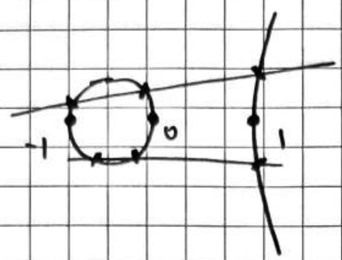
$$f(x,y) = y^2 - x(x-1)(x+1) \in \mathbb{Z}[x,y]$$

$$\mathbb{V}(f) = \{ (z_1, z_2) \mid f(z_1,z_2) = 0 \} \subseteq K^2$$

K is some field

If $K = \mathbb{R}$ then

$\mathbb{V}(f) \subseteq \mathbb{R}^2$



Observation: Lines appear to "regularly" meet $\mathbb{V}(f)$ at 3 pts.

$x^2+1=0$ over $\mathbb{R}$ is not so nice

In algebraic geometry, best to work over $K = \overline{K}$.

For the same polynomial $f \in \mathbb{Z}[x,y]$

$\mathbb{V}(f) \subseteq \mathbb{C}^2$

In this case $\mathbb{V}(f)$

Real torus
$S^1 \times S^1$

Set $C = \mathbb{V}(f) \subseteq \mathbb{C}^2$.

Guiding Question: What does it mean to study functions on C?

" Algebraic / polynomial - like "

Way to produce functions

$$C \subseteq \mathbb{C}^2 \xrightarrow[h \in \mathbb{C}[x,y]]{\text{polynomials}} \mathbb{C}$$

Whatever functions on C is

$$\underbrace{\mathbb{C}[X,Y]}_{\text{ring}} \longrightarrow \underbrace{\mathrm{Fun}(C)}_{?\ \text{Ring}}$$

? Ring $\longrightarrow$ Commutative ring with identity & contain $\mathbb{C}$

Concern: There may be many ways to express a function on C as the restriction of an element in $\mathbb{C}[x,y]$.

Ex: Take $h_1 \equiv 0 \in \mathrm{Fun}(C)$

Can express $0 \in \mathrm{Fun}(C)$ both as the restriction of the 0 function in $\mathbb{C}[x,y]$, or the restriction of f.

<u>Observe</u> Two polynomials $h_1, h_2 \in \mathbb{C}[x,y]$ restrict to the same function on C if $h_1 - h_2 \equiv 0$ on C.

Equivalently, $h_1 - h_2$ is divisible by f. (!)

In other words, the functions on C are
$$\frac{\mathbb{C}[x,y]}{(f)} = \mathrm{Fun}(C)$$

<u>3 main characters</u>

Vanishing sets in $\mathbb{C}^n$ or k^n of systems of polys in n-variables

$\updownarrow$

The ring $\mathbb{C}[x_1,\dots,x_n]/I$ where I is the ideal generated by "these" polynomials

Ideal itself

Fix $k = \bar{k}$ be an algebraically closed field.

<u>Def</u> <u>Affine space of dim^n n over k</u>, denoted $\mathbb{A}^n_k$ is the set k^n, equipped with the ring of polynomial functions $k[x_1,\dots,x_n]$.

"Ambient" space for varieties.

<u>Def</u> An <u>affine variety</u> is any subset of $\mathbb{A}^n_k$ of the form $\mathbb{V}(S)$ for $S \subseteq k[x_1,\dots,x_n]$.

$$\mathbb{V}(S) = \{\underline{P} \in \mathbb{A}^n_k \mid f(\underline{P}) = 0 \ \forall f \in S\}$$

<u>Initial observations</u>

- If S contains a non-zero element of k, then $\mathbb{V}(S) = \emptyset$.
- If $S \subseteq S'$ then $\mathbb{V}(S) \supseteq \mathbb{V}(S')$.
- If I_S is the ideal generated by S, then $\mathbb{V}(I_S) = \mathbb{V}(S)$.

[please convince yourself if unclear] [sic.]

Properties of $k[x_1,\dots,x_n]$

- Unique Factorization Domain + later
- Hilbert's Basis Theorem: every ideal in the polynomial ring is finitely generated, i.e. for all $I \subseteq k[\underline{x}]$ are $I = \langle f_1, \dots, f_r \rangle$, $f_i \in k[\underline{x}]$

⚠ Not a PID unless $n=1$

Geometric Consequences

$$\mathbb{V}(S) = \mathbb{V}(I_S) \underset{\text{HBT}}{=} \mathbb{V}(\{f_1, \dots, f_r\})$$

for some $f_1, \dots, f_r \in I_S$

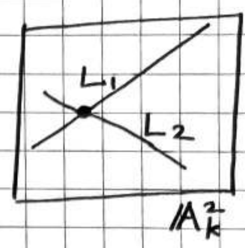
$$\mathbb{V}(f_1, \dots, f_r) \subseteq \mathbb{A}^n_k$$
$$\parallel$$
$$\bigcap_{i=1}^{r} \mathbb{V}(f_i)$$

WORD: $\mathbb{V}(f)$ is called a hypersurface

Every variety is the intersection of finitely many hypersurfaces.

Examples • Take $\ell_1(x,y) = a_1x + b_1y + c_1$

Consider $\mathbb{V}(\ell_1(x,y)) \subseteq \mathbb{A}^2_k$, a <u>line</u>



$\ell_2(x,y) = a_2x + b_2y + c_2$

$\mathbb{V}(\ell_2) : L_2$

$L_1 \cup L_2$ is also an affine variety

$\mathbb{V}(\ell_1\ell_2(x,y)) = L_1 \cup L_2$

"More" generally Take $\ell_1, \dots, \ell_d \in k[x,y]$ all linear (non-constant). Then $\mathbb{V}(\ell_1 \cdots \ell_d) \subseteq \mathbb{A}^2$ is a union of lines.

If the coefficients a_i, b_i, c_i are all chosen "randomly", then each line intersects each other line in exactly 1 point.

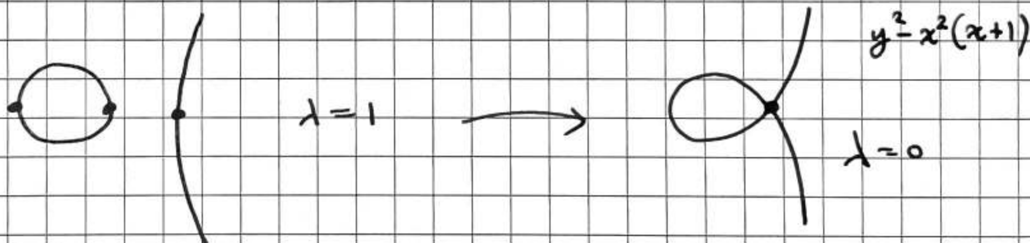
No 3 lines through any point.

Example 2 Last time $y^2 - x(x-1)(x+1)$

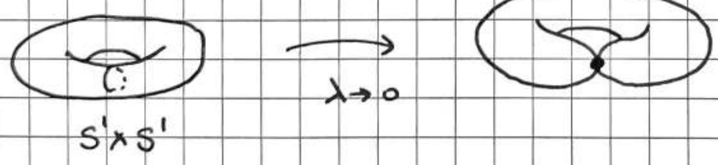
Start "varying" the "-1" i.e.

$$y^2 = x(x-\lambda)(x+1), \quad \lambda \to 0$$

Real Picture



Complex Picture (rough)



Ideals $\longrightarrow$ Affine varieties

$k[x_1, \dots, x_n] \qquad \mathbb{A}^n_k$

$I \longmapsto \mathbb{V}(I)$

Proposition (1) $\mathbb{V}(I+J) = \mathbb{V}(I \cup J) = \mathbb{V}(I) \cap \mathbb{V}(J)$

(2) $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$

$\ulcorner I, J$ are ideals in $k[\underline{x}]$ $\lrcorner$

Proof (1) Exercise

(2) • Say $p \in \mathbb{V}(I) \cup \mathbb{V}(J)$.

Wlog we can say $p \in \mathbb{V}(I)$.

Since $I \cap J \subseteq I$, $p \in \mathbb{V}(I \cap J)$.

By relabelling I, J, get $\mathbb{V}(I) \cup \mathbb{V}(J) \subseteq \mathbb{V}(I \cap J)$

• Conversely, say $p \in \mathbb{V}(I \cap J)$.

Assume $p \notin \mathbb{V}(I)$. We'll show that $p \in \mathbb{V}(J)$.

There exists $g \in I$ s.t. $g(p) \neq 0$.

But then for any $f \in J$, then

$fg \in I \cap J$ so $fg(p) = 0$

But $g(p) \neq 0$ so $f(p) = 0$.

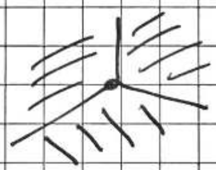
f arbitrary $\Rightarrow$ $p \in \mathbb{V}(J)$. $\square$

Definition (Irreducibility) A variety V in $\mathbb{A}^n_k$ is irreducible if $V \neq V_1 \cup V_2$ for $V_1, V_2 \neq V$.

V_i are varieties in $\mathbb{A}^n_k$

Variety that is not irreducible is reducible.

Ex $\mathbb{V}(xyz) \subseteq \mathbb{A}^3_k$



$\subseteq \mathbb{R}^3$

union of coord planes

This is reducible b/c

$$\underbrace{\mathbb{V}(xyz)}_{V} = \underbrace{\mathbb{V}(xy)}_{V_1} \cup \underbrace{\mathbb{V}(z)}_{V_2}$$

$$= \mathbb{V}(x) \cup \mathbb{V}(y) \cup \mathbb{V}(z)$$

Proposition Let V be an affine variety in $\mathbb{A}^n_k$. Then V can be written as a finite union of irreducibles.

Proof [Finiteness from Hilbert]

Suppose this isn't possible.

Write $V = V_1 \cup V_1'$. Wlog V_1 is not irreducible.

So $V_1 = V_2 \cup V_2'$ and so on

Produces $V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$

$$\begin{array}{ccc} V_0 & V_1 & V_2 \\ \updownarrow & \updownarrow & \updownarrow \\ \mathbb{V}(I_0) & \mathbb{V}(I_1) & \mathbb{V}(I_2) \cdots \end{array}$$

Say $I = \sum_j I_j \subseteq k[\underline{x}]$. But this is finitely generated.

So the chain terminates, which is a contradiction. $\square$

- Given a variety $A \subseteq \mathbb{A}^n_k$. Consider

$$I(V) = \{ f \in k[\underline{x}] \mid f(p) = 0 \ \forall p \in V \}$$

Make sure: this is an ideal.

Observations: • $V = \mathbb{V}(S)$; then $S \subseteq I(V)$.

- $V = \mathbb{V}(I(V))$ * Think this through
- $V = W \iff I(V) = I(W)$

Key fact (Hilbert's Nullstellensatz) If k is algebraically closed, then if $I \subsetneq k[\underline{x}]$, then $\mathbb{V}(I) \neq \emptyset$

[Proof: probably later]

Notice: $I(-)$ and $\mathbb{V}(-)$ are not precisely inverse

(Why?) Take $I = (x^2) \subseteq k[x]$

$V = \mathbb{V}(I) = 0 \in \mathbb{A}^1_k$

But $I(V) = (x)$.

Feature or Bug?

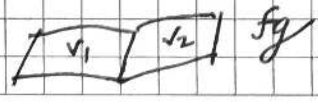
Nonetheless: $I(\mathbb{V}(I(V))) = I(V)$ b/c

if $f \in I(V)$ then $f^2 \in I(V)$

but also if $f^2 \in I(V)$; $f \in I(V)$

Proposition A variety V in $\mathbb{A}^n_k$ is irreducible if and only if $I(V)$ is a prime ideal.

If V is reducible:



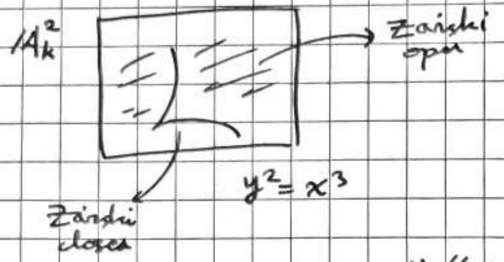
Ex Sheet 1

Two quick points

• Zariski Topology •

$\mathbb{V}(I) \subseteq \mathbb{A}^n_k$ act liek closed sets

Declare $\{ \mathbb{A}^n_k \setminus \mathbb{V}(I) \}$ as the open sets



• Functions always pull-back

$$\begin{cases} X \to Y & \text{"spaces"} \\ Fun(Y) \to Fun(X) & \leftarrow \text{Rings} \end{cases}$$

$$\text{Ideals in } k[x_1,\dots,x_n] \underset{\mathbb{I}(-)}{\overset{\mathbb{V}(-)}{\rightleftarrows}} \text{Algebraic varieties in } \mathbb{A}^n_k$$

$\left\lceil \mathbb{I}(V) = \{ f \in k[\underline{x}],\ f|_V = 0 \} \right\rfloor$

Theorem: (Nullstellensatz) $k = \bar{k}$

- If I is an ideal in $k[\underline{x}]$ then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ "Radical of I"
  where $\sqrt{I} = \{ f \in k[\underline{x}] \mid f^m \in I$ for some $m \geq 1 \}$
- Non-trivial ideals $(I \neq k[\underline{x}])$ have non-empty vanishing loci
- Every maximal ideal in $k[\underline{x}]$ has the form

$$m = ((x_1 - a_1), \dots, (x_n - a_n)) \quad \text{for} \quad a_i \in k$$

Remark The ideal associated to $(a_1, \dots, a_n)$ is exactly this m

Proposition If $V \subseteq \mathbb{A}^n_k$ is affine variety, then V is irreducible iff $\mathbb{I}(V)$ is prime.

Proof (by picture)

$(-)$ "$\Leftarrow$" If V is reducible,
say $V = V_1 \cup V_2$, then $\mathbb{I}(V) = \mathbb{I}(V_1) \cap \mathbb{I}(V_2)$

Neither V_i are equal to V. So choose

$$f_1 \in \mathbb{I}(V_1) \setminus \mathbb{I}(V), \quad f_2 \in \mathbb{I}(V_2) \setminus \mathbb{I}(V)$$

Then $f_1 f_2 \in I(V_1) \cap I(V_2) = I(V)$.

"$\Rightarrow$" Conversely assume $I(V)$ is not prime.

Pick $g_1, g_2 \notin I(V)$ s.t. $g_1 g_2 \in I(V)$.

Let $V_1 = \mathbb{V}(I(V), g_1)$, $V_2 = \mathbb{V}(I(V), g_2)$

$= V \cap \mathbb{V}(g_1)$ $= V \cap \mathbb{V}(g_2)$

Claim $V = V_1 \cup V_2$

Pf If $p \in V$, then $g_1 g_2(p) = 0$.

Then $g_1(p) = 0$ or $g_2(p) = 0$ so $p \in V_1 \cup V_2$.

And $V_1 \subsetneq V$, $V_2 \subsetneq V$. $\square$

Corollaries
- $\mathbb{A}^n_k$ is irreducible
- $\mathbb{V}(y^2 - x^3)$ is irreducible

▷ The irreducibility of $V \subseteq \mathbb{A}^n_k$ depends only on $k[\underline{x}]/I(V)$

(We know that I is prime in $R \Leftrightarrow R/I$ is integral)

WORD OF THE DAY For an affine variety $V \subseteq \mathbb{A}^n_k$, the coordinate ring is defined to be $k[x_1, \dots, x_n]/I(V)$

Notation ⚠
- Coord ring of $V \leftrightarrow k[V]$
- $\mathcal{O}_V$ or $\mathcal{O}(V)$ or $\mathcal{O}_V(V)$

What does $\mathcal{O}_V$ contain?

$$V \subseteq \mathbb{A}^n_k \longleftrightarrow I(V) \subseteq k[x_1, \dots, x_n]$$
$$\updownarrow$$
$$k[x_1, \dots, x_n] \twoheadrightarrow \frac{k[x_1, \dots, x_n]}{I(V)}$$

$\mathcal{O}_V$ on its own is simultaneously a ring & a k-vector space i.e. is a k-algebra

But if we're given $k[x_1, \dots, x_n] \overset{\pi}{\twoheadrightarrow} \mathcal{O}_V$ we also get the data of generators of this k-algebra i.e. $\pi(x_i) = f_i$

<u>Remark</u> $\mathcal{O}_V$ as a k-algebra can be presented as a quotient of a polynomial ring in many different ways

- Given a presentation:

$$k[y_1, \dots, y_m] \xrightarrow{\varphi} \mathcal{O}_V$$

$$\varphi^{-1}(0) = I \quad \text{and} \quad \mathbb{V}(I) \subseteq \mathbb{A}^m_k$$

So the same ring gives us affine varieties in different spaces

Given an element $f \in \mathcal{O}_V$ (and $V \subseteq \mathbb{A}^n_k$) we get a map $V \to \mathbb{A}^1_k$

<u>How?</u> Pick $\tilde{f} \in k[x_1, \dots, x_n]$ s.t. $\tilde{f} \mapsto f$ under π

Now $\tilde{f} : \mathbb{A}^n_k \to \mathbb{A}^1_k$, we can restrict $\tilde{f}|_V$

<u>Observe</u>: The set theoretic map $V \to \mathbb{A}^1_k$ doesn't depend on the choice of lift

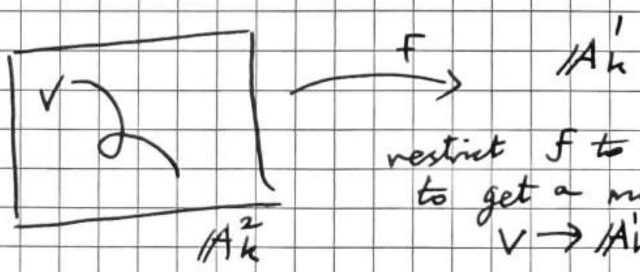
<u>Def</u> · A <u>morphism</u> from $V \subseteq \mathbb{A}^n_k$ to $\mathbb{A}^1_k$ is a map $V \to \mathbb{A}^1_k$ determined by some $f \in \mathcal{O}_V$

· A <u>morphism</u> from $V \to \mathbb{A}^m_k$ is similarly given by a collection $f_1, \dots, f_m \in \mathcal{O}_V$

Now, let $W \subseteq \mathbb{A}^m_k$ be another affine variety.

A <u>morphism</u> $V \to W$ is given by a morphism to $\mathbb{A}^m_k$ whose image is contained in W.

$$\left(\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, dashed] & \mathbb{A}^m_k \\ W \arrow[ur] & \end{tikzcd} \right)$$



restrict f to V to get a morphism $V \to \mathbb{A}^1_k$

<u>Def</u> An <u>isomorphism</u> of varieties is a morphism $V \to W$ w/ a two sided inverse

<u>Examples</u>

- $\mathbb{A}^2_k \longrightarrow \mathbb{A}^1$

$$\left.\begin{array}{rl} (x,y) & \longmapsto x \\ & \longmapsto x^2 \\ & \longmapsto x^2 y \end{array}\right\} \text{All morphisms}$$

- If $\mathbb{V}(x) \subseteq \mathbb{A}^2_k$, then

$$\mathbb{A}^2_k \to \mathbb{A}^1_k$$
$$(x,y) \mapsto y$$

restricts to an isomorphism $\mathbb{V}(x) \longrightarrow \mathbb{A}^1_k$

In <u>particular</u>, $\underbrace{\mathbb{V}(x_1, x_2, x_3)}_{= V} \subseteq \mathbb{A}^{30}_k$

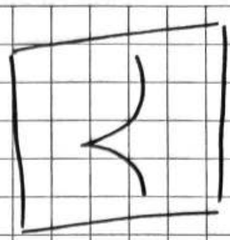
then $V \cong \mathbb{A}^{27}_k$

<u>Theorem</u> Let V, W be affine varieties in $\mathbb{A}^n_k$, $\mathbb{A}^m_k$ resp. Then there is a "natural" bijective correspondence:

$$\left\{ \begin{array}{c} \text{Morphisms} \\ V \to W \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Ring homomorphisms} \\ \mathcal{O}_W \to \mathcal{O}_V \\ \text{preserving } k \end{array} \right\}$$

<u>Example</u> $k[x^2, x^3] \subseteq k[x]$

$\cong k[y_1, y_2]/(y_1^3 - y_2^2)$



Affine varieties = Nilpotent free finitely-generated k-algebras

<u>How to ~~draw~~ visualise plane curves</u> (work over $\mathbb{C}$)

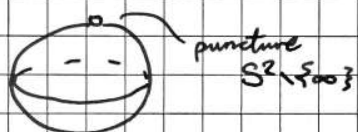
$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{C}} & \longleftrightarrow & \mathbb{R}^2 \times \mathbb{R}^2 \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ \mathbb{V}(f) & \longleftrightarrow & \mathbb{V}(I) \end{array}$$

Put Euclidean topology

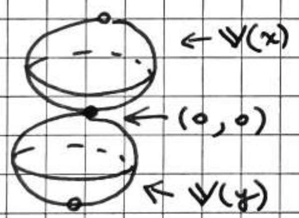
<u>This is the picture we can draw:</u>

1. $\mathbb{V}(\text{linear function}) \leftrightarrow \mathbb{V}(x) \subseteq \mathbb{C}^2$

$\mathbb{V}(x) \cong \mathbb{C}$



2. $\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$



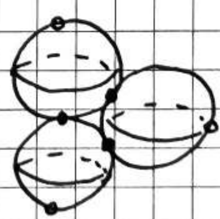
3. Similarly, if we take $l_1, \dots, l_d \in k[x_1, x_2]$ and $\mathbb{V}(l_1 \cdots l_d)$ = union of d lines

▷ Pick the l_i generically, then

$\mathbb{V}(l_1 \cdots l_d) \subseteq \mathbb{C}^2$

is the union of d once-punctured spheres meeting each other at unique points pairwise

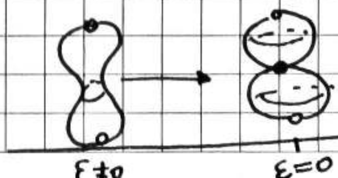
$d=3$



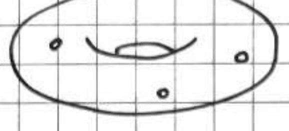
4. Given a polynomial f_d with degree d in 2-variables, with generic coefficients, how to draw?

$xy = \varepsilon$; ε small, $\varepsilon \to 0$

<u>Output</u>:



$\mathbb{V}(f_3)$ for f_3 w/ general coefficients:



Simple Exercise: $\mathbb{V}(f_d)$ is a topological surface with $\binom{d-1}{2}$ holes

$\boxed{\text{Degree-Genus formula}}$

Back to work: An abstract affine variety is a pair (V, R)
V is a topological space, R is a finitely generated nilpotent free k-algebra and $V \xrightarrow{\sim}$ the top space coming from R

Def The maximal spectrum $\mathrm{mspec}(R)$ is the topological space $\mathbb{V}(I) \subseteq \mathbb{A}^n_k$ for some representation $R = \dfrac{k[x_1, \dots, x_n]}{I}$ equipped with the Zariski topology.

Def A closed subvariety of an (abstract) affine variety V is a subset $W \subseteq V$ given by $\mathbb{V}(I)$ for $I \subseteq \mathcal{O}_V$ an ideal.

$\Updownarrow$ Notice this is well-def

$\mathcal{O}_V = R$

How about Zariski opens?

Zariski opens are the complements of Zariski closed.

What are the functions on Z-opens?

Example $\mathbb{A}^1_k \supseteq \mathbb{A}^1_k \setminus \{0\} = k^*$

Every function on $\mathbb{A}^1_k$ determines one on $\mathbb{A}^1_k \setminus \{0\}$ by restriction.

But some functions in $k[x]$ become invertible on $\mathbb{A}^1_k \setminus \{0\}$.

Precisely $\lambda x^n \in k[x]$, $\lambda \in k^*$

So the ring of functions on $\mathbb{A}^1_k \{0\}$ is $k[x, x^{-1}]$

If V is an affine variety and $f \in \mathcal{O}_V$ then $U = V \setminus \mathbb{V}(f)$

Then $\dfrac{1}{f}$ should be considered a function on U.

Assumption From now, V is irreducible.
Then it has a coordinate ring $\mathcal{O}_V$ which is an integral domain.
The field of fractions of $\mathcal{O}_V$ is well-defined
Ex: If $\mathcal{O}_V = k[x]$ then $FF(\mathcal{O}_V) = k(x)$

How to interpret elements of $FF(\mathcal{O}_V)$
Pick $f/g \in FF(\mathcal{O}_V)$. This is a well-defined function on $\underbrace{V \setminus \{g=0\}}_{\text{Zariski open}}$

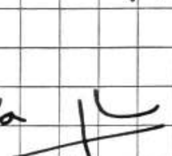
$FF(\mathcal{O}_V)$ is the ring of functions defined on some non-empty Zariski open in V
Notation $k[V]$ coordinate ring $\mathcal{O}_V$
$k(V)$ function field of V $\mathcal{O}_V(\eta)$

$\mathcal{O}_V(\eta) :=$ fraction field of $\mathcal{O}_V$

Given an open set $\mathcal{U} \subseteq V$, the ring of functions on $\mathcal{U}$ is
$$\mathcal{O}_V(\mathcal{U}) = \{ f/g \in \mathcal{O}_V(\eta) \mid g \text{ is non-vanishing on } \mathcal{U} \}$$

Word of the Day A quasi-affine variety is a pair $(\mathcal{U}, R)$ where $\mathcal{U} \subseteq V$ is Z-open, for V an affine variety and R is $\mathcal{O}_V(\mathcal{U})$.

Example: $k[x, x^{-1}] \simeq \dfrac{k[x,y]}{(xy-1)}$ rectangular hyperbola in $\mathbb{A}^2_k$



come up as the quasi-affine $\mathbb{A}^1_k \setminus \{0\} \subseteq \mathbb{A}^1_k$

Q: Take $\mathcal{U} = \mathbb{A}^2_k \setminus \{(0,0)\}$. Which rational functions in $k(X_1, X_2)$ are well-defined on $\mathcal{U}$?
Answer: No new functions

Algebraic Geometry

- Important point from last day: Dhruv is a DUMMY!
- Today: Finish our first pass of affine varieties.
- Next up: Projective geometry

Affine Varieties

- What is an affine variety?

PRACTICE: sol^{n}s to a multivariate poly system

THEORY: f.g. nilpotent free k-algebra

- Given the k-algebra R, what are the points?

PRACTICE: present $R = k[\underline{x}]/I$; $V = \mathbb{V}(I) \subset \mathbb{A}^k$

THEORY: the set of maximal ideals of R

⌈Why?⌋ Nullstellensatz, pts of $\mathbb{A}^n$ correspond exactly to maximal ideals of $k[x_1, \dots, x_n]$

$$(a_1, \dots, a_n) \longleftrightarrow ((x_1 - a_1), \dots, (x_n - a_n))$$

$$k[x_1, \dots, x_n] \twoheadrightarrow k[\underline{x}]/I$$

What is a morphism from $V = \mathrm{mSpec}\, R$?

PRACTICE:
- $V \to \mathbb{A}^1$ is an elt of $\mathcal{O}_V(R)$
- $V \to \mathbb{A}^m$ is m elts of $\mathcal{O}_V$
- $V \to W$ is $V \to \mathbb{A}^m$ landing in W
  $\quad\quad\;\; \cap\!\!\!\!\!\shortmid$ $\mathbb{A}^m$

THEORY: A morphism $V \xrightarrow{\varphi} W$ is exactly the data of $\mathcal{O}_W \xrightarrow{\varphi^*} \mathcal{O}_V$

Functions on W give functions on V by pullback.

What does it mean for a variety to be irreducible?

PRACTCE: You can't write V as a union of smaller varieties

THEORY: $\mathcal{O}_V$ is an integral domain

What is a rational function?

<u>PRACTICE</u>: It's a ratio of elts of $\mathcal{O}_V$ representing functions on opens in V

<u>THEORY</u>: It's an element of $FF(\mathcal{O}_V)$

· Let V be an affine variety and let $p \in V$ be a point. We'll assume V is irreducible.

$$\underset{\text{Coordinate ring}}{\mathcal{O}_V} \subseteq \underset{\text{local ring at } p}{\mathcal{O}_{V,p}} \subseteq \underset{\text{Function field}}{\mathcal{O}_V(\eta) = FF(\mathcal{O}_V)}$$

Say an element $h \in \mathcal{O}_V(\eta)$ is <u>regular</u> (or "defined") at a point p, if there exist representatives $f, g \in \mathcal{O}_V$

$h = f/g$ with $g(p) \neq 0$

Then $\mathcal{O}_{V,p} := \{ h \in \mathcal{O}_V(\eta) : h \text{ is regular at } p \}$ is the <u>local ring at p</u>.

<u>Observations</u> (1) The fraction field of $\mathcal{O}_{V,p}$ is $\mathcal{O}_V(\eta)$

(2) There is a natural ideal in $\mathcal{O}_{V,p}$, namely

$$\{ h \in \mathcal{O}_{V,p} \mid h(p) = 0 \} := m_p$$

(3) Consider the quotient $\mathcal{O}_{V,p}/m_p \cong K$ so m_p is maximal.

(4) Every element $h \in \mathcal{O}_{V,p} \setminus m_p$ is invertible in $\mathcal{O}_{V,p}$

<u>DEFINITION</u> A commutative ring R is called <u>local</u> if it has a unique maximal ideal.

<u>Proposition</u> A ring R is local if and only if $R \setminus R^*$ forms an ideal. If $R \setminus R^*$ is an ideal then it is the unique maximal ideal.

<u>Proof</u> An ideal $I \subseteq R$ is proper iff $I \cap R^* = \emptyset$.
So if $R \setminus R^*$ is an ideal then it contains any other proper ideal, i.e. is the unique maximal ideal.
Conversely, let R be local and $m \subseteq R$ be the maximal ideal. Then choose $x \in R \setminus R^*$.
Then $(x) \subsetneq R$.
<u>Fact</u> Every ideal is contained in a maximal ideal (Zorn's lemma)
So $(x) \subseteq m$ and $x \in m$. $\square$

<u>Example</u> Take $V = \mathbb{A}^1$, $p = 0$.

$$\mathcal{O}_{\mathbb{A}^1, p} = \{ f/g \mid f \in \mathcal{O}_V,\ g \in \mathcal{O}_V,\ g(0) \neq 0 \}$$

<u>Concretely</u> $\frac{1}{t-1} \in \mathcal{O}_{\mathbb{A}^1, 0}$

Over $\mathbb{C}$, expand f/g as a power series around 0.
Well-defined : power series starts with non-negative exponent.
Invertible if the constant term is non-zero.

$$\mathcal{O}_V \rightsquigarrow \underset{\substack{\text{Functions}\\ \text{defined in a}\\ \text{nghd of } p}}{\mathcal{O}_{V,p}} \supseteq \underbrace{m_p}_{\substack{\text{Functions}\\ \text{vanishing}\\ \text{at } p}}$$

m_p^2 = functions vanishing to order 2 ☺

<u>Def</u> For $p \in V$, the <u>Zariski cotangent space</u> to V at p is

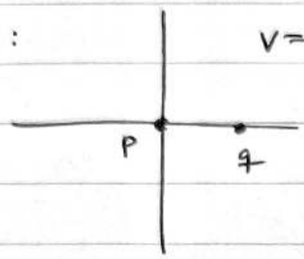
m_p / m_p^2, (as a k-vector space)

We think of these as linearizations of functions in m_p.

<u>Def</u> (smoothness) An affine variety V is smooth if $\dim_k \underbrace{m_p/m_p^2}_{T^*_{V,p}}$ is independent of p

L 5.4

Ex :



$V = \mathbb{V}(xy)$

$\dim T^*_{V,p} = 2$

$\dim T^*_{V,q} = 1$

#ad

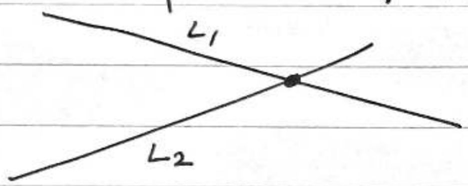
Projective Space

Look at the first part of Ex Sheet I.

"Loss of compactness"

· There exist parallel lines in $\mathbb{A}^2_k$

"Non-compactness" of $\mathbb{A}^2_k$



tilt L_2 until it becomes parallel to L_1

Fundamental *How to make parallel lines meet?*

We have to remember the slopes.

Slope of a line is an element of $k \cup \{\infty\} = \mathbb{P}^1_k$

set of possible slopes for lines (through the origin)

We think of $\mathbb{P}^2_k = \mathbb{A}^2_k \cup \mathbb{P}^1_k$

$\uparrow$ possible set of intersections pts of lines

$\uparrow$ possible set of slopes

In "3-dimensiones" $\mathbb{P}^3_k = \mathbb{A}^3_k \cup \mathbb{P}^2_k$

Official Definition $\mathbb{P}^n_k$ projective space of dim n over k is

$$\frac{k^{n+1} \setminus \{\underline{0}\}}{k^*}$$

i.e. modulo $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ for $\lambda \in k^*$.

Projective Space $\mathbb{P}^n_k$

Motivation: $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$

Definition: $\mathbb{P}^n_k$ is defined as $\mathbb{P}^n_k = \dfrac{k^{n+1} \setminus \{(0,\dots,0)\}}{k^*}$

Think: $\mathbb{P}^n_k = \{$ lines through 0 in $k^{n+1}\}$

Observations:

- To specify a point in $\mathbb{P}^n_k$ is equivalent to the data of $(a_0, \dots, a_n)$

∇ Specification is not unique b/c $(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n)$

∇ Not all a_is are allowed to be zero.

▶ Homogeneous coordinates on projective space.

A homogeneous coordinate is an equivalence class of vectors in k^{n+1} (equivalent under scaling)

$\mathbb{P}^2_k \ni p^* = [1:2:3] \sim [2:4:6] \sim [4:8:12]$

- Let $x_0, \dots, x_n$ be coordinates on $\mathbb{P}^n_k$.

Then the set $\{x_0 = 0\}$ is well defined.

Write $\mathbb{P}^n_k = \{X_0 = 0\} \cup \{X_0 \neq 0\}$

$$\{X_0 = 0\} = \frac{k^n \setminus \{(0,\dots,0)\}}{k^*} = \mathbb{P}^{n-1}_k$$

$*$: If $X_0 \neq 0$ then any point $P \in \{X_0 \neq 0\}$ can be written as $(a_0, \dots, a_n) \sim (1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})$

In particular the set $\{X_0 \neq 0\} \cong \mathbb{A}^n_k$

$*$: Observe, $\mathbb{P}^n_k = \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k$ "stuff at ∞"

$$= \mathbb{A}^n_k \cup \mathbb{A}^{n-1}_k \cup \cdots \cup \mathbb{A}^1 \cup pt$$

- In $\mathbb{P}^n_k$ w/ coords $X_0, \dots, X_n$ at every point $p \in \mathbb{P}^n$, $X_i(P) \neq 0$ for some i

Then $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ where $U_i = \{X_i \neq 0\}$

$*$ $\mathbb{P}^n$ is covered by $n+1$ copies of affine space

* <u>Test for "compactness"</u>

In affine space $\mathbb{A}^2$, the

$$\lim_{\substack{t\to\infty \\ 1/t \to 0}} (f(t), g(t)) \quad \text{need not exist}$$

<u>e.g.</u> $\lim_{t\to\infty} (t^2, t^3)$ DNE

<u>How does $\mathbb{P}^2$ fix this?</u>

Let's think about

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{P}^2 \\ \| & & \\ \{x_0 \neq 0\} & & \\ (x, y) & \longmapsto & [1 : x : y] \end{array}$$

$$\underset{\text{in } \mathbb{A}^2}{\lim_{t\to\infty} (t^2, t^3)} = \lim_{t\to\infty} [1 : t^2 : t^3] = \lim_{t\to\infty} \left[\frac{1}{t^3} : \frac{1}{t} : 1\right]$$

$$= \lim_{\frac{1}{t}\to 0} \left[\frac{1}{t^3} : \frac{1}{t} : 1\right] = [0 : 0 : 1] \in \mathbb{P}^2_k$$

<u>Remark</u>

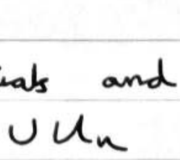
Projective space is "algebraically compact"

<u>Proper</u> in AG

- $\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$ can be equipped with the quotient topology coming from the Euclidean topology in $\mathbb{C}^{n+1}$

  This is compact in this topology

- $\mathbb{P}^1_{\mathbb{C}}$ (as above) is the Riemann sphere



<u>Towards defining projective varieties</u>

Two approaches:

1. Try to put a Zariski topology on $\mathbb{P}^n_k$
2. Define projective varieties using polynomials and ideals

<u>Sketch 1</u> We've seen that $\mathbb{P}^n_k = U_0 \cup \cdots \cup U_n$ with $U_i = \{x_i \neq 0\} \cong \mathbb{A}^n_k$

Each U_i can be given the Z-topology

Explicitly On U_i we have coordinates $\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}$
Polynomials in the variables $\{X_j/X_i\}_{\substack{j=0 \\ j\neq i}}^{n}$ give a Zariski topology on U_i.

Z-topology on $\mathbb{P}^n_k$ Declare a set $C \subseteq \mathbb{P}^n_k$ to be closed if $C \cap U_i$ is closed in the Zariski topology for all i

Could define A projective variety in $\mathbb{P}^n_k$ as a Z-closed subset of $\mathbb{P}^n_k$

Observe A polynomial in variables $X_0, \dots, X_n$ does not necessarily define a function on $\mathbb{P}^n$.

Take $X_0 + 1$. $([1:0]) \mapsto 2$
$([2:0]) \mapsto$ not 2

1. We didn't need polys to take on well-defined values $\rightsquigarrow$ important point was $\mathbb{V}(f)$

2. Definition A polynomial $f(X_0, \dots, X_n)$ is homogeneous if $\forall \lambda$

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n) \quad \text{for some } d \in \mathbb{Z}_{\geq 0}$$

Homogeneous of degree d.

Examples $\cdot$ $k[X_0, X_1, X_2]$

Good (homogeneous): X_0, X_0X_1, $X_0X_1 + X_2^2$, $X_0 + X_1 + X_2$

Bad (not homogeneous): $X_0^2 + X_1$, $X_0 + 1$

Definition A projective variety in $\mathbb{P}^n_k$ is the vanishing locus of a set $\mathbb{V}(S)$; $S \subseteq k[x_0, \dots, x_n]$ consisting entirely of homogeneous polynomials

⚡ • The sum of two homogeneous polynomials need not be homogeneous (need same degree)

• The ideal generated by S as above consists of / includes non-homogeneous elements

Theorem to come Let f, g be homogeneous polynomials in 3-vars X_1, X_2, X_0 of degrees d_1, d_2 respectively. Then $\mathbb{V}(f) \cap \mathbb{V}(g)$ consists of $d_1 \cdot d_2$ points subject to a genericity assumption. (Bézout)

Local Goal Understand $\mathbb{P}^n$

i.e. $\left\{ \begin{array}{l} \text{What does it look like?} \\ \text{How to change coordinates on it?} \\ \text{What are its subvarieties like?} \\ \text{How to define maps to it / from it?} \\ \text{How to adopt polynomial algebra to } \mathbb{P}^n \text{?} \end{array} \right.$

For intuition

$$\mathbb{P}^n_{\mathbb{C}} = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}_*}$$

$$\begin{tikzcd} \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2} \arrow[dr] & S^{2n+1} \arrow[l, hook'] \arrow[d] \\ & \mathbb{P}^n_{\mathbb{C}} \end{tikzcd}$$

unit norm

pure metric topology

$\mathbb{C}$ - dimension n
$\mathbb{R}$ - dimension $2n$

⚠ Special case $S^3 \longrightarrow S^2 = \mathbb{P}^1_{\mathbb{C}}$ Word of the day (Hopf Fibration)

⚠ $n \geq 2$ $\mathbb{P}^n_{\mathbb{C}} \not\cong S^{2n}$ [Not even homeomorphic!]

(Why?) They have different topological Euler characteristics
$\chi(\mathbb{P}^n_{\mathbb{C}}) = n+1$; $\chi(S^{2n}) = 2$

$\mathbb{P}^n$ is very symmetric

$$GL(n+1, k) \circlearrowright k^{n+1} \setminus \{0, \dots, 0\} / k^*$$
$$\|$$
$$\mathbb{P}^n \circlearrowleft PGL(n+1, k)$$

Any point can be sent to $[1:0:\cdots:0]$

Any $(n+\overset{2}{\cancel{1}})$ points in general position can be sent to
$[1:0:\cdots:0], [0:1:0:\cdots:0], \dots, [0:\cdots:0:1], [1:1:\cdots:1]$ (?) ✓

General position 3 points in $\mathbb{P}^2$ are in general position if they are not on a line.

<u>Subvarieties of $\mathbb{P}^n_k$</u> ; varieties defined by the vanishing loci of homogeneous poly's in $(n+1)$ variables

- Points in $\mathbb{P}^1$: (hypersurfaces in $\mathbb{P}^1$)

Let $f \in k[X_0, X_1]$ be homogeneous of degree d.

Then $\mathbb{V}(f) \subseteq \mathbb{P}^1$ consists of at most d points [Exercise]

There are <u>exactly</u> d points counted with <u>multiplicity</u>.

$$f(X_0, X_1) \left\{ \begin{array}{l} \text{Let } p \in \mathbb{P}^1 \text{ s.t.} \\ f(p) = 0 \end{array} \right.$$

Then either $p \in \mathbb{P}^1 \setminus \{X_0 = 0\}$

or $p \in \mathbb{P}^1 \setminus \{X_1 = 0\}$

For simplicity, assume $p \in \mathbb{P}^1 \setminus \{X_0 = 0\}$

Then identify $\mathbb{P}^1 \setminus \{X_0 = 0\} \cong \mathbb{A}^1$

$$[a_0 : a_1] \longmapsto a_1/a_0$$

So on this patch

$$f(X_0, X_1) = f(1, X_1/X_0) = f(y) \quad ; y = \frac{x_1}{x_0}$$

And the multiplicity f i.e.

$$\mathrm{mult}_p f(X_0, X_1) \equiv \mathrm{mult}_p f(y)$$

<u>Exercise</u> Use 1-variable polynomial facts to deduce

$$d = \sum_{p \in \mathbb{V}(f)} \mathrm{mult}_p f$$

- <u>Linear Subvarieties</u> A linear subvariety of $\mathbb{P}^n$ is produced as follows.

$$\begin{array}{c} k^{n+1} \setminus \{0\} \\ \downarrow \\ \mathbb{P}^n \end{array}$$

Choose a $\cancel{k}\,m+1$ dimensional linear subspace of k^{n+1}, say L.

Then $L \setminus \{0\} / k^*$ gives a copy of $\boxed{\mathbb{P}^m \hookrightarrow \mathbb{P}^n}$

<u>Claim</u> This is a projective variety

<u>Pf</u> produce equations writing $L \subseteq k^{n+1}$ as $\mathbb{V}(\ell_1, \dots, \ell_{-m+n})$ where $\ell_i : k^{n+1} \longrightarrow k$ are linear.

<u>Exercise</u> make sure you understand this

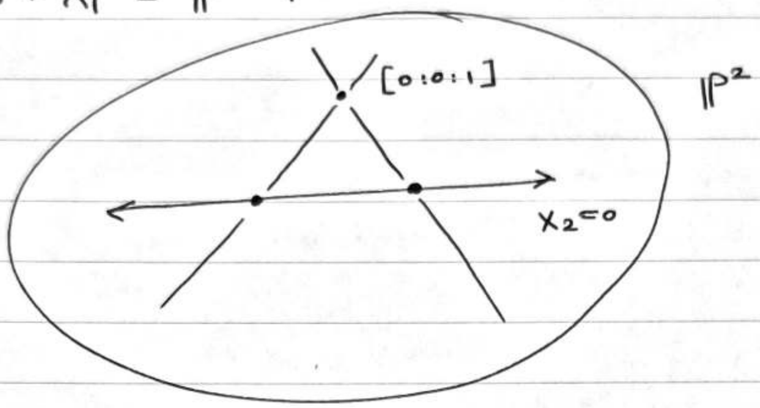
Quadric
<u>Quadratic Hypersurfaces</u>

$\mathbb{V}(f) \subseteq \mathbb{P}^n_k$ w/ f homogeneous of degree 2

<u>Ex</u>: • $X_0^2 + X_1^2 \subseteq \mathbb{P}^1$: 2 points

• $X_0^2 + X_1^2 \subseteq \mathbb{P}^2$:

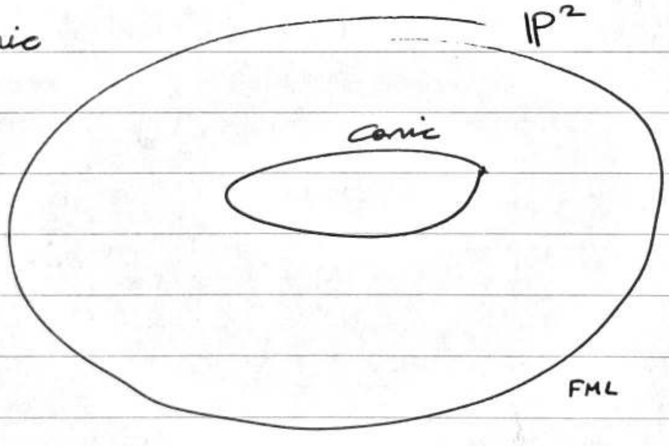
$\mathbb{V}(\cdot)$



Notice $X_0^2 + X_1^2 = (X_0 - \lambda X_1)(X_0 + \lambda X_1)$ where $\lambda^2 = -1$

• $\mathbb{V}(X_0^2 + X_1^2 + X_2^2) \subseteq \mathbb{P}^2_k$

Conic



* Characteristic of $k \neq 2$

Quadrics $\leftrightarrow$ Quadratic forms
Bilinear forms

Diagonalise a quadratic form

<u>Prop^n/Cor</u> Up to a change of coordinates on $\mathbb{P}^n_k$ every quadric hypersurface (non-degenerate) is equivalent to $\sum_{i=0}^{n} X_i^2$

- Cubic hypersurfaces: hard!
  - $n=2 \rightsquigarrow$ Elliptic curves
  - $n=3 \rightsquigarrow$ 1^{st} slide of Lecture 1
  - $n=4 \rightsquigarrow$ "27 lines on a cubic surface"
  - $n=5 \rightsquigarrow$ subject of ongoing research

Projective varieties: Projective variety is $\mathbb{V}(S)$; $S \subseteq k[X_0, \dots, X_n]$ $\hookrightarrow$ all homogeneous; $\mathbb{V}(S) \subseteq \mathbb{P}^n_k$

⚠ If S is entirely homogeneous, cannot be an ideal (basically)

Homogeneous Ideal

- $I \subseteq k[\underline{X}]$ is homogeneous if it is generated by homogeneous polys
- Want to avoid thinking about $\mathbb{V}(f) \subseteq \mathbb{P}^n_k$ if f is non-homogeneous.

Observe: Given any $f \in k[\underline{X}]$, it can be written as

$$f = \sum_i f_{[i]} \quad w/\ f_{[i]} \text{ hom of degree } i$$

$$f = \underbrace{X_0^3 + X_1^3}_{\text{cube } f_{[3]}} + \underbrace{X_2^2 + X_1X_2}_{f_{[2]}} + \underbrace{X_0}_{f_{[0]}}$$

~~Say~~ Call $f_{[i]}$ the degree i part of f.
Would be great: if for an ideal $I \subseteq k[\underline{X}]$, if $f \in I$ then $f_{[i]} \in I$ $\forall i$

Proposition An ideal I is homogeneous if and only if for any $f \in I$, the homogeneous parts of f also lie in I.

Proof "<=" Assume $I \trianglelefteq k[\underline{X}]$ is "great".
Take generators $h_1, \dots, h_r \in I$ and write

$$h_j = \sum_i h_{j[i]}$$

Then $h_{j[i]} \in I$ and $\{h_{j[i]} : i, j\}$ generate.

"$\Rightarrow$"

Conversely, say $f_1, \dots, f_r \in I$ are homogeneous, generate.
Take $g \in I$ and write

$$g = \sum g_{[i]}$$

Now observe that $g = \sum \underset{\text{hom}}{f_i} \underset{\text{general}}{h_i}$

for h_i not necessarily homogeneous.

Decompose $h_i = \sum_{e_i} h_{i[e_i]}$

Write $g_{[i]}$ in terms of f_j's and $h_{i[e_i]}$'s. $\square$

<u>Cor</u> If $S \subset k[\underline{X}]$ are homogeneous polynomials, then

$$\mathbb{V}(S) = \mathbb{V}(I_S) = \left\{ p \in \mathbb{P}^n_k \,\middle|\, \begin{array}{l} f(p) = 0 \\ f \in I_S \end{array} \right\}$$

could say homogeneous, but don't have to

Easy imports $V \subseteq \mathbb{P}^n$ a projective variety

- V is irreducible if $V \neq V_1 \cup V_2$ with V_i distinct from V
- V is irreducible iff $I(V)$ is prime,
  w/ $I(V) = \{ f \in k[\underline{X}] \mid f \text{ hom}, f(p)=0 \ \forall p \in V \}$
- Nullstellensatz (weak): if $\mathbb{V}(I) = \emptyset$ with $I \subset k[X]$ hom, then I contains the ideal $(X_0^m, \dots, X_n^m)$ for some $m>0$

Projective space $\mathbb{P}^n_k$ has a natural Z-topology defined by declaring closed sets to be of the form $\mathbb{V}(I)$ for I homogeneous ideal in $k[X_0, \dots, X_n]$

Remarks:

1. Equivalent to the "glued" affine Zariski topology wrt the cover $\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$, $U_i = \{X_i \neq 0\} \cong \mathbb{A}^n_k$
2. Write $\mathbb{P}^n = (k^{n+1} \setminus \{0\})/k^*$ the Z-topology is unchanged by linear change of coordinates on k^{n+1}.

Proposition Let $V \subseteq \mathbb{P}^n$ be irreducible, closed.
Let $W \subsetneq V$ proper closed, then $V \setminus W$ is Zariski dense.

Moral: Projective subvarieties are smaller.

Proof: Want to show that $\overline{V \setminus W} = V$ (in subspace topology)
Equivalently, want to show that if we have f homogeneous vanishing on $V \setminus W$ then it varishes on V.
Pick f vanishing on $V \setminus W$. Let $g \in I(W) \setminus I(V)$ [This exists by (?) $\mathbb{P}$-Nullstellensatz]
& hom
Then $fg \in I(V)$.
But $I(V)$ is prime since V is irreducible, so $f \in I(V)$. $\square$

Corollary Let $U_i = \{X_i \neq 0\} \subseteq \mathbb{P}^n$.
U_i is dense in $\mathbb{P}^n$.
Affine space is dense in projective space.

Let $V \subseteq \mathbb{P}^n$ be a projective subvariety.

<u>Proposition</u> Under the identification $\mathbb{P}^n \supset U_i \xrightarrow{\sim} \mathbb{A}^n_k$,

$V \cap U_i$ is identified with an affine variety.

If $V = \mathbb{V}(\underset{\text{hom}}{I})$, then $V \cap U_i = \mathbb{V}(\{ f(y_1, \dots, \hat{y}_i, \dots, y_n) \mid f \in I \})$

omit the ith entry?

<u>Proof</u> Trivial $\square$

<u>Corollary</u> A projective variety $V \subseteq \mathbb{P}^n$ is covered by $n+1$ affine varieties

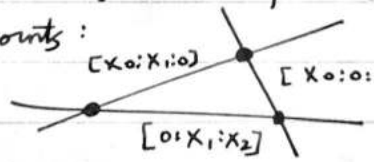
<u>Think</u> Manifold like

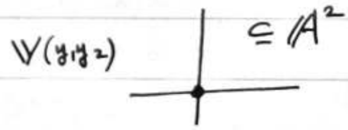
<u>Example</u> $V = \mathbb{V}(X_0 X_1 X_2) \subseteq \mathbb{P}^2_k$

$U_0 = \{X_0 \neq 0\}$. Say $V_0^{\text{aff}} = V \cap U_0 \subseteq \mathbb{A}^2_{y_1 y_2}$

$$V_0^{\text{aff}} = \mathbb{V}(X_1 X_2) \text{ on } \{X_0 \neq 0\}$$
$$= \mathbb{V}(y_1 y_2) \subseteq \mathbb{A}^2$$

V is a union of 3 copies of $\mathbb{P}^1$ meeting pairwise at distinct points:





$\triangledown$ The closure of V_0^{aff} is not V

in $\mathbb{P}^n_k$

How do we produce projectives from affines?

Algebraic Operation: Homogenization

Variables: $y_1, \dots, y_n$ affine variables; $X_0, \dots, X_n$ "proje variables"

Given a polynomial $f(y_1, \dots, y_n)$ its homogenization wrt X_0 is given by
$$f^h(X_0, \dots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

where d is the total degree of $f(y_1, \dots, y_n)$.

Examples : 1 : $f(y_1, y_2) = y_1^3 + y_1 y_2 + y_2^2$

$$f^h(X_0, X_1, X_2) = X_1^3 + X_0 X_1 X_2 + X_0 X_2^2$$

Notice : Given $g(X_0, \dots, X_n)$ homogeneous

$g(1, y_1, \dots, y_n)$ is typically non-homogeneous

This represents the polynomial g on U_0

Lemma Let $g(X_0, \dots, X_n)$ be homogeneous.
Let $f(y_1, \dots, y_n) = g(1, y_1, \dots, y_n)$. Then
setting $f^h(X_0, \dots, X_n)$ to be the homog wrt X_0. Then

$$g(X_0, \dots, X_n) = X_0^m f^h(X_0, \dots, X_m)$$

for some $m \geq 0$.

Proof Exercise in definitions. $\square$

Definition Let $I \subseteq k[y_1, \dots, y_n]$ be any ideal.
Then its homogenization is

$$I^h = \langle f^h(X_0, \dots, X_n) \mid f \in I \rangle$$

Remark / Warning

- If I is principal, i.e. $I = \langle f \rangle$ then $I^h = \langle f^h(X_0, \dots, X_n) \rangle$.
- If $I = \langle f_1, \dots, f_r \rangle \subseteq k[y_1, \dots, y_n]$ then I^h need not be generated by $f_i^h(X_0, \dots, X_n)$

Example : Take $V \subseteq \mathbb{A}^3$ to be

$$V = \{(t, t^2, t^3) \mid t \in k\}$$

Then $I(V) = \langle y_1^3 - y_3, y_1^2 - y_2 \rangle$.

But $I(V)^h \neq \langle X_1^3 - X_3 X_0^2, X_0 X_2 - X_1^2 \rangle$

[Example Sheet II]

(†) Proposition Let $V^{aff} \subseteq \mathbb{A}^n_k$ be an affine variety defined by an ideal I in $k[y_1, \dots, y_n]$. Then under the identification

$$\mathbb{A}^n_k \xrightarrow{\sim} U_0 \subseteq \mathbb{P}^n$$

the closure of V^{aff} in $\mathbb{P}^n$ is $\mathbb{V}(I^h)$.

Proof Let $V^{aff} \subseteq U_0$ be the vanishing locus of I.

Then the closure is

$$\overline{V_{aff}} = \bigcap_{\substack{f \in k[x_0, \dots, x_n] \\ f|_{V_{aff}} = 0}} \mathbb{V}(f) \quad \}*$$

If $f(x_0, \dots, x_n)$ has $f|_{V_{aff}} = 0$, then write coordinates on V^{aff} as $[1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}]$ and identify $\frac{x_i}{x_0}$ with y_i.

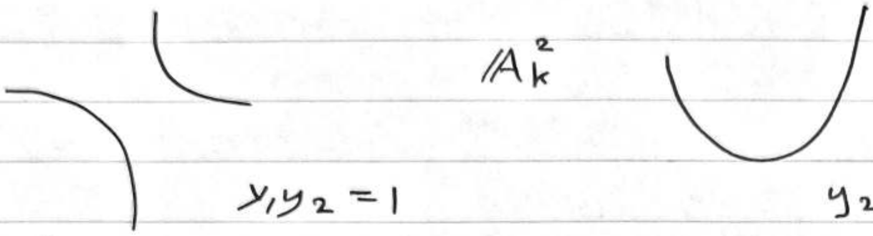
By def^n $f(1, y_1, \dots, y_n) \in I(V^{aff})^X$.

But by Lemma, f itself is given by X_0^m times the homogeniza^n of $f(1, y_1, \dots, y_n)$. Therefore f lies in $\sqrt{I^h}$.

Now apply the projective Nullstellensatz. $\square$

A tale of two conics

$\mathbb{A}^2_k$



C_1 $\quad y_1 y_2 = 1$ $\qquad\qquad$ $y_2 = y_1^2$ $\quad C_2$

<u>Identification</u> $\mathbb{A}^2 = \{X_0 \neq 0\} \subseteq \mathbb{P}^2$

$y_1 = X_1 / X_0$ & $X_2 = y_2 X_0$

Take Zariski (projective) closure

$$\left.\begin{aligned} \overline{C_1} &= \mathbb{V}(X_1 X_2 - X_0^2) \\ \overline{C_2} &= \mathbb{V}(X_0 X_2 - X_1^2) \end{aligned}\right\} \begin{array}{l} \text{"obviously" the same} \\ \text{[we saw this]} \end{array}$$

But $C_1 \not\cong C_2$: $\mathcal{O}_{C_1} \cong k[y_1, y_2]/(y_1 y_2 - 1) \cong k[y, y^{-1}]$, $\mathbb{A}^1 \setminus \{0\}$

whereas $\mathcal{O}_{C_2} = k[y_1, y_2]/(y_1^2 - y_2) \cong k[y]$, $\mathbb{A}^1$

<u>Today Central Tenet</u>: A homogeneous polynomial $f(X_0, \dots, X_n)$ is NOT a well-defined function on $\mathbb{P}^n$.

However if f & g are two such of the <u>same</u> degree, then

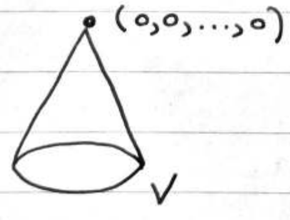
$$\frac{f}{g} : \mathbb{P}^n \setminus \mathbb{V}(g) \longrightarrow k$$

is a well-defined function (on a dense subset of $\mathbb{P}^n$)

Let V be a projective variety in $\mathbb{P}^n_k$.

Let $I(V)$ be the homogeneous ideal of polynomials vanishing on V.

Consider the vanishing locus of $I(V)$ inside $\mathbb{A}^{n+1}_k$.



Affine cone over V

$$\begin{array}{ccc} \mathbb{V}(I(V))^{\setminus\{0\}} & \subseteq & \mathbb{A}^{n+1}_k \setminus \{0\} \\ \downarrow & & \downarrow \\ V & \subseteq & \mathbb{P}^n_k \end{array}$$

<u>Definition</u> The <u>homogeneous coordinate ring</u> of V in $\mathbb{P}^n_k$ is
$k[X_0, \dots, X_n]/I(V)$, denoted $S(V)$.

It is the coordinate ring of the affine cone. (~~Nullstellensatz~~) not quite, just ~~##~~ =00

What are elements of $S(V)$? They're functions on the cone (V), but not on V itself.

$$k[X_0, \dots, X_n] \twoheadrightarrow S(V)$$

<u>Definition</u> The field of rational functions on a projective $V \subseteq \mathbb{P}^n_k$, with V <u>irreducible</u>, is defined as

Frac Field of $S(V)$ $\supseteq$ $$\mathcal{O}_V(\eta) := \left\{ \frac{\bar{P}}{\bar{Q}} \;\middle|\; \begin{array}{l} P, Q \text{ are homog. degree } d \text{ in} \\ k[X_0, \dots, X_n] \text{ and } \bar{P}, \bar{Q} \text{ images in } S(V), \bar{Q} \neq 0 \end{array} \right\}$$

<u>Why?</u> It will turn out that $\mathbb{P}^n_k$ (and any projective irred variety) has no non-constant globally defined functions (alg). [Liouville's Theorem]

<u>Simple Facts</u>

1. For V as above (irred proj) $\mathcal{O}_V(\eta)$ is a field extension of k
2. [Proposition] $\mathcal{O}_V(\eta)$ is a finitely generated field extension of k

<u>Proof</u> $V \subseteq \mathbb{P}^n$ but we can assume that $V \not\subseteq \{X_0 = 0\}$ by relabeling coordinates.

As we saw before, $V \cap U_0 = \{v : X_0 \neq 0\}$ "V_0^{aff}" is an affine variety.

Then we claim that

$$\mathcal{O}_V(\eta) \cong \underbrace{\mathcal{O}_{V_0^{aff}}(\eta)}_{\text{FField of coord ring of } V_0^{aff}}$$

To see this, observe that $\mathcal{O}_{V_0^{aff}}(\eta)$ is generated by the coordinates on U_0, $\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}$.

And similarly, writing $h \in \mathcal{O}_V(\eta)$ as $h = \frac{F}{G}$ with F, G homogeneous of degree d, dividing numerator and denominator by X_0^d, h is also a rational function in the $\frac{X_i}{X_0}$. We're done because we can view both $\mathcal{O}_V(\eta)$, $\mathcal{O}_{V_0^{aff}}(\eta)$ as subfields of the fraction field of $S(V)$. Π

WAT

Example If $V = \mathbb{P}^n_k$ then
$$\mathcal{O}_{\mathbb{P}^n}(\eta) = FF(\mathcal{O}_{\mathbb{A}^n}) \cong k(t_1, \dots, t_n)$$

Rational Maps & Morphisms : $V \subseteq \mathbb{P}^n_k$ irred proj

Given $h \in \mathcal{O}_V(\eta)$, we say that h is regular at a point $p \in V$ if there exist representatives $h = P/Q$ with $Q(p) \neq 0$.

Remark An element $h \in \mathcal{O}_V(\eta)$ gives a map $V \setminus \mathbb{V}(Q) \to k$

More generally, $V \supseteq \{p \in V \mid h \text{ is regular at } p\} \longrightarrow k$

Functions on open sets

As before, if $p \in V$, the local ring at p is $\mathcal{O}_{V,p} = \{h \in \mathcal{O}_V(\eta) \mid h \text{ reg at } p\}$

$\mathbb{P}^n \to \mathbb{P}^m$? Fix a collection $F_0, \dots, F_m$ of degree d hom polys in n variables.

$$\begin{aligned} \mathbb{P}^n \setminus \left(\bigcap_{i=1}^m \mathbb{V}(F_i)\right) &\longrightarrow \mathbb{P}^m \\ p = [a_0 : \dots : a_n] &\longmapsto [F_0(p) : \dots : F_m(p)] \end{aligned}$$

We write this with a broken arrow:
$$(F_0 : \dots : F_m) : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$$
partially defined

~~Def~~ Given ~~X~~ another homogeneous G in $X_0, \dots, X_n$, the tuple $[F_0 : \dots : F_m G]$ gives almost the same map. Precisely,

$[F_0 : \dots : F_m] : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$

$[F_0 G : \dots : F_m G] : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$

These maps agree on the set $U = \left(\bigcap_i \mathbb{V}(F_i) \cup \mathbb{V}(G)\right)^c$

Since $k[X_0, \dots, X_n]$ is a UFD, we can remove all common factors from a tuple $[F_0 : \dots : F_m]$ thus giving a 'best' representation of a rational map.

Def A rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is given by a m-tuple of homogeneous polynomials of fixed degree d.

Examples

• Projection from a point: Fix homogeneous coordinates on $\mathbb{P}^n_k$,
$[X_0 : \cdots : X_n]$

Then the projection from $[0 : \cdots : 0 : 1]$ is defined as
$$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$
$$[a_0 : \cdots : a_n] \longmapsto [a_0 : \cdots : a_{n-1}]$$
This map is undefined at $[0 : \cdots : 0 : 1]$

More generally, given $p \in \mathbb{P}^n$, choose a linear change of coords on k^{n+1} s.t. p is given by $[0 : \cdots : 0 : 1]$ and reuse previous def^n

• Veronese There are $N = \binom{n+d}{d}$ monomials of degree d in $n+1$ variables

Label them $F_0 : \cdots : F_{N-1}$

These define a map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{N-1}$

In fact this map is defined everywhere.

It's called the d^{th} Veronese.

$\boxed{\mathbb{P}^n \dashrightarrow \mathbb{P}^m}$

$X_0, \dots, X_n$ $\quad Y_0, \dots, Y_m$ homogeneous coordinates

Given $m+1$ homogeneous polynomials $F_0, \dots, F_m$ of the same degree in $X_0, \dots, X_n$ we get

$$\begin{array}{ccc} \mathbb{P}^n \setminus \mathbb{V}(F_0, \dots, F_m) & \longrightarrow & \mathbb{P}^m \\ \cap\! \mid & \nearrow & \\ \mathbb{P}^n & & \end{array}$$

Since $k[\underline{X}]$ is a UFD, there is a canonical represent^n of this rational map by $F_0, \dots, F_m$ w/o common factors

<u>Terminology</u> Given a rational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ we obtain a tuple $(F_0, \dots, F_m)$ of polys w/o common factors.

Base Locus $(\varphi) = \bigcap_i \mathbb{V}(F_i)$

<u>Ex</u>: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ (proj from $[0:0:1]$)

$[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$

Base locus is $\{[0:0:1]\}$

<u>Ex</u>: $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

$[x_0 : x_1] \mapsto [x_0^2 : x_1^2 , x_1 x_0]$

Map without base locus

<u>Def</u>: A rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is called a <u>morphism</u> if it has empty base locus.

[If BaseLocus is empty we use an unbroken arrow]

▷ <u>Rational Maps on Projective Varieties</u>

Let $V \subseteq \mathbb{P}^n_k$ be <u>IRREDUCIBLE</u>. We're going to define $V \dashrightarrow \mathbb{P}^m$

1. A tuple $(F_0, \dots, F_m)$ of homogeneous degree d polynomials in $\{X_i\}$ (in $\mathbb{P}^n$) is <u>said to determine a rational map</u> $\varphi : V \dashrightarrow \mathbb{P}^m$

   if BaseLocus$(F_0, \dots, F_m)$ does not contain V.

2. A tuple $(F_0, \dots, F_m)$ and $(G_0, \dots, G_m)$ <u>determine the same rational map</u> if $F_i G_j - F_j G_i \in I(V)^h$ for all i, j

3. A rational map $V \dashrightarrow \mathbb{P}^m$ is an equivalence class of m tuples under the equivalence in ②.

Remark • Straightforward to check that ② determines an equivalence rel^n.
• Base Locus of $(F_0, \dots, F_m)$ is allowed to have non-empty intersection with V

Unpack Condition ② Any $(F_0, \dots, F_m)$ determines $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$

$$\begin{array}{ccc} \mathbb{P}^n & \dashrightarrow & \mathbb{P}^m \\ \cup\!\!/ & \nearrow & \\ V \supseteq V^\circ & & \\ \| & & \\ V \cap \mathrm{BaseLocus}(\underline{F})^c & & \end{array}$$

Cover $\mathbb{P}^m$ by $\{Y_i \neq 0\}$.
Then on $\{Y_i \neq 0\}$ the tuple

$$(F_0, \dots, F_m) = \left(\frac{F_0}{F_i}, \dots, \frac{F_0}{F_i}\right)$$

When should $(F_0, \dots, F_m) = (G_0, \dots, G_m)$ on $\{Y_i \neq 0\}$?

Rewrite $\underbrace{\frac{F_0}{F_i} = \frac{G_0}{G_i}; \dots ; \frac{F_m}{F_i} = \frac{G_m}{G_i}}_{m \text{ conditions}}$

on V, i.e. $\frac{F_0}{F_i} - \frac{G_0}{G_i} \in \mathcal{I}(V^\circ)$

In practice : $\varphi : V \dashrightarrow \mathbb{P}^m_k$ we have a rational map represented by a homogeneous tuple, but we're allowed to use different tuples at different points.

Notions associated to $V \dashrightarrow \mathbb{P}^m$:

Fix $\varphi : V \dashrightarrow \mathbb{P}^m$. A point $p \in V$ is said to be regular (i.e. defined) if there is a representation $(F_0, \dots, F_m)$ for φ with $F_i(p)$ non-zero for at least one i

. The domain $\mathrm{dom}(\varphi)$ of a rational map is $\{p \in V \mid p \text{ reg for } \varphi\}$

{ A rational map $\mathbb{P}^n \supseteq V \overset{\varphi}{\dashrightarrow} W \subseteq \mathbb{P}^m$ is a rational map $\varphi : V \dashrightarrow \mathbb{P}^m$ s.t. $\varphi(\mathrm{dom}(\varphi)) \subseteq W$.

Warning: Composition of rational functions is tricky

$$V \underset{\varphi}{- - \to} W \underset{\psi}{- - \to} Z$$

Then $\psi \circ \varphi$ can only be defined if $\boxed{\varphi(\mathrm{dom}(\varphi)) \subseteq \mathrm{dom}(\psi) \subseteq W}$

So we dodge in the following way:
Say $\varphi : V \dashrightarrow W$ is dominant if
$\varphi(\mathrm{dom}(\varphi))$ is dense in W

If $V \overset{\varphi}{\dashrightarrow} W$ is dominant, then arbitrary compositions with $W \dashrightarrow Z$ exist.

Def $V \overset{\varphi}{\dashrightarrow} W$ is said to be a morphism if $\mathrm{dom}\,\varphi = V$
"Everywhere defined"

Examples

Projection from a linear subspace

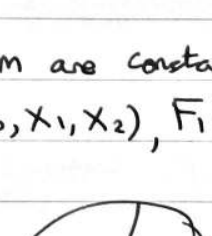
$$\mathbb{P}^n \dashrightarrow \mathbb{P}^k$$

$[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_k]$

Undefined on the locus $\underbrace{\{x_0 = \cdots = x_k = 0\}}_{\text{linear subspace}}$

- Fact All morphisms $\mathbb{P}^n \to \mathbb{P}^m$ with $n > m$ are constant

Reasoning Take $\mathbb{P}^2 \to \mathbb{P}^1$ given by $F_0(x_0, x_1, x_2), F_1(x_0, x_1, x_2)$
But $\mathbb{V}(F_0) \cap \mathbb{V}(F_1)$ is always non-empty



$\mathbb{P}^2$

Word of the Day

$\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ Cremona transformation

$[x_0 : \cdots : x_n] \mapsto \left[\frac{1}{x_0} : \cdots : \frac{1}{x_n}\right]$

Prototypes · $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ projection from a point [VIDEO!]

· $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ Cremona

$$[x_0 : x_1 : x_2] \longmapsto \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right]$$

$$\text{or } [x_1 x_2 : x_2 x_0 : x_0 x_1]$$

Crucial Example Second Veronese on $\mathbb{P}^2$:

$$\mathbb{P}^2 \xrightarrow{v_2} \mathbb{P}^5$$

$$[x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_1^2 : x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2]$$

Observe A linear combination in the $\mathbb{P}^5$ variables gives a quadratic form on the $\mathbb{P}^2$

Hyperplanes in $\mathbb{P}^5$ intersect $v_2(\mathbb{P}^2)$ in a conic

Ex Sh II $v_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ is an embedding

Proposition If X_d is a hypersurface of degree d in $\mathbb{P}^n$, then $U_d = \mathbb{P}^n \setminus X_d$ is an affine variety.

Proof

$$\begin{tikzcd} X_d \arrow[r, hook] \arrow[d, hook] & H \arrow[d, hook] \\ \mathbb{P}^n \arrow[r, hook] & \mathbb{P}^N \\ U_d \arrow[r, hook] \arrow[u, hook] & \mathbb{A}^N \arrow[u, hook] \end{tikzcd}$$

$\square$

Birational Equivalence V, W irred proj

Suppose $V \overset{\varphi}{\dashrightarrow} W$ is a dominant rational map

$\varphi(\mathrm{dom}(\varphi))$ is dense in W

If φ is dominant, we can compose φ w/ any rational map on W.

In particular, can compose with a rat^l f^n, i.e.

$$\varphi^* : \mathcal{O}_W(\eta) \to \mathcal{O}_V(\eta)$$

$$[\varphi^* : k(W) \longrightarrow k(V)]$$

is a homomorphism of field extensions of k.

Definition V and W are birational if there exist rational maps $\varphi : V \dashrightarrow W$, $\psi : W \dashrightarrow V$ s.t. $\varphi \circ \psi$, $\psi \circ \varphi$ are both defined and equal to the identity.

Think "weak" equivalence, says $V \simeq_{bir} W$ if V & W 'share' a common open (dense) set

Ex $\mathbb{A}^n_k \simeq_{bir} \mathbb{P}^n_k$

Theorem V and W are birational iff $\mathcal{O}_V(\eta)$ and $\mathcal{O}_W(\eta)$ are isomorphic as field extensions of k.

Pf Say $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ closed irreducible.

Then say $V \not\subseteq \{X_0 = 0\}$ and $W \not\subseteq \{Y_0 = 0\}$.

This means that $\mathcal{O}_V(\eta)$ is generated by $x_1, \dots, x_n$ where $x_i = X_i / X_0$ (modulo $I(V)$)

Similarly $\mathcal{O}_W(\eta)$ is generated by $y_1, \dots, y_m$; $y_i = \frac{Y_i}{Y_0}$

Given an abstract isomorphism:

$\omega : \mathcal{O}_W(\eta) \xrightarrow{\sim} \mathcal{O}_V(\eta)$ write

$\omega(y_i)$ as a rational function in the x_i's

So $\omega(y_i) = h_i(\underline{x})$ for rational h_i.

Now consider $[1 : h_1(\underline{x}) : \cdots : h_m(\underline{x})]$

Clear denominators, and homogenize wrt X_0 to get #CRINGE

$$[H_0(\underline{X}) : H_1(\underline{X}) : \cdots : H_m(\underline{X})]$$

This gives a rational map $V \dashrightarrow W$.

Similarly write x_i's in terms of the y_i's to get $W \dashrightarrow V$.

A tedious but easy calculation shows these are inverse. $\square$

Remark Straightforward to define birational map on affine varieties and birational equiv similarly.

Remarks The classification of algebraic varieties up to birational is equivalent to the classification for f.g. field extensions of k.

Ex • Any affine variety is birationally equivalent to any projective closure of it.

• $\mathbb{P}^2$ birational to $\underbrace{\mathbb{P}^1 \times \mathbb{P}^1}_{?}$

Products of projective varieties

Two approaches: Extrinsic & Intrinsic

Extrinsic Approach Put the structure of a projective variety on $\mathbb{P}^n \times \mathbb{P}^m$.

$[x_0 : \dots : x_n]$ $[y_0 : \dots : y_m]$

Given $[\underline{X}]$ and $[\underline{Y}]$, let $[\underline{X}] *_m [\underline{Y}] := \begin{bmatrix} X_0 \\ \vdots \\ X_n \end{bmatrix} [Y_0 : \dots : Y_m]$

Def The Segre embedding is given by $(n+1)\cdot(m+1)$ matrix

$$\Sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

$$\{Z_{ij}\}_{\substack{0 \le i \le n \\ 0 \le j \le m}}$$

sending $([\underline{X}], [\underline{Y}]) \longmapsto [\underline{X}] *_M [\underline{Y}]$

Theorem The set map $\Sigma_{n,m}$ defined above is injective with Zariski-closed image given by

$$\mathbb{V}(\{Z_{i,j} Z_{k,l} - Z_{i,l} Z_{kj}\}). \quad (*)$$

In other words, the image is the set of rank 1 matrices, the ideal is generated by the vanishing of all 2×2 minors in $[Z_{ij}]$.

Proof The vanishing locus $(*)$ consists of precisely those matrices $[Z_{ij}]$ that have rank 1, since those are the 2×2 minors. $\Rightarrow$ $\mathrm{im}(\Sigma_{n,m}) \subseteq \mathbb{V}(\cdots)$

Conversely, given a point in this vanishing locus $(*)$, write out the homogeneous coords in matrix form. Since the matrix is rank 1 all columns are scalar multiples.

Injectivity is easy linear algebra. □

Proposition The two projection maps

$$\begin{tikzcd} & \mathbb{P}^n \times \mathbb{P}^m \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathbb{P}^m \end{tikzcd}$$

are morphisms

Proof Exercise.

Corollary If V, W are projective varieties then so is $V \times W$

Proof $V \times V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is intersection of $\pi_1^{-1}(V)$, $\pi_2^{-1}(W)$ □

Remark Gives a way to produce morphism.
For instance, $\mathbb{P}^n \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(n+1)^2-1}$
restrict to the diagonal $\mathbb{P}^n \subseteq \mathbb{P}^n \times \mathbb{P}^n$ to get

$$\mathbb{P}^n \longrightarrow \mathbb{P}^{(n+1)^2-1}$$

This is a Veronese.

Intrinsic Approach Say $F(\underline{X}, \underline{Y})$ is bihomogeneous of degree (d, e) if it is homogeneous of degree d in X variables & degree e in Y variables.
Define the closed sets in $\mathbb{P}^n \times \mathbb{P}^m$ to be intersections of vanishing loci of bihomogeneous polys. We get intrinsic Z-topology on $\mathbb{P}^n \times \mathbb{P}^m$.

Proposition Intrinsic Z-topology coincides with Segre Z-topology.

Proof Straightforward but not obvious.

universal property?

A degree d hom poly $\mathbb{P}^{(n+1)(m+1)-1}$ and restrict to $\mathbb{P}^n \times \mathbb{P}^m$ we obtain a bihom poly of degree (d, d). □

A central question: (highly lucrative £££)

▸ When is a hypersurface of degree d in $\mathbb{P}^n$ birational to some projective space?

Comments on last time

- $\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$
- $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}$ but we haven't proved this
- For $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$: Every pair of curves in $\mathbb{P}^2$ intersect but this is false for $\mathbb{P}^1 \times \mathbb{P}^1$

* Commutative algebra can show this. I'll explain soon

Today Tangent spaces, smoothness, dimension.

Everything comes from linearizing polynomials!

Motivation Suppose $\mathbb{V}(f) \subseteq \mathbb{A}^n$. Given a line ℓ in $\mathbb{A}^n$, when is it tangent to $\mathbb{V}(f)$ at some point p.

Say $V = \mathbb{V}(f)$; $p = (a_1, \dots, a_n) = (\underline{a})$

$\ell = \{ (a_1 + b_1 t, \dots, a_n + b_n t) \mid t \in K, \underline{b} \neq \underline{0} \}$

Tangency $V \cap L$ is $f(a_1 + b_1 t, \dots, a_n + b_n t)$

If $p \in V$, well, this is a poly in 1-variable, $g(t)$ say

Write $g(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_r t^r$

- ℓ passes through $p \iff c_0 = 0$
- ℓ is tangent to V at $p \iff c_1 = 0$ (DEF^n)

Said differently: ℓ is tangent to V at p iff ℓ is contained in $\mathbb{V}\left(\sum_i \frac{\partial f}{\partial x_i}(p)\,(x_i - a_i) \right)$ (Check)

Notice $\sum_i \frac{\partial f}{\partial x_i}(p)\,(x_i - a_i)$ is the linear part of the Taylor polynomial of f at p

Rmk Purely formal differentiation

Def $V = \mathbb{V}(f) \subseteq \mathbb{A}^n$ The affine tangent space to V at $p \in V$ is the affine space $\mathbb{V}\left(\sum_i \frac{\partial f}{\partial x_i}(p)\,(x_i - a_i) \right) = T^{\mathrm{aff}}_{V,p}$

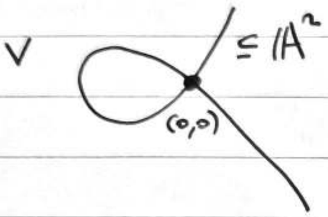
- Just for hypersurface, so far

For $V \subseteq \mathbb{P}^n$ projective hypersurface, the definition is similar.

<u>Def</u> An affine hypersurface $\mathbb{V}(f)$ is <u>smooth</u> if $\dim T^{aff}_{V,p}$ is independent of $p \in V$

For a hypersurface $V \subseteq \mathbb{A}^n$, $\dim T^{aff}_{V,p}$ is always either n or $n-1$

<u>Example</u> : Consider $\mathbb{V}(y^2 - x^2(x+1)) = V$



By calculating we find $T^{aff}_{V,(0,0)} = A^2$ and $\dim T^{aff}_{V,p} = 1$ for all other points

Let $V \subseteq \mathbb{A}^n$ be an affine variety and pick $p \in V$. Then

$$T_{V,p} = \left\{ \underline{v} \in k^n \;\middle|\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot v_i = 0 \text{ for all } f \in I(V) \right\}$$

<u>Def</u> If $V \subseteq \mathbb{P}^n$ is projective, then the tangent space to $p \in V$ is the tangent space p in (irred) any non-empty affine open containing p.

$\rightarrow p \in V$, $p \in V^{aff} \subseteq V$ $\{$ tangent space is local

<u>Def</u> (1) A variety V is smooth if $T_{V,p}$ has the same dimension for all $p \in V$

(2) If V is irreducible, then $\dim V = \min\limits_{p \in V} \dim T_{V,p}$

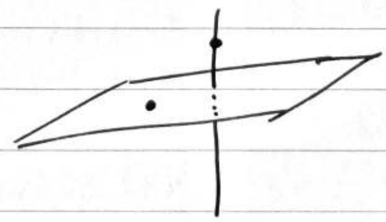
(3) If V is not necessarily irreducible,

then $\dim V = \max \dim V_i$ where V_i ranges over the irreducible components. (CRINGE)

<u>Pictures</u>

(1)

(2+3)



<u>Def</u> A point $p \in V$ is <u>smooth</u> if $\dim T_{V,p} = \dim V$ (V irred)

<u>Theorem</u> The set of smooth points of V form a non-empty open subset.

$\nparallel$ The set where the tangent space jumps is a set which solves some equation

Proof Obviously non-empty. Affine case suffices.
Let $V \subseteq \mathbb{A}^n_k$ be affine, let $I(V) = \langle f_1, \dots, f_r \rangle$
Then
$$\dim T_{V,p} = n - \operatorname{rank} \underbrace{\left(\frac{\partial f_j}{\partial x_i}(p) \right)_{i,j}}_{\substack{\text{matrix of linear} \\ \text{forms of the } f_j}}$$

But this rank being $< (n-r)$ is equal to the vanishing of the $(n-r) \times (n-r)$ minors.
So this set is Zariski closed. $\square$

Remark. In fact, the dimension of $T_{V,p}$ jumps on closed sets.
· If V is irreducible, the locus of $\dim T_{V,p}$ minimal is dense

We can also linearize maps
Let $\varphi : V \dashrightarrow W$ be a rational map, $p \in \operatorname{dom} \varphi$.
Then we can write $\varphi = (f_1, \dots, f_m) : V \dashrightarrow W$
Define $d\varphi_p : T_{V,p} \to T_{W,\varphi(p)}$ as follows

$$d\varphi_p(\underline{v}) = \left(\frac{\partial f_j}{\partial x_i}(p) \right)_{i,j} \cdot \underline{v} \in k^m$$

Proposition . The image of $v \in T_{V,p}$ under $d\varphi_p$ lies in $T_{W,\varphi(p)}$
· $d\varphi_p$ is well-defined & independent of the choice of $f_1, \dots, f_m$ expressing φ
· If $V \overset{\varphi}{\dashrightarrow} W \overset{\psi}{\dashrightarrow} Z$ with composition well-defined, then
$$d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p$$
for all $p \in \operatorname{dom}(\psi \circ \varphi)$

Proof First ~~two~~ & last follow immediately from multivariate chain rule
Second statement is a straightforward calculation.

<u>Corollary</u> Birational varieties have the same dimension

<u>Proof</u> Given $V \underset{\leftarrow - -}{\overset{- - \rightarrow}{}} W$ the differentials give an iso^m of tangent spaces where defined. $\square$

<u>Suggests</u>: Given finitely generated field extension of k, we should be able to do a field theory calculation to find dim of any variety with function field K.

Let K be a transcendental finitely generated field extension of k.

<u>Def</u> K is purely transcendental over k if
$$K \cong k(x_1, \dots, x_n) \text{ with } x_1, \dots, x_n$$
algebraically independent.

<u>Ex</u>: $\mathcal{O}_{\mathbb{P}^n}(y) \cong k(t_1, \dots, t_n)$ — Fraction field of the n-variable poly ring over k

<u>Proposition</u> Let K/k be transcendental and finitely generated. Then there exists a subextension $k \subseteq K_0 \subseteq K$ such that

K_0/k pure transcendental

K/K_0 finite and separable

⌈We will prove this in the case $\mathrm{char}(k) = 0$ but is true in general⌋

<u>Proof</u> (In characteristic 0) Choose a generating set $\{x_1, \dots, x_m\}$ for K over k.

Then there is a maximal algebraically indep subset (after reordering), say $\{x_1, \dots, x_n\}$

Now, $\underset{(i>n)}{x_i}$ is algebraic over $\underbrace{k(x_1, \dots, x_n)}_{\text{pure transcendental}}$

Then $K/k(x_1, \dots, x_n)$ is finite. In characteristic 0 separability is automatic. □

<u>Proposition</u> For K/k a f.g. field extension, we can always write
$$K = k(x_1, \dots, x_n, y) \quad \text{with } x_1, \dots, x_n \text{ alg indep}$$
and $y \in K$ algebraic over $k(x_1, \dots, x_n)$.

<u>Proof</u> Primitive element theorem. □

We now want to extract an algebraic variety from a f.g. field ext^n K/k.

<u>Proposition</u> Let K be transcendental over k, generated by $x_1, \dots, x_{n+1} \in K$. Assume $x_1, \dots, x_K$ alg indep; x_{n+1} alg over $k(x_1, \dots, x_n)$.

Consider $I = \{g \in k[W_1, \dots, W_{n+1}] \mid g(\underline{x_i}) = 0\}$. Then

① This ideal is principal, $I = (f)$; with f irreducible

② If f contains the variable W_j then $X_1, \dots, \hat{X_j}, X_{j+1}, \dots, X_{n+1}$ are alg indpt

<u>Proof</u> Consider x_{n+1} and consider its minimal irreducible polynomial over $k(x_1, \dots, x_n)$. ⌜smallest algebraic dependency⌟

This is a polynomial in $k(x_1, \dots, x_n)[T]$

Let $h \in k(x_1, \dots, x_n)[T]$ denote this poly.

Clear denominators to obtain poly in

$k[X_1, \dots, X_n][T]$; name it f

<u>Claim</u> f is irreducible b/c h is irred

<u>Proof</u> Gauss lemma [cf $\mathbb{Z} \rightsquigarrow \mathbb{Q}$]

We also claim that f generates the ideal I of alg dependencies:

~~Since~~ Given $g \in I$, since f is minimal, h divides g in $k(x_1, \dots, x_n)[T]$

Again by Gauss' Lemma, this means f divides g as well.

Final statement left as exercise. □

<u>Corollary</u> If V is irreducible then V is birational to a hypersurface in some $\mathbb{A}^w$.

<u>Proof</u> Take and let $K = \mathcal{O}_V(y)$. Then write

$K = FF(k[w_1, \dots, w_n]/f)$.

Then K is the function field of $\mathbb{V}(f)$.

We have proved that since V, $\mathbb{V}(f)$ have the same function field, they are birational. □

<u>Remark</u> Other similar statement (in spirit): if V is an irreducible variety then V is birational to a smooth projective variety in $char(k) = 0$ [Hironaka's Resolution of Singularities]. //

<u>Theorem</u> Let V be an irreducible variety.

Then $\dim_k V$ is equal to the transcendence <u>degree</u> of the function field of V, i.e. of $\mathcal{O}_V(y)$.

i.e. write $\mathcal{O}_V(y) = k(\underbrace{x_1, \dots, x_n}_{\text{alg indp}}, \underbrace{x_{n+1}}_{\text{algebraic}})$

then tr. deg. $\mathcal{O}_V(y) = n$.

Remark This theorem proves in particular that the tr. degree is well-defined.

Proof The dimension is a birational invariant, so assume $V = \mathbb{V}(f)$ with $f \in k[W_1, \ldots, W_{n+1}]$. Then by definition, the tr. degree of $k[\underline{W}]/(f)$ is n. But the dimension of a hypersurface is also n. $\square$

A bit more dimension theory

Let R be a finitely generated nilp. free k-alg (more generally R just Noetherian).

Let $\mathfrak{p} \subseteq R$ be a prime ideal.

Def (Height or Codimension) The height of $\mathfrak{p} \subseteq R$ is the supremum of the lengths of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

Ex In $k[x,y,z]$ the prime $\mathfrak{p} = (x,y,z)$ has height (at least 3) since

$$0 \subsetneq (x) \subsetneq (x,y) \subsetneq (x,y,z)$$

won't use $\Bigg\{$ Remark If $\mathfrak{p} \subseteq R$ is prime then the dimension of $\mathbb{V}(\mathfrak{p})$ is equal to the height of $\mathfrak{p}$. [In good circumstances]

Def If $I \subseteq R$ is an ideal then a minimal prime over I is a minimal element under inclusion of the set of all primes containing I.

⚠ Minimal prime not unique

Theorem (Krull's Height Theorem) If R is as above and I is an ideal generated by n elements, then every minimal prime containing I has height at most n.

Geometry The height constrains the number of elements needed to generate I.

Consequences · Every pair $\mathbb{V}(f)$, $\mathbb{V}(g)$ in $\mathbb{P}^2$ for f, g homogeneous intersect

- $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}$
- There are no non-constant morphisms

$$\mathbb{P}^n \to \mathbb{P}^m, \quad n > m$$

$xy \in k[x,y]$

$x' = x - xy$

$(x' + xy)\,y$

* Field Notes

Recall: Krull's Height Theorem Let R be a f.g. k-algebra and $I \subseteq R$ an ideal generated by n elements. The minimal primes containing ~~primes~~ I have height at most n.

{ Nonlinear version of linear algebra fact }

Corollary 1 Every pair of hypersurfaces in $\mathbb{P}^n$ for $n \geq 2$ intersect non-trivially.

Proof Suppose F, G are hom. polys in $(n+1)$ variables. WTS $\mathbb{V}(f) \cap \mathbb{V}(G) \neq \emptyset$ in $\mathbb{P}^n$.

Equivalently, we can show $\mathbb{V}(F) \cap \mathbb{V}(G)$ in $\mathbb{A}_k^{n+1}$ is not precisely the point $(0, \ldots, 0)$

But if F, G vanish both only on $(0, \ldots, 0)$ then

$\sqrt{(F, G)} = (X_0, \ldots, X_n)$

This has height at least $(n+1)$, which is impossible. $\square$

How does this radical come in? it's a minimal prime!

Corollary 2 All morphisms from $\mathbb{P}^n \to \mathbb{P}^1$ are constant. $(n \geq 2)$

Proof A morphism $\mathbb{P}^n \to \mathbb{P}^1$ is given by $[F : G]$ which by Corollary 1 have non-Trivial base locus. $\square$

Corollary 3 Any morphism $\mathbb{P}^n \to \mathbb{P}^m$ for $n > m$ is constant.

Proof Argument is similar. $\square$

Corollary 4 $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$

⌜However they are birational⌟

Proof $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ proj^n onto first factor is non-constant morphism. $\square$

Curves We will be intereseted in smooth projective irreducible curves over k

$\uparrow$ variety of dim^n 1

Remark (Working definition) An abstract projective smooth irreducible curve is an isomorphism class of curves.

Curves in the plane (i.e. not abstract curves)

Theorem (Bézout's Theorem) Let C & D be curves in $\mathbb{P}^2$ defined by homogeneous polynomials F & G of degrees c, d. Then if C, D do not share an irreducible component, then

$$\# C \cap D \leq cd$$

Moreover there exist integers m_p such that

$$\sum_{p \in C \cap D} m_p = cd.$$

~~Examples~~ Proposition [Bézout's theorem holds for $d=1$.] If L is a line and C is a curve defined by $\mathbb{V}(f)$ with $\deg f = c$ (in $\mathbb{P}^2$), & L not contained in C, then $\# L \cap C \leq C$ and $\exists\, m_p$ s.t. $\sum_{L \cap C} m_p = C$.

Proof (Polynomials in 1 variable) By projective change of coords, we can assume C does not contain $[0:1:0]$ and $L = \{X_2 = 0\}$.
Now, the intersection $L \cap C$ is contained in an $\mathbb{A}^2$ with L given by the vanishing of a coordinate.
Now apply FTA. m_p's are multiplicities of the roots $\square$

Proposition Every irreducible conic, i.e. $\mathbb{V}(f)$ w/ degree 2 is smooth. Moreover, $\exists$ quadratic forms Q_1, Q_2, Q_3 in 2 variables and an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathbb{V}(F) \subseteq \mathbb{P}^2$

$$(Y_0 : Y_1) \mapsto (Q_0(Y_0, Y_1), Q_1(Y_1, Y_2), Q_2(Y_1, Y_2))$$

Proof By changing coordinates on $\mathbb{P}^2$, since F is irreducible, we can write $F(X_0, X_1, X_2) = X_0 X_2 - X_1^2$

- This is smooth by a direct check
- After we've changed coordinates, take $(Q_0, Q_1, Q_2) = (Y_0^2, Y_0 Y_1, Y_1^2)$

⌜Ex Sh III shows the image is $\mathbb{V}(F)$ & F iso⌟

Remark In particular this shows that lines & smooth conics in $\mathbb{P}^2$ are isomorphic.
However, we'll see later that if F, G are homogeneous w/ $\deg F \geq 3$, $\deg F \neq \deg G$ then $\mathbb{V}(F) \not\cong \mathbb{V}(G)$. □

Another Baby Bezout Proposition
Let C and D be plane curves (w/out shared component) of degrees ≤ 2 in $\mathbb{P}^2$.
Then $\# C \cap D \leq 4$
Proof If $\deg C$ or $\deg D = 1$, done.
If C or D are a union of two lines, done.
So reduce to the case where C, D are irreducible degree 2.

- Parametrize $\mathbb{V}(F)$ as

$$\mathbb{P}^1 \longrightarrow \mathbb{V}(F) \subseteq \mathbb{P}^2$$
$$[Q_0 : Q_1 : Q_2]$$

Now as before we can assume the intersection lies in some fixed "coordinate patch" i.e. $\underset{\mathbb{A}^2}{\overset{\cong}{\mathcal{U}_0}} = \{X_0 \neq 0\} \subseteq \mathbb{P}^2$

Dehomogenise & solve the restriction of the equation defining D on the part of $\mathbb{P}^1$ contained in $\mathcal{U}_0$. □

Theorem Let $p_1, \dots, p_5$ be five points in $\mathbb{P}^2$, no 3 of which are colinear. Then $\exists$ unique smooth conic in $\mathbb{P}^2$ passing through these points.

"Five points determine a conic"

Proof Irreducibility of any such conic is clear, since a reducible conic would be a union of two lines (or worse), which breaks collinearity. $\Rightarrow$ smooth

- If there were two conics passing through $p_1, \dots, p_5$ this condratics Bézout's theorem
- To see it exists, consider the coefficients of a degree 2 hom poly in 3 variables. There is a 6-dim space of choices for the coeffs of degree 2 coeffs in 3 variables

$$\{ a_0 X_0^2 + a_1 X_1^2 + a_2 X_2^2 + a_3 X_0 X_1 + a_4 X_1 X_2 + a_5 X_2 X_0 \}$$

a_i's form coords for K^6

Imposing 5 linear conditions gives ≥ 1-dim subspace. $\square$

Aside "Where did it go?"

$\mathbb{A}^1 \setminus \{0\}$ has a lot more functions than $\mathbb{A}^1$.

But $\mathbb{A}^n \setminus \{0\}$ has no more functions than $\mathbb{A}^n$. $(n \geq 2)$

There should be some data that's gone missing! $\boxed{\text{Sheaf Cohomology}}$

Curves

Two sources: • $\mathbb{V}(F_d)$ homogeneous degree d poly, in $\mathbb{P}^2$

• $\mathbb{V}(G_{d_1,d_2})$ G_{d_1,d_2} is a bidegree (d_1, d_2) bihomog. poly, in $\mathbb{P}^1 \times \mathbb{P}^1$

"from now on, "curve" will refer to 1-dim irred variety →

Proposition Let C be a curve (smooth? projective irreducible)

Then every subvariety $W \subsetneq C$ is a finite union of points.

Proof Suffices to prove this on an affine open subset of C, call it $V \subset \mathbb{A}^2$.

Let $W \subsetneq V$ be irreducible proper subvariety. We'll show $W = \{pt\}$

$\varphi: W \hookrightarrow V$ induces a homomorphism of coord. rings

$$\varphi^*: \mathcal{O}_V \to \mathcal{O}_W$$

Claim $\mathcal{O}_W = k$ [this is equivalent to $W = \{pt\}$]

Suppose $\mathcal{O}_W \supsetneq k$. Choose $t \in \mathcal{O}_W \setminus k$.

Now inspect $\varphi^*: \mathcal{O}_V \twoheadrightarrow \mathcal{O}_W$

Choose $x \in \mathcal{O}_V$ s.t. $\varphi^*(x) = t$.

But since $I(V) \subsetneq I(W)$ we can find $y \in \mathcal{O}_V$ s.t. $\varphi^*(y) = 0$.
↑ non-zero

use $\mathcal{O}_W$ integral domain →

Observe that x & y must be algebraically independent (hom property)

get that if not y is unit in $\mathcal{O}_V(y)$ which it's not

But $\dim \mathcal{O}_V(y) = 1$ i.e. $\operatorname{tr deg}_k \mathcal{O}_V(\eta) = 1$. ⚡

Consequence If $C \subseteq \mathbb{P}^n$ a curve not contained in any hyperplane, then for any hyperplane $H \subseteq \mathbb{P}^n$, $H \cap C$ is a finite collection of points (Weak weak Bezout's theorem)

Soon we'll see that when counted with multiplicity, the number of intersection points is constant.

Local Structure Theorem

Fix an irreducible projective curve C, fix a point $p \in C$.

We have • $\mathcal{O}_C(\eta)$ $(= k(C))$ = fraction field of the coord ring of an affine patch

• $\mathcal{O}_{C,\eta}$ contains a local ring $\mathcal{O}_{C,p} \{ f \in \mathcal{O}_{C,\eta} \mid \text{reg } p \}$

• $\mathcal{O}_{C,p}$ contains $m_p = \{ f \in \mathcal{O}_{C,p} \mid f(p) = 0 \}$ max ideal

Theorem Let p be a smooth point of C.
Then the ideal m_p is a principal ideal in the local ring $\mathcal{O}_{C,p}$.

[in fact, this is an iff, so $\mathcal{O}_{C,p} \supset m_p$ is principal iff p smooth]

Terminology If $m_p = (t_p) \subseteq \mathcal{O}_{C,p}$ we call t_p a local coordinate / parameter

Note t_p is not unique, but if t_p' is another generator then $t_p = \lambda t_p'$ with λ a unit in $\mathcal{O}_{C,p}$

Trade Secrets

• On ExSh III you will prove that our two definitions of tangent spaces coincide.

1. Vanishing of linearised equations
2. $(m_p / m_p^2)^*$, i.e. linear dual

$\uparrow$ linear functionals on $T_{C,p}$

• (m_p / m_p^2) has dim 1 at all p iff C is smooth as defined.

• Say $R = \mathcal{O}_{C,p}$ functions defined near p

$ev_p : R \to R/m = K$

evaluation at p

Could also consider R/m^2. What is this?

$$R/m^2 \twoheadrightarrow R/m$$

What is the kernel?

Just m/m^2. So $R \to R/m$ keeps track of function value
$R \to R/m^2$ " plus linear part

• R/m^n for any n. Inductively, it's straightforward to show this is a vector space of dimension n over K.

The collection $\{ \bar{f}_n = \text{image of } f \text{ in } R/m^n \}$ 'power series' for f at p.

Expectations • Every rational function has a finite pole order
• The maximal ideal is the set of all elements in R which have zero constant in their Taylor series expansions

Ex: If $C = \mathbb{A}^1$, $p = 0$ then $\frac{1}{1-t^2} \in \mathcal{O}_{C,p}$
$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots$$

Proof of Theorem (i.e. p smooth $\Rightarrow m_p$ principal)
[we expect that $\cancel{\mathcal{O}_{C,p}}$ m_p should be generated by any element in $\mathcal{O}_{C,p}$ which is in m_p but not in m_p^2]

Lemma If R is a local ring w/ max ideal m and M is a f.g. R-module s.t. $mM = M$ then $M = 0$.
|Nakayama's Lemma| "zero detection"

Proof Let $M = (b_1, \ldots, b_n)$. Assume that $mM = M$.
By hypothesis, we can write $b_i = \sum_j a_{ij} b_j$ where $a_{ij} \in m$.
$$\sum_j (\delta_{ij} - a_{ij}) b_j = 0 \quad \text{[matrix form]}$$

· The diagonal entries $1 - a_{ii} \notin m$
· Off-diagonal entries $\in m$
$\Rightarrow \det(\delta_{ij} - a_{ij}) = 1 + d$ for some $d \in m$ is invertible! since ring is local.
This means $\forall j, b_j = 0$ and $M = 0$. □

Corollary If $t \in m \subset R$ s.t. $\bar{t} \in m/m^2$ non-zero and $\dim m/m^2 = 1$ then t generates m

Proof Apply Nakayama's lemma to $M = (t)$. Pass to the quotient to compare with m. □

not exactly, Nakayama on the quotient module $m/(t)$

Corollary Given $p \in C$ (smooth) there is a homomorphism $\mathrm{ord}_p : \mathcal{O}_{C,\eta}^\times \rightarrow \mathbb{Z}$ s.t. $\mathcal{O}_{C,p} = \{\mathrm{ord}(f) \geq 0\} \cup \{0\}$
$$m_p = \{ f \in \mathcal{O}_{C,\eta}^\times \mid \mathrm{ord}\, f > 0 \}$$

Proof Every element of

Corollary If C is smooth projective and $C \dashrightarrow \mathbb{P}^n$ is a rational map, then φ is a morphism.

A question to guide us:

Given an abstract smooth curve C, is there a canonical way to realize C inside a $\mathbb{P}^N$?

Last time: Smooth points on curves have principal maximal ideals in their local rings

[Why?] Find an element in m_p that spans m_p/m_p^2 as a vector space. + Nakayama

Motivation Given $f(t)$ a power series in t, its order of vanishing at $t=0$ is the exponent of its leading term:

$$\text{ord} : k[[t]] \longrightarrow \mathbb{Z}$$

e.g.
$$\begin{cases} 1+t^3+t^8+\cdots \mapsto 0 \\ t^2 \mapsto 2 \\ t^8+t^9+\cdots \mapsto 8 \end{cases}$$

Prop There is a group homomorphism ord_p

$$\mathcal{O}_{C,\eta}^\times \longrightarrow \mathbb{Z} \quad \text{s.t.} \quad \mathcal{O}_{C,p} = \{f \in \mathcal{O}_{C,\eta}^\times \mid \text{ord} f \geq 0\} \cup \{0\}$$
$$\cup\!\!\mid$$
$$m_p = \{f \in \mathcal{O}_{C,\eta}^\times \mid \text{ord} f > 0\} \cup \{0\}$$

Proof / construction First we examine $\mathcal{O}_{C,p}$

Note its fraction field is $\mathcal{O}_{C,\eta}$.

Given $f \in \mathcal{O}_{C,p}$ it lies in m_p^n for some $n \geq 0$.

Define $\text{ord}_p(f) = \overset{\max}{\cancel{\min}} \{n \mid f \in m_p^n\}$

Claim: This define the requisite hom

(• Want to know that no $f \in \mathcal{O}_{C,p}^\times$ is contained in $\bigcap_{n \geq 0} m^n$)

Let $M = \bigcap_{n\geq 0} m^n$. Notice $mM = M$.

So by Nakayama's lemma, $M \equiv 0$. $(\& f \notin m_p^{n+1})$

Since m_p is a principal ideal, if $f \in m_p^n$ then $f = t_p^n u$ where t_p generates m_p and $u \in \mathcal{O}_{C,p} \setminus m_p$

$\Rightarrow \text{ord}_p$ satisfies: $\text{ord}_p(fg) = \text{ord}_p(f) + \text{ord}_p(g)$

To define on $\mathcal{O}_{C,\eta}^\times$ extend linearly:

$$\text{ord}_p(f/g) = \text{ord}_p(f) - \text{ord}_p(g) \quad \text{for } f, g \in \mathcal{O}_{C,p} \setminus \{0\}.$$

$\square$

Tells us Given $f \in \mathcal{O}_{C,\eta}^{*}$ it is regular at a point p iff $\mathrm{ord}_p(f)$ is non-negative.

Ex Take $C = \mathbb{A}^1$, $p = 0$

$$\mathcal{O}_{\mathbb{A}^1,\eta} = FF(k[t]) = k(t)$$

$$\mathcal{O}_{\mathbb{A}^1,0} = \left\{ f/g \mid f \in k[t],\ g \in k[t] \setminus I(0) \right\}$$

$$\ni \frac{t^2+8t^3}{1+t^4} \ ; \text{ order of vanishing} = 2$$

$$\ulcorner (t^2+8t^3)(1-t^4+\cdots) \lrcorner$$

Corollary If C is a smooth irred curve, then for any $f \in \mathcal{O}_{C,\eta}$ either f or f^{-1} is regular at any point p

Contrast Take $f = \frac{x}{y}$ in $FF(\mathbb{C}[X,Y])$
neither f nor f^{-1} is regular at 0

Corollary If C is smooth projective irreducible then any rational map $C \dashrightarrow \mathbb{P}^n$ is a morphism.

[In fact $C \dashrightarrow V$ for V projective variety same conclusion holds]

Proof Let $C \dashrightarrow \mathbb{P}^n$ be a rational map. Fix $p \in C$.
Assume im of C is not contained in $\{X_0 = 0\}$.
Then $C \dashrightarrow \mathbb{P}^n$ determined by homogeneous polys

$$[G_0 : \cdots : G_n] \longmapsto [1 : g_1 : \cdots : g_n]$$

where g_i are rational functions on C.

The morphism is regular at p if $g_i \in \mathcal{O}_{C,p}$ $\forall i$.
Let m be the minimal order of vanishing among the g_i.
Multiply by t_p^{-m} to obtain an equivalent tuple of rational functions $\Rightarrow$ rational map regular at p. $\square$

Remark: Given $C \subseteq \mathbb{P}^2$ smooth conic & $p \in C$, the projection from a point gives $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Restriction of π to C gives a morphism by this corollary. (we did this explicitly).

Morphisms between curves

Let $C \xrightarrow{f} D$ be a morphism between irreducible curves.
Assume f is non-constant.

Proposition (1) For every point $q \in D$, the preimage $f^{-1}(q)$ is a finite set.
(2) f induces a homomorphism
$$f^*: \mathcal{O}_{D,\eta} \longrightarrow \mathcal{O}_{C,\eta}$$
This is a field extension; it is finite.

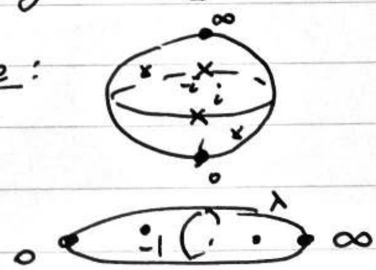
Proof: (1) q closed subvariety in D hence so is $f^{-1}(q)$, so since f non-constant, it's finite.
(2) C has infinitely many points, so $f(C)$ has to be dense.
So we get $f^*: \mathcal{O}_{D,\eta} \hookrightarrow \mathcal{O}_{C,\eta}$.
This is a field extension.
[Why finite?] Pick $t \in \mathcal{O}_{D,\eta} \setminus k$.
Now have a tower of extensions:
$$k \hookrightarrow k(t) \hookrightarrow \mathcal{O}_{D,\eta} \longrightarrow \mathcal{O}_{C,\eta}$$
But the image of t in $\mathcal{O}_{C,\eta}$ is transcendental, and $\mathcal{O}_{C,\eta}$ has tr deg 1.
So $\mathcal{O}_{C,\eta}$ is finite over $k(t) \Rightarrow$ over $\mathcal{O}_{D,\eta}$. $\square$

Given a morphism $f: C \to D$, define the degree of f to be the degree of the assoc. field extension
$$\mathcal{O}_{C,\eta}^{f^*} / \mathcal{O}_{D,\eta}.$$

Remark An algebraic version of the degree of a continuous map in topology.

Examples: Consider $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ sending $t \in \mathbb{A}^1 \subseteq \mathbb{P}^1$ to t^2
[I've told you on $\mathbb{A}^1 \to \mathbb{A}^1$ ergo $\mathbb{P}^1 \to \mathbb{P}^1$ by above]
This has degree 2 [Check]

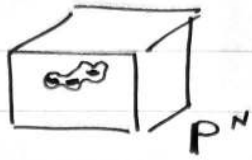
Picture: $/\mathbb{C}$



Next time: Given $C \to D$ we want to relate degree(f) to the number of preimages of points on D.
"Ramification" $\longleftrightarrow$ Smaller preimages

Big Goal:

C abstract curve $\rightsquigarrow$ realise C as an embedding in $\mathbb{P}^N$



Need rational functions $\leftsquigarrow$ Don't understand this well yet!
How to produce "canonical" elements

Theorem If $X \to Y$ is a morphism with X projective and Y arbitrary (affine, q affine, proj, q proj) variety, then the image of every closed set is closed.

Proof No proof; on ExSh III you will reduce this to the case where $X = \mathbb{P}^n \times \mathbb{A}^m$, $X \to Y$ is projection onto $\mathbb{A}^m$. $\square$

Corollary Let C, D be irreducible projective curves. Then if $\varphi: C \to D$ is non-constant, then φ is surjective.

Proof Last time we argued that $\mathrm{im}(C)$ is dense; but dense + closed $\Rightarrow$ image is all of D. $\square$

Last Time $\varphi: C \to D$ morphism of irred proj curves; non-constant then $\varphi^*: \mathcal{O}_{D,\eta} \hookrightarrow \mathcal{O}_{C,\eta}$ a finite extension

Def Degree of φ was defined as the degree of this field extension.

Say $C \xrightarrow{\varphi} D$ is as above. Let $p \in C$ and set $q = \varphi(p)$. Assume p & q are smooth points.

$\varphi^*: \mathcal{O}_{D,q} \to \mathcal{O}_{C,p}$ map b/w local rings

Def (Ramification Order) The ramification of φ at p is computed as: $e_p = \mathrm{ord}_p(\varphi^* t_q)$

where t_q is a local coordinate at q.

Ex: Take $\mathbb{A}^1 \to \mathbb{A}^1$; $p=0, q=0$
$z \mapsto z^2$

$e_p = 2$ (Why?) $\mathcal{O}_{\mathbb{A}^1,0} = \cancel{\mathbb{C}} \left\{ \frac{h(t)}{g(t)} \mid g(0) \neq 0 \right\}$

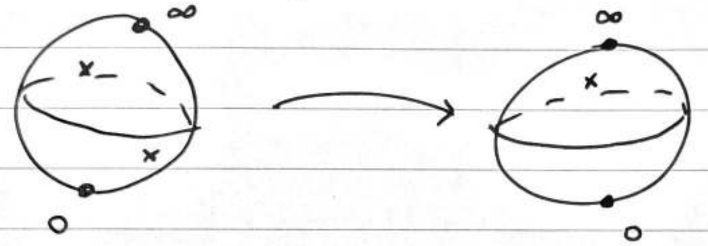
$m_0 = \{ h/g \mid g(0) \neq 0 \text{ but } h(0) = 0 \}$

<u>Theorem</u> (Ramifications compute degree)
Let $C \xrightarrow{\varphi} D$ be a non-constant morphism of smooth, projective, irred curves. Then for any $q \in D$,

$$\deg(\varphi) = \sum_{p \in \varphi^{-1}(q)} e_p$$

<u>Theorem</u> If in addition, k is characteristic 0, or if $\mathcal{O}_{C,\eta} / \mathcal{O}_{D,\eta}$ is separable, then $e_p = 1$ for all but finitely many $p \in C$

<u>Ex</u> ① $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$; $z \mapsto z^2$



(degree 2)

② Take $C^{aff} = \mathbb{V}(y^2 - (x-a)(x-b)(x-c))$

Take $C^{aff} \to \mathbb{A}^1$ given by projⁿ onto x-coord

$x = a \to$ unique y value giving a point on C^{aff}

x generic $\to$ two such points

<u>Proof of the Theorem</u> *non-examinable*

<u>Words about the proof</u>:

Take $C \xrightarrow{\varphi} D$. Trying to compute degree$(\varphi) = \dim_{\mathcal{O}_{D,\eta}} \mathcal{O}_{C,\eta}$

Take $q \in D$ and let $p_1, \dots, p_k$ be its preimages

$$\varphi^* : \underbrace{\mathcal{O}_{D,q}}_{A} \to \underbrace{\bigcap_j \mathcal{O}_{C,p_j}}_{B} \leftarrow (\text{inside } \mathcal{O}_{C,\eta})$$

<u>Proof involves the following</u>:

- The A-module B is torsion free
- Since A is a PID, B is free
- B is finitely generated $\Rightarrow$ B is a free module of finite rank over A, $B = A^{\oplus r}$

<u>Compute r in two ways</u>

1. Compute as degree $\mathcal{O}_{C,\eta}$ over $\mathcal{O}_{D,\eta}$ as a dimⁿ

2. Mod out $\mathcal{O}_{D,y}$ by m_q and mod out B by $(\varphi^*) m_q$. $\square$

<u>Corollary:</u> Let C be a smooth projective irreducible curve over k. Let $f \in \mathcal{O}_{C,y}$ be a rational function.

- If f is regular at all points $p \in C$, then it's constant
- $\{p \in C \mid \mathrm{ord}_p(f) \neq 0\}$ is finite and $\sum_{p \in C} \mathrm{ord}_p(f) = 0$

<u>Terminology</u> If $f \in \mathcal{O}_{C,y}$ and $p \in C$, then if $\mathrm{ord}_p(f) = n$;

$\begin{cases} n > 0 : f \text{ has a zero of order } n \\ n < 0 : f \text{ has a pole of order } -n \end{cases}$

<u>Proof of corollary</u> Given f as above

[Trick] Consider the morphism

$$\varphi : C \to \mathbb{P}^1$$
$$p \mapsto [1 : f(p)]$$

Since C smooth this defines a morphism

[First statement will be on ExSh IV]

We will compute ramification orders for φ.

Let $t = \frac{X_1}{X_0}$ be a local coordinate near $[1:0]$,

and $\frac{1}{t} = \frac{X_0}{X_1}$ " $[0:1]$, point "at ∞".

If $p \in C$ is s.t. $\varphi(p) = [1:0]$

$\Rightarrow$ f vanishes at p, $e_p = \mathrm{ord}_p(\varphi^* t)$
$= \mathrm{ord}_p(f)$

If $p \in C$ s.t. $\varphi(p) = [0:1]$ a similar calculation gives $e_p = \mathrm{ord}_p(\frac{1}{f}) = -\mathrm{ord}_p(f)$

If p maps neither to $[0:1], [1:0]$ then $\mathrm{ord}(\varphi^* t) = 0$.

We computed ramification at all points so the rational function f has non-zero order of vanishing precisely on $\varphi^{-1}(0) \cup \varphi^{-1}(\infty)$. Which are both finite.

Moreover, $\sum_{\mathrm{ord}_p > 0} e_p = \deg \varphi = \sum_{\mathrm{ord}_p < 0} e_p$

so conclude. $\square$

Remark We've seen here that a morphism to $\mathbb{P}^1$ from a smooth curve can be determined by a rational function.

In order to find morphisms $C \to \mathbb{P}^N$ we will use rational functions ???

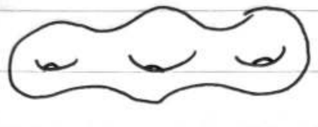
Divisors: Idea: organise the ∞-dim k-vector space $\mathcal{O}_{C,\eta}$ into finite dimensional subspaces; ~~by~~ organising principle will come from where a given $f \in \mathcal{O}_{C,\eta}$ has zeros & poles. //

Goal: Given an abstract curve C find a canonical realization $C \hookrightarrow \mathbb{P}^N$

Why? This will help us classify algebraic curves up to iso^{m}

State of affairs: The function theory of a curve C is totally mysterious!

Curve / $\mathbb{C}$



How to access the "genus" algebraically?

Local goal: Organise the function theory on a $\boxed{\text{smooth, irred proj}}$ curve C

↳ assumed from now on for all "curves"

Suppose we had a morphism $C \xrightarrow{\varphi} \mathbb{P}^N$ (non-constant)

Assume that image$(\varphi) \notin \{X_0 = 0\}$

Then an open subset $C^\circ \subseteq C$ has a morphism to $\mathbb{A}^N$

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}^N \\ \cup\!\! \mid & & \cup\!\! \mid \\ C^\circ & \longrightarrow & \mathbb{A}^N = \{X_0 \neq 0\} \end{array}$$

Each coordinate function $\frac{X_1}{X_0}, \dots, \frac{X_N}{X_0}$ gives a rational f^n on C

Call these rat^L functions $f_1, \dots, f_N \in \mathcal{O}_{C,\eta}$

Observe • If $p \in C^\circ$ then f_i is regular at p $\forall i$, i.e. $\operatorname{ord}_p(f_i) \geq 0$

SUS → • f_i all have poles at the points $C \setminus C^\circ$, i.e. if $q \in C \setminus C^\circ$ then $\operatorname{ord}_q(f_i) < 0$

But by explicit calculation (exercise based on last time) for all $q \in C \setminus C^\circ$, $\operatorname{ord}_q(f_i) = \operatorname{ord}_q(f_j)$ $\forall i,j$

↘ prove that f_i/f_j is invertible in $\mathcal{O}_{C,q}$

SUGGESTS: In order to build $C \to \mathbb{P}^N$ we should examine collections $\{f_1, \dots, f_N\}$ of rational functions which are regular on an open set $C^\circ \subseteq C$ and have the same pole orders at $C \setminus C^\circ$

DIVISORS: Let C be a curve.

Def A divisor on C is an element of the group $\bigoplus_{p \in C} \mathbb{Z}[p]$ or explicitly a finite $\mathbb{Z}$-combination of points.

Idea $D = a_1 p_1 + a_2 p_2 + a_3 p_3$
we will consider rational functions regular on $C - \{p_1, p_2, p_3\}$ with $\mathrm{ord}_{p_i}(\cdot) \geq -a_i$

- Given a divisor D, say $\sum a_i p_i$; $a_i \in \mathbb{Z}$, the degree of D is $\deg(D) = \sum a_i$.
- A divisor D is effective if all $a_i \geq 0$.

Let $f \in \mathcal{O}_{C,\eta}$ be a rational function.

Def The divisor associated to f, is
$$\mathrm{div}(f) = \sum_{p \in C} \mathrm{ord}_p(f)[p]$$

Proposition: degree of the divisor $\mathrm{div}(f)$ is always zero

Pf: Last time. □

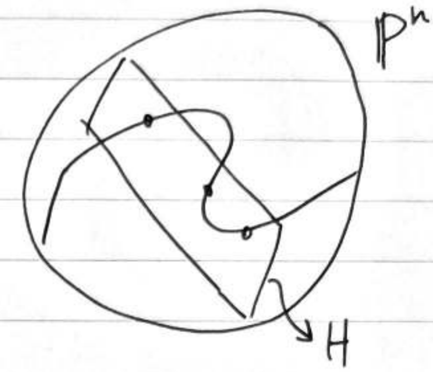
Question: Given D a divisor of degree 0, is $D = \mathrm{div}(f)$ for some $f \in \mathcal{O}_{C,\eta}$

Answer: No! But it will take time to see why.
If $C \cong \mathbb{P}^1$, the answer is yes; you'll do this on ExSh III.

Terminology: A divisor D is principal if $D = \mathrm{div}(f)$ for some $f \in \mathcal{O}_{C,\eta}$

Def: The divisor class group or Picard group of a curve C is the quotient of $\mathrm{Div}(C)$ by the subgroup of principal divisors.

Another source of divisors Let $C \subseteq \mathbb{P}^N$.
Let it be a curve, & $H = \mathbb{V}(L)$ for L a linear form.



Def the divisor $\mathrm{div}(L)$ on C is $\mathrm{div}(L) = \sum_{p \in C} n_p [p]$ where $n_p = \mathrm{ord}_p(L/X_i)$ for some X_i s.t. $X_i(p) \neq 0$.

Rmk This is independent of i.
Divisors of this form are called "hyperplane sections"

Def Divisors D, E are said to be linearly equivalent if $D - E$ is a principal divisor.

A few observations : If $C \subseteq \mathbb{P}^n$ and L, L' are linear forms, then L/L' is a rational function, [provided, $C \not\subseteq \mathbb{V}L, \mathbb{V}L'$]
Then $\mathrm{div}(L/L') = \mathrm{div}(L) - \mathrm{div}(L')$

Proof No proof. An exercise in definitions $\square$

Tautology If L, L' are linear forms as above, then $\mathrm{div}(L)$, $\mathrm{div}(L')$ are linearly equivalent.

An excellent source of divisors:
take C; embed $C \hookrightarrow \mathbb{P}^n$
$C \hookrightarrow \mathbb{P}^m$ ...
& take hyperplane sections.

Proposition If $C \subseteq \mathbb{P}^n$, then the degree of $\mathrm{div}(L)$ for L s.t. $\mathbb{V}(L) \not\supseteq C$ is independent of L.
This is called the degree of the curve $C \subseteq \mathbb{P}^n$.

Similarly, if G is a homogeneous degree m poly in $X_0, \dots, X_m$, we get $\mathrm{div}(G) = \sum_{p \in C} n_p [p]$, where $n_p = \mathrm{ord}_p(G/X_i^m)$ (some i)

Hypersurface sections

Theorem (Bézout) Given $C \subseteq \mathbb{P}^2$ and C' another curve (possibly not irreducible or smooth), the degree of

$$\# C \cap C' \leq \deg C \cdot \deg(C')$$

the degree of the poly G

provided it is finite.

Proof Degree of G s.t. $C' = \mathbb{V}(G)$.
Then $\# C \cap C' \leq$ degree $\mathrm{div}_C(G)$ $\square$

Let D be a divisor on C. Then

$L(D) =$ the vector space of meromorphic/rational functions on C w/ poles bdd by D

$:= \{ f \in \mathcal{O}_{C,\eta} \mid \mathrm{div}(f) + D$ is effective $\}$

Observe: (1) $L(D)$ is a k-vector space
(2) [We'll see] $\dim L(D) < \infty$, denote $\ell(D) := \dim_k L(D)$

We defined $L(D) = \{ f \in \mathcal{O}_{C,\eta} \mid \operatorname{div}(f) + D \geq 0 \}$

Ex : Take $D = [1:0]$ & $C = \mathbb{P}^1$

What is $L(D)$?

$$L(D) = \left\{ f/g \in k(t) \;\middle|\; \begin{array}{l} 1.\ \operatorname{ord}_p(\tfrac{f}{g}) \geq 0 \text{ for all } p \neq [1:0] \\ 2.\ \operatorname{ord}_{[1:0]}(\tfrac{f}{g}) \geq -1 \end{array} \right\}$$

Condition ① $\Rightarrow$ f/g has to be a polynomial on $\mathbb{P}^1 \setminus \{[1:0]\}$

i.e. a polynomial in $\frac{X_0}{X_1}$

Condition ② $\Rightarrow$ has the form $a\frac{X_0}{X_1} + b$, $a, b \in k$

$$L(D) = \left\{ a\frac{X_1}{X_0} + b \;\middle|\; a, b \in K \right\} \cong k^2$$

$= \{ \text{linear poly in } t \}$ t variable on affine patch $\mathbb{P}^1 \setminus [1:0]$

Similarly if $D = n \cdot [1:0]$ then $L(D) \cong k^{n+1}$ //

BASIC PROPN ON $L(D)$ Let D be a divisor on C.

Then 1. $\deg(D) < 0 \Rightarrow L(D) = 0$

2. $\deg(D) \geq 0 \Rightarrow \dim_k L(D) \leq \deg(D) + 1$

3. If p is any point of C, then
$$\dim_k L(D) \leq \dim_k L(D - p) + 1$$

Given $D = \sum a_p [p]$, $L(D)$ is the set of rational functions that:

- If $a_p > 0$, are allowed to have poles at p of order up to a_p
- If $a_p < 0$, are forced to vanish to order at least $-a_p$

Proof 1. Say $\deg(D) < 0$. If $L(D) \neq 0$, then pick $f \in L(D)$ s.t. $\operatorname{div}(f) + D \geq 0$ ※ since $\operatorname{div}(f)$ has degree zero.

3. Let n be the coeff of p in D.

Now consider the map
$$\alpha : L(D) \longrightarrow k$$
$$f \longmapsto (t_p^n f)(p)$$

⌈shifting power series ; get regular fn at p⌋

Then $L(D - p)$ is exactly the kernel of α. ☺

2. Follows from 3. □

Strategy Pick a basis $f_1, \dots, f_r$ on $L(D)$ for well-chosen D, & get

$$\begin{aligned} C &\longrightarrow \mathbb{P}^{r-1} \\ p &\longmapsto (f_i(p)) . \end{aligned}$$

NEED A CANONICAL CONSTRUCTION OF A DIVISOR UP TO LINEAR EQUIVALENCE

Fact If D & E are divisors w/ $D-E = \mathrm{div}(f)$ then $L(D)$ is isom to $L(E)$;

$$\begin{aligned} L(D) &\longrightarrow L(E) \\ g &\longmapsto g/f \end{aligned}$$

[Check this]

$\boxed{\text{WWWHATIWWATDC}}$

Differential forms in AG:

Let K/k be a transc. field extension of i.e. the function field $\mathcal{O}_V(\eta)$ for V an irred variety.

Calculus: $\{ \sum f_i \, dg_i \}$ subject to basic manipulations:

$$d(f+g) = df + dg, \quad d(fg) = f dg + g df$$

Def The vector space of Käler differentials is the quotient

$$\Omega_{K/k} := \frac{\{K\text{-vector space spanned by symbols } \delta f \mid f \in K\}}{\left\langle \begin{array}{c} \delta(x+y) = \delta(x) + \delta(y), \\ \delta(xy) = x\,\delta(y) + y\,\delta(x), \\ \delta\lambda = 0 \end{array} \right\rangle}$$

What is this?

Given $x, y \in K$, we can write dx, dy, $\underset{\substack{\parallel \\ 2x\,dx}}{d(x^2)}$, $\underset{\substack{\parallel \\ dx}}{d(x+1)} \in \Omega_{K,k}$

$\Omega_{K/k}$ is a K-vector space ;

it is equipped with a map

$$\begin{aligned} d : K &\longrightarrow \Omega_{K/k} \\ f &\longmapsto df \end{aligned}$$

[exterior derivative]

Def A k-linear derivation is a map of vector spaces $D: K \to U$ where U is a K-vector space, which is k-linear and obeys $D(xy) = x D(y) + y D(x)$

$D\lambda \overset{?}{=} 0$

Tautology $D : K \to U$ w/ U a vector space over K is a derivation if and only if $\exists$ factorisation of the sort

$$\begin{tikzcd} K \arrow[rr, "D"] \arrow[dr, "d"'] & & U \\ & \Omega_{K/k} \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Lemma Suppose $f = g/h \in k(W_1, \dots, W_r)$
Let $y = f(x_1, \dots, x_r)$ with $x_i \in K$
Then
$$dy = \sum_i \frac{\partial f}{\partial W_i}(x_1, \dots, x_r)\, dx_i$$

In particular, if $x_1, \dots, x_n$ generated K as a field, then $dx_1, \dots, dx_r$ span $\Omega_{K/k}$ as a K-vector space.

REDACTED

Theorem Let K/k be a transcendental field extension, as above. [ASSUME char $k = 0$]
Let $t \in K$ be transcendental. Then

$\Omega_{K/k}$ is spanned by the element dt over K

<u>Recall</u> $\Omega_{K/k}$, the K-vs of Kähler differentials

<u>Practically</u> · $\sum f_i dg_i$ subject to usual rules of calculus

· Not a field or ring ; a K-module

A <u>derivation</u> is any k-linear map $K \to U$ w/ U a K-vs satisfying product rule

<u>Facts</u> (by formal properties)

If $f = g/h \in k(W_1, \dots, W_n)$ and $y = f(x_1, \dots, x_n)$ then
$$dy = \sum \frac{\partial f}{\partial W_i}(x_1, \dots, x_n)\, dx_i$$

2. If $K = k(x_1, \dots, x_n)$ then $\{dx_i\}$ span $\Omega_{K/k}$

<u>From now</u>: $K = \mathcal{O}_{C,\eta}$ for C a curve,
$$\Omega_{K/k} = \Omega_{C,\eta}$$

<u>Theorem</u> Let $K/k(t)$ be a finite separable field extension (or assume char $K = 0$), w/ t transcendental over k.

Then $\Omega_{K/k}$ is 1 dimensional & spanned by dt

<u>Proof</u> First $K = k(t)$. Then by previous facts, dt spans $\Omega_{K/k}$, easy as. So $\dim_K \Omega_{K/k}$ is at most 1.

We need to show it is non-zero.

Enough to produce one non-zero derivation
$$\begin{tikzcd} k(t) \arrow[rr, "D"] \arrow[dr] & & k(t) \\ & \Omega_{K/k(t)} \arrow[ur] & \end{tikzcd}$$

$\frac{d}{dt} =: D_t$ i.e. differentiation wrt t is a derivation

<u>General case</u>: Say $K = k(t, \alpha)$ w/ α algebraic over $k(t)$ (primitive element). Let $h(W) \in k(t)[W]$ be the min poly of α. By separability, $h'(\alpha) \neq 0$. Then by formal calculus (previous facts), $\Omega_{K/k}$ is spanned by dt & $d\alpha$

But notice, by formal calculus,
$$0 = d(h(\alpha)) = (\underset{\substack{\uparrow \\ \text{diff coeffs}}}{D_t} h)(\alpha)\, dt + h'(\alpha) \cdot d\alpha$$

$\therefore$ dt spans $\Omega_{K/k}$

Need a non-zero derivation:

$$\frac{k(t)[W]}{(h(W))} = K \xrightarrow{D} K$$

Define a (nonzero) derivation: given $f \in K$,
if $f \in k(t)$ then $D(f) = D_t f$
and $D(W) = -\dfrac{D_t h(W)}{h'(W)}$ i.e. so it descends to K

extend by rules of calculus.

So $\Omega_{K/k}$ is nonzero. $\square$

<u>Remark</u> In char$(k) = p$, we can have
$d(x^p) = p x^{p-1} dx = 0$. This causes problems.

From now on, we will assume char$(k) = 0$; but could replace by separability hypotheses.

<u>Our situation</u> C is a smooth curve; refer to elements of $\Omega_{C,\eta}$ as RATIONAL DIFFERENTIALS on C

<u>Def</u> A rational differential ω on C is called <u>regular</u> at a point $p \in C$ if it can be expressed as $\sum f_i dg_i$ where $f_i, g_i \in \mathcal{O}_{C,p}$ (i.e. are regular at p)

<u>Notation</u>: $\Omega_{C,p} = \{ \omega \in \Omega_{C,\eta} \mid \omega \text{ regular at } p \}$
This has the structure of a module over $\mathcal{O}_{C,p}$.

<u>Theorem</u> The module $\Omega_{C,p}$ is a free module over $\mathcal{O}_{C,p}$; it is generated by dt_p for t_p any local coordinate at p.

<u>Consequences of this</u> 1. $\Omega_{C,p} = \{ f\,dt_p \mid f \in \mathcal{O}_{C,p} \}$

2. If t_p' is another local parameter at p, then
$dt_p = u\, dt_p'$ where u is a unit in $\mathcal{O}_{C,p}$

~~where u is a~~

<u>Def</u> If $\omega \in \Omega_{C,p}$ and $p \in C$ then $\text{ord}_p(\omega) = \text{ord}_p(f)$ for any expression $\omega = f\,dt_p$. "Order of vanishing of ω at p"

Pf of theorem (Sketch-a-sketch)

1. First check that $\Omega_{C,p}$ is finitely generated by reducing to an affine patch of C & using generators.
2. Apply Nakayama's lemma to show that the quotient

$$\frac{\Omega_{C,p}}{\mathcal{O}_{C,p}\, dt_p} \quad \text{is zero.}$$

$\square$

idea? $d(\text{unit}) = d(t_p^{\text{sth}}\text{ unit})$ by shifting

Def A divisor D on C is said to be canonical if

$$D = \sum_{p \in C} \mathrm{ord}_p(\omega)[p] = \mathrm{div}(\omega)$$

for some non-zero $\omega \in \Omega_{C,\eta}$.

Q: Why do $\mathrm{div}(\omega)$'s look any different from principal divisors?

A: They transform differently under coordinate changes.

Ex: $C = \mathbb{P}^1$; Let t be a local coord near 0; i.e. if we pick $[X_0 : X_1]$ as hom. coordinates and $0 = [0:1]$,

$$t = \frac{X_0}{X_1}$$

Let $p \in \mathbb{P}^1 \setminus \{[1:0]\}$; then the rational function $t - p$ is a local coordinate at p. But $d(t) = d(t-p)$ so

$$\mathrm{ord}_p(\omega) = 0$$

At ∞: local coordinate is $\frac{X_1}{X_0} = \frac{1}{t}$

Now, $dt = d\left(\frac{1}{u}\right) = -\frac{1}{u^2}\,du = -t^2 d\left(\frac{1}{t}\right)$

So $\mathrm{ord}_\infty(dt) = -2$.

Conclude $\mathrm{div}(dt)$ on $\mathbb{P}^1$ is $-2[\infty] = -2[1:0]$ ☺

Lemma For C a smooth curve and ω a rational differential, the quantities $\mathrm{ord}_p(\omega)$ is non-zero for at most finitely many p.

Proof essentially same as version for rational functions.

Def The canonical ~~divisors~~ class of a curve C is the class of $[\mathrm{div}(\omega)]$ for non-zero $\omega \in \Omega_{C,\eta}$ in the Picard group of C: $\ulcorner \mathrm{Pic}(C) = \mathrm{Div}(C) / \{\mathrm{div} f \mid f \in \mathcal{O}_{C,\eta}\} \lrcorner$

Apparently $\omega' = f\omega$ for $f \in k(C)$

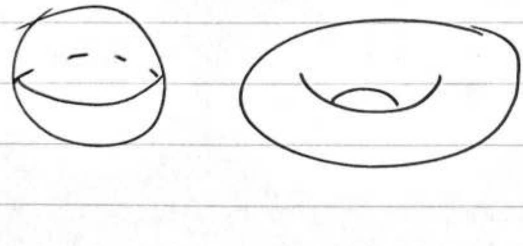
Exercise Check this is independent of ω.

Definition Let C be a smooth curve.
The genus of C is defined as $g(C) := \dim_k L(K_C)$ for $K_C = \mathrm{div}(\omega)$, $\omega \in \Omega_{C,\eta}$ non-zero.

Remarks: By the basic properties of $L(D)$, $L(D)$ is always finite dim^{l}. In particular the genus of an algebraic curve is finite.

Corollary: The genus of $\mathbb{P}^1$ is 0.
Pf As seen above, degeree of a canonical divisor is -2.
~~So~~ $L(K_C)$ is zero v.s. $\square$

Calculation If $C \subseteq \mathbb{P}^2$ is a smooth cubic; then $g(C) = 1$.
In particular, smooth plane cubics are not isomorphic to $\mathbb{P}^1$.



Guided (algebraic) meditation

0 : preparation

- $\mathbb{P}^1$: take $\omega = dt$; $\mathrm{ord}_p(dt) = 0$ for $p \neq \infty$
  $\mathrm{ord}_\infty(dt) = -2$

Thus $g(\mathbb{P}^1) = \dim_k L(-2[\infty]) = 0$.

- $C = \mathbb{V}(F(X,Y,Z)) \subseteq \mathbb{P}^2$ degree d

  $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ coordinates on $\mathbb{A}^2_{xy} \subseteq \mathbb{P}^2$

I. Notice the equation $f(x,y) = F(x,y,1) = 0$

On the curve: $f_x\, dx + f_y\, dy = 0$

II. Notice the smoothness of C

f_x & f_y cannot simultaneously vanish on C in this patch

dx vanishes $\Rightarrow$ f_y vanishes to the same order

III. Build a differential

$$\omega = \frac{dx}{f_y} = \frac{-dy}{f_x}$$ it has neither zeros nor poles in this patch

IV Embrace the part at infinity $\{X \neq 0\}$ say

write $z = \frac{Y}{X}$, $w = \frac{Z}{X}$.

the curve is cut out in $\mathbb{A}^2_{zw}$ by $F(1,\overset{z}{\cancel{y}},\overset{w}{\cancel{z}}) = g(z,w)$

we have $f(x,y) = x^d g\left(\frac{y}{x}, \frac{1}{x}\right)$ ✓

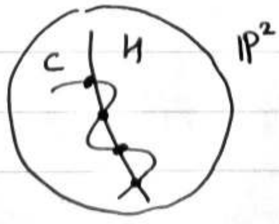
V Calculate like the locus

This is your task: change coords on $\frac{dx}{f_y} = -\frac{dy}{f_x}$ into (z,w)-coordinates; use that if

$g_z(p) \neq 0$ then $z - z(p)$ is a local parameter $\leftarrow$ sus!

Deduce:

Theorem If $C = \mathbb{V}(F_d) \subseteq \mathbb{P}^2$ then $K_C = (d-3)H$ for H a hyperplane section.



i.e. $(d-3)(H \cap C)$

Corollary A smooth plane curve of degree 3 in $\mathbb{P}^2$ has genus 1

Pf $K_C = (3-3)H = 0$.

And $L(0) = 1 = \{ f \in \mathcal{O}_{C,\eta} \mid \mathrm{div}(f) = 0 \}$

$= \{\text{const functions}\}$ $\square$

Corollary A plane curve of degree $d \geq 3$ is not isomorphic to $\mathbb{P}^1$.

Pf $L(K_C)$ is non-zero by the above calculation. $\square$

Riemann-Roch Theorem "duality"

$\sim$ Gauss-Bonnet, Atiyah-Singer index, ...

Let C be a smooth proj alg curve

The R-R problem: given a divisor D determine the dim^n of $L(D)$

"Understand the geometry of C"

Theorem (Riemann-Roch) Let g be the genus of C.

Let D be a divisor of degree d. Then

$$\ell(D) - \ell(K_C - D) = d - g + 1$$

Riemann: $\ell(D) \leq d - g + 1$

$\ell(K_C - D)$: correction term to a naive guess

Note: If $C = \mathbb{P}^1$ & D is effective then the inequality is an equality.

On $\mathbb{P}^1$: if D is effective of degree $n \geq 0$

then $D \sim n \cdot [\infty]$

i.e. $D - n \cdot [\infty] = \mathrm{div}(f)$, $f \in \mathcal{O}_{C,\eta}$

we calculated $L(n[\infty]) = \langle 1, t, t^2, \ldots, t^n \rangle$ for t local at zero

☒ Notation From now on write K for any divisor of the form $\mathrm{div}(\omega)$, $\omega \in \Omega_{C,\eta}$

Corollary Degree of K is $2g-2$ for g the genus of C

Proof Riemann-Roch with $D = K$. Calculate

$$\deg K = \ell(K) - \ell(K-K) + g - 1$$
$$= 2g - 2. \qquad \square$$

Corollary A plane curve C of degree d has genus $\binom{d-1}{2}$

Pf Degree of K is the degree of $(d-3)H$ where H is hyperplane slice. $\text{degree}(H) = d \Rightarrow \text{degree } (d-3)H$ is $d(d-3) = 2g-2$

$$\text{arithmetic} \Rightarrow g = \binom{d-1}{2}. \qquad \square$$

Remark If C is a curve are the $\mathbb{C}$ numbers, by $C \subseteq \mathbb{P}^N$, view C as having a $\mathbb{R}$ manifold structure. Then C is homeo to a genus g orientable surface.

Remark Curvature of genus g surface as above is +ve for $g=0$,

0 for $g=1$

$-$ve for $g \geq 2$

Notice degree of K_C is $2g-2$

& is $-$ve for $\mathbb{P}^1$, 0 for $g=1$, & +ve for $g \geq 2$.

Corollary D is divisor on C; then if $\deg(D) > 2g-2$ then $\ell(D) = \deg(D) - g + 1$

«No error term for large degree divisors»

Proof Apply R-R; $\ell(K_C - D)$ is zero; $K_C - D$ has negative degree. $\square$

Curves of genus 1 "Elliptic curves"

Let C be a curve of genus 1. Fix a point $p_0 \in C$.

Fix p, q points on C. Then examine

$$\ell(p+q-p_0) \overset{!}{=} 1$$

Corollary of RR If ~~x~~ D is a divisor of +ve degree on a genus 1 curve, then $\ell(D) = \text{degree}(D)$

so degree $(p+q-p_0) = 1$.

Since $\ell(p+q-p_0)$ is exactly 1, there is a unique effective degree 1 divisor $r \in C$ (i.e. a point) such that

$$p+q-p_0 \sim r$$

Declare: $p \oplus_C q = r$ to be a group law on the points of C (depending on C, p_0)

The pair (C, p_0) is therefore equipped w/ a group law.

Elliptic curves.

~~Theorem~~ For any curve C, we constructed a group:

$$\mathrm{Div}(C) = \bigoplus_{p \in C} \mathbb{Z}[p] / \sim \leftarrow \text{lin equiv}$$

i.e. the Picard group

$$\underset{\boxed{g(C)=1}}{\mathrm{Pic}(C)} \supseteq \mathrm{Jac}(C) = \{ [D] \in \mathrm{Pic}(C) \mid \text{degree}(D) = 0 \}$$

Theorem The operation $\oplus_C$ makes (C, p_0) into an abelian group with identity p_0.

Moreover, there is a natural isomorphism

$$C \longrightarrow \mathrm{Jac}(C)$$

Alg Geom

Last Time C : genus g, D : degree d

$$\underbrace{\ell(D) - \ell(K-D)}_{\text{geometry}} = \underbrace{d-g+1}_{\text{topology}}$$

Consequences : · $\deg K_C = 2g-2$

DEGREE GENUS · genus of $\mathbb{V}(F_d) \subseteq \mathbb{P}^2$ $\binom{d-1}{2}$

ELLIPTIC CURVES · if (C, P_0) w/ $g(C)=1$; $P_0 \in C$ we get a group law

Elliptic Curve : E is an elliptic curve (suppress P_0 from notation)

Basic fact : Let D be a divisor on a curve C s.t. $\dim_k L(D) = 1$. Then there is a unique effective divisor equivalent to D.

Equivalent : $E_1 \sim E_2 \iff E_1 - E_2 = \operatorname{div}(f)$, $f \in \mathcal{O}_{V,\eta}$

(Why?) There exists $F \in L(D)$

$$L(D) = \{ f \in \mathcal{O}_{C,\eta} \mid \operatorname{div} f + D \geqslant 0 \}$$

But $\dim L(D) = 1 \Rightarrow L(D) = \langle F \rangle$

So $\operatorname{div}(F) + D$ is the uniqe effective divisor equivalent to D.

Last time

Cor of RR If D is a divisor on C w/ $g(C)=1$, with $\deg(D) > 0$, then $\ell(D) = \deg(D)$.

Cor-Cor-RR Given $P, Q \in E$, there is $R \sim P + Q - P_0$

it is indeed a group → $\Rightarrow$ define the group law on E: $P \oplus_E Q = R$ as above.

Previously : C any curve, consider

$$\operatorname{Jac}(C) = \operatorname{Div}^0(C) / \text{Principal divisors}$$

Theorem : If E is an elliptic curve, there is a group isom

$$\beta : E \to \operatorname{Jac}(E)$$
$$P \mapsto [P - P_0]$$

Proof injective If $P, Q \in E$ s.t.

$\beta(P) = \beta(Q)$ then $P - P_0 \sim Q - P_0$

so $P \sim Q$

but $\ell(P) = 1$ so $P = Q$ (by Basic Fact)

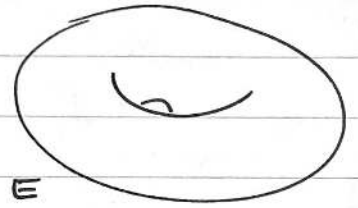
surjective pick $D \in \mathrm{Jac}(E)$. Then $D + P_0$ has degree 1

so $\ell(D + P_0) = 1$

so there is an effective divisor of degree 1, i.e. a point R s.t. $\beta(R) = R - P_0 \sim D$.

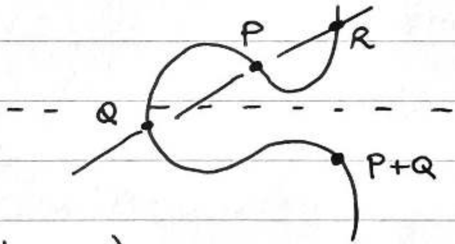
homomorphism exercise. $\square$

$g = 1$:



$\begin{cases} S^1 \times S^1 \\ \mathbb{C}/\mathbb{Z}^2 \end{cases}$

Ancients: Take C a plane smooth cubic



Theorem: (char $k \neq 2$)

Let E be $\mathbb{V}(F) \subseteq \mathbb{P}^2$ where $F(X_0, X_1, X_2) = X_0 X_2^2 - \prod_{i=1}^{3} (X_1 - \lambda_i X_0)$

$\lambda_i \neq \lambda_j$ for $i \neq j$

$P_0 = [0:0:1] \in E$

Then in the group law on E:

$$P \oplus_E Q \oplus_E R = O_E \iff P, Q, R \text{ colinear}$$

zero of the group i.e. P_0

Remark In L23, we will see that all genus 1 curves are isomorphic to plane cubics.

<u>Proof of Thm</u> (drop subscript E from $\oplus_E$)

$$P \oplus Q \oplus R = O_E \iff P+Q+R \sim 3P_0$$

The latter holds iff there is a function $f \in \mathcal{O}_{E,\eta}$ s.t.

$$\mathrm{div}(f) = P+Q+R-3P_0$$

$L(3P_0)$ has dimension 3 and contains the functions

CHECK!!

$1, \; X_1/X_0, \; X_2/X_0 \longrightarrow$ linearly indep

$\frac{X_0}{X_2}$ has zero order 3

So $L(3P_0) = \langle 1, \frac{X_1}{X_0}, \frac{X_2}{X_0} \rangle$

$\frac{X_1}{X_2}$ is local param

Therefore, such an f exists iff $f = \frac{G}{X_0}$ with G linear and $\mathrm{div}(G) = P+Q+R$. $\square$

"Chord-and-Tangent Process"

<u>Higher genus curves</u>

Few basic routes to higher genus curves:

1. Plane curves $\longrightarrow \binom{d-1}{2}$ (special values)

2. A degree (d_1, d_2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $(d_1-1)(d_2-1)$.
   [Takes all values]

[Proof on ExSh IIII] $\Rightarrow$ there exists a smooth curve of every genus

3. Non-constant maps between curves

Let $C \xrightarrow{\varphi} C'$ be a non-constant morphism of curves.
Given $P \in C$ & $Q = \varphi(P)$, we define the ramification of φ at P: $\mathrm{ord}_P(\varphi^* t_Q)$ where t_Q is a local coord at Q

Denote the ramif^n of φ at P as $e_P(\varphi) = e_P$ (when φ understood)

<u>Theorem</u> (Riemann-Hurwitz) Let $C \to C'$ be a morphism of curves of degree d. Then

$$2g(C) - 2 = d(2g(C')-2) + \sum_{P \in C} (e_P - 1)$$

[Remark $\exists$ quick proof via triangulations in topology]

$$C \to C'$$

Preliminaries

1. Write $\mathcal{O}_{C',\eta} / k(t)$ as a finite extension.

2. $\mathcal{O}_{C,\eta}$ is also a finite extension of $k(t)$
Moreover, the differential dt spans
$$\Omega_{C',\eta} \text{ and } \Omega_{C,\eta} \text{ by previous results}$$

3. Given $\varphi: C \to C'$ we get $\varphi^*: \Omega_{C',\eta} \to \Omega_{C,\eta}$
$$f\,dt \mapsto \varphi^* f \cdot d(\varphi^* t)$$

4. Given $C \to C'$, points $P \mapsto Q$, given t_Q a local coordinate,

i guess we didn't specify ord(ω) is well-defined?. i.e. change t_P

$$d(\varphi^* t_Q) = d(t_P^{e_P}) = \cancel{e_P} \; e_P t_P^{e_P - 1} dt_P$$

Gives a way of comparing $\operatorname{ord}_Q(dt_Q)$, $\operatorname{ord}_P(d\varphi^* t_Q)$

Already seen $\begin{matrix} C \to C' \\ P \mapsto Q \end{matrix}$ and $f \in \mathcal{O}_{C',\eta}$

s.t. $\operatorname{ord}_Q(f) = m$, then $\operatorname{ord}_P \varphi^* f = e_P m$

Proof of Riemann-Hurwitz

$$2g(C) - 2 = \text{degree of } \operatorname{div}(\omega) \quad \text{for any } \omega \in \Omega_{C,\eta}$$

Choose $\bar{\omega} \in \Omega_{C',\eta}$ a differential and take $\omega = \varphi^* \bar{\omega}$

Now compute degree of $\overset{\operatorname{div}}{}\omega$ via the preliminaries. $\square$

Hyperelliptic Curves A curve C of genus $g(C) \geq 2$ is called hyperelliptic if it admits a degree 2 morphism to $\mathbb{P}^1$.
Hyperelliptic curves of all $g \geq 2$ exist.
They are the simplest higher genus curves.

Proposition All genus 2 curves are hyperelliptic

Pf Take C s.t. $g(C) = 2$.

K_C divisor of degree 2

$\ell(K_C) = 2$ by RR

Pick a basis $L(K_C) = \langle 1, F \rangle$

and consider
$$C \longrightarrow \mathbb{P}^1$$
$$p \longmapsto [1 : F(p)]$$

This 2:1 & non-constant. $\square$

b/c it has ≥ 2 zeros ??
idk I guess ...

<u>Riemann–Hurwitz</u> $C \xrightarrow{\pi} C'$ a morphism of degree d:

$$2g(C)-2 = d(2g(C')-2) + \sum_{P \in C} (e_P - 1)$$

Just the statement that you can pullback differentials.

<u>Corollary</u> If $C \to C'$ non-constant then

$$g(C) \geq g(C')$$

No map from $\mathbb{P}^1$ to higher genus curves.

<u>Curves in $\mathbb{P}^N$</u>: always assume $C \subseteq \mathbb{P}^n$ is not contained in a hyperplane.

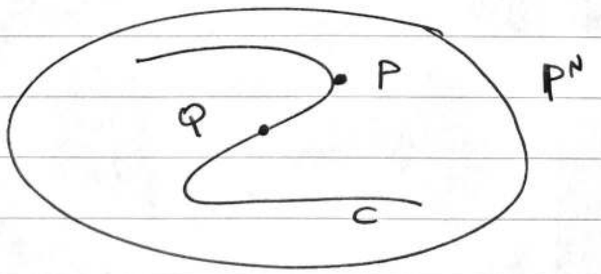
Given $C \hookrightarrow \mathbb{P}^N$, let $D = \mathrm{div}(X_0)$ where X_0 is a homog coordinate. We get the following map of k-vector spaces:

$$\left\{ \begin{matrix} \text{linear forms in} \\ X_0, \dots, X_N \end{matrix} \right\} \longrightarrow L(D)$$

$$F \longmapsto F/X_0$$

$C \hookrightarrow \mathbb{P}^N$ smooth irreducible. We make 2 observations:

(1) Given $P, Q \in C$, there exist hyperplanes H_1 & H_2 in $\mathbb{P}^N$ s.t. $P \in H_1$, $Q \notin H_1$; $Q \in H_2$, $P \notin H_2$.



(2) Given a point $P \in C$, there is a tangent (((line))) L_P to C at P. There exists a hyperplane $H \subseteq \mathbb{P}^N$ that contains P but doesn't contain L_P.

Let's turn (1) & (2) into statements about $L(D)$ using the linear map above:

(1) $\Longrightarrow$ for $P, Q \in C$ and $D = \mathrm{div}(X_0)$

[Assume $P, Q \notin V(X_0)$] then

$$\ell(D-P-Q) \leq \ell(D) - 2 \qquad \text{(general fact)}$$

by an earlier propⁿ, in fact

$$\ell(D-P-Q) = \ell(D) - 2.$$

(2) $\Rightarrow$ There exists a linear form G on $\mathbb{P}^N$ s.t. the rational function G/X_0 vanishes at P to order 1.

i.e. $\ell(D-2P) = \ell(D) - 2$

Taken together, these suggest a condition on a divisor D.

$\boxed{\text{CONDITION } *}$ Given a curve C and a divisor D, condition $*$ is satisfied if for any $P, Q \in C$ (not necessarily distinct)

$$\ell(D-P-Q) = \ell(D) - 2.$$

Given C & D as above, choose a basis $f_0, \dots, f_N$ for $\ell(D)$ as a k-vector space.

Denote $\varphi_D : C \longrightarrow \mathbb{P}^N$

$$P \longmapsto [f_0(P) : \dots : f_N(P)]$$

« The map to Projective space determined by D »

The map is unique up to change of coordinates on $\mathbb{P}^N$. ✓

~~Theorem~~ Def φ_D is an embedding if C is isomorphic to $\mathrm{Im}(\varphi_D)$

Theorem Given C & D as above, the morphism φ_D is an embedding iff D satisfies $\boxed{\text{condition } *}$

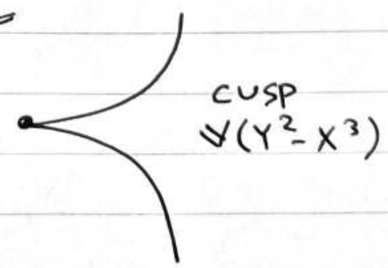
Injective Non-Isomorphism is given by the image of $\mathbb{A}^1$ under $\mathbb{A}^1 \to \mathbb{A}^2$

$t \mapsto (t^2, t^3)$ image is $\mathbb{V}(Y^2 - X^3)$

we know image of φ has coord ring not IM to $k[t]$.

· Explains the need for (2) in condition $*$

Corollary If D is a divisor of degree larger than $2g$, then D satisfies condⁿ $*$



Pf Apply Riemann-Roch. $\square$

Examples

g=0 Take $C = \mathbb{P}^1$; $D = n[\infty]$. Then

$L(D) = \langle 1, t, t^2, \dots, t^n \rangle$ where t is a local coordinate near 0. So we get

$$\mathbb{P}^1 \to \mathbb{P}^N$$
$$t \mapsto (1, t, t^2, \dots, t^n)$$
$$[x_0 : x_1] \mapsto [x_1^n ; x_1^{n-1} x_0 : \dots : x_0^n]$$

* every divisor of degree n on $\mathbb{P}^1$ is equivalent to this D.
* on $\mathbb{P}^1$, every divisor D with $\deg(D) > 0$ satisfies cond^n *

g=1 Let E be an elliptic curve and P_0 the distinguished point. Then inspect $L(nP_0)$ for various $n \geq 1$

R-R : $\ell(nP_0) = n$

$L(P_0) \cong k$ constant functions

$L(2P_0) \cong k^2$, is $\langle 1, x \rangle$ for x non-constant
$\mathrm{ord}_{P_0}(x) = -2$

$L(3P_0) = \langle 1, x, y \rangle$ for y non-constant
$\mathrm{ord}_{P_0}(y) = -3$

$$\left\{\begin{array}{l} E \to \mathbb{P}^2 \\ P \mapsto [1 : X(P) : Y(P)] \end{array}\right.$$

what can we say about this?

$L(4P_0) = \langle 1, x, y, x^2 \rangle$

$L(5P_0) = \langle 1, x, y, x^2, xy \rangle$

$L(6P_0) = \langle \underbrace{1, x, y, x^2, xy, y^2, x^3}_{\substack{\text{7 vectors in} \\ \text{a 6 dim } k\text{-VS}}} \rangle$

$\Rightarrow$ image of $\begin{array}{l} E \to \mathbb{P}^2 \\ P \mapsto [1 : X(P) : Y(P)] \end{array}$ lies on vanishing locus of an equation of the form

$$\boxed{y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6}$$

This is the Weierstrass equation for the elliptic curve E.

Why is the image of $\varphi_{3P_0} : E \to \mathbb{P}^2$ exactly the sol^n set to this eq^n?

This follows from our embedding criterion condition $*$.

<u>Remark</u> Using the above & more algebra,
in char $k \neq 2$, straightforward to show every E elliptic curve is given by (projectiviation of) the zero locus of an eq^n of the form

$$Y^2 = X(X-1)(X-\lambda) \quad , \quad \lambda \notin \{0,1\}$$

Legendre form of the Elliptic curve.

<u>Higher Genus Curves</u>

<u>Fact</u> $\exists$ a curve of every genus; e.g. (d_1, d_2)-degree curve in $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $(d_1-1)(d_2-1)$

<u>Much harder</u> In high genus there exist curves C of genus g that are not of the above form.

<u>How to embed a curve of genus ≥ 2</u>

$\deg K_C = 2g-2$; $g \geq 2 \Rightarrow \deg K_C$ is +ve

$L(K_C)$ has dim g

<u>Theorem</u> If C is a curve of genus g, then the morphism $\boxed{\varphi_K : C \to \mathbb{P}^{g-1}}$ is an embedding iff C is non-hyperelliptic!

<u>Proof</u> Apply R-R:

Suppose $\varphi_K : C \to \mathbb{P}^{g-1}$ is not an embedding; so fails condition $*$.

So there exist $P, Q \in C$ s.t. $\ell(\cancel{D}^{K} - P - Q) \geq g-1$

By R-R, on $D = P+Q$,

(!) $\Rightarrow$ $\ell(D) = 2$, so get a morphism $C \xrightarrow{\varphi_D} \mathbb{P}^1$
which is degree 2 since $\varphi_D^{-1}(p) = \{P, Q\}$. $\square$

<u>What about hyperelliptic?</u>

$0, K_C, 2K_C, \dots$

<u>Theorem</u> Take $D = 3K_C$ for C a curve of genus ≥ 2.
Then φ_D is an embedding

Riemann (1857): Riemann imagines the moduli space of genus g curves

<u>Observe</u> We have placed every genus g curve into the same projective space.

* Non-examinable *

Suppose X is a projective variety defined by homogeneous polynomials with integral coefficients.

$\{F_1, \dots, F_r\}$ homogeneous polys in $\mathbb{Z}[\underline{x}]$

World I: Let $X_{\mathbb{C}}$ be the projective variety $X = \mathbb{V}(F_1, \dots, F_r) \subseteq \mathbb{P}^N_{\mathbb{C}}$

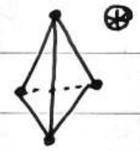
$X \subseteq \mathbb{P}^N_{\mathbb{C}} = \dfrac{\mathbb{C}^{N+1} \setminus \{0\}}{\mathbb{C}^*}$ consider $X_{\mathbb{C}}$ as a real topological space

Access: Is $X_{\mathbb{C}}$ connected?

path connected?

$\left\{ \begin{array}{l} \text{what is its Euler characteristic?} \\ \text{what are its Betti numbers?} \end{array} \right.$

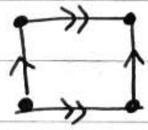
e.g. If X is $\mathbb{P}^1$,

$X_{\mathbb{C}} =$  $\underset{\text{homeo}}{\cong}$  ⊛

Extract from ⊛ a number:

$\sum (-1)^i \# \text{faces of dim } i \rightsquigarrow 2$

If X is an elliptic curve,

$X_{\mathbb{C}} =$  i.e. 

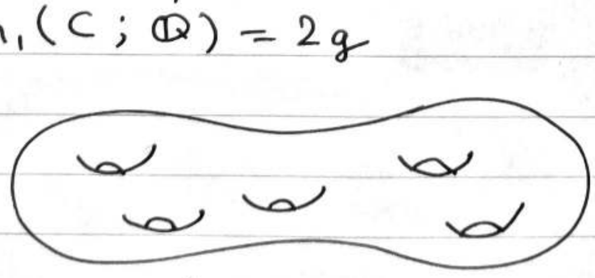
*: Topological Euler characteristic

Betti Numbers: dimensions of the homology groups $H_i(X_{\mathbb{C}}; \mathbb{Q})$

For a curve C of genus g these Betti numbers:

$\left\{ \begin{array}{l} h_0(C; \mathbb{Q}) = 1, \quad h_2(C; \mathbb{Q}) = 1 \\ h_1(C; \mathbb{Q}) = 2g \end{array} \right.$

C genus g



World 2 X mod p^n where p is a prime

$X(\mathbb{F}_{p^n}) = \{ \mathbb{F}_{p^n}\text{-solutions to the poly system defining } X \}$

FINITE

$X = \mathbb{A}^1$ then $X(\mathbb{F}_{p^m}) = \mathbb{F}_{p^m}$

$\# X(\mathbb{F}_{p^m}) = p^m$

$X = \mathbb{P}^1$ then $X(\mathbb{F}_{p^m})$ has size (p^m+1)

$X = \mathbb{P}^2$ $\quad \# X(\mathbb{F}_{p^m}) = \underset{\text{pt}}{1} + \underset{\mathbb{A}^1}{p^m} + \underset{\mathbb{A}^2}{p^{2m}}$

Take $\mathbb{F}_q$ to be a finite field

Given $X \rightsquigarrow N_m = \# X(\mathbb{F}_{q^m})$

Hasse-Weil Zeta Function :

$$Z_X(t) = \exp\left(\sum_{m} N_m \frac{t^m}{m}\right) \in \mathbb{Z}[[t]]$$

Weil Conjuctures [for curves conjectured by E. Artin] ??

(1) [RATIONALITY] For X a smooth variety (connected)

The function $Z_X(t)$ is a rational function in t.

$\Rightarrow$ The number of solutions for $m \gg 0$ is predictable from a finite set of initial values.

(2) [FUNCTIONAL EQN]

$$Z_X\left(\frac{1}{q^n t}\right) = \pm q^{nE/2} \cdot t^E Z_X(t)$$

$n = \dim X$ $\quad$ E is the topological Euler char of $X_{\mathbb{C}}$

(3) [RIEMANN HYPOTHESIS]

$$Z_X(t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

where P_i are polys, $\quad P_0(t) = 1-t, \quad P_{2n}(t) = 1-q^n t$

and for remaining i, $\quad P_i(t) = \prod_j (1-\alpha_{ij} t)$

where α_{ij} are algebraic integers $|\alpha_{ij}| = q^{i/2}$.

Substitute if $t = u^{-s}$, u formal symbol, s parameter
The zeros of $P_i(t)$ occurs at values of s with real part $\frac{i}{2}$.
(4) [BETTI NUMBERS]
The polynomials $P_i(t)$ have degree equal to the Betti numbers of $X_{\mathbb{C}}$.
$\rightsquigarrow$ If X is a curve then
$$Z_X(t) = \frac{P_1(t)}{(1-t)(1-qt)} \quad \text{where } P_1(t) \text{ has degree } 2g$$

The Weil conjectures minus RH were proved by Grothendieck in the 60's.
The RH was proved by Deligne.

For curves, we can sketch the proof of <u>rationality</u>:
1. Numbers N_m are determined by the first few
Relate $N_m = \# X(\mathbb{F}_{q^m})$ to the number of points over $\mathbb{F}_q$ on a different variety.
$$\mathrm{Sym}^k(X) = \overbrace{X \times \cdots \times X}^{k \text{ times}} / S_k$$

Points of $\mathrm{Sym}^k X$ are $\sum a_i p_i$ where $\sum a_i = k$
The function $Z_X(t) = \sum_k \# \mathrm{Sym}^k(X)(\mathbb{F}_q)\, t^k$

2. For k large, $k > 2g-2$, there is $\simeq$
For any k: $$\mathrm{Sym}^k(X) \xrightarrow{AJ} \underset{-k[p_0]}{\mathrm{Jac}(C)} \underset{\sim}{=} \frac{\{\sum a_i p_i \mid a_i \in \mathbb{Z}\}}{\mathrm{Princ}(X)}$$
<u>Fact</u> $\mathrm{Jac}(C)$ & $\mathrm{Sym}^k(X)$ are booth proj alg varieties
Given a divisor $[D] \in \mathrm{Jac}(X)$ what is $AJ^{-1}([D])$
$$\begin{array}{c} AJ^{-1}([D]) \\ \text{\rotatebox{90}{$\cong$}}? \\ \mathbb{P}(L(D)) \end{array}$$
(see Hulek)

3. For k large, $k > 2g-2$, the morphism
$$\mathrm{Sym}^k(X) \longrightarrow \mathrm{Jac}(X)$$
is a "piecewise product with $\mathbb{P}^N$"

i.e. we can cover $Jac(X)$ by U_i affine s.t.

$$AJ^{-1}(U_i) = \mathbb{P}^{k-g} \times U_i$$

4. Done because $\mathbb{P}^n$ satisfies rationality for its Zeta f^n

<u>Weil Conjectures for $\mathbb{P}^n$:</u>

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^1 \cup pt$$

<u>Calculate</u>:

$$Z_{\mathbb{P}^n}(t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}$$

<u>In the case of $\mathbb{P}^n$:</u> All the Weil conjectures can be checked by hand.

<u>In General</u>: via Lefschetz fixed point theorem (generalisation of Brouwer fixed point theorem)

<u>Grothendieck – Weil</u> Calculate the number of points of X over $\mathbb{F}_q$ by calculating the number of fixed points of a map

$$Frob_X : X \longrightarrow X$$
$$[z_1, \dots, z_N] \mapsto [z_1^q, \dots, z_r^q]$$

▶ Étale cohomology & Grothendieck – Lefschetz trace

~ FIN ~