SOME USEFUL REFERENCES

1. Recommended general textbooks

Many suitable texts relate to this mathematical methods course, but approaches vary (as well as the conventions—beware!). The following general textbooks are recommended because of their accessibility and strong overlap with the course.

Arfken, G., Weber, H.J., and Harris, F.E., *Mathematical Methods for Physicists* (Elsevier, 2012). The most suitable text for this course. Good coverage of almost all aspects with numerous worked examples. (Older editions in libraries by just Arfken should be fine.)

Riley, K., Hobson, M., and Bence, S.J., *Mathematical Methods for Physics and Engineering* (Cambridge, 2006). A useful and more elementary treatment of most material in the course.

Mathews, J. & Walker, R.L. *Mathematical Methods of Physics* (Benjamin, 1970). A readable account of the course material and much else besides.

Boas, M.L., *Mathematical Methods in the Physical Sciences* (Wiley, 2005). Very basic and simple coverage of all the topics in the course, except SL theory.

Kreyszig, E., *Advanced Engieering Mathematics* (Wiley, 2011).

2. Advanced and specialised texts

For those seeking greater depth, rigorous proofs, or further material on specific topics, then the following is a partial list of some other texts related to the course.

Jeffreys, H. & Jeffreys, B.S. *Mathematical Physics* (C.U.P., 1972). A rigorous and thoughtful approach to most of the course. There are many interesting applications to physics.

Sagan, H. *Boundary and Eigenvalue Problems in Mathematical Physics* (Wiley, 1990). Deals with Sturm-Liouville theory in a readable way with lots of examples.

Renardy, M. and Rogers, R. *An Introduction to Partial Differential Equations*, Springer (2004). A more advanced text going into much detail about the latter parts.

Körner, T., *Fourier Analysis*, Cambridge (1989). An advanced text on some aspects of the course with mathematical rigour and historical anecdote.

Courant, R. & Hilbert, D. *Methods of Mathematical Physics* (Vol 1., Springer, 1937). This classic text deals with eigenvalue problems and discusses existence theorems. Remarkable mathematical and physical insights abound.

3. Online Past Lecture Notes for Methods IB Course

Colm Caufield, Notes for Methods IB
`https://tinyurl.com/methods-caulfield`
The most recent presentation that covers all the material in a very readable way.

*David Skinner, Mathematical Methods, Part IB Lecture Notes
`http://www.damtp.cam.ac.uk/user/dbs26/1BMethods/All.pdf`
Towards a more rigorous exposition that will appeal to pure-minded students.

Richard Jozsa, IB Methods Lecture Notes
`http://www.damtp.cam.ac.uk/user/examples/B8La.pdf`
(This is Part I of the course and the remaining three are: B8Lb.pdf, B8Lc.pdf, B8Ld.pdf.) An excellent and very thorough write-up of all the material in the course.

Michaelmas term 2019 Professor E.P.S. Shellard

MATHEMATICAL METHODS IB

PART I - SELF-ADJOINT ODEs

1. Fourier series

- Periodic functions, properties of sine and cosine functions
- Definition of Fourier series, examples and Dirichlet conditions
- Complex and half-range Fourier series; Parseval's theorem

2. Sturm-Liouville theory

- Self-adjoint form of ODEs and boundary conditions
- Eigenfunction expansions, completeness and Parseval's identity
- Example: Legendre polynomials $P_n(x)$
- Inhomogeneous boundary value problem

PART II - PDEs ON BOUNDED DOMAINS & SEPARABILITY

3. The wave equation

- Waves on an elastic string; separation of variables and normal modes
- Oscillation energy; wave reflection and transmission
- Separation of variables in 2D polars; Bessel's equation and the vibrating drum

4. Diffusion equation

- Origin and applications; boundary conditions
- Separable solutions in cartesian and plane polar coordinates

5. Laplace's equation

- Applications and boundary conditions
- Separable solutions in cartesian and spherical polars (Legendre's equation)

PART III - INHOMOGENEOUS ODEs; FOURIER TRANSFORMS

6. Dirac Delta Function

- The Dirac Delta function $\delta(x)$ and generalized functions
- Fourier series and eigenfunction expansion of delta functions.

7. Green's functions

- Motivation: superposition for linear o.d.e.'s
- Green's function definition $G(x, \xi)$
- Constructing $G(x, \xi)$: boundary & initial value problems

8. Fourier transforms

- Fourier transforms and connection to Fourier series
- Definition, inverse, properties and convolution theorem
- Application to linear ODE solution, transfer functions
- Discrete Fourier transform

PART IV - PDEs ON UNBOUNDED DOMAINS

9. Characteristics

- Well-posed PDEs and introduction to characteristics
- Classification of second-order PDEs; solution by method of characteristics

10. Green's function solution of PDEs

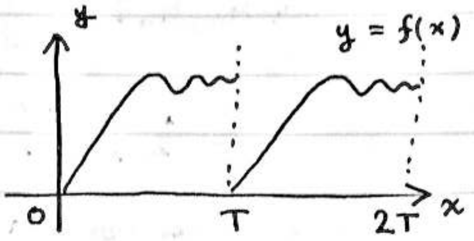
- Green's functions for inhomogeneous PDEs with several variables
- Examples: Fourier transform for diffusion equation, forced wave equation
- Poisson equation, Green's functions, and the method of images

Methods IB

Part 1: Self-adjoint ODEs

§1. Fourier Series

1.1 Periodic Functions



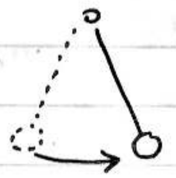
A function $f(x)$ is periodic if

$$\forall x, \ f(x+T) = f(x),$$

where T is the period.

Example: Simple harmonic motion

$$y = A \sin \omega t$$



where A is the amplitude and the period is $T = 2\pi/\omega$, with angular frequency ω (freq $f = 1/T$).

[In space, often refer to wavelength $\lambda = 2\pi/k$ and (ang.) wavenumber $k = 2\pi/\lambda$]

Properties of sine and cosine functions:

Consider the set of functions

$$g_n(x) = \cos(n\pi x/L), \quad h_n(x) = \sin(n\pi x/L), \quad 0 \leq n$$

which are periodic on the interval $0 \leq x \leq 2L$.

Recall the identities

$$\cos A \cos B = \tfrac{1}{2}(\cos(A-B) + \cos(A+B)),$$

$$\sin A \sin B = \tfrac{1}{2}(\cos(A-B) - \cos(A+B)),$$

$$\sin A \cos B = \tfrac{1}{2}(\sin(A+B) + \sin(A-B)).$$

Define an inner product $\langle f, g\rangle$ for two periodic functions f, g on the interval $0 \leq x < 2L$ as:

$$\langle f, g \rangle = \int_0^{2L} f(x) g(x) \, dx \ . \qquad (*)$$

The functions g_n, h_n are mutually orthogonal wrt $(*)$.

$$\langle h_n, h_m \rangle = \int_0^{2L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_0^{2L} \left(\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right) dx$$

$$= \frac{1}{2} \frac{L}{\pi} \left[\frac{\sin((n-m)\pi x/L)}{n-m} - \frac{\sin((n+m)\pi x/L)}{n+m} \right]_0^{2L}$$

$$= 0, \quad \text{for all } n \neq m.$$

For $n=m$,

$$\langle h_n, h_n \rangle = \int_0^{2L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^{2L}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$

$$= L \quad \text{for } n \neq 0.$$

Hence, $$\langle h_n, h_m \rangle = \begin{cases} L\delta_{mn} & \text{for } mn \neq 0, \\ 0 & \text{for } mn = 0. \end{cases} \qquad (1.1)$$

Similarly (exercise), (1.2)

$$\langle g_n, g_m \rangle = \int_0^{2L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\, dx = \begin{cases} L\delta_{mn} & \text{for } mn \neq 0, \\ 2L\delta_{mn} & \text{for } mn = 0 \end{cases}$$

$$\langle h_n, g_m \rangle = \int_0^{2L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\, dx = 0 \quad \forall\, n, m \qquad (1.3)$$

We assert that the g_n, h_n form a complete orthogonal set which spans the space of periodic f^{n}s on $0 \leq x < 2L$.

1.2 Definition of Fourier series

We can express any "well-behaved" periodic f^n $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (1.4)$$

where a_n, b_n are constants such that the RHS is convergent to $f(x)$ $\forall\, x$, where f is cts.

At a discontinuity, the fourier series approaches the midpoint

$$\frac{1}{2}\Big(\underbrace{f(x_+)}_{\text{limit above}} + \underbrace{f(x_-)}_{\text{limit below}}\Big)$$ (use this to replace LHS)

Fourier coefficients a_n, b_n

Consider $$\int_0^{2L} f(x)\cdot \sin\frac{m\pi x}{L}\, dx,$$

and substitute (1.4) into f.

This gives $$\sum_{n=1}^{\infty} L b_n \delta_{mn} = L b_m.$$

Thank the orthogonality relations.

Hence, we find

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cdot \sin\frac{n\pi x}{L}\,dx,$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cdot \cos\frac{n\pi x}{L}\,dx. \qquad (1.5)$$

Notes (i) a_n includes $n=0$, since $\frac{1}{2}a_0$ is the average

$$\langle f(x)\rangle = \frac{1}{2L}\int_0^{2L} f(x)\,dx.$$

(ii) range of integration is one period, so $\int_0^{2L} dx = \int_{-L}^{L} dx \ldots$

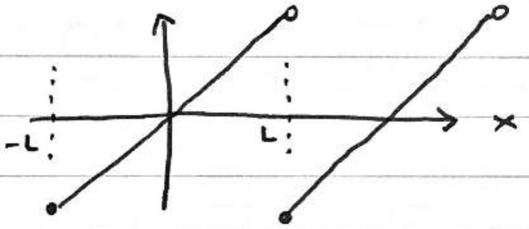
(iii) think of FS (1.4) as a decomposition into harmonics.

Simplest Fourier Series are sine and cosine functions.

Pure mode $\sin\frac{3\pi x}{L}$ has $b_3 = 1$, all other $a_n, b_n = 0$.

Classic example: Sawtooth wave

Consider $f(x) = x$ for $-L \leq x < L$ and periodic elsewhere



Here, we have

$$a_n = \frac{1}{L}\int_{-L}^{L} \underset{\substack{\uparrow \\ \text{odd}}}{x} \cos\Big(\underset{\substack{\uparrow \\ \text{even}}}{\frac{n\pi x}{L}}\Big)\,dx$$

$$= 0 \text{ for all } n.$$

$$b_n = \frac{2}{L}\int_{-L}^{L} x \sin\Big(\frac{n\pi x}{L}\Big)\,dx$$

L 2.1

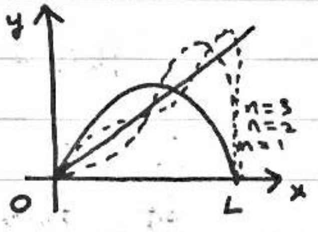
Example "sawtooth" (cont.)

$f(x) = x$ odd $\Rightarrow a_n = 0$

$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{L}\right]_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{2L}{n\pi}\cos n\pi = \frac{2L(-1)^{n+1}}{n\pi}$$



So "sawtooth" Fourier series

$$f(x) = \frac{2L}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} \qquad (1.6)$$

$$= \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{\pi 2x}{L} + \frac{1}{3}\sin\frac{\pi 3x}{L} - \dots\right)$$

which is slowly convergent.

Comments As n increases ($n \to \infty$):

(i) FS approx. improves (convergent where cts)

(ii) FS $\to 0$ at $x = L$, ie. midpnt of discty

(iii) FS has a persistent overshoot at $x = L$ (approx 9% known as Gibbs phenomenon, see Ex 1 Q5)

1.3 The Dirichlet conditions (Fourier's theorem)

Sufficiency conditions for a well-behaved f^n f to have a unique FS (1.4 – 5).

If f is a bounded periodic f^n (period $2L$) with a finite number of minima, maxima and discontinuities in $0 \le x < 2L$, then the FS converges to $f(x)$ at all points where f is cts; at discts. pts the series converges to the midpoint.

Notes (i) These are weak conditions (contrast Taylor series) but pathological functions are excluded, like $1/x$, $\sin(1/x)$, $\mathbb{1}_{\mathbb{Q}}$.

(ii) Converse is not true (e.g. $\sin 1/x$ has a FS)

(iii) Proof is difficult (see Jeffreys & Jeffreys)

Convergence of FS

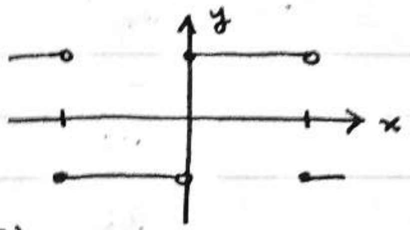
Rate of convergence depends on smoothness.

L2.2

<u>Thm</u> If f has cts derivatives up to $f^{(p)}$, which is discontinuous, then the FS converges at $\mathcal{O}(n^{-(p+1)})$ as $n \to \infty$

<u>Example (p = 0)</u> "Square" wave (see Ex1 Q5 here $L = 1$)

If $f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 1, & 0 \leq x < 1, \end{cases}$



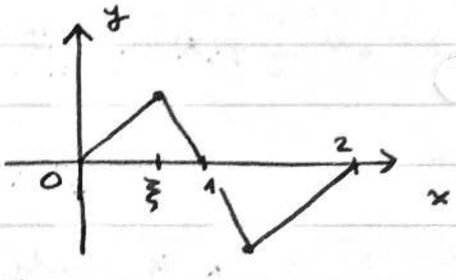
then FS is

$$f(x) = 4 \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi} \qquad (1.7)$$

<u>Exercise</u> (p = 1): General "see-saw" waves.

If $f(x) = \begin{cases} x(1-\xi) & 0 \leq x < \xi, \\ \xi(1-x) & \xi \leq x < 1, \end{cases}$

and odd for $-1 \leq x < 0$.



Show FS is

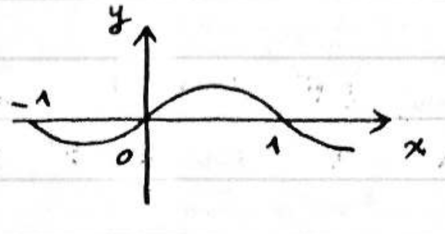
$$f(x) = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi\xi \sin n\pi x}{(n\pi)^2} \qquad (1.8)$$

<u>Exercise</u> (p = 2): Take $f(x) = \frac{1}{2}x(1-x) \qquad 0 \leq x < 1$

Show FS is (and odd else)

$$f(x) = 4 \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3} \qquad (1.9)$$

Why $\mathcal{O}(1/n^3)$?



<u>Example</u> (p = 3): $f(x) = (1-x^2)^2$ (see Ex1 Q1)

with FS $a_n = \mathcal{O}(1/n^4)$.

<u>Integration of FS</u>

It is always valid to integrate the FS (1.4) of $f(x)$ term-by-term to obtain $F(x) = \int_{-L}^{x} f(x)\,dx$, because F satisfies the Dirichlet conds (DCs) if f does (e.g. jump discts in f become cts in F).

<u>Differentiation of FS</u>

Take care with term-by-term diff

<u>Counter-example</u> Take square wave FS (1.7) and find

L2.3

$$f'(x) \overset{?}{=} 4 \sum_{m=1}^{\infty} \cos(2m-1)\pi x$$

which is divergent!

Theorem If f is cts and satisfies DCs, and f' also satisfies DCs, then f' has a FS obtain by differentiating that for f term-by-term.

1.4 Parseval's Theorem

Relation between integral of the square of a function and the square of its Fourier coeffs:

$$\int_0^{2L} [f(x)]^2 \, dx = \int_0^{2L} \left[\tfrac{1}{2}a_0 + \sum_n a_n \cos(\;) + \sum_n b_n \sin(\;) \right]^2 dx$$

$$= \int_0^{2L} \left[\tfrac{1}{4}a_0^2 + \sum_n a_n^2 \cos^2(\;) + \sum_n b_n^2 \sin^2(\;) \right] dx$$

by orthogonality (1.1-3)

$$= L \left[\tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \qquad (1.1\underset{0}{\cancel{x}})$$

Also called the completeness relation because LHS $\geq$ RHS if any base f^{n}'s are missing.

Example "Sawtooth" wave $f(x) = x, \; -L \leq x < L$

$$\text{LHS} = \int_{-L}^{L} x^2 \, dx = \frac{2}{3} L^{\cancel{2}3}$$

$$\text{RHS} = L \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} = \frac{4L^{\cancel{2}3}}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

PogChamp

L3.1

1.5 Alternative Fourier Series

Complex representation

$$\sin\frac{n\pi x}{L} = \frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)$$

$$\cos\frac{n\pi x}{L} = \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right)$$

so the FS becomes

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}. \qquad (1.11)$$

Here for $n>0$,

$$c_n = \tfrac{1}{2}(a_n - ib_n) \qquad c_{-n} = \tfrac{1}{2}(a_n + ib_n)$$

$$c_0 = \tfrac{1}{2}a_0.$$

Or, equivalently

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx. \qquad (1.11a)$$

Orthogonal: $\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L}\,dx = 2L\,\delta_{mn}$

Parseval: $\int_{-L}^{L}[f(x)]^2\,dx = 2L\sum_{n=-\infty}^{\infty}|c_n|^2$

Half-range series

Consider f defined only on $0 \leq x < L$. We extend its range over $-L \leq x < L$ in two simple ways:

(i) require to be odd $f(-x) = -f(x)$ (with period $2L$)

Then $a_n = 0$ (cosine even),

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (1.12)$$

e.g. sawtooth

So $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad (1.13)$$

which is a Fourier sine series.

(ii) or require even $f(-x) = f(x)$

Then $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad (1.14)$$

L3.2

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad (1.15)$$

1.6 Some FS motivations

Self-adjoint matrices

Suppose $\underline{u}, \underline{v}$ are complex N-vectors with inner product

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}^{\dagger} \underline{v} \qquad (1.16)$$

complex conj + transpose

The $N \times N$ matrix A is self-adjoint (or Hermitian) if

$$\langle A\underline{u}, \underline{v} \rangle = \langle \underline{u}, A\underline{v} \rangle \quad \forall\, \underline{u}, \underline{v} \quad \text{i.e. } A^{\dagger} = A$$

The eigenvalues λ_n and eigenvectors satisfying

$$A\underline{v}_n = \lambda_n \underline{v}_n \qquad (1.17)$$

have the following properties

(i) evals are real $\lambda_n^* = \lambda_n$
(ii) if evals are different, the evecs are orthogonal
(iii) we can rescale to make an orthonormal basis $\{\underline{v}_1, \dots, \underline{v}_N\}$

Given $\underline{b}$, we can solve for $\underline{x}$ in

$$A\underline{x} = \underline{b} \qquad (1.18)$$

Express $\underline{b} = \sum_{i=1}^{N} b_n \underline{v}_n$ (known) in evec basis.
Seek sol^n $\underline{x} = \sum_{i=1}^{N} c_n \underline{v}_n$ (unknown) substitute into (1.18) to find

$$A\underline{x} = \sum_{i=1}^{N} A c_n \underline{v}_n = \sum_{i=1}^{N} c_n \lambda_n \underline{v}_n \overset{?}{=} \sum_{i=1}^{N} b_n \underline{v}_n$$

By ((orthogonality)) $c_n \lambda_n = b_n \Rightarrow c_n = b_n / \lambda_n$.
So solution is

$$\underline{x} = \sum_{n=1}^{N} \frac{b_n}{\lambda_n} \underline{v}_n \qquad (1.19)$$

Solving inhomogeneous ODEs with FS

We wish to find $y(x)$ given $f(x)$

$$\mathcal{L} y \equiv -\frac{d^2 y}{dx^2} = f(x) \qquad (1.20) \qquad \text{with bcs } y(0) = y(L) = 0$$

driving term

L3.3

The related eigenvalue problem

$$\mathcal{L} y_n = \lambda_n y_n \quad , \quad y(0) = y(L) = 0$$

has eigenfunctions & eigenvalues

$$y_n = \sin\left(\frac{n\pi x}{L}\right) \quad , \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad (1.21)$$

Seek sol^{n}s as half-range sine series?

Try $y(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$ (unknown)

Expand $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ (known) with (1.12)

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx .$$

Substitute into (1.20)

$$\mathcal{L} y = -\frac{d^2}{dx^2} \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$
$$= \sum_{n=1}^{\infty} c_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L}$$
$$\overset{?}{=} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

By orthogonality (1.1-3) we have

$$c_n \left(\frac{n\pi}{L}\right)^2 = b_n \quad \text{i.e.} \quad c_n = \frac{b_n}{\lambda_n} .$$

So solution is

$$y(x) = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} y_n \qquad (1.22)$$

Example: 'Square' wave source ($L=1$)

$f(x) = 1, \quad 0 \le x < 1$ (odd f^n)

This has FS (1.7)

$$f(x) = \sum_{m=1}^{\infty} 4 \frac{\sin(2m-1)\pi x}{(2m-1)\pi}$$

So the sol^n (1.22) should be (with odd $n = 2m-1$)

$$y(x) = \sum_n \frac{b_n}{\lambda_n} y_n = 4 \sum_m \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3}$$

This has FS (1.9) for

$$y(x) = \tfrac{1}{2} x (1-x) \qquad (1.23)$$

<u>Exercise</u> Integrate $\mathcal{L} y = 1$ directly with b.c. $y(0) = y(1) = 0$ to verify sol^n (1.23).

<u>General integral solution</u>

Consider sol^n (1.22) with def^n (1.12) from b_n $(L = 1)$

$$y(x) = \sum_n b_n/\lambda_n \, y_n = \Sigma_n \frac{b_n}{n^2\pi^2} \sin^2 n\pi x$$

$$= \Sigma_n \frac{2}{(n\pi)^2} \int_0^1 f(x') \sin(n\pi x') \, dx' \, \sin(n\pi x)$$

use $x' = \xi$

$$= \int_0^1 2 \sum_n \frac{\sin(n\pi x)\sin(n\pi\xi)}{(n\pi)^2} f(\xi) \, d\xi$$

$$\equiv \int_0^1 \underset{\substack{\uparrow \\ \text{Green's} \\ \text{function}}}{G(x,\xi)} \, \underset{\substack{\uparrow \\ \text{source}}}{f(\xi)} \, d\xi \qquad (1.24)$$

where $G(x,\xi) = 2 \sum_{n=1}^{\infty} \dfrac{\sin(n\pi x)\sin(n\pi\xi)}{(n\pi)^2}$ (1.25)

But this FS (1.8) so the Green's f^n for (1.20)

$$\mathcal{L} y = -\frac{d^2y}{dx^2} = f(x)$$

is

$$G(x,\xi) = \begin{cases} x(1-\xi) & \text{for } 0 \le x \le \xi, \\ \xi(1-x) & \text{for } \xi \le x \le 1. \end{cases} \qquad (1.26)$$

L4.1

2. Sturm-Liouville Theory

2.1 Review of second-order linear ODEs

We wish to solve the general inhomogeneous ODE

$$\mathcal{L}y \equiv \alpha(x)y'' + \gamma(x)y + \beta(x)y' = f(x) \qquad (2.1)$$

· The homogeneous eq^{n} $\boxed{\mathcal{L}y = 0 \quad (2.2)}$ has two indep. sol^{n}s $y_1(x)$, $y_2(x)$ (besides trivial $y \equiv 0$), the complementary f^{n} y_c is general solution of (2.2) $y_c(x) = Ay_1(x) + By_2(x)$ (2.3) where A, B are arbitrary constants.

· The inhomogeneous eq^{n} $\boxed{\mathcal{L}y = f(x) \quad (2.4)}$ i.e. with driving force our source term $f(x)$ has a particular solution y_P, called the particular integral. The general sol^{n} of (2.4) is then

$$y(x) = y_P(x) + Ay_1(x) + By_2(x) \qquad (2.5)$$

· Two boundary or initial data are required to determine A, B in (2.5)

· (a) Boundary conditions Solve (2.4) on $a < x < b$ given y at $x = a, b$ – Dirichlet.

Neumann specifies y' at $x = a, b$

or mixed $y + ky' = 0$ etc.

Homogeneous b.c.s are often assumed, $y(a) = y(b) = 0$ (so admit trivial sol^{n} $y \equiv 0$)

Can be achieved by adding compl f^{n} (2.3)

$$\tilde{y} = y + Ay_1(x) + By_2(x)$$

s.t. $\tilde{y}(a) = \tilde{y}(b) = 0$.

OR (b) Initial conditions Solve (2.4) for $x \geq a$, given y, y' at $x = a$.

General eigenvalue problem

To solve (2.1) employing eigenf^{n} expansions (like FS (1.22)) we must first solve the related eval problem

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y = -\lambda\rho(x)y \qquad (2.6)$$

with specified bcs. This form often occurs after separation of variables in several dimensions.

L4.2

2.2 Self-adjoint operators

Inner product For two (complex valued) f^ns f, g on $a \le x \le b$, define as

$$\langle f, g \rangle = \int_a^b f^*(x)\, g(x)\, dx$$

Norm $\|f\| = \langle f, f \rangle^{1/2}$

(Later, we'll assume real f, g and drop cx conj.)

Sturm-Liouville equation

The eigenval problem (2.6) greatly simplifies if $\mathcal{L}$ is self-adjoint, i.e. it can be expressed in Sturm-Liouville form

$$\boxed{\mathcal{L} y \equiv -(p y')' + q y = \lambda w y \qquad (2.7)}$$

where the weight function $w(x)$ is non-negative $\forall x$.

Converting to SL form: Multiply (2.6) by an integrating factor $F(x)$ to find $F\alpha y'' + F\beta y' + F\gamma y = -\lambda F \rho y$.

$$\frac{d}{dx}(F\alpha y') - F'\alpha y' - F\alpha' y' + F\beta y' + F\gamma y = -\lambda F \rho y$$

Eliminate y' term $F'\alpha = F(\beta - \alpha')$

$$\therefore \frac{F'}{F} = \frac{\beta - \alpha'}{\alpha} \quad \text{so} \quad \boxed{F(x) = \exp \int^x \frac{\beta - \alpha'}{\alpha}\, dx \qquad (2.8)}$$

and $(F\alpha y')' + F\gamma y = -\lambda F \rho y$.

So $p(x) = F(x)\,\alpha(x)$; $q(x) = -F(x)\,\gamma(x)$ and
$w(x) = F(x)\,\rho(x)$ (note $F(x) \geq 0$)

Example Put Hermite eq^n for SHO

$y'' - 2xy' + 2ny = 0$ into SL form (2.7)

Compare (2.6) $\alpha = 1, \beta = -2x, \gamma = 0, \lambda\rho = 2n$

By (2.8) $F = \exp \int^x \frac{-2x-1}{1}\, dx = e^{-x^2}$.

Hence $\mathcal{L}y \equiv (-e^{-x^2} y')' = 2n e^{-x^2} y$ (2.9)

Self-adjoint definition

$\mathcal{L}$ is self-adjoint on $a \le x \le b$ for all pairs of y_1, y_2 satisfying appropriate bcs if

$\langle y_1, \mathcal{L} y_2 \rangle = \langle \cancel{y} \mathcal{L} y_1, y_2 \rangle$ or

L4.3

$$\boxed{\int_a^b y_1^{(*)}(x)\mathcal{L}y_2(x)\,dx = \int_a^b (\mathcal{L}\,y_1(x))^* y_2(x)\,dx \qquad (2.10)}$$

Boundary conditions: Substitute SL form (2.7) into (2.10) to find $\langle y_1, \mathcal{L}y_2\rangle - \langle \mathcal{L}y_1, y_2\rangle =$

$$= \int_a^b \left[-y_1(py_2')' + \cancel{qy_1y_2} + y_2(py_1')' - \cancel{qy_2y_1} \right] dx$$

ADD $-y_1'py_2' + y_1'py_2'$

$$= \int_a^b \left[-(py_1y_2')' + (py_1'y_2)' \right] dx$$

$$= \left[-py_1y_2' + py_1'y_2 \right]_a^b = 0 \qquad (2.11)$$

for given b.c.s at $x = a, b$

Self-adjoint compatible bcs include

- Hom $y(a) = y(b) = 0$ or $y'(a) = y'(b) = 0$ or mixed $y + ky' = 0$
- Periodic $y(a) = y(b)$, $y'(a) = y'(b)$
- Singular pts of ODE $p(a) = p(b) = 0$
- Appropriate combinations of the above

2.3 Properties of self-adjoint operators

1. Eigenvals are real
2. Eigenf^{n}s are orthogonal
3. Eigenf^{n}s are complete

Wronsk

Properties of self-adjoint operators (cont.)

1. Eigenvals λ_n are real
2. Eigenf^n's y_n are orthogonal
3. Eigenf^n's y_n form a complete set

1 Real eigenvalues

Given $\mathcal{L}y_n = \lambda w y_n$ (2.12) take c.c. $\mathcal{L}y_n^* = \lambda^* w y_n^*$.

Consider $\int_a^b (y_n^* \mathcal{L} y_n - y_n \mathcal{L} y_n^*)\, dx = (\lambda_n - \lambda_n^*) \int_a^b w y_n y_n^*\, dx$.

But LHS $= 0$ by self-adjoint (2.10) on RHS $\int w |y_n|^2 dx \geq 0$ so $\lambda_n = \lambda_n^*$ real.

(If λ_n non-degenerate / simple then $y_n^* = y_n$ real) ⊗ Doubt

Exercise* Prove that the regular SL problem (hom b.c.s) has simple eigenvalues (i.e. with a unique eigenf^n)

Hint Consider 2 eigenf^n's u, v for the same λ and use

$$u \mathcal{L} v - (\mathcal{L}u) v = [\underbrace{-p(uv' - u'v)}_{\text{Wronskian}}]'$$

2. Orthogonal eigenf^n's

Consider (2.12) with a 2nd eigenval $\lambda_m \neq \lambda_n$,

$$\mathcal{L} y_m = \lambda_m w y_m .$$

Then $\int_a^b (y_m \mathcal{L} y_n - y_n \mathcal{L} y_m)\, dx = (\lambda_n - \lambda_m) \int_a^b w y_n y_m\, dx$

is zero by (2.10).

But since $\lambda_n \neq \lambda_m$, $\boxed{\int_a^b w y_n y_m\, dx = 0 \quad \forall\, n \neq m. \quad (2.13)}$

So y_n, y_m are orthogonal wrt $w(x)$ on $a \leq x \leq b$.

Define inner product wrt weight $w(x)$ on $a \leq x \leq b$

$$\langle f, g \rangle_w = \int_a^b w(x) f(x)^* g(x)\, dx \quad (2.14)$$

$$= \langle wf, g \rangle = \langle f, wg \rangle$$

So orthog rel^n (2.13) becomes $\langle y_n, y_m \rangle_w = 0 \quad \forall\, n \neq m$ (2.15)

Aside : Watch the weight!

We can eliminate $w(x)$ by redefining

$$\tilde{y} = \sqrt{w}\, y \quad \text{and replacing } \mathcal{L}y \text{ by } \frac{1}{\sqrt{w}}\mathcal{L}\left(\frac{\tilde{y}}{\sqrt{w}}\right) = \tilde{\mathcal{L}}(\tilde{y})$$

but it's generally simpler to keep it!

Exercise For the Hermite eq^{n} (2.9) eliminate $w(x)$ with $\tilde{y} = e^{-x^2/2}y$ to find

$$\tilde{\mathcal{L}}\tilde{y} \equiv -\tilde{y}'' + (x^2-1)\,\tilde{y} = 2n\tilde{y}$$

3. Eigenf^{n} expansions

Completeness (not proven here) implies we can approximate any well-behaved f^{n} $f(x)$ on $a \leq x \leq b$ by the series

$$\boxed{f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \qquad (2.16)} \qquad \text{c.f (1.4)}$$

To find expansion coeffs do

$$\int_a^b dx\ w(x)\, y_n(x)\, f(x) = \sum_{m=1}^{\infty} a_m \int_a^b dx\ w(x)\, y_n(x)\, y_m(x)$$

$$= a_n \int_a^b dx\ w(x)\, y_n^2(x).$$

Hence

$$\boxed{a_n = \frac{\int_a^b dx\ w y_n f}{\int_a^b dx\ w y_n^2} \qquad (2.17)}$$

Eigenf^{n}s are normalised for convenience

Unit norm has $Y_n(x) = y_n(x) \big/ \left(\int_a^b w y_n^2\, dx\right)^{1/2}$

so orthonormal $\langle Y_n, Y_m \rangle_w = \delta_{mn}$ (2.18)

with $f(x) = \sum_{n=1}^{\infty} A_n Y_n(x)$.

Example I Recall FS (1.4) SL form

$$\mathcal{L}y_n \equiv -\frac{d^2}{dx^2} y_n = \lambda_n y_n \qquad (1.21)$$

with $\lambda_n = \left(\frac{\pi n}{L}\right)^2$ & orthog rel^{n} (1.1-3).

2.4 Completeness and Parseval's identity (compare FS (1.10))

Consider $\int_a^b [f(x) - \sum a_n y_n]^2 w \, dx$

$$= \int_a^b [f^2 - 2f \sum a_n y_n + \sum \underset{\nwarrow \text{by orthog}}{a_n^2 y_n^2}] w \, dx$$

$$= \int_a^b f^2 w \, dx - \sum a_n^2 \int w y_n^2 \, dx$$

because (2.17) $\Rightarrow \int f w y_n \, dx = a_n \int w y_n^2 \, dx$.

If the eigenf^{n}s are complete then series converges

$$\int_a^b w f^2 dx = \sum_{n=1}^{\infty} a_n^2 \int w y_n^2 \, dx \quad (2.19)$$

$$= \sum_{n=1}^{\infty} A_n^2 \quad (\text{for unit norm } Y_n \ (2.18))$$

Bessel's inequality if some eigenf^{n}s are missing

$$\int_a^b w f^2 dx \geqslant \sum_{n=1}^{\infty} A_n^2$$

Define partial sum $S_N(x) = \sum_{n=1}^{N} a_n y_n$ with $f = \lim_{N\to\infty} S_N(x)$. (2.20)

Mean square error

$$\varepsilon_N = \int_a^b w [f - S_N]^2 dx \xrightarrow{?} 0 \quad \text{as} \quad N \to \infty$$

This is a global def^n of convergence in the mean (unlike pointwise for FS)

Error in partial sum (2.20) is minimised by a_n (2.19) for $N = \infty$ expansion.

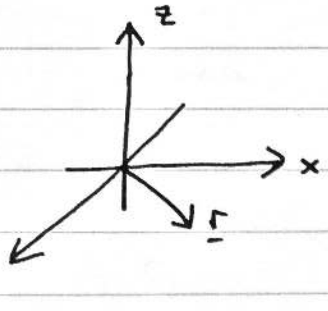
$$\frac{\partial \varepsilon_W}{\partial a_n} = -2 \int_a^b y_n [f - \sum_{m=1}^{N} a_m y_m] w \, dx$$

$$= -2 \int_a^b (w f y_n - a_n w y_n^2) \, dx$$

vanishes if a_n is given by (2.17)

More than just extremal

$$\frac{\partial^2 \varepsilon_N}{\partial a_n} = \int_a^b w y_n^2 dx > 0 .$$



L6.1

Exemplar II : Legendre polynomials

Consider Legendre's equation (arising from spherical polars $x = \cos\theta$)

$$(1-x^2)\,y'' - 2xy' + \lambda y = 0 \qquad (2.21)$$

on the interval $-1 \le x \le 1$ with y finite at $x = \pm 1$

(regular singular point of the ODE)

Eq^n (2.21) is in SL form (2.7) with

$$p = 1 - x^2, \quad q = 0, \quad w = 1$$

How to solve ? Seek power series about $x = 0$

$$y = \sum_n c_n x^n$$

Substitute

$$(1-x^2)\sum_n c_n n(n-1)x^{n-2} - 2x\sum_n c_n n x^{n-1} + \lambda \sum_n c_n x^n$$

Equate powers of x^n

$$(n+2)(n+1)\,c_{n+2} - n(n-1)\,c_n - 2nc_n + \lambda c_n = 0$$

$$\Rightarrow \quad c_{n+2} = \frac{(n+1)\cdot n - \lambda}{(n+1)(n+2)}\,c_n \qquad (2.22)$$

So specify c_0, c_1 to give 2 indep sol^{n}s (near $x = 0$)

$$y_{even} = c_0\left[1 + \frac{(-\lambda)}{2!}x^2 + \frac{(6-\lambda)(-\lambda)}{4!}x^4 + \cdots\right]$$

$$y_{odd} = c_1\left[x + \frac{2-\lambda}{3!}x^3 + \cdots\right]$$

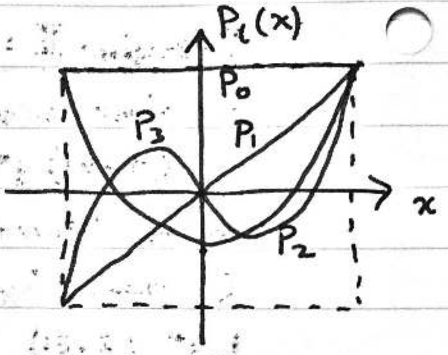
But as $n \to \infty$, $c_{n+2} \sim c_n$ so these are geometric series with radius of convergence $x = 1$.

So are divergent at $x = \pm 1$.

What can be done ? Finiteness.

Take $\lambda = l(l+1)$ with l integer. Then one of the series will terminate i.e. $c_n = 0 \;\; \forall\, n \geqslant l + 2$.

These are the <u>Legendre polynomials</u>, eigenf^{n}s of (2.21) on $-1 \le x \le 1$, with <u>normalisation</u> convention that $P_l(1) = 1$ (not unit norm)



$l=0$, $\lambda=0$, $P_0(x)=1$
$l=1$, $\lambda=2$, $P_1(x)=x$
$l=2$, $\lambda=6$, $P_2(x)=(3x^2-1)/2$
$l=3$, $\lambda=12$, $P_3(x)=(5x^3-3x)/2$

Notes · $P_l(x)$ has l zeros
· $P_l(x)$ is even or odd (à la l)

Alternative def^n (Generating function) – see later

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

$$= 1 + \tfrac{1}{2}(2xt - t^2) + \tfrac{3}{8}(2xt - t^2)^2 + \dots$$

$$= 1 + xt + \tfrac{1}{2}(3x^2-1)t^2 + \dots \qquad (2.23a)$$

Ex verify P_3, find P_4

Orthogonality

$$\int_{-1}^{1} P_n P_m \, dx = 0 \quad \text{for} \quad n \neq m$$

Normalisation

$$\int_{-1}^{1} P_n^2 \, dx = \frac{2}{2n+1} \qquad (2.24)$$

(Prove with Rodrigue's formula $P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$)

Eigenf^n expansion

Any $f(x)$ on $-1 \leq x \leq 1$ can be expressed as

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x) \qquad (2.25)$$

where $$a_l = \frac{2l+1}{2}\int_{-1}^{1} f(x) P_l(x)\, dx \qquad (2.26).$$

Exercise Verify explicitly from (2.26) that

$$f(x) = \frac{15}{2}x^2 - \frac{3}{2} = P_0(x) + 5P_2(x)$$

2.6 SL theory & inhom ODE

Consider the inhom prob (with hom bcs) on $a \leq x \leq b$

$$\mathcal{L}y = f(x) \equiv w(x)\, F(x) \qquad (2.27)$$

Given eigenf^{n}s $y_n(x)$ satisfying $\mathcal{L} y_n = \lambda_n w y_n$, expand as

$$y(x) = \sum_n c_n y_n(x) \qquad \text{(unkown)}$$

$$F(x) = \sum_n a_n y_n(x) \qquad \text{(known) with } a_n \text{ from (2.17)}$$

Substitute into (2.27)

$$\mathcal{L} y = \mathcal{L} \sum_n c_n y_n = \sum_n c_n \lambda_n w y_n = \sum_n a_n w y_n$$

By orthogonality (2.13), $c_n \lambda_n = a_n$, or $c_n = a_n / \lambda_n$.
So sol^n is

$$\boxed{y(x) = \sum_n \frac{a_n}{\lambda_n} y_n(x) \qquad (2.28)}$$

(assuming no $\lambda_n = 0$) Recall FS (1.22)

<u>Generalisation</u> driving forces often induce a linear response term $\tilde{\lambda}^{\xi} wy$ giving eq^n

$$\mathcal{L} y - \tilde{\lambda} w y = f(x) \qquad (2.29)$$

where $\tilde{\lambda}$ is <u>fixed</u>. The sol^n (2.28) becomes

$$y(x) = \sum_n \frac{a_n}{\lambda_n - \tilde{\lambda}} y_n(x) \qquad (2.30)$$

(assuming no $\lambda_n = \tilde{\lambda}$)

<u>Integral solutions and Green's functions</u>

Recall solution (II.28)

$$y(x) = \sum_n \frac{a_n}{\lambda_n} y_n(x) = \sum_n \frac{y_n(x)}{\lambda_n N} \int_a^b w(\xi)\, F(\xi)\, y_n(\xi)\, d\xi$$

$$= \int_a^b \underbrace{\sum_n \frac{y_n(x)\, y_n(\xi)}{\lambda_n N}}_{G(x,\xi)} \underbrace{w(\xi)\, F(\xi)}_{f(\xi)} d\xi$$

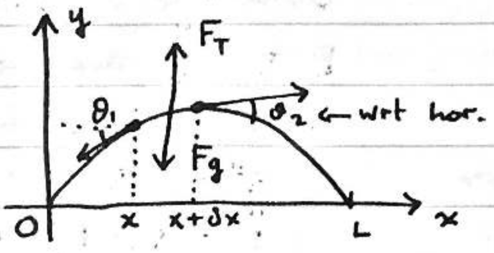
where $$G(x,\xi) = \sum_n \frac{y_n(x)\, y_n(\xi)}{\lambda_n N}$$

<u>Example</u> see sol^n (1.26) with FS

PART II – PDEs on bounded domains

3. The wave equation

3.1 Waves on an elastic string



Consider small displacements $y(x,t)$ on a stretched elastic string with fixed ends at $x=0$ and $x=L$, that is, with

$$\boxed{\text{boundary conditions} \quad y(0,t) = y(L,t) = 0 \qquad (3.1)}$$

Determine the string's motion for specified

$$\boxed{\text{initial conditions} \quad y(\underbrace{0,x}_{\text{swap}}) = p(x) \ \text{ and } \ \overset{\text{at } t=0}{\frac{\partial y}{\partial t}} = q(x) \qquad (3.2)}$$

Derive equation of motion Balance forces on string segment $(x, x+\delta x)$ and take $\delta x \to 0$

Assume $\left|\frac{\partial y}{\partial x}\right| \ll 1$, $\forall x$, so θ_1, θ_2 are small ← no need really, see Skinner

- Resolve in x-direction $T_1 \cos\theta_1 = T_2 \cos\theta_2 \overset{T}{=},$
  but $\cos\theta = 1 - \theta^2/2$ so $T_1 \simeq T_2 = T$. Hence tension T is constant, indep of x up to $O(|\frac{\partial y}{\partial x}|^2)$.
- Resolve in y-direction $F_T = T_2 \sin\theta_2 - T_1 \sin\theta_1$
  $$\simeq T\left(\partial y/\partial x|_{x+\delta x} - \partial y/\partial x|_x\right) \quad \text{small angles}$$
  $$\simeq T\, \partial^2 y/\partial x^2 \cdot \delta x$$

Thus $F = ma = (\mu\,\delta x)\,\partial^2 y/\partial t^2 = F_T + F_g$
$$= T\,\partial^2 y/\partial x^2 \cdot \delta x - (\mu\,\delta x)\, g,$$
where μ is mass per unit length (linear mass density).

Define the wave speed $c = \sqrt{T/\mu}$ (const.) and we find

$$\frac{\partial^2 y}{\partial \cancel{x}^2_t} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g = c^2 \frac{\partial^2 y}{\partial x^2} - g \qquad (3.3)$$

Assuming gravity is negligible, we have 1D wave equation ($\ddot{y} = c^2 y''$)

$$\boxed{\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \qquad (3.4)}$$

3.2 Separation of variables

We wish to solve (3.4) subject to bcs (3.1) and ics (3.2)

Consider possible solution of separable form (ansatz)

$$y(x,t) = X(x)\,T(t) \qquad (3.5)$$

Substitute in (3.4) $\frac{1}{c^2} X\ddot{T} = X''T \Rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}$

But $\frac{\ddot{T}}{T}$ depends only on t, while $\frac{X''}{X}$ depends only on x so both sides must be equal to a constant, say $-\lambda$.

$X'' + \lambda X = 0$ (3.6)

$\ddot{T} + \lambda c^2 T = 0$ (3.7)

3.3 Boundary conditions and normal modes

Three possibilities for λ (−, 0, +) in spatial ODE (3.6) but restricted by b.c.s

(i) $\lambda < 0$. Take $\chi^2 = -\lambda$, then $X(x) = Ae^{\chi x} + Be^{-\chi x}$

$= \tilde{A}\cosh\chi x + \tilde{B}\sinh\chi x$

but bcs $X(0) = X(L) = 0$ $\Rightarrow \tilde{A} = \tilde{B} = 0$ (trivial sol^n)

$\Downarrow \tilde{A} = 0$ $\Downarrow \tilde{B} = 0$

(ii) $\lambda = 0$, then $X(x) = Ax + B \Rightarrow A = B = 0$, by (3.1)

(iii) $\lambda > 0$ then $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

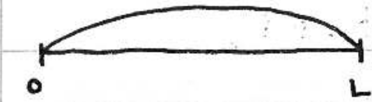
Here bcs (3.1) imply $A = 0$, $\sqrt{\lambda} = n\pi/L$,

$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $(n>0)$ (3.8)

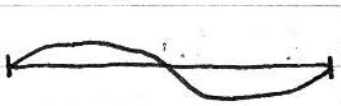
i.e. eigenfunctions and eigenvalues of the system

These are normal modes because spatial shape in x does not change in time t.

Fundamental mode, $\lambda_1 = \pi^2/L^2$ lowest frequency of vibration or first harmonic.



Second mode (n=2), $\lambda_2 = 4\pi^2/L^2$ second harmonic or overtone.



Third mode (n=3) etc.

3.4 Initial conditions and temporal solutions

Substitute evals $\lambda_n = (n\pi/L)^2$ into time ODE (3.7)

$\ddot{T} + \frac{n^2\pi^2c^2}{L^{\cancel{4}2}} T = 0$ which has sol^{n}s

$$T_n(t) = C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L} \qquad (3.9)$$

Thus, a specific solution to (3.4) satisfying (3.1) is

$$y_n(x,t) = T_n(t)\, X_n(t) = \left(C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{\cancel{\pi}L} \right) \sin \frac{n\pi x}{L}$$

(absorbing B_n into C_n, D_n) Ex verify solves (3.4)

Since wave eq^n (3.4) is homogeneous and linear (and have hom bcs (3.1)) we can add solutions together to find general string solution

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{\cancel{\pi}L} + D_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L} \qquad (3.10)}$$

General string solution (cont.)

(3.10) $$y(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}$$

By construction (3.10) satisfies the bcs (3.1), so now impose ics (3.2). For $t = 0$ we have

$$y(x,0) = p(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \quad \text{by } (3.10)$$

$$\frac{\partial y}{\partial t}(x,0) = q(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}.$$

So the coeffs. are those for the Fourier sine series given by

(1.12) $$C_n = \frac{2}{L} \int_0^L p(x) \sin \frac{n\pi x}{L}\, dx,$$

$$D_n = \frac{2}{n\pi c} \int_0^L q(x) \sin \frac{n\pi x}{L}\, dx \qquad (3.11)$$

Hence, (3.10-11) is the sol^n of (3.4) satisfying (3.1-2).

Example Pluck string at $x = \xi$.

$$y(x,0) = p(x) = \begin{cases} x(1-\xi) & \text{for } 0 \le x \le \xi, \\ \xi(1-x) & \text{for } \xi < x \le 1. \end{cases}$$

Then FS (1.8) $$C_n = 2\frac{\sin n\pi\xi}{(n\pi)^2}, \quad D_n = 0.$$

So we have sol^n

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \sin n\pi\xi \sin n\pi x \cos \pi nct$$

Take $\xi = \frac{1}{2}$, $C_{2m} = 0$, $C_{2m-1} = \dfrac{2(-1)^{m+1}}{((2m-1)\pi)^2}$ (odd only)

e.g. guitar $\frac{1}{4} \le \xi \le \frac{1}{3}$, violin $\xi \approx 1/7$

3~~2~~.5 Oscillation energy

A vibrating string has kinetic energy due to its motion (e.g. particle $\frac{1}{2}mv^2$)

$$KE = \frac{1}{2}\mu \int_0^L \left(\frac{\partial y}{\partial t} \right)^2 dx$$

and potential energy due to stretching Δx.

$$PE = T\Delta x = T\int_0^L \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right) dx$$

$$\simeq \tfrac{1}{2} T \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \quad \text{for} \quad \left|\frac{\partial y}{\partial x}\right| \ll 1$$

The total summed energy ($c^2 = T/\mu$)

$$E = \tfrac{1}{2}\mu \int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right] dx \qquad (3.13)$$

by orthog

Substitute (3.10) and use orthogonality (1.1-3)

$$E = \tfrac{1}{2}\mu \sum_{n=1}^{\infty} \int_0^L \left[\left(-\frac{n\pi c}{L} C_n \sin\frac{n\pi ct}{L} + \frac{n\pi c}{L} D_n \cos\frac{n\pi ct}{L}\right)^2 \sin^2\frac{n\pi x}{L}\right.$$

$$\left. + c^2\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)^2 \frac{n^2\pi^2}{L^2} \cos^2\frac{n\pi x}{L}\right] dx$$

$$= \tfrac{1}{4}\mu \sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{L^{\cancel{2}}}\left(C_n^2 + D_n^2\right) \qquad (3.14)$$

use $\cos^2 + \sin^2 = 1$

$\int \cos^2 = \tfrac{1}{2}L$

$$= \sum_{\substack{\text{normal}\\\text{modes}}} \left[\text{energy in } n^{\text{th}} \text{ mode}\right]$$

This is constant, conserved in time (no dissipation)

<u>Aside</u> <u>Solution in characteristic coordinates</u>

Recall sine / cosine summation identities (before (1.1)) which means (3.10) becomes

$$y(x,t) = \tfrac{1}{2}\sum_{n=1}^{\infty}\left[\left(C_n \sin\frac{n\pi}{L}(x-ct) + D_n \cos\frac{n\pi}{L}(x \mp ct)\right)\right.$$

$$\left. + \left(C_n \sin\frac{n\pi}{L}(x+ct) - D_n \cos\frac{n\pi}{L}(x+ct)\right)\right]$$

$$= f(x-ct) + g(x+ct) \qquad (3.12)$$

The standing wave solⁿ as sum of right moving wave (along characteristic $x - ct = \eta$, const.) and a left moving mode (along $x + ct = \xi$, const.) i.e. general solⁿ for (3.4) with arbitrary f, g (see later)

<u>Special case</u> $q(x) = 0$ in (3.1) $\Rightarrow f = g = \tfrac{1}{2}p$ with

$$y(x,t) = \tfrac{1}{2}\left[p(x-ct) + p(x+ct)\right]$$

L8.3

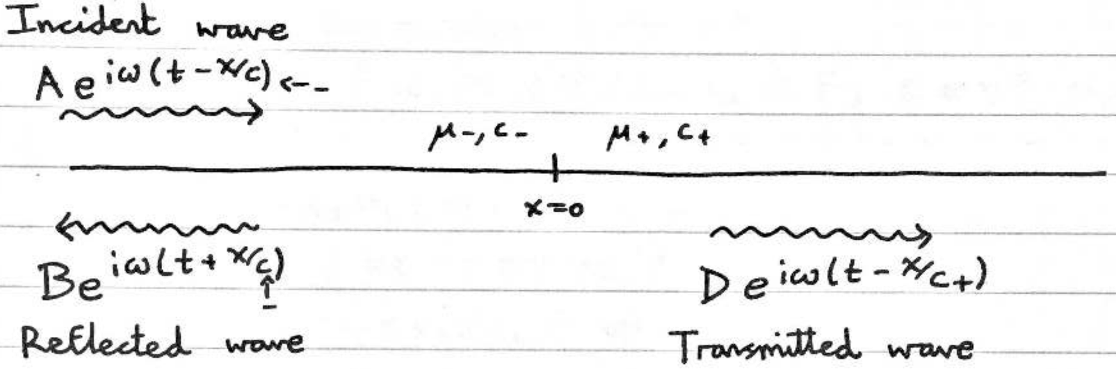
3.6 Wave reflection and transmission

A simple harmonic travelling wave is

$$y(x,t) = \mathrm{Re}\left[A e^{\underset{\text{out}}{i}(\omega t - x/c)}\right] \underset{\substack{\text{assume}\\ \text{real part}}}{=} |A| \cos\left[\omega\left(t - \frac{x}{c}\right) + \phi\right] \quad (3.15)$$

where the phase is $\phi = \arg A$ and wavelength is $\frac{2\pi c}{\omega}$.

Consider a density discontinuity on a string at $x = 0$,
with $\mu = \mu_-$ for $x < 0 \Rightarrow c_- = \sqrt{T/\mu_-}$
and $\mu = \mu_+$ for $x > 0 \Rightarrow c_+ = \sqrt{T/\mu_+}$



Boundary (or junction) conditions at $x = 0$

- String does not break at $x = 0 \Rightarrow A + B = D \quad (*)$
- Forces balance $T\left(\frac{\partial y}{\partial x}\right)_{x=0_-} = T\left(\frac{\partial y}{\partial x}\right)_{x=0_+}$ i.e. $\frac{\partial y}{\partial x}$ cts

$$\Rightarrow -\frac{i\omega A}{c_-} + \frac{i\omega B}{c_-} = -\frac{i\omega D}{c_+} \quad (+)$$

$$(*) - \frac{c_-}{i\omega}(+) \Rightarrow 2A = D + D\frac{c_-}{c_+} = \frac{D}{c_+}(c_- + c_+)$$

So given A, we have sol^n

Transmitted amplitude $\qquad$ Reflected amp

$$D = \frac{2c_+}{c_+ + c_-} A \qquad B = \frac{-c_- + c_+}{c_+ + c_-} A \qquad (3.16)$$

In general, different phase shifts in ϕ are possible.

Limiting cases

1) Continuity $c_- = c_+ \Rightarrow D = A,\ B = 0$

L9.1

Limit Cases 1) Continuity $c_- = c_+ \Rightarrow D = A$, $B = 0$ (no reflection)
2) Dirichlet bcs $\mu_+/\mu_- \to \infty$ (fixed ends at $x = 0$, $y(0,t) = 0$)
then $c_+/c_- \to 0 \Rightarrow D = 0$, $B = -A$ i.e. total reflection with opposite phase ($\phi = \pi$)
3) Neumann bcs $\mu_+/\mu_- \to 0$ (free end of string, $\frac{\partial y}{\partial x}\big|_{x=0} = 0$)
then $c_+/c_- \to \infty \Rightarrow D = 2A$, $B = A$ i.e. total reflection with same phase ($\phi = 0$)

3.7 Wave equation in 2D plane polar coords

The wave eqⁿ for $u(r,\theta,t)$ becomes

$$\boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \quad (3.17)} \qquad u(1,\theta,t) = 0 \ \forall \theta, t \quad (3.18)$$

with b.c.s at $r=1$ on unit disk (drum)
and i.c.s for $t = 0$,

$$u(r,\theta,0) = \phi(r,\theta)$$
$$\frac{\partial u}{\partial t}(r,\theta,0) = \Psi(r,\theta) \qquad (3.19)$$

Temporal separation Substitute $u(r,\theta,t) = T(t)V(r,\theta)$ into (3.17) to find

$$\ddot{T} + \lambda c^2 T = 0 \quad (3.21)$$
$$\nabla^2 V + \lambda V = 0 \quad (3.22)$$

which in polar coords

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \lambda V = 0.$$

Spatial separation Now try $V(r,\theta) = \Theta(\theta)R(r)$ in (3.22)

$$\Theta'' + \mu\Theta = 0 \quad (3.23)$$

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \quad (3.24)$$

where λ, μ are separation constants.

Polar solution Configuration implies periodic bcs $\Theta(0) = \Theta(2\pi)$ with $\mu > 0$, so the eval $\mu = m^2$ ($m \in \mathbb{Z}$) with solⁿ

$$\Theta_m(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta). \quad (3.25)$$

(or $C_m e^{im\theta}$ $-\infty < m < \infty$)

Radial eqⁿ Multiply (3.24) by r^{-1} to bring to SL form (2.7)
($\mu = m^2$)

$$\frac{d}{dr}(rR') - \frac{m^2}{r}R = -\lambda r R \quad (0 \le r \le 1) \quad (3.26)$$

where $p(r) = r$; $q(r) = m^2/r$ and weight $w(r) = r$ with self-adjoint bcs with $R(1) = 0$ and bounded at 0, since $p(0) = 0$, a regular singular point.

3.8 Bessel's equation

Substitute $z = \sqrt{\lambda}\, r$ in (3.26) with

$$\boxed{z^2 \frac{d^2R}{dz^2} + z\frac{dR}{dz} + (z^2 - m^2)R = 0 \quad (3.27)}$$

Frobenius solution Substitute power series
$R = z^p \sum_{n\geq 0} a_n z^n$ to obtain

$$\sum_{n\geq 0} a_n\left((n+p)(n+p-1)z^{n+p} + (n+p) z^{n+p} + z^{n+p+2} - m^2 z^{n+p}\right) = 0$$

Equate powers of z, get indicial eq^n $p^2 - m^2 = 0$, $p = \pm m$
Regular sol^n $p = m$, has recursion relation

$$(n+m)^2 a_n + a_{n-2} - m^2 a_n = 0$$

$$\Rightarrow a_n = \frac{-1}{n(n+2m)} a_{n-2}$$

Stepping up from a_0,

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)(n+m)\cdots(n+1)} a_0$$

Take $a_0 = \dfrac{1}{2^m m!}$ (convention) to find

$$\boxed{J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!}\left(\frac{z}{2}\right)^{2n} \quad (3.28)}$$

Also works for $m = \nu \notin \mathbb{Z}$, $(n+m)! \to \Gamma(n+m+1)$
Second sol^n $p = -m$, Neumann fn $Y_m(z)$
Exercise* Use (3.28) to show that

$$\frac{d}{dz}\left(z^m J_m(z)\right) = z^m J_{m-1}(z)$$

and hence $J_m'(z) + \frac{m}{z} J_m(z) = J_{m-1}(z)$ (3.29)
Repeat with z^{-m} to find

$$\left.\begin{aligned} J_{m-1}(z) + J_{m+1}(z) &= \frac{2m}{z} J_m(z) \\ -J_{m+1}(z) + J_{m-1}(z) &= 2J_m'(z) \end{aligned}\right. \quad (3.30)$$

<u>Asymptotic behaviour</u> $J_m(z), Y_m(z)$

- <u>Small $z \to 0$</u> $J_0(z) \to 1$, $J_m(z) \to \frac{1}{m!}\left(\frac{z}{2}\right)^m$ for $m > 0$

(3.31) $$Y_0(z) \to \frac{2}{\pi}\ln\left(\frac{z}{2}\right) , \; Y_m(z) \to -\frac{(m-1)!}{\pi}\left(\frac{2}{z}\right)^m$$

i.e. Y_m is divergent as $z \to 0$

- <u>Large $z \to \infty$</u> oscillatory

$$J_m(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

(3.32)

$$Y_m(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

<u>Exercise</u> Use $y = \sqrt{z}R$ into (3.27) to find

$$y'' + y\left(1 + \frac{4}{z} - \frac{m^2}{z^2}\right) = 0$$

as $z \to \infty$, $y'' \approx -y$ so we have $R \approx \frac{1}{\sqrt{z}}(A\cos z + B\sin z)$

<u>Zeros of Bessel f^n $J_m(z)$</u>

Define j_{mn} to be the n^{th} zero of $J_m(z)$ i.e. $J_m(j_{mn}) = 0$

This occurs approximately from (3.32) when

$$\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) = 0$$

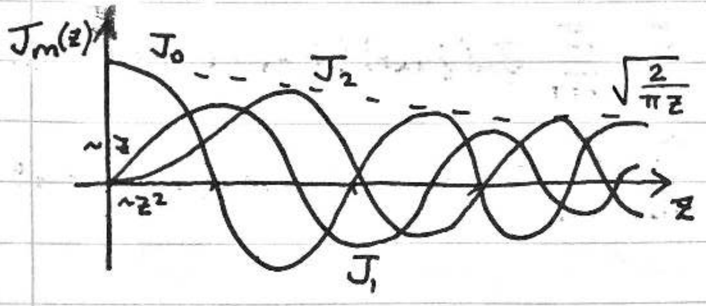
i.e. $z = +\frac{m\pi}{2} + \frac{\pi}{4} + n\pi - \frac{\pi}{2}$

So zero $$z \approx n\pi + \frac{m\pi}{2} - \frac{\pi}{4} \equiv \tilde{j}_{mn} \quad (3.32)$$

(Accuracy $\left|\frac{j_{mn} - \tilde{j}_{mn}}{j_{mn}}\right| < 1/10$ for $n > m^2/2$)

Bessel f^ns (cont.) Zeros of $J_0(z)$, $j_{01} \simeq 2.405$, $j_{02} \simeq 5.520$

$j_{0n} \simeq n\pi - \frac{\pi}{4}$ (precision $\sim \frac{1\%}{n}$)



3.9 2D Wave eq^n (cont.): Vibrating drum

From § 3.8, radial sol^ns to (3.26) are

yikes notation

$$R_m(z) = R_m(\sqrt{\lambda} r) = A J_m(\sqrt{\lambda} r) + B Y_m(\sqrt{\lambda} r)$$

Impose bcs: · Regularity at $r=0 \Rightarrow B=0$ by (3.31)

~ divergent $Y_m(0)$

· Unit disk $r=1$ with $R=0$ implies $J_m(\sqrt{\lambda} r) = 0$

But these zeros occur at j_{mn} $(\approx \tilde{j}_{mn} = n\pi + \frac{m\pi}{2} - \frac{\pi}{4})$ so our eigenvalues must be $\lambda_{mn} = j_{mn}^2$ (3.34).

With the polar mode (3.26), the spatial sol^n is

$$V_{mn}(r,\theta) = \Theta_m(\theta) \underbrace{R_{mn}}_{oof}(\sqrt{\lambda_{mn}}\, r)$$

$$= (A_{mn} \cos m\theta + B_{mn} \sin m\theta)\, J_m(j_{mn} r) \qquad (3.25)$$

The temporal solution to (3.21) $\ddot{T} = -\lambda_{mn} c^2 T$ are

$$T_{mn}(t) = \underset{\substack{\uparrow \\ A, Bs}}{\cos(j_{mn} c t)} \text{ and } \underset{\substack{\uparrow \\ C, Ds}}{\sin(j_{mn} c t)}$$

For our linear, homogeneous PDE (3.17) we can sum together to obtain the general solution (noting special case $m=0$)

$$\boxed{\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} J_0(j_{0n} r)\left(A_{0n} \cos(j_{0n} c t) + C_{0n} \sin(j_{0n} c t)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn} r)\left(A_{mn} \cos m\theta + B_{mn} \sin m\theta\right) \cos(j_{mn} c t) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn} r)\left(C_{mn} \cos m\theta + D_{mn} \sin m\theta\right) \sin(j_{mn} c t) \qquad (3.36) \end{aligned}}$$

Now impose <u>initial conditions</u> (3.19) at $t=0$:

$$u(r,\theta,0) = \phi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)\left(A_{mn}\cos m\theta + B_{mn}\sin m\theta\right)$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \psi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} j_{mn}\, c\, J_m(j_{mn}r)\left(C_{mn}\cos m\theta + D_{mn}\sin m\theta\right)$$

(3.37)

<u>Orthogonality</u> Find coeffs by multiplying by J_m, cos, sin and exploiting orthogonality (1.1-3) and E1Q8

$$\boxed{\begin{aligned}\int_0^1 J_m(j_{mn}r)\,J_m(j_{mk}r)\,r\,dr &= \tfrac{1}{2}\left[J_m'(j_{mn})\right]^2\delta_{nk}\\ &= \tfrac{1}{2}\left[J_{m+1}(j_{mn})\right]^2\delta_{nk}\end{aligned}}$$

by recursion (3.29)

Now integrate to obtain A_{mn}

$$\int_0^{2\pi} d\theta\,\cos p\theta \int_0^1 r\,dr\; J_p(j_{pq}r)\,\phi(r,\theta)$$

$$= \frac{\pi}{2}\left[J_{p+1}(j_{pq})\right]^2 A_{pq}$$

becomes 2π for $p=0$

<u>Ex</u>: Write expressions for B_{mn}, C_{mn}, D_{mn}

<u>Eg</u>: Initial radial profile $u(r,\theta,0) = \phi(r) = 1-r^2$

$$\Rightarrow m=0,\; B_{mn}=0,\; A_{\underset{m\neq 0}{m}n}=0$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = 0 \Rightarrow C_{mn} = D_{mn} = 0$$

We only need to find:

$$A_{0n} = \frac{2}{J_1(j_{0n})^2}\int_0^1 J_0(j_{\underset{0}{\cancel{m}}n}(r))\,(1-r^2)\,r\,dr$$

$$= \frac{2}{J_1(j_{0n})^2}\cdot\frac{J_2(j_{0n})}{j_{0n}^{\,2}} \simeq \frac{J_2(j_{0n})}{n} \text{ as } n\to\infty$$

Exercise * using (2.29-30)

Solution is

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n}\,J_0(j_{0n}r)\cos(j_{0n}ct)$$

Fundamental mode

$$\omega_d = j_{0nc}\left({}^{2}/{}_{d}\right) \simeq 4.8\,{}^{c}/{}_{d}$$ diameter d

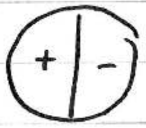
String length d

$$\omega_s = {}^{\pi c}/{}_{d} \simeq 0.77\,\omega_d$$

Nodal lines $\{m,n\}$ modes

(0,1)

(1,1)



$f_2 = 1.59\, f_1$

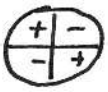
(0,2)

$f_4 = 2.3\, f_1$

(1,2)

$f_5 = 2.92\, f_1$

(2,1)



$f_3 = 2.14\, f_1$

(0,3)

$f_{10} = 3.59\, f_1$

4. THE DIFFUSION EQUATION

4.1 Physical origin of heat equation

Applies to processes which diffuse due to spatial gradients. An early example was Fick's law with flux

$$\underset{\sim}{J} = -D\,\underset{\sim}{\nabla}\, c$$

with concentration c and diffusion coeff. D. For heat flow we have Fourier's law

$$\underset{\substack{\nearrow\\\text{heat}\\\text{flux}}}{q} = -\underset{\substack{\uparrow\\\text{thermal}\\\text{conducc}}}{k}\,\nabla\,\underset{\substack{\nwarrow\\\text{temperature}}}{\theta} \qquad (4.1)$$

L 11.1

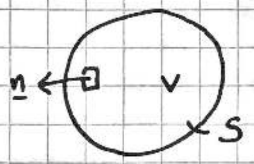
Origin of heat equation (cont.)

For heat flow, we have Fourier's law

$$\underset{\substack{\uparrow\\ \text{heat flux}}}{\underset{\sim}{q}} = -\underset{\substack{\uparrow\\ \text{thermal}\\ \text{conductivity}}}{k}\, \underset{\sim}{\nabla}\, \theta \quad \leftarrow \text{temperature } \theta = T \qquad (4.1)$$

In a volume V, the overall heat energy Q is

$$Q = \int \underset{\substack{\uparrow\\ \text{specific}\\ \text{heat}}}{c_v}\, \underset{\substack{\nwarrow\\ \text{mass}\\ \text{density}}}{\rho}\, \theta \, dV \qquad (4.2)$$



so rate of change due to heat flow is

$$\frac{dQ}{dt} = \int c_v \rho \frac{\partial \theta}{\partial t}\, dV \qquad (*)$$

Now integrate (4.1) over the surface S enclosing V

$$\underset{\substack{\uparrow\\ \text{out}\\ \text{normal}}}{-} \frac{dQ}{dt} = \int \underset{\sim}{q}\, \cancel{\underline{n}} \cdot d\underline{S} = \int_S (-k \underset{\sim}{\nabla} \theta)\, \cancel{\underline{n}} \cdot d\underline{S} \underset{\substack{\uparrow\\ \text{by Gauss}}}{=} \int_V (-k \nabla^2 \theta)\, dV \qquad (+)$$

Equating $(*)$ and $(+)$ $\displaystyle\int \left(c_v \rho \frac{\partial \theta}{\partial t} - k \nabla^2 \theta \right) dV = 0$

True for all V, so integrand must vanish, so with $D = \dfrac{k}{c_v \rho}$

$$\boxed{\frac{\partial \theta}{\partial t} = D \nabla^2 \theta \qquad (4.3)}$$

Brownian motion (random walk)

Gas particles diffuse by scattering every Δt with probability density function $p(\xi)$ where ξ is distance. So by symmetry $\langle \xi \rangle = 0$.

Suppose the pdf after $N\,\Delta t$ steps is $P_{n\Delta t}(x)$, then after the $(N+1)\,\Delta t$ step

$$\begin{aligned} P_{(N+1)\Delta t} &= \int_{-\infty}^{\infty} p(\xi)\, P_{N\Delta t}(x - \xi)\, d\xi \\ &\approx \int_{-\infty}^{\infty} p(\xi) \left[P_{N\Delta t}(x) + P'_{N\Delta t}(x)(-\xi) + P''_{N\Delta t}(x) \left(\frac{\xi^2}{2} \right) \right] d\xi \\ &= P_{N\Delta t}(x) - P'_{N\Delta t}(x) \underset{\substack{\uparrow\\ \text{zero}}}{\langle \xi \rangle} + P''_{N\Delta t}(x) \frac{\overset{\substack{\text{variance}\\ \nearrow}}{\langle \xi^2 \rangle}}{2} \end{aligned}$$

Identify $P_{N\Delta t}(x) = P(x, N\Delta t)$, then we have

$$P(x,(N+1)\Delta t) - P(x, N\Delta t) = \frac{\partial^2}{\partial x^2} P(x, N\Delta t) \frac{\langle \xi^2 \rangle}{2}$$

Assume $\frac{\langle \xi^2 \rangle}{2} = D\Delta t$, then as $\Delta t \to 0$, we find

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \qquad ☺ \quad (4.4)$$

4.2 Similarity solutions

The characteristic relationship between the variance and time suggests seeking solutions with the dimensionless parameter $\eta = {}^{x}\!/_{2\sqrt{Dt}}$ (4.5)

Can we find solutions $\theta(x,t) = \theta(\eta)$?

Change variables in (4.3)

$$\text{LHS} \quad \frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{1}{4}\frac{x}{\sqrt{D}\,t^{3/2}}\,\theta' = -\frac{1}{2}\frac{\eta}{t}\,\theta'$$

$$\text{RHS} \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\frac{\partial \theta}{\partial \eta}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{Dt}}\,\theta'\right) = \frac{1}{4Dt}\,\theta''$$

We deduce $\theta'' = -2\eta\theta'$ (4.6). Take $\psi = \theta'$

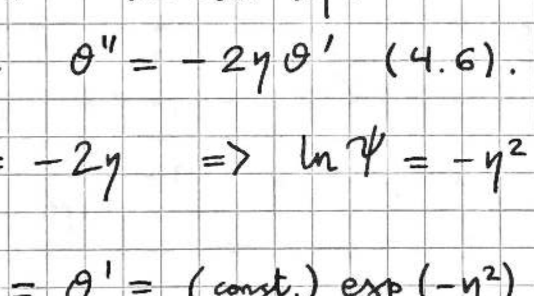
$$\frac{\psi'}{\psi} = -2\eta \quad \Rightarrow \quad \ln \psi = -\eta^2 + \text{const.}$$

$$\Rightarrow \quad \psi = \theta' = (\text{const.}) \exp(-\eta^2)$$

Integrate to find $\theta = C \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du = C \,\text{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$

where the <u>error function</u> is $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$.

This describes discontinuous initial conditions at $t = 0$ that spread over time :



L11.3

4.3 Heat conduction in a finite bar

Suppose we have a bar of length $2L$ with $-L \leq x \leq L$ and initial temperature $\theta(x,0) = H(x) = \begin{cases} 1, & 0 \leq x \leq L, \\ 0, & -L \leq x < 0. \end{cases}$ (4.8)

with bcs $\theta(L,t) = 1$, $\theta(-L,t) = 0$ (4.9).

Transforming boundary conditions The bcs (4.9) are not homogeneous. Can we identify steady state solutions (time indep) that exhibit the late time behaviour of the solution?

Try $\theta_s(x) = Ax + B$, satisfies $\dfrac{\partial^2 \theta}{\partial x^2} = 0$.

To satisfy (4.9) $A = \frac{1}{2L}$, $B = \frac{1}{2}$, $\theta_s(x) = (x+L)/2L$ (4.10).

Transform $\hat{\theta}(x,t) = \theta(x,t) - \theta_s(x)$ with hom b.c.s

$$\hat{\theta}(-L,t) = \hat{\theta}(L,t) = 0$$

and initial condition $\hat{\theta}(x,0) = H(x) - (x+L)/2L$ (4.11)

Separation of variables

Try $\hat{\theta}(x,t) = X(\cancel{t}\, x)\, T(t) \Rightarrow X'' = -\lambda X$, $\dot{T} = -D\lambda T$ (4.12)

B.c.s imply $\lambda > 0$, with $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$.

For $\cos(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda_m} = \dfrac{m\pi}{2L}$, $m = 1, 3, 5, \ldots$

$\sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda_n} = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

But ics are odd so take $X_n = B_n \sin\left(\dfrac{n\pi x}{L}\right)$, $\lambda_n = \dfrac{n^2\pi^2}{L^2}$.

Put λ_n into (4.12) to find

$$T(t) = C_n \exp\left(-D\frac{n^2\pi^2}{L^2}t\right)$$

so general sol^n is

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-D\frac{n^2\pi^2}{L^2}t\right) \quad (4.13)$$

L12.1

2D Heat equation general sol^n (odd ics)

$$\hat{\theta}(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2 D}{L^2}t}$$

Now impose ics (4.11) at $t=0$

$$b_n = \frac{1}{L}\int_{-L}^{L} \hat{\theta}(x,0)\sin\frac{n\pi x}{L}\,dx \quad \text{where } \hat{\theta}(x,0) = H(x) - \frac{x+L}{2L}$$

$$= \frac{2}{L}\underbrace{\int_0^L \Big(H(x) - \overset{\frac{1}{2}}{\frac{1}{2}}\Big)\sin\frac{n\pi x}{L}\,dx}_{\text{square wave FS }(1.7)/2} - \frac{2}{L}\underbrace{\int_0^L \frac{x}{2L}\sin\frac{n\pi x}{L}}_{\substack{\text{sawtooth}\\ \text{FS }(1.6)/2L}}\,dx$$

$$= \frac{2}{(2m-1)\pi} - \frac{(-1)^{n+1}}{n\pi} \overset{✓}{=} \frac{1}{n\pi}$$

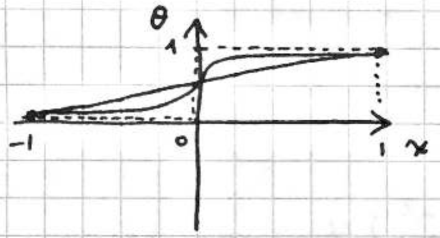
↑ $n = 2m-1$ odd

$= 0$ even

Solution $\hat{\theta}(x,t) = \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin\frac{n\pi x}{L}\, e^{-\frac{Dn^2\pi^2}{L^2}t}$

or with original bcs $\theta(x,t) = \hat{\theta}(x,t) + \frac{x+L}{2L}$ (4.14)

Plot with $L=1$ and $D=1$



Approx sol^n $\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$ fits well for small t

5. THE LAPLACE EQUATION

Laplace's equation $\boxed{\nabla^2\phi = 0 \quad (5.1)}$ has wide application in math. physics, applied and pure maths. Examples include

- steady-state heat flow
- potential theory $\underset{\sim}{F} = -\underset{\sim}{\nabla} V$ (also with $\nabla^2\phi = \rho$)
- incompressible fluid flow $\underset{\sim}{v} = \underset{\sim}{\nabla}\phi$ (irrotational)
- complex analysis

We solve (5.1) in a domain D subject to bcs

Dirichlet ϕ given on boundary surface ∂D

Neumann $\underline{n}\cdot\underline{\nabla}\phi$ " " ∂D

5.1 3D Cartesian coordinates

Eq^n (5.1) becomes $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ (5.2).

Seek separable sol^n's $\phi = X(x)Y(y)Z(z)$.

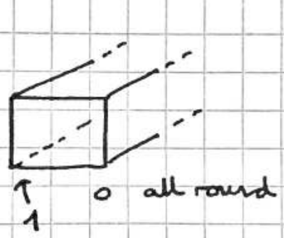
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$
$$\underset{-\lambda_L}{\uparrow} \qquad \underset{-\lambda_m}{\uparrow} \qquad \underset{\lambda_n = \lambda_L + \lambda_m}{\uparrow} \quad (5.3)$$

General solⁿ from eigenmodes $\phi(x, y, z) = \sum_{L,m} a_{Lm} X_L(x) Y_m(y) Z_n(z)$, (5.4)

Example: Steady heat conduction

i.e. (4.3) with $\theta_t = 0 \Rightarrow (5.1)$

Consider a semi-infinite rectangular bar with bcs



$\phi = 0$ at $x = 0, a$ and $\phi = 0$ at $y = 0, b$

and $\phi = 1$ at ~~x~~ $z = 0$ and $\phi \to 0$ as $z \to \infty$

Solve for eigenodes succesively

- $\underline{X'' + \lambda_L X = 0}$ w/ $X(0) = X(a) = 0$

  $\lambda_L = \frac{L^2\pi^2}{a^2}$, $X_L = \sin\frac{L\pi x}{a}$, $L = 1, 2, \ldots$

- $\underline{Y'' + \lambda_m Y = 0}$ w/ bcs

  $\lambda_m = \frac{m^2\pi^2}{b^2}$, $Y_m = \sin\frac{m\pi y}{b}$, $m = 1, 2, \ldots$

- $\underline{Z'' = -\lambda_n Z} = (\lambda_L + \lambda_m) Z = \pi^2\left(\frac{L^2}{a^2} + \frac{m^2}{b^2}\right) Z$ and bcs

  $Z_{Lm} = \exp\left[-\left(L^2/a^2 + m^2/b^2\right)^{1/2} \pi z\right]$

So our general solⁿ is $\phi(x, y, z) = \sum_{L,m} a_{Lm} \sin\frac{L\pi x}{a} \sin\frac{\pi m y}{b} \exp\left(-\left(\frac{L^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2} \pi z\right)$

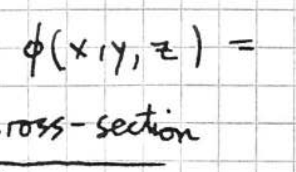
Now fix a_{Lm} using $\phi(x, y, 0) = 1$ thanks to Fourier sin series

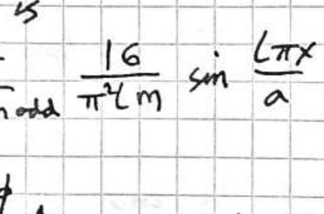
$$a_{Lm} = \frac{2}{b}\int_0^b dy \, \frac{2}{a}\int_0^a dx \; \underbrace{1 \cdot \sin\frac{L\pi x}{a} \cdot \sin\frac{m\pi y}{b}}_{\text{two square waves}} = \underset{\text{w/ } L, m \text{ odd}}{\frac{16}{\pi^2 l m}}$$

So heat flow solⁿ is

$$\phi(x, y, z) = \sum_{L,m \text{ odd}} \frac{16}{\pi^2 l m} \sin\frac{L\pi x}{a} \sin\frac{\pi m y}{b} \exp\left(-\left(\frac{L^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2} \pi z\right)$$

Cross-section





low l, m dominate

5.2 2D Plane polar coordinates

Recall $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$ (5.6)

and try $\phi = R(r)\,\Theta(\theta)$ to find

L12.3

$$\Theta'' + \lambda \Theta = 0 \quad \text{and} \quad r(rR')' - \lambda R = 0$$

- Polar eq^n & as before (3.25) periodic bcs $\Rightarrow \lambda = m^2$

$$\Theta_m = \cos m\theta \quad \text{and} \quad \sin m\theta$$

- Radial eq^n $\quad r(rR')' - m^2 R = 0 \quad (5.7)$

Try $R = \alpha r^\beta \Rightarrow \beta^2 - m^2 = 0 \Rightarrow \beta = \pm m$

$R_m = r^m$ and r^{-m}

If $m=0$, $(rR')' = 0 \Rightarrow rR' = \text{const} \Rightarrow R = \log r$

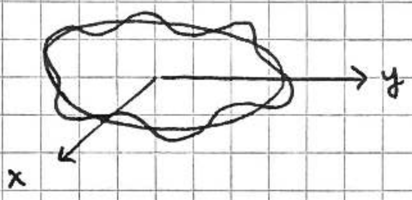
$R_0 = \text{const}$ and $\log r$

General solution

$$\phi(r,\theta) = \frac{a_0}{2} + c_0 \log r + \sum_{m=1}^{\infty} \left[(a_m \cos m\theta + b_m \sin m\theta)\, r^m\right] + \left[(c_m \cos m\theta + d_m \sin m\theta\right] r^{-m}] \qquad (5.8)$$

Example Soap film on a unit disk

Solve (5.6) with a distorted wire of radius 1 and of given boundary conditions $\phi(1,\theta) = f(\theta)$ to find $\phi(r,\theta)$ for $r<1$.



L13.1

Soap film on a circular wire (cont.)

Solve Laplace (5.6) with bcs $\phi(1,\theta) = f(\theta) \quad (r \leq 1)$

Regularity at $r=0$ implies $c_m = d_m = 0 \;\; \forall\, m$. At $r=1$

$$\phi(1,\theta) = f(\theta) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta)$$

so the FS coeffs are (1.5)

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta \, d\theta , \qquad b_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta \, d\theta$$

Sol^n $\phi(r,\theta) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta)\, r^m$.

Note high harmonics are confined near edges $r \simeq 1$ because of r^m.

5.3 3D Cylindrical Polar Coords

Here $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (5.9)$

Say $\phi = R\,\Theta\,Z$ we get

$$Z'' = \lambda Z , \quad \Theta'' = -\mu\,\Theta , \quad r(rR')' + (\lambda r^2 - \mu)R = 0$$

Polar (as before) $\mu_m = m^2$, $\Theta = \cos m\theta , \sin m\theta$ $\qquad k = \sqrt{\lambda}$

Radial (Bessel 3.26) $r(rR')' + (\lambda r^2 - m^2)R = 0$ with sol^{n}s

$R = J_m(kr)$ and $Y_m(kr)$. Setting bcs $R=0$ at $r=a$

means $J_m(ka) = 0 \Rightarrow k = \frac{j_{mn}}{a}$ where j_{mn} is the n^{th} zero

Radial ef^n $R_{mn} = J_m\left(\frac{j_{mn}}{a}r\right) \qquad (5.10) \qquad$ NB Y_m sol^n if $r \neq 0$

Z eq^n $Z'' = k^2 Z$ implies $Z = e^{-kz}$ and e^{kz} $\left(\begin{smallmatrix}\text{usually}\\ Z\to 0 \text{ as } z\to\infty\end{smallmatrix}\right)$

General sol^n is $\phi(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{mn}\cos m\theta + b_{mn}\sin m\theta) \times J_m\left(\frac{j_{mn}}{a}r\right)e^{-\frac{j_{mn}z}{a}}$

Exercise Describe steady heat flow in a semi-infinite circular wire with bcs $\phi = 0$ at $r=a$, $\phi = T_0$ at $z=0$ and $\phi \to 0$ as $z\to\infty$ (see §3.9 §5.1)

Show sol^n is $\phi(r,\theta,z) = \sum_{n=1}^{\infty}\frac{2T_0}{j_{0n}J_1(j_{0n})}J_0\left(\frac{j_{0n}}{a}r\right)e^{-\frac{j_{0n}z}{a}}$

5.4 3D Spherical Polar Coords

Recall $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$.

Laplace (5.1) becomes

L13.2

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \qquad (5.12)$$

Axisymmetric case (no ϕ dependence)

Seek separable sol^{n}s $\Phi(r,\theta) = R(r)\,\Theta(\theta)$.

$$(\sin\theta\,\Theta')' + \lambda\sin\theta\,\Theta = 0, \quad (r^2R')' - \lambda R = 0 \qquad (5.13)$$

Polar (Legendre) Substitute $x = \cos\theta$, $-1 \leq x \leq 1$

with $\frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\Theta}{d\theta} = -\sin\theta\frac{d\Theta}{dx}$

$$-\cancel{\sin}\,\theta\frac{d}{dx}\Big[-\underset{\substack{\uparrow\\ 1-\cos^2\theta}}{\sin^2\theta}\,\frac{d\Theta}{dx}\Big] + \lambda\cancel{\sin}\,\theta\,\Theta = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \lambda\Theta = 0$$ which is Legendre's eq^n (2.21)

with evals $\lambda = \ell(\ell+1)$ and ef^{n}s $\Theta = P_\ell(x) = P_\ell(\cos\theta)$ (5.14)

(see §2.5)

Radial eq^n $(r^2R')' - \ell(\ell+1)R = 0$

Seek sol^{n}s $R = \alpha r^\beta$: $\beta(\beta+1) - \ell(\ell+1) = 0$

with sol^{n}s $\beta = \ell$ or $-\ell-1$ so $R_\ell = r^\ell$ and $r^{-\ell-1}$.

General axisymmetric sol^n $\Phi(r,\theta) = \sum_{\ell=1}^{\infty}(a_\ell r^\ell + b_\ell r^{-\ell-1})P_\ell(\cos\theta)$ (5.15)

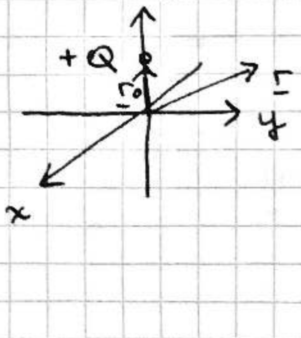
where a_ℓ, b_ℓ are given by bcs usually at fixed $r = a$.

Use orthogonality conditions for P_ℓ see (2.24)

Unit sphere Solve $\nabla^2\Phi = 0$ with axisym bcs at $r=1$, $\Phi(1,\theta,\phi) = f(\theta)$

$f(\theta) = F(x)$ so $F(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$ with $a_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} F(x)P_\ell(x)\,dx$

Generating function for $P_\ell(x)$ (2.23a)



Place a charge at $\underline{r}_0 = (0,0,1)$. Potential is

$$\Phi(\underline{r}) = \frac{1}{|\underline{r}-\underline{r}_0|} = \frac{1}{(x^2+y^2+(z-1)^2)^{1/2}}$$

$$= \frac{1}{(r^2\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + 1)^{1/2}}$$

$$= \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \frac{1}{\sqrt{r^2 - 2rx + 1}}$$

Any axisym sol^n can be expressed as

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)\,r^\ell$$

L13.3

With norm $P_\ell = 1 \Rightarrow \frac{1}{1-r} = \sum a_\ell r^\ell \Rightarrow$ given series $a_\ell = 1$

Thus generating f^n

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{\ell=0}^{\infty} P_\ell(x) r^\ell \qquad (5.16)$$

PART III - INHOMOGENEOUS ODEs, FOURIER TRANSFORMS

6. THE DIRAC DELTA FUNCTION

6.1 Definition of $\delta(x)$

Define a generalized function $\delta(x - \xi)$ with the follaring properties

$$\boxed{\begin{aligned} &\delta(x-\xi) = 0 \quad \forall\, x \neq \xi \\ &\int_{-\infty}^{\infty} \delta(x-\xi)\, dx = 1 \end{aligned} \qquad (6.1)}$$

This acts as a linear operator $\int dx\, \delta(x-\varepsilon)$ on an arbitrary $f(x)$ to produce a number $f(\xi)$, i.e.

$$\boxed{\int_{-\infty}^{\infty} dx\, \delta(x-\xi) f(x) = f(\xi) \qquad (6.2)}$$

provided f is "well-behaved" at $x = \xi, \pm\infty$

Notes: • The delta f^n $\delta(x)$ is classified as a distribution (not a f^n)

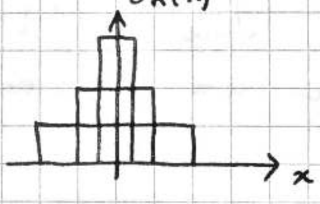
- $\delta(x)$ always appears inside an integrand as a linear operator, where it is well-defined
- Represents a unit point source (e.g. mass, charge)

Some limiting approximations

Discrete

(limit $n \to \infty$)

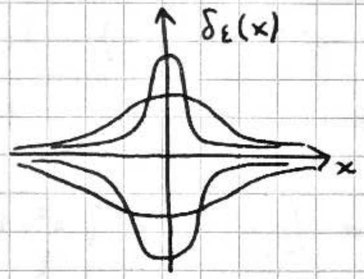
$$\delta_n(x) = \begin{cases} 0, & x > 1/n, \\ n/2, & |x| \leq 1/n, \\ 0, & x < -1/n. \end{cases}$$



Continuous

(limit $\varepsilon \to 0$)

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2}$$



Verify (6.2)

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx$$

$$= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2} f(x)\, dx$$

$$= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} f(\varepsilon y)\, dy \quad = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \left[f(0) + \varepsilon y f'(0) + \cdots \right] dy$$

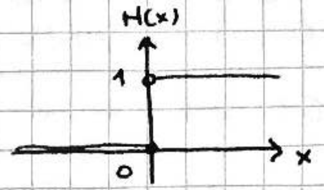
$$\simeq f(0) \quad \forall\, f \text{ well-behaved at } x = 0, \pm\infty$$

Further examples $\quad \delta_n(x) = \dfrac{\sin nx}{\pi x} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ikx}\, dk \qquad (6.4)$

$$\delta_n(x) = \frac{n}{2}\operatorname{sech}^2 nx \quad (6.5)$$

6.2 Properties of $\delta(x)$

Heaviside function $H(x)$



The unit step function $$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (6.6)$$

is the integral of $\delta(x)$: $$H(x) = \int_{-\infty}^{x} \delta(u)\, du \quad (6.7)$$

and we can identify H'.

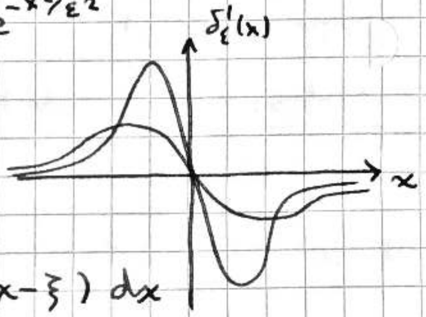
Ex Verify using (6.5)

Derivative of $\delta(x)$ Define $\delta'(x)$ through integration by parts

$$\int_{-\infty}^{\infty} \delta'(x-\xi) f(x)\, dx = \left[\delta(x-\xi) f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-\xi) f'(x)\, dx$$

$$= -f'(\xi) \quad (6.8)$$ for all smooth f at ε.

E.g. Consider Gaussian approximation $\delta'_\varepsilon(x) = -\frac{2x}{\varepsilon^3\sqrt{\pi}} e^{-x^2/\varepsilon^2}$



Sampling property

$$\int_a^b f(x)\, \delta(x-\xi)\, dx = \begin{cases} f(\xi) & \text{if } a<\xi<b, \\ 0 & \text{otherwise}. \end{cases}$$

Even property $\int_{-\infty}^{\infty} f(x)\, \delta(-(x-\xi))\, dx = \int_{-\infty}^{\infty} f(x)\, \delta(x-\xi)\, dx$

$$\text{LHS} = \int_{\infty}^{-\infty} f(\xi-u)\, \delta(u)\, (-du) = \int_{-\infty}^{\infty} f(\xi-u)\, \delta(u)\, du = f(\xi).$$

Scaling property $\int_{-\infty}^{\infty} f(x)\, \delta(a(x-\xi))\, dx = \frac{1}{|a|} f(\xi) \quad (6.11)$

Ex Show using $u = ax$ (take care if $a<0$)

Advanced scaling Suppose $g(x)$ has n isolated zeros at $x_1, \ldots, x_n$.

Then (with $g'(x_i) \neq 0$) $$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|} \quad (6.12)$$

Ex Show for g with single root

Example $$I = \int_{-\infty}^{\infty} f(x)\, \delta(\underset{\uparrow \text{ roots at } \pm 1}{x^2-1})\, dx = \int_{-1-\varepsilon}^{-1+\varepsilon} f(x) \frac{\delta(x+1)}{2}\, dx + \int_{1-\varepsilon}^{1+\varepsilon} f(x) \frac{\delta(x \overset{\overset{-}{\downarrow}}{+} 1)}{2} \underset{dx}{\uparrow}$$

$$= \frac{1}{2}\left(f(1) + f(-1) \right)$$

Isolation property If $g(x)$ is cts at $x = 0$ then $g(x)\,\delta(x) = g(0)\,\delta(x)$ $\underset{(6.13)}{\uparrow}$

Ex Evaluate and show $\int_0^\infty \delta'(x^2-1)\, x^2\, dx = -1/4$??

using $u = x^2-1$ and (6.8), (6.12)

get $\frac{1}{2}$

oh fair it's '

6.3 Eigenfunction expansions of $\delta(x)$

Fourier series (complex)

FS coeffs (1.11) are $\quad c_n = \frac{1}{2L}\int_{-L}^{L} \delta(x) e^{-inx\pi/L} dx = \frac{1}{2L}$

So $\quad \delta(x) = \frac{1}{2L} \sum_{n\in\mathbb{Z}} e^{in\pi x/L}$.

Take $\quad f(x) = \sum_{n\in\mathbb{Z}} d_n e^{in\pi x/L}$, then

$$\int_{-L}^{L} f(x)\,\overline{\delta(x)}\,dx = \frac{1}{2L}\sum_{n\in\mathbb{Z}} \int_{-L}^{L} \overset{\text{orthog}}{d_n}\, dx = \sum d_n = f(0)$$

General eigenf^{n}s and $\delta(x)$

Suppose $\delta(x-\xi) = \sum_{n=1}^{\infty} a_n y_n(x)$, $a \leq \xi \leq b$ with coeffs.

$$a_n = \int_a^b w(x)\, y_n(x)\, \delta(x-\xi)\, dx \Big/ \int_a^b w(x)\, y_n^2(x)\, dx$$

$$= w(\xi)\, y_n(\xi) / \int_a^b w(x)\, y_n^2(x)\, dx = w(\xi)\, Y_n(\xi)$$

for unit norm (2.18).

Then
$$\delta(x-\xi) = w(\xi) \sum_{n=1}^{\infty} Y_n(\xi)\, Y_n(x),$$
$$= W(x) \sum_{n=1}^{\infty} Y_n(\xi)\, Y_n(x),$$

since $\frac{W(x)}{W(\xi)}\delta(x-\xi) = \delta(x-\xi)$ by (6.13).

Hence
$$\boxed{\delta(x-\xi) = w(x) \sum_{n=1}^{\infty} y_n(\xi)\, y_n(x) \Big/ \int_a^b w y_n^2\, dt \qquad (6.15)}$$

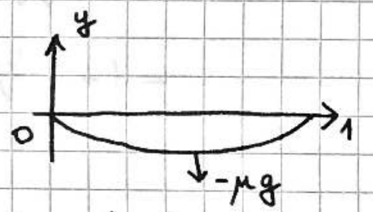
Example Consider FS for $y(0) = y(1) = 0$ with $y_n(x) = \sin n\pi x$.

Here $\delta(x-\xi) = 4 \sum_{n=1}^{\infty} \sin n\pi\xi \sin n\pi x$.

Ex Integrate twice and compare to $-G(x,\xi)$ (1.25)

7. GREEN'S FUNCTIONS

7.1 Physical motivation: Static forces on a string



Consider a massive elastic string (tension T, linear mass density μ) suspended with $y(0) = y(1) = 0$. (7.1)

By resolving forces in §3.1 this satisfies

(3.3) $T\frac{d^2y}{dx^2} - \mu g = 0$

So solve inhom. ode subject to (7.1)

$$-\frac{d^2y}{dx^2} = f(x) \qquad (7.2)$$

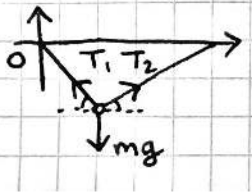
with $f(x) = -\frac{\mu g}{T}$.

Soln 1 Direct integration for uniform massive string

ODE (7.2) $\Rightarrow$ $-y = \left(-\frac{\mu g}{T}\right)\left(\frac{1}{2}x^2\right) + k_1 x + k_2$

BCS (7.1) $\Rightarrow$ $y = \left(-\frac{\mu g}{T}\right) \cdot \frac{1}{2}x(1-x)$ (7.3)

Soln 2 Superposition of point masses on a light string



Consider pt mass m suspended at $x = \xi$ on a very light string ($\mu \to 0$).

(Interpret as $N >> 1$ segments $m = \mu \delta x$ with $\delta x = 1/N$)

Resolve in y-dir^n to find $y_i(\xi_i)$

$$0 = T(\sin\theta_1 + \sin\theta_2) - mg$$

$$\simeq T\left(-\frac{y_i}{\xi_i} + -\frac{y_i}{1-\xi_i}\right) - mg$$

$$\Rightarrow \quad -T(y_i(1-\xi_i) + y_i\xi_i) = mg\,\xi_i(1-\xi_i)$$

$$y_i(\xi_i) = \left(-\frac{mg}{T}\right)\xi_i(1-\xi_i)$$

Hence sol^n is

comma ?

$$y_{\xi_i}(x) = \left(-\frac{mg}{T}\right)\begin{cases} x(1-\xi_i), & x < \xi_i \\ \xi_i(1-x), & x > \xi_i \end{cases} \equiv \underset{\substack{\uparrow \\ \text{source} \\ \text{meq}}}{f_i}\;\underset{\substack{\nwarrow \\ \text{sol}^n\text{ for unit} \\ \text{source at } x=\xi_i \\ \text{(Green's f}^n\text{)}}}{G}(x - \xi_i) \qquad (7.4)$$

By linearity sum N point masses at

$$y(x) = \sum_{i=1}^{N} f_i\,G(x, \xi_i)$$

maybe ξ_i? ✓

or in continuum limit with $f_i = -\frac{mg}{T} = -\frac{\mu\delta x}{T}g = \underset{\substack{\uparrow \\ \text{source per} \\ \text{unit length}}}{f(x)}\,\delta x$

we have $y(x) = \int_0^1 f(\xi)\,G(x,\xi)\,d\cancel{x}\xi$

$$= \left(-\frac{\mu g}{T}\right)\left[\underset{\substack{\uparrow \\ 2^{nd}\text{ sol} \\ \xi < x}}{\int_0^x \xi(1-x)\,d\xi} + \underset{\substack{\uparrow \\ 1^{st}\text{ sol} \\ x < \xi}}{\int_x^1 x(1-\xi)\,d\xi}\right]$$

$$= \left(-\frac{\mu g}{T}\right)\left(\left[\frac{\xi^2}{2}(1-x)\right]_0^x + \left[x\left(\xi - \frac{\xi^2}{2}\right)\right]_x^1\right)$$

$$= \left(-\frac{\mu g}{T}\right)\cdot\frac{1}{2}x(1-x) \qquad \text{as seen in } (7.3)$$

Compare with eq^{n}s (1.23 − 26)

7.2 Definition of Green's function

We wish to solve inhom ODE (recall § 2.1)

$$\mathcal{L}y \equiv \alpha(x)y'' + \beta(x)y' + \gamma(x)y = f(x) \qquad (7.6)$$

on $a \leq x \leq b$ with α (non-zero), β, γ cts, and with f all bounded subject to hom b.c.s $y(a) = y(b) = 0$ (or similar)

The Green's function $G(x,\xi)$ for the diff operator $\mathcal{L}$ is the sol^n for the unit pt mass at $x = \xi$:

$$\mathcal{L}\,G(x,\xi) = \delta(x-\xi) \qquad (7.7)$$

L15.3

which satisfies hom bcs $G(a,\xi) = G(b,\xi) = 0$ (or similar)

By linearity we can construct sol^{n}s to (7.6) by integrating over source $f(x)$ with the Green's f^n:

$$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi \qquad (7.8)$$

where y satisfies the hom b.c.s.

Formally verify

$$\mathcal{L} y = \int \mathcal{L} G(x,\xi)\, f(\xi) d\xi = \int \delta(x-\xi) f(\xi)\, d\xi = f(x),$$

so sol^n (7.8) is given by the inverse operator $\mathcal{L}^{-1}$ where

$$\mathcal{L}^{-1} = \int d\xi\ G(x,\xi)$$ (N.B. G only depends on $\mathcal{L}$ and b.c.s, not on f)

Defining properties (summary)

The Green's function splits into two parts:

$$G(x,\xi) = \begin{cases} G_1(x,\xi), & a \leq x < \xi, \\ G_2(x,\xi), & \xi < x \leq b, \end{cases} \qquad (7.9)$$

such that

1. Hom sol^{n}s G solves hom. eq^n for $x \neq \xi$

so $\mathcal{L} G_1 = \mathcal{L} G_2 = 0$ (7.10)

2. Hom bcs G satisfies hom bcs so

$$G_1(a,\xi) = 0, \quad G_2(b,\xi) = 0$$

L 16.1

Defining properties (cont.)

$$G(x,\xi) = \begin{cases} G_1(x,\xi), & a \leq x < \xi, \\ G_2(x,\xi), & \xi < x \leq b. \end{cases} \qquad (7.9)$$

1. Hom sol^{n}'s G solves hom eq^{n} $\forall x \neq \xi$

so $\mathcal{L} G_1 = 0$, $\mathcal{L} G_2 = 0$ (7.10)

2. Hom bcs G satisfies hom bcs

so $G_1(a,\xi) = 0$, $G_2(b,\xi) = 0$ (7.11)

3. Continuity cond^{n} G is cts at ξ

so $G_1(\xi,\xi) = G_2(\xi,\xi)$ (7.12)

4. Jump cond^{n} Derivative G' is discts at ξ

$$[G']_{\xi^-}^{\xi^+} = \frac{1}{\alpha(\xi)} \qquad (7.13)$$

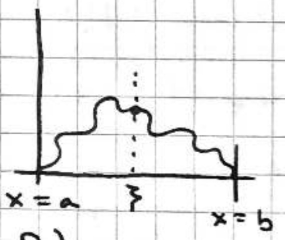
where α is defined in $\mathcal{L}$

Ex: verify (1) − (4) true for (7.4)

7.3 Constructing $G(x,\xi)$: Boundary value problems

Solve $\mathcal{L} G(x,\xi) = \delta(x-\xi)$ with $G(a,\xi) = G(b,\xi) = 0$

1 & 2) Solves hom eq^{n} with hom bcs



Assume 2 indpt sol^{n}s are known $y_1(x)$, $y_2(x)$.

For $a \leq x < \xi$, $G(x,\xi) = Ay_1(x) + By_2(x)$

s.t. $Ay_1(a) + By_2(a) = 0$ (i.e. choose suitable A, B)

Thus define a compl f^{n} (2.3) $y_-(x)$ s.t. $y_-(a) = 0$

So general form with $G_1(a) = 0$,

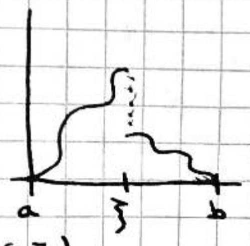
$$\boxed{G_1 = \underset{\uparrow}{C}\, y_-(x) \quad \text{with} \quad y_-(a) = 0 \quad (7.14)}$$

arb constant (defined later $C = (C(\xi))$)

For $b \geq x > \xi$, similarly find

$$\boxed{G_2 = D y_+(x) \quad \text{with} \quad y_+(b) = 0 \quad (7.15)}$$

where y_+ is compl f^{n} s.t. $y_+^{*}(b) = 0$ (2.3)



3) Why is G cts at $x = \xi$?

Suppose G were discts, so $G \propto H(x-\xi)$ see (6.7) which satisfies $G' \propto \delta(x-\xi)$ and $G'' \propto \delta'(x-\xi)$.

But $\mathcal{L}G \propto \alpha(x)\,\delta'(x-\xi) + \beta(x)\,\delta(x-\xi) + \gamma(x)\,H(x-\xi) = \delta(x-\xi)$

i.e. there is no term $\delta'(x-\xi)$ ※

Hence, we have $[G]_{\xi^-}^{\xi^+} = 0$, so we have

$$\boxed{C y_-(\xi) = D y_+(\xi) \quad (7.16)}$$

4. Why the jump condition for G' at ξ?

Integrate $\mathcal{L}G = \delta(x-\xi)$ across $x = \xi$:

$$\text{LHS} = \int_{\xi_-}^{\xi_+} \mathcal{L}G\,dx = \int_{\xi_-}^{\xi_+} (\underbrace{\alpha G'' + \beta G'}_{\text{by parts}} + \gamma G)\,dx$$

$$= \alpha(\xi)\,[G']_{\xi_-}^{\xi_+} + (\beta - \alpha')\,\underbrace{\underset{\substack{\downarrow\\ \text{zero}}}{[G]_{\xi_-}^{\xi_+}} + \underset{\substack{\searrow\\ \text{zero}}}{\int_{\xi_-}^{\xi_+}(\gamma - \beta' + \alpha'')\,G\,dx}}_{\text{by continuity of } \alpha, \beta, \gamma,\ G}$$

$$\text{RHS} = \int_{\xi_-}^{\xi_+} \delta(x-\xi)\,dx = 1$$

Thus, $[G']_{\xi_-}^{\xi_+} = \dfrac{1}{\alpha(\xi)}$ from which

$$\boxed{D y'_+(\xi) - C y'_-(\xi) = \frac{1}{\alpha(\xi)} \quad (7.17)}$$

Wronskian $W(\xi)$

Solving (7.16) and (7.17) we find

$$C(\xi) = \frac{y_+(\xi)}{\alpha(\xi)W(\xi)}, \quad D(\xi) = \frac{y_-(\xi)}{\alpha(\xi)W(\xi)} \qquad (7.18)$$ ㉒

where $W(\xi) = y_-(\xi)\,y'_+(\xi) - y_+(\xi)\,y'_-(\xi)$ (7.19)

$\neq 0$ since y_-, y_+ are lin indep

$$\boxed{G(x,\xi) = \begin{cases} \dfrac{y_-(x)\,y_+(\xi)}{\alpha(\xi)W(\xi)}, & \alpha \leq x < \xi, \\[2ex] \dfrac{y_+(x)\,y_-(\xi)}{\alpha(\xi)W(\xi)}, & \xi < x \leq \beta. \end{cases} \qquad (7.20)}$$

So the solution (7.6) with $y(a) = y(b) = 0$, is

$$y(x) = \int_a^b G(x,\xi)\,f(\xi)\,d\xi = \int_a^x G_2(x,\xi)\,f(\xi)\,d\xi + \int_x^b G_1(x,\xi)\,f(\xi)\,d\xi$$

$$= y_+(x)\int_a^x \frac{y_-(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\,d\xi + y_-(x)\int_x^b \frac{y_+(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\,d\xi \qquad (7.21)$$

Notes 1. If $\mathcal{L}$ is in SL form, i.e. $\beta = \alpha'$, the denominator

$\alpha(\xi)W(\xi)$ is a constant and G is symmetric $G(x,\xi) = G(\xi,x)$.

Ex Show $\frac{d}{dx}(\alpha(x)W(x)) = 0$ using $\alpha' = \beta$ and (2.10) (self adjoint eq)

2. Usually take $\alpha = 1$ (but SL form $\alpha < 0$)

3. Indefinite integrals $\int_x$ in (7.21) are particular integrals in general solution (2.5)

Ex For $-y'' = f(x)$, $y(0) = y(1) = 0$ directly construct $G(x,\xi)$ (7.4)

Example Solve $y'' - y = f(x)$, with $y(0) = y(1) = 0$

Construct $G(x,\xi)$, 1 & 2) hom sol^{n}s $y_1 = e^x$, $y_2 = e^{-x}$

so with hom bcs (by inspection)

$$G = \begin{cases} C\sinh x & (0 \le x < \xi), \\ D\sinh(1-x) & (\xi < x \le 1). \end{cases}$$

3) Cty at $\xi \Rightarrow C\sinh\xi = D\sinh(1-\xi)$

4) $[G']_-^+ = 1 \Rightarrow -D\cosh(1-\xi) - C\cosh(\xi) = 1$

$$\therefore -D\left[\cosh(1-\xi)\sinh\xi + \sinh(1-\xi)\cosh\xi\right] = \sinh\xi$$

$$\therefore D = -\sinh\xi/\sinh 1 \quad , \quad C = -\sinh(1-\xi)/\sinh 1$$

So the solution is

$$y = -\frac{\sinh(1-x)}{\sinh 1}\int_0^x \sinh\xi \, f(\xi)\, d\xi - \frac{\sinh x}{\sinh 1}\int_x^1 \sinh(1-\xi)\, f(\xi)\, d\xi \qquad (7.22)$$

L17.1

Green's f^n BVP (Cont)

Inhomogeneous BCs

Find y_P sol^n to $\mathcal{L}y = 0$ satisfying bcs $y(a) = c$, $y(b) = d$.

Find Green's f^n for $\mathcal{L}y_q = f(x)$ with $y_q(a) = y_q(b) = 0$

using $y_q(x) = y(x) - y_P(x)$

Example $y'' - y = f(x)$ with $y(0) = 0$, $y(1) = 1$

$y_P = A\sinh x + B\cosh x$, $y_P(0) = 0 \Rightarrow \cancel{y}\ B = 0$

$y_P(1) = 1 \Rightarrow A = 1/\sinh 1$

Solve for $y_q = y - y_P$ with $y_q(a) = y_q(1) = 0$

Sol^n is $y(x) = \dfrac{\sinh x}{\sinh 1} + y_q(x)$ (same sol^n as (7.22))

Higher order ODEs

If $\mathcal{L}y = f(x)$ to n^{th} order (coeff $\alpha(x)\dfrac{d^n y}{dx^n}$) with hom b.c.s, then generalise Green's f^n $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ with properties

1 & 2) G_1, G_2 hom sol^ns satisfying hom bcs

3) Cty $G_1 = G_2$, $G_1' = G_2'$, …, $G_1^{(n-2)} = G_2^{(n-2)}$ at $x = \xi$

4) Jump in $(n-1)$st derivative

$$\left[G^{(n\pm1)}\right]_{\xi_-}^{\xi_+} = \frac{1}{\alpha(\xi)}$$

(SEE Ex 3 Q 4)

Eigenfunction expansions

Suppose $\mathcal{L}$ in SL form (2.7) with eigenfunctions y_n and evals λ_n.

Then seek $G(x,\xi) = \sum_{n=1}^{\infty} A_n(\xi)\, y_n(x)$ satisfying $\mathcal{L}G(x,\xi) = \delta(x-\xi)$.

$$\mathcal{L}G = \sum_n A_n \mathcal{L}_n y_n = \sum_n A_n \lambda_n w y_n$$

$$= \delta(x-\xi) = w \sum_n y_n(\xi)\, y_n / \mathcal{N} \quad \text{by (6.15) with } \mathcal{N} = \int w y_n^2\, dx.$$

So $A_n(\xi) = \dfrac{y_n(\xi)}{\lambda_n \mathcal{N}}$ by orthog.

Thus $$\boxed{G(x,\xi) = \sum_{n=1}^{\infty} \frac{y_n(\xi)\, y_n(x)}{\lambda_n \mathcal{N}} \quad (7.23)}$$

we obtained without δ in (2.31), refer to § 2.6 of SL theory

Simplifies if $\mathcal{N} = 1$.

7.4 Constructing $G(x,\xi)$ for Initial Value Problem

Solve $\mathcal{L}y = f(t)$ for $t \geqslant a$ with $y(a) = y'(a) = 0$ (7.24)

using $G(t,\tau)$ satisfying $\mathcal{L}G = \delta(t-\tau)$ with same bcs.

For $t<\tau$ $G_1 = Ay_1(t) + By_2(t)$ with $Ay_1(a) + By_2(a) = 0$

$Ay_1'(a) + By_2'(a) = 0$

Since y_1, y_2 indep, $W(a) \neq 0$ and $A = B = 0$.

So $G_1(t,\tau) = 0$, i.e. no change until impulse at τ.

For $t>\tau$, by cty of G, $G_2(\tau,\tau) = 0$ so choose $G_2 = D\overset{\text{depends on }\tau}{\overset{\swarrow}{y_+}}(t)$

with $y_+(\overset{t}{a}) = \tilde{A}\, y_1(\underset{t}{a}) + \tilde{B}\, y_2(\underset{t}{a})$ & $y_+(\tau) = 0$.

But discty in G' (7.13) : $[G']_{\tau^-}^{\tau^+} = G_2'(\tau,\tau) - G_1'(\tau,\tau) = \frac{1}{\alpha(\tau)}$

i.e. $\tilde{A}y_1'(\tau) + \tilde{B}y_2'(\tau) \underset{\substack{\uparrow \\ \text{no, WTF}}}{=} \frac{1}{\alpha(\tau)} \Rightarrow D(\tau) = \frac{1}{\alpha(\tau)\,y_+'(\tau)}$

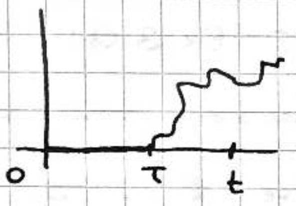
Hence, we have

$$G(t,\tau) = \begin{cases} 0 & t<\tau, \\ \dfrac{y_+(t)}{\alpha(\tau)\,y_+'(\tau)} & t>\tau. \end{cases} \qquad (7.25)$$

The solution to IVP (7.24) is

$$y(t) = \int_a^t G_2(t,\tau) f(\tau)\,d\tau + \int_t^\infty G_1(t,\tau) f(\tau)\,d\tau$$

$$= \int_a^t \frac{y_+(t) f(\tau)}{\alpha(\tau)\,y_+'(\tau)}\,d\tau \qquad (7.26)$$



Causality is "built in" as y depends only on f for values prior in time.

Example Solve $y'' - y = f(t)$ with $y(0) = y'(0) = 0$

1 & 2) Hom sol^ns & ics $\cdot\ t<\tau \Rightarrow G_1 = 0$

$\cdot\ t>\tau \Rightarrow G_2 = Ae^t + Be^{-t}$

3) Cty $\Rightarrow G_2(\tau) = 0$ so $G_2 = D\sinh(t-\tau)$

4) Jump $[G'] = 1 \Rightarrow G_2' = D\cosh(t-\tau) = D = 1$ at τ

Hence sol^n is

$$y(t) = \int_0^t f(\tau)\sinh(t-\tau)\,d\tau.$$

8 FOURIER TRANSFORMS

8.1 Introduction

Def^n The Fourier transform (FT) of function $f(x)$ is

$$\tilde{f}(k) = \mathcal{F}(f)(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad (8.1)$$

and the inverse Fourier transform is

$$f(x) = \mathcal{F}^{-1}(\tilde{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk \qquad (8.2)$$

(Beware these are different conventions)

The Fourier inversion theorem states that

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x) \qquad (8.3)$$

with a sufficient condition for this to be true is that f and $\tilde{f}$ are absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |f(x)|\, dx = M < \infty \;, \quad \overset{?}{\text{so}} \quad f \to 0 \text{ as } x \to \pm\infty$$

Gaussian example Find the Fourier Transform of $f(x) = \underset{\frac{1}{\sigma\sqrt{\pi}}}{\uparrow} e^{-x^2/\sigma^2}$ (8.4)

$$\tilde{f}(k) = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2} \cos kx\, dx \qquad \text{(as sin is odd)}$$

Consider
$$\frac{d\tilde{f}}{dk} = \tilde{f}'(k) = \frac{-1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x\, e^{-x^2/\sigma^2} \sin kx\, dx$$

$$= \frac{-1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{k\sigma^2}{2} e^{-x^2/\sigma^2} \cos kx\, dx \qquad \text{by parts}$$

$$= -\frac{k\sigma^2}{2}\tilde{f}(k) \qquad \ddot{o}$$

Solve for $\tilde{f}$ and deduce

$$\tilde{f}(k) = C e^{-\frac{k^2\sigma^2}{4}}$$

But $\tilde{f}(0) = 1$ so $C = 1$ and $\boxed{\tilde{f}(k) = e^{-k^2\sigma^2/4} \qquad (8.5)}$

L18.1

Exponential exercise Show that $f(x) = e^{-a|x|}$, $a \geqslant 0$

has FT $\hat{f}(k) = \frac{2a}{a^2+k^2}$ (8.6) in two ways

(i) Integrate $2\int_0^\infty e^{-ax} \cos kx \, dx$ by parts twice

(ii) Integrate $\int_0^\infty e^{-(a-ik)x} dx + \int_{-\infty}^0 e^{(a+ik)x} dx$ directly

Note that if $f(x) = e^{-ax} H(x)$

then $\tilde{f}(k) = \frac{1}{ik+a}$ (8.6a)

8.2 FT relation to Fourier series

We can write (1.11) as $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$ (*)

where $k_n = \frac{n\pi}{L}$ so write $\Delta k = \frac{k_n}{n} = \frac{\pi}{L}$, then

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-ik_n x} dx = \frac{\Delta k}{2\pi}\int_{-L}^{L} f(x) e^{-ik_n x} dx$$

So (*) becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x} \int_{-L}^{L} f(x') e^{-ik_n x'} dx'$$

But $\sum_{n=-\infty}^{\infty} \Delta k \, g(k_n) \longrightarrow \int_{-\infty}^{\infty} g(k) \, dk$

where $g(k_n) = \frac{e^{ik_n x}}{2\pi}\int_{-L}^{L} f(x') e^{-ik_n x'} dx'$.

So take limit $L \to \infty$ and obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, e^{ikx} \left[\int_{-\infty}^{\infty} f(x') e^{-ikx'} dx'\right] = \mathcal{F}^{-1}(\mathcal{F}(f))(x) \quad \text{i.e. } (8.3)$$

Note when f is discts at x, the FT gives the midpoint

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right). \quad (8.7)$$

8.3 FT Properties

1 **Linearity** $h(x) = \lambda f(x) + \mu g(x) \iff \hat{h}(k) = \lambda \hat{f}(k) + \mu \hat{g}(k)$ (8.8)

2 **Translation** $h(x) = f(x-\lambda) \iff \tilde{h}(k) = e^{-i\lambda k}\tilde{f}(k)$ (8.9)

3 **Frequency** $h(x) = e^{i\underset{\lambda}{\cancel{k}}x} f(x) \iff \tilde{h}(k) = \tilde{f}(k-\lambda)$ (8.10)

4 **Scaling** $h(x) = f(\lambda x) \iff \tilde{h}(k) = \frac{1}{|\lambda|}\tilde{f}(k/\lambda)$ (8.11) SELF DUAL

5 **Times x** $h(x) = x f(x) \iff \tilde{h}(k) = i\tilde{f}'(k)$ (8.12)

because $\int_{-\infty}^{\infty} x f(x) e^{-ikx} dx = -\frac{1}{i}\frac{d}{dk}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

6 **Derivative** $\boxed{h(x) = f'(x) \iff \tilde{h}(k) = ik\tilde{f}(x) \quad (8.13)}$

because $\tilde{h}(k) = \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx = ik\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

7 Duality Consider $f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{-ikx}\, dk$

so $k \leftrightarrow x \Rightarrow f(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(x)\, e^{-ikx}\, dx$

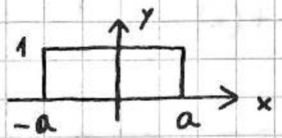
Thus if $\boxed{g(x) = \tilde{f}(x) \iff \tilde{g}(k) = 2\pi f(-k) \quad (8.14)}$

We have $f(-x) = \frac{1}{2\pi}\mathcal{F}(\tilde{f})(x) = \frac{1}{2\pi}\mathcal{F}^2(f)(x)$

repeating $\mathcal{F}^4(f)(x) = 4\pi^2 f(x)$. Ex Verify 1 − 7

"Top hat" example

Find FT for $f(x) = \begin{cases} 1, & |x| \leq a, \\ 0, & |x| > a. \end{cases}$



$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \int_{-a}^{a} \cos kx\, dx = 2\,\frac{\sin ka}{k} \quad (8.15)$$

Fourier inversion thm (8.3) implies $\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{\sin ka}{k}\, dk = \begin{cases} 1, & |x| \leq a, \\ 0, & |x| > a. \end{cases}$

Set $x = 0$, then take dummy $k \to x$ to obtain Dirichlet's discts formu

$$\int_0^{\infty} \frac{\sin ax}{x}\, dx = \begin{cases} \pi/2, & a > 0 \\ 0, & a = 0 \\ -\pi/2, & a < 0 \end{cases} = \frac{\pi}{2}\,\mathrm{sgn}(a) \quad (8.16)$$

Here we allow $a < 0$, so $\sin(-ax) = -\sin ax$. (see RJ notes for direct inverse)

8.4 Convolution & Parseval's Theorem

We want to multiply FTs in frequency domain $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$, so consider

$$\begin{aligned} h(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,\tilde{g}(k)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy\right)\tilde{g}(k)\, e^{ikx}\, dk \\ &= \int_{-\infty}^{\infty} f(y)\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(k)\, e^{ikx}\, e^{-iky}\, dk\right) dy \\ &= \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = f * g\,(x) \quad (8.17) \end{aligned}$$

By duality (8.14) we also have

$$h(x) = f(x)\,g(x) \iff \tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(p)\,\tilde{g}(k-p)\, dp$$

Parseval's Theorem Consider $h(x) = g^*(-x)$, then

$$\begin{aligned} \tilde{h}(k) &= \int_{-\infty}^{\infty} g^*(-x)\, e^{-ikx}\, dx = \left[\int_{-\infty}^{\infty} g(-x)\, e^{ikx}\, dx\right]^* \\ &= \left[\int_{-\infty}^{\infty} g(y)\, e^{-iky}\, dy\right]^* = \tilde{g}^*(k) \end{aligned}$$

Substitute into (8.17) $\Rightarrow \int_{-\infty}^{\infty} f(y)\, g^*(\overset{\leftrightarrow}{x-y})\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\,\tilde{g}^*(k)\, e^{ikx}\, dk.$

Parseval (cont.) Conv. thm (8.17) $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$

$$\Rightarrow h(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy = f * g\,(x)$$

Substitute $g^*(-x) \Longleftrightarrow \tilde{g}^*(k)$

$$\int_{-\infty}^{\infty} f(y)\,g^*(y-x)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}^*(k)\,e^{ikx}\,dk$$

Take $x=0$, then dummy $y \to x$ on LHS

$$\int_{-\infty}^{\infty} f(x)\,g^*(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,\tilde{g}^*(k)\,dk \quad \text{(8.19)} \quad \text{i.e. } \langle g, f\rangle = \frac{1}{2\pi}\langle \tilde{g}, \tilde{f}\rangle$$

Now set $g^* = f^*$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk \quad (8.20)}$$ Parseval's theorem

8.4 FT of generalized functions

(Note caveat)

Delta f^n $\delta(x)$ Consider Fourier inversion

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-iku}\,du\right]e^{ikx}\,dk$$

$$= \int_{-\infty}^{\infty} f(u)\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}\,dk\right]}_{\delta(x-u)}du$$

So identity

$$\delta(x-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}\,dk.$$

- If $f(x) = \delta(x)$, then $\tilde{f}(k) = \int_{-\infty}^{\infty}\delta(x)\,e^{ikx}\,dx = 1$. (8.21)
- If $f(x) = 1$, then $\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\,dx = 2\pi\,\delta(k)$ (8.22)
- If $f(x) = \delta(x-a)$, then $\tilde{f}(k) = e^{-ika}$ (8.23)

Trig functions

$\frac{1}{2}(e^{i\omega x} + e^{-i\omega x})$

$f(x) = \cos\omega x \Rightarrow \tilde{f}(k) = \pi(\delta(k+\omega) + \delta(k-\omega))$

$f(x) = \sin\omega x \Rightarrow \tilde{f}(k) = i\pi(\delta(k+\omega) - \delta(k-\omega))$

Ex Find $\mathcal{F}^{-1}$ for $\sin\omega k$, $\cos\omega k$ using duality

Heaviside functions Subtle derivation require central value $H(0) = \frac{1}{2}$.

Then $H(x) + H(-x) = 1 \quad \forall x$, continuous at zero.

By (8.22) $\tilde{H}(k) + \tilde{H}(-k) = 2\pi\,\delta(k)$ $(*)$

Recall (6.7) $H'(x) = \delta(x)$ which implies $ik\tilde{H}(k) = 1$ $(\dagger)$

by (8.13), (8.21)

But $k\delta(k) = 0$, so $(*)$ and $(+)$ are consistent if

$$\tilde{H}(k) = \pi\delta(k) + \frac{1}{ik}$$

Dirichlet discts formula (8.16) Rewrite as $\frac{1}{2}\operatorname{sgn}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{ik}\,dk$

so $f(x) = \frac{1}{2}\operatorname{sgn}(x) \iff \tilde{f}(k) = \frac{1}{ik}$

8.5 Applications of Fourier Transforms

Motivation: for ODE and BVP

Consider $y'' - y = f(x)$ with hom bcs $y \to 0$ as $x \to \pm\infty$

Take the FT $(-k^2 - 1)\tilde{y} = \tilde{f}$ by (8.13) so the solution is

$$\tilde{y}(k) = -\frac{\tilde{f}(k)}{1+k^2} \equiv \tilde{f}(k)\,\tilde{g}(k) \quad \text{where } \tilde{g}(k) = -\frac{1}{1+k^2}$$

But $\tilde{g}$ is FT of $g(x) = -\frac{1}{2}e^{-|x|}$ (see (8.6)). Thus conv thm (8.17) implies

$$y(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du = -\frac{1}{2}\int_{-\infty}^{\infty} f(u)\,e^{-|x-u|}\,du$$
$$= -\frac{1}{2}\int_{-\infty}^{x} f(u)\,e^{u-x}\,du - \frac{1}{2}\int_{x}^{\infty} f(u)\,e^{x-u}\,du$$

which is in the form of a BVP Green's f^n (7.20)

Ex Verify by constructing $G(x,u)$ directly.

Motivation: Signal processing (IVP)

Suppose (given) input $J(t)$ acted upon by linear operator $\mathcal{L}_{in}$ to yield output $\mathcal{O}(t)$. The FT $\tilde{J}(\omega)$ is denoted the resolution

$\tilde{J}(\omega) = \int_{-\infty}^{\infty} J(t)\,e^{-i\omega t}\,dt$ (8.27).

In freq domain, action $\mathcal{L}_{in}$ $J(t)$ means $\tilde{J}(\omega)$ is multiplied by a transfer function $\tilde{R}(\omega)$ to yield the output $\mathcal{O}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(\omega)\,\tilde{J}(\omega)\,e^{i\omega t}\,d\omega$ (8.28)

with response function given by

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(\omega)\,e^{i\omega t}\,d\omega \quad (8.29)$$

By the convolution thm (8.17) output is

$$\mathcal{O}(t) = \int_{-\infty}^{\infty} J(\underset{\uparrow\, u}{\cancel{x}})\,R(t-u)\,du.$$

Now assume no input $J(t) = 0$ for $\overset{t}{\cancel{x}} < 0$ and zero output $R(t) = 0$ for $t < 0$. (i.e. $R(t-u)$ has source $\delta(t-u)$) so we require by causality $0 < u < t$.

L19.3

$$\vartheta(t) = \int_0^t J(\underset{u}{\cancel{t}})\, R(t-u)\, du \quad (8.30)$$

Have the same form as the IVP for Green's function.

General transfer functions for ODEs

Suppose input / output rel^n given by a linear operator (n^{th} order DE)

$$\mathcal{L}\, O(t) = \sum_{\underset{\uparrow \text{ const.}}{i=1}}^{n} a_i \frac{d^i}{dt^i} O(t) = \underset{\uparrow \text{ (here set } \mathcal{L}_{in} = 1)}{J(t)} \qquad (8.31)$$

Take the Fourier transform

$$(a_0 + a_1 (i\omega) + \cdots + a_n (i\omega)^n)\, \tilde{O}(\omega) = \tilde{J}(\omega)$$

So the transfer function (8.28) is

FTA

$$\tilde{R}(\omega) = \frac{1}{a_0 + a_1 (i\omega) + \cdots + a_n (i\omega)^n} \qquad (8.32)$$

Factorise n^{th} degree poly into linear terms $(i\omega - c_j)^{k_j}$ with multiplicities $\sum k_j = \ast\, n$.

L 20.1

General transfer f^n (cont.)

Factorise n^{th} deg poly into roots $(i\omega - c_j)^{k_j}$, with multiplicity k_j.

Then
$$\tilde{R}(\omega) = \frac{1}{(i\omega - c_1)^{k_1} \cdots (i\omega - c_J)^{k_J}}$$
$$= \sum_{j=1}^{J} \sum_{m=1}^{k_j} \frac{\Gamma_{jm}}{(i\omega - c_j)^{\cancel{k}m}} \quad (8.33)$$

since it can be expanded in <u>partial fractions</u> (const Γ_{jm}).

For repeated roots $\frac{1}{(i\omega - c_j)^{k_j}} \rightarrow \frac{\Gamma_{j1}}{i\omega - c_j} + \cdots + \frac{\Gamma_{jk_j}}{(i\omega - c_j)^{k_j}}$.

To solve we invert $\frac{1}{(i\omega - a)^m}$, $m \geqslant 1$.

We know (8.6a)
$$\mathcal{F}^{-1}\left(\frac{1}{i\omega - a}\right) = \begin{cases} 0, & t < 0, \\ e^{at}, & t > 0 \end{cases}$$

for $Re(a) < 0$, so we assume $Re(c_j) < 0 \ \forall j$ to elim exp growth.

For $m = 2$, note
$$i\frac{d}{d\omega}\left(\frac{1}{i\omega - a}\right) = \frac{1}{(i\omega - a)^2}$$

and recall (8.12)
$$\mathcal{F}(t f(t)) = i\tilde{f}'(\omega)$$

so
$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^2}\right) = \begin{cases} 0, & t < 0, \\ t e^{at}, & t > 0. \end{cases}$$

By induction
$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^m}\right) = \begin{cases} 0, & t < 0, \\ \frac{t^{m-1}}{(m-1)!} e^{\overset{a}{\cancel{k}}t}, & t > 0. \end{cases} \quad (8.34)^?$$

The response function takes the form
$$R(t) = \sum_j \sum_m \Gamma_{jm} \frac{t^{m-1}}{(m-1)!} e^{\cancel{k}t}_{c_j}, \quad t > 0 \quad (8.35)$$

We can solve (8.31) in Green's function form (8.30) or directly invert $\tilde{R}(\omega)\tilde{J}(\omega)$ for polynomial $\tilde{J}(\omega)$.

<u>Example</u> (Damped oscillator)

Solve $\mathcal{L}y \equiv y'' + 2py' + (p^2 + q^2)y = f(t)$ with damping $p > 0$.

↖ hom b.c.s $y(0) = y'(0) = 0$

FT is $(i\omega)^2\tilde{y} + 2ip\omega\tilde{y} + (p^2 + q^2)\tilde{y} = \tilde{f}$

so
$$\tilde{y} = \frac{\tilde{f}}{-\omega^2 + 2ip\omega + (p^2 + q^2)} = \tilde{R}(\omega)\tilde{f}(\omega)$$

Inverting with convolution theorem (8.17)

$$y(t) = \int_0^t R(t-\tau) f(\tau)\, d\tau$$

and response

$$R(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-\tau)}}{-\omega^2 + 2ip\omega + p^2 + q^2}\, d\omega$$

Ex Show $\mathcal{L} R(t-\tau) = \delta(t-\tau)$ using (8.23) i.e. the response function $R(t-\tau)$ is the Green's f^n (see Ex III Q4)

8.6 Discrete Fourier Transform

Discrete sampling and Nyquist frequency

Sample a signal $h(t)$ at equal times $t_n = n\Delta$, with time-sampling Δ with values $h_n = h(n\Delta)$, $n = \ldots, -1, 0, 1, \ldots$ (8.36)

i.e. with sampling frequency $f_s = 1/\Delta \neq (\omega_s = 2\pi f_s = 2\pi/\Delta)$.

The Nyquist freq $f_c = \frac{1}{2\Delta}$ is the highest freq actually sampled with Δ. Suppose we have a signal with a given freq f

$$g_f(t) = A\cos(2\pi f t + \phi) = \mathrm{Re}\left(A e^{2\pi i f t + i\phi}\right)$$

$$= A^{*} e^{i\phi} e^{2\pi i f t} + A e^{-i\phi} e^{-2\pi i f t} \quad \leftarrow \text{Times } \tfrac{1}{2} \qquad (8.38)$$

(i.e. for complex FS, sum of pos freq f and neg freq $-f$)

What if we sample at Nyquist freq f_c?

$$g_{f_c}(t_n) = A\cos\left(2\pi\left(\tfrac{1}{2\Delta}\right) n\Delta + \phi\right) = A\cos\pi n\cos\phi + \text{zero}$$

$$= A'\cos(2\pi f_c t_n) \quad \text{with } A' = A\cos\phi \qquad (8.39)$$

so phase / ampl info is lost (no distinction) and we can identify i.e. (8.38) and (8.39) are aliased together.

What happens if we sample at $f > f_c$?

Exercise Take $f = f_c + \delta f > f_c$ $(\delta f < f_c)$ and show that

$$g_f(t_n) = A\cos\left(2\pi(f_c + \delta f)t_n + \phi\right)$$

$$= A\cos\left(2\pi(f_c - \delta f)t_n - \phi\right) \qquad (8.40)$$

actually $-(f_c - \delta f)$

So the effect is to alias a "ghost signal" to freq $f_c - \delta f$

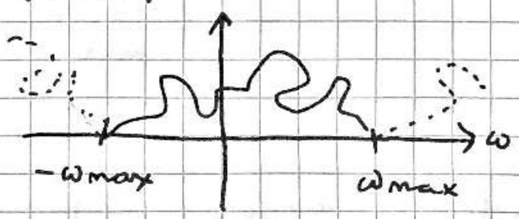
Sampling theorem A signal $g(t)$ is bandwidth limited if it contains no frequency above $\omega_{max} = 2\pi f_{max}$ i.e. $\tilde{g}(\omega) = 0$ for $|\omega| > \omega_{max}$

So $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_{max}}^{\omega_{max}} \tilde{g}(\omega) e^{i\omega t} d\omega$ (8.41)

Set sampling to satisfy Nyquist cond^n $\Delta = \frac{1}{2f_{max}}$, then $g_n \equiv g(t_n) = \frac{1}{2\pi}\int_{-\omega_{max}}^{\omega_{max}} \tilde{g}(\omega)\, e^{i\pi n\omega/\omega_{max}} d\omega$ which is a complex FS coeff (1.11) $c_n \times \frac{\omega_{max}}{\pi}$

This FS represents a periodic f^n (period $2\omega_{max}$)

$$\tilde{g}_{per}(\omega) = \frac{\pi}{\omega_{max}} \sum_{n=-\infty}^{\infty} g_n\, e^{-in\pi\omega/\omega_{max}} \qquad (8.42)$$



The actual FT $\tilde{g}(\omega)$ is found by multiplying by "top hat".

$$\tilde{h}(\omega) = \begin{cases} 1 & \text{for } |\omega| < \omega_{max}, \\ 0 & \text{for } |\omega| > \omega_{max}. \end{cases}$$

I.e. $\tilde{g}(\omega) = \tilde{g}_{per}(\omega)\, \tilde{h}(\omega)$ (8.43) which is exact.

Inverting with (8.42)

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}_{per}(\omega)\, \tilde{h}(\omega)\, e^{i\omega t} d\omega$$

$$= \frac{1}{2\omega_{max}} \sum_{n=-\infty}^{\infty} g_n \int_{-\omega_{max}}^{\omega_{max}} \exp\left(i\omega t - i\omega \frac{n\pi}{\omega_{max}}\right) d\omega$$

$$= \sum_{n=-\infty}^{\infty} g_n \frac{\sin(\omega_{max} t - n\pi)}{\omega_{max} t - n\pi} \qquad (8.44)$$

So $g(t)$ is exactly represented after discrete sampling at discrete times (sampling thm).

Discrete Fourier Transform

Suppose we have a finite no. N of samples

$$h_m = h(t_m), \quad t_m = m\Delta, \quad m = 0, 1, \ldots, N-1 \qquad (8.45)$$

Approx the FT at N freq within Nyquist freq ($f_c = \frac{1}{2\Delta}$) equally spaced $\Delta f = \frac{1}{N\Delta}$ in the range $-f_c \leq f \leq f_c$. We could take $f_n = n\Delta f = \frac{n}{N\Delta}$ with $n = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2}$.

This has N+1 freq, but f_c and $-f_c$ are aliased (8.39)

Note also that $(\frac{N}{2}+m)\Delta_f = f_c + \delta f$ is aliased back to $-(\frac{N}{2}-m)\Delta_f$

$= -(f_c - \delta f)\Delta f$

so we choose instead

$$f_n = \frac{n}{N\Delta} \quad \text{with} \quad n = \underset{f=0}{0},\ \underset{\Delta f}{1},\ 2, \dots,\ \underset{f_c}{\frac{N}{2}}, \dots,\ \underset{-\Delta f}{N-1} \qquad (8.46)$$

The <u>discrete FT</u> at freq f_n becomes

$$\tilde{h}(f_n) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f_n t} dt \approx \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i f_n t_m}$$

$$= \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i m n/N} \equiv \Delta \tilde{h}_d(f_{\cancel{m}\,n}) \qquad (8.47)$$

So the matrix $[DFT]_{mn} = e^{-2\pi i m n/N}$ $\quad m, n = 0, \dots, N-1$

defines discrete FT for $h = \{h_m\}$ as $\tilde{h}_d = [DFT]\, h$.

The inverse of $[DFT]$ is $\frac{1}{N}[DFT]^{\dagger}$ (unitary $)^{ish}$

and its built from roots of unity $\omega = e^{-2\pi i/N}$

e.g. $N = 4, \omega = -i,$ $\quad DFT = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$

The <u>inverse DFT</u> is

$$h_m = h(t_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(\omega) e^{i\omega t_m} d\omega = \int_{-\infty}^{\infty} \tilde{h}(f) e^{2\pi i f t_m} df$$

$$\simeq \sum_{n=0}^{N-1} \frac{1}{N} \tilde{h}_d(f_n) e^{2\pi i m n/N} \qquad (8.48)$$

<u>Ex</u> Establish Parseval's theorem for DFT

$$\sum_{m=0}^{N-1} |h_m|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{h}_m|^2 \qquad (8.49)$$

The convolution theorem has g_m, h_m

$$c_k = \sum_{m=0}^{N-1} g_m h_{k-m} \iff \tilde{c}_k = \tilde{g}_k \tilde{h}_{k \leftarrow d?}$$

PART IV : PDEs on UNBOUNDED DOMAINS

9 CHARACTERISTICS

9.1 Well-posed Cauchy problems

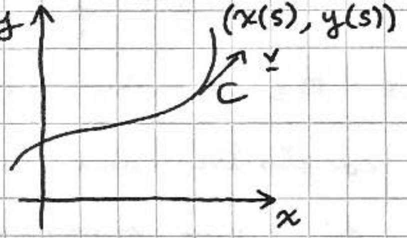
Solving PDEs depends on the nature of the equations in combination with the boundary and/or initial conditions.

A Cauchy problem is the PDE for ϕ together with the auxiliary data (i.e. ϕ and its derivatives) specified on a surface (or curve in 2D) which is called Cauchy data.

A Cauchy problem is well-posed if

1) A solution exists,
2) The solution is unique,
3) The solution depends continuously on auxiliary data.

9.2 Method of characteristics



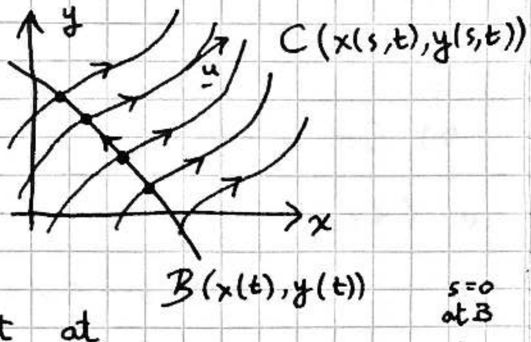
Consider a parametrised curve C given by $(x(s), y(s))$ with tangent vector $\underline{v} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$.

For a function $\phi(x,y)$ we can define a directional derivative along C

$$\left.\frac{d\phi}{ds}\right|_C = \frac{dx}{ds}\frac{\partial\phi}{\partial x} + \frac{d\overset{y}{x}}{ds}\frac{\partial\phi}{\partial y} = \underline{v}\cdot\nabla\phi\big|_C \qquad (9.1)$$

If $\underline{v}\cdot\nabla\phi = 0$, then $\frac{d\phi}{ds} = 0$ and $\phi = \text{const.}$ along C.

Now suppose we have a vector field $\underline{u} = (\alpha(x,y), \beta(x,y))$ (9.2) with a family of integral curves C which are non-intersecting and fill $\mathbb{R}^2$, i.e. at a point (x,y) the integral curve has tangent $\underline{u}$.



Define a curve B by $(x(t), y(t))$ transverse to $\underline{u}$, such that its tangent $\underline{w} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ is nowhere parallel to $\underline{u}$.

s=0 at B ↓

Label each integral curve C of $\underline{u}$ using t at the intersection with B, then take s to parametrise along the curve.

L21.2

Our integral curves satisfy

$$\frac{dx}{ds} = \alpha(x,y) \qquad \frac{dy}{ds} = \beta(x,y) \qquad (9.3)$$

Solve these to find a family of characteristic curves along which t remains constant (i.e. new coordinate system).

9.3 Characteristics of a first order PDE

Consider the linear 1^{st} order PDE

$$\alpha(x,y)\frac{d\phi}{dx} + \beta(x,y)\frac{\partial\phi}{\partial y} = 0 \qquad (9.4)$$

with specified Cauchy data on an initial curve B $(x(t), y(t))$

$$\phi(x(t), y(t)) = f(t) \qquad (9.5).$$

Note from (9.1) and (9.2) that $\alpha\phi_x + \beta\phi_y = \underline{u}\cdot\nabla\phi = \frac{d\phi}{ds}\Big|_C$, which is the dir der along integral curves C of $\underline{u} = (\alpha, \beta)$, called the <u>characteristic curves</u> of the PDE (see (9.3)). Since $\frac{d\phi}{ds} = 0$ by (9.4), the f^n $\phi(x,y)$ will be constant along the curves C.

I.e. the Cauchy data $f(t)$ defined on B at $s=0$ will be propagated constantly along the curves C to give the solution

$$\phi(s,t) = \phi(x(s,t), y(s,t)) = f(t) \qquad (9.6).$$

To obtain $\phi(x,y)$, transform coordinates from $\phi(s,t)$ using

$$s = s(x,y), \quad t = t(x,y) \qquad (\text{provided Jacobian } x_t y_s - x_s y_t \neq 0)$$

to finally obtain $\phi(x,y) = f(t(x,y))$. (9.7)

<u>Prescription</u> To solve (9.4) with (9.5)

1) Find characteristic eq^ns (9.3).

2) Parametrise initial conditions on B $(x(t), y(t))$ (9.8).

3) Solve characteristic eq^n to find $x = x(s,t)$, $y = y(s,t)$ subject to i.c.s (9.8) at $s=0$ $x(0,t) = x(t)$, $y(0,t) = y(t)$.

4) Solve (9.4) with (9.1) $\frac{d\phi}{ds} = \alpha\phi_x + \beta\phi_y = 0$ (or γ) i.e. (9.6) $\phi(s,t) = f(t)$ (or solve ODE in inhom case).

5) Invert rel^ns $s = s(x,y)$, $t = t(x,y)$.

6) Change coords to find (9.7).

L21.3

Simple example Solve $\frac{\partial\phi}{\partial x}(x,y) = 0$ with $\phi(0,y) = f(y)$.

(Sol^n is clearly $\phi(x,y) = f(y)$)

1) $\frac{dx}{ds} = \alpha = 1$, $\frac{dy}{ds} = 0$ $(*)$

2) y-axis $(x(t), y(t)) = (0,t)$

3) From $(*)$, $x = s + c$, $y = d$, but at $s=0$, $x=0$, $y=t$

so $(x,y) = (s,t)$ ö

4) $\frac{d\phi}{ds} = 0 \Rightarrow \phi(s,t) = f(t)$ ← ~~ok?~~ yes

5) Invert $s = x$, $t = y$

6) Sol^n $\phi(x,y) = f(t) = f(y)$

Example Solve $e^x\phi_x + \phi_y = 0$ with $\phi(x,0) = \cosh x$

1) $\frac{dx}{ds} = e^x$, $\frac{dy}{ds} = 1$ $(*)$

2) x-axis $(x(t), y(t)) = (t, 0)$ $(+)$

3) From $(*)$, $-e^{-x} = s + c$, $y = s + d$.

At ~~t~~ $s = 0$, $-e^{-t} = c$, $0 = d$.

$\Rightarrow e^{-x} = e^{-t} - s$, $y = s$

4) $\frac{d\phi}{ds} = 0 \Rightarrow \phi(s,t) = \cosh t$

5) $s = y$, $e^{-t} = y + e^{-x} \Rightarrow t = -\log(y + e^{-x})$

6) $\phi(x,y) = \cosh\log(y + e^{-x})$

Inhomogeneous 1^{st} order PDEs

Solve $\alpha(x,y)\,\phi_x + \beta(x,y)\,\phi_y = \gamma(x,y)$ (9.9)

with Cauchy data $\phi(x(t), y(t)) = f(t)$ on curve B.

The characteristic curves C are identical to hom. case (9.4) with $\phi = f(t)$ at $s = 0$ on B.

I.e. $f(t)$ no longer propagated constantly as sol^n and must solve ODE (9.10). So upgrade Point 4 in prescription to integrate $\phi(s,t)$ before reverting to $\phi(x,y)$.

Example Solve $\phi_x + 2\phi_y = y e^x$ with $\phi = \sin x$ on $y = x$.

1) Char eq^n $\frac{dx}{ds} = 1$, $\frac{dy}{ds} = 2$ $(*)$

2) ICs on $y = x$, take $(x(t), y(t)) = (t, t)$ $(+)$

3) From $(*)$, $(+)$, $x = t + s$, $y = t + 2s$.

4) Solve $\frac{d\phi}{ds} = (t+2s)e^{t+s}$ subject to $\phi(0) = \sin(t + \cancel{s})$.

So $\phi(s,t) = (2s - 2 + t)e^{s+t}$ + const.

$\Rightarrow \phi(0,t) = (t-2)e^{t}$ + const. $= \sin t$

$\therefore \phi(s,t) = (2s - 2 + t)e^{s+t} + \sin t + (2-t)e^t$

5) Invert $s = y - x$, $t = 2x - y$

$\Rightarrow \phi(x,y) = (y-2)\,e^x + (y - 2x + 2)\,e^{2x-y} + \sin(2x - y)$

9.4 Second-order PDE classification

In two dimensions, the general 2^{nd} order linear PDE is

$$\mathcal{L}\phi \equiv a(x,y)\frac{\partial^2\phi}{\partial x^2} + 2b(x,y)\frac{\partial^2\phi}{\partial x\partial y} + c(x,y)\frac{\partial^2\phi}{\partial y^2} + d(x,y)\frac{\partial\phi}{\partial x} + e(x,y)\frac{\partial\phi}{\partial y} + f(x,y)\,\phi = 0 \quad (9.11)$$

The principal part is given by a

$$\sigma_p(x, y, k_x, k_y) = \underline{k}^T A\, \underline{k} = (k_x, k_y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

The PDE is classified by the evals of A.

- $b^2 - ac < 0$ elliptic (λ_1, λ_2 same sign)
- $b^2 - ac > 0$ hyperbolic (" diff sign)
- $b^2 - ac = 0$ parabolic ($\lambda_1 \lambda_2 = 0$)

L22.2

Examples · Wave eq^n (3.4) $a = \frac{1}{c^2}$, $b = 0$, $c = -1$ hyperbolic

· Heat eq^n (4.3) $a = 0$, $b = 0$, $c = -D$ parabolic

· Laplace's eq^n (5.1) $a = 1$, $b = 0$, $c = 1$ elliptic

Characteristic curves

A curve $f(x, y) = 0$ will be characteristic if $(f_x \ f_y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix}$ vanishes. The curve "can" be re-written as $y = y(x)$ where

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{f_x}{f_y} = -\frac{dy}{dx} \quad (9.13)$$

Substitute into (9.12) to yield the quadratic — which we may solve to find

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - ac}}{a} \quad (9.14)$$

If hyperbolic, $b^2 - ac > 0$, then 2 characteristics.

If parabolic, $b^2 - ac = 0$, one sol^n.

If elliptic, ☹

Transforming to char coords (u, v) will set $a = c = 0$ in (9.11) so PDE takes canonical form

$$\frac{\partial^2 \phi}{\partial u \partial v} + \underset{\uparrow \text{ lower order}}{\ldots} = 0$$

Example Consider $-y\phi_{xx} + \phi_{yy} = 0$ $(*)$

Have $a = -y$, $b = 0$, $c = 1$, so hyperbolic for $y > 0$
(elliptic for $y < 0$, parabolic $y = 0$)

Find characteristics for $y > 0$ satisfying (9.14)

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{y}} \quad \Rightarrow \quad \frac{2}{3}y^{3/2} \pm \cancel{d}x = c_{\pm}$$

So char curves are $u = \frac{2}{3}y^{3/2} + x$, $v = \frac{2}{3}y^{3/2} - x$.

Derivatives are $u_x = 1$, $u_y = y^{1/2}$, $v_x = -1$, $v_y = y^{1/2}$.

Hence $\phi_x = \phi_u - \phi_v$, $\phi_y = y^{1/2}(\phi_u + \phi_v)$

$$\phi_{xx} = \phi_{uu} - 2\phi_{uv} + \phi_{vv}, \quad \phi_{yy} = y(\phi_{uu} + 2\phi_{uv} + \phi_{vv}) + \frac{1}{2y^{1/2}}(\phi_u + \phi_v)$$

From $(*)$ $-y\phi_{xx} + \phi_{yy} = y\left(4\phi_{uv} + \frac{1}{2y^{3/2}}(\phi_u + \phi_v)\right) = 0$.

So canonical form

$$\phi_{uv} + \frac{1}{6(u+v)}(\phi_u + \phi_v) = 0.$$

L22.3

9.5 General solution to Wave Equation (D'Alembert)

Solve (3.4) $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0$ with initial conditions

$$\phi(x,0) = f(x)\,,\ \phi_t(x,0) = g(x)$$

With $a = \frac{1}{c^2}$, $b=0$, $c=-1$ the char eq^n is (9.16)

$\frac{dx}{dt} = \pm c$ so choose $u = x \overset{-}{\cancel{+}} ct$, $v = x + ct$

which yields simple canonical form $\frac{\partial^2\phi}{\partial u \partial v} = 0$ (9.17)

Integrate wrt u, $\frac{\partial\phi}{\partial v} = F(v)$ and wrt v

$$\phi = G(u) + \int^v F(y)\,dy = G(u) + H(v).$$

Impose initial conditions with $u = v = x$.

$$\phi(x,0) = G(x) + H(x) = f(x) \quad (*)$$

$$\phi_t(x,0) = -\underset{c}{\uparrow} G'(x) + c\,\underset{H}{\cancel{G}}'(x) = g(x) \quad (\dagger)$$

Differentiating $(*)$, $G'(x) + H'(x) = f'(x)$ $(\ddagger)$ which combined with $(\dagger)$ gives $H'(x) = \frac{1}{2}\left(f'(x) + \frac{1}{c}g(x)\right)$.

Integrate to give $H(x) = \frac{1}{2}(f(x) - f(0)) + \frac{1}{2c}\int_0^x g(y)\,dy$.

From $(*)$, $G(x) = \frac{1}{2}(f(x) + f(0)) - \frac{1}{2c}\int_0^x g(y)\,dy$.

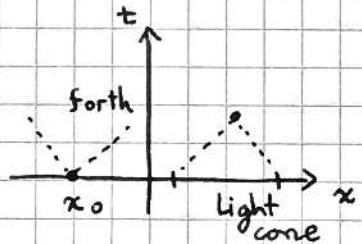
Putting this together

$$\phi(x,t) = G(x-ct) + H(x+ct)$$

$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

Domain of dependence

Waves propagate at speed c, so sol^n at (x,t) is wholly determined by f, g on the interval $[x-ct, x+ct]$.



While $x = x_0$ influences regions forth in time $x_0 - ct \le x \le x_0 + ct$.

L23.3
L23.1

10 Solving PDEs with Green's Functions

10.1 Diffusion equation and Fourier Transform

Recall heat eq^n (4.3) for a conducting wire

$$\frac{\partial\theta}{\partial t}(x,t) = D\frac{\partial^2\theta}{\partial x^2}(x,t) \qquad (10.1) \quad \text{with ICs } \theta(x,0) = h(x)$$

Take the FT wrt x using (8.13) and $\theta \to 0$ as $x \to \pm\infty$

$$\frac{\partial}{\partial t}\hat{\theta}(k,t) = -Dk^2\hat{\theta}(k,t)$$

Integrate $\hat{\theta}(k,t) = Ce^{-Dk^2t}$ with ICs $\hat{\theta}(k,0) = \tilde{h}(k)$

$$\therefore \hat{\theta}(k,t) = \tilde{h}(k)\,e^{-Dk^2t}$$

Now invert

$$\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{h}(k)\,\underset{\uparrow \tilde{g} \text{ for Gaussian}}{e^{-Dk^2t}}\,e^{ikx}\,dk$$

$$= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}h(u)\exp\left(-\frac{(x-u)^2}{4Dt}\right)du \qquad \text{by convolution thm (8.17)}$$

$$= \int_{-\infty}^{\infty}h(u)\,S_d(x-u,t)\,d\cancel{x}u \qquad (10.2)$$

where the fundamental solution $S_d(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$ (10.3)

Also known as diffusion kernel or source function.

(Know how to derive (10.3) via Gaussian)

Note With localised ICs $\theta(x,0) = \theta_0\,\delta(x)$ then

$$\theta(x,t) = \theta_0\,S_d(x,t) = \frac{\theta_0}{\sqrt{4\pi Dt}}e^{-x^2/4Dt \leftarrow \eta^2} \qquad (10.4)$$

where $\eta = \frac{x}{2\sqrt{Dt}}$ is the similarity parameter (4.5)

So for $t \geqslant 0$ spreads smoothly as a Gaussian

Example (Gaussian pulse)

Suppose initially $h(x) = \sqrt{\frac{a}{\pi}}\,\theta_0\,e^{-ax^2}$ then (10.2) implies

$$\theta(x,t) = \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2Dt}}\int_{-\infty}^{\infty}\exp\left[-au^2 - \frac{(x-u)^2}{4Dt}\right]du$$

$$\exp\left[\frac{-ax^2}{1+4aDt}\right] \swarrow$$

$$= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2Dt}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{1+4aDt}{4Dt}\right)\left(u - \frac{x}{1+4aDt}\right)^2\right]du$$

L23.3
L23.2

Hence

$$\theta(x,t) = \theta_0 \sqrt{\frac{a}{\pi(1+4aDt)}} \exp\left[-\frac{ax^2}{1+4aDt}\right] \qquad (10.5)$$

Here, width depends on SD $\propto \sqrt{t}$ with constant area.

10.2 Forced Diffusion Equation

Consider $\frac{\partial\theta}{\partial t} - D\frac{\partial^2\theta}{\partial x^2} = f(x,t)$ (10.6)

with $\theta(x,0) = 0$.

Construct a 2D Green's function

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x-\xi)\,\delta(t-\tau)$$

with $G(x,0) = 0$. Actually $G = G(x,t;\xi,\tau)$.

Take FT wrt x using (8.23)

$$\frac{\partial\tilde{G}}{\partial t} + Dk^2\tilde{G} = e^{-ik\xi}\,\delta(t-\tau)$$

Use mult. factor e^{Dk^2t}

$$\frac{\partial}{\partial t}\left(e^{Dk^2t}\tilde{G}\right) = e^{-ik\xi + Dk^2t}\,\delta(t-\tau)$$

Integrate WRT t using $G=0$ at $t=0$

$$e^{Dk^2t}\tilde{G} = e^{-ik\xi}\int_0^t e^{Dk^2t'}\,\delta(t'-\tau)\,dt'$$

$$= H(t-\tau)\,e^{-ik\xi}\,e^{Dk^2\tau} \quad \text{by } (6.7)$$

$$\Rightarrow \tilde{G}(k,t;\xi,\underset{\tau}{\cancel{t}}) = H(t-\tau)\,e^{-ik\xi}\,e^{-Dk^2(t-\tau)}$$

So inverting we get Green's function

$$G(x,t;\xi,\tau) = \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-Dk^2(t-\tau)}\,dk \qquad \begin{matrix} t'=t-\tau \\ x'=x-\xi \end{matrix}$$

$$= \frac{H(t')}{2\pi}\int_{-\infty}^{\infty} e^{ikx'}e^{-Dk^2t'}\,dk$$

$$= \frac{H(t')}{\sqrt{4\pi Dt}}\,e^{-x'^2/4Dt'}$$

$$G(x,t;\xi,\tau) = H(t-\tau)\,S_d(x-\xi,\,t-\tau) \qquad (10.8)$$

where S_d is the fundamental solution (10.3). Solution is

$$\theta(x,t) = \int_{-\infty}^{\infty} d\overset{\xi}{\cancel{x}} \int_{-\infty}^{\infty} d\tau\; G(x,t;\xi,\tau)\,f(\xi,\tau)$$

$\nwarrow$ CRINGE

L23.3

Hence by form of G

$$\theta(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} d\xi \; f(\xi,\overset{\tau}{t})\; S_d(x-\xi,\, t-\tau) \qquad (10.9)$$

This is an example of <u>Duhamel's principle</u> relating
(i) solution of forced equation with hom bcs (10.6)
to (ii) sol^n's of hom eq^n with inhom bcs (10.1)
Recall that the sol^n of (10.1) with ICs at $t=\tau$ is (10.2)

$$\theta(x,t) = \int_{-\infty}^{\infty} f(u)\, S_d(x-u,\, t-\tau)\, du \qquad (t>\tau)$$

So forcing term at (x,t) acts as an IC for subsequent evolution. The integral (10.9) is a superposition of all these IC effects from $0<\tau<t$.

10.3 Forced Wave Equation

Consider $\dfrac{\partial^2\phi}{\partial t^2} - c^2\dfrac{\partial^2\phi}{\partial x^2} = f(x,t)$ with $\phi(x,0)=0$, $\phi_t(x,0)=0$. ← (10.10)

Construct Green's f^n via FT

$$\frac{\partial^2\tilde{G}}{\partial t^2} + c^2k^2\tilde{G} = \cancel{f(k,t)}\; e^{-ik\xi}\,\delta(t-\tau)$$

Recall §7.4 for IVP Green's f^n, so by inspection

$$\tilde{G} = \begin{cases} 0 & \text{for } t<\tau, \\ e^{-ik\xi}\sin[kc(t-\tau)]\cdot\dfrac{1}{kc} & \text{for } t>\tau, \end{cases}$$

$$= e^{-ik\xi}\,\frac{\sin[kc(t-\tau)]}{kc}\,H(t-\tau).$$

Invert FT

$$G(x,t;\xi,\tau) = \frac{H(t-\tau)}{2\pi c}\int_{-\infty}^{\infty} \underset{\substack{\uparrow\\ \text{by parity}\\ \cos kA}}{e}^{ik\overbrace{(x-\xi)}^{A}}\,\frac{\sin[k\overbrace{c(t-\tau)}^{B}]}{k}\,dk$$

$$= \frac{H(t-\tau)}{\pi c}\int_0^{\infty}\cos kA\,\frac{\sin kB}{k}\,dk$$

$$= \frac{H(t-\tau)}{2\pi c}\int_0^{\infty}\frac{1}{k}\left[\sin k(A+B) - \sin k(A-B)\right]dk$$

$$= \frac{H(t-\tau)}{4\underset{\nwarrow ?}{\cancel{\pi}}c}\left[\operatorname{sgn}(A+B) - \operatorname{sgn}(A-B)\right] \qquad \text{by } (8.16)$$

L 23.4

Now with $H(t-\tau) \Rightarrow B = (t-\tau) > 0$, so only non-zero if $|A| < B$ i.e. $|x-\xi| < c(t-\tau)$.

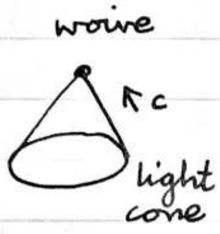
So Green's f^n or <u>fundamental sol^n</u>

$$G(x,t;\xi,\tau) = \frac{1}{2c} H\big(c(t-\tau) - |x-\xi|\big) \qquad (10.11)$$

The solution is

$$\phi(x,t) = \int_0^\infty \int_{-\infty}^\infty G(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau$$

$$= \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi\, d\tau \qquad (10.12)$$



<u>Ex</u> Relate (10.12) to D'Alembert's sol^n with initial conditions at $t=0$, $\phi=0$, $\phi_t = f(x)$ as an example of <u>Duhamel's principle.</u>

L 24.1

10.5 Poisson's Equation

Solve $\nabla^2 \phi = -\rho(\underline{r})$ (10.13) on domain D with Dirichlet BCs

$\phi = 0$ on ∂D.

Fundamental solution

The Dirac $\delta(\underline{r})$ on $\mathbb{R}^3$ has the following properties:

$$\delta(\underline{r}-\underline{r}') = 0, \quad \forall\, \underline{r} \neq \underline{r}'$$

$$\int_D \delta(\underline{r}-\underline{r}')\, d^3\underline{r} = \begin{cases} 1 & \text{if } \underline{r}' \in D \\ 0 & \text{otherwise} \end{cases} \qquad (10.4) \qquad \sim \text{open}$$

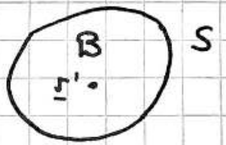
$$\int_D f(\underline{r})\, \delta(\underline{r}-\underline{r}')\, d^3\underline{r} = \begin{cases} f(\underline{r}') & \text{if } \underline{r}' \in D \\ 0 & \text{otherwise} \end{cases}$$

The free-space Green's function is defined as

$$\nabla^2 G_{FS}(\underline{r};\underline{r}') = \delta(\underline{r}-\underline{r}') \qquad (10.15)$$

This is spherically symmetric $G(\underline{r},\underline{r}') = G|\underline{r}-\underline{r}'|$.

WLOG $\underline{r}' = \underline{0}$, so $G = G(\underline{r}) = G(r)$.



Integrate over ball B (radius r) around $\underline{r}' = \underline{0}$.

$$\int_B \nabla^2 G_{FS}\, d^3\underline{r} = \int_S \nabla G_{FS} \cdot \underline{n}\, dS$$

$$= \int_\Omega \frac{\partial G}{\partial r} \cdot r^2\, d\Omega = 4\pi r^2 \frac{\partial G}{\partial r}.$$

And on the other side

$$\int_B \delta(\underline{r})\, d^3\underline{r} = 1.$$

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi r^2} \Rightarrow G = -\frac{1}{4\pi r}$$ since require $G \to 0$ at ∞

Free space solution

$$\boxed{G(\underline{r};\underline{r}') = -\frac{1}{4\pi|\underline{r}-\underline{r}'|} \qquad (10.16)}$$

Green's identities

Two scalar f^ns ϕ, ψ twice diff on D. Consider

$$\int_D \nabla\cdot(\phi\nabla\psi)\, d^3\underline{r} = \int_D (\phi\nabla^2\psi) + (\nabla\phi)(\nabla\psi)\, d^3\underline{r}$$

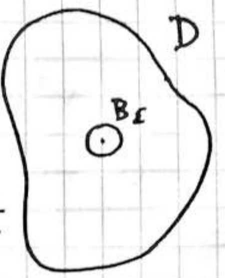
$$= \int_{\partial D} \phi\, \underline{\nabla}\psi \cdot \underline{n}\, dS \qquad (10.17)$$

(10.18) ↓

is Green's first identity. Now swap ϕ, ψ and subtract to get

Green's second identity $\int_{\partial D} (\phi\underline{\nabla}\psi - \psi\underline{\nabla}\phi)\cdot\underline{n}\, dS = \int_D (\phi\nabla^2\psi - \psi$

L24.2



Take ϕ in (10.18) s.t. $\nabla^2\phi = -\rho$ and ψ to be $G_{FS}(\underline{r};\underline{r}')$. Excise a small ball $B_\varepsilon(\underline{r}')$.

From (10.18), RHS $= \int_{D\setminus B_\varepsilon} (\phi \underset{\text{zero}}{\underset{\uparrow}{\nabla^2 G_{FS}}} - G_{FS}\underset{-\rho}{\underset{\uparrow}{\nabla^2\phi}})\, d^3\underline{r}$

$$= \int_{D\setminus B_\varepsilon} G_{FS}\cdot\rho\, d^3\underline{r}$$

$$\text{LHS} = \int_{\partial D}\Big(\phi\frac{\partial G_{FS}}{\partial n}\underset{-G_{FS}\frac{\partial\phi}{\partial n}}{\uparrow}\Big)\,dS + \int_{S_\varepsilon}\Big(\phi\frac{\partial G_{FS}}{\partial n} - G_{FS}\frac{\partial\phi}{\partial n}\Big)\,dS \qquad (*)$$

Second integral over small sphere S_ε, take limit $\varepsilon \to 0$.

$$(*) = \Big(-\overline{\phi}\cdot\frac{1}{4\pi\varepsilon^2} + \frac{1}{4\pi\varepsilon}\overline{\frac{\partial\phi}{\partial n}}\Big)4\pi\varepsilon^2 \longrightarrow -\phi(\underline{0})$$

Combining, get Green's third identity (10.19) ↓

$$\phi(\underline{r}') = -\int_D G_{FS}(\underline{r};\underline{r}')\,\rho\, d^3\underline{r} + \int_{\partial D}\Big(\phi\frac{\partial G(\underline{r};\underline{r}')}{\partial n} - G_{FS}\frac{\partial\phi}{\partial n}\Big)\,dS$$

Dirichlet Green's Functions

Solve $\nabla^2\phi = -\rho$ on D with inhom. BCs $\phi(\underline{r}) = h(\underline{r})$ on ∂D.

Dirichlet Green's f^n satisfies

(1) $\nabla^2 G(\underline{r};\underline{r}') = 0 \quad \forall\, \underline{r} \neq \underline{r}'$

(2) $G(\underline{r};\underline{r}') = 0$ on ∂D

(3) $G(\underline{r};\underline{r}') = G_{FS}(\underline{r};\underline{r}') + \overset{\text{harmonic ö}}{H(\underline{r};\underline{r}')} \quad \forall\, \underline{r}$

Green's 2nd identity (10.18) with $\nabla^2\phi = -\rho$, $\nabla^2 H = 0$

$$\int_{\partial D}\Big(\phi\frac{\partial H}{\partial n} - H\frac{\partial\phi}{\partial n}\Big)\,dS = \int_D H\rho\, d^3\underline{r}\ .$$

Now, $G_{FS} = G - H$ in Green's 3rd identity (10.19)

$$\phi(\underline{r}') = \int(G-\cancel{H})(-\rho)\,d^3\underline{r} + \int_{\partial D}\Big(\phi\frac{\partial(G-\cancel{H})}{\partial n} - (G-\cancel{H})\frac{\partial\phi}{\partial n}\Big)\,dS$$

By above, H terms vanish, also $G = 0$, $\phi = h$ on boundary

$$\boxed{\phi(\underline{r}') = \int_D G(-\rho)\,d^3\underline{r} + \int_{\partial D} h(\underline{r})\frac{\partial G}{\partial n}\,dS \qquad (10.20)}$$

L24.3

(See summary slides for Neumann BCs, see also Josza)

Ex Use (10.18) to show GF symmetric

$G(\underline{r};\underline{r}') = G(\underline{r}';\underline{v}) \;\; \forall \, \underline{v} \neq \underline{v}'.$

10.6 Method of Images - Laplace Eq^n

For symmetric domains D can construct Green's functions with $G=0$ on ∂D by cancelling the boundary signal with an opposite mirror source / image source placed outside D.

Laplace's equation on the half-plane

Solve $\nabla^2\phi = 0$ on $D = \{(x,y,z) : z \geqslant 0\}$ with

$$\phi(x,y,0) = h(x,y), \quad \phi \to 0 \text{ as } |\underline{v}| \to 0$$

Now $G_{FS}(\underline{r};\underline{r}') \longrightarrow 0$ as $|\underline{r}| \to \infty$ but $G_{FS} \neq 0$ at $z=0$.

So for $G_{FS}(\underline{r};\underline{v}')$ at $\underline{v}' = (x',y',z')$ subtract an image

$G_{FS}(\underline{r};\underline{r}'')$ with $\underline{v}'' = (x'',y'',z'') = (x',y',-z')$.

Then $G(\underline{r};\underline{v}') = -\dfrac{1}{4\pi|\underline{r}-\underline{r}'|} + \dfrac{1}{4\pi|\underline{r}-\underline{r}''|} = \text{UGH}$

vanishes on $z=0$, so have Dirichlet BCs.

We have $\left.\dfrac{\partial G}{\partial n}\right|_{z=0} \underset{\substack{\uparrow \\ \text{minus?}}}{=} \left.\dfrac{\partial G}{\partial z}\right|_{z=0} = -\dfrac{1}{4\pi}\left(\dfrac{z-z'}{|\underline{r}-\underline{r}'|^3} - \dfrac{z+z'}{|\underline{r}-\underline{r}''|^3}\right)$

Sol^n from (10.20)

$$\phi(x',y',z') = \frac{z'}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\cancel{(}(x-x')^2+(y-y')^2+z'^2\right)^{-3/2} h(x,y)\,dx\,dy$$

See summary for more images.