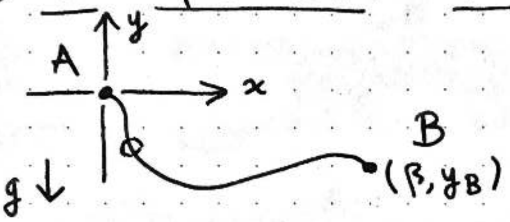
L1.1

Variational Principles

§0. Motivation

Example 0.1 The Brachistochrone



A particle starts at rest and slides along the wire from A to B under the influence of gravity.

Find the shape of the wire st the travel time is minimal.

1696 Johan Bernoulli

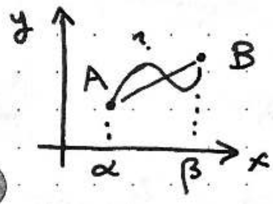
$T+V = \text{const.}$ $\quad \frac{mv^2}{2} + mgy = 0$ by computing at A

$$\text{Time} = \int_A^B dt = \int_0^\beta \frac{dl}{v(x,y)} = \int_0^\beta \frac{\sqrt{1+(y')^2}\,dx}{v}$$

Now, $v = \sqrt{-2gy}$, to give

$$\text{Time} = \int_0^\beta \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$$

Example 0.2 Geodesic = shortest path (if it exists) between two points on a surface $\Sigma \subset \mathbb{R}^3$. Take $\Sigma = \mathbb{R}^2$



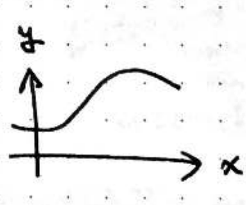
$$D = \int_\alpha^\beta \sqrt{1+(y')^2}\,dx$$

minimise D with $y(\alpha) = y_A$, $y(\beta) = y_B$

Maths In both cases we aim to minimise

$$F[y] = \int_\alpha^\beta f(x, y, y')\,dx \qquad (0.1)$$

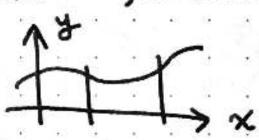
(0.1) is an example of a *functional*



function : number $\longrightarrow$ number

functional : function $\longrightarrow$ number

e.g.



$\longrightarrow$ area

$y(x) \longrightarrow F[y]$

Books

Gelfand & Fomin "Calculus of Variations"

DAMTP notes (Townsend)

This course (diff order 5)

L1.2

<u>Calculus of variations</u> finding critical points of functionals (a part of functional analysis)

Need to specify a class of functions $y(x)$ for which (0.1) is to be minimised.

e.g. $C(\mathbb{R})$ = space of continuous functions on $\mathbb{R}$

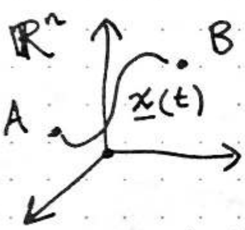
$C^k(\mathbb{R})$ = " continuously k-times diff'ble

$C^k_{(\alpha,\beta)}(\mathbb{R})$ = " " supported on (α,β)

<u>Variational principles</u> (?) = principles of nature where the laws reduce to minimising (maximising) a functional

<u>Example 0.3</u> <u>Fermat's principle</u> "Light travels between A and B along a path which minimises travel time"

<u>Example 0.4</u> <u>Principle of least action</u>



T = kinetic energy (e.g. $\frac{m|\dot{\underline{x}}|^2}{2}$)

V = potential energy (e.g. $V(\underline{x})$)

$$\text{Action } S[\underline{x}] = \int_{t_1}^{t_2} (T - V)\, dt$$

Then action is stationary along the paths taken by the particle. $\Rightarrow \ddot{\underline{x}} = -\underline{\nabla} V$

<u>Leibniz on Example 0.3</u> "We live in the best of all worlds"

Theology alert! ⚠

<u>Feynman on Example 0.3</u> "This is wrong in Quantum Theory"

A particle takes all possible paths with different probabilities.

<u>This course</u>

1) Find necessary conditions for $y(x)$ to minimise $F[y]$
   → Euler-Lagrange equations
2) Lots of examples (geometry, physics)
   Problems with constraints (e.g. find area maximising shape)
3) Some sufficient conditions for min/max (2^{nd} variation)

L2.1

§ 1 Calculus on $\mathbb{R}^n$

$f \in C^2(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$, continuous $\overset{\text{2nd}}{\downarrow}$ partial derivatives

$\underline{a} \in \mathbb{R}^n$ critical point i.e. $\underline{\nabla} f|_a \equiv (\partial_1 f, \dots, \partial_n f)|_a = \underline{0}$

where $\partial_i = \partial / \partial x_i$ for $i = 1, \dots, n$

Expand around $\underline{a}$: $$f(\underline{x}) = f(\underline{a}) + (\underline{x} - \underline{a}) \cdot \overbrace{\underline{\nabla} f|_a}^{\text{zero}} + \frac{1}{2} \sum_{i,j} (x^i - a^i)(x^j - a^j) \frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_{\underline{a}} + O(|\underline{x} - \underline{a}|^3)$$

Hessian matrix H, $H_{ij} \equiv \dfrac{\partial^2 f}{\partial x^i \partial x^j} = H_{ji}$ since C^2

Shift the origin to $\underline{a} = \underline{0}$. Diagonalise $H(\underline{0})$ via an orthogonal transformation R:

$$H'(\underline{0}) = R^T H(\underline{0}) R = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

Go back to Taylor expansion:

$$f(\underline{x}) - f(\underline{0}) = \frac{1}{2} \sum_i \lambda_i (x'^i)^2 + O(|\underline{x}|^3)$$

i) If all $\lambda_i > 0$, then $f(\underline{x}) - f(\underline{0}) > 0 \quad \forall\, \underline{x} \neq \underline{0}$ (to 2^{nd} o)

$\underline{x} = \underline{0}$ is a local minimum

ii) If all $\lambda_i < 0$ have a local maximum

iii) Some $\lambda_i > 0$, some $\lambda_i < 0$ give saddle point

iv) Some $\lambda_i = 0$, look at higher order terms in series

Special case $n = 2$. $\det(H) = \lambda_1 \lambda_2$, $\operatorname{Tr}(H) = \lambda_1 + \lambda_2$

(i) $\det H > 0$, $\operatorname{Tr} H > 0$ local minimum

(ii) $\quad\quad$, $\operatorname{Tr} H < 0$ local maximum

(iii) $\det H < 0$ saddle point

(iv) $\det H = 0$ look at higher order terms

Remark: If $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ then need to examine the boundary of D for global max/min

L2.2

Examples 1.1) Harmonic functions on $D \subseteq \mathbb{R}^2$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \det H < 0 \text{ so min/max on } \partial D$$

1.2) $f(x,y) = x^3 + y^3 - 3xy$

$$\underline{\nabla} f = \left(3(x^2-y),\ 3(y^2-x) \right) \quad \text{so get } (0,0) \text{ and } (1,1)$$

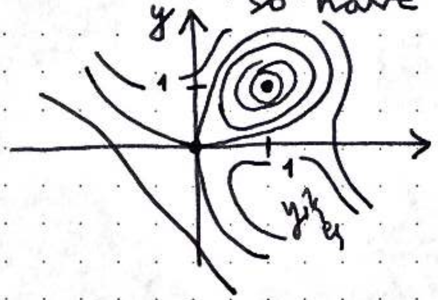
$$H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$H|_{(0,0)}$ has det -9, so saddle

$H|_{(1,1)}$ has det 27, trace 12

so have a minimum point

Contour plot

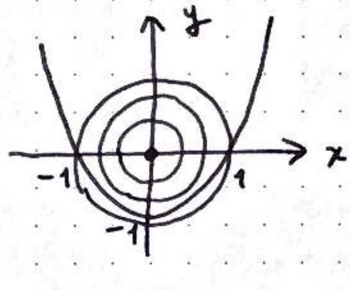


near $(0,0)$

$f \sim -3xy$

§ 1.1 Constraints and Lagrange Multipliers

Example Find a circle in $\mathbb{R}^2$, centred at the origin which has minimal radius and intersects the parabola $y = x^2 - 1$



Two approaches

(1) Solve the constraint: $y = x^2 - 1$

$$f(x,y) = x^2 + y^2 = x^2 + (x^2-1)^2$$
$$= x^4 - x^2 + 1 = \hat{f}(x)$$

$$\frac{d\hat{f}}{dx} = 4x^3 - 2x = 2x(2x^2-1) \quad \text{is zero at } x = 0, \pm \sqrt{2}/2$$

$f = 1$ (at $x=0$), $f = \frac{3}{4}$ (at $x = \pm\sqrt{2}/2$) ☺

(2) Lagrange multipliers

Define $h(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ where λ is the Lagrange multiplier, and $g(x,y) = 0$ is the constraint.

Extremise h wrt x, y, λ without constraints

$$h = x^2 + y^2 - \lambda(y - x^2 + 1)$$

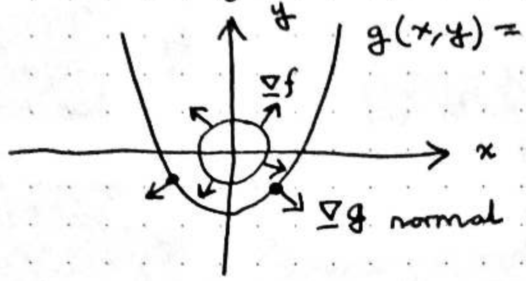
$$\frac{\partial h}{\partial x} = 2x + 2x\lambda \qquad \frac{\partial h}{\partial y} = 2y - \lambda \qquad \frac{\partial h}{\partial \lambda} = x^2 - y - 1$$

L2.3

$2x(1+2y) = 0$ so $x = 0$ or $y = -\frac{1}{2}$

Then in third equation, $x=0 \Rightarrow y=-1$ and $y = -\frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Why do Lagrange multipliers work? Geometry



Then at critical points,

$\underline{\nabla} f$ and $\underline{\nabla} g$ are parallel

So $\underline{\nabla} f = \lambda \underline{\nabla} g$ and

$\underline{\nabla}(f - \lambda g) = 0$.

$\underline{\nabla} h = 0$ follows through

In general $f : \mathbb{R}^n \to \mathbb{R}$, subject to K constraints $g_\alpha(\underline{x}) = 0$

where $g_\alpha : \mathbb{R}^n \to \mathbb{R}$, $\alpha = 1, \dots, K$

Define $h(x_1, \dots, x_n, \underbrace{\lambda_1, \dots, \lambda_K}_{K \text{ Lagrange bois}}) = f - \sum_{\alpha=1}^{K} \lambda_\alpha g_\alpha$

Extremise h: $\frac{\partial h}{\partial x^i} = 0$ and $\frac{\partial h}{\partial \lambda_\alpha} = 0$ $\quad n+K$ eq^{n}s

Eliminate λ_α and solve for critical points

L3.1

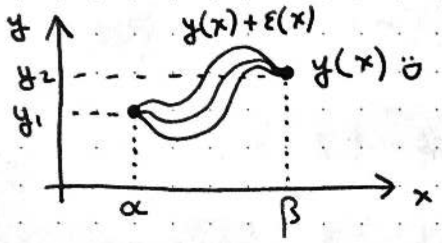
§ 2 The Euler–Lagrange Equations

Find extrema of the functional $(0,1)$: $y : \mathbb{R} \to \mathbb{R}$

$y \in C^2_{[\alpha,\beta]}(\mathbb{R})$ just "on" $[\alpha,\beta]$

$$F[y] = \int_\alpha^\beta f(x, y, y')\, dx \qquad (2.1)$$

subject to $y(\alpha) = y_1$, $y(\beta) = y_2$



<u>Lemma</u> Let $g : [\alpha,\beta] \to \mathbb{R}$ be continuous, and such that

$\int_\alpha^\beta g(x)\eta(x)\, dx = 0$ for all continuous $\eta : [\alpha,\beta] \to \mathbb{R}$

such that $\eta(\alpha) = \eta(\beta) = 0$.

Then $g(x) \equiv 0$ on $[\alpha,\beta]$.

<u>Proof</u> Assume $\exists\, \bar{x} \in [\alpha,\beta]$ such that $g(\bar{x}) \neq 0$ (say $g(\bar{x}) > 0$)

Then $\exists\, c > 0$, $[x_1, x_2] \subset [\alpha,\beta]$ such that $\bar{x} \in [x_1, x_2]$

and $g(x) > c$ on $[x_1, x_2]$.

Define
$$\eta(x) = \begin{cases} (x - x_1)(x_2 - x) & \text{if } x \in [x_1, x_2] \\ 0 & \text{if } x \notin [x_1, x_2] \end{cases} \qquad (2.2)$$

Then $\int_\alpha^\beta g(x)\eta(x)\, dx > \int_{x_1}^{x_2} c\,(x - x_1)(x_2 - x)\, dx > 0$. ⨳

So no such $\bar{x}$ exists, and $g \equiv 0$. □

Assume that (2.1) admits an extremum, and replace y by $y(x) + \varepsilon\eta(x)$ in (2.1), where $\eta(\alpha) = \eta(\beta) = 0$.

$$F[y + \varepsilon\eta] = \int_\alpha^\beta f(x, y + \varepsilon\eta, y' + \varepsilon\eta')\, dx$$

$$= F[y] + \varepsilon \int_\alpha^\beta \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right) dx + O(\varepsilon^2)$$

def^n ? of derivative

For an extremum, the linear term in ε should vanish,

i.e. $\left.\dfrac{dF}{d\varepsilon}\right|_{\varepsilon=0} = 0$

So $0 = \int_\alpha^\beta \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right) dx$. Integrate second term.

$$0 = \underbrace{\int_\alpha^\beta \left(\frac{\partial f}{\partial y}\eta - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\right) dx}_{\text{zero } \forall \eta} + \underbrace{\eta \frac{\partial f}{\partial y'}\Big]_\alpha^\beta}_{\substack{\text{zero as } \eta \\ \text{vanishes on } \alpha, \beta}}$$

By Lemma, with $g = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$, obtain

<u>Theorem</u> If an extemum $y \in C^2(\mathbb{R})$ of (2.1) exists, then it satisfies

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.} \qquad (2.3)$$

<u>Remarks</u> 1) (2.3) is a 2^{nd} order ODE for $y(x)$, subject to boundary conditions $y(\alpha) = y_1$, $y(\beta) = y_2$

It is called the Euler-Lagrange equation (✉: Lagrange $\downarrow$ Euler 1745)

2) η in (2.2) is an example of a "bump" function which was continuous. A $C^k(\mathbb{R})$ bump function could be

$$\eta(x) = \begin{cases} [(x-x_1)(x-x_2)]^{k+1} & \text{if } x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}$$

3) Notation and terminology. LHS of (2.3) is called the functional derivative and denoted by $\frac{\delta F[y]}{\delta y(x)}$. Some books / notes (e.g. P Townsend) replace $\varepsilon\eta(x) \to \delta y(x)$. "a small variation"

$$F[y + \delta y] = F[y] + \delta F[y] \quad \text{where} \quad \delta F[y] = \int_\alpha^\beta \frac{\delta F[y]}{\delta y(x)} \underset{\swarrow \delta y}{dx}$$

4) Be careful with derivatives.

In (2.3) $\frac{\partial}{\partial x}\Big|_{y,y'}$ $\frac{\partial}{\partial y}\Big|_{x,y'}$ $\frac{\partial}{\partial y'}\Big|_{x,y}$

Take $h(x, y, y')$. Then

$$\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx} + \frac{\partial h}{\partial y'}\frac{dy'}{dx}$$

L3.3

So $\frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'}$, the total derivative.

For example, take $h(x,y,y') = x\left((y')^2 - y^2\right)$.

Then $\frac{\partial h}{\partial x} = (y')^2 - y^2 \quad \frac{\partial h}{\partial y} = -2xy \quad \frac{\partial h}{\partial y'} = 2xy'$, and

$$\frac{dh}{dx} = (y')^2 - y^2 - 2y'xy + 2y''xy'$$

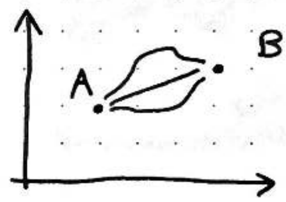
§2.1 The first integrals of E−L (Euler-Lagrange)

In some cases, it is possible to reduce (2.3) to a 1^{st} order ODE. (i) Assume $\frac{\partial f}{\partial y} = 0$, i.e. $f = f(x, y')$

Then (2.3) implies $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$,

so $\frac{\partial f}{\partial y'} = \text{const.}$ (a first integral of 2.3) (2.4)

Example 2.1 geodesics on $\mathbb{R}^2$



$$F[y] = \int_\alpha^\beta \underbrace{\left(1 + (y')^2\right)^{1/2}}_{f(y')} dx$$

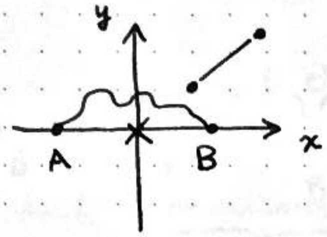
So have, by (2.4), $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = \text{const}$

So $y' = m$ (const.)

And y is a straight line

Recall the Euler-Lagrange equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$ (2.3)

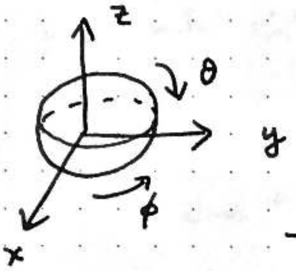
- Geodesics on $\mathbb{R}^2$ = straight lines $y = mx + c$



Consider $\mathbb{R}^2 \setminus \{0\}$

Then no geodesic $A \to B$

<u>Example</u> Geodesics on a sphere $S^2 \subset \mathbb{R}^3$, $x^2+y^2+z^2 = 1$



Parametrise via θ and ϕ so

$x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$

$$ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2(\theta)\,d\phi^2$$

Treat ϕ as a function of θ to get

$$F[\phi] = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2(\theta)(\phi')^2}\,d\theta$$

(Could have also done $\theta = \theta(\phi)$)

Integrand independent of ϕ, so via first integral (2.4)

$$\frac{\partial}{\partial\phi'}(\text{integrand}) = \text{const} = K$$

$$\frac{\phi'\sin^2(\theta)}{\sqrt{1+\sin^2(\theta)(\phi')^2}} = K$$

- Square to obtain

$$(\phi')^2\sin^4(\theta) = K^2 + K^2\sin^2(\theta)(\phi')^2$$

$$\Rightarrow \phi' = \pm\frac{K}{\sin\theta\sqrt{\sin^2\theta - K^2}}$$

$$\Rightarrow \phi - \phi_0 = \pm K\int \frac{1}{\sin\theta\sqrt{\sin^2\theta - K^2}}\,d\theta$$

$\pm$ means two possible directions along "great circle"

$\cot\theta = u$ gives $\boxed{\pm\frac{\sqrt{1-K^2}}{K}\cos(\phi-\phi_0) = \cot\theta}$

- Plug x and y back in to obtain $z = \alpha x + \beta y$ for some $\alpha, \beta \in \mathbb{R}$, i.e. we indeed have a great circle

(ii) Another type of first integral. Assume $f = f(y, y')$ in (2.3) i.e. $\partial f / \partial x = 0$. Then

<u>Claim</u> $\quad f - y' \frac{\partial f}{\partial y'} = \text{const}$ (along solution curves) (2.5)

<u>Proof</u>

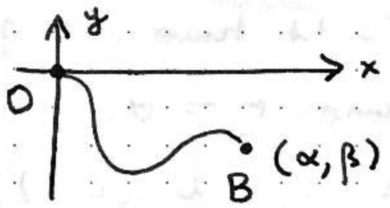
$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \underbrace{\frac{\partial f}{\partial x}}_{\text{zero}} + y'\frac{\partial f}{\partial y} + \underbrace{y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'}}_{\text{zero}} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

We are left with

$$y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) = 0$$

which is true by Euler-Lagrange. □

<u>Example</u> The Brachistochrone problem.



Minimise travel time

$$T = \int_A^B dt = \int_A^B \frac{dl}{v(x,y)}$$

$$F[y] = \frac{1}{\sqrt{2g}} \int_0^\beta \underbrace{\frac{\sqrt{1+(y')^2}}{\sqrt{-y}}}_{f(y,y')} dx$$

Using (2.5) we deduce

$$K = f - y'\frac{\partial f}{\partial y'} = \frac{\sqrt{1+(y')^2}}{\sqrt{-y}} - y' \cdot \frac{y'}{\sqrt{-y}\sqrt{1+(y')^2}}$$

$$= \frac{1 + \cancel{(y')^2} - \cancel{(y')^2}}{\sqrt{-y}\sqrt{1+(y')^2}} \quad \Rightarrow \quad (y')^2 + 1 = \frac{1}{-yK^2}$$

$$\Rightarrow y' = \pm\sqrt{\frac{1}{-yK^2} - 1}$$

Integrate to deduce

$$x = \pm K \int \frac{\sqrt{-y}}{\sqrt{1+K^2 y}}\, dy$$

Now let $y = -\frac{1}{K^2} \sin^2\left(\frac{\theta}{2}\right)$ so that

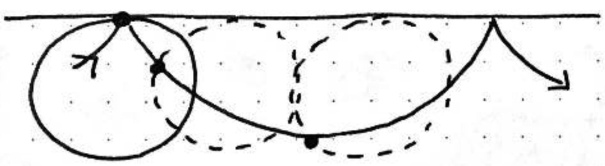
$$x = \pm \cancel{K} \int \frac{\sin(\theta/2)}{\cancel{\cos(\theta/2)}} \cdot -\frac{1}{K^2} \sin(\theta/2) \cancel{\cos(\theta/2)} \, d\theta$$

$$= \mp \frac{1}{2K^2} (\theta - \sin\theta) + c$$

Initial conditions $A = (0,0) \longrightarrow c = 0$

$$\begin{cases} x = \dfrac{\theta - \sin\theta}{2K^2} \\ y = -\dfrac{1}{K^2} \sin^2(\theta/2) \end{cases}$$

parametric equations of <u>cycloid</u>



The curve traced by a point on the rim of a circle, as the circle rolls on a straight line.

Brachistochrone is a cycloid (Bernoulli 1697)

§ 2.2 Fermat's principle

"Light / sound moves along paths which minimise the time of the journey"

Ray ("light") $y = y(x)$. Speed of light / sound $c(x, y)$.

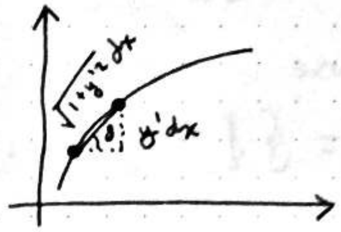
$$F[y] = \text{Time} = \int_A^B \frac{dl}{c(x,y)} = \int_\alpha^\beta \frac{\sqrt{1 + (y')^2}}{c(x,y)} \, dx$$

Assume that $c = c(x)$.

LS.1

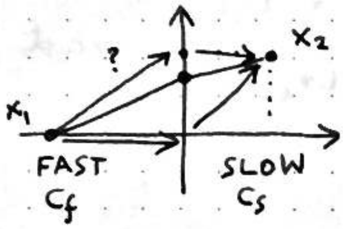
Fermat's principle, with the assumption $c = c(x)$ allows via (2.4)

$\Rightarrow \dfrac{\partial f}{\partial y'} = \text{const.}$ $\qquad \dfrac{y'}{\sqrt{1+y'^2}\,c} = \text{const.}$



$$\Rightarrow \frac{\sin\theta}{c(x)} = \text{const.} \qquad (\text{Snell's Law}) \qquad (2.6)$$

Consider medium of the form



More such that angles related via Snell's

§3 Extensions of the Euler-Lagrange equations

§3.1 E-L equations with constraints

Extermise $\quad F[y] = \int_\alpha^\beta f(x, y, y')\,dx$

subject to $\quad G[y] = \int_\alpha^\beta g(x, y, y')\,dx = \text{some constant}$

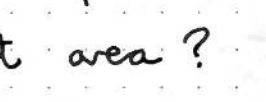
Extremise $\quad \Phi[y, \lambda] = F[y] - \lambda G[y]$

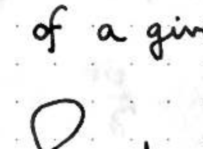
In EL equations (2.3) replace f by $h(x, y, y', \lambda) = f - \lambda g$

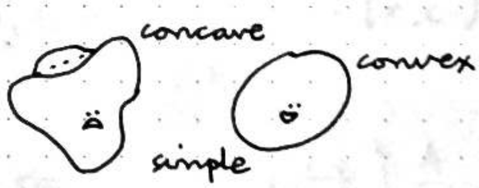
E-L become
$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}(f - \lambda g)\right) = \frac{\partial}{\partial y}(f - \lambda g) \qquad (3.1)$$

Example The Dido problem (aka 'isoperimetric' problem)

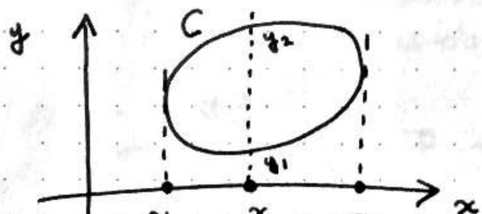
What simple closed curve of a given perimeter encloses the largest area?







Assume WLOG that our curve is convex.



$\alpha < x < \beta$ corresponds to two values of y

$dA = [y_2(x) - y_1(x)]\,dx$ area element

x is monotonically increasing between α, β

" decreasing " β, α

$$A[y] = \int_\alpha^\beta (y_2(x) - y_1(x))\,dx = \oint_C y(x)\,dx$$

Impose constraint

$$L = \oint \sqrt{1+y'^2}\,dx$$

λ = Lagrange multiplier

E-L equations on h

$$h(x,y,y',\lambda) = y - \lambda\sqrt{1+(y')^2}$$

Use 1^{st} integral (2.5) to obtain

$$h - y'\frac{\partial h}{\partial y'} = \text{const.} \Rightarrow y - \lambda\sqrt{1+y'^2} + y'\cdot\frac{\lambda y'}{\sqrt{1+y'^2}} = \text{const.} = K$$

$$\Rightarrow y - K = \frac{\lambda}{\sqrt{1+y'^2}} \qquad 1^{st} \text{ order ODE}$$

Solution is

$$(x-x_0)^2 + (y-y_0)^2 = \lambda^2$$

where $y_0 = K$.

Circle of radius λ, but $2\pi\lambda = L$ so $\lambda = \frac{L}{2\pi}$

<u>Example</u> <u>The Sturm-Liouville problem</u>

Let $\rho(x) > 0$ for $x \in [\alpha,\beta]$.

Consider

$$F[y] = \int_\alpha^\beta \left[\rho(x)\cdot(y')^2 + \sigma(x)y^2\right]dx$$

subject to $G[y] = \int_\alpha^\beta y^2\,dx = 1$

$$h(x,y,y',\lambda) = \left(\rho(y')^2 + \sigma y^2\right) - \lambda\left(y^2 - \frac{1}{\beta-\alpha}\right)$$

E-L (3.1): $\frac{\partial h}{\partial y'} = 2\rho y' \qquad \frac{\partial h}{\partial y} = 2\sigma y - 2\lambda y$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{\partial h}{\partial y} \Rightarrow \boxed{-\frac{d}{dx}(\rho y') + \sigma y = \lambda y} \qquad (3.2)$$

Sturm-Liouville eigenvalue problem

$\mathcal{L}(y) = \lambda y$ where $\mathcal{L} = -\frac{d}{dx}(\rho\,\cdot) + \sigma\,\cdot$ Sturm-Liouville operator

λ is an eigenvalue

E.g. If $\rho(x) = 1$ and $\sigma(x) > 0$ "Potential"
obtain the Schrödinger equation with energy λ

- If $\sigma > 0$, then $F[y] > 0$

<u>Claim</u> Positive minimum is equal to lowest eigenvalue in (3.2)

<u>Proof</u> Integrate (3.2) $\times\, y$ from α to β:

$$\int_{\alpha}^{\beta} \left(\left[-\frac{d}{dx}(\rho y') \right] y + \sigma y^2 \right) dx = \lambda \int_{\alpha}^{\beta} y^2 \, dx$$

Integrate by parts and get $F[y] = \lambda G[y]$

so $\lambda_{min} = F[y]\big|_{min}$

L6.1

§ 3.2 Several dependant variables

$$(y_1(x), \dots, y_n(x)) = \underline{y}(x)$$

$$F[\underline{y}] = \int_\alpha^\beta f(x, y_1, \dots, y_n, y_1', \dots, y_n')\, dx$$

$\underline{y}(\alpha) = \underline{y}_1$ and $\underline{y}(\beta) = \underline{y}_2$

Follow the derivation of (2.3): $\underline{y}(x) \to \underline{y}(x) + \underline{\eta}(x) \cdot \varepsilon$

Where $\underline{\eta}(\alpha) = \underline{\eta}(\beta) = 0$.

$$F[\underline{y} + \varepsilon \underline{\eta}] = F[\underline{y}] + \varepsilon \int_\alpha^\beta \Big(\sum_i \frac{\partial f}{\partial y_i} \eta_i + \underbrace{\frac{\partial f}{\partial y_i'} \eta_i'}_{-\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\eta_i} \Big)\, dx$$

Apply Lemma: + boundary terms] = 0

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \quad \text{for } i = 1, \dots, n \qquad (3.3)$$

System of n 2^{nd} order E-L equations

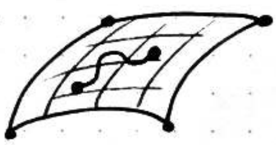
First integrals

a) $\dfrac{\partial f}{\partial y_j} = 0$ for $1 \leq j \leq n$ $\overset{(3.3)}{\Rightarrow}$ $\dfrac{\partial f}{\partial y_j'} = \text{const.}$

b) $\dfrac{\partial f}{\partial x} = 0$ $\overset{(3.3)}{\Rightarrow}$ $f - \sum_{i=1}^n y_i' \dfrac{\partial f}{\partial y_i'} = \text{const.}$

Example Geodesics (shortest paths) on a surface

$\Sigma \subseteq \mathbb{R}^3$



Σ given by $g(\underline{x}) = 0$

Curve $t \to \underline{x}(t)$ $\quad \underline{x}\,\cancel{\xi}(\alpha) = \underline{x}_1$

$\uparrow$ indep variable $\quad \underline{x}\,\cancel{\xi}(\beta) = \underline{x}_2$

$$\tilde{F}[\underline{x}] = \int_\alpha^\beta |\underline{x}'(t)|\, dt$$ but introduce $\lambda(t)$ so

$$F[\underline{x}] = \int_\alpha^\beta \left[|\underline{x}'(t)| - \lambda(t)\, g(\underline{x}) \right] dt$$

EL with λ $\quad \dfrac{\partial f}{\partial \lambda} - \dfrac{d}{dx}\underbrace{\left(\dfrac{\partial f}{\partial \lambda'}\right)}_{\text{zero}} = 0 \Rightarrow g(\underline{x}) = 0$

EL with $\underline{x}$

give geodesic ODEs

§3.3 Several independent variables

$\Phi : \mathbb{R}^3 \to \mathbb{R}$ i.e. $\phi(x,y,z)$
(or $\mathbb{R}^n$)

$$F[\phi] = \int_{D \subseteq \mathbb{R}^3} f(x, y, z, \phi, \underbrace{\phi_x}_{\frac{\partial \phi}{\partial x}}, \phi_y, \phi_z)\, dx\, dy\, dz$$

Perturb to 1^{st} order in ε : $\phi \to \phi + \varepsilon \eta(x,y,z)$

$\eta = 0$ on ∂D

$$F[\phi + \varepsilon\eta] - F[\phi]$$
$$= \varepsilon \int_D \left(\frac{\partial f}{\partial \phi}\eta + \frac{\partial f}{\partial \phi_x}\eta_x + \text{cyc} \right) dx\, dy\, dz$$
$$= \varepsilon \int_D \left[\frac{\partial f}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \phi_x} \right) - \text{cyc} \right] \eta\, dV$$

$\boxed{\cancel{\nabla \cdot (f \eta)} \quad \nabla \cdot \left(\frac{\partial f}{\partial \phi_i} \eta \right) \text{ \& use div theorem}}$

Apply Lemma to deduce

$$\frac{\partial f}{\partial \phi} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial \phi_{x_i}} \right) = 0 \qquad (3.4)$$

PDE

Remains valid for $\phi : \mathbb{R}^n \to \mathbb{R}$

<u>Example</u> $\phi : \mathbb{R}^2 \to \mathbb{R}$, $\phi(x,y)$

Extremise $F[\phi] = \int_D \frac{1}{2}(\phi_x^2 + \phi_y^2)\, dx\, dy$

$$\frac{\partial f}{\partial \phi} = 0, \quad \frac{\partial f}{\partial \phi^x} = \phi_x, \quad \frac{\partial f}{\partial \phi_y} = \phi_y$$

$$(3.4) \Rightarrow \frac{\partial}{\partial x}\phi_x + \frac{\partial}{\partial y}\phi_y = 0 \Rightarrow \phi_{xx} + \phi_{yy} = 0$$

<u>Example</u> Minimal surfaces

Construct $\Sigma \subseteq \mathbb{R}^3$ with prescribed boundary and minimal area (soap film)

$$\Sigma = \{ \underline{x} \in \mathbb{R}^3 : g(\underline{x}) = 0 \}$$

write Σ as a graph $z = \phi(x,y)$

L6.3

$$ds^2 = dx^2 + dy^2 + dz^2 = (1+\phi_x^2)\,dx^2 + (1+\phi_y^2)\,dy^2 + 2\phi_x\phi_y\,dx\,dy$$

Riemannian metric ☺

$$\underbrace{\begin{pmatrix} 1+\phi_x^2 & \phi_x\phi_y \\ \phi_x\phi_y & 1+\phi_y^2 \end{pmatrix}}_{G}$$

Area element

$$\sqrt{\det G}\,dx\,dy = \sqrt{1+\phi_x^2+\phi_y^2}\,dx\,dy$$

$$F[\phi] = \int_D \sqrt{1+\phi_x^2+\phi_y^2}\,dx\,dy$$

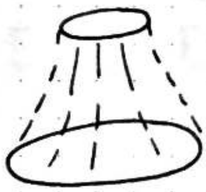
$$\frac{\partial f}{\partial \phi} = 0 \qquad \frac{\partial f}{\partial \phi_x} = \frac{\phi_x}{\sqrt{1+\phi_x^2+\phi_y^2}} \qquad \text{similar for } \frac{\partial f}{\partial \phi_y}$$

Expand (3.4) to obtain

$$(1+\phi_y^2)\,\phi_{xx} + (1+\phi_x^2)\,\phi_{yy} - 2\phi_x\phi_y\,\phi_{xy} = 0 \qquad (3.5)$$

« Pretty hard »

Assume circular symmetry $\phi(x,y) = z(r)$



Exercise: (3.5) $\Rightarrow$ ODE

$\cancel{z = r_0}$ $\quad r = r_0 \cosh\left(\frac{z-z_0}{r_0}\right)$

§3.4 Higher derivatives

$$F[y] = \int_\alpha^\beta f(x, y, y', \dots, y^{(n)})\, dx \qquad y^{(k)} = \frac{d^k y}{dx^k}$$

Given boundary conditions for $y, y', \dots, y^{(n-1)}$ $\quad$ $2n$ constants

Proceed as in §2, $y \to y + \varepsilon\eta$

$\eta, \eta', \dots, \eta^{(n-1)}$ vanish at α, β

$$F[y+\varepsilon\eta] - F[y] = \varepsilon \int_\alpha^\beta \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \dots \frac{\partial f}{\partial y^{(n)}}\eta^{(n)} \right) dx$$

$$-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta + \underset{\text{zero}}{\cancel{\text{boundary}}} \qquad (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right)\eta$$

Integrate by parts n times

Apply the Lemma, so

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \dots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0 \qquad (3.6)$$

Generalised E-L equation: ODE of order $2n$

<u>Example</u> take $n = 2$, assume $\frac{\partial f}{\partial y} = 0$. Then

$$\frac{d}{dx}\Bigg(\underbrace{\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)}_{\text{constant}} \Bigg) = 0$$

<u>Example</u> $n = 2$, minimise

$$F[y] = \int_0^1 (y'')^2\, dx \quad \text{subject to} \quad \begin{array}{l} y'(0) = y(0) = 0 \\ y(1) = 0,\ y'(1) = 1 \end{array}$$

Use first integral:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = \text{const.} \Rightarrow y''' = K \Rightarrow y = \frac{kx^3}{6} + ax^2 + \underset{\uparrow \atop \text{zero}}{b}x + \underset{\uparrow \atop \text{zero}}{c}$$

$$\Rightarrow y = x^3 - x^2$$

Claim this is a minimum of F.

$$F[y+\eta] - F[y] = \int_0^1 [(y''+\eta'')^2 - (y'')^2]\, dx$$

$$= \int_0^1 [(\eta'')^2 + 2(\eta'')(y'')]\, dx$$

$$= \underbrace{\int_0^1 (\eta'')^2\, dx}_{>0 \text{ if } \eta \neq 0} + 2\int_0^1 \underbrace{(\eta'')(y'')}_{\frac{d}{dx}(\eta' y'') - \underbrace{\eta' y'''}_{6\eta' = \frac{d}{dx}(6\eta)}}$$

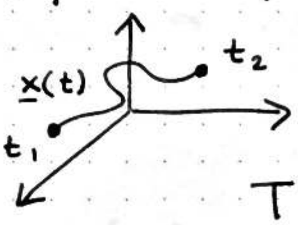
$\frac{d}{dx}(\eta' y'')$ giveth 0; $\frac{d}{dx}(6\eta)$ giveth 0

$\therefore F[y+\eta] > F[y]$ ☺

§4 Least action principle and Noether's theorem

§4.1 Variational formulation of dynamics

A particle of unit mass moves in $\mathbb{R}^3$.



t = independent variable

3 dependent variables

T = kinetic energy

V = potential energy

$$L = T - V = \text{Lagrangian} \quad (4.1)$$

$$S[\underline{x}] = \int_{t_1}^{t_2} L\, dt = \text{action} \quad (4.2)$$

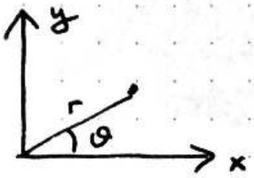
Hamilton's principle (aka "Least action principle"):
The action is stationary along the path taken
So L satisfies the E-L equations

<u>Example</u> $T = \frac{1}{2}|\dot{\underline{x}}|^2$, $V = V(\underline{x})$

$$\frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i, \quad \frac{\partial L}{\partial x_i} = -\frac{dV}{\partial x_i}$$

E–L eqⁿ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \Rightarrow \ddot{\underline{x}} = -\underline{\nabla} V$ (Newton)

<u>Example</u> particle in $\mathbb{R}^2$ moving in radial potential



$$L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

E–L eqⁿ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

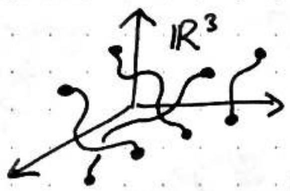
explore first integrals

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} = \text{const.}$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{r}\frac{\partial L}{\partial \dot{r}} - L = \underbrace{\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)}_{\text{Energy}}$$

is constant as well

<u>Example</u> (Configuration space)



Consider N particles in $\mathbb{R}^3$

$\underline{x}_1, \dots, \underline{x}_N$ positions

Work in $C = \mathbb{R}^{3N}$ where $q \in C$ has

$(q_1^j, q_2^j, q_3^j, \dots, q_{3N}^j)$ as the components of the particles

N paths in $\mathbb{R}^3 \longrightarrow 1$ path in C

Now $L = L(q^i, \dot{q}^i, t)$ and $S[q] = \int_{t_1}^{t_2} L \, dt$.

q^i: dependent, t: indep

§4.2 Noether's theorem

$$F[\underline{q}] = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t)\,dt$$

Example $L = \frac{1}{2}(\dot{q}^i \dot{q}^i) - V(q^1 - q^2)$

$Q^1 = -q^2,\ Q^2 = -q^1 \Rightarrow L(Q^1, Q^2, \dot{Q}^1, \dot{Q}^2) = L(q^1, q^2, \dot{q}^1, \dot{q}^2)$

This is a symmetry of L, albeit a discrete symmetry.

Insted take $Q^1 = q^1 + s,\ Q^2 = q^2 + s,\ s \in \mathbb{R}$

Also a symmetry for any s, so have a continuous symmetry.

In general, a continuous symmetry of L is a 1-parameter family of transformations $q^i(t) \to Q^i(s,t)$ such that

$Q^i(0,t) = q^i(t)$, Q^i is C^1 in s, and

$L(Q^i, \dot{Q}^i, t) = L(q^i, \dot{q}^i, t)\ \forall s$

<u>Theorem</u> Given a continuous symmetry of L,

$$\sum_i \frac{\partial L}{\partial \dot{q}^i} \frac{\partial Q^i}{\partial s}\Big|_{s=0} \text{ is a } 1^{st} \text{ integral of the E-L equations.} \tag{4.3}$$

<u>Proof</u>

$$0 = \frac{d}{ds} L(Q^i(s,t), \dot{Q}^i(s,t), t)\Big|_{s=0}$$

$$= \left(\frac{\partial L}{\partial Q^i} \frac{\partial Q^i}{\partial s} + \frac{\partial L}{\partial \dot{Q}^i} \frac{\partial \dot{Q}^i}{\partial s}\right)\Big|_{s=0}$$

$$\overset{E\text{-}L}{=} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\frac{\partial Q^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i} \frac{d}{dt}\left(\frac{\partial Q^i}{\partial s}\right)\Big|_{s=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial Q^i}{\partial s}\right) \text{ as desired}$$

$\square$

Back to the example,

$\frac{\partial Q^i}{\partial s}\Big|_{s=0} = 1$, $\cancel{\partial Q}$ $\frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i$

$\Rightarrow \dot{q}_1 + \dot{q}_2$ is a 1^{st} integral (momentum)

L7.2

§5 Convex functions

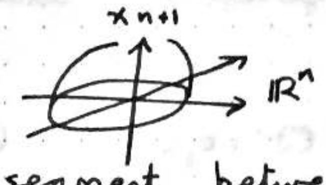
Go back to calculus in $\mathbb{R}^n$. Study a class of functions where finding min / max is easy.

Def^n a) A set $S \subseteq \mathbb{R}^n$ is convex if

$$\forall \underline{x}, \underline{y} \in S,\ t \in [0,1],\ (1-t)\underline{x} + t\underline{y} \in S$$

b) A graph of a function $f: \mathbb{R}^n \to \mathbb{R}$ is a surface Σ in $\mathbb{R}^{n+1}$ given by $x_{n+1} - f(x_1, \dots, x_n) = 0$.

E.g. $f(\underline{x}) = \sqrt{1 - |\underline{x}|^2}$ gives a hemisphere

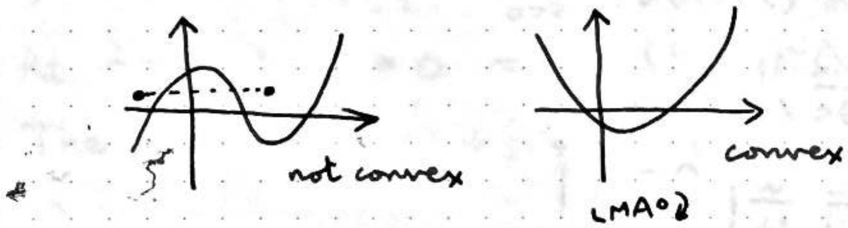


c) A chord of $f: \mathbb{R}^n \to \mathbb{R}$ is a straight line segment between two points on the graph of f.

Def^n A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f((1-t)\underline{x} + t\underline{y}) \leq (1-t)f(\underline{x}) + tf(\underline{y}) \quad (5.1) \qquad \Leftrightarrow \text{ convex epigraph}$$

$\forall \underline{x}, \underline{y} \in \mathbb{R}^n,\ t \in [0,1]$



Note 1) f is concave if $-f$ is convex, i.e. (5.1) has $\geq$

2) f is strictly convex if for $t \in (0,1)$, $\underline{x} \neq \underline{y}$, (5.1) has $<$

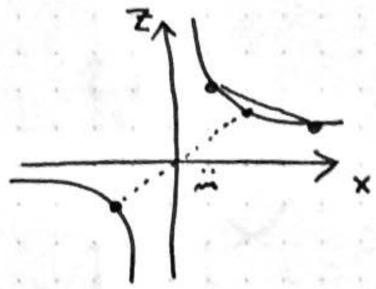
Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$\begin{aligned} &f((1-t)x + ty) - (1-t)f(x) - tf(y) \\ &= (1-t)^2x^2 + 2t(1-t)xy + t^2y^2 - (1-t)x^2 - ty^2 \\ &= -t(1-t)x^2 - t(1-t)y^2 + 2t(1-t)xy \\ &= -t(1-t)(x-y)^2 < 0 \text{ for } t \in (0,1),\ x \neq y \end{aligned}$$

So f is strictly convex

L8.3

<u>Example</u> $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 1/x$



f not convex

But $f|_{\mathbb{R}_+}$ is convex

§5.1 Conditions for convexity

3 different tests for convexity

a) If f is once diff'ble, then f is convex iff $\forall\, \underline{x}, \underline{y}$

$$f(\underline{y}) \geq f(\underline{x}) + (\underline{y} - \underline{x}) \cdot \underline{\nabla} f(\underline{x}) \qquad (5.2)$$

<u>Proof</u> 1) Assuming (5.2), have that $\forall\, \underline{x}, \underline{y}, \underline{z}$

$$f(\underline{x}) \geq f(\underline{z}) + (\underline{x} - \underline{z}) \cdot \underline{\nabla} f(\underline{z})$$

$$f(\underline{y}) \geq f(\underline{z}) + (\underline{y} - \underline{z}) \cdot \underline{\nabla} f(\underline{z})$$

$1 \cdot (1-t) + 2 \cdot t$ gives

$$(1-t) f(\underline{x}) + t f(\underline{y}) \geq f(\underline{z}) + [(1-t)\underline{x} + t\underline{y} - \underline{z}] \cdot \underline{\nabla} f(\underline{z})$$

Take $\underline{z} = (1-t)\underline{x} + t\underline{y}$ and be done. "$\Rightarrow \square$"

2) Conversely, assume (5.1) and define, for $\underline{x}, \underline{y}$ fixed,

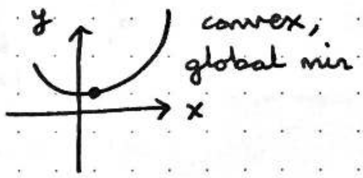
$h(t) = (1-t) f(\underline{x}) + t f(\underline{y}) - f((1-t)\underline{x} + t(\underline{y})) \geq 0$ on $[0,1]$

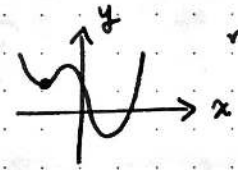
At $t=0$, $h(t) = 0 \Rightarrow h'(t) \geq 0$ by diff'bility? yes!

Then obtain $-f(\underline{x}) + f(\underline{y}) - (\underline{y} - \underline{x}) \cdot \underline{\nabla} f(\underline{x}) \geq 0$, as desired.

"$\Leftarrow \square$"

Claim If $\underline{x}_0$ is a stationary point of a convex function, then it is an absolute minimum of the function





Proof Follows directly from (5.2) $f(\underline{y}) \geq f(\underline{x}) + (\underline{y}-\underline{x})\cdot \underline{\nabla}(f(\underline{x}))$ $\square$

b) Another 1st order test. $f: \mathbb{R}^n \to \mathbb{R}$ is convex iff

$$(\underline{y}-\underline{x})\cdot(\underline{\nabla} f(\underline{y}) - \underline{\nabla} f(\underline{x})) \geq 0 \qquad \text{[Exercise]} \qquad (5.3)$$

c) Second order test. Assume f is a C^2 function $\mathbb{R}^n \to \mathbb{R}$.

Then f is convex iff all eigenvalues of the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{are non-negative } \forall\, \underline{x} \in \mathbb{R}^n.$$

[If evals strictly positive, f is strictly convex]

Proof Assume f is convex, use (5.2) with $\underline{y} = \underline{x} + \underline{h}$.

$$f(\underline{x}+\underline{h}) \geq f(\underline{x}) + \underline{h}\cdot\underline{\nabla} f(\underline{x})$$

Expand LHS: $f(\underline{x}+\underline{h}) = f(\underline{x}) + \underline{h}\cdot\underline{\nabla} f(\underline{x}) + \frac{1}{2} h_i h_j H_{ij}(\underline{x}) + o(h^2)$

We deduce $\frac{1}{2} h_i h_j H_{ij}(\underline{x}) + o(h^2) \geq 0$

This implies evals of H_{ij} are all non-negative.

"<=" See Townsend's notes

Example $f: \mathbb{R}^2_+ \to \mathbb{R}$

$$(x,y) \to \frac{1}{xy} \qquad H_{ij} = \frac{1}{xy}\begin{pmatrix} 2/x^2 & 1/xy \\ 1/xy & 2/y^2 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Determinant } \frac{3}{x^3y^3} > 0 \\ \text{Trace } > 0 \end{array}\right\} \text{ So } f \text{ is strictly convex}$$

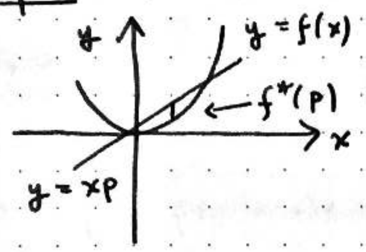
§6 Legender transform

Def^n The Legendre transform of $f: \mathbb{R}^n \to \mathbb{R}$ is

$$f^*(\underline{p}) = \sup_{\underline{x}\in\mathbb{R}^n} (\underline{p}\cdot\underline{x} - f(\underline{x})) \qquad (6.1)$$

The domain of f^* are all $\underline{p}$ such that the sup exists

Example $n=1$



$$\frac{\partial}{\partial x}(px - f(x)) = 0$$

gives x_0 s.t.

$$p = \frac{\partial f}{\partial x}\bigg|_{x_0}$$ Find $x(p)$ and substitute in (6.1)

Take $f(x) = ax^2$ with $a > 0$.

$$\frac{\partial}{\partial x}(px - ax^2) = 0 \Rightarrow p = 2ax, \; x = \frac{p}{2a}$$

$$\therefore f^*(p) = p \cdot \frac{p}{2a} - a \cdot \frac{p^2}{4a^2} = \frac{p^2}{4a}$$

$$f^{**}(x) = \frac{x^2}{4(\frac{1}{4a})} = ax^2 \cdot$$ Relation holds $\forall$ strictly convex functions f

Note that if $a < 0$, f^* undefined everywhere

Prop Legendre transforms are convex functions with convex domain

Proof $f^*((1-t)\underline{p} + t\underline{q}) = \sup_{\underline{x} \in \mathbb{R}^n} ((1-t)\underline{p} \cdot \underline{x} + t\underline{q} \cdot \underline{x} - f(\underline{x}))$

$$= \sup_{\underline{x} \in \mathbb{R}^n} ((1-t)\underline{p} \cdot \underline{x} + t\underline{q} \cdot \underline{x} - f(\underline{x}) + tf(\underline{x}) - tf(\underline{x}))$$

$$= \sup_{\underline{x} \in \mathbb{R}^n} ((1-t)\underline{p} \cdot \underline{x} - (1-t)f(\underline{x}) + t\underline{q} \cdot \underline{x} - tf(\underline{x}))$$

$$\leq (1-t) \sup_{\underline{x} \in \mathbb{R}^n} (\underline{p} \cdot \underline{x} - f(\underline{x})) + t \sup_{\underline{x} \in \mathbb{R}^n} (\underline{q} \cdot \underline{x} - f(\underline{x}))$$

$$= (1-t)f^*(p) + tf^*(q)$$

This shows f^* defined on a convex set, and is convex. □

How to find f^* in practice? Assume f is strictly convex, diff'ble

To find sup in (6.1), $\underline{\nabla}(\underline{p} \cdot \underline{x} - f(\underline{x})) = 0 \Rightarrow \underline{p} = \underline{\nabla} f(\underline{x})$

Now if f strictly convex, can invert for $\underline{x}(\underline{p})$

Substitute back to yield $f^*(\underline{p}) = \underline{p} \cdot \underline{x}(\underline{p}) - f(\underline{x}(\underline{p}))$

It is true that $f^{**} = f$ (No proof given) for f strictly convex

L9.3

§ 6.1 Applications in thermodynamics

* Dunajski not like how this is presented *

Complex system, like a gas, can be described in terms of a few macroscopic parameters, $T =$ temperature, $V =$ volume
$p =$ pressure, $S =$ entropy

Internal energy $U(S, V)$ satisfies 1st law of thermo:

$$dU = T\,dS - p\,dV$$

L10.1

$U(S,V)$ and $dU = T\,dS - p\,dV$

Free energy $F = \min\limits_S (U - TS) = F(T,V)$

$$= -\max_S (TS - U) = -U^*(T)$$

Legendre transform wrt S regarding V as a parameter

$$\frac{\partial}{\partial S}(TS - U)\Big|_V = 0 \Rightarrow T = \frac{\partial U}{\partial S}\Big|_V$$

Enthalpy $H(p, S) = \underset{V}{\min\,\cancel{\max}}\,(U + pV)$

$$= -\max_V\big((-p)V - U\big) = -U^*(-p)$$

Legendre transform wrt V

$$\frac{\partial}{\partial V}(U + pV)\Big|_S = 0 \Rightarrow p = -\frac{\partial U}{\partial V}\Big|_S$$

In both cases Legendre transform swaps roles of thermodynamic quantities:

$$\begin{tikzcd} U(S,V) \arrow[r, "S"] \arrow[d, "V"] & F(T,V) \\ H(p, S) \arrow[ur, dashed] & \end{tikzcd}$$

§7 Hamiltonian mechanics

Recall (§4.1) the Lagrangian $L(q^i, \dot{q}^i, t) = T - V$

q^i = "positions" $\quad v^i = \dot{q}^i$ = "velocities"

H = Legendre transform of L wrt the velocities

$$H(p_i, q^i, t) = \sup_{\underline{v}}\big(p_i v^i - L(q^i, v^i, t)\big)$$

$p_i = \dfrac{\partial L}{\partial v_i}$ at sup $\Rightarrow$ invert relation

$$H = p_i v^i(\underline{p}, \underline{q}) - L(\underline{q}, \underline{v}(\underline{p}, \underline{q}), t) \qquad (7.1)$$

<u>Example</u> $L = m\dfrac{|\underline{v}|^2}{2} - V(\underline{q}) \qquad p_i = \dfrac{\partial L}{\partial v_i} = m v_i$

$$\therefore H(\underline{p}, \underline{q}, t) = -\frac{m}{2}\frac{|\underline{p}|^2}{m^2} + V(\underline{q}) + \underline{p}\cdot\frac{\underline{p}}{m} = \frac{|\underline{p}|^2}{2m} + V(\underline{q})$$

total energy

$\underline{p}$ = momenta

L10.2

What about E-L equations? $\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i}$

Compute dH in two ways.

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial t} dt$$

$$= d\left(p_i \dot{q}^i - L(q^i, \dot{q}^i, t)\right)$$

$$= \dot{q}^i dp_i + \underbrace{p_i d\dot{q}^i}_{} - \underbrace{\frac{\partial L}{\partial q^i}}_{\dot{p}_i \text{ by E-L}} dq^i - \underbrace{\boxed{\frac{\partial L}{\partial \dot{q}^i}} d\dot{q}^i}_{=p_i} - \frac{\partial L}{\partial t} dt$$

$$= \dot{q}^i dp_i - \dot{p}_i dq^i - \frac{\partial L}{\partial t} dt$$

Now compare coeffs to deduce

$$\underbrace{\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}}_{\substack{\textit{Hamilton's equations} \\ 2n \text{ first order ODEs}}}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (7.2)$$

↑ note $\partial/\partial t$ means different

$\exists$ unique solution once p_i, q^i at $t=0$ are specified

initial data $\rightarrow$ $p_i(t), q^i(t)$ is a curve in $2n$-dimensional phase space

Hamiltonian formulation links with quantum dynamics (PQM)

<u>Example</u> Take $n=1$, $H(p,q) = \frac{p^2}{2m} + V(q)$

$$(7.2) \rightarrow \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -V'$$

$$p = m\dot{q} \longrightarrow m\ddot{q} = -V' \quad \text{ö N2}$$

Hamilton's equations from the least action principle

$L = \underline{p} \cdot \underline{\dot{q}} - H(\underline{p}, \underline{q}, t)$ where ...

$$S[\underline{p}, \underline{q}, t] = \int_{t_1}^{t_2} (\underbrace{p_i \dot{q}^i - H(\underline{p}, \underline{q}, t)}_{f}) \, dt$$

E-L equations for $2n$ dependant variables $\underline{q}, \underline{p}$

$$\frac{\partial f}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}^i}\right), \quad \frac{\partial f}{\partial p_i} = \frac{d}{dt}\Big(\underbrace{\frac{\partial f}{\partial \dot{p}_i}}_{\text{zero}}\Big)$$

L10.3

So $\dot{q}^i - \frac{\partial H}{\partial p_i} = 0$.

Other eq^{ns} give

$$-\frac{\partial H}{\partial q^i} = \frac{d}{dt} p_i$$

gives (7.2)

oh boy

§8 The second variation

Look at the nature of stationary points of a functional (2.1)

$$F[y] = \int_\alpha^\beta f(x, y, y')\, dx \qquad y \to y + \varepsilon\eta(x) \text{ s.t. } \eta(\alpha) = \eta(\beta) = 0$$

$$F[y+\varepsilon\eta] - F[y] = \int_\alpha^\beta [f(x + \cancel{\varepsilon\eta}, y + \varepsilon\eta, y' + \varepsilon\eta') - f(x, y, y')]\, dx$$

Expand in ε up to quadratic terms. Assume $y(x)$ satisfies E-L

$$F[y+\varepsilon\eta] - F[y] = \varepsilon \int_\alpha^\beta \eta \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)}_{\text{zero by E-L}} dx$$

$$+ \frac{1}{2}\varepsilon^2 \int_\alpha^\beta \left\{ \eta^2 \frac{\partial^2 f}{\partial y^2} + 2\eta\eta' \frac{\partial^2 f}{\partial y \partial y'} + \eta'^2 \frac{\partial^2 f}{\partial y'^2} \right\} dx \quad + o(\varepsilon^2)$$

Note
$$2\frac{\partial^2 f}{\partial y \partial y'}\eta\eta' = \frac{\partial^2 f}{\partial y \partial y'}\frac{d}{dx}(\eta^2) = \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\eta^2\right) - \eta^2 \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\right)$$

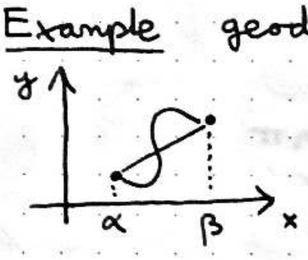
vanishes on integration ☺

$$\left.\begin{aligned} &\delta^2 F[y] = \frac{1}{2}\int_\alpha^\beta (Q\eta^2 + P(\eta')^2)\, dx \quad \text{the second variation} \\ &P = \frac{\partial^2 f}{\partial y'^2} \qquad Q = \frac{\partial^2 f}{\partial y^2} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\right) \end{aligned}\right\} (8.1)$$

We proved

<u>Prop</u> If $y(x)$ is a solution to the E-L equation, and if $Q\eta^2 + P(\eta')^2 > 0 \;\forall\, \eta$ s.t. $\eta(\alpha) = \eta(\beta) = 0$ [and P, Q as in 8.1] then $y(x)$ is a local minimiser of (2.1)

<u>Example</u> geodesics on $\mathbb{R}^2$



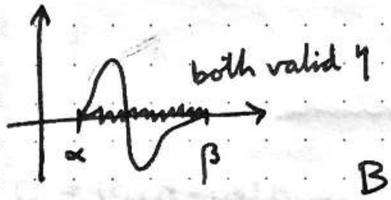
$$F[y] = \int_\alpha^\beta \sqrt{1 + (y')^2}\, dx$$

$$Q = 0, \quad P = \frac{\partial^2 f}{\partial y'^2} = \frac{\partial}{\partial y'}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) \overset{\dots}{=} \frac{1}{(1+(y')^2)^{3/2}} > 0$$

It follows that $\int P(\eta')^2 > 0$ for all non-zero η

Go back to (8.1)

<u>Prop</u> If $y_0(x)$ is a solution to E-L, then $P = \frac{\partial^2 f}{\partial y'^2} \geq 0$ (8.2) is a necessary condition for y_0 to be a local minimum



"P is more important than Q"

<u>Proof</u> Assume that $\exists\, x_0 \in (\alpha,\beta)$ s.t. $P<0$

By continuity $\exists\, \delta>0$ s.t. $P<-c/2$ on $(x_0-\delta, x_0+\delta)$

Take η s.t. $\eta=0$ outside $(x_0-\delta, x_0+\delta)$ and is small but oscillates rapidly inside $(x_0-\delta, x_0+\delta)$. (See Gelfand-Fomin for details)

For this choice of η, second variation becomes

$$\delta^2F[y] = \int_{x_0-\delta}^{x_0+\delta} \Big(\underbrace{-\tfrac{c}{2}(\eta')^2}_{\text{large}} + \underbrace{Q\eta^2}_{\text{bounded}}\Big)\,dx < 0$$

$\square$

§8.1 Associated eigenvalue problem

Try to see if condition $P>0$ is sufficient for $\delta^2F[y]>0$

Look at (8.1)

$$Q\eta^2 + P\eta'^2 = Q\eta^2 + \underbrace{\frac{d}{dx}(P\eta'\eta)}_{\text{vanishes on } \int} - (P\eta')'\eta$$

So $\delta^2F[y] = \int_\alpha^\beta \eta\big(-(P\eta')' + Q\eta\big)\,dx$

$$= \frac{1}{2}\int_\alpha^\beta \eta\cdot\mathcal{L}(\eta)\,dx \qquad (8.\pi),$$

where $\mathcal{L}(\eta) = -(P\eta')' + Q\eta$ is the Sturm-Liouville operator

If $\exists\, \eta$ s.t.
$$\begin{cases} \mathcal{L}(\eta) = -\omega^2\eta & (\omega \text{ real}, \neq 0) \\ \eta(\alpha) = \eta(\beta) = 0 \end{cases} \qquad (8.4)$$

then y cannot be a local minimisor, as then

$$\delta^2F[y] = \int_\alpha^\beta -\omega^2\eta^2 < 0 \quad \text{if } \eta \neq 0$$

<u>Example</u> Take $f(y) = \frac{1}{2}[(y')^2 - (y^2)]$, $\alpha=0$, $\beta \underbrace{\neq n\pi}_{\downarrow \text{ no } \sin nx}$

Minimise (2,1) subject to $y(0)=y(\beta)=0$

E-L equations $y''+y=0 \longrightarrow y=0$ is solution

Is this a local minimisor?

$$\delta^2F[y] = \frac{1}{2}\int_0^\beta \{\eta'^2 - \eta^2\}\,dx \qquad P=1,\ Q=-1$$

$+\eta'' + \eta = +\omega^2 \eta$ $\quad \eta \overset{?}{=} A \sin\left(\frac{\pi x}{\beta}\right)$ so $\eta(0) = \eta(\beta) = 0$

$-\left(\frac{\pi}{\beta}\right)^2 + 1 \overset{?}{=} \omega^2 \iff 1 - \omega^2 = \left(\frac{\pi}{\beta}\right)^2$ POG! This gets you real if $\beta > \pi$

In general, problems may arise if $[\alpha, \beta]$ is [illegible]
We shall now make this precise

[Remark: Compare (3.2) The minimum of $\delta^2 F[y = y_0, \eta]$ as a functional of η is the lowest eigenvalue of $\mathcal{L}$]

§ 8.2 Jacobi condition

Legendre tried (and failed) to prove that $P > 0$ is sufficient for positivity of $\delta^2 F$. His idea:

Take any $\phi \in C^1_{[\alpha,\beta]}$. Then

$$\underbrace{\int_\alpha^\beta (\phi \eta^2)' \, dx}_{\text{zero } \forall \phi} = \int_\alpha^\beta (\phi' \eta^2 + 2\phi \eta \eta') \, dx$$

Add this to (8.1) to obtain

$$\delta^2 F[y, \eta] = \int_\alpha^\beta (P\eta'^2 + 2\phi\eta\eta' + Q\eta^2 + \phi'\eta^2) \, dx$$

$$= \int_\alpha^\beta \Big(\underset{>0}{P} \underbrace{\left(\eta' + \frac{\phi\eta}{P}\right)^2}_{\geq 0} + \underbrace{\left(Q + \phi' - \frac{\phi^2}{P}\right)}_{\text{take } \phi \text{ s.t. this vanishes}} \eta^2 \Big) \, dx$$

$\phi^2 = P(Q + \phi')$ $\quad \sim$ Ricatti type equation $\quad$ (8.5)

Transform into 2nd order linear ode: $\phi = -P\frac{u'}{u}$ $\leftarrow$ need $u \neq 0$

$$\cancel{P^2\left(\frac{u'}{u}\right)^2} = P\left(Q - P'\frac{u'}{u} - P\frac{u''u - \cancel{u'^2}}{u^2}\right)$$

$$Q - P'\frac{u'}{u} - P\frac{u''}{u} = 0 \implies -(Pu')' + Qu = 0 \qquad (8.6)$$

Jacobi's accessory equation

Need solution to (8.6) which is non-zero on $[\alpha, \beta]$

Such solution could not exist if $[\alpha, \beta]$ is too large

L12.2

Go back to $P=1$, $Q=-1$. Then (8.6) is

$u'' + u = 0 \rightarrow u = A\sin(x - x_0)$ so rekt for large bois

So ö