = a^2$

$F = \langle \sigma_u, \sigma_v \rangle = 0$ so FFF $= a^2 du^2 + a^2\cos^2 u\, dv^2$

$G = \langle \sigma_v, \sigma_v \rangle = a^2\cos^2 u$

(2) Cylinder of radius a



$\sigma(u,v) = (a\cos v, a\sin v, u)$, $u \in \mathbb{R}, v \in (0, 2\pi)$

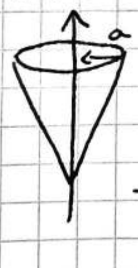
parametrises complement of vertical line.

So $\sigma_u = (0,0,1)$, $\sigma_v = (-a\sin v, a\cos v, 0)$

& FFF is $du^2 + a^2 dv^2$.

NB: We could rescale $(u,v) \mapsto (u, \frac{v}{\sqrt{a}})$ & thus see that the cylinder and the plane are locally isometric: it admits a param$^{\underline{n}}$ with FFF $du^2 + dv^2$.

(3)



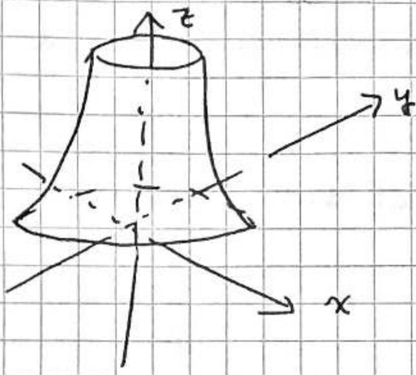
Cone (minus origin) is a sfc with param$^{\underline{n}}$

$\sigma(u,v) = (au\cos v,\ au\sin v,\ u)$

Then FFF is $(1+a^2)\,du^2 + a^2u^2\,dv^2$.

L8.4

(4) Take a curve $y = f(z)$ & rotate it about the z-axis.



The we have a param$^{\underline{n}}$

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, u),$$

and the FFF is

$$(1 + (f'(u)^2))\, du^2 + f(u)^2\, dv^2.$$

Remark $\sigma_u = (f'(u)\cos v, f'(u)\sin v, 1)$

$\sigma_v = (-f(u)\sin v, f(u)\cos v, 0)$

$\sigma_u \times \sigma_v \neq \underline{0}$ provided $f(\cancel{y}^{u}) \neq_{>} 0$ & $f'(u) \neq 0$ as we standardly assume, σ_u & σ_v are linearly independent & this was an allowable parametrisation.

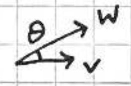
The FFF also lets us talk about angles between curves on surfaces.

If $v, w \in \mathbb{R}^3$, $v \cdot w = |v||w| \cos\theta$ defines angle θ between v and w

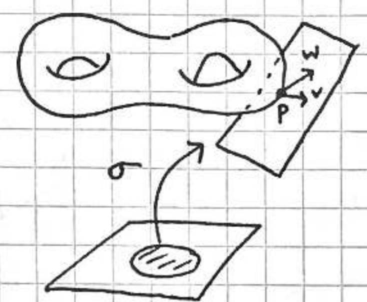
as $\cos\theta = \dfrac{\langle v, w\rangle}{\langle v,v\rangle^{1/2}\langle w,w\rangle^{1/2}} = \dfrac{\Gamma(v,w)}{\Gamma(v,v)^{1/2}\Gamma(w,w)^{1/2}}$ $\quad \Gamma = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ in param$^{\underline{n}}$

Corollary Let Σ be a smooth surface & $(U, \varphi : U \to V \subseteq_{\text{open}} \mathbb{R}^2)$ an allowable chart. Then φ is conformal (i.e. angle preserving) iff the corresponding FFF has $E = G$, $F = 0$.

L9.1 <u>Isometry</u>

Last time: 

Recall $\cos(\theta) = \dfrac{\langle v, w\rangle}{\|v\|\,\|w\|}$.



If $v, w \in T_p\Sigma$ for Σ smooth surface in $\mathbb{R}^3$, & we could write $v = D\sigma(\hat{v})$, $w = D\sigma(\hat{w})$ & then
$$\cos\theta = \frac{\Gamma(\hat{v},\hat{w})}{\Gamma(\hat{v},\hat{v})^{1/2}\,\Gamma(\hat{w},\hat{w})^{1/2}} \quad \text{for} \quad \Gamma = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

More concretely still, if $t \mapsto (u(t), v(t))$ are two
$t \mapsto (\tilde{u}(t), \tilde{v}(t))$
curves in the domain of σ, the angle between these at $t = 0$ is given by
$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}.$$

<u>Corollary</u> A chart is conformal (preserves angles) iff $E = G$, $F = 0$

<u>Proof</u> If the chart is conformal, the "coordinate line curves" $t \mapsto (0,t)$
$t \mapsto (t,0)$
should be orthogonal, & this exactly says $F = 0$. Similarly if $t \mapsto (t,t)$
$t \mapsto (t,-t)$
are defining orthogonal curves, then the formula $\Rightarrow E = G$.

Conversely, if $E = G$, $F = 0$, the metric is just a <u>rescaling</u> of the flat Euclidean metric. □

<u>Remark</u> If a surface has an atlas of conformal charts, then in particular you understand how to rotate by $\pi/2$ in $T_p\Sigma$ at any $p \in \Sigma$ "intrinsically", this sets up a link to complex analysis via "multiplication by i"

<u>Before</u> We saw the FFF captures information about surfaces being locally isometric.

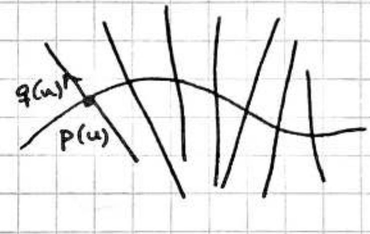
Basic question: which surfaces in $\mathbb{R}^3$ are locally isometric to the flat plane $\mathbb{R}^2 = \{z = 0\}$?

(True of cylinder, also true for the cone away from its vertex)

L9.2

<u>Def^n</u> A <u>ruled surface</u> is a surface swept out by a line in space, i.e. we have a smooth curve $p : \mathbb{R} \to \mathbb{R}^3$, & for each time u, we have a direction $q(u)$, & the surface is given by the param$^{\underline{n}}$

$$\sigma(u,v) = p(u) + v\,q(u)$$

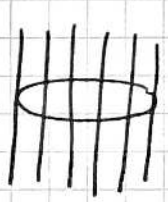


<u>Note</u> This does not necessarily define a surface!

Note $\sigma_u = p_u + v q_u$, $\sigma_v = q$.

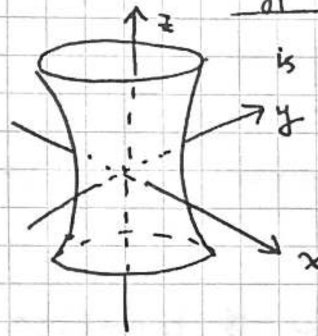
So for $D\sigma$ to have rank 2, need $(p_u + v q_u) \times q \neq 0$.

<u>Example</u> (i) A cylinder is clearly a ruled surface:



Let $p(t) = (\cos t, \sin t, 0)$, $q(t) = (0,0,1)$.

(ii) The <u>hyperboloid</u> of 1 sheet



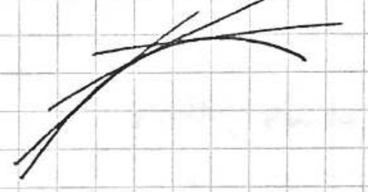
is $\{x^2 + y^2 = 1 + z^2\} \subseteq \mathbb{R}^3$.

If $t \in \mathbb{R}$, the line $\left\{ \begin{array}{l} x - z = t(1-y) \\ x + z = 1/t\,(1+y) \end{array} \right\}$ lies entirely on Σ. Indeed, if $\sigma(u,v) = (\cos u - v\sin u, \sin u + v\cos u, v)$

$$= (\cos u, \sin u, 0) + v(-\sin u, \cos u, 1),$$

then $\text{image}(\sigma) \subseteq \Sigma$.

<u>Def^n</u> A ruled surface is <u>developable</u> if it is swept out by the tangent lines to a curve in $\mathbb{R}^3$, i.e. admits a param^n $\sigma(u,v) = p(u) + v\,p'(u)$.



Again, this <u>needn't</u> be a surface, but often does, at least under suitable hypotheses on p & where $\overset{v}{\cancel{\kappa}} \neq 0$.

Here: $\sigma_u = p_u + v\,p_{uu}$ & if p was param^d by arc length, so $\|p_u\| = 1$
$\sigma_v = p_u$
everywhere, then $p_u \cdot p_{uu} = 0$ & then $\sigma_u \times \sigma_v \neq 0$ if $p_{uu} \neq 0$ & $v \neq 0$

(NB $\|p_{uu}\| = \|p''(u)\|$ is called <u>curvature</u> of the space curve p)

<u>Proposition</u> A developable sfc is locally isometric to the plane

<u>Proof</u> We have Σ locally param^d as $p(u) + v\,p'(u)$, $u \in \mathbb{R}$, $v \in I \subseteq \mathbb{R}\setminus\{0\}$ open

We assume p is param$^{\underline{d}}$ by arc length.

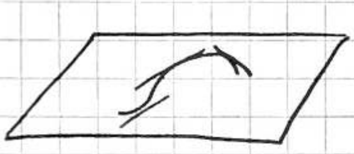
$\sigma_u = p' + v p''$, $\|p'\| = 1 \Rightarrow p' \cdot p'' = 0$ so $\sigma_u \cdot \sigma_u = \|p'\|^2 + v^2 \|p''\|^2$

$\sigma_v = p'$ $\qquad = 1 + \kappa^2 v^2,$

where $\kappa = \|\rho''\|$ is the curvature, & $\sigma_u \cdot \sigma_v = 1, \sigma_v \sigma_v = 1$.

So FFF is $(1+v^2\kappa^2)\,du^2 + 2\,du\,dv + dv^2$ & is completely determined by κ.

Suppose now I have a plane curve $\rho(u) = (x(u), y(u), 0)$ which also has curvature given (locally) by κ. (We assume $\kappa \neq 0$ locally so our developable surface was a surface)



If image $(\rho) \subseteq \{z=0\}$, the associated developable sfc lies in $\mathbb{R}^2$ too.

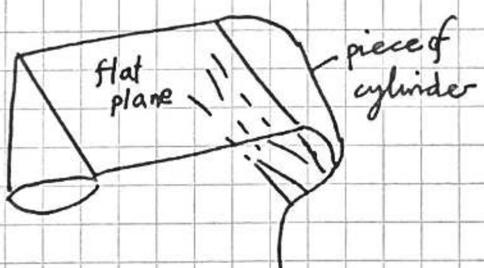
So then Σ_ρ (developable for ρ) locally sweeps an open subset of $\mathbb{R}^2$ & admits some param^{n} $\tau(s,t)$ with FFF $ds^2 + dt^2$.

So we want to find a piece of plane curve ρ with prescribed non-zero curvature.

If we solve the equations $\ddot{x} = \kappa \dot{x}$?? then the curvature of this
$\ddot{y} = \kappa \dot{y}$
curve (param^{d}) by arc-length ~~x~~ is κ.

This is a system of 1st order ODEs so admits a local sol^{n} by the Picard–Lindelöf theorem (it is true that the resulting cure is smooth, see "geodesics" chapter later. $\square$

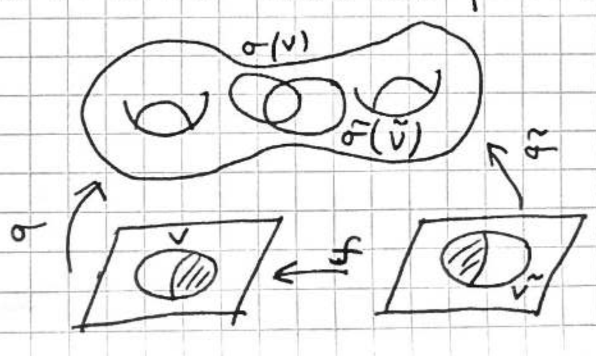
Remark What does the general surface in $\mathbb{R}^3$ locally isometric to a plane look like?



Glue together pieces of plane, cone, cylinder, developable along straight lines.

We may later show that this is the most general thing you can have.

(In) Dependence: Suppose we have two allowable parametrisations for a patch of surface $\sigma : V \to \Sigma$
$\tilde{\sigma} : \tilde{V} \to \Sigma$



$f = \sigma^{-1} \circ \tilde{\sigma}$ defined on open subset of V.

L 9.4

Lemma If $A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is FFF wrt σ, $\tilde{A} = \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}$ is FFF wrt $\tilde{\sigma}$, then $\tilde{A} = (Df)^T A (Df)$

Proof Note $A = (D\sigma)^T D\sigma$ by def^n of FFF.

Now use $f = \sigma^{-1} \circ \tilde{\sigma}$ & the chain rule. □

L10.1

The second fundamental form

Last time we saw the curvature $\|\gamma''(s)\|$ of a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ parametrised by arc-length. If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a plane curve param^d by arc-length, then

$$\gamma(t+h) = \gamma(t) + h\gamma'(t) + \frac{h^2}{2}\gamma''(t) + O(h^3),$$

we can measure its deviation from the tangent line $\mathbb{R}\gamma'(t)$ by evaluating

$$\underline{n} \cdot (\gamma(t+h) - \gamma(t)) = \frac{h^2}{2}\underline{n}\cdot\gamma''(t) = \frac{1}{2}h^2 \underbrace{\kappa}_{\text{curvature}} + O(h^3)$$

choosing $\underline{n}$ s.t. $\langle \gamma'(t), \underline{n} \rangle$ form an o.n. basis for $\mathbb{R}^2$.

Now let $\sigma : V \to U \subseteq \Sigma \subseteq \mathbb{R}^3$ be an allowable param^n of a patch of smooth surface in $\mathbb{R}^3$.

$$\begin{aligned}\sigma(u+h, v+l) = \ & \sigma(u,v) + (\sigma_u(u,v)h + \sigma_v(u,v)l) \\ & + \frac{1}{2}\{\sigma_{uu}(u,v)h^2 + 2\sigma_{uv}(u,v)hl + \sigma_{vv}(u,v)l^2\} \\ & + O(h^3, l^3)\end{aligned}$$

Recall $T_p\Sigma =$ $\overset{\text{wh?}}{\mathbb{R}}\langle \sigma_u, \sigma_v \rangle$ at $p = \sigma(u,v)$, so if we pick a unit normal $\underline{n}$ at p, s.t. $\{\sigma_u, \sigma_v, \underline{n}\}$ is a right-handed basis of $\mathbb{R}^3$, then $\underline{n} \cdot (\sigma(u+h, v+l) - \sigma(u,v))$

$$= \frac{1}{2}\{\langle n, \sigma_{uu}\rangle h^2 + 2\langle n, \sigma_{uv}\rangle hl + \langle n, \sigma_u \rangle l^2\} + O(h^3, l^3)$$

Def^n The second fundamental form of a smooth surface in $\mathbb{R}^3$ with respect to the allowable param^n σ is the expression

$$L\,du^2 + 2M\,du\,dv + N\,dv^2$$

where $L = \langle n, \sigma_{uu}\rangle$, $M = \langle n, \sigma_{uv}\rangle$, $N = \langle n, \sigma_{vv}\rangle$ are smooth functions on V, & $\{\sigma_u, \sigma_v, n\}$ form a right-handed basis with $|\underline{n}| = 1$.

Note The 2nd FF is again naturally a symmetric bilinear form $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ on $T_p\Sigma$. (But unlike the FFF, this need not be non-deg^t)

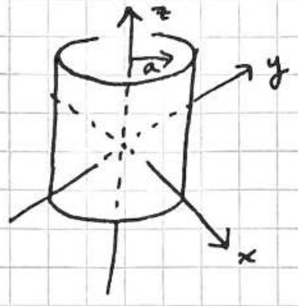
Example (0) For flat $\mathbb{R}^2$, i.e. $\sigma(u,v) = (u,v,0)$ in $\mathbb{R}^3$, then

$$\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0, \ \& \ \mathrm{II}_p = 0 \ \forall p \in \Sigma$$

(We sometimes write II_p for 2nd FF on $T_p\Sigma$)

L10.2

(1) The cylinder has param^n $\sigma(u,v) = (a\cos u, a\sin u, v)$, $u \in (0, 2\pi)$, $v \in \mathbb{R}$



$\sigma_u = (-a\sin u, a\cos u)$ $\quad$ $\sigma_{uu} = (-a\cos u, -a\sin u, 0)$

$\sigma_v = (0,0,1)$ $\quad$ $\sigma_{uv} = 0$

$\underline{n} = (\cos u, \sin u, 0)$ $\quad$ $\sigma_{vv} = 0$

& FFF : $a^2 du^2 + dv^2$

2FF : $-a\,du^2$

<u>Lemma</u> Let $\sigma : V \longrightarrow \Sigma \subseteq \mathbb{R}^3$ be an allowable param^n for a patch of smooth surface in $\mathbb{R}^3$. Assume V is a <u>connected</u> open subset of $\mathbb{R}^2$.
If the 2FF of Σ vanishes identically in V, then $\sigma(V)$ is contained in a plane $\subseteq \mathbb{R}^3$.

<u>Proof</u> Let $\underline{n} : V \to \mathbb{R}^3$ be our unit normal vector.
Since $\langle n, n\rangle = 1$, we see $\langle n, n_u\rangle = 0 = \langle n, n_v\rangle$.
Also $\langle n, \sigma_u\rangle = 0 \Rightarrow \langle n_u, \sigma_u\rangle + \langle n, \sigma_{uu}\rangle = 0 \Rightarrow L = -\langle n_u, \sigma_u\rangle$

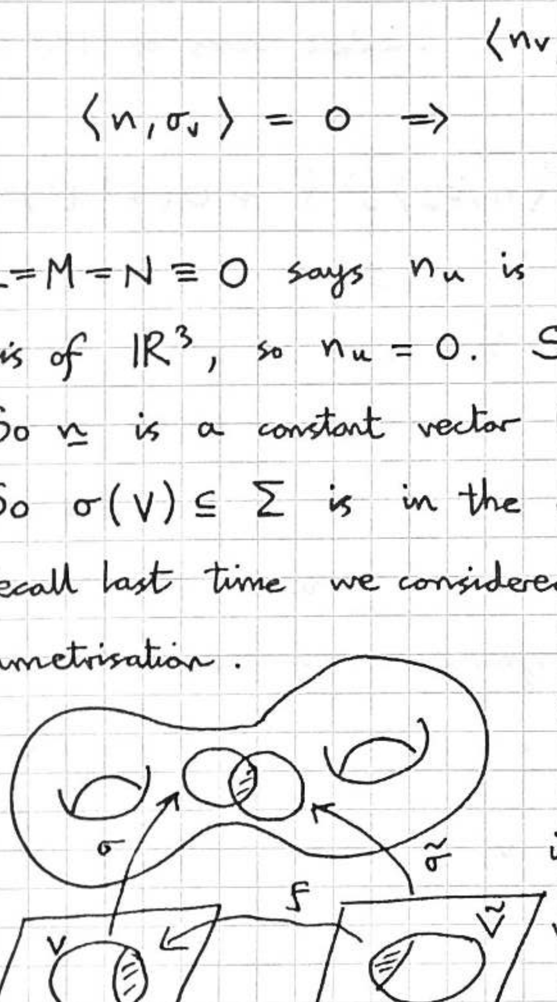
$\langle n_v, \sigma_u\rangle + \langle n, \sigma_{uv}\rangle = 0 \Rightarrow M = -\langle n_v, \sigma_u\rangle$

$\langle n, \sigma_v\rangle = 0 \Rightarrow$ $\quad$ $= -\langle n_u, \sigma_v\rangle$

$N = -\langle n_v, \sigma_v\rangle$

$L = M = N \equiv 0$ says n_u is orthogonal to $\sigma_u, \sigma_v, \underline{n}$ which form a basis of $\mathbb{R}^3$, so $n_u = 0$. Similarly $n_v = 0$.
So $\underline{n}$ is a constant vector (locally constant by MVT, & V connected).
So $\sigma(V) \subseteq \Sigma$ is in the affine plane $\{\underline{x} \in \mathbb{R}^3 : \underline{n}\cdot\underline{x} = 0\}$. $\square$

Recall last time we considered how FFF changes under a change of parametrisation.



$f = \sigma^{-1} \circ \tilde{\sigma}$ is the transition map

We observed that the FFF can be expressed in param^n σ as $(D\sigma)^T(D\sigma)$, so wrt $\tilde{\sigma}$ we get $(Df)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} (Df)$.

The 2FF $\begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} \langle n_u, \sigma_u \rangle & \langle n_v, \sigma_u \rangle \\ \langle n_u, \sigma_v \rangle & \langle n_v, \sigma_v \rangle \end{pmatrix}$

so if we write $n_\sigma : V \to \mathbb{R}^3$ for the unit normal,

$$\mathrm{II} = -(Dn_\sigma)^T D\sigma = -\begin{pmatrix} n_u \\ n_v \end{pmatrix}(\sigma_u, \sigma_v)$$

<u>Corollary</u> If $\sigma : V \to \Sigma$ & $\tilde{\sigma} : \tilde{V} \to \Sigma$ are allowable parametrisations with unit normal functions n_σ and $n_{\tilde{\sigma}}$ and transition function f as above, then $\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = \pm (Df)^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} (Df)$

<u>Proof</u> The normal vector is intrinsic up to sign, so

$$n_{\tilde{\sigma}}\big|_{f(x,y)} = n_\sigma \circ f\big|_{(x,y)}$$

if f has +ve Jacobian (preserves local idea of orientation) & same with opposite sign otherwise. □

<u>Remark</u> Some authors only discuss 2FF for an oriented surface, note a patch $\sigma(V) \subseteq \Sigma$ is always oriented as $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$

<u>Def^n</u> The <u>Gauss curvature</u> of a smooth surface Σ in $\mathbb{R}^3$ at a point $p \in \sigma(V) \subseteq \Sigma$ is the ratio

$\kappa_\Sigma = \dfrac{LN - M^2}{EG - F^2}$ of the determinants of 2^{nd} & 1^{st} FF.

<u>Note</u> $\kappa_\Sigma : \Sigma \to \mathbb{R}$ defines a smooth function on Σ

(a) FFF is non-degenerate, so $EG - F^2$ is never zero

(b) since both FFF & 2FF transform in the same way under change of param$^{\hat{}}$ (up to sign) & $\det(-A) = \det A$ for a 2×2 mx, κ is INDEPENDENT of param$^{\hat{}}$

<u>Example</u> If Σ is given locally as the graph of a function

$\Sigma = \{(x, y, F(x,y))\}$, obvious param$^{\hat{n}}$ $(x,y) \overset{\sigma}{\mapsto} (x, y, F(x,y))$

Then $\sigma_x = (1, 0, F_x)$, $\sigma_{xx} = (0, 0, F_{xx})$

$\sigma_y = (0, 1, F_y)$, $\sigma_{xy} = (0, 0, F_{xy})$

$\sigma_{yy} = (0, 0, F_{yy})$

$n = \dfrac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} = \dfrac{(-F_x, -F_y, 1)}{\text{norm}}$

One computes $\kappa_\Sigma = \dfrac{F_{xx}F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}$ so is a scaled <u>Hessian</u> of F.

<u>Example</u> Round sphere of radius r is the graph of

$$F(x,y) = \sqrt{r^2 - x^2 - y^2}.$$

Then $F_x = -\dfrac{x}{\sqrt{r^2 - x^2 - y^2}}$, so $F_{xx}\big|_0 = -\dfrac{1}{r}$ & by symmetry

$$F_{yy}\big|_0 = -\frac{1}{r}, \quad F_{xy}\big|_0 = 0.$$

So at $(0,0,r) \in S^2$, then Gauss curvature is $\frac{1}{r^2}$

<u>NOTe</u> Isometries of $\mathbb{R}^2$, namely $SO(3)$ presere S^2 and act transitively, so by symmetry $\kappa_{S^2} = \frac{1}{r^2}$ everywhere.

L11.1 Curvature

Recall We introduced the Gauss curvature κ of a smooth surface in $\mathbb{R}^3$

$$\kappa = \frac{LN - M^2}{EG - F^2}$$

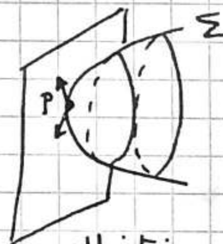
(& this is intrinsic, independent of the allowable parametrization)

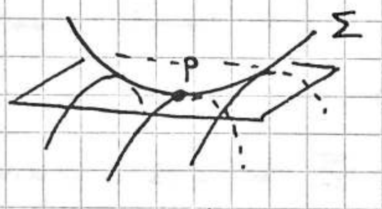
Def^n For Σ a smooth surface in $\mathbb{R}^3$, we say a point $p \in \Sigma$ is elliptic if $\kappa > 0$, hyperbolic if $\kappa < 0$, parabolic if $\kappa = 0$.

Lemma In a ngbd of an elliptic point p, the surface Σ lies on one side of $T_p\Sigma \cong \mathbb{R}^2 \subseteq \mathbb{R}^3$.

In any ngbd of a hyperbolic point p, Σ lies on both sides of $T_p\Sigma$.

Pictures





Proof Take a local param^n $\sigma : V \to \mathcal{U} \subseteq \Sigma$ near p.

Recall, if $p = \sigma(0,0)$, $(0,0) \in V \subseteq \mathbb{R}^2$ & if $(u,v) \in V$, then distance from $\sigma(u,v)$ to $T_p\Sigma$ is

$$\langle \sigma(u,v) - \sigma(0,0), \underline{n} \rangle = \tfrac{1}{2}(Lu^2 + 2Muv + Nv^2) + O(u^3, v^3)$$
$$= \tfrac{1}{2}\,\mathrm{II}_p(\omega) + O(u^3, v^3)$$

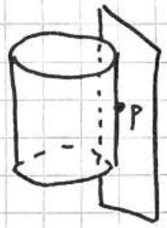
where $\omega = u\sigma_u + v\sigma_v \in T_p\Sigma$.

If p elliptic, then $\kappa > 0$ near p says II_p is +ve/-ve definite near p. In particular, $\mathrm{II}_p(\omega)$ & hence d, the (signed) distance to $T_p\Sigma$ has the same sign in a ngbd of p, so Σ lies locally on one side of $T_p\Sigma$.

Conversely, if p is hyperbolic, then II_p is <u>indefinite</u>: so $\exists$ points (u,v) and $(u',v') \in T_p\Sigma$ s.t. $\mathrm{II}_p(u\sigma_u + v\sigma_v)$ and $\mathrm{II}_p(u'\sigma_u + v'\sigma_v)$ have different signs, so the signed distance d also takes both signs. (cf $\kappa \propto \mathrm{Hess}(f)$ if $\Sigma \underset{\text{locally}}{\cong} \mathrm{Graph}(f)$) $\square$

L11.2

Examples of the parabolic case

(i) 

Last time we computed 2FF: saw that $LN - M^2 = 0$

(All points are parabolic)

(ii) We consider the "monkey saddle"

$$\sigma(u,v) = (u, v, u^3 - 3v^2u).$$

Then $\sigma_u = (1, 0, 3u^2 - 3v^2)$ $\quad \sigma_{uu} = (0,0,6u)$ $\quad \sigma_{uv} = (0,0,-6v)$

$\sigma_v = (0, 1, -6uv)$ $\quad \sigma_{vv} = (0,0,-6u)$

$\underline{n} \propto \sigma_u \times \sigma_v = (?, ?, \cancel{?}^{\,1})$ $\quad \left\{ \begin{array}{l} 3 \text{ lobes go up} \\ 3 \text{ go down} \end{array} \right\}$

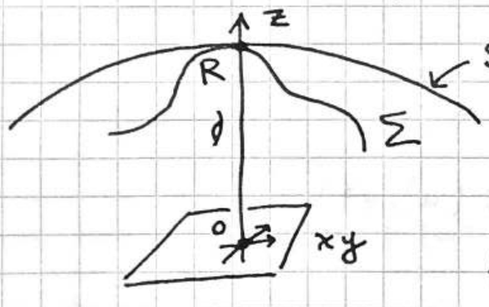
are you sure about that? ok it's 1

$\Rightarrow \kappa = 0$ at 0

<u>Proposition</u> Let $\Sigma \subseteq \mathbb{R}^3$ be a smooth surface which is <u>compact</u>. Then Σ has an elliptic point.

<u>Proof</u> Being compact $\Rightarrow$ Σ is closed and bounded.

So I can find a smallest closed ball $\overline{B(0,N)}$ s.t. Σ is contained inside the ball. Then (after a global isometry of $\mathbb{R}^3$ i.e. using an element of $O(3)$), I can reduce to



$S^2(R)$ = boundary of $\overline{B(0,R)}$

So locally, Σ is given by the graph of a function f s.t. $f(x,y) - \sqrt{R^2 - x^2 - y^2} \leq 0$, $\quad (*)$

$f(0,0) = R$,

$f_x(0,0) = f_y(0,0) = 0$.

$$(*) \Rightarrow \frac{1}{2}(f_{xx}x^2 + 2f_{xy}xy + f_{xy}y^2) + \frac{1}{2R}(x^2+y^2) \leq 0$$

(taking Taylor series for order 2) with (x,y) sufficiently small

$$Lx^2 + 2Mxy + Ny^2 \leq -\frac{1}{R}(x^2+y^2)$$

so locally the 2FF is negative definite near 0, so $\kappa > 0$. $\square$

L11.3

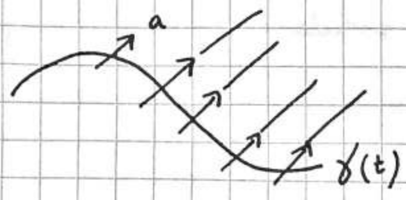
Def^n A surface is flat if $\kappa_\Sigma \equiv 0$.

Recall a ruled surface is that swept by a moving line

$\sigma(u,v) = \gamma(u) + v \cdot a(u)$ $\qquad$ $t \mapsto \gamma(t)$ smooth curve,

a gives direction of lines.

We say Σ is non-cylindrical (near a point) if locally $a'(t) \neq 0$.



Cylindrical ruled surfaces (which include cylinder) are swept by parallel lines, & can be analysed directly.

Proposition A non-cylindrical ruled surface Σ in $\mathbb{R}^3$ is flat iff it is developable (swept by tangent lines to a curve).

Proof Parametrise $\sigma(u,v) = \gamma(u) + v\,a(u)$.

Then $\sigma_{uu} = \gamma'' + v a''$, $\qquad$ In 2FF, $N = 0$.

$\sigma_{uv} = a'$, $\qquad$ So $\kappa = 0$ ($\Leftrightarrow LN = M^2$) iff $M = 0$.

$\sigma_{vv} = 0$.

equivalent chain:

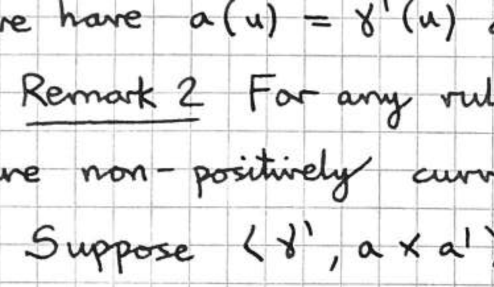
$$\left\{\begin{array}{l} \text{So } \langle \sigma_u \times \sigma_v , a' \rangle = 0. \\ \text{So } \langle (\gamma' + \underbrace{v a'}_{\text{drops}}) \times a , a' \rangle = 0. \\ \text{So } \langle \gamma' \times a , a' \rangle = 0. \\ \text{So } \langle \gamma' , a \times a' \rangle = 0. \quad (+) \end{array}\right.$$

Remark If Σ is developable, then (for a suitable param$^{\hat{=}}$) we have $a(u) = \gamma'(u)$ & (+) holds; so developable $\Rightarrow$ flat

Remark 2 For any ruled surface, $\kappa \propto -M^2 \leq 0$, the surfaces are non-positively curved.

Suppose $\langle \gamma', a \times a' \rangle = 0$. Note Σ is also swept by

$\tilde{\gamma}(u) + v\,a(u)$ whenever $\tilde{\gamma}(u) = \gamma(u) + \alpha(u)\,a(u)$



We want to choose $\tilde{\gamma}$ so that $\tilde{\gamma}'$ is parallel to a, i.e. if $\tilde{\gamma}'(t) = \beta(t)\,a(t)$ for a function β,

$$\beta(t)\,a(t) = \gamma'(t) + \alpha'(t)\,a(t) + \alpha(t)\,a'(t).$$

L11.4

Here γ, γ', a, a' all given, I would like to solve for α, β.

Suppose $a(t)$ always had length 1 (WLOG), so $a(t) \cdot a'(t) = 0$.

Then $\beta = \gamma' \cdot a + \alpha'$ by dotting eqⁿ with a.

So $(\gamma' \cdot a)\, a - \gamma' = \alpha a'$ feeding back in.

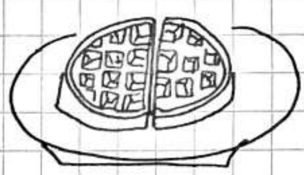
Non-cylindrical: $a'(t) \neq 0$. So α is now determined.

The eqⁿ becomes an ODE for β, so $\exists$ local solution. ooh... $\square$

no need for ODE dood,
like just choose that α,
it's ok, then $\langle \tilde{\gamma}', a \times a' \rangle = 0$

$$\langle \tilde{\gamma}', a' \rangle = 0$$

MASSIVE



L12.1 <u>Area</u>

<u>Corrigendum</u> Last time, we saw the "monkey saddle"

$(u,v) \mapsto (u, v, u^3 - 3v^2u)$

$\sigma_u = (1, 0, 3u^2 - 3v^2)$ $\quad\quad$ $\sigma_{uu} = (0,0,6u)$

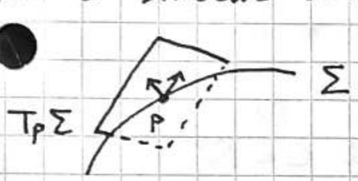
$\sigma_v = (0, 1, -6uv)$ $\quad\quad$ $\sigma_{vv} = (0,0,-6u)$

$\underline{n} = (?, ?, 1)$ $\quad\quad$ $\sigma_{uv} = (0,0,-6v)$

<u>Note</u> $\kappa\big|_{(0,0)} = 0$

So $(0,0,0) \in \Sigma$ is parabolic

The first fundamental form also gives a way of measuring <u>area</u> for a smooth surface $\Sigma \subset \mathbb{R}^3$.



Think about an "infinitesimal parallelogram" on Σ spanned by σ_u, σ_v for σ allowable.

If $a, b \in \mathbb{R}^2$, the parallelogram spanned by vectors a, b has area $\|a \times b\|$. Using $\langle a\times b, a\times b\rangle = \langle a,a\rangle\langle b,b\rangle - \langle a,b\rangle^2$,

$$\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$$

<u>Def^n</u> If $\sigma: V \to \mathcal{U} \subseteq \Sigma$ is allowable, & if $V' \subseteq V$, $\sigma(V') = \mathcal{U}'$, the area of $\mathcal{U}'$ is $\int_{V'} \sqrt{EG - F^2}\, du\, dv$ (in open $\mathbb{R}^2$)

<u>Lemma</u> The area is independent of the choice of local param$^{\underline{n}}$

<u>Proof</u> If $\sigma: V \to \Sigma$ & $\tilde{\sigma}: \tilde{V} \to \Sigma$ are allowable param$^{\underline{n}}$ whose images contain $\mathcal{U}'$, we have a transition map $f = \sigma^{-1} \circ \tilde{\sigma}$ from (subset of $\tilde{V}$) to (subset of V).

Know $\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (Df)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} (Df)$.

So $\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = |\det Df| \sqrt{EG - F^2}$.

Whenever $W \subseteq \mathbb{R}^2$, $H: W \to \mathbb{R}$ continuously diff and $\phi: W' \to W$ a diffeo$^{\underline{m}}$, then $\int_W H\, du\, dv = \int_{\phi^{-1}(W)} (H \circ \phi)\, |\mathrm{Det}(D\phi)|\, d\tilde{u}\, d\tilde{v}$. $\square$

<u>Example</u> If we have a graphical surface $\{z = f(x,y)\} \subseteq \mathbb{R}^3$ for smooth f, then $\sigma_x = (1, 0, f_x)$ for $\sigma: (x,y) \mapsto (x, y, f(x,y))$.

$\sigma_y = (0, 1, f_y)$

L12.2

So $\sqrt{EG-F^2} = \left((1+f_x^2)(1+f_y^2) - f_x^2 f_y^2\right)^{1/2} = \sqrt{1+f_x^2+f_y^2}$.

If $R \subseteq \mathbb{R}^2$ is a region, e.g. "$R = B(0,R)$", then note

$$\text{Area}(\sigma(R)) \geqslant \text{Area}_{\text{Eucl}}(R)$$

with equality iff f is constant on R.

<u>Ex</u> Monkey saddle & "$R = B(0,R)$", $\text{Area}(\overset{\sigma}{R}) = \pi R^2 + \frac{3}{2}\pi R^4$!

One can use area to give a different geometric interpretation to the Gauss curvature.

<u>Def^n</u> Let $\Sigma \subseteq \mathbb{R}^3$ be a smooth <u>oriented</u> surface, so I have a chosen unit normal vector everywhere (continuously) on Σ.

Then the map $\underline{n} : \Sigma \to S^2 = \{x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3$

$p \mapsto n(p)$ normal vector to Σ at p, viewed as based at $0 \in \mathbb{R}^3$

This is a smooth map.

<u>Lemma</u> For Σ oriented in $\mathbb{R}^3$ as above, the Gauss curvature at a point p vanishes $\Leftrightarrow$ n_u, n_v are linearly dependent at p.

(Here fix an allowable param$^{\underline{n}}$ $\sigma : V \to U \subseteq \mathbb{R}^3$ near p)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \mapsto p$

<u>Proof</u> By def^n, n has length 1,

so $\langle n, n\rangle = 1 \Rightarrow \langle n, n_u\rangle = \langle n, n_v\rangle = 0$.

So $n_u \times n_v$ is parallel to n, in fact

$$\|n_u \times n_v\| = \langle n, n_u \times n_v\rangle \quad \text{(up to sign)}$$

$$= \frac{\langle \sigma_u \times \sigma_v, n_u \times n_v\rangle}{\|\sigma_u \times \sigma_v\|}$$

$$= \frac{\langle \sigma_u, n_u\rangle\langle \sigma_v, n_v\rangle - \langle \sigma_u, n_v\rangle\langle \sigma_v, n_u\rangle}{\|\sigma_u \times \sigma_v\|}.$$

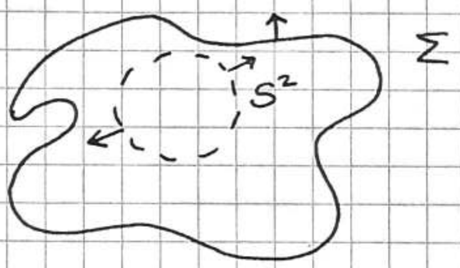
Recall $\langle \sigma_u, n_u\rangle = -\langle \sigma_{uu}, n\rangle$ sine $\langle \sigma_u, n\rangle = 0$.

Using this identity and its cousins,

$$\|n_u \times n_v\| = \frac{LN - M^2}{\sqrt{EG-F^2}} = \kappa\sqrt{EG-F^2}. \quad \text{(up to sign) (conclusion still holds)} \quad \square$$

L12.3

<u>Proposition</u> For $\Sigma \subseteq \mathbb{R}^3$ oriented and $p \in \Sigma$, consider a family of ngbds $\{U_i\}$ of p which shrink down to p.
(e.g. $\Sigma \cap B_{\mathbb{R}^3}(p, \varepsilon) \supseteq U_i$ for i large enough)
Suppose $\kappa(p) \neq 0$. Then $\lim_{i \to \infty} \dfrac{\text{area}_{S^2}(n(U_i))}{\text{area}_\Sigma(U_i)}$ exists, and is equal to $\kappa(p)$ if locally n preserves orientation, & $-\kappa(p)$ if n locally reverses orientation, where we orient S^2 with the outward normal vector.



<u>Proof</u> We work in a param$^{\underline{n}}$ σ locally near p. We know

$$\text{Area}_\Sigma(U) = \int_V \|\sigma_u \times \sigma_v\| \, du\, dv$$
$$= \int_V \sqrt{EG - F^2} \, du\, dv$$

where $U = \sigma(V)$ lies in image(σ).

We also have the map $n \circ \sigma \equiv n : V \to S^2$ the composite of param$^{\underline{n}}$ with the Gauss map.

Lemma $\Rightarrow$ n_u, n_v linearly independent near p

So $n : V \to S^2$ is an allowable param$^{\underline{n}}$.

So $\text{area}_{S^2}(n(U)) = \int_V \|n_u \times n_v\| \, du\, dv$
$= \int_V |\kappa(u,v)| \, \|\sigma_u \times \sigma_v\| \, du\, dv$ by Lemma.

Write $\kappa(u,v) = \kappa|_p + (\kappa(u,v) - \kappa|_p)$.

Then $\text{area}_{S^2}(n(U)) = |\kappa|_p| \, \text{area}_\Sigma(U) + \int_V |\kappa(u,v) - \kappa|_p| \, \|\sigma_u \times \sigma_v\| \, du\, dv$

$$\therefore \left| \frac{\text{area}(n(U))}{\text{area}(U)} - |\kappa|_p| \right| \leq \frac{1}{\text{area}(U)} \max_{q \in U} |\kappa|_q - \kappa|_p| \int_V \|\sigma_u \times \sigma_v\| \, du\, dv.$$

Now continuity of κ as a function near p yields the result. $\square$

<u>Remark</u> Result is true even if $\kappa|_p = 0$, but takes more work.

L 13.1 **Rigidity**

Recall $\text{Area}(R) = \int_R \sqrt{EG-F^2}\, du\, dv = \int_R \|\sigma_u \times \sigma_v\|\, du\, dv$

is necessarily positive, & so $\lim_{U \to p} \dfrac{\text{Area}_{S^2}(n(U))}{\text{Area}_\Sigma(U)} = |\kappa(p)|$.

Recall: we showed $n_u \times n_v$ is proportional to n,

$$\langle n, n_u \times n_v \rangle = \kappa \sqrt{EG-F^2}$$

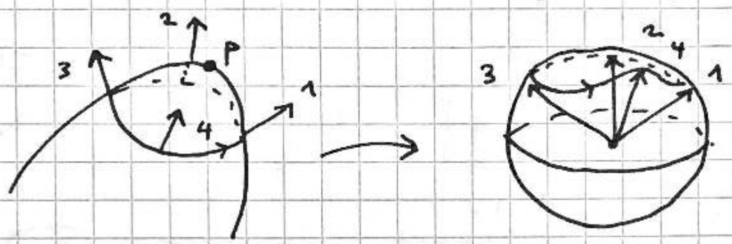
$$\therefore \|n_u \times n_v\| = |\kappa| \sqrt{EG-F^2}$$

Using $|\kappa(u,v)| = |\kappa_p + (\kappa(u,v) - \kappa_p)| \leq |\kappa_p| + |\kappa(u,v) - \kappa_p|$,

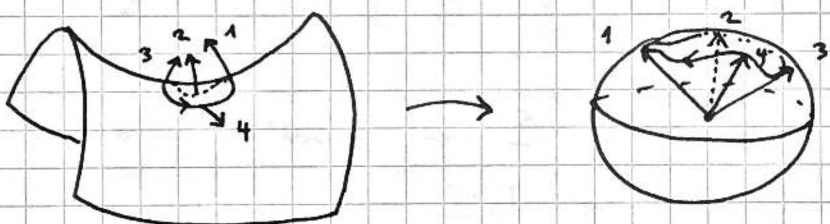
our argument last time proves $\lim_{U \to p} \dfrac{\text{Area}_{S^2}(n(U))}{\text{Area}_\Sigma(U)} = |\kappa_p|$.

Note $\{\sigma_u, \sigma_v, n\}$ is a positively oriented basis for $\mathbb{R}^3$

but $\{n_u, n_v, n\}$ need not be positively oriented, it is precisely if κ is positive.



orientation preserved

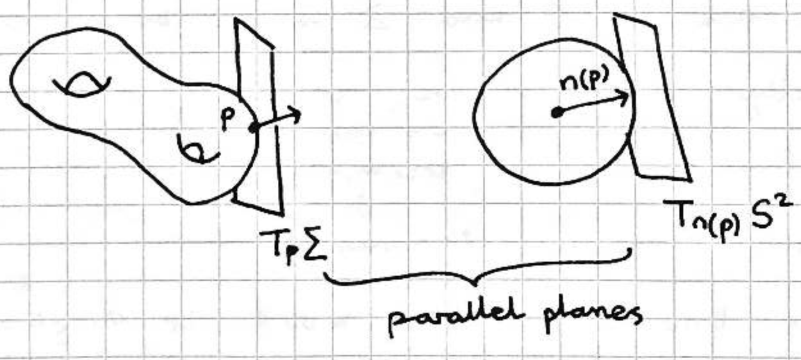


orientation reversed

see Do Carmo

We see that the Gauss map changes orientation based on κ.

Remark



If we view the derivative of the Gauss map at p,

$$D(\text{Gauss})|_p : T_p\Sigma \to T_{n(p)}S^2$$
$$\quad\quad\quad\quad\quad \text{\rotatebox{90}{$\cong$}} \quad\quad \text{\rotatebox{90}{$\cong$}}$$
$$\quad\quad\quad\quad\quad V \longrightarrow V \leftarrow \text{plane through } 0 \text{ in } \mathbb{R}^3$$

Then our def^n of what it means for Gauss map to preserve

L13.2

orientation matches our notion of a linear map preserving orientation.

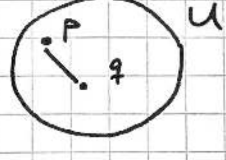
Convention If $\kappa(p) < 0$, many authors declare $\mathrm{Area}_{S^2}(n(\varphi\mathcal{U}))$ & $\mathrm{Area}_\Sigma(\mathcal{U})$ to have opposite sign. With that convention, one writes

$$\lim_{\mathcal{U} \to p} \frac{\mathrm{Area}_{S^2}(n(\mathcal{U}))}{\mathrm{Area}_\Sigma(\mathcal{U})} = \kappa|_p .$$

Note that if $\kappa_p \neq 0$, then n_u & n_v are linearly independent, so that $D(\text{Gauss})$ is locally invertible ?✓ So the Gauss map is locally invertible by the Inverse Function Theorem.

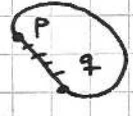
Consider the case of a compact surface $\Sigma \subseteq \mathbb{R}^3$ for which $\kappa > 0$ is everywhere positive. Then $n : \Sigma \to S^2$ is everywhere a local diffeo$^{\underline{m}}$.

Let's furthermore assume that Σ is the boundary of a strictly convex region $\mathcal{U} \subseteq \mathbb{R}^3$, i.e.

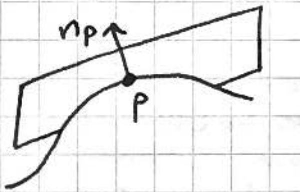


$\forall p, q \in \mathcal{U}$, the line segment $[p,q] \subseteq \mathcal{U}$,

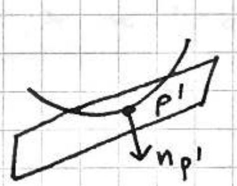
& $\forall p, q \in \mathcal{U}$, $[p,q] \cap \Sigma \subseteq \{p,q\}$.



Exercise For such a Σ, the Gauss map is actually a bijection.



Key claim: $\forall p \in \Sigma$, Σ lies globally on one side of the affine plane $T_p\Sigma$



Corollary If $\Sigma \subseteq \mathbb{R}^3$ is a compact smooth surface such that $\kappa > 0$ and Σ is the boundary of a strictly convex region, then $\int_\Sigma \kappa \, dA = 4\pi$.

Here dA is the area form on Σ, i.e. locally $\sqrt{EG - F^2}$.

(This is the convex Gauss-Bonnet theorem)

Pf Area elt on S^2 is $\|n_u \times n_v\| = \kappa \|\sigma_u \times \sigma_v\|$ i.e. $dA_{S^2} = \kappa \, dA_\Sigma$.

Since the Gauss map is smooth, bijective, and locally a diffeo$^{\underline{m}}$, the change of variables formula for integration says

$$\int_\Sigma \kappa \, dA_\Sigma = \int_{S^2} 1 \, dA_{S^2} = 4\pi$$

□

Remarks Later we will see the general Gauss-Bonnet theorem, which says $\int_\Sigma \kappa \, dA = 2\pi \chi(\Sigma)$ ← Euler characteristic

(ii) In II Algebraic Topology, we'll see that if S is a surface & $f: S \to S^2$ is a local diffeo$^{\underline{m}}$, then necessarily $S \cong S^2$ and f is a bijection.

So if $\kappa(\Sigma) > 0$ for Σ a compact surface in $\mathbb{R}^3$, in fact the Gauss map is a bijection w/o any convexity type hypotheses.

(iii) (Convex) Gauss-Bonnet is a rigidity statement, in the sense that it contains the possible functions $\kappa: S^2 \to \mathbb{R}$ which are the curvature functions of smooth (convex) surfaces in $\mathbb{R}^3$.

In fact

Theorem (Rigidity) If $\Sigma, \tilde{\Sigma}$ are connected smooth surfaces in $\mathbb{R}^3$ with parametrizations $\sigma: V \to \Sigma$ & $\tilde{\sigma}: \tilde{V} \to \tilde{\Sigma}$ for which the 1st and 2nd fundamental forms agree as functions on V, then, on the images of σ and $\tilde{\sigma}$, Σ and $\tilde{\Sigma}$ agree up to a rigid motion of $\mathbb{R}^3$ (i.e. after translation and $O(3)$).

Proof Ommited, apparently not too hard. □

There's another rigidity theorem, "theorema egregium"

Theorem (Gauss) The FFF determines the Gauss curvature κ.

Hence, if two surfaces are isometric they have the same κ.

Note Any family of inner products $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ parametrised by points of a disc D gives me a way of measuring lengths of curves in that disc D.

This gives a notion / definition of an abstract Riemannian metric on a disc (needn't come from a map $D \to \mathbb{R}^3$).

Energy

Let $\gamma : [a,b] \to \mathbb{R}^3$ a smooth curve.

The length of γ is by def^n $\int_a^b \|\dot\gamma(t)\|\, dt$.

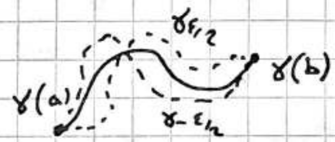
The energy of γ is by def^n $\int_a^b \|\dot\gamma(t)\|^2\, dt$.

(This is not independent of the parametrisation)

We approach geodesics via the calculus of variations for energy.

Def^n A one-parameter variation of γ with fixed end-points is a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to \mathbb{R}^3$ where if $\gamma_s(\cdot) = \Gamma(s, \cdot)$, then $\gamma_0 = \gamma$ and for each s, γ_s is a smooth curve from $\gamma(a)$ to $\gamma(b)$.

Suppose now $\gamma : [a,b] \to \Sigma \subseteq \mathbb{R}^3$ has image in a smooth surface, & in fact image in the image of an allowable param^n $\sigma : V \to \Sigma$. ($V \subseteq \mathbb{R}^2$)



So small perturbations of γ lie inside $\sigma(V)$, so we can analyse the condition $\frac{d}{ds}\big|_{s=0} E(\gamma_s) = 0$ in terms of the FFF wrt σ.

If FFF is $E\,du^2 + F\,2\,du\,dv + G\,dv^2$, & we set

$R = E\dot u^2 + 2F\dot u\dot v + G\dot v^2$ then if we have $\{\gamma_s\}$ as above,

$\frac{d}{ds} E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\, dt$ where $\gamma_s(t) = (u(s,t), v(s,t))$.

Here
$$\begin{aligned}\frac{\partial R}{\partial s} &= (E_u \dot u^2 + 2F_u \dot u\dot v + G_u \dot v^2)\frac{\partial u}{\partial s}\\ &+ (E_v \dot u^2 + 2F_v \dot u\dot v + G_v \dot v^2)\frac{\partial v}{\partial s}\\ &+ 2(E\dot u + F\dot v)\frac{\partial \dot u}{\partial s} + 2(F\dot u + G\dot v)\frac{\partial \dot v}{\partial s}.\end{aligned}$$

Note here $\dot u = \partial u/\partial t$, $\dot v = \partial v/\partial t$.

We now integrate by parts, & use that $\partial u/\partial s$ & $\partial v/\partial s$ vanish at the end-points a, b.

Thus $\frac{d}{ds}\big|_{s=0} E(\gamma_s) = \int_a^b \left(A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s}\right) dt$

where
$$\left.\begin{aligned}A &= E_u \dot u^2 + 2F_u \dot u\dot v + G_u \dot v^2 - 2\tfrac{d}{dt}(E\dot u + F\dot v)\\ B &= E_v \dot v^2 + 2F_v \dot u\dot v + G_v \dot v^2 - 2\tfrac{d}{dt}(F\dot u + G\dot v).\end{aligned}\right\}\text{ evaluated at } s=0$$

Def^n A curve $\gamma : [a,b] \to \sigma(V) \subseteq \Sigma \subseteq \mathbb{R}^3$ is a geodesic if (A) & (B) both vanish.

<u>Remark</u> The geodesic equations

$$\tfrac{d}{dt}(E\dot{u} + F\dot{v}) = \tfrac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$$
$$\tfrac{d}{dt}(F\dot{u} + G\dot{v}) = \tfrac{1}{2}(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2)$$

for a curve $\gamma(t) = (u(t), v(t))$ on Σ are <u>local</u> & depend only on γ.

<u>Remark</u> If $f, g : [a,b] \to \mathbb{R}$ are continuous, then the Cauchy-Schwartz inequality says $\left(\int_a^b fg\right)^2 \leq \int_a^b f^2 \cdot \int_a^b g^2$.

If $f = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}$ & $g = 1$, then we see

$$\text{Length}(\gamma)^2 \leq \text{Energy}(\gamma)\,(b-a)$$

with equality $\Leftrightarrow$ $f = cg$ for a <u>constant</u> c.

<u>Corollary</u> If $\gamma : [a,b] \to \sigma(V) \subseteq \Sigma$ globally minimises energy, then it globally minimises length & has constant speed.

If γ locally minimises length and has constant speed, then γ locally minimises energy & is therefore a geodesic.

<u>Example</u> the plane

If $\sigma(u,v) = (u,v,0)$, FFF is $du^2 + dv^2$, $E = 1 = G$, $F = 0$.

The geodesic eq^{n}s reduce to $\tfrac{d}{dt}(\dot{u}) = 0$, $\tfrac{d}{dt}(\dot{v}) = 0$, i.e. $\ddot{u} = \ddot{v} = 0$. So geodesics are straight lines, parametrised at constant speed.

<u>Example</u> the sphere

$\sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$, FFF: $du^2 + \cos^2 u\, dv^2$.

The equations become $\tfrac{d}{dt}(\dot{u}) = -\cos u \sin u\, \dot{v}^2$,

$$\tfrac{d}{dt}(\cos^2 u\, \dot{v}) = 0.$$

So $\ddot{u} + \sin u \cos u\, \dot{v}^2 = 0$ & $\ddot{v} - 2\tan u\, \dot{u}\dot{v} = 0$. $\quad (+)$

Constant speed param^n says $\dot{u}^2 + \cos^2 u\, \dot{v}^2 = 1$. $\quad (++)$

$(+)$: $\ddot{v}/\dot{v} = 2\tan u\, \dot{u} \Rightarrow \ln(\dot{v}) = -2\ln(\cos u) + \text{constant}$

$$\Rightarrow \dot{v} = C/\cos^2 u \quad \text{for constant } C$$

Now $(++)$ says $\dot{u}^2 = 1 - C^2/\cos^2 u \Rightarrow \dot{u} = \sqrt{\dfrac{\cos^2 u - C^2}{\cos^2 u}}$,

so $\dfrac{\dot{v}}{\dot{u}} = \dfrac{dv}{du} = \dfrac{C}{\cos u \sqrt{\cos^2 u - C^2}} \Rightarrow v = \displaystyle\int \frac{C \sec^2 u}{\sqrt{1 - C^2 \sec^2 u}}\, du.$

$\Rightarrow \quad v = \int \frac{dw}{\sqrt{1-w^2}} = \sin^{-1}(w) + \text{constant} = \sin^{-1}\Big(\underbrace{\frac{C\tan u}{\sqrt{1-C^2}}}_{w}\Big) + \underbrace{\text{const.}}_{\delta}$

So $\sin(v-\delta) = \lambda \tan u$, $\qquad \lambda = \dfrac{c}{\sqrt{1-c^2}}$.

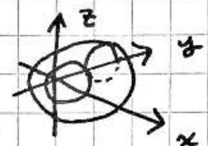
$\Rightarrow \quad \sin v \cos\delta - \cos v \sin\delta - \lambda \tan u = 0$

$\Rightarrow \quad a(\underbrace{\sin v \cos u}_{y}) + b(\underbrace{\cos v \cos u}_{x}) + c\,\underbrace{\sin u}_{z} = 0 \qquad a, b, c$ constants

So γ lies in a plane through O, i.e. γ is an arc of a great circle.

(Any such arc is indeed a geodesic on $S^2 \subseteq \mathbb{R}^3$)

<u>Example</u> the torus of revolution



Rotate $(x-a)^2 + z^2 = 1$ about z-axis.

$\sigma(u,v) = ((a+\cos u)\cos v, (a+\cos u)\sin v, \sin u)$.

FFF is $du^2 + (a+\cos u)^2\, dv^2$

Geodesic eq^{n}s : $\ddot{u} + (a+\cos u)\sin u\, \dot{v}^2 = 0$

$$\ddot{v} - \frac{2\sin u}{a+\cos u}\, \dot{u}\dot{v} = 0$$

Use the same strategy as in previous case.

We find (via arc-length condition) $\dfrac{dv}{du} = \dfrac{C\sqrt{a+\cos u}}{\sqrt{(a+\cos u)^2 - c^2}}$ const. c

which cannot be integrated in terms of elementary functions.

<u>Strategy</u> Use a combination of information on local existence results, & in special cases to use global symmetry. Note if $f : \Sigma \to \Sigma$ is an isometry, then f sends geodesics to geodesics.

Geodesics

A straight line in $\mathbb{R}^2$ is a geodesic.

A straight line is (a) locally shortest, but also

(b) locally straightest, i.e. its tangent vector changes as little as possible (not at all)

We derived the geodesic eq^{n}s in the framework of (a), but there is an analogue of (b).

Proposition A curve $\gamma : [a,b] \to \Sigma$, for $\Sigma \subseteq \mathbb{R}^3$ a smooth surface, & image$(\gamma) \subseteq$ image(σ) for an allowable param^n, is geodesic iff its acceleration $\ddot{\gamma}(t)$ is everywhere normal to Σ, i.e. the derivative of the tangent vectors to γ is normal to Σ.

Proof Let $\gamma(t) = \sigma(u(t), v(t))$ so $t \mapsto (u(t), v(t)) \in V = \text{domain}(\sigma)$.

Then $\dot{\gamma}(t) = \sigma_u \dot{u} + \sigma_v \dot{v}$ & at $p \in \sigma(V) \subseteq \Sigma$, $T_p\Sigma = \langle \sigma_u, \sigma_v \rangle$,

so $\ddot{\gamma}(t)$ is normal to $\Sigma \iff \langle \ddot{\gamma}, \sigma_u \rangle = \langle \ddot{\gamma}, \sigma_v \rangle = 0$,

i.e. $\langle \underbrace{\frac{d}{dt}(\sigma_u \dot{u} + \sigma_v \dot{v}), \sigma_u}_{} \rangle = \langle \dots, \sigma_v \rangle = 0$.

$$\frac{d}{dt}\langle \sigma_u \dot{u} + \sigma_v \dot{v} \rangle, \sigma_u - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \frac{d}{dt}\sigma_u \rangle$$
$$= \frac{d}{dt}(E\dot{u} + F\dot{v}) - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_{uu}\dot{u} + \sigma_{uv}\dot{v} \rangle$$
$$= \frac{d}{dt}(E\dot{u} + F\dot{v}) - \tfrac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$$

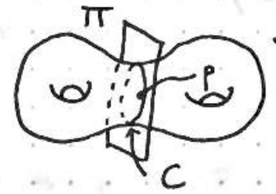
noting $\sigma_u \cdot \sigma_{uu} = \frac{1}{2}E_u$, etc.

The second geodesic equation arises from expanding $\langle \ddot{\gamma}, \sigma_v \rangle = 0$. □

Remarks (i) The proposition says the tangent vector to a geodesic changes only in the direction it must change for the curve to stay on Σ. The geodesic equations are the equations of motion of a "free particle", constrained to stay on Σ but not otherwise. This viewpoint is essential in classical dynamics & general relativity.

<u>Example</u> Let $\Sigma \subseteq \mathbb{R}^3$ be smooth, let $\Pi \cong \mathbb{R}^2 \subseteq \mathbb{R}^3$ be an affine plane, & suppose $\Pi \cap \Sigma = C$ is a smooth curve, & that reflection in Π preserves Σ. Then C is a geodesic on Σ (i.e. when param^d by arc-length, C is a geodesic)

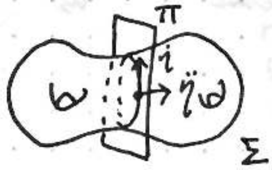
<u>Proof</u>



Let $p \in \Sigma \cap \Pi = C$.

If $\eta(t)$ is a local param^n of C through p, then $\dot{\eta}(t)$, $\ddot{\eta}(t)$ lie in Π.

If we take the arc-length param^n of η, then $\langle \dot{\eta}(t), \ddot{\eta}(t) \rangle = 0$. $(+)$



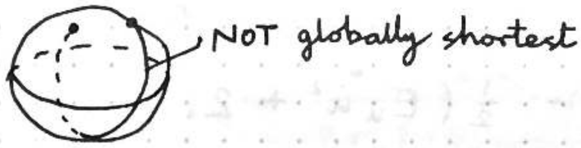
If Refl(Π) preserves Σ & fixes p, then it preserves the decomposition $\mathbb{R}^3 = T_p\Sigma \oplus \mathbb{R}\langle n_p \rangle$.

Since $\Pi \cap \Sigma = C$, Refl(Π) fixes $\dot{\eta}(t)$ & n_p & reverses signs of one possible basis vector for $T_p\Sigma$.

So $\ddot{\eta}(t)$ is fixed by Refl(Π) as η is fixed pointwise, & $(+)$, so $\ddot{\eta}$ is in the direction of n_p, as desired. $\square$

<u>NB</u> So (a) great circles in S^2 are geodesics

(b) geodesics in general are <u>local</u> but not <u>global</u> length minimisers.



Consider a surface of revolution:

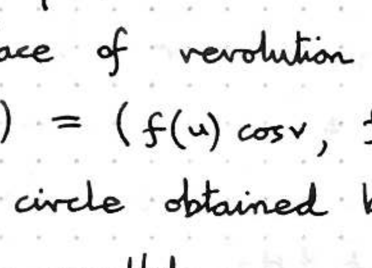
We take $\eta(u) = (f(u), 0, g(u))$, η smooth & embedded, $f > 0$

The surface of revolution has local param^n

$$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u)) \qquad \begin{matrix} a < u < b \\ v \in [0, 2\pi) \end{matrix}$$

<u>Def^n</u> A circle obtained by rotating a fixed point of η (i.e. u fixed) is a <u>parallel</u>.

A curve obtained from η by rotating by a fixed angle is a <u>meridian</u>.



L15.3

Meridians are (components of) intersections $\Sigma \cap \Pi$ for planes Π containing the axis of revolution, which are planes of symmetry, so meridians are geodesics.

Lemma A parallel $u = u_0$ is a geodesic (when param^d by arc length) exactly when $f'(u_0) = 0$.

Proof The FFF of σ is $((f')^2 + (g')^2)\,du^2 + f^2\,dv^2$.

So if η is param^d by arc-length, this is $du^2 + f^2 dv^2$.

If $\gamma(t) = \sigma(u(t), v(t))$, then the geodesic eq^{n}s are

$$\tfrac{d}{dt}(E\dot u + F\dot v) = \tfrac12 (E_u \dot u^2 + 2F_u \dot u \dot v + G_u \dot v^2) \rightarrow \ddot u = f f' \dot v^2$$

$$\tfrac{d}{dt}(F\dot u + G\dot v) = \tfrac12 (E_v \dot u^2 + 2F_v \dot u \dot v + G_v \dot v^2) \searrow \tfrac{d}{dt}(f^2 \dot v) = 0$$

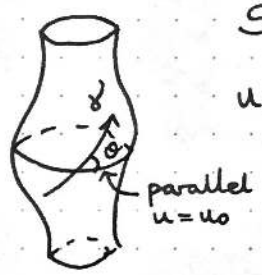
Unit speed / arc-length for the geodesic

$\dot u^2 + f^2 \dot v^2 = 1$.

So if $u = u_0$ is constant, $f^2 \dot v^2 = 1$, so $\dot v = \pm \frac{1}{f(u_0)}$ is const.

So the second equation holds.

The first equation holds $\iff f'(u_0) = 0$, recalling $f > 0$. $\square$



Suppose γ is a geodesic which meets the parallel at u_0 at angle θ.

Then $\cos(\theta) = \dfrac{\langle \sigma_v, \sigma_u \dot u + \sigma_v \dot v\rangle}{\|\sigma_v\| \underbrace{\|\sigma_u \dot u + \sigma_v \dot v\|}_{1 \text{ by arc-length}}}$

where $\gamma(t) = \sigma(u(t), v(t))$.

$F = 0$ in FFF as $\langle \sigma_u, \sigma_v\rangle = 0$, so $\cos(\theta) = \dfrac{\langle \sigma_v, \sigma_v\rangle \dot v}{\|\sigma_v\|} = f\dot v$.

The geodesic equations say $f^2 \dot v$ is constant.

If we write r for the radius of the parallel (i.e. $r = f(u_0)$), we get $\underline{r\cos\theta}$ is constant along γ (Clairaut's relation).

Trap!

e.g. Σ is an ellipsoid. Say γ makes an angle $\theta_0 \neq \frac{\pi}{2}$ with the waist curve, of radius r_0. So $r\cos\theta = r_0 \cos\theta_0$ along γ. So r is bounded below unless $\theta_0 = \pi/2$ i.e. meridian

« fattest waist » So γ is trapped away from the poles,

Existence of geodesics

The geodesic equations

$$\frac{d}{dt}(E\dot{u}+F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2),$$
$$\frac{d}{dt}(F\dot{u}+G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2).$$

These equations imply that $E\ddot{u}+F\ddot{v}$ respectively $F\ddot{u}+G\ddot{v}$ can be expressed as (quadratic) functions of $\dot{u},\dot{v}$ with coefficients smooth functions of u,v, i.e.

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = \left(\text{some quadratic}\right)$$

& since $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is invertible, we can write this system

$$\ddot{u} = A(u,v,\dot{u},\dot{v})$$
$$\ddot{v} = B(u,v,\dot{u},\dot{v}) \qquad \text{for smooth } A, B.$$

So if $p=\dot{u}$ & $q=\dot{v}$, then we get a system of ODEs

$$\dot{u}=p,\quad \dot{v}=q,$$
$$\dot{p}=A(u,v,p,q),\quad \dot{q}=B(u,v,p,q).$$

Recall from IB Analysis & Topology that you studied ODEs of the following general shape: we have $I=[t_0-a,\,t_0+a]\subseteq\mathbb{R}$,

study $\dfrac{dx}{dt}=f(t,x)$, $x(t_0)=x_0$ for $x: I\to\mathbb{R}^n$.

$B=\overline{B}(x_0,\delta)\subseteq\mathbb{R}^n$ closed ball,
$f: I\times B\to\mathbb{R}^n$, f unif Lipschitz
$\|f(t,x_1)-f(t,x_2)\|\leq N\|x_1-x_2\|$,

<u>Picard-Lindelöf theorem</u> This ODE has a unique sol^n for small time, e.g. for $|t-t_0|\leq h$ where $h\leq\min\{a,\ \cancel{b}/\sup\|f\|\}$. δ

One immediately concludes

<u>Corollary</u> If $\Sigma\subseteq\mathbb{R}^3$ is a smooth surface, & if $p\in\Sigma$ & $v\in T_p\Sigma\setminus\{0\}$ then there is a geodesic $\gamma:(-\varepsilon,\varepsilon)\to\Sigma$, $\gamma(0)=p$, $\gamma'(0)=v$.

<u>Remark</u> This is a <u>local</u> existence theorem: the question of when there is a geodesic param^d by arc-length & defined on all of $\mathbb{R}$ is a global, non-trivial question.

General fact: If (X, d) is a complete metric space & $\{f_\mu\}$ is a family of contractions depending on $\mu \in \mathbb{R}^k$ (cts'ly), with uniformly bounded contraction parameter, then the fixed point x_μ of f_μ varies continuously with μ. (Exercise!) If we consider the problems:

Lipschitz implicit function theorem

$$\frac{dx}{dt} = f(t,x) \qquad \text{or} \qquad \frac{dx}{dt} = f(t,x,\mu) \quad , \mu \text{ parameter}$$
$$x(t_0) = a \qquad\qquad\qquad x(t_0) = a$$

Natural to ask: does the local solⁿ vary nicely with a (LHS) or μ (RHS)?

<u>Remark</u> These are basically the same problem.

If x solves LHS & $\tilde{x} = x - a$, $\tilde{f}(t, \tilde{x}, a) = f(t, \tilde{x} + a)$, then $\tilde{x}$ solves $\begin{cases} \dfrac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x}, a) \\ \tilde{x}(t_0) = 0 \end{cases}$ so now a appears as a param.

<u>Theorem</u> Let $\underset{(t,\, x,\, \mu)}{f : I \times B \times \mathbb{R}^k} \to \mathbb{R}^n$ which is continuously diff (or smooth) with respect to x, μ. Then the solⁿ $x(\cdot, \mu)$ of

$$\frac{dx}{dt} = f(t,x,\mu), \quad x(t_0) = x_0$$

depends continuously differentiably (smoothly) on μ.

<u>Corollary</u> The unique geodesic on Σ through p and in dirⁿ $v \in T_p\Sigma \setminus \{0\}$ depends smoothly on p, v.

<u>AIM</u> outline the proof of the theorem in the special case $n = k = 1$, i.e.

$$\underset{(t,\, x,\, \mu)}{f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}} \to \mathbb{R} \quad \text{which is continuously diff in } x, \mu$$

Want to show our solution $x(t)$ to the ODE is continuously diff in μ

<u>Key idea</u> One would hope $y = \dfrac{\partial x}{\partial \mu}$ satisfies

$$\frac{dy}{dt} = f_x(t, x(t,\mu), \mu)\, y + f_\mu(t, x(t,\mu), \mu), \quad y(t_0) = 0 \qquad (**)$$

The RHS of $(**)$ is continuous, locally Lipschitz wrt y, & so $\exists$ some solution $y(t, \mu)$ by the Picard–Lindelöf theorem.

L16.3

Let $\omega(t,\mu,\delta) = \dfrac{x(t,\mu+\delta) - x(t,\mu)}{\delta}$

& try to show $\omega(t,\mu,\delta) \xrightarrow{\delta\to 0} y(t,\mu)$.

Let $z(t,\mu,\delta) = \omega(t,\mu,\delta) - y(t,\mu)$. Consider

$$\frac{dz}{dt}(t,\mu,\delta) = \frac{d\omega}{dt}(t,\mu,\delta) - f_x(t,x,\mu)\,y(t,\mu) - f_\mu(t, x(t,\mu),\mu)$$

& $\frac{d\omega}{dt}(t,\mu,\delta) = \big(f(t,x(t,\mu+\delta),\mu+\delta) - f(t,x(t,\mu),\mu)\big)/\delta$

$$= \frac{f(t,x(t,\mu+\delta),\mu+\delta) - f(t,x(t,\mu),\mu+\delta)}{\delta} + \frac{f(t,x(t,\mu),\mu+\delta) - f(t,x(t,\mu),\mu)}{\delta}$$

$$\underset{MVT}{=} f_x(t, x(t,\mu)+\theta_1\Delta, \mu+\delta)\,w(t,\mu,\delta) + f_\mu(t, x(t,\mu), \mu+\theta_2\delta)$$

where $\theta_i \in [0,1]$ & $\Delta = x(t,\mu+\delta) - x(t,\mu)$

$$\therefore \left|\frac{dz}{dt}(t,\mu,\delta)\right| \le \left|f_\mu(t,x(t,\mu),\mu+\theta_2\delta) - f_\mu(t,x(t,\mu),\mu)\right|$$
$$+ |f_x(t,x+\theta_1\Delta,\mu+\delta)|\,|z(t,\mu,\delta)|$$
$$+ |f_x(t,x+\theta_1\Delta,\mu+\delta) - f_x(t,x,\mu)|\,|y(t,\mu)|$$
$$+ |f_x(t,x,\mu+\delta) - f_x(t,x,\mu)|\,|y(t,\mu)|$$

Therefore, $\left|\frac{dz}{dt}(t,\mu,\delta)\right| \le p(t,\mu,\delta) + \big(|f_x(t,x,\mu)| + p(t,\mu,\delta)\big)\,|z(t,\mu,\delta)|$

where $p \to 0$ as $\delta \to 0$, uniformly on bounded sets.

Recall the classical Gronwall inequality, which says

if $X : [t_0, T] \to \mathbb{R}$ is continuous & $X(t) \le C + \kappa\int_{t_0}^{t} X(s)\,ds$

for some $C, \kappa > 0$, then $X(t) \le Ce^{\kappa(t-t_0)}\ \forall t$.

[Proof: $F'(t) = \kappa X(t) \le \kappa F(t)$ so $F'/F \le \kappa$]

For us, let $X(t) = \varepsilon + (K+\varepsilon)|z|$, choosing K s.t.

$|f_x(t, x(t,\mu),\mu)| \le K$ & $|p(t,\mu,\delta)| < \varepsilon$.

Gronwall $\Rightarrow$ $|z| \le \varepsilon\big(\exp[(K+\varepsilon)(t-t_0)] - 1\big)/(K+\varepsilon)$.

At time t, having picked K as above, then I can pick ε as small as I like (as $\delta \to 0$ & $p \to 0$) to bound (†)

So as $\delta \to 0$, $z \to 0$ as req^d $\square$

L17.1 <u>Model Spaces</u>

Recall: we saw that for an ODE $\begin{cases} \dfrac{dx}{dt} = f(t,x,\underbrace{\mu}_{\text{parameter}}) \\ x(t_0) = x_0, \end{cases}$

if f is continuously diff in x,μ, then the solution $x(\cdot,\mu)$ is also diff in μ.

We saw that, for $y = \dfrac{dx}{d\mu}$, y solved $\begin{cases} \dfrac{dy}{dt} = f_x y + f_\mu \\ y(t_0) = 0, \end{cases}$

so if f is twice continuously diff in x,μ then y solves an ODE of the same shape with diff RHS, so x is twice diff in μ.

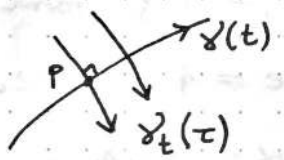
So if f is smooth in x,μ then the sol^n $x(t,\mu)$ is smooth in μ.
"jointly" smooth in fact

Let $\Sigma \subseteq \mathbb{R}^3$ be a smooth surface, & $p \in \Sigma$.

We fix a small geodesic arc $\gamma(t)$ centred at p & param^d by arc length, & for small $|t|$, construct a geodesic γ_t s.t.

(i) $\gamma_t(0) = \gamma(t)$

(ii) $\gamma_t'(0)$ is orthogonal to $\gamma'(t)$



again param^d by arc length.

Define $\sigma(u,v) = \gamma_v(u)$ for suff small $|u|,|v|$.

<u>Lemma</u> This defines an allowable param^n of Σ near p

<u>Proof</u> The smooth dependence of geodesics on starting point shows that σ is a smooth param^n. At $(0,0)$, by def^n, σ_u and σ_v are orthogonal, so stay lin indep in some ngbd of $(0,0)$. $\square$

<u>Remark</u> These are called <u>geodesic normal</u> coordinates near p.

<u>Lemma</u> In geodesic normal coordinates, the FFF of Σ has the form $du^2 + G(u,v)\,dv^2$, i.e. $E \equiv 1$, $F \equiv 0$.

<u>Proof</u> The curves $v = \text{const.}$, $u \mapsto \gamma_v(u)$ are by def^n geodesics param^d by arc length. That exactly says $E \equiv 1$.

The geodesic equation

$$\frac{d}{dt}\left(F\dot{u} + G\dot{v}\right) = \frac{1}{2}\left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right)$$

is solved by $u=t$, $v=\text{const.}$ (and $E=1$) so $\dfrac{dF}{dt} = F_u = 0$.

(Here we used $u=t$ & $\dot{v}=0$)

So F is independent of u, but when $u=0$, then γ_v meets γ orthogonally, so $F \equiv 0$. $\square$

<u>Proposition</u> If $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface with an allowable param^n σ for which $E=1$ & $F=0$, then $\kappa = \dfrac{-\sqrt{G}_{uu}}{\sqrt{G}}$ where κ is the Gauss curvature.

<u>Remark</u> This gives the "theorema egregium", i.e. if $\Sigma \subseteq \mathbb{R}^3$ and $\Sigma' \subseteq \mathbb{R}^3$ are smooth, locally isometric near p, p' then they have the same Gauss curvature functions.

<u>Remark</u> There are related but more conceptual proofs.

<u>Proof of Proposition</u>

At $p = \sigma(0) \in \Sigma$, let $e = \sigma_u$, $f = \sigma_v/\sqrt{G}$, $n = $ +ve unit normal, so that $\{e, f, n\}$ form an orthonormal basis of $\mathbb{R}^3$.

Since $e \cdot e = 1$, $e \cdot e_u = e \cdot e_v = 0$, we'll write

$$e_u = \alpha f + \lambda_1 n, \quad e_v = \beta f + \mu_1 n \qquad \alpha, \beta, \alpha', \beta'$$
$$f_u = -\alpha' e + \lambda_2 n, \quad f_v = -\beta' e + \mu_2 n \qquad \lambda_i, \mu_i \text{ scalars}$$

Also $e \cdot f = 0$ (as $F \equiv 0$) so $e_u \cdot f + e \cdot f_u = e_v \cdot f + e \cdot f_v = 0$.

Hence $\alpha = \alpha'$, $\beta = \beta'$.

$$\alpha = e_u \cdot f = \sigma_{uu} \cdot \sigma_v / \sqrt{G} = \underbrace{\frac{(\sigma_u \cdot \sigma_v)_u}{\sqrt{G}}}_{\substack{\text{zero since} \\ F \equiv 0}} - \underbrace{\frac{(\sigma_u \cdot \sigma_u)_v}{2\sqrt{G}}}_{\substack{\text{zero since} \\ E \equiv 1}} = 0$$

$$\beta = e_v \cdot f = \sigma_{uv} \cdot \sigma_v / \sqrt{G} = \frac{\frac{1}{2} G_u}{\sqrt{G}} = (\sqrt{G})_u$$

Next, $\lambda_1 \mu_2 - \lambda_2 \mu_1 = e_u \cdot f_v - f_u \cdot e_v = (e \cdot f_v)_u - \underbrace{(e \cdot f_u)_v}_{\text{zero}}$

$$= -\beta_u = -(\sqrt{G})_{uu}.$$

Finally, recall from our discussion of the Gauss map,

$$n_u \times n_v = \kappa\, \sigma_u \times \sigma_v.$$

L17.3

Here $n = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \frac{\sigma_u \times \sigma_v}{\sqrt{G}} = e \times f$

$$\Rightarrow K\sqrt{G} = (n_u \times n_v) \cdot n$$
$$= (n_u \times n_v) \cdot (e \times f)$$
$$= (n_u \cdot e)(n_v \cdot f) - (n_u \cdot f)(n_v \cdot e)$$
$$= (n \cdot e_u)(n \cdot f_v) - (n \cdot f_u)(n \cdot e_v)$$
$$= \lambda_1 \mu_2 - \mu_1 \lambda_2 = (-\sqrt{G})_{uu}. \qquad \square$$

Rmk Even Gauss thought that was pretty cool.

Theorem Let Σ be a smooth surface with constant curvature.
Then (i) if $\kappa = 0$ then Σ is locally isometric to a plane
(ii) if $\kappa = +1$ then Σ is locally isometric to the round sphere $(S^2, du^2 + \cos^2 u\, dv^2)$ in $\mathbb{R}^3$
(iii) if $\kappa = -1$ then Σ is locally isometric to $\left(h, \frac{dx^2 + dy^2}{y^2}\right)$ where h is the upper half-plane

Proof We take a param^n of Σ such that FFF is $du^2 + G\,dv^2$, coming from geodesic normal form.

If $\kappa \equiv 0$, then $(\sqrt{G})_{uu} = 0$, so $\sqrt{G} = A(v)\,u + B(v)$.

Claim: we also have $G(0,v) = 1$, $G_u(0,v) = 0$.

Recall, σ_v has unit length at $u=0$ by arc-length condition.

Also $\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$

Applying this to $u=0$ shows $0 = \frac{1}{2} G_u(0,v)$.

So now $B = 1$, $A = 0$ and FFF is $du^2 + dv^2$.

If $\kappa \equiv 1$, then $(\sqrt{G})_{uu} + \sqrt{G} = 0$; solved by
$$\sqrt{G} = A(v)\sin u + B(v)\cos u.$$

The same boundary conditions show $A(v) \equiv 0$, $B(v) \equiv 1$ and so the FFF is $du^2 + \cos^2 u\, dv^2$.

If we parametrise S^2 via $\sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$ we get the same FFF.

L17.4

If $\kappa = -1$, we get $\sqrt{G}_{uu} = \sqrt{G}$ & the resulting FFF comes out as $du^2 + \cosh^2(u)\, dv^2$.

If we set $x = e^v \tanh u$, $y = e^v \operatorname{sech} u$, then $(u,v) \mapsto (x,y)$ is a locally invertible coordinate change which transforms the 3rd model into $\frac{dx^2 + dy^2}{y^2}$ on $\mathcal{H}$. $\square$

Example for $\kappa = -1$ is less familiar & will be revisited at length!

Abstract Riemannian metrics

Recall For $\Sigma \subseteq \mathbb{R}^3$ a smooth surface, & $\sigma : V \to U \subseteq \Sigma$ allowable param^n, we saw that (V in $\mathbb{R}^2$ open)

$$E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad E = \langle \sigma_u, \sigma_u \rangle, \quad F = \langle \sigma_u, \sigma_v \rangle,$$
$$G = \langle \sigma_v, \sigma_v \rangle$$

enabled us to compute lengths, areas, angles of curves, regions, vectors in $\sigma(V) \subseteq \Sigma$.

Def^n An abstract Riemannian metric on a disc D is a triple of smooth functions E, F, G on D, valued in $\mathbb{R}$, s.t.

$$E > 0, \quad G > 0, \quad EG - F^2 > 0 \quad \text{everywhere,}$$

i.e. we have a smooth map $z \mapsto \begin{pmatrix} E(z) & F(z) \\ F(z) & G(z) \end{pmatrix}$ valued in non-degenerate symmetric bilinear forms.

We write g (or ds^2) for $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Given $\gamma(t) = (u(t), v(t))$ a smooth map $I \to D$, (I in $\mathbb{R}$)
we define

$$\text{Length}(\gamma) = \int_I (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\,dt$$

Def^n Let Σ be an abstract smooth surface, i.e. Σ has an atlas of charts $\{(U_i, \varphi_i)\}_{i \in I}$, $U_i \underset{\text{open}}{\subseteq} \Sigma$, $\bigcup_{i \in I} U_i = \Sigma$ for which the transition maps are smooth diffeo^{m}s of open subsets of $\mathbb{R}^2$.

We define an abstract Riemannian metric on Σ to be

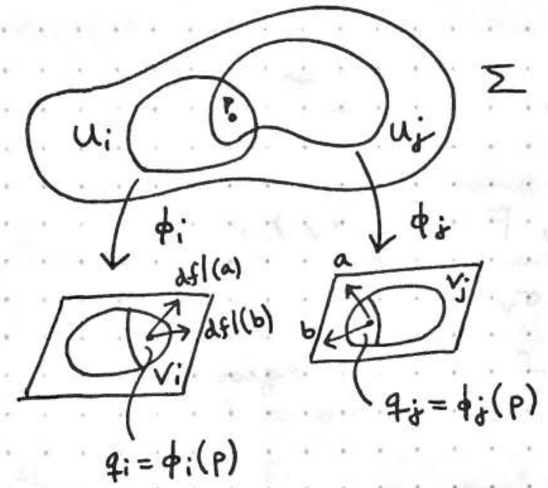
(i) abstract metrics on the open subsets $\varphi_i(U_i) = V_i \subseteq \mathbb{R}^2$
(the V_i needn't be discs, but we could make them so)

(ii) such that if $ds^2_{V_i} = \begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix}$ are two local metrics,
$ds^2_{V_j} = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}$

& we have the transition map $f : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$, then we have

$$(Df)^T \begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix} (Df) = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}.$$

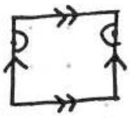
L18.2



If a, b are vectors at q_j,

$$\langle df|_{q_j}(a), df|_{q_j}(b) \rangle_{V_i, q_i}$$
$$= \langle a, b \rangle_{V_j, q_j},$$

i.e. $df|_{q_j}$ is an isometry from $\left(\mathbb{R}^2, \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}\right)$ to $\left(\mathbb{R}^2, \begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix}\right)$

Example  $T^2 = [0,1]^2/\sim$

Recall this had an atlas s.t. the transition functions were locally translations of (open subsets of) $\mathbb{R}^2$.

Consider the usual Euclidean metric $du^2 + dv^2$ on $\mathbb{R}^2$.

Since locally $df = Id$ & $\begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix} = Id$ (for any i), this defines a global abstract metric on the torus T^2.

Note This metric ds^2_{Eucl} is locally isometric to $(\mathbb{R}^2, du^2 + dv^2)$.

Note We can define geodesics for an abstract Riemannian metric as solutions to the geodesic equations; construct geodesic local coordinates, and define the Gauss curvature of an abstract metric via $\kappa = \frac{-\sqrt{G}_{uu}}{\sqrt{G}}$ for $du^2 + G\,dv^2$ the metric in geodesic coordinates.

Either this way, or by saying "whatever κ is, it's isometry invariant", we see that κ for (T^2, ds^2_{Eucl}) vanishes. So this flat abstract metric doesn't arise from any embedding of $T^2 \hookrightarrow \mathbb{R}^3$ (compact embedded surfaces have elliptic points).

For $p, q \in (\Sigma, ds^2)$, & if $\gamma : [0,1] \to \Sigma$ has $\gamma(0) = p$, $\gamma(1) = q$ is a piecewise smooth path from p to q, then I can define its length by adding up expressions $\int_{t_0}^{t_1} (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2} dt$ where $0 = t_0 \le t_1 \le \cdots \le t_N = 1$ is a dissection of $[0,1]$ with

$\gamma([t_i, t_{i+1}]) \subset U_\alpha$ for some U_α in the atlas of Σ.

Consistency of the abstract metrics on the V_i makes this independent of our choice of dissection.

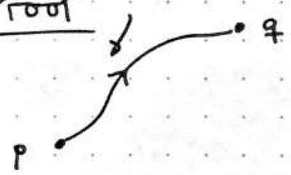
<u>Lemma</u> If Σ is a connected smooth surface with an abstract Riemannian metric, & if I let

$$d(p,q) = \inf\{\text{Length}(\gamma) \mid \gamma \text{ piecewise smooth path from } p \text{ to } q\}$$

then this defines a metric on Σ in the sense of metric spaces.

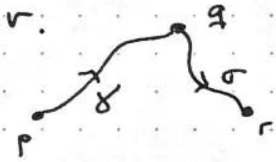
(The metric topology <u>agrees</u> with the existing topology of Σ)

<u>Proof</u>



$\hat{\gamma}(t) = \gamma(1-t)$ is a smooth path from q to p if γ is a smooth path from p to q.

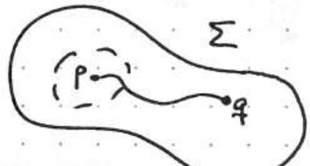
Similarly, if γ, σ are paths from p to q, q to r, then their concatenation is a piecewise smooth path from p to r.



$\therefore\ d(p,q) = d(q,p)$, $d(p,r) \leq d(p,q) + d(q,r)$

Key: $d(p,q) > 0$ if $p \neq q$

Pick a chart at p. Any path from p to q must escape the open ball $B_\varepsilon(p)$ for some ε suff small s.t. $q \notin B_p(\varepsilon)$, by Hausdorff. *



STP that for our unknown metric Riemannian metric, the distance from p to $\partial B_p(\varepsilon)$ is strictly positive.

On the compact set $\overline{B_p(\varepsilon/2)}$, the pointwise non-degenerate metric $\begin{pmatrix} E(z) & F(z) \\ F(z) & G(z) \end{pmatrix}$ & the Euclidean inner product $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are unif. related, such that $\exists\ \lambda > 0$ s.t.

$\begin{pmatrix} E(z)-\lambda & F(z) \\ F(z) & G(z)-\lambda \end{pmatrix}$ would still be a metric on the disc.

This is true since $E, G, EG - F^2$ bdd below.

So for a path inside $\overline{B_p(\varepsilon/2)}$, $\text{length}_{ds^2}(\gamma) \geq \lambda\, \text{length}_{\text{Eucl}}(\gamma)$.

This proves the claim, so indeed $d(p,q) > 0$. $\square$

The same argument shows the metric topology on Σ is the given Locally Euclidean one.

L18.4

Natural question When is the metric space (Σ, d from ds^2) complete?

E.g. If $\Sigma \subseteq \mathbb{R}^2$ smooth we know that if Σ is closed as a subspace then it's complete as a metric space.

E.g. The flat torus, or the torus of revolution is complete.

Fact Σ is complete when our local geodesics are defined on all of $\mathbb{R}$. No complete surface in $\mathbb{R}^3$ has $K \equiv -1$.

L19.1 Möbius maps

Recall Möb $= GL(2,\mathbb{C}) / \text{Scalars} = \mathbb{P}SL_2\mathbb{C}$

$= \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}) \right\}$

acting on $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}} = \mathbb{C}_\infty$.

Recall: for $\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\alpha/\infty = 0$, $\frac{\infty}{\infty} = 1$, $\frac{\infty}{\alpha} = \infty$

Properties / features of the Möbius group

(i) Möb acts triply transitively: given distinct α, β, γ & u, v, w $\exists!$ Möbius map taking $\alpha \mapsto u$, $\beta \mapsto v$, $\gamma \mapsto w$.

(ii) So if $(a,b,c,d) \in \mathbb{C}_\infty$ are distinct, $\exists! \, \tau \in$ Möb s.t.

$\tau(a) = 0$, $\tau(b) = 1$, $\tau(c) = \infty$, $\tau(d) := [a:b:c:d]$

is the cross-ratio of the four points. Explicitly $T(z) = \frac{z-a}{z-c} \cdot \frac{b-c}{b-a}$.

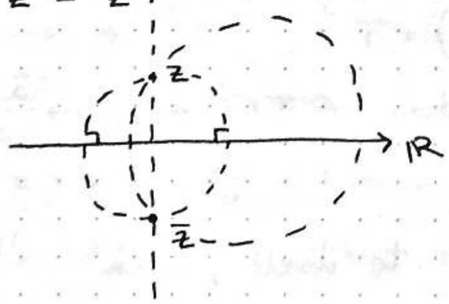
(iii) Möb is generated by $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, $z \mapsto z + a$, $a \in \mathbb{C}$, $z \mapsto \frac{1}{z}$.

(iv) If $T \in$ Möb, then T preserves the set {circles, straight lines} in $\mathbb{C}$.

Let $\Gamma \subseteq \mathbb{C}$ be a circle or line. We say $z, z' \in \mathbb{C} \cup \{\infty\}$ are inverse points for Γ if every circle which contains z and is orthogonal to Γ also contains z'. If $z \in \Gamma$, then z is self-inverse by definition.

Example If $\Gamma = \mathbb{R} \cup \{\infty\}$, then z and z' are inverse for Γ iff $z' = \bar{z}$.

(Since circles are orthogonal to $\mathbb{R}$ iff their centre lies on $\mathbb{R}$)



Lemma If $T \in$ Möb & $\Gamma \subseteq \mathbb{C}_\infty$ is a circle, with z, z' inverse for Γ, then Tz and Tz' are inverse points for $T(\Gamma)$.

Proof Möbius maps send circles to circles and preserve angles. ($z \mapsto \lambda z$ & $z \mapsto z + a$ obviously preserve angles)

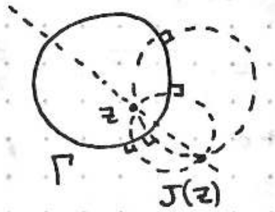
Need only check for $\frac{1}{z}$. This is an easy exercise, cf the idea of a conformal map. ⌟

Corollary For any circle Γ & point z, there is a unique point which is inverse to z for Γ.

Proof If $\Gamma = \mathbb{R}$, we know this : we set $C(z) = \bar{z}$ (C for conjugation & that works).

For any Γ, $\exists\, T \in$ Möb s.t. $T(\mathbb{R}) = \Gamma$ (since a circle is determined by 3 points)

Define $J(z) = T \circ C \circ T^{-1}(z)$, this works by the previous Lemma. □



Def^n The map $z \mapsto J(z)$ is called inversion in Γ. Clearly, $J^2 = id$, $\mathrm{Fix}(J) = \Gamma$.

Example Inversion in S^1 is the map $z \mapsto 1/\bar{z}$.

(E.g. note $z \mapsto -i\left(\frac{z+i}{z-i}\right)$ sends $S^1 \to \mathbb{R}_\infty$)

Proposition A composition of two inversions is a Möbius maps.

Proof Take circles Γ_1 & Γ_2 with inversions J_{Γ_1} & J_{Γ_2}.

Note, if C is $z \mapsto \bar{z}$ as above, that

$$J_{\Gamma_1} \circ J_{\Gamma_2} = (J_{\Gamma_1} \circ C) \circ (C \circ J_{\Gamma_2})$$
$$= (C \circ J_{\Gamma_1})^{-1} \circ (C \circ J_{\Gamma_2}).$$

STP $C \circ J_\Gamma$ is a Möbius map.

Moreover, if $T \in$ Möb sends $\mathbb{R}$ to Γ, then $J_\Gamma = T \circ C \circ T^{-1}$, so $C \circ J_\Gamma = C \circ T \circ C \circ T^{-1} = (C \circ T \circ C) \circ T^{-1}$!

If T is represented by $z \mapsto \frac{az+b}{cz+d}$, then CTC is $z \mapsto \frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}$ which lies in Möb. □

Proposition If $T \in$ Möb maps $D = \{|z|<1\}$ to itself, then

$$Tz = \frac{az+b}{\bar{b}z+\bar{a}} \quad \text{with } |a|^2+|b|^2 = 1.$$

Call subgroup of such maps Möb$(D) = \mathbb{P}SU(1,1)$.

Proof If T sends D to D it sends S^1 to S^1 & hence inverse points for S^1 to inverse points. So if $Jz = 1/\bar{z}$, then

- $J \circ T \circ J = T$

If represent T by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc=1$, the LHS is

$$z \longmapsto \frac{\bar{d}z+\bar{c}}{\bar{b}z+\bar{a}} \quad \text{so} \quad \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If the sign was negative, we get $T = \begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix}$ with $-|a|^2+|b|^2 = 1$, & $T(0) = -b/\bar{a} \notin D$.

So have a $+$ sign, then get the map we want. $\square$

Proposition If $H \in \mathbb{C}$ is the upper half-plane $\{z \mid \mathrm{Im}(z) > 0\}$

- $$\text{Möb}(h) = \{T \in \text{Möb} \mid Th = h\} = PSL_2(\mathbb{R})$$ not quite! need +ve det
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Möb} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \right\}.$$

Proof As before, replacing $Jz = 1/\bar{z}$ with $Cz = \bar{z}$. $\square$

Proposition There is a unique (up to scale) metric in the sense of metric spaces $d : D \times D \to \mathbb{R}_{\geq 0}$, $D = \{|z| < 1\}$ which is invariant under the Möbius group $\text{Möb}(D)$, i.e. $\forall\, T \in \text{Möb}(D)$,

$$d(Tz, Tw) = d(z, w), \quad z, w \in D$$

and which makes the real axis $\mathbb{R} \cap D$ a length minimising path.

- The metric d is induced from the abstract Riemannian metric with 1^{st} FF $\quad 4\left(\frac{du^2+dv^2}{(1-u^2-v^2)^2}\right)$.

This is the hyperbolic metric on D.

Proof If d exists, since $\text{Möb}(D)$ acts transitively, if $a \in D$,

$$z \mapsto \frac{z-a}{1-\bar{a}z} \quad \text{sends } a \to 0.$$

It's determined by $d(0,a)$ for $a \in D$.

And $z \to e^{i\theta}z$ is in $\text{Möb}(D)$, so enough to know $d(0,a)$ for $a \in D \cap \mathbb{R}_+$. Call this $p(a) := d(0,a)$, $a \in \mathbb{R}_+ \cap D$.

- If $0 < a < b < 1$ in $\mathbb{R}_+ \cap D$, then $d(0,b) = d(0,a) + d(a,b)$ if $\mathbb{R} \cap D$ is a geodesic.

$$\therefore\ p(b) = p(a) + p\left(\frac{b-a}{1-ab}\right)$$

L19.4

Differentiate wrt b & set $b=a$, to get

$$p'(a) = \frac{p'(0)}{1-a^2}.$$ By convention, we set $p'(0) = 2$. Then $p(a) = 2\tanh^{-1}(a)$, &

$$d(a,b) = 2\tanh^{-1}\left(\left|\frac{b-a}{1-\bar{a}b}\right|\right).$$

(To be continued!)

L 20.1 <u>The hyperbolic plane I</u>

<u>Recall</u> We showed that if there is a metric $d : D \times D \to \mathbb{R}_{\geq 0}$, $D = \{ z \in \mathbb{C} : |z| < 1 \}$, which is invariant under Möb(D),

then $d(p,q) = 2 \tanh^{-1}\left(\left| \frac{p-q}{1-\bar{p}q} \right| \right)$. + diameter geodesic

(Since there is a unique $T \in$ Möb(D) sending an ordered pair (p,q) to $(0, a \in \mathbb{R}_+ \cap D)$ this expression <u>is</u> Möb-invariant)

<u>Lemma</u> d satisfies the triangle inequality

<u>Proof</u> By Möb(D)-invariance, STP $\underbrace{d(a,b)}_{\gamma} \leq \underbrace{d(a,0)}_{\alpha} + \underbrace{d(0,b)}_{\beta}$.

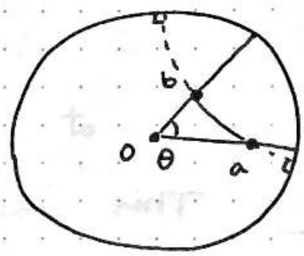
<u>CLAIM</u> $\cosh(\gamma) = \cosh(\alpha)\cosh(\beta) - \sinh(\alpha)\sinh(\beta)\cos(\theta)$ for $\theta = \arg(b/a)$

(the "cosine rule" for hyperbolic triangles)

CLAIM $\Rightarrow$ Δ-inequality for d

$\cos(\theta) \geq -1$, so $\cosh(\gamma) \leq \cosh(\alpha+\beta)$

On $\mathbb{R}_+$ cosh is increasing, so $\gamma \leq \alpha + \beta$.



In fact, therefore we see

$d(a,b) \leq d(a,0) + d(0,b)$ & we only get equality if a, b lie on the same line through 0 (i.e. $\theta = \pi$)

<u>Proof of CLAIM</u>

We'll treat the case $a \in \mathbb{R}_+ \cap D$, & b on the diameter at angle θ as shown. ~~Let~~ Note ~~x~~ $a = \tanh\left(\frac{\alpha}{2}\right)$, $b = e^{i\theta}\tanh(\beta/2)$

Use that if $t = \tanh\left(\frac{\lambda}{2}\right)$ then $\cosh\lambda = \frac{1+t^2}{1-t^2}$, $\sinh\lambda = \frac{2t}{1-t^2}$.

Thus $\cosh(\alpha) = \frac{1+a^2}{1-a^2}$, $\cosh(\beta) = \frac{1+|b|^2}{1-|b|^2}$, $\tanh(\gamma/2) = \left| \frac{b-a}{1-\bar{a}b} \right|$.

Therefore $\cosh(\gamma) = \frac{|1-\bar{a}b|^2 + |b-a|^2}{|1-\bar{a}b|^2 - |b-a|^2} \underset{(ex)}{=} \frac{(1+|a|^2)(1+|b|^2) - 2(\bar{a}b + a\bar{b})}{(1-|a|^2)(1-|b|^2)}$

& $a \in \mathbb{R}$, $b + \bar{b} = 2\mathrm{Re}(b) = 2|b|\cos\theta$

$\therefore \cosh(\gamma) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta\cos\theta$. $\square$

<u>Remark</u> If α, β, γ are small, $\sinh\alpha \approx \alpha$, $\cosh\alpha \approx 1 + \frac{\alpha^2}{2}$.

Then hyp cos formula becomes $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta$ cf Euclid

L 20.2

<u>2nd approach</u> If $z, z+\delta z$ are close in D, then

$$d(z, z+\delta z) \approx 2\tanh^{-1}\left(\frac{|\delta z|}{1-|z|^2}\right) \approx \frac{2|\delta z|}{1-|z|^2}$$

<u>So</u> introduce the abstract Riemannian metric

$$\frac{4|dz|^2}{(1-|z|^2)^2} = \frac{4(du^2+dv^2)}{(1-u^2-v^2)^2} = g_{hyp} = ds^2_{hyp}$$

If $\gamma:[0,1] \to D$ has $\gamma(0)=0$, $\gamma(1)=a \in \mathbb{R}_+ \cap D$, γ smooth,

$$L(\gamma) = \int_0^1 \frac{2|\dot\gamma(t)|}{1-|\gamma(t)|^2}\,dt = \int_0^1 \frac{2(\dot u^2+\dot v^2)^{1/2}}{1-u^2-v^2}\,dt$$

$$\geqslant \int_0^1 \frac{2(\dot u^2)^{1/2}}{1-u^2}\,dt$$ attained for some γ, noting $v(0)=v(1)=0$

$$\geqslant \int_0^1 \frac{2\dot u}{1-u^2}\,dt = 2\tanh^{-1}(u(1)) = 2\tanh^{-1}(a) = d(0,a)$$

<u>CLAIM</u> This abstract Riemannian metric is invariant under Möb(D)

Given this, we see d is just the metric associated to g_{hyp} (i.e. $d(a,b) = \inf_\gamma L(\gamma)$), & a posteriori d satisfies Δ.

<u>Proof of CLAIM</u>

Möb(D) is generated by $z \mapsto e^{i\theta}z$, $z \mapsto \frac{z-a}{1-\bar a z}$, $|a|<1$

Clearly, $z \mapsto e^{i\theta}z$ preserves g_{hyp}.

If $w = \frac{z-a}{1-\bar a z}$, $dw = \frac{dz}{1-\bar a z} + \frac{(z-a)}{(1-\bar a z)^2}\bar a\,dz = \frac{dz(1-|a|^2)}{(1-\bar a z)^2}$

so $$\frac{|dw|}{1-|w|^2} = \frac{|dz|(1-|a|^2)}{|1-\bar a z|^2\left(1-\left|\frac{z-a}{1-\bar a z}\right|^2\right)} = \frac{|dz|(1-|a|^2)}{|1-\bar a z|^2-|z-a|^2} = \frac{|dz|}{1-|z|^2} \quad \square$$

<u>Def^n</u> The <u>hyperbolic disc</u> (or disc model of the hyperbolic plane) is (D^2, g_{hyp}, d) LMAO

<u>Note</u> We could also derive a model for the hyperbolic plane by fixing any disc in $\mathbb{C}_\infty$ & considering a metric invt under the corresponding gp of Möbius maps.

Taking $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ we get the <u>upper half-plane model</u> for the hyperbolic plane.

L 20.3

<u>Def^n</u> The <u>hyperbolic upper half-plane</u> is h with the abstract Riemannian metric $\frac{dx^2+dy^2}{y^2}$

<u>Remark</u> To check this is the right thing,

(i) There are inverse iso^{m}s $D \to h$, inverse $w \mapsto \frac{w-i}{w+i}$

$$z \mapsto i\left(\frac{1-z}{1+z}\right)$$

Check these relate $\frac{|dz|^2}{(1-|z|^2)^2}$ on D and $\frac{|dw|^2}{|\mathrm{Im}\, w|^2}$ on h.

(ii) Alternatively, recall $\text{Möb}(h) = \mathbb{P}SL_2\mathbb{R}$

$$= \left\{ T \in \text{Möb} \;\middle|\; T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R} \right\}$$

This is generated by $w \mapsto w + \underset{\text{real}}{c}$, $w \mapsto \underset{>0}{a}w$, $w \mapsto -\frac{1}{w}$

Now check the third preserves the given FFF / abstract metric.

This shows we have the right metric on h up to scale.

Now check for one pair of points.

<u>Remark</u> When we discussed model spaces, we wrote down an explicit coordinate change which was a local isometry from (h, g_{hyp}) & $du^2 + \cosh^2 u\, dv^2$.

So indeed, hyperbolic space has Gauss curvature -1.

<u>Properties of hyperbolic space</u>

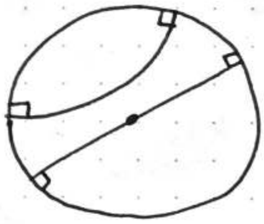
(i) For any $a, b \in \mathbb{H}^2$ (= hyperbolic plane)

$\exists!$ a geodesic between a, b

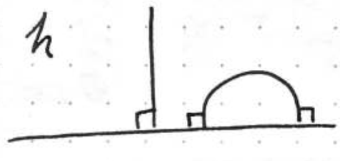
In D, geodesics are diameters and circles orthogonal to $\partial D = S^1$.

In h, " vertical half-lines and " $\partial h = \mathbb{R}$.

D



h



(ii) The isometry group of $\mathbb{H}^2$ is generated by Möb$(\mathbb{H})$ <u>and</u> inversion in geodesics (in this context <u>reflections</u>)

Indeed, $\frac{|dz|^2}{(1-|z|^2)^2}$ inv't under conjugation, while $\frac{dx^2+dy^2}{y^2}$ is invariant under reflection in $i\mathbb{R}_+$.

L20.4

and $\text{Möb}(\mathbb{H})$ is transitive on geodesics.

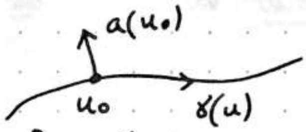
[Check: why is this now <u>all</u> isometries?]

Digression Recall (L11)

non-conical!

Prop^n A non-cylindrical ruled surface $\Sigma \subset \mathbb{R}^3$ is flat $\Leftrightarrow$ it is developable

Proof $\sigma(u,v) = \gamma(u) + v\,a(u)$



Special situation : $a'(u) = 0$ & all lines are parallel "gen cylinder"

or $\gamma'(u) = 0$  then the surface locally lies on a cone, "gen cone"

In proof, we studied $\tilde{\gamma}(t) = \gamma(t) + \underline{\alpha(t)}\, a(t) \in \Sigma$ & under the hypothesis $a'(t) \neq 0$ we solve for

$$\tilde{\gamma}'(t) = \underline{\beta(t)}\, a(t) \qquad (\text{Solve for } \alpha, \beta)$$

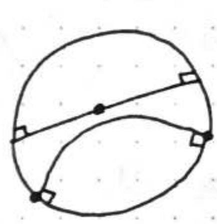
Showed: if $a'(t) \neq 0$, $\exists$ sol^n for α, and determines β

But could have $\tilde{\gamma}(t) = \text{const}$. So $\tilde{\gamma}'(t)$ is parallel to $a(t)$!

The case is the cone, which is not developable!

The hyperbolic plane II

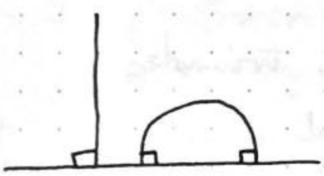
Recall The disc model



$$d_{hyp}(a,b) = 2\tanh^{-1}\left(\left| \frac{b-a}{1-\bar{a}b} \right| \right)$$

metric $\quad g_{hyp} = \dfrac{4\,(du^2 + dv^2)}{(1-u^2-v^2)^2}$

The half-plane model



$$d_{hyp} = 2\tanh^{-1}\left(\left| \frac{b-a}{b-\bar{a}} \right| \right) \quad (\text{check})$$

$$g_{hyp} = \frac{dx^2 + dy^2}{y^2}$$

recall $z \mapsto \frac{1}{i}\left(\frac{z-i}{z+i}\right)$

$h \cong D$

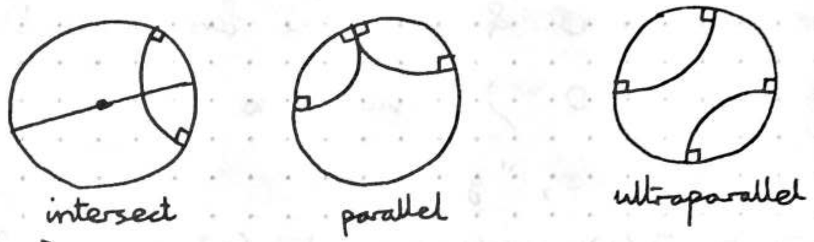
We call the circle at infinity the boundary of the disc D, $|z| = 1$ or of h, $\mathbb{R}e \cup \{\infty\}$.

People usually refer to geodesics in the hyp plane as hyperbolic lines.

L21.2

<u>Def^n</u> Two geodesics / lines in $\mathbb{H}^2$ (hyp plane) are <u>parallel</u> if they share one end-point at infinity, & <u>ultraparall</u> if they don't meet <u>either</u> in $\mathbb{H}^2$ <u>or</u> at infinity. (Otherwise, they intersect in $\mathbb{H}^2$.

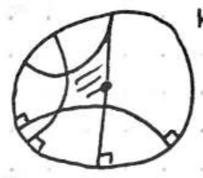
<u>In pictures</u>

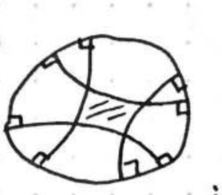


at most one point, consider putting the intersection point at $0 \in D$ (by isometry of $\mathbb{H}^2$) & both geodesics become diameters.

<u>Remark</u> Given one line ℓ, & $p \notin \ell$ in $\mathbb{H}^2$, $\exists$ many ultraparallel lines through p (& not meeting ℓ) cf Euclid was doomed :(

<u>Def^n</u> A hyperbolic <u>polygon</u> is the region bound by a collection of hyperbolic lines.





We allow <u>ideal</u> vertices (ones at ∞) but not as in the very right.

i.e. lines are pairwise cyclically intersecting or parallel.

<u>Lemma</u> Let T be a hyperbolic triangle (perhaps with ideal vertices) & with internal angles α, β, γ [internal angle at ∞ zero]

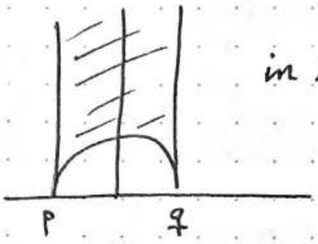
Then $\text{Area}(T) = \pi - (\alpha + \beta + \gamma)$.

<u>Remarks</u> (i) This is Gauss-Bonet for hyperbolic triangles

(ii) Note g_{hyp} has $E = G$ & $F = 0$ so it's conformal, angles in the hyperbolic plane agree with Euclidean angles.

<u>Proof of Lemma</u> Since Möbius group acts triply transitively on ∂D or ∂h (check in h, $z \mapsto az + b$, $a, b \in \mathbb{R}$, $a > 0$ fix ∞ and move 0 to b, then we can rescale) Up to isometry, there is a unique <u>ideal</u> hyp triangle (<u>all</u> vertices are ideal)

e.g. take

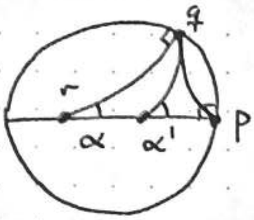


in h

$$\text{Area}(T) = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$$

We reduce all other cases to this one.



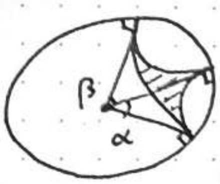
Let $A(\alpha)$ be the area of a triangle with 2 vertices at ∞ & one internal angle α

α-triangle $\supseteq$ α'-triangle

presume?

$A(\alpha)$ is decreasing in α, goes to π as $\alpha \to 0$

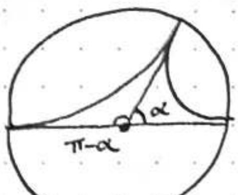
goes to 0 as $\alpha \to \pi$



This picture says

$$A(\alpha) + A(\beta) = A(\alpha+\beta) + \pi$$

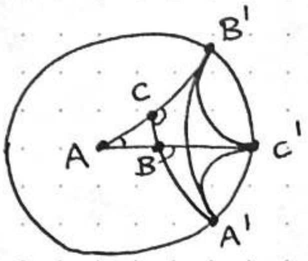
Let $F(\alpha) = \pi - A(\alpha)$. Then



$$\left.\begin{array}{l} \text{(i) } F(\alpha+\beta) = F(\alpha) + F(\beta) \\ \text{(ii) } F \text{ is increasing \& continuous} \end{array}\right\} \Rightarrow F(\alpha) = \lambda\alpha$$

Also, $A(\alpha) + A(\pi-\alpha) = \pi \Rightarrow \lambda = 1$ so $A(\alpha) = \pi - \alpha$.

Consider



For areas,

$$ABC + A'CB' + A'B'C'$$
$$= AB'C' + A'BC'$$
$$\Rightarrow ABC + (\pi - \pi + \gamma) + \pi = (\pi - \alpha) + (\pi - \beta)$$
$$\Rightarrow ABC = \pi - (\alpha+\beta+\gamma)$$

$\square$

<u>Corollary</u> Given a hyperbolic n-gon with internal angles $\alpha_1, \dots, \alpha_n$, then it has area $(n-2)\pi - \sum_{i=1}^{n} \alpha_i$

<u>Proof</u> Cut the hyperbolic n-gon into hyperbolic triangles noting that hyperbolic polygons are <u>convex</u> (they contain the geodesic segments between the vertices) $\square$

doubt, need to actually prove

L22.1 Compact hyperbolic surfaces

Recall The unit sphere $S^2 \subseteq \mathbb{R}^3$ has constant Gauss curvature 1

The torus $T^2 =$  admits an abstract Riemannian metric of constant curvature 0.

Every compact ~~smooth~~ orientable abstract smooth surface is Σ_g  $g=3$ for some g (This is not obvious)

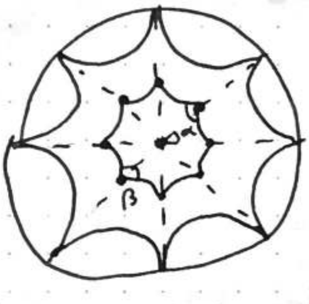
Theorem For each $g \geq 2$, there is an abstract smooth surface Σ_g with an abstract Riem. metric of constant curvature -1, i.e. a hyperbolic metric.

We'll give (outline) two constructions of such metrics.

Construction 1 (Gluing polygons)

Lemma If $g \geq 2$, there is a regular geodesic $4g$-gon in $\mathbb{H}^2$ with all internal angles $\frac{\pi}{2g}$.

Proof Start with an ideal regular $4g$-gon in (D, g_{hyp})



(vertices at $4g$-th roots on the circle at infinity)

Slide vertices inwards symmetrically

Angle $\alpha = \frac{2\pi}{4g}$.

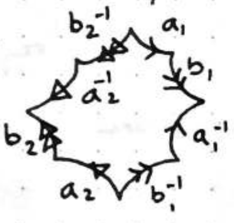
The original ideal $4g$-gon has area $(4g-2)\pi$.

As we slide vertices, this area varies monotonically to zero.

The internal angle β varies continuously from 0 ("ideal case") to β_{min}, where $4g\pi - 2\pi - 4g\,\beta_{min} = 0$ (limiting case of area 0).

So internal angles in $(0, \beta_{min})$ are related by actual regular $4g$-gons. But $\beta_{min} = \pi(1 - 1/2g)$ so if $g \geq 2$, $\frac{\pi}{2g} \in (0, \beta_{min})$ $\square$

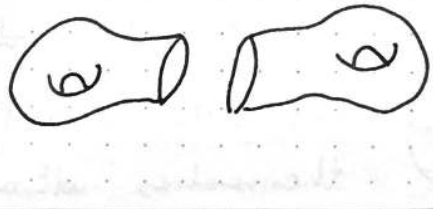
Recall



The $4g$-gon labelled as

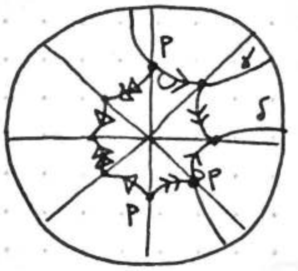
$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

is a pattern s.t. the associated surface after gluing is an (orientable) Σ_g.



Remark Given 2 geodesics in (D, g_{hyp}), & points on the geodesics, & an orientation (direction along geodesic), & a choice of side of geodesic, then there is an isometry taking one geodesic to the other compatible with data.

(To arrange last condition, use inversion if necessary)

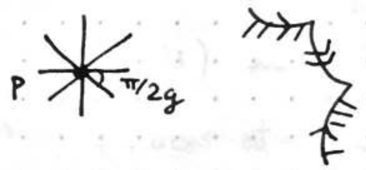


$\exists$ isometry φ (associated to $\rightarrow$)
taking γ to δ, taking $p \in \gamma$ to $p \in \delta$,
taking direction $\underset{\gamma}{\rightarrow}$ to that of δ,
& taking sides of γ, δ as indicated.

Now I can perform the gluing of sides using the isometries φ and cousins, one for each pair of edges

So a ngbd of any point $z \in$ interior of an edge inherits a ngbd isometric to a small disc in D.

At p, all the vertices of the polygon are identified, so our gluing pattern is taking $4g$ sectors of angle $\frac{\pi}{2g}$ in the hyperbolic plane & gluing edges of the sectors by isometries

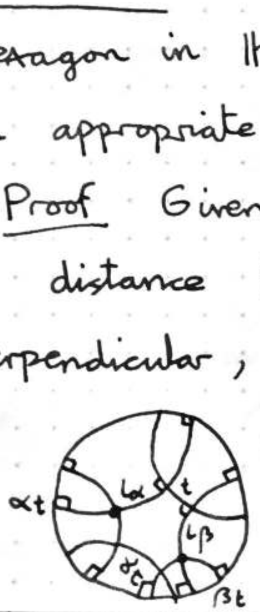


I see that since $4g\,(\cancel{2}\pi/2g) = 2\pi$, the sectors exactly close to a disc in $\mathbb{H}$. $\square$

Construction II: gluing hexagons

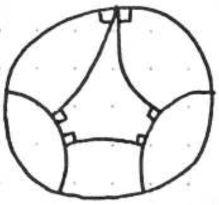
Lemma For each $\ell_\alpha, \ell_\beta, \ell_\gamma \in \mathbb{R}_+$ there is a right-angled hexagon in $\mathbb{H}^2$ whose sides have lengths $\ell_\alpha, ?, \ell_\beta, ?, \ell_\gamma, ?$ in appropriate cyclic order.

Proof Given $t > 0$, take geodesics α_t, β_t which are ultraparallel & distance t apart (as measured along the unique common perpendicular, cf Ex Sheet 4)



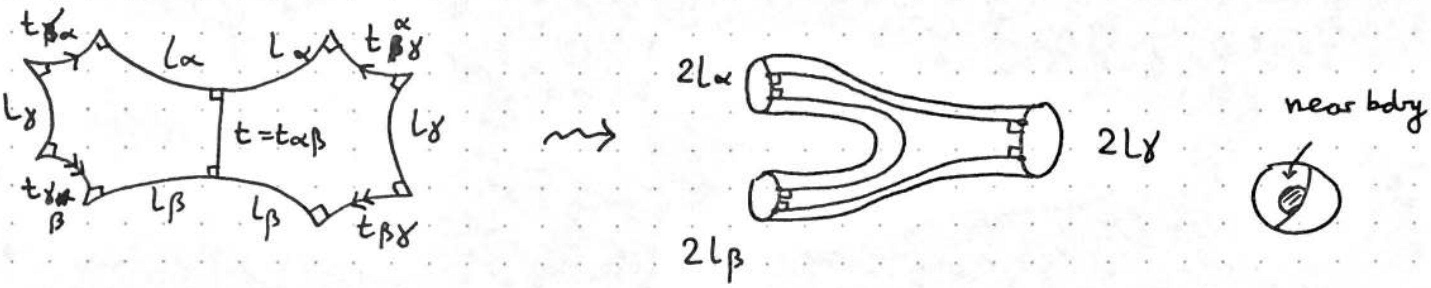
Shoot new geodesics at distances ℓ_α, ℓ_β as shown. If $t \gg 0$ is suff large, these new geodesics are themselves ultraparallel, & have a ! common $\perp$.

Threshold: The new ones are ultraparallel for $t \in (t_0, \infty)$ for some critical t_0.



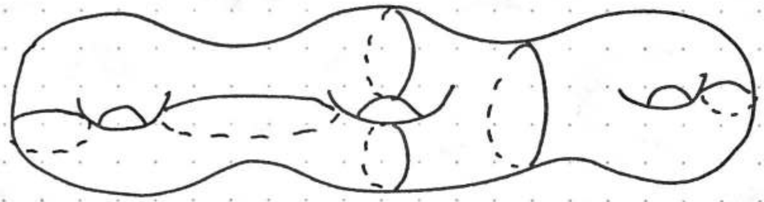
Now for $t \in (t_0, \infty)$, the length of γ_t increases monotonically from 0 to ∞, so $\exists$ a value of t such that $\text{length}(\gamma_t) = \ell_g$. $\square$

The <u>pair-of-pants</u>  is the surface with boundary $S^2 \setminus 3$ open disjoint discs.

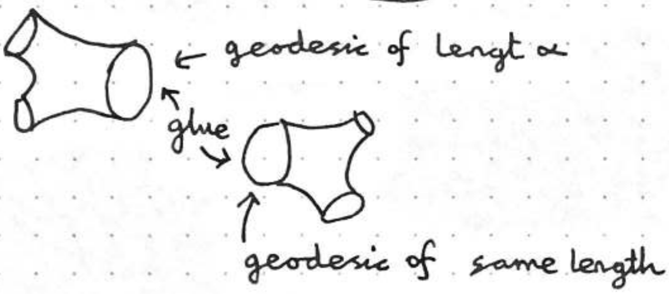


Gluing 2 hexagons with the same parameters, get a hyperbolic structure on the pair-of-pants such that the boundary circles are <u>geodesics</u>.

If I draw



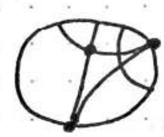
pants



$\therefore$ Another construction of curvature -1 metrics on Σ_g if $g \geq 2$.

L23.1 Gauss–Bonnet

Recall (Example sheet 2)

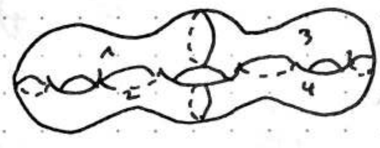


A spherical triangle has area $\alpha+\beta+\gamma-\pi$ (α,β,γ internal angles)

In L21 we saw for a hyperbolic triangle area $= \pi-(\alpha+\beta+\gamma)$.

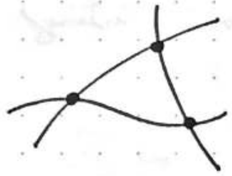
Also: torus of revolution in $\mathbb{R}^3$, then $\int_{T^2} \kappa \, dA = 0$.

And a compact hyperbolic surface Σ_g, we had $2g-2$ pants so $(4g-4)$ right angled hexagons, each of area π.



So although our gluing construction had parameters (Lengths of the hexagons) the resulting metric on Σ_g always has area $(4g-4)\pi$.

If (Σ, g) is a smooth surface with an abstract Riemannian metric, then a <u>geodesic polygon</u> on (Σ, g) is a closed disc whose boundary is composed of finitely many geodesic arcs.



geodesic triangle

NOT a geodesic triangle

Proposition (Gauss–Bonnet for geodesic polygons)

Given a surface (Σ, g), a geodesic polygon P on Σ with n sides & internal angles $\alpha_1, \dots, \alpha_n$ then

$$\int_P \kappa \, dA_\Sigma = \sum_{i=1}^n \alpha_i - (n-2)\pi$$

(Here κ is Gauss curvature)

Example On S^2, $\kappa \equiv 1$, so area$(\Delta) = \alpha+\beta+\gamma-\pi$

In $\mathbb{H}^2$, $\kappa \equiv -1$, so area$(\Delta) = \pi-\alpha-\beta-\gamma$

Globally, we have

Theorem (Gauss–Bonnet for compact surfaces)

If (Σ, g) is a smooth compact surface with an abstract metric g, then $\int_\Sigma \kappa \, dA_\Sigma = 2\pi \cdot \chi(\Sigma)$.

Euler characteristic of subdivision

L23.2

<u>Ex</u> $\chi(\Sigma_g) = \chi(g \text{ holes}) = 2-2g$, so $\text{Area}(\Sigma_g, g_{hyp}) = (4g-4)\pi$.

<u>Remark on G-B</u>

It is amazing : (1)

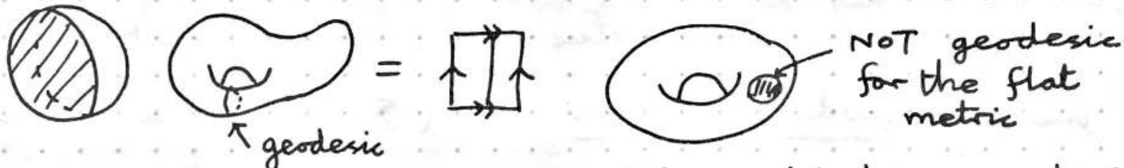
(ii) $\chi(\Sigma)$ is independent of the subdivision, globally the curvature of any metric satisfies a topological constraint.

(iii) E.g. if Σ is compact & orientable & admits a metric with $\kappa|_p > 0$ for all $p \in \Sigma$, then $\Sigma \cong S^2$ is diffeo^m to a sphere.

(iv) Suppose (Σ, g) is flat, i.e. $\kappa|_p \equiv 0$ wrt the metric g.

Then if $\gamma \in \Sigma$ is a <u>closed</u> geodesic, i.e. γ is the image of a smooth map from S^1 to Σ which locally satisfies the Euler-Lagrange equations, then γ <u>cannot</u> bound a disc in Σ.

<u>Contrast</u>



It would be a geodesic Δ with all angles π, contradicting polygonal GB.

<u>Note</u> Suppose (Σ, g) admits a subdivision s.t. all faces are geodesic polygons.

Then $\int_\Sigma \kappa_g \, dA_\Sigma$

$$= \sum_{\text{polygons}} \int_P \kappa \, dA_\Sigma$$

Suppose furthermore that all polygons have at least 3 edges, e.g. take a triangulation of Σ.

So $\displaystyle\int_\Sigma \kappa \, dA = \sum_{\substack{n \geq 3 \\ P \text{ n-gon}}} \left(\sum_{i=1}^{n} \alpha_i(P) - (n-2)\pi \right)$ by "polygonal GB"

$$= 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(\Sigma). \qquad \square$$

So the structure of proof of GB is

(a) Show geodesic polygonal division exists

L 23.3

(b) Prove GB for a geodesic polygon

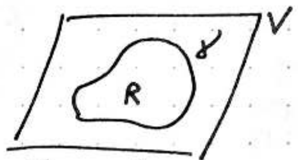
- One does (a) using ideas of "geodesic convexity"

For (b), sketch. Suppose $\Sigma \subseteq \mathbb{R}^3$, & the polygon lies in the image of a chart? $\sigma : V \to \Sigma$ for V an open disc in $\mathbb{R}^2$.

Recall Green's theorem in the plane: if P, Q are smooth functions on V, & if γ is a piecewise smooth closed curve in V, then

$$\int_\gamma P\,du + Q\,dv = \int_R (Q_u - P_v)\,du\,dv$$

where γ "bounds" R.



On V, let e, f form an o.n. basis of $T_{\sigma(p)}\Sigma$ for $p \in V$.

- (e.g. in geodesic normal coordinates, where FFF $= du^2 + G(u,v)\,dv^2$, could take $e = \sigma_u$ & $f = \sigma_v/\sqrt{G}$.

Let $I = \int_\gamma \langle e, \dot f\rangle\,dt$ where $\gamma = \gamma(t)$ is param^d by arc length on its smooth segments.

<u>One hand</u> $\dot f = f_u \dot u + f_v \dot v$ so $P = \langle e, f_u\rangle$, $Q = \langle e, f_v\rangle$

$$\& \quad Q_u - P_v = \langle e_u, f_v\rangle + \cancel{\langle e, f_{uv}\rangle} - \langle e_v, f_u\rangle \cancel{-} \cancel{\langle e, f_{uv}\rangle}$$
$$= \langle e_u, f_v\rangle - \langle e_v, f_u\rangle$$

In L17, we showed this was $-\sqrt{G}_{uu}$ (cf "$\lambda_1\mu_2 - \mu_1\lambda_2$")

- $\overset{?}{\underset{(L17)}{\Rightarrow}}$ is $K\sqrt{G} = K\sqrt{EG - F^2}$ as $E = 1, F = 0$ in these coords

$$= K\,dA_\Sigma \quad \text{so RHS of Green is } \int_R K\,dA_\Sigma$$

<u>Also</u>, let $\theta(t)$ be the angle from $\dot\gamma(t)$ to e

So $\dot\gamma(t) = \cos\theta \cdot e + \sin\theta \cdot f$

$\Rightarrow \ddot\gamma = \dot e \cos\theta + \dot f \sin\theta + \eta(\dot\theta)$, for $\eta = -\sin\theta \cdot e + \cos\theta \cdot f$

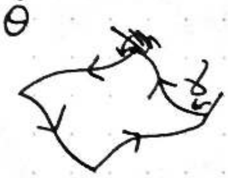
γ is made of geodesic arcs $\Rightarrow \ddot\gamma$ is <u>normal</u> to Σ

so $\langle \ddot\gamma, \eta\rangle = 0$

And $\langle \ddot\gamma, \eta\rangle = \dot\theta - \langle e, \dot f\rangle(\cos^2\theta + \sin^2\theta)$

- $\Rightarrow \langle e, \dot f\rangle = \dot\theta$

LHS of Green's thm



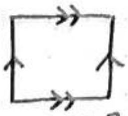
Now $\int_\gamma \dot\theta\,dt = 2\pi - (\sum_i \text{ext angle})$

$\Rightarrow$ Result! $= \cancel{\sum(\pi - \text{internal})}$

□

L24.1

Maps and Moduli

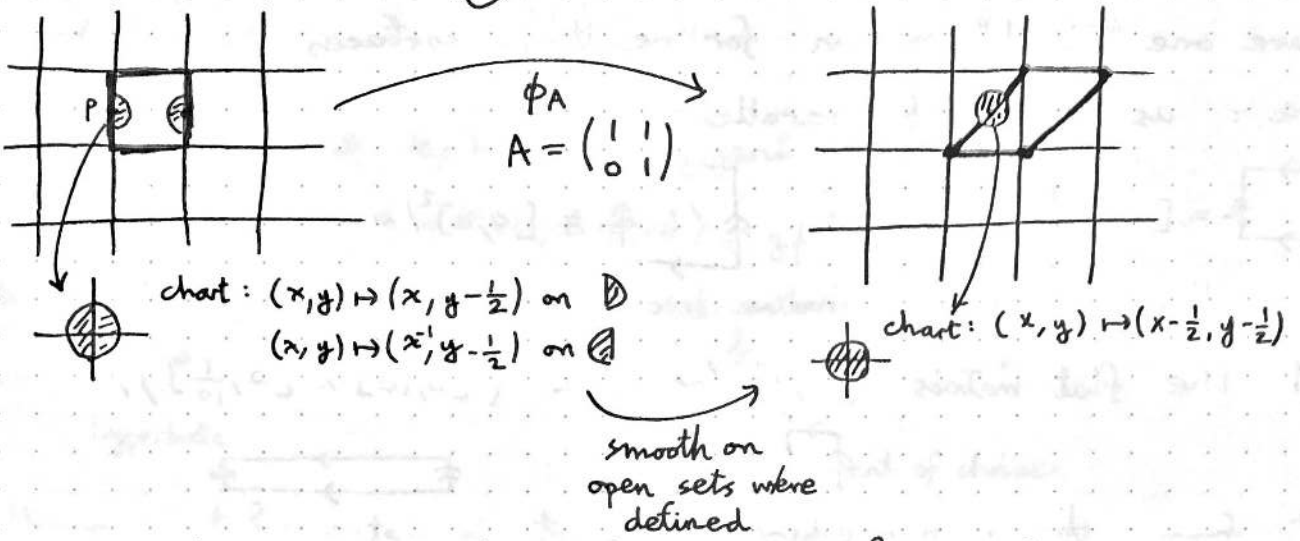
The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2/\sim$ 

If $A \in SL(2,\mathbb{Z})$, then A defines a map on $\mathbb{R}^2$ which preserves $\mathbb{Z}^2$, so induces a map $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{array}{ccc} \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ \| & & \| \\ T^2 & \xrightarrow{\phi_A} & T^2 \end{array}$$

ϕ_A is a homeomorphism of T^2

In fact, with respect to our usual smooth atlas, ϕ_A is a diffeo^m of T^2 (smooth) & with smooth inverse.



<u>Upshot</u> The group $SL(2,\mathbb{Z})$ acts on T^2 by diffeomorphisms.

<u>Different setting</u> We also know $SL(2,\mathbb{Z}) \leq SL(2,\mathbb{R})$ acts on

$h = \{ z \mid \operatorname{Im} z > 0 \}$ by hyperbolic isometries

<u>Definition</u> We say an element of $SL(2,\mathbb{R})$ is

<u>elliptic</u> if $|\text{trace}| < 2$,

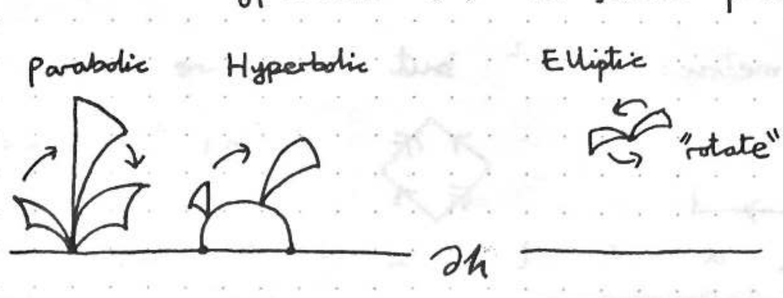
<u>parabolic</u> if $|\text{trace}| = 2$,

<u>hyperbolic</u> if $|\text{trace}| > 2$.

<u>Exercise</u> Elliptic $\Leftrightarrow$ 2 fixed poinst, one in h

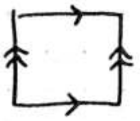
Parabolic $\Leftrightarrow$ 1 fixed point in $\partial h = \mathbb{R} \cup \{\infty\}$

Hyperbolic $\Leftrightarrow$ 2 fixed points, both on ∂h

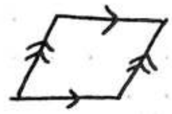


L24.2

When we constructed hyp metrics on Σ_g for $g \geq 2$, there were parameters we could choose. We had analogous flexibility in building the <u>flat</u> metric on T^2.



We could take <u>any</u> parallelogram

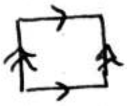
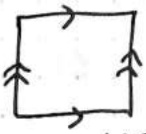


& glue opposite sides by translations.

isometries of Eucl metric

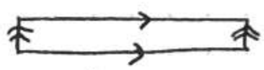
$\rightsquigarrow$ flat metric on a surface which is topologically T^2

We have one "trivial" reason for resulting surfaces to not be isometric: use <u>area</u> of parallelogram

 $= [0,1]^2/\sim$ vs  $= [0,2]^2/\sim$

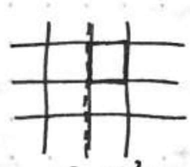
The resulting surfaces have different areas.

<u>CLAIM</u> The flat metrics $[0,1]^2/\sim$ and $([0,10] \times [0,\frac{1}{10}])/\sim$

on T^2 from these procedures are not isometric.

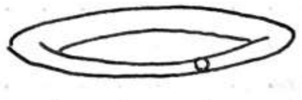
<u>Why?</u> A geodesic on (T^2/g_{flat}) is a piece of straight line in a chart which identifies an open set in T^2 with an open set in $(\mathbb{R}^2, g_{Eucl})$.

 $= \mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2/\sim$ 

closed geodesic

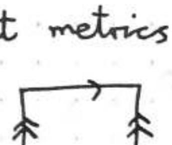
$\mathbb{R} \subset \mathbb{R}^2$

shortest closed geodesic has Euclidean length 1 (minimal distance between points of $\mathbb{Z}^2$)

 $= ([0,10] \times [0,\frac{1}{10}])/\sim$ 

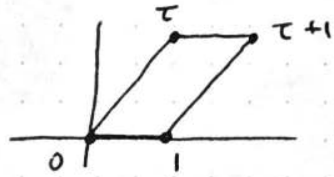
$\}\, 1/10$

contains closed geodesic of length $1/10$

Suppose we care about flat metrics on T^2 but we ignore global rescaling. Note also that  &  define the same flat metric on T^2, but with a different view of "North".

L 24.3

Modulo scale & this rotation, we can make a parallelogram have sides $1, \tau$ where $\tau \in \mathfrak{h}$.



This defines a map

$$\mathfrak{h} \to \{\text{flat metrics on } T^2 \text{ up to scale}\}$$

If S is a surface, & g is a metric on S, & $\phi : S \to S$ is a diffeo^m, ϕ may not preserve g (may not be an isometry), but I can define a <u>new</u> metric g_ϕ on S s.t.

$$(S, g) \xrightarrow{\phi} (S, g_\phi)$$

is tautologically an isometry.

If $\phi(p) = q$ & v, w are tangent vectors at p,

$$\langle v, w \rangle_{p,g} =: \langle d\phi(v), d\phi(w) \rangle_{q, g_\phi}$$

<u>So</u> $\mathfrak{h} / SL(2, \mathbb{Z}) \to \{$ Flat metrics on T^2 up to scale <u>&</u> up to action of diffeo^{m}s $\}$

hyperbolic

independent of choices

<u>Where next?</u>

Alg Top: Top. aspects of spaces, Euler characteristic, "coverings": any surface is of form $\mathbb{R}^2/\Gamma$ (or S^2, $\mathbb{RP}^2$)

Diff Geo: Studied $\kappa \sim \det(\mathrm{II}\ 2FF)$

Trace $\sim$ <u>mean</u> curvature, soap bubbles

naturally complex analysis appears

Riem Surf: Study surfaces as locally $\mathbb{C}$: $\mathrm{Isom}(D^2, g_{hyp})$ <u>are</u> holomorphic maps

GR: Gravity as curvature & the study of geodesics