IB Linear Algebra

Chap I Vector spaces

F - arbitrary field (schedules $\mathbb{R}$ or $\mathbb{C}$)

Def An F-vector space (or vector space over F) is an abelian group $(V, +)$ equipped with a function ← scalar mult.

$$F \times V \to V$$
$$(\lambda, v) \mapsto \lambda v$$

s.t.
$$\left.\begin{aligned} \lambda(v_1 + v_2) &= \lambda v_1 + \lambda v_2 \\ (\lambda_1 + \lambda_2) v &= \lambda_1 v + \lambda_2 v \\ \lambda_1(\lambda_2 v) &= (\lambda_1 \lambda_2) v \\ 1 \cdot v &= v \end{aligned}\right\} \forall \begin{array}{l} \lambda_i \in F \\ v_i \in V \end{array}$$

scalars

Additive unit denoted by $\underline{0}$.

Examples

① $\forall n \in \mathbb{N}$, F^n i.e. column vectors with entries in F

② For any set X, $\mathbb{R}^X = \{ f : X \to \mathbb{R} \}$ the set of $\mathbb{R}$-valued functions on X.

Addition and scalar mult. are defined pointwise, e.g.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

③ $M_{n,m}(F)$, the space of $n \times m$ F-valued matrices

Ex 1. The above examples satisfy the axioms

2. $0 \underline{v} = \underline{0}$ for all $\underline{v}$

$(-1) v = -v$

Def Let V be a vector space over F. The subset $U \subseteq V$ is a subspace ($U \leq V$) if:

- $\underline{0} \in U$ ← redundant

$$\left\{\begin{array}{ll} \cdot\ u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U & \forall u_1, u_2 \in U \\ \cdot\ u \in U \Rightarrow \lambda u \in U & \forall u \in U, \lambda \in F \end{array}\right.$$

$\hookrightarrow \lambda_1 u_1 + \lambda_2 u_2 \in U \quad \forall u_i \in U, \lambda_i \in F$

Ex If U is a subspace of V (v. space over F), then U is a v. space over F.

Eg $V = \mathbb{R}^{\mathbb{R}}$, space of functions $\mathbb{R} \to \mathbb{R}$

$\vee$

$\mathcal{C}(\mathbb{R})$, space of cts real functions

$\vee$

$P(\mathbb{R})$, " real polynomials

<u>E.g. 2</u> $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = t \right\}$

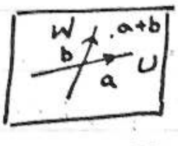
Check that this is a subspace of $\mathbb{R}^3$ iff $t = 0$.

<u>Prop</u> Let V be an F-vector space, and $U, W \leq V$.
Then $U \cap W \leq V$.

<u>Proof</u> $\left.\begin{matrix} \underline{0} \in U \\ \underline{0} \in W \end{matrix}\right\} \Rightarrow \underline{0} \in U \cap W$

- Suppose $u, v \in U \cap W$. Fix $\lambda, \mu \in F$.
  U subspace $\Rightarrow \lambda u + \mu v \in U$ ✓
  W " $\Rightarrow \lambda u + \mu v \in W$ $\square$

⚠ Union of subspaces almost never a subspace

$\mathbb{R}^2 = V$  $a + b \notin U \cup W$

<u>Def</u> Let V be an F-vector space, and $U, W \leq V$.
The sum of U and W is the set
$$U + W := \{ u + w \mid u \in U, w \in W \}.$$

<u>Prop</u> $U + W \leq V$

<u>Proof</u> $\underline{0} \in U, W \Rightarrow \underline{0} + \underline{0} = \underline{0} \in U + W$

- $u_i \in U$, $w_i \in W$
  $$(u_1 + w_1) + (u_2 + w_2) = \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W} \in U + W$$
  & similarly for scalar mult. $\square$

<u>Note</u> $U + W$ is the smallest subspace of V containing both

<u>Def</u> V an F-vector space, $U \leq V$.
The quotient space V/U is the abelian group V/U with scalar multiplication
$$F \times V/U \to V/U$$
$$(\lambda, v + U) \mapsto \lambda v + U$$

<u>Prop</u> This is well-defined, and V/U is an F-vector space

<u>Proof</u> · Well-defined: $v_1 + U = v_2 + U \in V/U$
$\Rightarrow v_1 - v_2 \in U \Rightarrow \lambda(v_1 - v_2) \in U$
$\Rightarrow \lambda v_1 + U = \lambda \overset{U \leq V}{v_2} + U$ ☺ ✓

· Vector space axioms now follow from those for V,

e.g. $\lambda(\mu(\underset{v}{u} + \underset{U}{V})) = \lambda(\mu \underset{v}{u} + \underset{U}{V}) = \lambda(\mu \underset{v}{u}) + U$

$= (\lambda\mu)v + U = (\lambda\mu)(v + U)$ $\square$

<u>Def</u> V an F-vector space, S a sub<u>set</u> of V.

S may not be finite.

Span of S: $\langle S \rangle := \{ \underset{\text{finite sum}}{\sum} \lambda_s s : \lambda_s \in F \}$

<u>Remarks</u> 1) For a general v space, one cannot make sense of infinite sums.

2) $\langle S \rangle$ is the smallest subspace of V containing S

<u>Convention</u> $\langle \emptyset \rangle = \{ \underline{0} \}$

<u>Eg 1</u> $V = \mathbb{R}^3$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$

$$\langle S \rangle = \left\{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

<u>Eg 2</u> X set. For $x \in X$, $\delta_x : X \to \mathbb{R}$

$$y \mapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \delta_x : x \in X \rangle = \{ f \in \mathbb{R}^X : f \text{ has finite support} \}$$

(Support of a f^n to $\mathbb{R}/\mathbb{C}$ = set where $f \neq 0$)

Def V is an F-vector space; $S \subset V$. S spans V if

$$\langle S \rangle = V.$$

Def V is finite dimensional over F if it is spanned by a finite set.

Def The vectors $v_1, \dots, v_n$ in V are linearly independent over F if $\sum_{i=1}^{n} \lambda_i v_i = \underline{0} \Rightarrow \forall i, \lambda_i = 0$

$S \subset V$ is linearly independent if any finite subset of it is. Otherwise, say S is linearly dependent.

Note A linearly independent set can't contain $\underline{0}$.

Def S is a basis for V if S is linearly independent and spans V.

Eg 1 F^n has standard basis $e_1, \dots, e_n$ where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ entry}$$

Eg 2 $V = \mathbb{C}$ over $\mathbb{C}$ has natural basis $\{1\}$

" $\mathbb{R}$ " $\{1, i\}$

Eg 3 $V = \mathcal{P}(\mathbb{R})$ space of all polys has basis $\{1, x, x^2, \dots\}$

Lem V an F-vector space. The vectors $v_1, \dots, v_n$ form a basis for V iff each vector $v \in V$ has a unique expression $v = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \in F$.

Proof ($\Rightarrow$) Fix $v \in V$. The v_i span V, so $\exists \lambda_i \in F$ s.t.

$$v = \sum \lambda_i v_i.$$

Suppose that also $v = \sum \mu_i v_i$, then

$$\sum (\lambda_i - \mu_i) v_i = \underline{0},$$

thus by lin independence $\lambda_i = \mu_i \ \forall i$.

($\Leftarrow$) Where u at

<u>Lem</u> Let $v_1, \dots, v_n$ span V over F. Then some subset of them is a basis for V over F.

($\Leftarrow$) Baby's back. The v_i span V, since each $v \in V$ is a linear combination of them.

If $\sum \lambda_i v_i = \underline{0}$, then by uniqueness of the expression for $\underline{0}$ ($= \sum 0 v_i$), $\lambda_i = 0 \ \forall i$. □

<u>Proof</u> If $v_1, \dots, v_n$ are linearly independent, done.

Otherwise: for some l, there exist $\alpha_1, \dots, \alpha_{l-1} \in F$ s.t.

$$v_l = \alpha_1 v_1 + \dots + \alpha_{l-1} v_{l-1}$$

⌈If $\sum \lambda_i v_i = \underline{0}$ with not all $\lambda_i = 0$, take l ~~minimal~~ max with $\lambda_l \neq 0$⌋

The $v_1, \dots, v_{l-1}, v_{l+1}, \dots, v_n$ still span V.

Now continue until you get linear independence. □

<u>Thm</u> (Steinitz exchange lemma)

Let V be a finite dimensional vector space over F.

Take $v_1, \dots, v_m$ to be linearly independent, and $w_1, \dots, w_n$ to span V. Then $m \leq n$, and rearranging the w_i if needed, $v_1, \dots, v_m, w_{m+1}, \dots, w_n$ span V.

<u>Pf</u> Induction. Suppose we've replaced l (≥ 0) of the w_i.

Reordering the w_i if needed, $v_1, \dots, v_l, w_{l+1}, \dots, w_n$ span V.

If $m = l$, done.

Assume $l < m$. The $v_{l+1} = \sum_{i \leq l} \alpha_i v_i + \sum_{i > l} \beta_i w_i$ $\quad \alpha_i, \beta_i \in F$

As the v_i are linearly independent, $\beta_i \neq 0$ for some i.

After reordering, may assume $\beta_{l+1} \neq 0$.

Then $w_{l+1} = \frac{1}{\beta_{l+1}} \left(v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i > l+1} \beta_i w_i \right)$

Then V is spanned by $v_1, \dots, v_l, v_{l+1}, w_{l+2}, \dots, w_n$.

After m steps, we will have replaced m of the w_i with v_i's.

$\therefore m \leq n$ and done. □

<u>Main theorem</u> If V is a finite dimens v. space over F, then any two bases have the same number of elements.
We call this the dimension of V, $\dim_F V$.
<u>Pf</u> If $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are bases for V over F, then the v_i are lin indep & w_i span, so $n \leq m$.
Conversely, w_i " & v_i ", so $m \leq n$. $\square$
<u>Eg</u> $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

<u>Lem</u> V f. dim v. space over F. If $w_1, \ldots, w_l$ is a l. indep set of vectors, we can extend it to a basis.
<u>Pf</u> Apply Steinitz to $w_1, \ldots, w_l$ and any basis $v_1, \ldots, v_n$.
<u>Cor</u> Let V be an F- v. space w/ finite dim n
i) Any independent set of vectors has at most n elements, with equality iff it's a basis
ii) Any spanning set has at least n elements, with equality iff it's a basis.
<u>Slogan</u> "Choose the best basis for the job"

L 3.1

Thm Let U, W be subspaces of V. If U and W are finite dimensional, so is $U+W$, and

pick basis iteratively & use Steinitz

$$\dim(U+W) = \dim U + \dim W - \dim U \cap W$$

Pf Pick basis $v_1, \dots, v_l$ of $U \cap W$
Extend it to a basis of U, say $v_1, \dots, v_l, u_1, \dots, u_m$
— W, say $v_1, \dots, v_l, w_1, \dots, w_n$

Claim $v_1, \dots, v_l, u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for $U+W$

· lin indep $\quad \sum \alpha_i v_i + \sum \beta_i u_i + \sum \gamma_i w_i = 0$

$$\Rightarrow \underbrace{\sum \alpha_i v_i + \sum \beta_i u_i}_{\in U} = \underbrace{-\sum \gamma_i w_i}_{\in W} \quad \text{so in } U \cap W$$

$$= \sum \delta_i v_i \quad \text{some } \delta_i$$

$\Rightarrow \gamma_i = \delta_i = 0$ by lin indep of $\{v_i, \dots, w_i, \dots\}$

$\Rightarrow \alpha_i = \beta_i = 0$ by lin indep of $\{v_i, \dots, u_i, \dots\}$

· span $\quad$ for $u \in U$, $u = \sum \alpha_i v_i + \sum \beta_i u_i$

$$w \in W, \quad w = \sum \gamma_i v_i + \sum \delta_i w_i$$

So $u + w = \sum (\alpha_i + \gamma_i) v_i + \sum \beta_i u_i + \sum \delta_i w_i$ $\square$

Thm If V is a finite dim v. space over F, and U is a subspace then U and V/U are also finite dim, and

$$\dim V = \dim U + \dim V/U$$

Pf ex: Let $u_1, \dots, u_l$ be a basis for U
Extend to a basis for V, say $u_1, \dots, u_l, w_{l+1}, \dots, w_n$.
Check that $w_{l+1} + U, \dots, w_n + U$ form a basis for V/U. $\square$

<u>Cor</u> If U is a proper subspace of a finite dim v space V, then $\dim U < \dim V$

<u>Pf</u> $V/U \neq \{\underline{0}\} \Rightarrow \dim V/U > 0 \Rightarrow \dim U < \dim V$ □

<u>Def</u> V vector space over F, $U, W \leq V$

Then $V = U \oplus W$ if every element of V can be written

(internal) direct sum

uniquely as $v = u + w$ with $u \in U$ and $w \in W$.

We say that W is a direct complement of U in V

<u>Lem</u> $U, W \leq V$. TFAE

i) $V = U \oplus W$

ii) $V = U + W$ and $U \cap W = \{\underline{0}\}$

iii) For any (some) bases $\mathcal{B}_1$ of U $\mathcal{B}_2$ of W, the union

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \text{ is a basis for } V$$

← disjoint union / multi-setty

<u>Pf</u> ii) ⇒ i)

- Any $v \in V$ is $u + w$ for some $u \in U$, $w \in W$
- $u_1 + w_1 = u_2 + w_2$ (with $u_i \in U$, $w_i \in W$)

$\Rightarrow u_1 - u_2 = w_2 - w_1 \in U \cap W = \{\underline{0}\}$

So $u_1 = u_2$, $w_1 = w_2$

i) ⇒ iii)

i) implies ∀ bases $\mathcal{B}_1, \mathcal{B}_2$ ⇓ for some $\mathcal{B}_1, \mathcal{B}_2$

- $\mathcal{B}$ spans: any $v \in V$ is $u + w$ for some $u \in U$, $w \in W$

We can write u in terms of $\mathcal{B}_1$, w in terms of $\mathcal{B}_2$, so v is a lin combi of elements of $\mathcal{B}$

- $\mathcal{B}$ lin indep: $$\sum_{v \in \mathcal{B}} \lambda_v v = \underline{0} = \underline{0}_U + \underline{0}_W$$

$$= \sum_{u \in \mathcal{B}_1} \lambda_u u + \sum_{w \in \mathcal{B}_2} \lambda_w w$$

By uniqueness of expression,

$$\sum_{u \in \mathcal{B}_1} \lambda_u u = \underline{0}_U \quad \text{and} \quad \sum_{w \in \mathcal{B}_2} \lambda_w w = \underline{0}_W$$

As each of the $\mathcal{B}_i$ is a lin indep set, all λ_u, λ_w are zero.

iii) ⇒ ii) If $v \in V$, $v = \sum_{x \in \mathcal{B}} \lambda_x x = \sum_{u \in \mathcal{B}_1} \lambda_u u + \sum_{w \in \mathcal{B}_2} \lambda_w w$

$\Rightarrow v \in U + W$

- If $v \in U \cap W$, $v = \sum_{u \in \mathcal{B}_1} \lambda_u u = \sum_{w \in \mathcal{B}_2} \lambda_w w$.

$$\Rightarrow \sum_{u \in \mathcal{B}_1} \lambda_u u - \sum_{w \in \mathcal{B}_2} \lambda_w w = \underline{0} \Rightarrow \text{all } \lambda_u, \lambda_w = 0$$

and so $U \cap W = \{\underline{0}\}$ □

__Lem__ V f. dim v. space over F, $U \leq V$. Then $\exists$ direct complement to U in V

⚠ not unique

__Pf__ Let $u_1, \dots, u_l$ be a basis for U.
Extend to a basis of V, say $u_1, \dots, u_l, w_{l+1}, \dots, w_n$.
Then $\langle w_{l+1}, \dots, w_n \rangle$ is a direct complement to U. □

__Def__ $V_1, \dots, V_l \leq V$ $\quad \sum V_i = V_1 + \dots + V_l$

$$= \{ v_1 + \dots + v_l : v_i \in V_i \}$$

The sum is direct if such expressions are unique.

__Notation__ $V_1 \oplus \dots \oplus V_l = \bigoplus_i V_i$

__Ex__ $V_1, \dots, V_l \leq V$ TFAE

i) The sum is direct

ii) $V_i \cap \left(\sum_{j \neq i} V_j \right) = \{\underline{0}\} \ \forall i$ $\quad$ & $\sum V_i = V$

iii) For some (any) bases $\mathcal{B}_i$ of V_i, the union $\mathcal{B} = \bigcup_i \mathcal{B}_i$ is a basis for $\sum V_i$

__Def__ U, W v. spaces over F. External direct sum

$$U \oplus W = \{ (u, w) : u \in U, w \in W \}$$

with operations component-wise.
We won't distinguish between internal & external.

§2 Linear Maps

<u>Def^n</u> V, W F-vector spaces. A map $\alpha: V \to W$ is linear if $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$
$\alpha(\lambda v) = \lambda \alpha(v)$ $\quad \forall v, v_i \in V, \lambda \in F$

or $\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)$

<u>E.g.</u> 1) $A \in M_{n;m}(F)$, $\alpha: F^n \to F^m$ indices
$x \mapsto Ax$

2) $D: P \to P$
$f \mapsto f'$

3) Fix $x \in [a,b]$. $C([a,b]) \to \mathbb{R}$
$f \longmapsto f(x)$

<u>Note</u> U, V, W v. spaces over F
i) $id_V : V \to V$ is linear
ii) $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ then $U \xrightarrow{\alpha \circ \beta} W$ is linear
linear

<u>Lemma</u> V, W F-vector spaces, $\mathcal{B}$ basis for V
If $\alpha_0 : \mathcal{B} \to W$ is <u>any</u> map, then there is a unique linear map $\alpha: V \to W$ extending α_0.

<u>Proof</u> Let $v \in V$. Then $v = \sum_{v_i \in \mathcal{B}} \lambda_i v_i$ uniquely.

Linearity forces $\alpha(v) = \sum_{v_i \in \mathcal{B}} \lambda_i \alpha_0(v_i)$ linear well-def $\square$

<u>Note</u> i) True if V is infinite dim
ii) If $\alpha_1, \alpha_2 : V \to W$ linear & agree on a basis, then they are equal.

<u>Def^n</u> V, W over F. The map $\alpha: V \to W$ is an isomorphism if it's linear and a bijection. Write $V \simeq W$.

<u>Lemma</u> $\simeq$ is an equivalence relation on the class of all vector spaces over F
i) $i_V : V \to V$ is an iso
ii) $\alpha: V \to W$ an iso $\Rightarrow$ $\alpha^{-1}: W \to V$ is linear so an iso
iii) If $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ then $U \xrightarrow{\alpha \circ \beta} W$ an iso
iso

Pf ii) α bijection $\Rightarrow \alpha^{-1}: W \to V$ exists
Let $w_1, w_2 \in W$. Then $w_i = \alpha(v_i)$ for unique $v_i \in V$.

$$\begin{aligned}\alpha^{-1}(w_1 + w_2) &= \alpha^{-1}(\alpha(v_1) + \alpha(v_2)) \\ &= \alpha^{-1}(\alpha(v_1 + v_2)) \\ &= v_1 + v_2 \\ &= \alpha^{-1}(w_1) + \alpha^{-1}(w_2)\end{aligned}$$

Similarly $\alpha^{-1}(\lambda w) = \lambda \alpha^{-1}(w)$.
i) & iii) immediate $\square$

Thm If V is a vector space over F of f. dim n, then V is isomorphic to F^n.
Pf Choose a basis $\mathcal{B}$ of V, say $v_1, \dots, v_n$.

$$\begin{aligned} V &\longrightarrow F^n \\ \Sigma \lambda_i v_i &\longmapsto (\lambda_i)\end{aligned}$$

is linear and a bijection, so an iso $\square$

Rk Choosing an iso $V \cong F^n$ is equivalent to choosing a basis for V

Thm V, W f. dim v. spaces over F, are iso iff they have the same dimension
Pf Trivial. $\square$

Claim $\alpha(\mathcal{B})$ is a basis for $\mathrm{Im}(\alpha)$.
Check · $\alpha(\mathcal{B})$ spans from surjectivity
· $\alpha(\mathcal{B})$ lin indep from injectivity

Def $\alpha: V \to W$ linear
$N(\alpha) = \{ v \in V : \alpha(v) = \underline{0} \} = \ker \alpha \leq V$
$\mathrm{Im}(\alpha) = \{ w \in W : \exists v \in V \text{ s.t. } w = \alpha(v) \} \leq W$

Thm (First isomorphism thm)
Let $\alpha: V \to W$ be a linear map.
It induces an isomorphism

$$\begin{aligned}\bar{\alpha} : V/\ker\alpha &\longrightarrow \mathrm{Im}\,\alpha \\ v + \ker\alpha &\longmapsto \alpha(v)\end{aligned}$$

Pf · $\bar{\alpha}$ well-def: $v + \ker\alpha = v' + \ker\alpha$
$\Rightarrow v - v' \in \ker\alpha \Rightarrow \alpha(v - v') = 0 \Rightarrow \alpha(v) = \alpha(v')$.

L4.3

- $\tilde{\alpha}$ is linear follows from α linear
- $\tilde{\alpha}$ bijection – $\tilde{\alpha}(v + \ker\alpha) = 0 \Rightarrow \alpha(v) = 0$
  $\Rightarrow v \in \ker\alpha$ so injective
  – surjective by def^n of $\operatorname{Im}\alpha$ $\square$

Def $r(\alpha) = rk(\alpha) = \dim \operatorname{Im}(\alpha)$ rank of α
$n(\alpha) = \dim N(\alpha)$ nullity of α

Thm (Rank-nullity thm)
Let U, V be v. spaces over F, $\dim_F U, V < \infty$
Let $\alpha : U \to V$ be linear. ...
Then $r(\alpha) + n(\alpha) = \dim U$.

Pf $U/\ker\alpha \cong \operatorname{Im}\alpha$
$\Rightarrow \dim U - \dim\ker\alpha = \dim\operatorname{Im}\alpha$ $\square$

Lem Let V, W v spaces over F of equal finite dim
Let $\alpha : V \to W$. TFAE:
i) α injective
ii) α surjective
iii) α ~~surj~~ isomorphism

E.g. $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$ $\dim V$?

$\alpha : \mathbb{R}^3 \to \mathbb{R}$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x+y+z$

$\ker\alpha = V$
$\operatorname{Im}\alpha = \mathbb{R}$
By rank-nullity $\dim V = 2$.

The space of linear maps from $V \to W$ is a vector space
V, W v. spaces$/F$. $\mathcal{L}(V, W) = \{\alpha : V \to W \mid \alpha \text{ lin}\}$

Prop $\mathcal{L}(V, W)$ is a v.space $/F$ under operations
$(\alpha_1 + \alpha_2)(v) := \alpha_1(v) + \alpha_2(v)$ $v \in V$, $\alpha_i \in \mathcal{L}(V, W)$
$(\lambda\alpha)(v) := \lambda\alpha(v)$

If both V, W have finite dimension over F, then so does $\mathcal{L}(V, W)$ and $\dim\mathcal{L}(V, W) = (\dim V)(\dim W)$.

Pf $\alpha_1 + \alpha_2$, $\lambda\alpha$ defined as above are well-def lin. maps, and s.t. v space axioms are satisfied.
Claim above dim: see later

Matrices

<u>Def^n</u> An $m \times n$ matrix over F is an array with m rows and n columns, entries in F.

$$A = (a_{ij}) \quad 1 \le i \le m,\ 1 \le j \le n$$

$M_{m,n}(F)$ = set of all such matrices

<u>Prop</u> $M_{m,n}(F)$ is an F-vector space under operations

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$\lambda(a_{ij}) = (\lambda a_{ij})$$

with dimension over F of $m \times n$.

<u>Proof</u> · v space clear (see L1)
· a standard basis is given by

$$E_{ij} = \begin{array}{c} \\ i \rightarrow \end{array}\begin{pmatrix} 0 & & 0 \\ \cdots & 1 & \\ 0 & \vdots & 0 \end{pmatrix}$$
$$\hspace{2em}\uparrow j$$

Indeed, $\sum a_{ij} E_{ij} = (a_{ij})$ so we have span, lin indep.
The claim about dimension follows. $\square$

<u>Representation of linear maps by matrices</u>

V, W v. spaces over F, $\alpha : V \longrightarrow W$ linear.

Bases $\mathcal{B} = (v_1, \dots, v_n)$ of V,
$\mathcal{C} = (w_1, \dots, w_m)$ of W

If $v \in V$, $v = \sum \lambda_i v_i$, write $[v]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

coordinate vector of v

Similarly for $w \in W$, $[w]_{\mathcal{C}} \in F^m$.

<u>Define</u> $[\alpha]_{\mathcal{B},\mathcal{C}}$ matrix rep for α wrt $\mathcal{B}, \mathcal{C}$

$$[\alpha]_{\mathcal{B},\mathcal{C}} = \Big([\alpha(v_1)]_{\mathcal{C}} \Big| \cdots \Big| [\alpha(v_n)]_{\mathcal{C}} \Big)$$

$$= (a_{ij}) \text{ is } m \times n$$

We have $\alpha(v_j) = \sum a_{i\underline{j}} w_{\underline{i}}$

<u>Lemma</u> For all $v \in V$, $[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}}$ matrix multi

<u>Pf</u> Fix $v \in V$, $v = \sum \lambda_j v_j$ so $[v]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$.

$$\alpha(v) = \alpha\Big(\sum_j \lambda_j v_j\Big) = \sum_j \lambda_j \alpha(v_j) = \sum_j \lambda_j \sum_i a_{ij} w_i$$

$$= \sum_i \underbrace{\Big(\sum_j a_{ij}\lambda_j\Big)}_{i^{\text{th}} \text{ entry of } [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}}} w_i$$

$\square$

Lemma $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ linear, with U, V, W f. dim.
Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be bases for U, V, W.
Then $[\alpha \circ \beta]_{\mathcal{A},\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [\beta]_{\mathcal{A},\mathcal{B}}$.

Pf $(\alpha\circ\beta)(\underset{\text{in } \mathcal{A}}{u_l}) = \alpha(\beta(u_l)) = \alpha\Big(\sum_j b_{jl} \underset{\text{in } \mathcal{B}}{v_j}\Big)$

$$= \sum_j b_{jl}\,\alpha(v_j) = \sum_j b_{jl} \sum_i a_{ij} \underset{\text{in } \mathcal{C}}{w_i}$$

$$= \sum_i \underbrace{\Big(\sum_j a_{ij} b_{jl}\Big)}_{(i,l) \text{ entry of } AB} w_i$$

$\square$

Prop If V, W are v. spaces over F with dims n, m, then $\mathcal{L}(V,W) \cong M_{m,n}(F)$.

Proof Fix bases $\mathcal{B}$ of V, $\mathcal{C}$ of W.

Claim $\theta : \mathcal{L}(V,W) \longrightarrow M_{m,n}(F)$ is an iso.

$$\alpha \longmapsto [\alpha]_{\mathcal{B},\mathcal{C}}$$

- θ is linear $[\lambda_1\alpha_1 + \lambda_2\alpha_2]_{\mathcal{B},\mathcal{C}} = \lambda_1[\alpha_1]_{\mathcal{B},\mathcal{C}} + \lambda_2[\alpha_2]_{\mathcal{B},\mathcal{C}}$
- θ surjective. Given $A = (a_{ij})$ Let

$$\alpha : v_j \to \sum_{i=1}^m a_{ij} w_i \quad \text{\& extend linearly}$$

- θ injective $[\alpha]_{\mathcal{B},\mathcal{C}} = 0$ matrix $\Rightarrow \alpha$ the zero map $\square$

Corollary $\dim \mathcal{L}(V,W) = \dim V \cdot \dim W$.

$\boxed{\text{Change of bases}}$

	V	W
bases	$\mathcal{B} = (v_1, \dots, v_n)$	$\mathcal{C} = (w_1, \dots, w_m)$
new bases	$\mathcal{B}' = (v_1', \dots, v_n')$	$\mathcal{C}' = (w_1', \dots, w_n')$

Def The change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$ is $P = (p_{ij})$ given by $\boxed{v_j' = \sum p_{ij} v_i}$.

$$P = \left([v_1']_{\mathcal{B}} \mid \cdots \mid [v_n']_{\mathcal{B}} \right) = [id]_{\mathcal{B}',\mathcal{B}}$$

Lem $[v]_{\mathcal{B}} = P \cdot [v]_{\mathcal{B}'}$

Pf $P \cdot [v]_{\mathcal{B}'} = [id]_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}'} = [id(v)]_{\mathcal{B}} = [v]_{\mathcal{B}}$. □

Note P is an invertible $n \times n$ matrix and P^{-1} is the change of basis matrix from B to B'.

⌈ $[id]_{\mathcal{B},\mathcal{B}'} \cdot [id]_{\mathcal{B}',\mathcal{B}} = [id]_{\mathcal{B}',\mathcal{B}'} = I_{n}$

$[id]_{\mathcal{B}',\mathcal{B}} \cdot [id]_{\mathcal{B},\mathcal{B}'} = [id]_{\mathcal{B},\mathcal{B}} = I_n$ POG ⌋

Let Q be the change-of-basis matrix from $\mathcal{C}'$ to $\mathcal{C}$.
Q is also invertible, $m \times m$

Prop Let $\alpha : V \to W$ linear, $A = [\alpha]_{\mathcal{B},\mathcal{C}}$
$A' = [\alpha]_{\mathcal{B}',\mathcal{C}'}$

Then $A' = Q^{-1} A P^{+1}$.

Proof $[\alpha]_{\mathcal{B}',\mathcal{C}'} = [id]_{\mathcal{C},\mathcal{C}'} [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [id]_{\mathcal{B}',\mathcal{B}}$

or $[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}} = A \cdot \underbrace{[id]_{\mathcal{B}',\mathcal{B}}}_{P} \cdot [v]_{\mathcal{B}'}$

$$\| $$

$$\overbrace{[id]_{\mathcal{C}',\mathcal{C}}}^{Q} \cdot [\alpha(v)]_{\mathcal{C}'}$$

$$\|$$

$$Q \cdot [\alpha]_{\mathcal{B}',\mathcal{C}'} \cdot [v]_{\mathcal{B}'}$$

$\therefore QA' = AP$ □

<u>Def</u> The matrices $A, A' \in M_{m,n}(F)$ are <u>equivalent</u> if $A' = Q^{-1}AP$ for invertible matrices P $n\times n$, Q $m \times m$.

<u>Note</u> This defines an equivalence relation on $M_{m,n}(F)$

<u>e.g.</u> $A' = Q^{-1}AP$, $A'' = Q'^{-1}A'P'$

$\Rightarrow A'' = (QQ')^{-1}A\,PP'$

<u>Prop</u> Let V, W be two F-vector spaces, of dimensions n, m repsectively. Let $\alpha : V \to W$ linear map.
There exist bases $\mathcal{B}$ of V, $\mathcal{C}$ of W s.t.

$$[\alpha]_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad I_r = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{r\times r}$$

<u>Pf</u> Fix r so that $N(\alpha)$ has dim $n-r$.
Fix a basis for $N(\alpha)$, say $v_{r+1}, \dots, v_n$.
Extend it to a basis of V, say $v_1, \dots, v_n$, say $\mathcal{B}$.
Now $\alpha(v_1), \dots, \alpha(v_r)$ is a basis for $\mathrm{Im}(\alpha)$.

<u>Claim</u> · span: $\alpha(v_1), \alpha \dots (v_r), \underbrace{\dots, \alpha(v_n)}_{\text{zero}}$ span $\mathrm{Im}(\alpha)$ ✓

· lin indep: $\sum_{i=1}^{r} \alpha(v_i)\lambda_i = 0 \Rightarrow \sum_{i=1}^{r} \alpha(\lambda_i v_i) = 0$

$\Rightarrow \alpha\left(\sum_{i=1}^{r} \lambda_i v_i\right) = 0 \Rightarrow \sum_{i=1}^{r} \lambda_i v_i = \sum_{j=r+1}^{n} \lambda_j v_j$

Since v_i a basis, deduce $\lambda_i = 0 \ \forall i$. ✓

Extend to a basis $\mathcal{C}$ of W.
By construction, $[\alpha]_{\mathcal{B},\mathcal{C}}$ is as desired. □

<u>Rk</u> $r = r(\alpha)$ rank of α
This gives a different proof of the rank-nullity thm.

<u>Cor</u> Any $m\times n$ matrix is equivalent to $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$ for some r.

<u>Def</u> Let $A \in M_{m,n}(F)$. • The column rank of A, $r(A)$, is the dimension of the subspace of F^n spanned by its colums.
• The row rank of A is the column rank of A^T.

ch <u>Note</u> If α is linear map, represented by A wrt some bases of F^n & F^m, then $r(\alpha)$ = column rank

<u>Prop</u> Two $m \times n$ matrices A, A' are equivalent iff $r(A) = r(A')$

<u>Pf</u> ($\Leftarrow$) Both A, A' are equivalent to $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

($\Rightarrow$) Let α be the linear map represented by A wrt

the standard bases for F^m & F^n. $r(A) = r(\alpha)$.
$A' = Q^{-1} A P$

represents α wrt different bases of F^m & F^n
$r(\alpha)$ is defined in a basis independent way,
so $r(A) = r(A')$. □

<u>Thm</u> $r(A) = r(A^T)$ i.e. row rank = col rank.

<u>Pf</u> $Q^{-1}AP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$, some r, P, Q invertible

Transpose: $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times m} = P^T A^T Q^{T^{-1}}$

$\Rightarrow A^T$ equivalent to $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times m}$ so $r(A^T) = r$ □

<u>Elementary operations, elementary matrices</u>

<u>Def</u> Elementary column operations on an $m \times n$ matrix A:
i) swap two columns i, j (wlog $i \neq j$)
ii) replace col i by $\lambda \times$ col i $(\lambda \neq 0)$
iii) add $\lambda \times$ col i to col j ($i \neq j$, wlog $\lambda \neq 0$)

Elementary row operations are defined analogously.
All of these are reversible.

Corresponding elementary matrices: effect of performing the operation on I_n (or I_m)

i) $T_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$ (ith, jth)

ii) $M_{i,\lambda} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ (ith)

iii) $C_{i,j,\lambda} = I_n + \lambda E_{ij}$ — 1 in entry (i,j), 0 elsewhere

An elementary ~~column~~ row operation can be performed by multiplying A by the corresponding elementary matrix from the ~~right~~ left.

<u>Constructive proof</u> that any $m \times n$ matrix is equivalent to $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$ for some r.

L6.3

- Start with A. If $A \equiv 0$, done.
- If $a_{ij} = \lambda \neq 0$
    - swap rows $1, i$ cols $1, j$ $\Rightarrow \lambda = a'_{11}$
    - now multiply col 1 by $1/\lambda$ $\Rightarrow$ get 1 in $(1,1)$
    - now clear the entries in first row, col using type iii) operations $\Rightarrow$ get $\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \tilde{A}^{1} \\ 0 & \end{array}\right)$
- Continue with $\tilde{A}^{1}$, is $(m-1) \times (n-1)$
- Iterating, get $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \underbrace{E'_1 \cdots E'_l}_{\substack{\text{row ops} \\ Q^{-1}}} A \underbrace{\overbrace{E_1 \cdots E_c}^{\text{col ops}}}_{P}$

Yatta $\square$

Variations

① If you only use elementary row operations, can get the row echelon form of a matrix (cf optimisation)

② If inv A is an $n \times n$ matrix, we can obtain I_n by using only elementary col operations (or only row)

Lem If A is an $n \times n$ invertible matrix, then we can obtain I_n using only elementary col. (OR row) ops.

Pf Induction on the number of rows ← we're using col. operations

Suppose we have $\left(\begin{array}{cc} I_k & 0 \\ \hline \multicolumn{2}{c}{*} \end{array}\right)$ } $k \geq 0$ rows ← has non-zero elt

Claim There exists $j \geq k+1$ with $a_{k+1,j} = \lambda \neq 0$

⌈Else $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$ ← $k+1$ doesn't lie in the column span, contradicting 'invertibility ⌋

Swap cols $k+1$ & j, then divide col. $k+1$ by λ, to get

$\left(\begin{array}{ccc} \multicolumn{2}{c}{I_k} & 0 \\ \hline * \cdots * & 1 & * \cdots * \end{array}\right)$ Now clear out the rest of row $k+1$ using type iii) col. operations. □

Upshot: $A E_1 \cdots E_c = I_n$

$\Rightarrow A^{-1} = E_1 \cdots E_c$ recipe for inverses

Chap III Dual spaces & dual maps

V vector space over F

Def $V^* = L(V, F) = \{\alpha : V \to F \text{ linear}\}$ is the dual of V.

V^* is a vector space over F. Its elements are sometimes called linear functionals.

E.g. ① $V = \mathbb{R}^3$, $\theta : V \to \mathbb{R}$, $\theta \in V^*$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a - c$$

② $\operatorname{tr} : M_{n,n}(F) \to F$, $\operatorname{tr} \in (M_{n,n}(F))^*$

$$A \mapsto \sum_i a_{ii}$$

③ $V = C[0,1]$, $\theta : V \to \mathbb{R}$

$$f \mapsto \int_0^1 \sin(16t) f(t)\, dt$$

Lem: Let V be a f. dim v. space $/F$ with bais $\{e_1, \dots, e_n\}$ $\mathcal{B}$"

Then there's a basis for V^* given by

$\mathcal{B}^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ where $\varepsilon_j\left(\sum_i \lambda_i e_i\right) = \lambda_j$.

$\mathcal{B}^*$ is called the dual basis to $\mathcal{B}$.

$\varepsilon_j(e_i) = \delta_{ij}$

Pf · Indepedence $\sum_j \lambda_j \varepsilon_j = 0 \Rightarrow \left(\sum_j \lambda_j \varepsilon_j\right)(e_i) = 0$

$\Rightarrow \lambda_i = 0 \quad \forall i$ ✓

· Span if $\alpha \in V^*$, then α is determined by its action on e_i.

$\alpha = \sum_j \alpha(e_j)\varepsilon_j$ ✓ □

Cor $\dim V^* = \dim V$ if V is finite dim.

Rk. Sometimes useful to think of V^* as the space of row vectors of length n ($= \dim V$) over F. ($M_{1,n}(F)$) ← m before

$e_1, \dots, e_n$ basis of V $\quad x = \sum x_i e_i \in V$

$\varepsilon_1, \dots, \varepsilon_n$ dual basis $\quad \alpha = \sum a_i \varepsilon_i \in V^*$

$$\alpha(x) = \sum a_i x_i = (a_1, \dots, a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \boxed{\text{mad}}$$

Def If $U \subseteq V$, $U^\circ = \{\alpha \in V^* \text{ s.t. } \forall u \in U, \alpha(u) = 0\}$

subset $\quad U^\circ$ is the annihilator of U

Lem 1) $U^\circ \leq V^*$

subspace

2) If $U \leq V$ and $\dim V = n < \infty$, then

$\dim V = \dim U + \dim U^\circ$

Pf 1) If $\alpha, \alpha' \in U^\circ$, then $(\alpha + \alpha')(u) = \alpha(u) + \alpha'(u) = 0$

& similarly $(\lambda\alpha) \in U^\circ$ for all α, α', λ. ✓

2) Let $e_1, \dots, e_k$ be a basis for U. Extend it to a basis for V, say $\mathcal{B} = \{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis $\mathcal{B}^*$ dual to $\mathcal{B}$.

Claim $U^\circ = \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle$ where $k' \leq k$

· If $i > k$, $\varepsilon_i(e_{k'}) = 0$ so $\varepsilon_i \in U^\circ$

· If $\alpha \in U^\circ$, say $\alpha = \sum a_i \varepsilon_i$. Now for $i \leq k$,

$0 = \alpha(\underbrace{e_i}_{\in U}) = a_i \Rightarrow \alpha = \sum_{i \geq k+1} a_i \varepsilon_i$ ✓ □

Lem V, W vector spaces / F. Let $\alpha \in L(V, W)$.

Then the map $\alpha^* : W^* \to V^*$ is linear

$\varepsilon \to \varepsilon \circ \alpha$

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d] & W \arrow[d] \\ F & F \end{tikzcd}$$

We'll call α^* the dual of α.

L7.3 $\varepsilon(\alpha(v)) = \alpha^*(\varepsilon)(v)$

Pf · $\varepsilon \circ \alpha$ is linear, so in V^*

· α^* linear: fix $\theta_1, \theta_2 \in W^*$.

$$\begin{aligned}\alpha^*(\theta_1+\theta_2) &= (\theta_1+\theta_2)\circ\alpha \\ &= \theta_1\circ\alpha + \theta_2\circ\alpha \\ &= \alpha^*(\theta_1) + \alpha^*(\theta_2)\end{aligned}$$

Similarly $\alpha^*(\lambda\theta) = \lambda\alpha^*(\theta)$. □

Prop Let V, W be f. dim v. spaces $/\mathbb{F}$, w/ bases $\mathcal{B}, \mathcal{C}$. Let $\mathcal{B}^*, \mathcal{C}^*$ be the dual bases of V^*, W^*. Then

$$[\alpha^*]_{\mathcal{C}^*,\mathcal{B}^*} = [\alpha]_{\mathcal{B},\mathcal{C}}^T$$

Pf Say $\mathcal{B} = (b_1, \dots, b_n)$ $\mathcal{C} = (c_1, \dots, c_m)$

$\mathcal{B}^* = (\beta_1, \dots, \beta_n)$ $\mathcal{C}^* = (\gamma_1, \dots, \gamma_m)$.

Say $[\alpha]_{\mathcal{B},\mathcal{C}} = (a_{ij})$.

Then $\alpha^*(\gamma_r)(b_s) = (\gamma_r\circ\alpha)(b_s) = \gamma_r(\alpha(b_s))$

$$= \gamma_r\left(\sum_t a_{ts} c_t\right) = \sum_t a_{ts}\gamma_r(c_t) = a_{rs}$$

$$= \left(\sum_i a_{ri}\beta_i\right)(b_s)$$

$\Rightarrow \alpha^*(\gamma_r) = \sum_i a_{ri}\beta_i \Rightarrow [\alpha^*]_{\mathcal{C}^*,\mathcal{B}^*} = A^T$

where $A = (a_{ij})$. □

L 8.2

So $\mathrm{Im}(\alpha^*) = (N(\alpha))^\circ$. $\square$

Double duals V vector space $/F$

$V^* = L(V,F)$ dual of V

$V^{**} = L(V^*, F)$ dual of V^*, double dual of V

Thm If V f. dim v. space $/F$, then the map

$$\wedge : V \longrightarrow V^{**} \qquad \text{where } \hat{v}(\underset{\uparrow \text{in } V^*}{\varepsilon}) = \varepsilon(v).$$
$$v \longmapsto \hat{v}$$

is an isomorphism.

Pf • First, for $v \in V$, the map $\hat{v} : V^* \to F$ is linear: ö

So $\wedge$ is indeed a map $V \longrightarrow V^{**}$.

• $\wedge$ is linear: if $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in F$, $\varepsilon \in V^*$ then

massive hat

$$(\widehat{\lambda_1 v_1 + \lambda_2 v_2})(\varepsilon) = \varepsilon(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \varepsilon(v_1) + \lambda_2 \varepsilon(v_2)$$
$$= \lambda_1 \hat{v}_1(\varepsilon) + \lambda_2 \hat{v}_2(\varepsilon) = (\lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2)(\varepsilon) \qquad ✓$$

• $\wedge$ is injective: let $e \neq 0$ in V, want $\hat{e} \neq 0$.

Pick a basis $e_1, e_2, e_3, \dots$ of V

Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the dual basis

Then $\hat{e}(\varepsilon) = \varepsilon(e) = 1 \Rightarrow \hat{e} \neq 0$ so $\wedge$ inj ✓

• Since $\dim V = \dim V^* = \dim V^{**}$, $\wedge$ iso. $\square$

${}^\circ U^\circ$ Rks 1) For V f. dim, showing $V \simeq V^*$ required showing a basis, whereas $V \simeq V^{**}$ is natural, don't rely on basis.

2) Compare: if $v \in F^n$ column vector,

v^T row vector

$(v^T)^T$ back to col vector

3) In fact when V is infinite dimensional, $\wedge$ as defined still is injective, so $V \leq V^{**}$.

Lem Let V be a f. dim v. space $/F$, and $U \leq V$.

Then $\hat{U} = U^{\circ\circ}$, so after identification of V with V^{**}, $U^{\circ\circ} = U$.

Pf Show that $\hat{U} \leq U^{\circ\circ}$.

Let $u \in U \Rightarrow \varepsilon(u) = 0 \ \forall \ \varepsilon \in U^\circ$.

$\Leftrightarrow \hat{u}(\varepsilon) = 0 \ \forall \varepsilon \in U^\circ$.

$\Leftrightarrow \hat{u} \in U^{\circ\circ}$

$\dim U^{\circ\circ} = \dim V - \dim U^\circ = \dim U$ ☺ $\square$

Bases: $(e_1, \dots, e_n) = \mathcal{E}$ — Change of basis matrix
for V — $(f_1, \dots, f_n) = \mathcal{F}$ — $P = [id]_{\mathcal{F},\mathcal{E}}$

Dual bases: $(\varepsilon_1, \dots, \varepsilon_n) = \mathcal{E}^*$
for V^* — $(\eta_1, \dots, \eta_n) = \mathcal{F}^*$

<u>Lem</u> Change of basis matrix from $\mathcal{F}^*$ to $\mathcal{E}^*$ is $(P^{-1})^T$

<u>Pf</u> $[id]_{\mathcal{F}^*,\mathcal{E}^*} = [id]_{\mathcal{E},\mathcal{F}}^T = \left([id]_{\mathcal{F},\mathcal{E}}^{-1} \right)^T$ □

CAUTION Let $V = P$, the space of all real polynomials.
Then $P = \langle p_0, p_1, \dots \rangle$; $p_r(t) = t^r$.
ES2 $P^* \simeq \mathbb{R}^{\mathbb{N}}$
$\varepsilon \mapsto (\varepsilon(p_0), \varepsilon(p_1), \dots)$
<u>But:</u> $P \not\simeq \mathbb{R}^{\mathbb{N}}$ ← no countable basis
(ES1)

<u>Lem</u> Let V, W vector spaces / F. Let $\alpha \in L(V,W)$.
Let $\alpha^* \in L(W^*, V^*)$ be its dual. Then

1) $N(\alpha^*) = (\text{Im}(\alpha))^\circ$ (so α^* inj $\Leftrightarrow$ α surj)

2) $\text{Im}(\alpha^*) \leq (N(\alpha))^\circ$ w/ equality if V, W finite dim
(so α^* surj $\Leftrightarrow$ α inj)

<u>Pf</u> 1) Let $\varepsilon \in W^*$.
Then $\varepsilon \in N(\alpha^*) \Leftrightarrow \alpha^*(\varepsilon) = 0 \Leftrightarrow \varepsilon \circ \alpha = 0$
$\Leftrightarrow \varepsilon \in (\text{Im}(\alpha))^\circ$ ✓

2) First show inclusion.
Let $\varepsilon \in \text{Im}(\alpha^*)$. So $\varepsilon = \alpha^*(\varphi) = \varphi \circ \alpha$ with $\varphi \in W^*$.
Let $u \in N(\alpha)$. Then $\varepsilon(u) = \varphi(\alpha(u)) = \varphi(0) = 0$.
So $\varepsilon \in (N(\alpha))^\circ$.
Now show equality when finite dim.
$\dim(\text{Im}\,\alpha^*) = r(\alpha^*) \underset{\text{row rank}}{=} r(\alpha) = \dim V - \dim N(\alpha)$
$= \dim((N(\alpha))^\circ)$

So $\mathrm{Im}(\alpha^*) = (N(\alpha))^\circ$. $\square$

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So $\wedge$ is indeed a map $V \longrightarrow V^{**}$.

• $\wedge$ is linear: if $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in F$, $\varepsilon \in V^*$ then

massive hat $\rightarrow$ $(\widehat{\lambda_1 v_1 + \lambda_2 v_2})(\varepsilon) = \varepsilon(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \varepsilon(v_1) + \lambda_2 \varepsilon(v_2)$

$= \lambda_1 \hat{v}_1(\varepsilon) + \lambda_2 \hat{v}_2(\varepsilon) = (\lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2)(\varepsilon)$ ✓

• $\wedge$ is injective: let $e \neq 0$ in V, want $\hat{e} \neq 0$.

Pick a basis $e_1, e_2, e_3, \dots$ of V

Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the dual basis

Then $\hat{e}(\varepsilon) = \varepsilon(e) = 1 \Rightarrow \hat{e} \neq 0$ so $\wedge$ inj ✓

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Pf Show that $\hat{U} \leq U^{\circ\circ}$.

Let $u \in U \Rightarrow \varepsilon(u) = 0 \quad \forall \varepsilon \in U^\circ$.

$\Leftrightarrow \hat{u}(\varepsilon) = 0 \quad \forall \varepsilon \in U^\circ$.

$\Leftrightarrow \hat{u} \in U^{\circ\circ}$

$\dim U^{\circ\circ} = \dim V - \dim U^\circ = \dim U$ ☺ $\square$

L8.3

<u>Lem</u> Let V be a f. dim v. space $/F$, let $U_1, U_2 \leq V$.

Then 1) $(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ$

2) $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$

<u>Pf</u> 1) Let $\theta \in V^*$.

$$\begin{aligned} \theta \in (U_1+U_2)^\circ &\iff \theta(u_1+u_2) = 0 \quad \forall u_1, u_2 \in U_1, U_2 \\ &\iff \theta(u) = 0 \quad \forall u \in U_1 \cup U_2 \\ &\iff \theta \in U_1^\circ \cap U_2^\circ \end{aligned}$$

2) Apply $^\circ$ to 1) & relable. $\square$

Chap 4 Bilinear forms I

U, V vector spaces over F

Def $\varphi: U \times V \to F$ is a bilinear form if it's linear in both arguments: $\forall u \in U,\ \varphi(u, \cdot): V \to F$ lies in V^*

$\forall v \in V,\ \varphi(\cdot, v): U \to F$ lies in U^*

E.g. 1) $V \times V^* \longrightarrow F$

$(v, \theta) \mapsto \theta(v)$

2) $U = V = \mathbb{R}^n$, $\varphi(x, y) = \sum x_i y_i$

3) $A \in M_{m,n}(F)$, $\varphi: F^m \times F^n \to F$

$(u, v) \mapsto u^T A v$

4) $U = V = C([0,1])$, $\varphi(f, g) = \int_0^1 f(t) g(t)\, dt$

Def $\mathcal{B} = (e_1, \dots, e_m)$ basis for U $\qquad \varphi: U \times V \to F$

$\mathcal{C} = (f_1, \dots, f_n)$ " V $\qquad$ bilinear

The matrix of φ wrt $\mathcal{B}$ and $\mathcal{C}$ is

$$[\varphi]_{\mathcal{B},\mathcal{C}} = \big(\varphi(e_i, f_j)\big)_{m \times n}$$

Lem $\varphi(u, v) = [u]_{\mathcal{B}}^T [\varphi]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{C}}$

Pf Say $u = \sum \lambda_i e_i$, $v = \sum \mu_j f_j$

$$\varphi(u, v) = \varphi\Big(\sum \lambda_i e_i, \sum \mu_j f_j\Big)$$

$$= \sum_{i,j} \lambda_i \mu_j \varphi(e_i, f_j) \quad ✓$$

$\square$

Note $[\varphi]_{\mathcal{B},\mathcal{C}}$ is the unique matrix with this property

DNote $\varphi: U \times V \to F$ bilinear determines linear maps

$$\varphi_L: U \to V^* \quad \text{and} \quad \varphi_R: V \to U^*$$

given by $\varphi_L(u)(v) = \varphi(u, v)$

$\varphi_R(v)(u) = \varphi(u, v)$ $\qquad$ ☺

Lem $\mathcal{B} = (e_1, \dots, e_m)$ for U

dual $\mathcal{B}^* = (\varepsilon_1, \dots, \varepsilon_m)$ for U^*

$\mathcal{C} = (f_1, \dots, f_n)$ for V

dual $\mathcal{C}^* = (\eta_1, \dots, \eta_n)$ for V^*

If $A = [\varphi]_{\mathcal{B},\mathcal{C}}$, then $[\varphi_L]_{\mathcal{B},\mathcal{C}^*} = A^T$
$\quad = (a_{ij})$ $\qquad [\varphi_R]_{\mathcal{C},\mathcal{B}^*} = A$

Pf $\varphi_L(e_i)(f_j) = a_{ij} \Rightarrow \varphi_L(e_i) = \sum_j a_{ij} \eta_j$

$\varphi_R(f_j)(e_i) = a_{ij} \Rightarrow \varphi_R(f_j) = \sum_i a_{ij} \varepsilon_i$ $\square$

Def $\ker \varphi_L$ left kernel of φ
$\ker \varphi_R$ right kernel "
φ is called non-degenerate if $\ker\varphi_L = \{0\}$ and $\ker \varphi_R = \{0\}$. Otherwise, φ is degenerate.

Lem Let $U, \mathcal{B}, V, \mathcal{C}$ be as before.
Then $\varphi: U \times V \to F$ bilinear, U, V f. dim imply
- φ is non-degenerate iff $A = [\varphi]_{\mathcal{B},\mathcal{C}}$ is invertible

(Cor: If φ is non-degenerate, $\dim U = \dim V$)

Pf φ non-deg iff $\begin{cases} \ker\varphi_L = \{0\} \\ \ker \varphi_R = \{0\} \end{cases} \Leftrightarrow \begin{cases} n(A^T) = 0 \\ n(A) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} r(A^T) = \dim U \\ r(A) = \dim V \end{cases} \Leftrightarrow A$ invertible $\square$

Cor When U & V are of f. dim, choosing a non-degenerate bilinear form $\varphi: U \times V \to F$ is equivalent to choosing an isomorphism $\varphi_L : U \to V^*$

Def For $T \subset U$, $T^\perp = \{v \in V : \varphi(t,v) = 0 \ \forall t \in T\}$
For $S \subset V$, ${}^\perp S = \{u \in U : \varphi(u,s) = 0 \ \forall s \in U\}$

Prop U bases $\mathcal{B}, \mathcal{B}'$ $\qquad P = [id]_{\mathcal{B}',\mathcal{B}}$
V bases $\mathcal{C}, \mathcal{C}'$ $\qquad Q = [id]_{\mathcal{C}',\mathcal{C}}$
Let $\varphi: U \times V \to F$ be a bilinear map. Then

$$[\varphi]_{\mathcal{B}',\mathcal{C}'} = P^T [\varphi]_{\mathcal{B},\mathcal{C}} Q$$

Pf $\varphi(u,v) = [u]_{\mathcal{B}}^T [\varphi]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{C}}$

$$= \Big(\underbrace{[id]_{\mathcal{B}',\mathcal{B}}}_{P} [u]_{\mathcal{B}'} \Big)^T [\varphi]_{\mathcal{B},\mathcal{C}} \Big(\underbrace{[id]_{\mathcal{C}',\mathcal{C}}}_{Q} [v]_{\mathcal{C}'} \Big)$$

$$= [u]_{\mathcal{B}'}^T \cancel{\{} P^T [\varphi]_{\mathcal{B},\mathcal{C}} Q [v]_{\mathcal{C}'}$$

$$= [u]_{\mathcal{B}'}^T [\varphi]_{\mathcal{B}',\mathcal{C}'} [v]_{\mathcal{C}'} \qquad \forall u \in U, v \in V$$

$\square$

Def The rank of φ is the rank of any matrix representing it.
Well-defined as $r(P^T A Q) = r(A)$ for inv^tble P, Q.
Note $r(\varphi) = r(\varphi_L) = r(\varphi_R)$

Chap 5 Determinant & trace

$\boxed{\text{Trace}}$ Def For $A \in M_n(F)$, $\operatorname{tr} A = \sum_i a_{ii}$ (trace of A)

Lem For $A, B \in M_n(F)$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Pf $\operatorname{tr}(AB) = \sum_i \sum_j a_{ij} b_{ji} = \sum_j \sum_i b_{ji} a_{ij} = \operatorname{tr}(BA)$. $\square$

Def $A, B \in M_n(F)$ are similar if there exists an invertible $P \in M_n(F)$ s.t. $A = P^{-1} \cancel{A} B P$.

Lem Similar matrices have the same trace.

Pf $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(PP^{-1}A) = \operatorname{tr}(A)$ $\square$

~~Lem~~ Def If $\alpha : V \to V$ linear, $[\alpha]_{\mathcal{B}} = [\alpha]_{\mathcal{B},\mathcal{B}}$

$\mathcal{B}, \mathcal{B}'$ bases of V $\qquad P = [\mathrm{id}]_{\mathcal{B}',\mathcal{B}}$

$$[\alpha]_{\mathcal{B}'} = P^{-1} [\alpha]_{\mathcal{B}} P \qquad \ddot{o}$$

Cor If $\alpha : V \to V$ linear, can define $\operatorname{tr} \alpha = \operatorname{tr} [\alpha]_{\mathcal{B}}$ for any basis $\mathcal{B}$, and is well-defined.

L10.1

Lem Let $\alpha : V \to V$ linear, $\alpha^* : V^* \to V^*$ dual.
Then $\operatorname{tr} \alpha = \operatorname{tr} \alpha^*$.

Pf $\operatorname{tr} \alpha = \operatorname{tr} [\alpha]_{\mathcal{B}} = \operatorname{tr} [\alpha]_{\mathcal{B}}^{T} = \operatorname{tr} \alpha^*$. $\square$

$\boxed{\text{Determinants}}$

S_n = group of permutations of $\{1, 2, \dots, n\}$

$$\varepsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ a product of evenly many transpositions} \\ -1 & \text{if } \sigma \text{ '' oddly ''} \end{cases}$$

Def $A \in M_n(F)$, $A = (a_{ij})$

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

$n!$ summands, each a product of n elements
one from each row, column

Eg $n = 2$ $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Lem If $A = (a_{ij})$ is upper triangular, $\det A = a_{11} \cdot a_{22} \cdots a_{nn}$

Pf For $a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}$ to be non-zero, need $\sigma(j) \leq j \ \forall j$.
$\Rightarrow \sigma = id$. $\square$

Lem $\det A = \det A^T$

Pf $\det A = \sum_{\sigma} \varepsilon(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$

$$= \sum_{\sigma} \underbrace{\varepsilon(\sigma)}_{= \varepsilon(\sigma^{-1})} \underbrace{a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}}_{\text{reordered}}$$

$$= \sum_{\tau} \varepsilon(\tau)\, a_{1\tau(1)} \cdots a_{n\tau(n)} = \det A^T$$ $\square$

Def A volume form d on F^n is a function
$\underbrace{F^n \times F^n \times \cdots \times F^n}_{n \text{ copies}} \to F$ s.t.

1) d is multilinear: for any i, fixing any $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n \in F^n$
$d(v_1, \dots, v_{i-1}, \cdot, v_{i+1}, \dots, v_n) : F^n \to F$ is linear

2) d is alternating: whenever $v_i = v_j$ $(i \neq j)$
$d(v_1, \dots, v_n) = 0$

Notation $A = (a_{ij}) = (A^{(1)} | \cdots | A^{(n)})$ columns
e.g. $I = (e_1 | \cdots | e_n)$

Lem $\det : F^n \times F^n \times \cdots \times F^n \to F$ is a volume form

$$(A^{(1)}, \ldots, A^{(n)}) \mapsto \det A$$

Pf 1) Multilinear: for a fixed σ, $\prod_{i=1}^{n} a_{\sigma(i),i}$ is multilinear as there's one term from each column in the product. Now use that the sum of multilinear functions is multilinear. ✓

2) Alternating: suppose that $A^{(k)} = A^{(l)}$ with $k \neq l$, Let $\tau = (k\ l)$.

So $a_{ij} = a_{i\tau(j)}$ $\forall i, j$. $\quad S_n = \underbrace{A_n}_{\text{evenly many } \tau s} \sqcup \tau A_n$ ↖ trick

TRANSPOSE

$$\det A = \sum_{\sigma \in A_n} \varepsilon(\sigma) a_{1\sigma(1)*} \cdots a_{n\sigma(n)*}$$

↕ but they match

$$+ \sum_{\sigma \in A_n} (-1)\varepsilon(\sigma) a_{1\tau\sigma(1)} \cdots a_{n\tau\sigma(n)}$$

$$= 0. \quad \square$$

Lem Let d be a volume form. Swapping two entries changes the sign (hence the label "alternating")

Pf $0 = d(v_1, \ldots, \underset{i^{th}\text{ place}}{v_i + v_j}, \ldots, \underset{j^{th}\text{ place}}{v_j + v_i}, \ldots, v_n)$

$$= \cancel{d(\cdots, v_i, \ldots, v_i, \cdots)} + d(\cdots, v_i, \ldots, v_j, \cdots)$$
$$+ d(\cdots, v_j, \ldots, v_i, \cdots) + \cancel{d(\cdots, v_j, \ldots, v_j, \cdots)} \quad \square$$

Cor If $\sigma \in S_n$, $d(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \varepsilon(\sigma)\, d(v_1, \ldots, v_n)$

Pf Immediate $\square$

Thm Let d be a volume form on F^n. $A = (A^{(1)} | \cdots | A^{(n)})$

Then $d(A^{(1)}, \ldots, A^{(n)}) = \det A \times d(e_1, \ldots, e_n)$

Pf $d(A^{(1)}, \ldots, A^{(n)}) = d\left(\sum_{i=1}^{n} a_{i1} e_i, A^{(2)}, \ldots, A^{(n)}\right)$

$$= \sum_{i=1}^{n} a_{i1}\, d(e_i, A^{(2)}, \ldots, A^{(n)})$$
$$= \sum_{i,j} a_{i1} a_{j2}\, d(e_i, e_j, A^{(3)}, \ldots, A^{(n)})$$
$$= \sum_{i_1, \ldots, i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \underbrace{d(e_{i_1}, e_{i_2}, \ldots, e_{i_n})}_{\text{zero unless } (i_1, \ldots, i_n) \text{ a perm}^n \text{ of } (1, \ldots, n)}$$

L10.3

Now use $d(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \varepsilon(\sigma) d(e_1, \dots, e_n)$ $\square$

Cor det is the unique volume form d s.t. $d(e_1, \dots, e_n) = 1$.

Prop $A, B \in M_n(F)$. Then $\det(AB) = \det A \cdot \det B$

Proof Monday ü

Def A is singular if $\det A = 0$. Otherwise A is non-singular.

Lem If A is invertible, then A is non-singular, and $\det(A^{-1}) = 1/\det A$

Pf $1 = \det I = \det(A \cdot A^{-1}) = \det A \cdot \det A^{-1}$. $\square$

Lem $\det(AB) = \det A \cdot \det B$

Pf Let $d_A : \underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{n \text{ copies}} \longrightarrow \mathbb{F}$

$$(v_1, \dots, v_n) \longmapsto \det(Av_1 | \cdots | Av_n)$$

- d_A is multilinear (det multi, $v \to Av$ linear)
- alternating ($v_i = v_j \Rightarrow Av_i = Av_j$, det alternating)

$\Rightarrow$ d_A is a volume form. Thus

$$d_A(Be_1, \dots, Be_n) \underset{\substack{\uparrow \\ \text{vol form}}}{=} \det B \cdot d_A(e_1, \dots, e_n)$$

$$= \det B \cdot \det A = \det(ABe_1, \dots, ABe_n)$$

$$= \det(AB) \det(e_1, \dots, e_n)$$ □

Rk Alternatively, could expand as in the proof that det is vol form

Thm Let $A \in M_n(\mathbb{F})$. TFAE

1) A invertible 2) A non-singular 3) $r(A) = n$

Pf 3) $\Rightarrow$ 1) this is rank nullity

1) $\Rightarrow$ 2) last time

2) $\Rightarrow$ 3) Suppose $r(A) < n$. By rank-nullity, $n(A) > 0$, and $\exists$ $\lambda \neq 0$ s.t. $A\lambda = \underline{0}$. So $\sum_{i=1}^{n} \lambda_i A_{ji} = \underline{0}$. Say $\lambda_k \neq 0$.

Let

$$B = \left(e_1 | e_2 | \cdots | e_{k-1} | \lambda | e_{k+1} | \cdots | e_n \right).$$

NIFT

Then $\det B = \lambda_k \neq 0$.

Note AB has k^{th} column zero $\Rightarrow \det(AB) = 0$.

So $\det A = 0$. □

<u>Determinants of linear maps</u>

Lem Conjugate matrices have the same determinant

Pf $\det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P) = \det(A)$ □

Def Let $\alpha : V \to V$ be linear. Define $\det \alpha = \det [\alpha]_{\mathcal{B}}$.

This is well-defined by the previous lemma.

L11.2

Determinants of block triangular matrices

Lem $A \in M_k(F)$, $B \in M_l(F)$, $C \in M_{k,l}(F)$.

Then $$\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \cdot \det B$$

Pf Set $n = k+l$, $X = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in M_n(F)$.

$$\det X = \sum_{\sigma \in S_n} \varepsilon(\sigma) \left(\prod_{i=1}^{n} x_{\sigma(i)\, i} \right)$$

Note $X_{\sigma(i) i} = 0$ if $\sigma(i) > k$, $i \leq k$

So we're summing over σ acting on $1, \dots, k$ and $k+1, \dots, n$

Get $x_{\sigma(j)\, j} = a_{\sigma_1(j) j}$ where $\sigma_1 = \sigma|_{[1,k]} \in S_k$ for $j \leq k$,

and $x_{\sigma(j)\, j} = b_{\sigma_2(j) j}$ where $\sigma_2 = \sigma|_{[k+1,n]} \in S_l$ for $j > k$.

shitty?

Noting that $\varepsilon(\sigma) = \varepsilon(\sigma_1)\, \varepsilon(\sigma_2)$, get

$$\det X = \left(\sum_{\sigma_1 \in S_k} \varepsilon(\sigma_1) \prod_{i=1}^{k} a_{\sigma_1(i)\, i} \right) \left(\sum_{\sigma_2 \in S_l} \varepsilon(\sigma_2) \prod_{j=k+1}^{n} b_{\sigma_2(j) j} \right)$$

$$= \det A \cdot \det(B)$$ $\square$

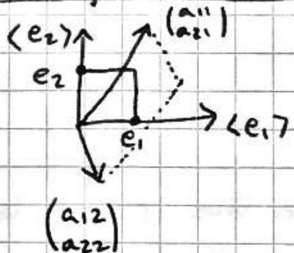
Cor For square matrices $A_1, \dots, A_k$, we have

$$\det \begin{pmatrix} A_1 & & \text{stuff} \\ & A_2 & \\ 0 & & \ddots \; A_k \end{pmatrix} = \prod_{i=1}^{k} \det A_i$$ Pf: inducción $\square$

⚠ In general $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq \det A \det D - \det B \det C$

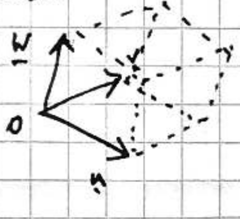
$\boxed{\text{Aside}}$ Volume interpretation of determinants

$\mathbb{R}^2$: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$



det = signed area of parallelogram

$\mathbb{R}^3$: $\det(u \,|\, v \,|\, w)$



det = signed volume of parallelepiped

L11.3

$A \in M_n(\mathbb{R})$

hypercube $H = [0,1]^n \subseteq \mathbb{R}^n$

$$\left\{ \sum_{i=1}^n t_i e_i \mid t_i \in [0,1] \right\} \longrightarrow A(H) = \left\{ \sum_{i=1}^n t_i A^{(i)} \mid t_i \in [0,1] \right\}$$

generalised volume $= 1$ $\longrightarrow$ generalised volume $= \det A$

Column expansion & adjugate matrices

Lem Let $A \in M_n(F)$, $A = (a_{ij})$

Define $A_{\hat{i}\hat{j}} \in M_{n-1}(F)$ by deleting row i, col j from A.

1) For fixed j, $\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A_{\hat{i}\hat{j}}$ "expansion in col j"

2) For fixed i, $\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{\hat{i}\hat{j}}$ "expansion in row i"

Pf of 1) (2) by transpose)

$$\det A = \det\left(A^{(1)} \mid \cdots \mid \sum_{i=1}^n a_{ij} e_i \mid \cdots \mid A^{(n)}\right)$$

$$= \sum_{i=1}^n a_{ij} \det\left(A^{(1)} \mid \cdots \mid \underset{\substack{\uparrow \\ j^{th}\text{ col}}}{e_i} \mid \cdots \mid A^{(n)}\right)$$

$$= \sum_{i=1}^n a_{ij} (-1)^{(i-1)+(j-1)} \det \begin{pmatrix} 1 & \\ 0 & A_{\hat{i}\hat{j}} \\ 0 & \end{pmatrix}$$

col, row swaps

$$= \sum_{i=1}^n a_{ij} (-1)^{i+j} \det A_{\hat{i}\hat{j}}$$

□

Def Let $A \in M_n(F)$. The adjugate matrix $\mathrm{adj}(A)$ is the $n \times n$ matrix with i,j entry $(-1)^{i+j} A_{\hat{j}\hat{i}} \leftarrow \det(\cdot)$

Thm 1) $(\mathrm{adj}\, A) \cdot A = (\det A) \cdot I$

2) If A is invertible, $A^{-1} = \frac{1}{\det A} \mathrm{adj}\, A$

Pf 1) • $\det A \underset{\substack{\uparrow \\ \text{last time}}}{=} \sum_i (\mathrm{adj}\, A)_{ji}\, a_{ij} = (j,j)$ entry of $(\mathrm{adj}\, A) \cdot A$

• For $j \neq k$, $\det(A^{(1)} | \cdots | \underset{\substack{\uparrow \\ j^{\text{th}} \text{col}}}{A^{(k)}} | \cdots | A^{(k)} | \cdots | A^{(n)}) = 0$

$= \sum_i (\mathrm{adj} A)_{ji}\, a_{ik} = (j,k)$ entry of $(\mathrm{adj}\, A) \cdot A$ ✓

2) immediate from 1 □

Chap 6 Endomorphisms

V v. space $/ F$, $\dim V = n < \infty$, $\mathcal{B} = (v_1, \dots, v_n)$ basis

$\alpha \in L(V) = L(V,V)$ an endomorphism of V

Pb Change $\mathcal{B}$ so that $[\alpha]_{\mathcal{B}}$ has "nice" form

Equivalent pb $A \in M_n(F)$, want A' conjugate to A w/ nice form

Def $\alpha \in L(V)$ is diagonalisable if $\exists\, \mathcal{B}$ w/ $[\alpha]_{\mathcal{B}}$ diagonal

$\alpha \in L(V)$ is triangleable if $\exists\, \mathcal{B}$ w/ $[\alpha]_{\mathcal{B}}$ upper triangular

$A \in M_n(F)$ is diagonalisable if conjugate to a diagonal matrix etc.

Def 1) λ is an eigenvalue [eval] of α if there exists $v \in V \setminus \{\underline{0}\}$ s.t. $\alpha v = \lambda v$

2) v is an eigenvector [evec] of α if $\alpha v = \lambda v$ for some $\lambda \in F$

3) $V_\lambda = \{ v \in V : \alpha v = \lambda v \} \leq V$ is λ-eigenspace of α

Rk 1) λ eval $\Leftrightarrow$ $\alpha - \lambda\, \mathrm{id}$ singular $\Leftrightarrow$ $\det(\alpha - \lambda\, \mathrm{id}) = 0$

$V_\lambda = \ker(\alpha - \lambda\, \mathrm{id})$

2) If $\alpha v_j = \lambda v_j$ then the j^{th} column of $[\alpha]_{\mathcal{B}}$ is $\begin{pmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}}$

3) $[\alpha]_{\mathcal{B}}$ diagonal $\Leftrightarrow$ $\mathcal{B}$ consists of eigenvectors

$[\alpha]_{\mathcal{B}}$ upper-triangular $\Leftrightarrow$ $\alpha(v_j) \in \langle v_1, \dots, v_j \rangle$

so v_1 is an evec

Facts on polynomials

$F[t] = \{$ polynomials w/ coeff.s in $F\}$ Convent^n $\deg 0 = -\infty$

Lem If $\lambda \in F$ is a root of f, then $(t-\lambda)$ divides $f(t)$

$f(t) = (t-\lambda)\, g(t)$ some poly g with $\deg g = \deg f - 1$

Sketch $f(t) = a_n t^n + \cdots + a_1 t + a_0$, $a_i \in F$, $a_n \neq 0$

$$f(t) = f(t) - f(\lambda) = a_n \underbrace{(t^n - \lambda^n)}_{(t-\lambda)(t^{n-1}+\cdots+\lambda^{n-1})} + \cdots + a_1 (t-\lambda)$$ & pull out $t-\lambda$ □

Def λ is a root of p with multiplicity e if $(t-\lambda)^e$ divides p and $(t-\lambda)^{e+1}$ does not

Cor A polynomial of degree n ($n \geq 0$) has at most n roots, counted with multiplicities □

Cor If f_1, f_2 polys of $\deg < n$ agree for n elements, then $f_1 = f_2$

Fundamental thm of algebra [complex analysis]

Any polynomial $f \in \mathbb{C}[t]$ of positive degree has a root (hence $\deg f$ roots counted w/ multiplicity)

Def The characteristic poly: $\chi_\alpha(t) = \det(\alpha - t\,\mathrm{id})$

$\alpha \in L(V)$, $A \in M_n(F)$ $\quad \chi_A(t) = \det(A - tI)$

Conjugate matrices have the same characteristic poly χ_A

Thm α is triangleable iff $\chi_\alpha(t)$ can be written as a product of linear factors over F. In particular, if $F = \mathbb{C}$ then every matrix is triangleable.

Pf ((⇒)) Suppose α is triangleable & represented by $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$ wrt some basis. Then $\chi_\alpha(t) = \det \begin{pmatrix} a_1 - t & & * \\ & a_2 - t & \\ 0 & & \ddots\, a_n - t \end{pmatrix} = \prod_{i=1}^{n} (a_i - t)$ ✓

((⇐)) Induction on $n = \dim V$. • $n = 0$ or 1 ✓

• Suppose $n > 1$, & thm holds for endomorphisms of space of smaller dimension. By assumption, $\chi_\alpha(t)$ has a root in F, say λ. Call $U = V_\lambda \neq \{0\}$. Have $\alpha(U) \leq U \Rightarrow \alpha$ induces

$\bar{\alpha} : V/U \to V/U$

Pick basis $v_1, \dots, v_k$ for U, extend it to $\mathcal{B} = (v_1, \dots, v_n)$ for V

$$[\alpha]_{\mathcal{B}} = \begin{pmatrix} \lambda I_k & * \\ 0 & C \end{pmatrix} \quad \text{where } C = [\bar{\alpha}]_{\bar{\mathcal{B}}}$$

$$\bar{\mathcal{B}} = v_{k+1} + U, \dots, v_n + U$$

So $\chi_\alpha(t) = (\lambda - t)^k \chi_{\bar{\alpha}}(t)$.

Thus $\chi_{\bar{\alpha}}$ is a product of linear factors.

By induction hypothesis, $\exists$ basis for V/U, say

$w_{k+1} + U, \dots, w_n + U$ wrt which $\bar{\alpha}$ has upper triangular matrix

Now wrt $v_1, \dots, v_k, w_{k+1}, \dots, w_n$, α itself is upper triangular. $\square$

__Lem__ $n = \dim V$, $\chi_A(t) = (-1)t^n + c_{n-1}t^{n-1} + \dots + c_0$

Then $c_0 = \det A$

for $F = \mathbb{R}$ or $\mathbb{C}$, $c_{n-1} = (-1)^{n-1} \operatorname{tr} A$ $\square$

__Pf__ $c_0 = \chi_\alpha(0) = \det(A - 0I) = \det A$

For $F = \mathbb{R}$, note $[\alpha]_{\mathcal{B}}$ can be thought of as a matrix over $\mathbb{C}$ that happens to have real entries.

Either way, $[\alpha]_{\mathcal{B}}$ triangleable over $\mathbb{C}$

$$\chi_\alpha(t) = \det \begin{pmatrix} a_0 - t & & * \\ & \ddots & \\ 0 & & a_n - t \end{pmatrix} = \prod (a_i - t) \quad \& \quad \sum a_i = \operatorname{tr} \alpha \qquad \square$$

⚠ $F = \mathbb{R}$, $V = \mathbb{R}^2$, a rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \chi(t) = t^2 - 2\cos\theta\, t + 1$$

__not__ triangleable $/\mathbb{R}$

over $\mathbb{C}$ conjugate to diag

Notation $p(t) = a_n t^n + \cdots + a_0 \in F[t]$

For $A \in M_n(F)$, $p(A) = a_n A^n + \cdots + a_0 I \in M_n(F)$

$\alpha \in L(V)$, $p(\alpha) = a_n \alpha^n + \cdots + a_0 \mathrm{id} \in L(V)$

V: v. space $/F$

Thm V vector space $/F$, $\dim V < \infty$. Let $\alpha \in L(V)$. Then α is diagonalisable iff $p(\alpha) = 0$ for some poly p a product of distinct linear factors.

Pf ($\Rightarrow$) Suppose α diagonalisable, distinct evals $\lambda_1, \dots, \lambda_k$. Let $p(t) = (t-\lambda_1)\cdots(t-\lambda_k)$. Let $\mathcal{B}$ be a basis of evecs. Let $v \in \mathcal{B}$, $\alpha v = \lambda_i v$ some i

$\Rightarrow (\alpha - \lambda_i \mathrm{id}) v = 0 \Rightarrow p(\alpha) v = 0$

This is true $\forall v \in \mathcal{B}$, so $p(\alpha) = 0$. ✓

($\Leftarrow$) Let's assume $p(\alpha) = 0$ for $p(t) = (t-\lambda_1)\cdots(t-\lambda_k)$ with λ_i pairwise distinct.

Claim $V = \bigoplus_i V_{\lambda_i}$

Pf Let $q_j(t) = \prod_{j \neq i} \frac{t-\lambda_i}{\lambda_j - \lambda_i}$ for $j = 1, \dots, k$

Then $q_j(\lambda_i) = \delta_{ij}$.

Let $q(t) = q_1(t) + \cdots + q_k(t)$ has degree $\leq k-1$, and is 1 for k different values (λ_i). So $q(t) = 1$.

Let $\pi_j = q_j(\alpha) : V \to V$.

By construction $\sum_j \pi_j = \mathrm{id}$.

So given $v \in V$, $v = q(\alpha) v = \sum \pi_j(v)$

Also, $(\alpha - \lambda_j \mathrm{id}) \pi_j(v) = \frac{1}{\prod_{i \neq j} (\lambda_i - \lambda_j)} p(\alpha) v = 0 v = 0$

So $\pi_j(v) \in \ker(\alpha - \lambda_j \mathrm{id})$. So $V = \sum_i V_i$. # EPIC

To see the sum is direct, suppose $v \in V_{\lambda_j} \cap \left(\sum_{i \neq j} V_{\lambda_i}\right)$.

$v \in V_{\lambda_j} \Rightarrow \pi_j(v) = \prod_{i \neq j} \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i} v = v$

$v \in \sum_{i \neq j} V_{\lambda_i} \Rightarrow \pi_j(v) = 0$ so $v = 0$ and sum is direct $\square$

Rk L13.2

The sum $\sum V_{\lambda_j}$ is always direct, the way that diagonalisation fails is if $\sum V_{\lambda_j} \lneq V$,

Cor If $A \in M_n(\mathbb{R}/\mathbb{C})$ has finite order ($A^m = I$, some m) then A is diagonalisable over $\mathbb{C}$.

Pf $A^m = I$ so $p(A) = 0$ with $p(t) = t^m - 1 = \prod_{i=0}^{m-1}(t - \xi_i)$ ← roots of unity

Thm (Simultaneous diagonalisation)

Let $\alpha, \beta \in L(V)$ be diagonalisable. Then α, β are simultaneously diagonalisable iff they commute.

Pf ($\Rightarrow$) Suppose that $\exists$ basis $\mathcal{B}$ s.t. $A = [\alpha]_{\mathcal{B}}$ and $B = [\alpha]_{\mathcal{B}}$ are diagonal. Diagonal matrices commute, so $AB = BA$ and $\alpha\beta = \beta\alpha$

($\Leftarrow$) Suppose α, β commute & both are individually diagonalisable.

Then $V = V_1 \oplus \cdots \oplus V_k$ where V_i are espaces for α with λ_i.

Claim $\beta(V_i) \leq V_i$

Pf Suppose $v \in V_i$. Then $\alpha\beta(v) = \beta\alpha(v) = \lambda_i \beta(v)$ ☺

So $\beta(v) \in V_i$ too. $\square_1$

As β is diagonalisable, $\exists$ poly p a product of distinct linear factors such that $p(\beta) = 0$.

So $p(\beta|_{V_i}) = 0$. So $\beta|_{V_i} \in L(V_i)$ is diagonalisable.

Pick basis for V_i, say $\mathcal{B}_i$ of evectors for β. By construction, these are also evecs for α. So wrt $\mathcal{B} = \bigcup_i \mathcal{B}_i$, both α and β are diagonal. ☺ $\square$

Lem (Euclidean algorithm for polynomials)

Given polys a, b over $\mathbb{F}$, with $b \neq 0$, there exist polys q, r with $\deg r < \deg b$ and $a = qb + r$.

Pf By GMR. $\square$

Def V an f. dim. $\mathbb{F}$-vector space, $\alpha \in L(V)$. The minimal poly of α, m_α is the non-zero monic poly $m_\alpha(t)$ of smallest degree such that $m_\alpha(\alpha) = 0$.

Rk Say $\dim_F V = n < \infty \Rightarrow \dim_F (L(V)) = n^2$, so $id, \alpha, \alpha^2, \dots, \alpha^{n^2}$ must be lin. dep in $L(V)$, so $a_{n^2}\alpha^{n^2} + \dots + \alpha a_1 + a_0 id = 0$ for some $a_i \in F$ not all zero. So minimal polys exist.

Lem Let $\alpha \in L(V)$, $p \in F[t]$. Then $p(\alpha) = 0$ iff $m_\alpha \mid p$.

Pf • Have $q, r \in F[t]$ s.t. $p(t) = m_\alpha(t) q(t) + r(t)$. deg r < deg m_α

$0 = p(\alpha) = \underbrace{m_\alpha(\alpha) q(\alpha)}_{\text{zero too}} + r(\alpha) \Rightarrow r(\alpha) = 0$ but minimal deg so $r = 0$ □

Cor m_α minimal uniquely determined

Pf Say m_1, m_2 minimal. Then $m_1 \mid m_2$, $m_2 \mid m_1$ and both are monic, so $m_1 = m_2$ □

Cayley-Hamilton thm Let V be f. dim vector space $/F$. Let $\alpha \in L(V)$. Then $\chi_\alpha(\alpha) = 0 \in L(V)$.

Corollary $m_\alpha \mid \chi_\alpha$

Pf over $\mathbb{C}$ For some basis $\mathcal{B}$, $(v_1, \dots, v_n)$ say, $[\alpha]_\mathcal{B}$ triangular. $= \begin{pmatrix} a_1 & \ddots & * \\ 0 & & a_n \end{pmatrix}$

Let $U_j = \langle v_1, \dots, v_j \rangle$. So $(\alpha - a_j id) U_j \leq U_{j-1}$.

$$\underbrace{(\alpha - a_1 id) \cdots \underbrace{\underbrace{(\alpha - a_n id) V}_{\leq U_{n-1}}}_{\leq U_{n-2}}}_{\leq U_0 = \{0\}}$$

Thus $\chi_\alpha(\alpha) = 0$. □

L14.1

Def^n λ an eval of $\alpha \in L(V)$

- a_λ : algebraic multiplicity of λ, is its multiplicity as a root of χ_α
- $g_\lambda = n(\alpha - \lambda\, id)$: geometric multiplicity of λ

Lem If λ is an eval, then $1 \leq g_\lambda \leq a_\lambda$

Pf · $1 \leq g_\lambda$ since $\alpha - \lambda\, id$ is singular

· $g_\lambda \leq a_\lambda$? Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V with $v_1, \dots, v_g$ a basis of $N(\alpha - \lambda\, id)$

$$[\alpha]_{\mathcal{B}} = \begin{pmatrix} \lambda I_g & * \\ 0 & A_1 \end{pmatrix} \quad \text{for some } A_1$$

$\chi_\alpha(t) = (\lambda - t)^g \chi_{A_1}(t)$ so $g \leq a_\lambda$ □

Lem Let λ be an eval. Let c_λ be the mulliplicity of λ as a root of m_α. Then $1 \leq c_\lambda \leq a_\lambda$.

Pf · $m_\alpha \mid \chi_\alpha \Rightarrow c_\lambda \leq a_\lambda$

· For $c_\lambda \geq 1$, λ an eval so $\alpha v = \lambda v$, some $v \neq 0$

$\underbrace{m_\alpha(\alpha)}\, v = m_\alpha(\lambda) v \Rightarrow m_\alpha(\lambda) = 0$

So $t - \lambda \mid m_\alpha$ and $c_\lambda \geq 1$. □

E.g. $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\chi_A(t) = (t-1)^2(t-2)$

Choices for m_A : a) $(t-1)^2(t-2)$

b) $(t-1)(t-2)$

Check $(A - I)(A - 2I) = 0$ so b) holds, & A diag^{ble} □

E.g. 2) $A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_n(F)$ Check: $g_\lambda = 1$, $a_\lambda = n$, $c_\lambda = n$

$A = \lambda I_n \in M_n(F)$ $g_\lambda = n = a_n$, $c_\lambda = 1$

Prop $\boxed{F = \mathbb{C}}$, $\alpha \in L(V)$. TFAE

a) α diag^{ble}

b) $a_\lambda = g_\lambda$ for all evals λ

c) $c_\lambda = 1$ "

<u>Pf</u> (a) $\Leftrightarrow$ (b) Let $\lambda_1, \dots, \lambda_k$ evals of α.

$$\alpha \text{ diag}^{\text{ble}} \Leftrightarrow \underset{\substack{\uparrow \\ \dim = n \\ = \sum a_{\lambda_i} \\ \uparrow \\ \text{by FTA}}}{V} = \underbrace{\bigoplus_{i=1}^{k} V_{\lambda_i}}_{\dim = \sum g_{\lambda_i}}$$

$g_i \leq a_i \ \forall i$

so $g_i = a_i \ \forall i$

(b) $\Leftrightarrow$ (c) By FTA, m_α is a product of linear factors. But α is diag$^{\text{ble}}$ iff these are distinct i.e. $c_\lambda = 1 \ \forall \lambda$. $\square$

<u>Over $\mathbb{C}$</u> $\chi_\alpha(t) = \pm (t-\lambda_1)^{a_1} \cdots (t-\lambda_k)^{a_k}$

$m_\alpha(t) = (t-\lambda_1)^{c_1} \cdots (t-\lambda_k)^{c_k}$

λ_i evals

$1 \leq c_i \leq a_i$

$g_i = n(\alpha - \lambda_i \, id)$

<u>Jordan Normal Form</u> $F = \mathbb{C}$

<u>Def</u> $A \in M_n(\mathbb{C})$ is in JNF if it is block diagonal of the form

$$A = \begin{pmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{pmatrix}$$

$k \geq 1$, $n_i \in \mathbb{N}$, $\sum n_i = n$

$\lambda_i \in \mathbb{C}$, need not be distinct

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_n(\mathbb{C})$$

"Jordan block"

<u>Thm</u> Every square $A \in M_n(\mathbb{C})$ is conjugate to a matrix in JNF. Moreover, JNF is unique up to reordering the blocks.

<u>Pf</u> Non-examinable (csq of main thm on modules in GRM)

<u>Ex</u> Possible JNFs for $A \in M_2(\mathbb{C})$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

m_α: $(t-\lambda_1)(t-\lambda_2)$, $(t-\lambda)$, $(t-\lambda)^2$

- $A \in M_3(\mathbb{C})$, λ_i distinct

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{pmatrix}$$

$(t-\lambda_1)(t-\lambda_2)(t-\lambda_3)$ $\qquad$ $(t-\lambda_1)(t-\lambda_2)$ $\qquad$ $(t-\lambda_1)(t-\lambda_2)^2$

$$\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

$(t-\lambda)$ $\qquad$ $(t-\lambda)^2$ $\qquad$ $(t-\lambda)^3$

L14.3

Thm Generalized eigenspace decomposition

V f. dim / $\mathbb{C}$, $\alpha \in L(V)$ Suppose $m_\alpha(t) = \prod_{i=1}^{k} (t-\lambda_i)^{c_i}$

Then $V = \bigoplus_{j=1}^{k} V_j$, where $\underbrace{V_j = N\left((\alpha - \lambda_j\, id)^{c_j}\right)}_{\text{generalised eigenspace}}$

Pf on ES3 (guided – non examin.)

Rk Could use to reduce JNF proof to a single eval

Rk
- a_λ sum of sizes of Jordan blocks w/ λ
- q_λ number of blocks w/ λ
- c_λ max size among blocks w/ λ

Eg Basis wrt which α given by $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ in JNF

$\chi_A(t) = t^2 - 2t + 1 = (t-1)^2$

Note: if $A \sim I$, then $A = I$ ※

So $m_A(t) = (t-1)^2$ and JNF is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Espace? $A - I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ so $\ker = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = \langle v_1 \rangle$

v_2 satisfies $(A-I)v_2 = v_1$ $\quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so take $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$$A = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}^{-1}}_{P}$$

$$A^n = P^{-1} J^n P = P^{-1} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} P$$

Ö

Chap 7 Bilinear form II

$\varphi : \underbrace{V \times V}_{\text{same space}} \longrightarrow \mathbb{F}$ bilinear form

This chapter same basis $\mathcal{B}$ for both copies of V, $[\varphi]_{\mathcal{B}} = [\varphi]_{\mathcal{B},\mathcal{B}}$.

Lem $\varphi : V \times V \to F$ bilinear, $\mathcal{B}, \mathcal{B}'$ bases for V.
Let $P = [id]_{\mathcal{B}',\mathcal{B}}$. Then $[\varphi]_{\mathcal{B}'} = P^T [\varphi]_{\mathcal{B}} P$.

Pf L9 □

Def $A, B \in M_n(F)$ are congruent if there is some invertible matrix P s.t. $A = P^T B P$ (this is an equivalence relation)

Def A bilinear form is symmetric if

$$\varphi(u,v) = \varphi(v,u) \quad \forall\, u, v \in V.$$

Note • $A \in M_n(F)$ is symmetric if $A^T = A$

φ symmetric $\iff$ $[\varphi]_{\mathcal{B}}$ is symmetric for any $\mathcal{B}$

• To be able to represent ~~f~~ φ by a diagonal matrix, φ needs to be symmetric. $P^T A P = D$ (D diagonal)

$\Rightarrow D^T = P^T A^T P = D \Rightarrow A^T = A$ (P invertible)

Def A map $Q : V \to F$ is a quadratic form if there exists a bilinear form $\varphi : V \times V \to F$ s.t.

$$Q(u) = \varphi(u,u) \quad \forall\, u \in V.$$

E.g. $V = \mathbb{R}^2$ $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto (x \ y) \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + (b+c)xy + dy^2$

This wouldn't have changed if we replaced A with $\frac{1}{2}(A + A^T)$.

Prop (Assume $1+1 \neq 0$) If $Q : V \to F$ is a quadratic form then there is a unique symmetric bilinear form $\varphi : V \times V \to F$ s.t.

$$Q(u) = \varphi(u,u) \quad \forall\, u \in V.$$

Pf • Let ψ bilinear be s.t. $Q(u) = \psi(u,u)$ $\forall\, u \in V$.

Let $\varphi(u,v) = \frac{1}{2}(\psi(u,v) + \psi(v,u))$. ($\leftarrow$ idea)

This is symmetric, and $\varphi(u,u) = \psi(u,u) = Q(u)$ $\forall u$.

• Uniqueness of φ? If φ is such a symm. bilinear form, for $u, v \in V$

$$Q(u+v) = \varphi(u+v, u+v) = \varphi(u,u) + \varphi(u,v) + \varphi(v,u) + \varphi(v,v)$$
$$= Q(u) + 2\varphi(u,v) + Q(v)$$
$$\therefore \boxed{\varphi(u,v) = \tfrac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right)}$$ polarisation identity

$\Rightarrow \varphi$ uniquely determined □

Thm (Assume $1+1 \neq 0$) Let $\varphi: V \times V \to F$ be a symmetric bilinear form ($\dim_F V < \infty$). There's a basis β of V s.t. $[\varphi]_\beta$ is diagonal.

Pf Induction on $n = \dim V$

- $n = 0, 1$ ✓ Suppose thm holds for $\dim < n$.
- If $\varphi(u,u) = 0 \ \forall u$, then by the polarisation identity, φ is identically zero and we're done.
- Otherwise, $\varphi(e_1, e_1) \neq 0$ for some $e_1 \in V$.

Let $U = \langle e_1 \rangle^\perp = \{u \in V : \varphi(e_1, u) = 0\}$
$= \ker \varphi_L(e_1)$ $\qquad \varphi_L = \varphi_R$

By rank-nullity, U has dimension $n-1$

Moreover, $V = \langle e_1 \rangle \oplus U$ $\left(\begin{array}{l} \langle e_1 \rangle \cap U = \{0\} \\ \dim(\langle e_1 \rangle + U) = n \end{array} \right)$

Consider $\varphi|_U : U \times U \to F$.

By the induction hypothesis, there's a basis $e_2, \dots, e_n$ of U wrt which $\varphi|_U$ is diagonal. Now φ is diagonal wrt $e_1, \dots, e_n$ □

Worked example $V = \mathbb{R}^3$, std basis e_1, e_2, e_3

$$Q(\underbrace{x_1, x_2, x_3}_{\sum x_i e_i}) = x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$$

Want a basis f_1, f_2, f_3 of $\mathbb{R}^3$ s.t. $Q(af_1 + bf_2 + cf_3)$ diag., i.e.
$$Q(af_1 + bf_2 + cf_3) = \lambda a^2 + \mu b^2 + \nu c^2.$$

Matrix wrt std basis
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $f_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $f_3 = \begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix}$

Method 1 Complete the square

$$Q(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2 + x_3^2 - 4x_2x_3$$
$$= (\underbrace{x_1 + x_2 + x_3}_{X_1})^2 + (\underbrace{x_3 - 2x_2}_{X_2})^2 - (\underbrace{2x_2}_{X_3})^2$$

L15.3

$P^T A P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ where to get P we use $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Method 2 Follow steps of pf of diagonalisation

Cor Let φ be sym bilinear form on f. dim $\mathbb{C}$ v. space V.

There's a basis $\mathcal{B} = (v_1, \ldots, v_n)$ of V s.t.

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } r = r(\varphi)$$

Equivalently, $Q(\sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{r} \lambda_i^2$.

Pf Pick basis $\mathcal{E} = (e_1, \ldots, e_n)$ s.t. $[\varphi]_{\mathcal{E}} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

Reorder basis s.t. $a_i \neq 0$ for $i = 1, \ldots, r$

$a_i = 0$ for $i > r$

For $i \leq r$, let $\sqrt{a_i}$ be a choice of cx square root of a_i

Now let $v_i = \begin{cases} e_i / \sqrt{a_i} & \text{for } i \leq r, \\ e_i & \text{for } i > r. \end{cases}$ $\square$

Cor Let φ be a sym bilinear form on f. dim $\mathbb{R}$ v. space V.

Then there's a basis $\mathcal{B} = (v_1, \ldots, v_n)$ of V s.t.

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix} \quad \text{where } p, q \geq 0, \ p + q = r(\varphi).$$

Equivalently, $Q(\sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{p} \lambda_i^2 - \sum_{i=p+1}^{q} \lambda_i^2$,

<u>Cor</u> φ sym. bil. form / $\mathbb{R}$, $\exists$ basis $\mathcal{B} = (v_1, \dots, v_n)$ s.t.

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

<u>Pf</u> Pick $\mathcal{E} = (e_1, \dots, e_n)$ s.t. $[\varphi]_{\mathcal{E}} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

Reorder indices so that

$$\begin{cases} a_i > 0 & \text{for } i = 1, \dots, p, \\ a_i < 0 & \text{for } i = p+1, \dots, p+q, \\ 0 & \text{else.} \end{cases}$$

Let $v_i = \begin{cases} e_i/\sqrt{a_i} & \text{for } i = 1, \dots, p, \\ e_i/\sqrt{-a_i} & \text{for } i = p+1, \dots, p+q \end{cases}$

leave rest unchanged. $\square$

Note $r(\varphi) = p + q$.

<u>Def</u> φ sym. bil. form / $\mathbb{R}$

- φ is positive definite if $\varphi(u,u) > 0 \quad \forall u \neq 0$
- φ is positive semidefinite if $\varphi(u,u) \geq 0 \quad \forall u$
- φ is negative definite if $-\varphi$ is positive definite
- φ is negative semidefinite if $-\varphi$ is positive semidefinite

Else, indefinite. Same terminology for quadratic forms.

<u>E.g.</u> $\begin{pmatrix} I_p & \\ & 0 \end{pmatrix}_{n \times n}$ +ve def if $p = n$, +ve semi def if $p < n$

<u>Def</u> The signature of φ, $s(\varphi) = p - q$ (or of the associated quadratic form)

<u>Thm</u> (Sylvester's law of inertia)

If a real symmetric bilinear form is represented by

$$\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix} \text{ and } \begin{pmatrix} I_{p'} & & \\ & -I_{q'} & \\ & & 0 \end{pmatrix}$$

wrt $\mathcal{B}$, $\mathcal{B}'$ then $p = p'$, $q = q'$.

<u>Cor</u> The signature is well-defined

<u>Pf</u> For uniqueness of p, enough to show that p is the largest possible dimension of a subspace on which φ is pos. def.

Say $\mathcal{B} = (v_1, \dots, v_n)$. Let $X = \langle v_1, \dots, v_p \rangle$. φ + def on X.

Suppose that φ is +ve def on some X'.

Let $Y = \langle v_{p+1}, \dots, v_n \rangle$. As φ is −ve semi-definite on Y,

$X' \cap Y = \{0\}$ by considering the sign of $\varphi(u,u)$.

So $\dim Y + \dim X' \leq n \Rightarrow \dim X' \leq n - (n-p) = p$ ✓

Similarly for q with −ve definite subspaces. $\square$

<u>Def</u> $K = \{ v \in V : \varphi(u,v) = 0 \text{ for all } u \in V \}$

kernel of a sym. bilinear form φ

<u>Note</u> $\dim K + r(\varphi) = n$

<u>Rk</u> Notation as above. There's a subspace T of dimension

$n - (p+q) + \min(p,q)$ s.t. $\varphi|_T = 0$

Assume $p \geq q$. $T = \langle v_1 + v_{p+1}, \dots, v_q + v_{p+q}, v_{p+q+1}, \dots, v_n \rangle$

Moreover, can check that this is the largest possible dimension of such a subspace.

$\boxed{\text{Sesquilinear forms}}$

Standard inner product on $\mathbb{C}^n$: $\langle x, y \rangle = \sum x_i \overline{y_i}$ NOT BILINEAR

<u>Def</u> V, W $\mathbb{C}$-vector spaces. A sesquilinear form is a f^n

$\varphi : V \times W \to \mathbb{C}$ s.t.

- $\varphi(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \varphi(v_1, w) + \lambda_2 \varphi(v_2, w)$
- $\varphi(v, \lambda_1 w_1 + \lambda_2 w_2) = \overline{\lambda_1} \varphi(v, w_1) + \overline{\lambda_2} \varphi(v, w_2)$

<u>Def</u> Notation as above, $\mathcal{B} = (v_1, \dots, v_m)$ of V

$\mathcal{C} = (w_1, \dots, w_n)$ of W

$\{ \ddot{\cdot}$

$[\varphi]_{\mathcal{B},\mathcal{C}} = (\varphi(v_i, w_j))_{i,j}$

<u>Lem</u> $\varphi(u,v) = [u]^T_{\mathcal{B}} [\varphi]_{\mathcal{B},\mathcal{C}} \overline{[v]_{\mathcal{C}}}$

<u>Pf</u> Meh $\square$

<u>Lem</u> $\mathcal{B}, \mathcal{B}'$ bases for V $\qquad P = [id]_{\mathcal{B}',\mathcal{B}}$

$\mathcal{C}, \mathcal{C}'$ bases for W $\qquad Q = [id]_{\mathcal{C}',\mathcal{C}}$

Then $[\varphi]_{\mathcal{B}',\mathcal{C}'} = P^T [\varphi] \overline{Q}$ ← new hat

<u>Pf</u> Eh $\square$

Def A sesquilinear form $\varphi : V \times V \to \mathbb{C}$ is called <u>Hermitian</u> if

$$\varphi(u,v) = \overline{\varphi(v,u)} \quad \text{for all } u,v \in V$$

Note 1) For φ Hermitian, $\varphi(u,u) \in \mathbb{R}$

$$\varphi(\lambda u, \lambda u) = |\lambda|^2 \varphi(u,u)$$

$\rightsquigarrow$ Can talk about positive / negative (semi) def Hermitian forms

2) $[\varphi]_{\mathcal{B}}$ when same basis $\mathcal{B}$ for both sides of ~~$\mathcal{B}$~~ V

Lem A sesquilinear form φ is Hermitian iff for any basis $\mathcal{B}$,

$$[\varphi]_{\mathcal{B}} = \overline{[\varphi]}_{\mathcal{B}}^{T} \quad \text{new}$$

Pf $\Rightarrow$ Let $A = [\varphi]_{\mathcal{B}} = (a_{ij})$. $\quad a_{ij} = \varphi(v_i, v_j) = \overline{\varphi(v_j, v_i)}$

$$= \overline{a_{ji}}.$$

$\Leftarrow$ $\varphi(\sum \lambda_i v_i, \sum \mu_i v_i) = \lambda^T A \overline{\mu}$

$$= \lambda^T \overline{A}^T \overline{\mu} = \overline{\mu}^T \overline{A} \lambda = \overline{(\mu^T A \overline{\lambda})}$$

$$= \overline{\varphi(\sum \mu_i v_i, \sum \lambda_i v_i)} \quad \text{so Hermitian.}$$

$\square$

<u>Polarisation identity</u> A Hermitian form φ on a v. space $V/\mathbb{C}$ is determined by $Q : V \to \mathbb{R}$, $v \mapsto \varphi(v,v)$ via the following

$$\varphi(u,v) = \frac{1}{4}\Big[Q(u+v) - Q(u-v) + i\,Q(u+iv) - i\,Q(u-iv) \Big]$$

Pf Exercise $\square$

Thm Hermitian version of Sylvester's Law & diagonalisation

V of f.dim $\mathbb{C}$-v space, $\varphi : V \times V \to \mathbb{C}$ a Hermitian form.

Then $\exists$ basis $\mathcal{B} = (v_1, \dots, v_n)$ of V s.t.

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

Moreover, p and q only depend on φ.

<u>Sketch</u> (nearly identical to $\mathbb{R}$ case)

- $\exists$: if $\varphi \equiv 0$, done

By Polarisation, $\exists\, e_1 \in V$ s.t. $\varphi(e_1, e_1) \neq 0$

Rescale to $e_1 / \sqrt{|\varphi(e_1,e_1)|} = v_1$ so $\varphi(v_1, v_1) = \pm 1$.

L16-4

Let $W = \{ w \in V : \varphi(v_1, w) = 0 \}$. Check $V = W \oplus \langle v_1 \rangle$.

Now apply inductive hypothesis to W. ✓

!: For p, similar argument, Look at maximal dimension of a subspace on which φ is pos def. Same for q

Chap 8 Inner Product spaces $F = \mathbb{R}$ or $\mathbb{C}$

Def V v. space / $\mathbb{R}$ [or $\mathbb{C}$]. An inner product on V is a positive definite symmetric [Hermitian] form φ on V

Notation $\langle u, v \rangle$ for $\varphi(u,v)$

V is called a real [complex] inner product (IP) space and a Euclidean [unitary] space

Egs · dot product on $\mathbb{R}^n / \mathbb{C}^n$

· $V = \mathcal{C}([0,1], \mathbb{C})$, $w : [0,1] \to \mathbb{R}_{>0}$ cts weight f^n e.g. 1

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, w(t) \, dt$$

Length $\|v\| = \sqrt{\langle v, v \rangle}$, $\|v\| = 0 \iff v = 0$

$\|\cdot\|$ determines the whole inner product via polarisation identity

Lem (Cauchy – Schwartz) $|\langle u, v \rangle| \leq \|u\| \|v\| \quad \forall u, v$

Pf WLOG $u \neq 0$. For all $t \in F$,

$$0 \leq \|tu - v\|^2 = |t|^2 \|u\|^2 - t\langle u, v \rangle - \bar{t}\, \overline{\langle u, v \rangle} + \|v\|^2$$

Now set $t = \overline{\langle u, v \rangle} / \|u\|^2$ $\square$

Cor Triangle inequality $\|u + v\| \leq \|u\| + \|v\|$

Pf Square + C-S. $\square$

Cor $\|\cdot\|$ is a norm

Def Vectors $e_1, \dots, e_k$ in V are orthogonal if $\langle e_i, e_j \rangle = 0$ for $i \neq j$

orthonormal if $\langle e_i, e_j \rangle = \delta_{ij}$
(o, n)

Lem If $e_1, \dots, e_k$ are orthogonal, non-zero, then they lin indep

Moreover, if $v = \sum_{i=1}^{k} \lambda_i e_i$, then $\lambda_i = \dfrac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle}$.

Pf $\langle v, e_i \rangle = \left\langle \sum_{j=1}^{k} \lambda_j e_j, e_i \right\rangle = \lambda_i \langle e_i, e_i \rangle$. $\square$

Cor (Parseval's identity) V f. dim v. space w/ on basis $e_1, \dots, e_n$

Then $\langle u, v \rangle = \sum_{j=1}^{n} \langle u, e_j \rangle \overline{\langle v, e_j \rangle}$

Thm (Gram – Schmidt) V IP space and $v_1, \dots, v_n, \dots$ set of lin indep vectors (ctble). There's a sequence $e_1, e_2, \dots$ of o.n. vectors such that $\langle v_1, \dots, v_k \rangle = \langle e_1, \dots, e_k \rangle$ for all k.

Pf Induction on k · $k=1$ ✓ · Say we found $e_1, \dots, e_k$

$$e'_{k+1} = v_{k+1} - \underbrace{\sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle e_i}_{\substack{\in \operatorname{span}\{e_1,\dots,e_k\} \\ = \operatorname{span}\{v_1,\dots,v_k\}}}$$

· $\langle e'_{k+1}, e_i \rangle = 0$ for $i=1,\dots,k$ by construction

· $\operatorname{span}\{v_1, \dots, v_{k+1}\} = \operatorname{span}\{e_1, \dots, e_{k+1}\}$

· $e'_{k+1} \neq 0$ by lin indep of v_i. Set $e_{k+1} = \dfrac{e'_{k+1}}{\|e'_{k+1}\|}$. □

Cor V f. dim IP space. Any orthonormal set of vectors can be extended to an orthonormal basis

Pf Suppose $e_1, \dots, e_k$ o.n. Then they're lin indep.

So can extend to a basis $e_1, \dots, e_k, v_{k+1}, \dots, v_n$.

Now apply Gram-Schmidt to this set (no effect on first k vecs) □

Note $A \in M_{m,n}(\mathbb{R})$ has orthonormal columns if $A^T A = I$

$\mathbb{C}$ $\qquad A^\dagger A = I$

Def $A \in M_n(\mathbb{R})$ is orthogonal if $A^T A = I$ (iff $A^T = A^{-1}$)

$\mathbb{C}$ unitary if $A^T \bar{A} = I$ (iff $A^\dagger = A^{-1}$)

Prop $A \in M_n(\mathbb{R})$ [or $\mathbb{C}$] is non-singular, it can be written as

$A = RT$ where · T upper triangular

· R orthogonal [unitary]

Pf Apply Gram-Schmidt to columns of ~~(a)~~ A □

Orthogonal complements & projections

Def V an IP space, $V_1, V_2 \leq V$.

V is the orthogonal direct sum of V_1, V_2 if

1) $V_1 + V_2 = V$

2) $\forall v_1, v_2 \in V_1, V_2$, $\langle v_1, v_2 \rangle = 0$ ($\Rightarrow V_1 \cap V_2 = \{0\}$)

implies $V_2 = V_1^\perp$

Write $V = V_1 \perp V_2$ (or $V_1 \oplus V_2$)

Def $W \leq V$. $W^\perp = \{v \in V : \langle v, w \rangle = 0 \ \forall w \in W\} \leq V$

$W^\perp$ is the orthogonal complement of W in V

<u>Lem</u> V f. dim IP space, $W \leq V$, then $V = W \perp W^{\perp}$ (true for ∞ if W finite)

<u>Pf</u> · If $w \in W$, $u \in W^{\perp}$, then $\langle w, u \rangle = 0$ so 2) holds

· Remains to show 1) i.e. $V = W + W^{\perp}$. Let $e_1, \dots, e_k$ be an orthonormal basis for W, extend it to an orthonormal basis $e_1, \dots, e_n$ of V

Note $e_{k+1}, \dots, e_n \in W^{\perp}$ □

<u>Rk</u> Orthogonal complements are unique (cringe) Jeez poorly phrased

<u>Def</u> Suppose $V = U \oplus W$ (U is $\underline{a}$ complement of W in V)

$\pi : V \to W$ for all u, w · linear
$u + w \mapsto w$ · well def
· $\pi^2 = \pi$

π is $\underline{\underline{a}}$ projection from V to W (depends on U)

If $U = W^{\perp}$ (the orthogonal complement of W in V)

then π is the orthogonal projection onto W.

<u>Note</u> $\pi' = id - \pi$ is the othogonal projection of V onto $W^{\perp}$

<u>Lem</u> Let V be a IP space, $W \leq V$ f. dim subspace w/ orthonormal basis $e_1, \dots, e_k$, π the orthogonal projection onto W. Then

a) $\pi(v) = \sum_{i=1}^{k} \langle v, e_i \rangle e_i$

b) $\| v - \pi(v) \| \leq \| v - w \|$ $\forall v, \overset{V}{w} \in W$ with equality iff $w = \pi(v)$

($\pi(v)$ is closest point to v)

<u>Pf</u> a) Need to show $v - \sum_{i=1}^{k} \langle v, e_i \rangle e_i \in W^{\perp}$

Gets killed $\left\langle v - \sum_{i=1}^{k} \langle v, e_i \rangle e_i, e_j \right\rangle = 0$ $\forall j = 1, \dots, k$

b) $\| v - w \|^2 = \| \underbrace{\overset{v}{\cancel{w}} - \pi(v)}_{\in W^{\perp}} + \underbrace{\pi(v) - w}_{\in W} \|^2$

$= \| v - \pi(v) \|^2 + \| \pi(v) - w \|^2$

So done. □

Adjoint

Prop Let V, W f. dim. IP spaces, $\alpha \in L(V,W)$. Then there's a unique linear map $\alpha^*: W \to V$ s.t.

$$\langle \alpha(v), w \rangle = \langle v, \alpha^*(w) \rangle \quad \forall v \in V, w \in W.$$

If $\mathcal{B}$ is an orthonormal basis for V, $\mathcal{C}$ " W, then $[\alpha^*]_{\mathcal{C},\mathcal{B}} = \left(\overline{[\alpha]_{\mathcal{B},\mathcal{C}}}\right)^T$

α^* is called the adjoint of α

Pf Say $\mathcal{B} = (v_1, \dots, v_n)$ $\quad A = [\alpha]_{\mathcal{B},\mathcal{C}} = (a_{ij})$

$\mathcal{C} = (w_1, \dots, w_m)$

- Existence: say α^* is the linear map s.t. $[\alpha^*]_{\mathcal{C},\mathcal{B}} = \overline{A}^T = (c_{ij})$

$$\langle \alpha(\Sigma \lambda_i v_i), \Sigma \mu_j w_j \rangle = \langle \sum_{i,k} \lambda_i a_{ki} w_k, \Sigma \mu_j w_j \rangle$$
$$= \sum_{i,j} \lambda_i a_{ji} \overline{\mu_j}$$

$$\langle \Sigma \lambda_i v_i, \alpha^*(\Sigma \mu_j w_j) \rangle = \langle \Sigma \lambda_i v_i, \sum_{j,k} \mu_j c_{kj} v_k \rangle$$
$$= \sum_{i,j} \lambda_i \overline{c_{ij}} \overline{\mu_j} \quad \text{(have swap)}$$

By specialising the above to basis elements, uniqueness is clear. $\square$

Notation $A^\dagger = \overline{A}^T$

Caution Same notation (α^*) for adjoint & dual of α

Rk V, W real IP spaces, $\alpha \in L(V,W)$

$$\psi_{R,V}: V \xrightarrow{\simeq} V^* \qquad \psi_{R,W}: W \xrightarrow{\simeq} W^*$$
$$v \mapsto \langle \cdot, v \rangle \qquad w \mapsto \langle \cdot, w \rangle$$

Check the adjoint of α is given by

$$W \xrightarrow[\psi_{R,W}]{} W^* \xrightarrow{\text{dual}} V^* \xrightarrow[\psi_{R,V}^{-1}]{} V \qquad \text{checks out!}$$

Self-adjoint maps & isometries ($V = W$)

Def V an IP space, $\alpha \in L(V)$, α^* be its adjoint

Condition	Equivalent	Adjective for α
$\langle \alpha v, w \rangle = \langle v, \alpha w \rangle$ $\forall v, w$	$\alpha = \alpha^*$	self-adjoint — /$\mathbb{R}$: symm /$\mathbb{C}$: hermitian
$\langle \alpha v, \alpha w \rangle = \langle v, w \rangle$	$\alpha^* = \alpha^{-1}$	isometry — /$\mathbb{R}$: orthog /$\mathbb{C}$: unitary

<u>Pf</u> (of equivalence of second conditions)

($\Rightarrow$) $\|\alpha v\|^2 = \|v\|^2$, so α injective, can define α^{-1} ← look at basis

$\forall v, w, \ \langle v, \alpha^* w \rangle = \langle \alpha v, w \rangle = \langle v, \alpha^{-1} w \rangle$ so $\alpha^{-1} = \alpha^*$

($\Leftarrow$) $\langle \alpha v, \alpha w \rangle = \langle \alpha^* \alpha v, w \rangle = \langle v, w \rangle$ by inverse $\square$

<u>Rk</u> By polarisation identity, $\|\alpha v\| = \|v\| \ \forall v \in V$

$\iff$ α is an isometry

<u>Lem</u> V f. dim real [complex] IP space

- $\alpha \in L(V)$ is self-adjoint iff for any <u>orthonormal</u> basis β, $[\alpha]_\beta$ is symmetric (or Hermitian)
- α is an isometry iff ——— $[\alpha]_\beta$ is orthogonal (or Unitary)

<u>Pf</u> For an orthonormal β, $[\alpha^*]_\beta = \overline{[\alpha]_\beta}^T$

- self-adjoint $[\alpha]_\beta = \overline{[\alpha]}_\beta^T$ · isometry $[\alpha]_\beta^{-1} = \overline{[\alpha]}_\beta^T$ $\square$

<u>Def</u> V an f dim IP space

$F = \mathbb{R}$: $O(V) = \{\alpha \in L(V) : \alpha \text{ an isometry}\}$ ← group under composition

the orthogonal group of V

$F = \mathbb{C}$: $U(V) = \{\alpha \in L(V) : \quad " \quad \}$ the unitary group of V

<u>Lem</u> Let V be an IP space w/ orthonormal basis $(e_1, \dots, e_n)$

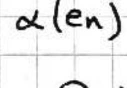
$F = \mathbb{R}$: $O(V) \longleftrightarrow \{\text{o.n. bases of } V\}$

$\alpha \longleftrightarrow \{\alpha(e_1), \dots, \alpha(e_n)\}$

$F = \mathbb{C}$: $U(V \longleftrightarrow \{\text{o.n. bases of } V\}$

$\alpha \longleftrightarrow \{\alpha(e_1), \dots, \alpha(e_n)\}$

<u>Spectral theory for self-adjoint maps</u>



<u>Lem</u> V IP space ← no dim assumptions

If $\alpha \in L(V)$ is self-adjoint, then a) α has real evalues
b) orthogonal evecs for diff evals

<u>Pf</u> a) Suppose $\alpha v = \lambda v$, $v \in V \setminus \{0\}$, $\lambda \in \mathbb{C}$.

Then $\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle \alpha v, v \rangle = \langle v, \alpha v \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$.

Since $v \neq 0$, $\lambda = \bar{\lambda}$ i.e. $\lambda \in \mathbb{R}$

Suppose $\alpha v = \lambda v$, $\alpha w = \mu w$ with $\lambda \neq \mu \in \mathbb{R}$

$$\lambda \langle v, w\rangle = \langle \lambda v, w\rangle = \langle \alpha v, w\rangle = \langle v, \alpha w\rangle = \langle v, \mu w\rangle = \bar{\mu}\langle v, w\rangle$$
$$= \mu \langle v, w\rangle$$

So $\lambda \neq \mu \Rightarrow \langle v, w\rangle = 0$. $\square$

<u>Thm</u> Let V be f. dim IP space, $\alpha \in L(V)$ self-adjoint. Then V has an orthonormal basis of eigenvectors of α.

<u>Pf</u> $F = \mathbb{R}$ or $\mathbb{C}$ Induction on $n = \dim V$

- $n = 0, 1$ ✓
- Inductive step. Say $[\alpha]_{\mathcal{B}} = A$ wrt some orthonormal basis.

By the fundamental thm of algebra, $\chi_A(t)$ has a root in $\mathbb{C}$.

So A, viewed as a complex matrix, has an eval. By previous Lemma, this eval is real. Thus for either field, α has an eval, say λ. Pick $v_1 \in V \setminus \{0\}$ s.t. $\alpha v_1 = \lambda v_1$.

if real, need a bit of justification to find evec (real part)

Let $U = \langle v_1 \rangle^{\perp} \leq V$. Want $\alpha(U) \leq U$.

If $u \in U$, $\langle \alpha u, v_1\rangle = \langle u, \alpha v_1\rangle = \lambda \langle u, v_1\rangle = 0$ so $\alpha u \in U$.

So $\alpha|_U$ makes sense, is self-adjoint. By induction hypothesis, $\exists$ o.n. basis for U of evecs of α, say $v_2, \dots, v_n$.

Then $\frac{v_1}{\|v_1\|}, \dots, v_n$ is an orthonormal basis of evecs of α. $\square$

<u>Cor</u> V f dim IP space. If $\alpha \in L(V)$ self-adjoint, then V is the orthogonal direct sum of its eigenspaces.

L19.1 (The End)

Spectral theory for unitary maps

Lem V complex IP space $\leftarrow$ no dim assumptions!

$\alpha \in L(V)$ unitary (i.e. isometry). Then

a) all evals of α lie on unit circle

b) evecs with diff. eval are orthogonal

Pf a) Say $\alpha v = \lambda v$, $v \neq 0$, $\lambda \neq 0$ as injective

$\lambda \langle v, v \rangle = \langle \alpha v, v \rangle \underset{\alpha^* = \alpha^{-1}}{=} \langle v, \lambda^{-1} v \rangle = \overline{\lambda^{-1}} \langle v, v \rangle \Rightarrow \lambda \bar{\lambda} = 1$

what if $v \notin \operatorname{im} \alpha$? eh, it doth

b) Say $\alpha v = \lambda v$, $\alpha w = \mu w$, $\lambda \neq \mu$

$\lambda \langle v, w \rangle = \langle \alpha v, w \rangle = \langle v, \alpha^{-1} w \rangle = \overline{\mu^{-1}} \langle v, w \rangle = \mu \langle v, w \rangle$

So if $\lambda \neq \mu$ then $\langle v, w \rangle = 0$ $\square$

Thm V f. dim. complex IP space. Let $\alpha \in L(V)$ be unitary.
Then V has an o.n. basis of ~~evals~~ evecs for α.

Pf By FTA, χ_α has at least one root, so α an eval λ.
Fix $v_1 \in V \setminus \{0\}$ s.t. $\alpha v_1 = \lambda v_1$ and $\|v_1\| = 1$. Let $U = \langle v_1 \rangle^\perp$.
For all $u \in U$, $\langle \alpha u, v_1 \rangle = \langle u, \alpha^{-1} v_1 \rangle = \overline{\lambda^{-1}} \langle u, v_1 \rangle = 0$.
$\Rightarrow \alpha(U) \leq U$ and can consider $\alpha|_U$
By induction (on dim V) U has an o.n. basis of evecs of $\alpha|_U$, say $v_2, \dots, v_n$. Then $v_1, \dots, v_n$ is what we wanted. $\square$

⚠ $A \in M_n(\mathbb{R})$ orthogonal need not be diagonalisable e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
However, may find o.n. basis wrt which A is block-diagonal with blocks of sizes ≤ 2.

Rmk Generalisation of $\begin{cases} \text{self-adjoint} \\ \text{unitary} \end{cases}$ $\alpha^* \alpha = \alpha \alpha^*$, normal map

Application to bilinear forms

Note For an orthogonal change of basis matrix P, then

$P^T A P = P^{-1} A P$ [similar for unitary]

<u>Cor</u> Let $A \in M_n(\mathbb{R})$ [$\mathbb{C}$] be a symmetric [Hermitian] matrix.
Then there's an orthogonal [unitary] matrix P s.t.
$P^T A P$ [$P^\dagger A P$] is diagonal w/ real entries

<u>Pf</u> $F = \mathbb{R}$ or $\mathbb{C}$. Let $\langle \ , \ \rangle$ be the standard inner product on F^n.
$A \in L(F^n)$ is self-adjoint, so there is an o.n. basis of F^n consisting of evecs of A, w/ real evals say $v_1, \dots, v_n$
Let $P = (v_1 | \cdots | v_n)$ orthogonal [unitary]
Then $P^{-1} A P = P^T A P$ [$P^\dagger A P$] is diagonal with real entries.

<u>Cor</u> V f. dim real [cx] IP space, $\varphi : V \times V \to F$ symm bilinear form [Hermitian]. Then there's an o.n. basis of V s.t. φ is represented by a diagonal matrix.

<u>Pf</u> Let $\mathcal{B} = (v_1, \dots, v_n)$ be any o.n. basis, $A = [\varphi]_{\mathcal{B}}$
$A = A^T$ [$A^\dagger$] and there's an orthogonal [unitary] matrix P s.t.
$P^T A P$ [$P^\dagger A P$] is diagonal, say D.
Let w_i be i^{th} row of P^T[$P^\dagger$]. Then for $(w_1, \dots, w_n)$, an o.n. basis $\mathcal{B}'$, we have $[\varphi]_{\mathcal{B}'} = D$.
w_i seeing v_i as st. basis $\square$

<u>Rk</u> diagonal entries of $P^T A P$ are evals of A

$s(\varphi) = \#$ +ve evals $- \#$ −ve evals
signature

<u>Cor</u> (Simultaneous diagonalisation) V f. dim real [cx] v. space
Let φ, ψ be symmetric bilinear [Hermitian] forms.
Assume φ is <u>+ve definite</u>. Then there's a basis $(v_1, \dots, v_n)$ of V wrt which both forms are represented by a diagonal matrix.

<u>Pf</u> V equipped with φ is an IP space. Thus there exists an orthonormal (wrt this inner product) basis wrt which ψ is represented by a diagonal matrix. Moreover, φ is represented by the identity $\ddot\smile$ $\square$

L19.3

<u>Cor</u> Suppose $A, B \in M_n(\mathbb{R})$ [$\mathbb{C}$] symmetric [Hermitian]

Suppose $\bar{x}^T A x > 0$ for all $x \neq 0$. Then there exists $Q \in M_n(\mathbb{R})$ [$\mathbb{C}$] invertible s.t.

$Q^T A Q$ [$Q^\dagger A Q$]

$Q^T B Q$ [$Q^\dagger B Q$] both diagonal

Michaelmas Term 2019

Linear Algebra

Not lectured compared with Michaelmas 2017:

1. Proof of the Cayley-Hamilton theorem over a general field. *Only examinable over* $\mathbb{C}$. *See ES3 Q6.*
2. Proof of the Generalised Eigenspace Decomposition theorem. *Usually lectured but labeled as non-examinable. See ES3 Q4(iii).*
3. Smaller remark about ideas used in the (non-examinable) proof of JNF. *See ES3 Q9, slightly rephrased from previously.*
4. Skew-symmetric bilinear forms. *Not in schedules, and not lectured e.g. by Dr. Wadsley in 2016. See ES4 Q4.*
5. Proof of the triangle inequality :)