II Algebraic Topology

Basic question: When are two topological spaces the same? i.e. homeomorphic

$X =$  $Y =$ 

Basic strategy: Suppose there is a way of associating a group $H(X)$ to any topological space X, and associating to any continuous f^n $f: X \to Y$ a homomorphism of groups $H(f): H(X) \to H(Y)$

Then (with a few assumptions) if $X \cong Y$, we expect $H(X) \cong H(Y)$.

The extension problem Let X be a top space, $A \subseteq X$ a subspace, and $f: A \to Y$ a continuous map into a top space Y.

Does there exist a map $F: X \to Y$ with $F|_A = f$?

$$\begin{tikzcd} X \arrow[rr, dashed, "F"] & & Y \\ & A \arrow[ul, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Does there exist F s.t. $f = F \circ i$

"The diagram commutes"

Example Theorem There is no continuous function

$$F: D^n \to S^{n-1}$$

$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ n-disk

$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ (n-1)-sphere

such that the composition

$$S^{n-1} \xhookrightarrow[\text{inclusion}]{} D^n \xrightarrow{F} S^{n-1}$$

is the identity. ☺

"Proof" Construct an invariant H such that

$H(S^{n-1}) \cong \mathbb{Z}$ and $H(D^n) \cong \{0\}$.

If we had such an H, we get

$$\begin{array}{ccccc} H(S^{n-1}) & \longrightarrow & H(D^n) & \longrightarrow & H(S^{n-1}) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{Z} & \longrightarrow & \{0\} & \longrightarrow & \mathbb{Z} \end{array}$$

with composition being zero.

An additional property: H(composition of maps) = composition of Hs of maps

i.e. $H(F \circ i) = H(F) \circ H(i) = 0$

Another property: $H(\text{identity}) = \text{identity}$

Thus we can't have $F \circ \iota = \text{identity}$, as otherwise

$$H(F \circ i) = \text{identity}_{\mathbb{Z}} \neq 0_{\mathbb{Z}}$$ ▨

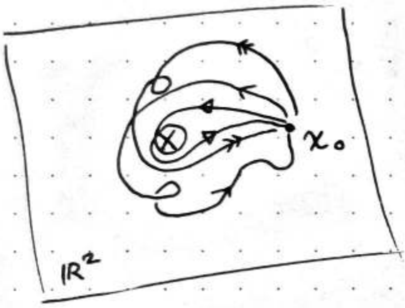
Conventions: A space means a topological space.

A map $f : X \to Y$ means a cts function between spaces.

§1 Fundamental group

Idea Fix space X and a point $x_0 \in X$. Consider loops based at x_0, i.e. maps $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

e.g. $X = \mathbb{R}^2 \setminus \{0\}$



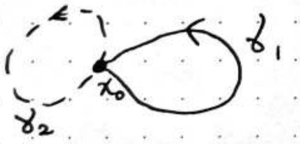
Will define a notion of equivalence of loops based on the idea of perturbing the loop continuously.

Define the fundamental group of X

$$\pi_1(X, x_0) = \{\text{ equiv classes of loops }\}$$

To multiply loops γ_1, γ_2, first traverse γ_1 and then γ_2.

Technical details



Def Let $f_0, f_1 : X \to Y$ be maps.
A homotopy between f_0, f_1 is a map

$$F : X \times I \longrightarrow Y \quad \text{where } I = [0,1]$$

such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$, $\forall x \in X$.

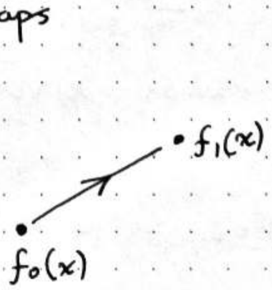
We often write $f_t(x) = F(x,t)$ for $t \in I$.

If such an F exists, we say f_0 is homotopic to f_1 and write

$$f_0 \underset{F}{\simeq} f_1 \quad \text{or} \quad f_0 \simeq f_1$$

Example If $Y \subseteq \mathbb{R}^2$ is convex, then any maps $f_0, f_1 : X \longrightarrow Y$ are homotopic, via

$$F(x,t) = t f_1(x) + (1-t) f_0(x) \in \mathbb{R}^2$$



Variation If $f_0 \underset{F}{\simeq} f_1$ with $f_0, f_1 : X \to Y$ and if $Z \subseteq X$ such that $F(z,t) = f_0(z) = f_1(z) \quad \forall z \in Z, \forall t \in I$

we say $f_0 \simeq f_1$ relative to Z.

Lemma Let $Z \subseteq X$, Y be spaces.

Then $\simeq$ (relative to Z) is an equivalence relation on the set of continuous maps $X \to Y$.

Pf Reflexive $f_0 \simeq f_0$ via $F(x,t) = f_0(x) \quad \forall t \in I$

Symmetry Given $f_0 \underset{F}{\simeq} f_1$, take $F'(x,t) = F(x, 1-t)$

Then $f_1 \underset{F'}{\simeq} f_0$. $\quad (F'_0 = F_1 = f_1, \; F'_1 = \underset{f_0}{F_0})$

Transitivity If $f_0 \underset{F_0}{\simeq} f_1$, $f_1 \underset{F_1}{\simeq} f_2$, take

$$F(x,t) = \begin{cases} F_0(x, 2t), & 0 \le t \le \frac{1}{2}, \\ F_1(x, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then F is continuous and defines a homotopy between f_0 and f_2. ▨

Recall $f : X \to Y$ is a homeomorphism if $\exists g : Y \to X$ with

$f \circ g = id_Y, \quad g \circ f = id_X$

Replace "=" with "$\simeq$"

Def A homotopy equivalence $f : X \to Y$ is a map f with a homotopy inverse, i.e. a map $g : Y \to X$ such that

$$f \circ g \simeq id_Y, \quad g \circ f \simeq id_X$$

If such an f exists, we say X is homotopy equivalent to Y.

Example  is homotopy equivalent to 

Remark All the invariants of this course are homotopy invariants, i.e. only depend on the homotopy equiv class of the space.

Example Let $*$ be the one-point space, $f : \mathbb{R}^n \to *$ the map,

$g : * \to \mathbb{R}^n, \; * \mapsto 0.$

Then $f \circ g = id_*$, so $f \circ g \simeq id_*$.

$g \circ f$ is the constant map $\mathbb{R}^n \mapsto 0$.

So $g \circ f \neq id_{\mathbb{R}^n}$.

But $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(x,t) = tx$.

$F_0 = g \circ f$, $F_1 = \mathrm{id}_{\mathbb{R}^n}$

Thus $\mathbb{R}^n$, $*$ are homotopy equivalence.

Def If X is homotopy equivalent to $*$, we say X is contractible.

Example Let $f: S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$, inclusion,

$$g: \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}, \quad x \longmapsto \frac{x}{\|x\|}$$

Then $g \circ f = id_{S^{n-1}}$,

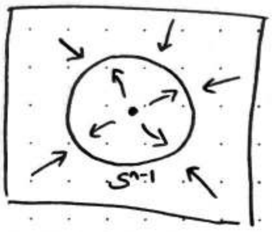
$f \circ g \neq id_{\mathbb{R}^n \setminus \{0\}}$.

Consider

$$F(x,t) = (1-t)x + t\frac{x}{\|x\|},$$

a homotopy between $f \circ g$ and $id_{\mathbb{R}^n \setminus \{0\}}$. (other way)

Thus $f \circ g \underset{F}{\simeq} id_{\mathbb{R}^n \setminus \{0\}}$ and hence f, g are homotopy equivalences, with g the homotopy inverse to f



Def Let $f: X \to Y$, $g: Y \to X$ be maps.

If $g \circ f = id_X$, we say X is a retract of Y, and g is a retraction.

If, in addition, $f \circ g \simeq id_Y$ relative to $f(X)$, then X is a deformation retract of Y.

Lemma Homotopy equivalence is an equivalence relation on spaces

Proof Reflexive, symmetric ✓

Transitive: Suppose $X \simeq Y$, $Y \simeq Z$, via pairs of maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

WTS $\begin{tikzcd} X \arrow[r, shift left, "f' \circ f"] & Z \arrow[l, shift left, "g \circ g'"] \end{tikzcd}$ induce a homotopy equivalence

Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$

Have $g' \circ f' \underset{F'}{\simeq} id_Y$, so

$$\underset{X \times I}{\overset{\cap}{(x,t)}} \longmapsto g \circ F'(f(x), t)$$

gives a homotopy between $g \circ (g' \circ f') \circ f$ and $g \circ f$

Thus $g \circ (g' \circ f') \circ f \simeq g \circ f \simeq id_X$

Hence by transitivity of homotopy equivalence of maps, we see

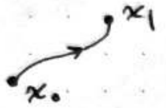
$$(g \circ g') \circ (f' \circ f) \simeq id_X$$

Similarly, $(f' \circ f) \circ (g \circ g') \simeq id_Z$ ▨

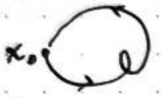
Loops and π_1

Def If X is a space, a path in X is a map $\gamma : I \to X$.

It is a path from $x_0 \in X$ to $x_1 \in X$ if $\gamma(0) = x_0$, $\gamma(1) = x_1$.

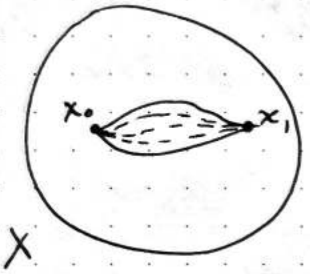


A loop in X based at x_0 is path from x_0 to x_0.



Def Let $\gamma_1, \gamma_2 : I \to X$ be paths from x_0 to x_1.

We say γ_1, γ_2 are homotopic if they are homotopic relative to $\{0,1\} \subseteq I$.

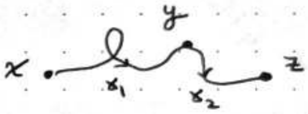


We write $[\gamma]$ for the homotopy (equivalence) class of γ.

Def X a space, $x, y, z \in X$,

γ_1 a path from x to y,

γ_2 a path from y to z.



① The concatenation of γ_1, γ_2 is the path from x to z defined by

$$(\gamma_1 . \gamma_2)(t) = \begin{cases} \gamma_1(2t) & , 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1) & , \frac{1}{2} \leq t \leq 1. \end{cases}$$

② The constant path at x is the path c_x with

$$c_x(t) = x, \quad \forall t \in I$$

③ The inverse of γ_1 is the path $\overline{\gamma}_1$ defined by

$$\overline{\gamma}_1(t) = \gamma_1(1-t),$$

a path from y to x.

Theorem Let X be a space, $x_0 \in X$.

Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops based at x_0.

Then $\pi_1(X, x_0)$ has a group structure with binary operation

$$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 . \gamma_2]$$

and identity $[c_{x_0}]$,

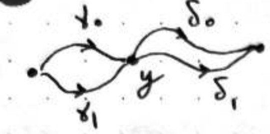
and inverse $[\gamma]^{-1} = [\overline{\gamma}]$.

Call $\pi_1(X, x_0)$ the fundamental group of X (based at x_0)

What we need to check: Need operations to be well-defined; group axioms satisfied

<u>Lemma</u> If $\gamma_0 \simeq \gamma_1$ are paths to y and $\delta_0 \simeq \delta_1$ are paths from y

Then $\gamma_0 . \delta_0 \simeq \gamma_1 . \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$

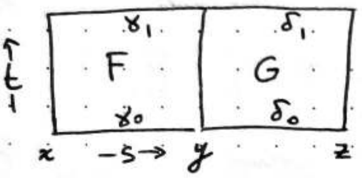


<u>Proof</u> $\gamma_0 \underset{F}{\simeq} \gamma_1$, $\delta_0 \underset{G}{\simeq} \delta_1$

Define

$$H(s,t) = \begin{cases} F(2s, t), & 0 \leq s \leq \frac{1}{2}, \\ G(2s-1, t), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Here s is the coordinate on the domain of γ_0, γ_1



If $\gamma_0 \underset{F}{\simeq} \gamma_1$ then

$F'(s,t) = F(1-s, t)$ gives

$\overline{\gamma_0} \underset{F'}{\simeq} \overline{\gamma_1}$ ▨

Thus product operation; inverse operation are well-defined.

<u>Lemma</u> Let

$$\begin{tikzcd} w \arrow[r, "\alpha"] & x \arrow[r, "\beta"] & y \arrow[r, "\gamma"] & z \end{tikzcd}$$

be paths in X.

Then ① $(\alpha . \beta) . \gamma \simeq \alpha . (\beta . \gamma)$

② $\alpha . c_x \simeq c_w . \alpha \simeq \alpha$

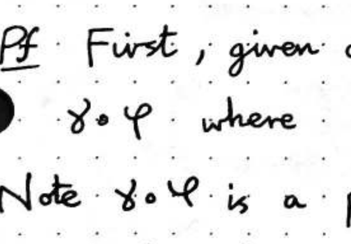
③ $\alpha . \overline{\alpha} \simeq c_w$

<u>Pf</u> First, given a path γ, a <u>reparametrisation</u> of γ is a composition $\gamma \circ \varphi$ where $\varphi : I \to I$ is a map with $\varphi(0) = 0$, $\varphi(1) = 1$.

Note $\gamma \circ \varphi$ is a path with the same endpoints as γ.

Note $\gamma \simeq \gamma \circ \varphi$ via $F(s,t) = \gamma(t\varphi(s) + (1-t)s)$

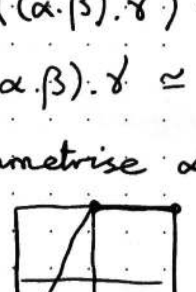
Now reparametrise $(\alpha . \beta) . \gamma$ via the function φ with graph



Then $((\alpha . \beta) . \gamma) \circ \varphi = \alpha . (\beta . \gamma)$

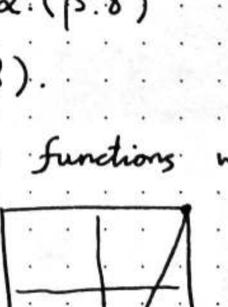
Thus $(\alpha . \beta) . \gamma \simeq \alpha . (\beta . \gamma)$.

② Reparametrise α by two functions whose graphs are either



$\alpha \simeq \alpha . c_x$

or



$c_w . \alpha \simeq \alpha$

③ Use homotopy F given by

$$F(s,t) = \begin{cases} \alpha(2s) , & 0 \leq s \leq t/2 \\ \alpha(t) , & t/2 \leq s \leq 1 - t/2 \\ \alpha(2-2s) , & 1 - t/2 \leq s \leq 1 \end{cases}$$

$t=0$ gives c_w, $t=1$ gives $\alpha . \bar{\alpha}$

Thus $\alpha . \bar{\alpha} \simeq c_w$ ▨

This lemma immediately implies the operations on $\pi_1(X, x_0)$ satisfy the group axioms.

L3.1

Example $X = \mathbb{R}^n$, $x_0 = 0$

If γ is a loop at x_0, we have $\gamma \simeq c_{x_0}$ via straight line homotopy

Thus $\pi_1(X, x_0) = \{c_{x_0}\}$.

Formal properties

~~Let~~ Lemma Let $f : X \to Y$ be a map, $x_0 \in X$, $y_0 = f(x_0)$.

Then there is an induced map
$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f\circ\gamma]$$

This is a group homomorphism. Further

① If $f \simeq f'$ rel x_0, then $f_* = f'_*$

② If $g : Y \to Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g\circ f)_*$

③ $(id_X)_* = id_{\pi_1(X,x_0)}$

Proof Well defined: If $\gamma_1 \underset{F}{\simeq} \gamma_2$, then $f\circ\gamma_1 \underset{f\circ F}{\simeq} f\circ\gamma_2$

Group hom:
$$\underset{f_*(\gamma_1.\gamma_2)}{\underset{\text{"}\|\text{"}}{f\circ(\gamma_1.\gamma_2)}} = \underset{f_*(\gamma_1)\,f_*(\gamma_2)}{\underset{\|}{(f\circ\gamma_1).(f\circ\gamma_2)}}$$

① If $f \underset{F}{\simeq} f'$ rel x_0, then for a loop based at x_0, we want a homotopy between $f\circ\gamma$ and $f'\circ\gamma$. (rel y_0) $\leftarrow$ rel $\{0,1\}$ I think

But $(s,t) \mapsto F(\gamma(s), t)$ is just that.

Thus $f_*[\gamma] = [f\circ\gamma] = [f'\circ\gamma] = f'_*[\gamma]$.

②, ③ Obvious □

Dependence on base point

Lemma Let X be a space. A path from x_0 to x_1 defines a group isomorphism
$$\alpha_\# : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$
$$[\gamma] \longmapsto [\bar{\alpha}.\gamma.\alpha]$$

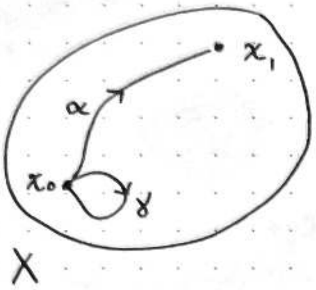
Further ① If $\alpha \simeq \alpha'$, then $\alpha_\# = \alpha'_\#$

② $(c_{x_0})_\# = id_{\pi_1(X,x_0)}$

③ If β is a path from x_1 to x_2, then
$$(\alpha.\beta)_\# = \beta_\# \circ \alpha_\#$$

④ If $f : X \to Y$ is a map with $y_i = f(x_i)$, then

$$(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$$



Proof $\alpha_\#$ is a group hom:

For loops γ, δ based at x_0,

$$\alpha_\#(\gamma) . \alpha_\#(\delta) \simeq (\bar\alpha . \gamma . \alpha).(\bar\alpha . \delta . \alpha)$$
$$\simeq \bar\alpha . \gamma . (\alpha . \bar\alpha) . \delta . \alpha$$
$$\simeq \bar\alpha . \gamma . \delta . \alpha$$
$$\simeq \alpha_\#(\gamma . \delta) \quad ✓$$

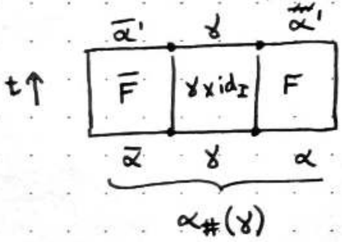
Note also $\bar\alpha_\# : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ verifies

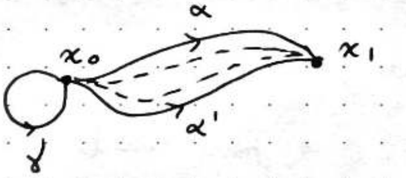
$$\bar\alpha_\# \circ \alpha_\# = id_{\pi_1(X, x_0)} ,$$

$$\left[\; \bar\alpha_\# \alpha_\# (\gamma) \simeq \bar\alpha_\# (\bar\alpha \gamma \alpha) \right.$$
$$\simeq (\alpha\bar\alpha) \gamma (\alpha\bar\alpha)$$
$$\left. \simeq \gamma \; \right]$$

So $\bar\alpha_\#$ is inverse to $\alpha_\#$, so $\alpha_\#$ is an isomorphism.

① If $\alpha \simeq_F \alpha'$, then given γ,





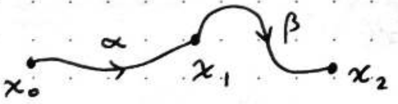
gives a homotopy between $\alpha_\#(\gamma)$, $\alpha'_\#(\gamma)$

thus $\alpha_\# = \alpha'_\#$

② Immediate because c_{x_0} is identity in $\pi_1(X, x_0)$,

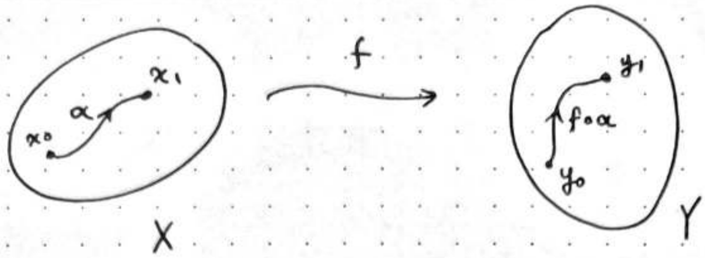
i.e. $\bar c_{x_0} . \gamma . c_{x_0} \simeq \gamma$

③



$$(\alpha . \beta)_\# (\gamma) \simeq \overline{\alpha . \beta} . \gamma . \alpha . \beta$$
$$\simeq \bar\beta . \bar\alpha . \gamma . \alpha . \beta$$
$$\simeq \bar\beta \, \alpha_\#(\gamma) \beta$$
$$\simeq \beta_\# \circ \alpha_\# (\gamma)$$

④



$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\alpha_\#"] \arrow[d, "f_*"] & \pi_1(X,x_1) \arrow[d, "f_*"] \\
\pi_1(Y,y_0) \arrow[r, "(f\circ\alpha)_\#"'] & \pi_1(Y,y_1)
\end{tikzcd}$$

WTS the diagram is commutative, i.e. $(f\circ\alpha)_\# \circ f_* = f_* \circ \alpha_\#$

$$\begin{aligned}
((f\circ\alpha)_\# \circ f_*)(\gamma) &\simeq (f\circ\alpha)_\#(f\circ\gamma) \\
&\simeq (\overline{f\circ\alpha})(f\circ\gamma)(f\circ\alpha) \\
&\simeq f\circ(\bar{\alpha}\gamma\alpha) \\
&\simeq f_*(\alpha_\#(\gamma)) \quad \checkmark
\end{aligned}$$

□

Remark If α is a loop based at x_0, then $\alpha_\#$ is conjugation by $[\alpha]$ in $\pi_1(X,x_0)$. This may not be the identity

Def If X is path connected and $\pi_1(X,x_0)$ is trivial for some (hence all) $x_0 \in X$; we say X is simply connected.

L4.1

① $f : X \to Y, \quad x_0 \in X$

get $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$

② α path in X connecting x_0 to x_1

get $\alpha_\# : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

<u>Lemma</u> Let $x_0 \in X$, $f, g : X \to Y$ with $f \underset{F}{\simeq} g$. Set $\alpha(t) = F(x_0, t)$.

α is a path from $f(x_0)$ to $g(x_0)$.

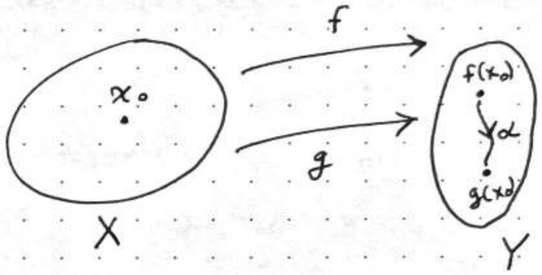
Then

$$\begin{tikzcd} & \pi_1(\overset{Y}{\cancel{X}}, f(x_0)) \arrow[d, "\alpha_\#"] \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[r, "g_*"'] & \pi_1(Y, g(x_0)) \end{tikzcd}$$

commutes,

i.e. $\alpha_\# \circ f_* = g_*$.

<u>Proof</u> Need to check for a $\overset{\gamma}{\downarrow}$ loop based at x_0

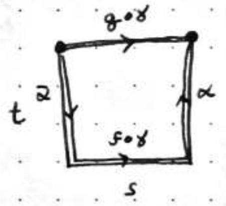


$\alpha_\# \circ f_*[\gamma] = g_*[\gamma]$

$\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha \simeq g \circ \gamma$

Let $G : I \times I$ be defined by

$G(s,t) = F(\gamma(s), t)$



Consider two paths in $\underset{s}{I} \times \underset{t}{I}$

$a(t) = (t, 1)$, $\quad b = b_1 \cdot b_2 \cdot b_3$ for

$b_1(t) = (0, 1-t)$

$b_2(t) = (s, 0)$

$b_3(t) = (1, t)$

$(G \circ \overset{a}{\cancel{\gamma}})(s) = G(s, 1) = F(\gamma(s), 1) = (g \circ \gamma)(s)$

So $G \circ a = g \circ \gamma$.

$G \circ b$: $(G \circ b_1)(s) = G(0, 1-s) = F(\gamma(0), 1-s) = F(x_0, 1-s) = \bar{\alpha}(s)$

$(G \circ b_2)(s) = G(s, 0) = F(\gamma(s), 0) = (f \circ \gamma)(s)$

$(G \circ b_3)(s) = G(1, s) = F(\gamma(1), s) = F(x_0, s) = \alpha(s)$

Thus $G \circ b = \bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$

Since $I \times I$ is convex, have a straight line homotopy $a \simeq_H b$.

Then $G \circ H$ is the homotopy we need. □

L4.2

Theorem If $f: X \to Y$ is a homotopy equivalence, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

is an isomorphism of groups for every $x_0 \in X$.

Proof Need to show f_* is a bijection.

Let $g : Y \to X$ be a homotopy inverse, i.e. $\exists$ homotopies F, G s.t.

$$\mathrm{id}_X \simeq_F g \circ f \quad , \quad \mathrm{id}_Y \simeq_G f \circ g$$

Let $\alpha(t) = F(x_0, t)$, a path from $\mathrm{id}_X(x_0) = x_0$

to $(g \circ f)(x_0)$.

Then $g_* \circ f_* = (g \circ f)_*$

$= \alpha_{\#} \circ (\mathrm{id}_X)_*$ from Lemma

$= \alpha_{\#}$

Since $\alpha_{\#}$ is an isomorphism, hence a bijection, this shows in particular that f_* is an injection.

Let $\beta(t) = G(f(x_0), t)$, a path from $f(x_0)$

to $\cancel{g}(f \circ g \circ f)(x_0)$.

Then $f_* \circ g_* = (f \circ g)_*$

$= \beta_{\#} \circ (\mathrm{id}_Y)_*$ ← kinda dodgy?

$= \beta_{\#}$,

$$\begin{tikzcd}
x_0 \arrow[r, "f_1"] \arrow[d, shift right, "\alpha_{\#}"'] & fx_0 \arrow[d, shift left] \arrow[dl, "g_*"] \\
gfx_0 \arrow[u, shift right] \arrow[r, "f_2"'] & fgfx_0 \arrow[u, shift left, "\beta_{\#}"']
\end{tikzcd}$$

so that f_* is surjective, so bijective. □

Cor Contractible spaces are simply connected.

Pf If X is contractible, $\exists\, x_0 \in X$ and a homotopy F between id_X and the constant map $X \to \{x_0\} \subseteq X$.

So $F(x, \cdot) : I \to X$ is a path from $x \in X$ to x_0.

Thus X is path-connected.

Now $\pi_1(X, x_0) \cong \pi_1(\cancel{X_+} \{x_0\}, x_0) = 0$. □

L4.3

§2 Covering spaces

Def Let $p : \hat{X} \to X$ be a map.

An open subset $\mathcal{U} \subseteq X$ is evenly covered if there exists a set $\Delta_{\mathcal{U}}$ with discrete topology and a homeomorphism

$$p^{-1}(\mathcal{U}) \overset{\psi}{\cong} \Delta_{\mathcal{U}} \times \mathcal{U}$$

such that the diagram

$$\begin{tikzcd}
p^{-1}(\mathcal{U}) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & \Delta_{\mathcal{U}} \times \mathcal{U} \arrow[dl, "(\delta{,}u) \mapsto u"] \\
& \mathcal{U} &
\end{tikzcd}$$

is commutative.

Write $\mathcal{U}_\delta := \{\delta\} \times \mathcal{U} \subseteq \Delta_{\mathcal{U}} \times \mathcal{U}$

$$\begin{tikzcd}
\{\delta\} \times \mathcal{U} \arrow[r, phantom, "\subseteq"] \arrow[dr, bend right] & \Delta_{\mathcal{U}} \times \mathcal{U} \arrow[d, "\psi^{-1}"] \\
& p^{-1}(\mathcal{U})
\end{tikzcd}$$

We can canonically identify $\Delta_{\mathcal{U}}$ with $p^{-1}(\{x\})$ for any $x \in \mathcal{U}$.

Note $p^{-1}(\mathcal{U}) \cong \coprod_{\delta \in \Delta_{\mathcal{U}}} \mathcal{U}_\delta$

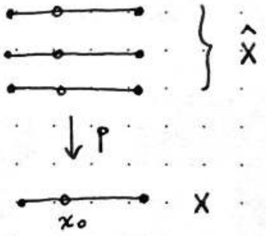
If every point in X has an evenly covered neighbourhood, then p is a covering map and $\hat{X}$ is a covering space of X.

Examples ① $X = I = [0,1]$

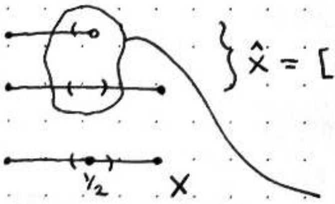
$\Delta_X = \{1, 2, 3\}$

$$\begin{tikzcd}
\hat{X} = \Delta_X \times X \arrow[d, "p"] \\
X
\end{tikzcd}$$

, $p(n;t) = t$



Something which is not a covering map :



$\hat{X} = [0,1] \amalg [0,\frac{1}{2})$

not a copy of the interval with some $\Delta_{\mathcal{U}}$.

(restriction more to do with projection commuting.)

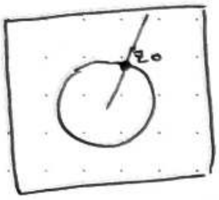
L4.4

② Let $\hat{X} = \mathbb{R}$, $X = S^1 \subseteq \mathbb{C}$, the unit circle

Take $p : \hat{X} \longrightarrow X$, $t \mapsto e^{2\pi i t}$

Claim p is a covering map

Pf If $\mathcal{U} \subsetneq S^1$ is a proper open subset,
pick some point $z_1 \in S^1 \setminus \mathcal{U}$, and choose a branch of the logarithm well-defined on $S^1 \setminus \{z_0\}$.



Write this choice as $z \mapsto \log(z)$, well-defined on $\mathcal{U}$.

Now every point $\hat{z} \in f^{-1}(\mathcal{U})$ can be uniquely written as

$$\hat{z} = k + \frac{\log z}{2\pi i} \quad \text{for some } k \in \mathbb{Z},\ z \in \mathcal{U}.$$

So $p^{-1}(\mathcal{U}) \cong \mathbb{Z} \times \mathcal{U}$

$$k + \frac{\log z}{2\pi i} \mapsfrom (k, z)$$

Thus p is a covering map.

Note we can't write $\tilde{X} = \mathbb{R}$ as $\Delta \times S^1$ for any set Δ.

③ Let $\hat{X} = X = S^1 \subseteq \mathbb{C}$ the unit circle.

Take $p(z) = z^n$ for some positive integer n.

Claim $p : S^1 \longrightarrow S^1$ is also a covering map

Choose a local branch for $\sqrt[n]{\ }$, and then can uniquely writen

$$\hat{z} = e^{2\pi i k/n} \cdot \sqrt[n]{z}$$

④ Let $\hat{X} = S^2$, $G = \mathbb{Z}/2\mathbb{Z}$ acts on S^2 via the antipodal map.

$X = \hat{X}/G$ is called the real projective plane $\mathbb{R}\mathbb{P}^2$, and the quotient map $p : \hat{X} \to X$ is a covering map.

Pf If $x \in \hat{X}$, take $\mathcal{U}$ to be a suff small neighbourhood of x such that $(-\mathcal{U}) \cap \mathcal{U} = \emptyset$

Let $V = p(\mathcal{U})$. Then $p^{-1}(V) = \mathcal{U} \cup (-\mathcal{U})$,
so V is evenly covered.

L5.1

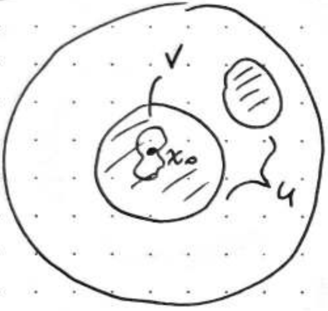
Def A covering map $p : \hat{X} \to X$ is said to be n-sheeted if $\# p^{-1}(x) = n$ (this includes the case $n = \infty$), $\forall x \in X$. We call n the degree of p.

Lifting properties [key for connecting the fundamental group to covering spaces]

Def Let $p : \hat{X} \to X$ be a covering map, $f : Y \to X$ a map.
A lift of f to $\hat{X}$ is a map $\hat{f} : Y \to \hat{X}$ with $f = p \circ \hat{f}$,

i.e.
$$\begin{tikzcd} & \hat{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$
is commutative.

Def A space X is locally path connected if $\forall x \in X$, and $\mathcal{U}$ an open nbgd of x, then there exists a path connected nbgd V of x with $V \subseteq \mathcal{U}$.



Example (never mind) perverse

Lemma "Uniqueness of lifting"

Let $p : \hat{X} \to X$ a covering map and $\hat{f}_1, \hat{f}_2 : Y \to \hat{X}$ lifts of the same $f : Y \to X$, with Y connected and locally path connected.

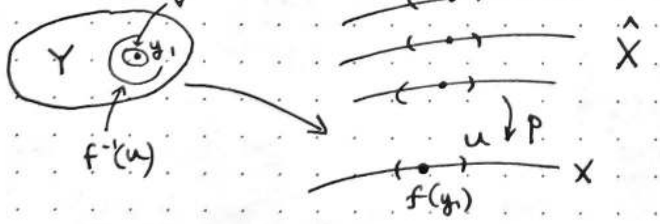
If $\exists\, y_0 \in Y$ s.t. $\hat{f}_1(y_0) = \hat{f}_2(y_0)$ then $\hat{f}_1 = \hat{f}_2$. ☺

Proof We will show the set

$$S = \{ y \in Y \mid \hat{f}_1(y) = \hat{f}_2(y) \} \text{ is } Y,$$

by showing S is open and closed in Y. Since Y is connected and $y_0 \in S$, we must have $Y = S$. [usual argument]

Let $y_1 \in Y$ be arbitrary, let $\mathcal{U}$ be an evenly covered open nbgd of $f(y_1)$.



i.e. $p^{-1}(u) \cong \mathcal{U} \times \Delta_u$

Let $V \subseteq f^{-1}(u)$ be a path-connected open neighbourhood of y_1.

Will show $\circledast$ $y_1 \in S \Rightarrow V \subseteq S$ (so S open)

$(\ast\ast)$ $y_1 \notin S \Rightarrow V \subseteq Y \setminus S$ (so S closed)

L5.2

Let $y \in V$, let α be a path in V connecting y_1 to y.

Then $\hat{f}_i \circ \alpha$ is a path from $\hat{f}_i(y_1)$ to $\hat{f}_i(y)$.

Note $p \circ \hat{f}_i \circ \alpha(t) \underset{\substack{\uparrow \\ \text{since } \hat{f}_i \\ \text{lifts } f}}{=} f \circ \alpha(t)$,

and $f(\alpha(t)) \in f(V) \subseteq \mathcal{U}$.

Thus $\hat{f}_i \circ \alpha$ is a path in $p^{-1}(\mathcal{U})$.

Thus $\hat{f}_i(y_1)$, $\hat{f}_i(y)$ must lie in the same path component of $\cancel{\hat{X}}$ $p^{-1}(\mathcal{U})$, and in particular lie in $\mathcal{U}_{\delta_i} = \mathcal{U} \times \{\delta_i\}$, for $\delta_1, \delta_2 \in \Delta_\mathcal{U}$.

$\circledast$ If $y_1 \in S$, then $\hat{f}_1(y_1) = \hat{f}_2(y_1)$ so $\delta_1 = \delta_2$, ~~as~~ as distinct copies of $\mathcal{U}$ are disjoint.

Thus $\hat{f}_1(y) = p_{\delta_1}^{-1} \circ f(y) = p_{\delta_2}^{-1} \circ f(y) = \hat{f}_2(y)$.

⌈ here, $p_{\delta_1} : \mathcal{U} \longrightarrow \mathcal{U}_{\delta_1}$ is homeomorphism inverse to $p|_{\mathcal{U}_{\delta_1}} : \mathcal{U}_{\delta_1} \to \mathcal{U}$ ⌋

Thus $y \in S$. Thus $V \subseteq S$.

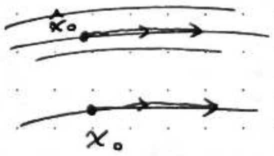
$(**)$ If $y_1 \notin S$, then $\hat{f}_1(y_1) \neq \hat{f}_2(y_1)$.

As each $\mathcal{U}_\delta$ contains a unique point of $p^{-1}(f(y_1))$, it follows that $\delta_1 \neq \delta_2$.

Since $\hat{f}_1(y)$, $\hat{f}_2(y)$ now lie in disjoint $\mathcal{U}_{\delta_1}, \mathcal{U}_{\delta_2}$, necessarily $\hat{f}_1(y) \neq \hat{f}_2(y)$ so $y \notin S$. Thus $V \subseteq Y \setminus S$. ☐

Def Let $\gamma : I \to X$ be a path from x_0, $p : \hat{X} \longrightarrow X$ a covering map. A lift of γ at $\hat{x}_0 \in \hat{X}$ is a lift $\hat{\gamma}$ of γ with $\hat{x}_0 = \hat{\gamma}(0)$.

(In particular, $p(\hat{x}_0) = x_0$)



Path-lifting Lemma Let $p : \hat{X} \longrightarrow X$ be a covering map and let $\gamma : I \to X$ be a path from x_0. For any $\hat{x}_0 \in p^{-1}(x_0)$, there exists a unique lift of γ from $\hat{x}_0$

Proof Uniqueness follows from the uniqueness Lemma.

Will show

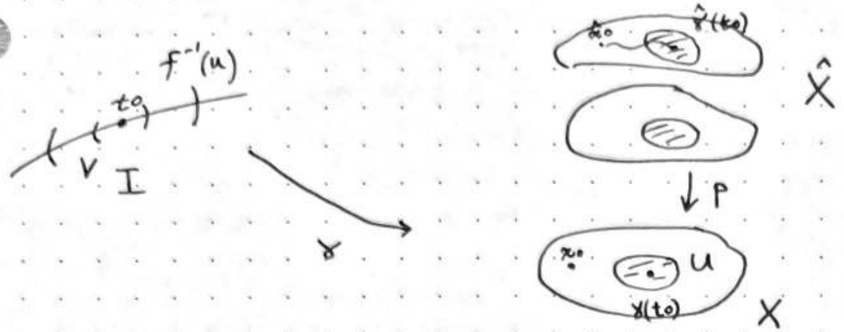
$$S = \{ t \in I \mid \gamma|_{[0,t]} \text{ lifts at } \hat{x}_0 \text{ to } \hat{X} \}$$

is open and closed.

Note $0 \in S$ so it will follow that $S = I$.

L 5.3

Let $t_0 \in I$, and let U be an evenly covered neighbourhood of $\gamma(t_0)$, and let $V \subseteq \gamma^{-1}(U)$ be an open interval containing t_0



We will show $\boxed{t_0 \in S \text{ iff } V \subseteq S}$ (uh...)

actually showed
$t_0 \in S \Rightarrow V \subset S$
$t_0 \notin S \Rightarrow V \subset I \setminus S$

Let $t \in V$ and suppose first that $t_0 \in S$.

If $t \le t_0$, then $t \in S$ (since $\gamma|_{[0,t_0]}$ lifting from $\hat{x}_0$ implies $\gamma|_{[0,t]}$ too).

If $t > t_0$, then $\hat\gamma$ makes sense, lifting $\gamma|_{[0,t_0]}$ at $\hat{x}_0$, and in particular have $\hat\gamma(t_0)$

Since $[t_0, t] \subseteq V$, we must have $\gamma([t_0, t]) \subseteq U$.

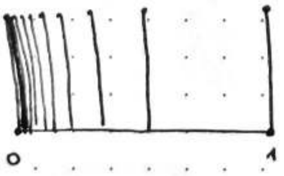
Thus the path
$$\underset{[0,t]}{\underset{\cap}{s}} \longmapsto \begin{cases} \hat\gamma(s), & 0 \le s \le t_0, \\ p_\delta^{-1} \circ \gamma(s), & t_0 \le s \le t; \end{cases}$$
where $\delta \in \Delta_U$ is chosen so $\hat\gamma(t_0) \in U_\delta$, lifts $\gamma|_{[0,t]}$.

Thus $t \in S$. Thus $V \subseteq S$.

If $t_0 \notin S$, $t \in V$ and $t \in S$, well if $t \geq t_0$ we contradict $t_0 \notin S$, and if $t < t_0$ we reverse the role of t_0, t above (for the case $t_0 \in S$), and deduce $t_0 \in S$ ⨳.

So $V \subseteq I \setminus S$ and we are done. □

« Topologist's Comb »



problems at y-axis

L6.1

Lemma If $p : \hat{X} \to X$ is a covering map and X is path connected, then p is an n-sheeted cover for some $n \in \mathbb{N} \cup \{\infty\}$

[Stronger: If $x, y \in X$, then there is a bijection between $p^{-1}(x)$, $p^{-1}(y)$]

Pf Let γ be a path in X from x to y.

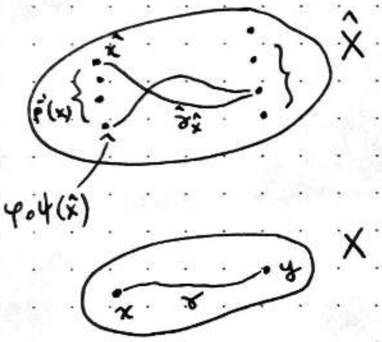
For any $\hat{x} \in p^{-1}(x)$, there is a unique path $\hat{\gamma}_{\hat{x}}$ lifting γ with start point $\hat{x}$, i.e. $\hat{\gamma}_{\hat{x}}(0) = \hat{x}$.

This gives a map
$$\psi : p^{-1}(x) \longrightarrow p^{-1}(y)$$
$$\hat{x} \longmapsto \hat{\gamma}_{\hat{x}}(1)$$

Let $\varphi : p^{-1}(y) \to p^{-1}(x)$ be the map similarly constructed using $\bar{\gamma}$, i.e. if $\hat{x} \in p^{-1}(y)$ then $\varphi(\hat{x}) = \widehat{(\bar{\gamma})}_{\hat{x}}(1)$

Claim $\psi \circ \varphi = id_{p^{-1}(y)}$, $\varphi \circ \psi = id_{p^{-1}(x)}$

E.g. Let's show $\varphi \circ \psi = id_{p^{-1}(x)}$



Note $\varphi \circ \psi(\hat{x}) = \varphi(\hat{\gamma}_{\hat{x}}(1))$
$$= \widehat{(\bar{\gamma})}_{\hat{\gamma}_{\hat{x}}(1)}(1)$$

Note that $\hat{\gamma}_{\hat{x}} \cdot \widehat{(\bar{\gamma})}_{\hat{\gamma}_{\hat{x}}(1)}$ is a lift of $\gamma \cdot \bar{\gamma}$,
and so is $\hat{\gamma}_{\hat{x}} \cdot \overline{(\hat{\gamma}_{\hat{x}})}$

By uniqueness of lifts, these two paths are equal, and hence
$$\widehat{(\bar{\gamma})}_{\hat{\gamma}_{\hat{x}}(1)}(1) = \overline{(\hat{\gamma}_{\hat{x}})}(1) = \hat{x},$$
so $\varphi(\psi(\hat{x})) = \hat{x}$. ☐

Homotopy lifting lemma Let $p : \hat{X} \to X$ be a covering map, and $f_0 : Y \to X$ a map from a locally path connected space.

Suppose $F : Y \times I \to X$ is a homotopy with $F(y, 0) = f_0(y)$, $\forall y \in Y$, and let $\hat{f}_0 : Y \to \hat{X}$ be a lifting of f_0. Then there exists a unique lifting $\hat{F} : Y \times I \to \hat{X}$ with $\hat{F}(y, 0) = \hat{f}_0(y)$, $\forall y \in Y$.

Remark The path lifting lemma is a special case, with Y is a one-point space.

LG.2

Proof of homotopy lifting

For each $y \in Y$, the homotopy F defines a path $\gamma_y(t) = F(y,t)$ from $f_0(y)$. By the path lifting lemma, each γ_y has a unique lift from $\hat{f}_0(y)$, say $\hat{\gamma}_y$.

By the uniqueness of lifts of paths, if $\hat{F}$ exists, we must have

$$\hat{F}(y,t) = \hat{\gamma}_y(t)$$

We can simply define $\hat{F}$ using this formula, so just have to show $\hat{F}$ is continuous.

Trick Will construct on open subsets of $Y \times I$ a differently constructed lift $\tilde{F}$ which a priori continuous and then show $\tilde{F}$ agrees with $\hat{F}$ on these open sets

Fix $y_0 \in Y$. The goal is to find a ngbd V of y_0 and a lifting of F to $\tilde{F}$ on $V \times I$.

For any $t \in I$, $F(y_0, t) \in X$ has an evenly covered ngbd $U_t \subseteq X$.

Then $F^{-1}(U_t)$ contains an open ngbd of $(y_0, t) \in Y \times I$ of the form

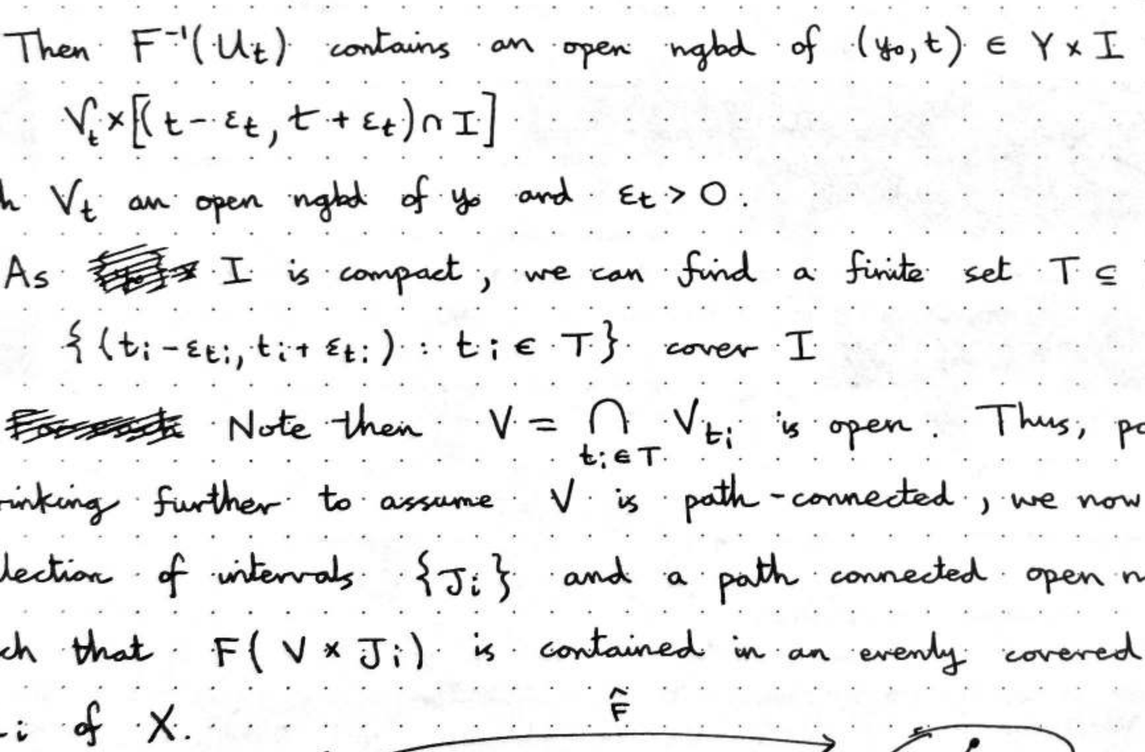
$$V_t \times [(t - \varepsilon_t, t + \varepsilon_t) \cap I]$$

with V_t an open ngbd of y_0 and $\varepsilon_t > 0$.

As I is compact, we can find a finite set $T \subseteq I$ such that

$$\{(t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}) : t_i \in T\} \text{ cover } I$$

Note then $V = \bigcap_{t_i \in T} V_{t_i}$ is open. Thus, possibly after shrinking further to assume V is path-connected, we now have a finite collection of intervals $\{J_i\}$ and a path connected open ngbd V of y_0 such that $F(V \times J_i)$ is contained in an evenly covered open subset U_i of X.



L6.3

Let $\mathcal{U}_{\delta_i}$ be the unique copy of $\mathcal{U}_i$ in $p^{-1}(\mathcal{U}_i)$ such that

$$\hat{F}(\{y_0\} \times J_i) \subseteq \mathcal{U}_{\delta_i}$$

For $(y,t) \in V \times I$, define $\tilde{F}(\overset{y}{\cancel{x}},t) = p_{\delta_i}^{-1} \circ F(y,t)$ if $t \in J_i$

where $p_{\delta_i} : \mathcal{U}_{\delta_i} \xrightarrow{\cong} \mathcal{U}_i$ is $p_{\delta_i} = p|_{\mathcal{U}_{\delta_i}}$

<u>Well-defined</u> Suppose $t \in J_i \cap J_j$

Let α be a path in V from y_0 to y, and define $\alpha_t(s) = F(\alpha(s),t)$

Then $p_{\delta_i}^{-1} \circ \alpha_t$ is a lift of α_t from $p_{\delta_i}^{-1} \circ \alpha_t(0) = p_{\delta_i}^{-1}(F(\alpha(0),t))$

$$= p_{\delta_i}^{-1}(F(y_0,t))$$

and $p_{\delta_j}^{-1} \circ \alpha_t$ is a lift of α_t from $p_{\delta_j}^{-1}(F(y_0,t))$

But $p_{\delta_i}^{-1}(F(y_0,t)) = \hat{F}(y_0,t) = p_{\delta_j}^{-1}(F(y_0,t))$

Thus $p_{\delta_i}^{-1} \circ \alpha_t = p_{\delta_j}^{-1} \circ \alpha_t$ by uniqueness of lifts,

so $p_{\delta_i}^{-1} \circ \alpha_t(1) = \tilde{F}(y,t) = p_{\delta_j}^{-1} \circ \alpha_t(1)$

$$= p_{\delta_i}^{-1} \circ F(y,t) = p_{\delta_j}^{-1} \circ F(y,t)$$

Hence $\tilde{F}$ is well-defined.

As V is open path connected and $\tilde{F}(\cdot,0)$ is a lift of f_0 that agrees with $\hat{f}_0$ at y_0, we have $\tilde{F}(y,0) = \hat{f}_0(y)$ $\forall$ $y \in V$ by uniqueness of lifts.

For each $y \in V$, $\tilde{F}(y,\cdot)$ is a lift of $\gamma_{\cancel{x}y}$ $\lceil \gamma_{\underset{y}{\cancel{x}}}(t) = F(y,t) \rfloor$ at $\hat{f}_0(y)$, thus $\tilde{F}(y,t) = \hat{\gamma}_y(t)$ by uniqueness of lifts.

Thus by def^n of $\hat{F}$, $\tilde{F} = \hat{F}$ on $V \times I$

But $\tilde{F}$ is continuous by construction, so $\hat{F}$ is also continuous. $\square$

L7.1

$p : \hat{X} \to X$ covering map

- path lifting lemma

$$\begin{tikzcd} & \hat{X} \arrow[d] \\ \gamma : I \arrow[r] \arrow[ur, dashed, "\exists \hat{\gamma}"] & X \end{tikzcd}$$

$\hat{\gamma}$ is unique given a choice at $\hat{\gamma}(0)$

- Homotopy lifting lemma

$$\begin{tikzcd} & \hat{X} \arrow[d] \\ F : Y \times I \arrow[r] \arrow[ur, dashed, "\exists \hat{F}"] & X \end{tikzcd}$$

$F(\cdot, 0) = f_0$

we assume given a lift $\hat{f}_0$ at f_0

<u>Lemma</u> Let $p : \hat{X} \to X$ be a covering map and $F : I \times I \to X$ be a homotopy of paths. Then any lift $\hat{F}$ of F is also a homotopy of paths

<u>Proof</u> Since F is a homotopy of paths, $F(0, \cdot)$ and $F(1, \cdot)$ are constant paths in X.

Thus $\hat{F}(0, \cdot)$, $\hat{\hat{F}}(1, \cdot)$ are paths lifting the constant paths, hence constant (by uniqueness). Thus $\hat{F}$ is a homotopy of paths. $\square$

<u>Applications to π_1</u>

<u>Lemma</u> Let $p : \hat{X} \to X$ be a covering map with $\hat{x} \in \hat{X}$, $x = p(\hat{x})$.

Then the induced map

$$p_* : \pi_1(\hat{X}, \hat{x}) \longrightarrow \pi_1(X, x)$$
$$[\gamma] \longmapsto [p \circ \gamma]$$

is injective.

<u>Proof</u> Suppose $[\hat{\gamma}] \in \ker p_*$, i.e. $p \circ \hat{\gamma} =: \gamma$ is homotopic to the constant path c_x.

If F is a homotopy between γ and c_x, we can lift the homotopy $\hat{F}$ from $\hat{\gamma}$. By previous lemma, this is a homotopy of paths.

$\hat{F}$ must be a homotopy between $\hat{\cancel{F}}_{\gamma}$ and $c_{\hat{x}}$. Thus $[\hat{\gamma}] =$ id elt $\square$

<u>Hence</u> We will now view $\pi_1(\hat{X}, \hat{x})$ as the subgroup $p_* \pi_1(\hat{X}, \hat{x})$ of $\pi_1(X, x)$. Note given $[\gamma] \in \pi_1(X, x)$, we get a map

[note well def]

$$p^{-1}(x) \longrightarrow p^{-1}(x)$$
$$\hat{x} \longmapsto \hat{\gamma}_{\hat{x}}(1)$$

i.e. lift γ to a path from $\hat{x}$

which is a bijection.

L7.2

Thus this defines an action of $\pi_1(X,x)$ on $p^{-1}(x)$. This is a <u>right</u> action.

We write for $\hat{x} \in p^{-1}(x)$, γ a loop based at x,

$$\hat{x} \cdot \gamma = \hat{\gamma}_{\hat{x}}(1) \quad , \text{ so that}$$

$$(\hat{x} \cdot \gamma) \cdot \delta = \hat{x} \cdot (\gamma \cdot \delta) \qquad \text{[easy check]}$$

<u>Lemma</u> Let $p : \hat{X} \to X$ be a covering map, and suppose $\hat{X}$ is path connected. Let $x \in X$. Then the map

$$\underset{\substack{\uparrow \\ \text{set of right cosets} \\ p_*(\pi_1(\hat{x},\hat{x}))\cdot\gamma}}{p_*(\pi_1(\hat{X},\hat{x})) \backslash \pi_1(X,x)} \longrightarrow p^{-1}(x)$$

$$p_*(\pi_1(\hat{X},\hat{x}))\cdot[\gamma] \longmapsto \hat{x}\cdot\gamma$$

is a bijection for any choice of $\hat{x} \in p^{-1}(x)$.

Further, this bijection satisfies

$$p_* \pi_1(\hat{X},\hat{x})\,[\gamma][\delta] \longmapsto \hat{x}\cdot(\gamma\cdot\delta)$$

$\forall$ loops γ, δ based at x.

<u>Proof</u> (The given map is well-defined)

If $[\delta] \in p_* \pi_1(\hat{X},\hat{x})$, note $\hat{x}\cdot\delta = \hat{x}$

Thus $\hat{x}\cdot[\delta][\gamma] = (\hat{x}\cdot\delta)\cdot\gamma = \hat{x}[\gamma]$

(Map is a bijection)

We use the orbit-stabiliser theorem, applied to the right action of $\pi_1(X,x)$ on $p^{-1}(x)$. The stabiliser of $\hat{x}$ is the set of loops $[\gamma]$ at x such that

$\hat{\gamma}_{\hat{x}}(1) = \hat{x}$ where $\hat{\gamma}_{\hat{x}}$ is a lift of γ from $\hat{x}$,

i.e. $\hat{\gamma}_{\hat{x}}$ is a loop, so $\gamma = p \circ \hat{\gamma}_{\hat{x}} \in p_* \pi_1(\hat{X},\hat{x})$.

Thus the stabiliser of $\hat{x}$ is $p_* \pi_1(\hat{X},\hat{x})$.

(Action is transitive)

Let $\hat{y} \in p^{-1}(x)$, $\exists$ path $\hat{\gamma}$ in $\hat{X}$ from $\hat{x}$ to $\hat{y}$. (path connected)

Then $\gamma = p \circ \hat{\gamma}$ is a loop based at x. Thus $\hat{x}\cdot\gamma = \hat{y}$.

Orbit-stabiliser theorem then gives the desired bijection $\square$

Remark The degree of $p : \hat{X} \to X$ in the lemma is just the index of the subgroup, i.e. $\deg(p) = \# p^{-1}(x) = [\pi_1(X, x) : p_* \pi_1(\hat{X}, \hat{x})]$

Example Consider $\pi_1(S^1, x)$.

We have covers $\mathbb{R} \to S^1$ which is infinite degree, and degree n covers $S^1 \to S^1$ for all $n \geq 1$.

Thus $\pi_1(S^1, x)$ is an infinite group with subgroups with every possible finite index. ⌜Can you guess it?⌟

Def If $p : \hat{X} \to X$ is a covering map and $\hat{X}$ is simply connected, then $\hat{X}$ is a universal cover of X.

e.g. $\mathbb{R} \to S^1$

Cor If $p : \hat{X} \to X$ is a universal cover, then for any choice of $\hat{x} \in p^{-1}(x)$, there is a bijection

$$\begin{aligned} \pi_1(X, x) &\longrightarrow p^{-1}(x) \\ [\gamma] &\longmapsto \hat{x} \cdot \gamma \end{aligned}$$

and the group structure is determined by

$$\hat{x} \cdot (\gamma \cdot \delta) = (\hat{x} \cdot \gamma) \cdot \delta$$

Example Have $p : \mathbb{R} \to S^1$ universal cover and hence a bijection

$$\pi_1(S^1, 1) \longrightarrow p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$$

For $n \in \mathbb{Z}$, define $\hat{\gamma}_n(t) = nt$, a path from 0 to n in $\mathbb{R}$, and let $\gamma_n = p \circ \hat{\gamma}_n$ a loop based at 1.

Clearly $\underset{p^{-1}(1)}{\underset{\cap}{0}} \cdot \gamma_n = \hat{\gamma}_n(1) = n$

Thus any loop in S^1 based at 1 is homotopic to some γ_n, and the bijection is

$$\begin{aligned} \pi_1(S^1, 1) &\longrightarrow \mathbb{Z} \\ [\gamma_n] &\longmapsto n \end{aligned}$$

Now for any $m \in \mathbb{Z}$, $m + \hat{\gamma}_n$ is a lift of γ_n from m (to $m+n$)

Thus $(0 \cdot \gamma_m) \cdot \gamma_n = m \cdot \gamma_n = m + n = 0 \cdot \gamma_{m+n}$

Thus $\gamma_m \cdot \gamma_n = \gamma_{m+n}$

Thus $\pi_1(S^1, 1) \cong \mathbb{Z}$.

L7.4

This captures the winding number of a loop.

L8.1

Applications

Theorem The identity $id_{S^1} : S^1 \to S^1$ does not extend to a map
$r : D^2 \to S^1$ (here S^1 is viewed as the boundary of D^2)
i.e. S^1 is not a retract of D^2

Pf Suppose $\iota : S^1 \hookrightarrow D^2$ inclusion.

If r exists, then $r \circ \iota = id_{S^1}$.

Since D^2 is contractible, $\pi_1(D^2, 1) = 0$, so we get a factorisation

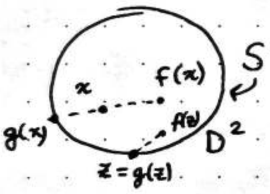
$$(id_{S^1})_* = (r \circ \iota)_* = r_* \circ \iota_* : \underset{\underset{\mathbb{Z}}{\uparrow}}{\pi_1(S^1, 1)} \xrightarrow[\iota_*]{} \underset{\underset{\text{zero}}{\uparrow}}{\pi_1(D^2, 1)} \xrightarrow[r_*]{} \underset{\underset{\mathbb{Z}}{\uparrow}}{\pi_1(S^1, 1)}$$

a contradiction. ▨

Brouwer fixed point theorem

Every map $f : D^2 \to D^2$ has a fixed point, i.e. $\exists x \in D^2$ with $f(x) = x$

Pf Suppose not. Let $g : D^2 \to S^1$ be the map given by projecting $f(x)$ through x onto S^1.



Note $g(x) = x$ if $x \in S^1$

So $g : D^2 \to S^1$ is a retract, contradicting previous result. ▨

The fundamental theorem of algebra

Every non-constant polynomial $p : \mathbb{C} \to \mathbb{C}$ has a zero.

Pf Let $r : \mathbb{C} \setminus \{0\} \to S^1$ defined by $r(z) = z/|z|$, a retraction.

Let $\lambda_R : S^1 \to \mathbb{C}$ be given by $z \in S^1 \mapsto Rz$ $(R \geqslant 0)$

If p has no zero, then

$$f_R := r \circ p \circ \lambda_R : S^1 \to S^1$$

In particular, for any R_1, R_2; f_{R_1} and f_{R_2} are homotopic.

r(p(R)), not 1

So induce the same* map $f_{R_1 *} = f_{R_2 *} : \underset{\underset{\mathbb{Z}}{\shortparallel}}{\pi_1(S^1, 1)} \longrightarrow \underset{\underset{\mathbb{Z}}{\shortparallel}}{\pi_1(S^1, 1)}$

given by multiplication by some number.

For $R = 0$, $r \circ p \circ \lambda_R$ takes the value $r \circ p(0)$, so f_0 is constant and hence f_{0*} is the zero map.

For R very large, the leading term $z^{\deg p}$ dominates p, and thus

f_{R*} is given by multiplication by deg p.

$p \circ \lambda_R \simeq$ to the nth

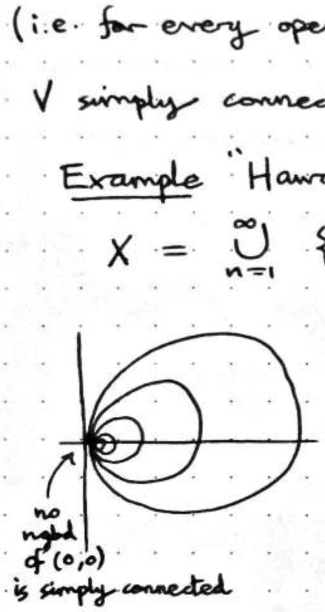
Thus $\deg p = 0$, i.e. p is a constant polynomial ▨

Theorem (Existence of universal covers)

Let X be a path-connected space such that X is locally simply connected (i.e. for every open ngbd $\mathcal{U}$ of a point $x \in X$, $\exists$ a ngbd $x \in V \subseteq \mathcal{U}$ with V simply connected) Then there is a universal covering space $p: \hat{X} \to X$

Example "Hawaiian earring"

$$X = \bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^2 \mid (x - \tfrac{1}{n})^2 + y^2 = \tfrac{1}{n^2}\}$$



Pf (sketch, non-examinable)

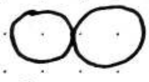
Fix $x_0 \in X$ and consider the set of all paths γ starting at x_0, i.e. $\mathcal{X} := \{\gamma: I \to X \mid \gamma(0) = x_0\}$

Define $\hat{X} = \mathcal{X} / \cong$ where $\gamma \cong \gamma'$ if γ, γ' homotopic paths.

The map $p: \hat{X} \to X$ is given by $p([\gamma]) = \gamma(1)$.

Tricky bit is defining the topology and showing p has the right properties ▨

Example

$X =$ 

"Figure eight"

§ The Galois correspondence

Idea We can classify all covering spaces using subgroups of the fundamental group.

Def Let X be a path-connected space and $p_1: \hat{X}_1 \to X$, $p_2: \hat{X}_2 \to X$ be covering maps. An isomorphism of covering spaces is a homeo^m $\phi: \hat{X}_1 \to \hat{X}_2$ making

$$\begin{tikzcd}
\hat{X}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & \hat{X}_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

commute.

Note then that ϕ^{-1} is also an isomorphism of covering spaces.

If $\hat{X}_i$ is equipped with a base point $\hat{x}_i \in \hat{X}_i$, $i=1,2$ and $\phi(\hat{x}_i) = \hat{x}_2$, we say that ϕ is based

Remark Note ϕ is alift of p_1 to $\hat{X}_2$. So by uniqueness of lifting, a based isomorphism is uniquely determined by $\phi(\hat{x}_1) = \hat{x}_2$.

($\hat{X}_1$ should be connected, locally path connected for this)

Theorem (Galois correspondence with basepoints)

Let X be a path connected, locally simply connected space with basepoint x_0.

The map which sends a covering space $p: \hat{X} \to X$ equipped with a basepoint $\hat{x}_0 \in p^{-1}(x_0)$ to the subgroup $p_*\pi_1(\hat{X}, \hat{x}_1) \leq \pi_1(X, x)$ induces a bijection between the set of based isomorphosim classes of path connected covering spaces with base point and the set of subgroups of $\pi_1(X, x_0)$.

Proof omitted (non-examinable)

Example As $\pi_1(S^1, 1) = \mathbb{Z}$, each subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for n a natural number (possibly 0).

Then $p: \mathbb{R} \to S^1 \quad \longleftrightarrow \quad 0 \leq \mathbb{Z}$ (since $\pi_1(\mathbb{R}, 0) = 0$)

$$\underset{z \mapsto z^n}{p: S^1 \to S^1} \quad \longleftrightarrow \quad n\mathbb{Z} \leq \mathbb{Z}$$

Thus, up to based isomorphism, these are the only path connected covering spaces of S^1.

Corollary Let X be a path-connected, locally path-connected space.

Any two universal covers $p_1: \hat{X}_1 \to X$, $p_2: \hat{X}_2 \to X$ are isomorphic.

Pf Choose a basepoint $x_0 \in X$, $\hat{x}_i \in \hat{X}_i$ with $\hat{x}_i \in p_i^{-1}(x_0)$.

Since $\hat{X}_1, \hat{X}_2$ are simply connected, they both correspond to the 0 subgroup of $\pi_1(X, x_0)$ hence are based isomorphic. ▨

L9.1

Cor (Galois correspondence without basepoints)

Let X be a path connected, locally simply connected space with basepoint x_0.

Then the map that sends a covering space $p: \hat{X} \to X$ equipped with a base point $\hat{x}_0 \in p^{-1}(x_0)$ to the $p_* \pi_1(\hat{X}, \hat{x}_0) \leq \pi_1(X, x_0)$ induces a bijection between isomorphism classes of path-connected covering spaces of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Pf The map is surjective by Galois correspondence.

To see the map is injective, we need to show that if $p_{1*}(\pi_1(\hat{X}_1, \hat{x}_1))$ and $p_{2*}(\pi_1(\hat{X}_2, \hat{x}_2))$ are conjugate subgroups of $\pi_1(X, x_0)$, then there's an isomorphism $\phi: \hat{X}_1 \to \hat{X}_2$ of covering spaces (not necessarily based).

So suppose $p_{1*}\pi_1(\hat{X}, \hat{x}_1) = [\gamma]\, p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)\, [\bar{\gamma}]$ for some $[\gamma] \in \pi_1(X, x_0)$.

Let $\hat{\bar{\gamma}}$ be the lift of $\overset{\text{in } \hat{X}_2}{\bar{\gamma}}$ at $\hat{x}_2$ and let $\hat{x}_2'$ be the other endpoint of $\hat{\bar{\gamma}}$. Thus

$$
\begin{aligned}
&[\gamma]\, p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)\, [\bar{\gamma}] \\
&\underset{\substack{\uparrow \\ \text{by def}^n \\ \text{of } \bar{\gamma}_\#}}{=} \bar{\gamma}_\# (p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)) \\
&= p_{2*}\, \hat{\bar{\gamma}}_\#\, \pi_1(\hat{X}_2; \hat{x}_2) \qquad \text{since } \bar{\gamma} = p_2 \circ \hat{\bar{\gamma}} \\
&= p_{2*}\, \pi_1(\hat{X}_2, \hat{x}_2').
\end{aligned}
$$

Thus $p_{1*}\pi_1(\hat{X}_1, \hat{x}_1) = p_{2*}\pi_1(\hat{X}_2, \hat{x}_2')$.

Thus by the Galois correspondence, there is a based isomorphism of covering spaces $(\hat{X}_1; \hat{x}_1) \cong (\hat{X}_2; \hat{x}_2')$

Thus $\hat{X}_1 \cong \hat{X}_2$ as covering spaces. ☒

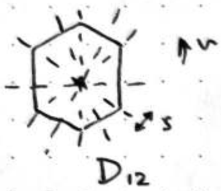
L9.2

§3 The Seifert - Van Kampen theorem

Some group theory

Recall ! you might have seen presentations of groups, e.g. the dihedral group $D_{2n} = \langle r, s \mid s^2 = r^n = 1,\ srs = r^{-1} \rangle$

i.e. the group generated by r, s subject to the given relations.



Def Let A be a set. $F(A)$ a group and $A \to F(A)$ a map of sets. We say that $F(A)$ is the free group on A (or 'generated by A') if it satisfies the following property :

$\forall$ group G and any set map $f : A \to G$,

$\exists!$ homom $\tilde{f} : F(A) \to G$ making the diagram commute.

$$\begin{tikzcd} & F(A) \arrow[dr, "\tilde{f}"] & \\ A \arrow[ur, "\iota"] \arrow[rr, "f"'] & & G \end{tikzcd}$$

Ex Let $A = \{\alpha\}$, $u: A \to \mathbb{Z}$ the map $\alpha \mapsto 1$.

Given a map $h: A \to G$, $\alpha \mapsto g$, we get a map $f : \mathbb{Z} \to G$, $n \mapsto g^n$

This is the only group hom satisfying $\underset{f(1)}{\underset{\|}{f(u(\overset{\alpha}{*}))}} = \underset{g}{\underset{\|}{h(\alpha)}}$

Thus $\mathbb{Z}$ is the free group on $\{\alpha\}$.

Remarks ① This definition is an example of a definition via a universal property, and as such $A \to F(A)$ is unique if it exists.

Pf Suppose $\varphi : A \to F(A)$, $\varphi' : A \to F'(A)$ are both free groups on A. Take $G = F'(A)$ in the def^n for $F(A)$.

$$\begin{tikzcd} & F(A) \arrow[d, dashed, "f \text{ unique}"] \\ A \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & F'(A) \end{tikzcd}$$

We get f as shown.

Taking instead $G = F(A)$ in the def^n of free group for $F'(A)$, get

$$\begin{tikzcd} & F(A) \\ A \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & F'(A) \arrow[u, dashed, "g \text{ unique}"'] \end{tikzcd}$$

Claim f, g are inverse to each other. Look at $f \circ g$: this a comm diagram

$$\begin{tikzcd} & F'(A) \arrow[d, "f \circ g"] \\ A \arrow[ur, "\varphi'"] \arrow[r, "\varphi'"'] & F'(A) \end{tikzcd}$$

But

$$\begin{tikzcd} & F'(A) \arrow[d, "id"] \\ A \arrow[ur, "\varphi'"] \arrow[r, "\varphi'"'] & F'(A) \end{tikzcd}$$

is comm as well.

Thus $f \circ g = id_{F'(A)}$ by the uniqueness part of the definition of free, applied to the diagram

$$\begin{tikzcd} & F'(A) \arrow[d, dashed] \\ A \arrow[ur, "\varphi'"] \arrow[r, "\varphi'"'] & F'(A) \end{tikzcd}$$

Ditto for $g \circ f = id_{F(A)}$ ☐

L9.3

$f: F(A) \to F'(A)$ is a canonical isomorphism

We say $F(A)$ is unique up to unique isomorphism.

② We don't know $F(A)$ exists yet!

③ If A is a finite set of cardinality r, then we write $F_r := F(A)$, the free group of rank r.

We can write words in A by identifying elements of A with their image in $F(A)$, so e.g. if $a, b \in A$, then $w = aba^{-1}bba^{-1}a^{-1}b \in F(A)$

Note the subgroup of $F(A)$ generated by the image of $A \to F(A)$ also satisfies the universal property [Check!], so by the uniqueness, $F(A)$ is generated by A, i.e. is the set of words in A (with some identifications e.g. $aa^{-1}b = b$).

Def A presentation (of a group) is a set A and a subset of relations $R \subseteq F(A)$. It presents the group $\langle A \mid R \rangle := F(A) / \langle\langle R \rangle\rangle$ where $\langle\langle R \rangle\rangle$ denotes the normal closure of R, i.e. the smallest normal subgroup of $F(A)$ containing R. This is the subgroup generated by

$$\{ srs^{-1} \mid s \in F(A), r \in R \}.$$

The presentation is finite if both A and R are finite.

Lemma (Universal property of group presentations)

Let $q: F(A) \longrightarrow \langle A \mid R \rangle$ be the quotient map. Whenever $f: F(A) \to G$ is a group hom such that $R \subseteq \ker f$, there is a unique hom $g: \langle A \mid R\rangle \to G$ making

$$\begin{tikzcd}
\langle A \mid R \rangle \arrow[rd, dashed, "g"] & \\
F(A) \arrow[u, "q"] \arrow[r, "f"'] & G
\end{tikzcd}$$

commute

Proof As $\langle\langle R \rangle\rangle$ is generated by elements of the form srs^{-1} with $r \in R$, $s \in F(A)$

and
$$\begin{aligned} f(srs^{-1}) &= f(s)\, f(r)\, f(s)^{-1} \\ &= f(s)\, f(s)^{-1} \\ &= e \quad \text{identity element} \end{aligned}$$

Thus $\langle\langle R \rangle\rangle \subseteq \ker f$ and hence g is obtained as

$g(w\langle\langle R \rangle\rangle) = f(w)$ well-defined, clearly unique $\square$

Example $F(\{a\}) \cong \mathbb{Z}$,

and every subgroup is normal, i.e. $\langle\langle a^n \rangle\rangle = \langle a^n \rangle \cong n\mathbb{Z} \leq \mathbb{Z}$

$\langle a \mid a^n \rangle = \langle a \rangle / \langle a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$

Example $\langle r, s \mid r^n, s^2, rsrs \rangle$

Claim $G \cong D_{2n}$ the dihedral group

Pf The hom $F(\{r,s\}) \longrightarrow D_{2n}$

$r \longmapsto$ rotation of an n-gon by $2\pi/n$ "using universal property"

$s \longmapsto$ flip of n-gon

takes $R = \{r^n, s^2, rsrs\}$ to the identity in D_{2n}.

Thus by universal property, get a hom

$G \longrightarrow D_{2n}$

This map is easily seen to be surjective.

We can use the relations to write any element of G as one of

$1, r, \dots, r^{n-1};\ s, sr, sr^2, \dots, sr^{n-1}$

Thus $O(G) \leq 2n$ (order of G)

By surjectivity, $O(G) = 2n$ and have a group isomorphism. $\square$

Example Every group has a presentation. The identity map of sets $G \to G$ induces a group hom $F(G) \to G$,

clearly surjective. This map has a Kernel R, and then

$$G \cong F(G)/\langle\langle R \rangle\rangle = \langle G \mid R \rangle$$

(since $\langle\langle R \rangle\rangle = R$ and iso thm)

Def (Pushout) Consider a commutative square of groups.

$$\begin{tikzcd}
\Gamma & A \arrow[l, "k"'] \\
B \arrow[u, "l"] & C \arrow[l, "j"] \arrow[u, "i"']
\end{tikzcd}$$

The diagram is called a pushout if it satisfies the following universal property:

for G a group with homs $f : A \to G$
$g : B \to G$

such that $f \circ i = g \circ j$, there exists a unique map $\phi : \Gamma \longrightarrow G$ with $\phi \circ l = g$, $\phi \circ k = f$

$$\begin{tikzcd}
 & A \arrow[dr, "k"] \arrow[drr, bend left, "f"] & & \\
C \arrow[ur, "i"] \arrow[dr, "j"'] & & \Gamma \arrow[r, dashed, "\exists ! \phi"] & G \\
 & B \arrow[ur, "l"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

If given
$$\begin{tikzcd}
 & A \\
B & C \arrow[l, "j"] \arrow[u, "i"']
\end{tikzcd}$$
and we can fill in to get a pushout
$$\begin{tikzcd}
\Gamma & A \arrow[l, "k"'] \\
B \arrow[u, "l"] & C \arrow[l, "j"] \arrow[u, "i"']
\end{tikzcd}$$

then we write $\Gamma = A \amalg_C B$.

Special cases

① If $C = \{e\}$ is the trivial group, then $A \amalg_C B$ is called the free product of A and B, sometimes called $A * B$.

② If i and j are injective, the pushout $A \amalg_C B$ is often called the amalgamated product and is written $A *_C B$.

Lemma For $i : C \longrightarrow A$, $j : C \longrightarrow \{e\}$ we have $A \amalg_C \{e\} \cong A / \langle\langle i(C) \rangle\rangle$

Proof Check universal property:

$$\begin{tikzcd}
G & & \\
 & A/\langle\langle i(C)\rangle\rangle \arrow[ul, dotted] & \{e\} \arrow[l] \arrow[ull, bend right, "g"'] \\
 & A \arrow[u, "\text{quotient}"] \arrow[uul, bend left, "f"] & C \arrow[l, "i"] \arrow[u]
\end{tikzcd}$$

commutative

By commutativity, $f \circ i$ is the constant map / hom

Thus $f(i(C)) = \{e\} \subseteq G$.

Thus $i(C) \subseteq \ker f$

Since $\ker f$ is normal, $\langle\langle i(C) \rangle\rangle \subseteq \ker f$

L10.3

So we obtain a well-defined map

$$\phi : A/\langle\langle i(C)\rangle\rangle \longrightarrow G$$

(unique!) as desired. $\square$

$$\begin{tikzcd} B \\ C \arrow[u, "j"] \arrow[r, "i"'] & A \end{tikzcd}$$

Lemma Let $A = \langle S_1 \mid R_1\rangle$, $B = \langle S_2 \mid R_2\rangle$ and let $T \subseteq C$ be a generating set for C.

Let $\tilde{i} : T \to F(S_1)$ be a lift of maps of sets of $T \xrightarrow{i} A$

and let $\tilde{j} : T \to F(S_2)$ be a lift of maps of sets of $T \xrightarrow{j} B$

Then $\Gamma = \langle S_1 \amalg S_2 \mid R_1 \cup R_2 \cup \{\tilde{i}(t)^{-1}\tilde{j}(t) \mid t \in T\}\rangle$

is a presentation for $A \amalg_C B$

Pf Check universal property

$$\begin{tikzcd}
F(S_2) \arrow[rr, "q_2"] & & B \arrow[dr, "l"] \arrow[rr, bend left, "g"] & & G \\
T \arrow[u, "\tilde{j}"] \arrow[r, hook] \arrow[d, "\tilde{i}"'] & C \arrow[ur, "j"] \arrow[dr, "i"'] & & \Gamma \arrow[r, dashed, "?"] & \\
F(S_1) \arrow[rr, "q_1"'] & & A \arrow[ur, "k"'] \arrow[uurr, bend right, "f"'] & &
\end{tikzcd}$$

k, l are induced by the maps of sets

$$S_1 \hookrightarrow S_1 \amalg S_2$$
$$S_2 \hookrightarrow S_1 \amalg S_2$$

$\ulcorner$ $S_1 \hookrightarrow S_1 \amalg S_2$ induces a map of free groups

$$F(S_1) \longrightarrow F(S_1 \amalg S_2) \longrightarrow \Gamma$$

induces a hom $A \to \Gamma$ $\lrcorner$ ← use R_1 killed in Γ

The inner square commutes: enough to check on generators (not obvious at first!)

Let $\tilde{f} = f \circ q_1$, $\tilde{g} = g \circ q_2$, killing R_1, R_2 respectively

Also, $\tilde{f} \circ \tilde{i} = \tilde{g} \circ \tilde{j}$ (chase)

Now construct $\phi : F(S_1 \amalg S_2) \longrightarrow G$ to be induced by $\tilde{f}$ on S_1, $\tilde{g}$ on S_2

This induces a hom $\phi : \Gamma \to G$ if all relations are sent to $\{e\}$.

But $\phi(R_1) = \tilde{f}(R_1) = \{e\}$, $\phi(R_2) = \tilde{g}(R_2) = \{e\}$

$$\phi(\tilde{i}(t)^{-1}\tilde{j}(t)) = \tilde{f}(\tilde{i}(t)^{-1})\,\tilde{g}(\tilde{j}(t)) = e$$

So by universal property of presentations, obtain desired map ϕ. (unique!) $\square$

Further help:

$$\begin{tikzcd}
 & T \arrow[dl, bend right, "\tilde{j}"'] \arrow[dr, bend left, "\tilde{i}"] & \\
F(S_2) \arrow[r, hook] \arrow[d, two heads] & F(S_1 \amalg S_2) \arrow[d, two heads] & F(S_1) \arrow[l, hook'] \arrow[d, two heads] \\
B \arrow[r, "l"'] & \Gamma & A \arrow[l, "k"]
\end{tikzcd}$$

§ Seifert - van Kampen for wedges

Def Let X, Y be spaces ; $x_0 \in X$, $y_0 \in Y$ basepoints.

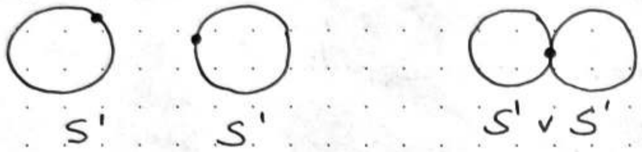
Then the wedge of X and Y is

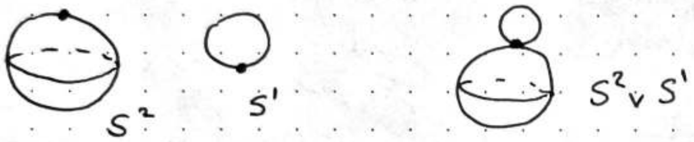
$$X \vee Y = (X \amalg Y)/\sim$$

where $\sim$ is the smallest equivalence relation such that $x_0 \sim y_0$.

The equivalence class $[x_0] = [y_0]$ in $X \vee Y$ is called the wedge point

Example





Theorem Suppose $X = Y_1 \vee Y_2$ with Y_1, Y_2 both path-connected, and $x_0 \in X$ the wedgepoint. Then

$$\pi_1(X, x_0) = \pi_1(Y_1, x_0) * \pi_1(Y_2, x_0)$$

Pf [Sketch, non-examinable]

Need to show $\pi_1(X, x_0)$ satisfies the universal property.

Suppose given $f_1 : \pi_1(\cancel{X}\,Y_1, x_0) \to G$, $f_2 : \pi_1(Y_2, x_0) \to G$; for some group G.

We need to show that there is a unique map

$$g : \pi_1(X, x_0) \to G$$

with the compositions $\pi_1(Y_1, x_0) \xrightarrow{i_{1*}} \pi_1(X, x_0) \xrightarrow{g} G$

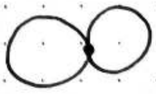
$$\pi_1(Y_2, x_0) \xrightarrow{i_{2*}} \pi_1(X, x_0) \xrightarrow{g} G$$

agreeing with f_1 and f_2 respectively.

Here $i_1 : Y_1 \to X_0$, $i_2 : Y_2 \to X$ are the natural inclusions.

Given a loop γ in X, write it as a concatenation

$$\gamma = \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_n \beta_n$$



with α_i a loop in Y_1, β_i a loop in Y_2. (wave hands)

No choice but to define

$$g(\gamma) = f_1(\alpha_1) f_2(\beta_1) \cdots f_1(\alpha_n) f_2(\beta_n)$$

Need to check this map g is well-defined, independent of the representative for γ in $[\gamma]$. Welp. ▨

<u>Example</u> ① $\pi_1(S^1 \vee S^1, x_0) = \pi_1(S^1, x_0) * \pi_1(S^1, x_0)$

$= \mathbb{Z} * \mathbb{Z}$

$= F_2$ (recall construction of pushout)

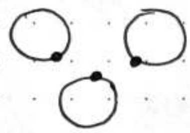
② Let A be any <u>finite</u> set.

Define $\bigvee_A S^1 = (A \times S^1)/\sim$

where $\sim$ is the smallest equivalence relation such that

$(a, 1) \sim (a', 1) \ \forall\, a, a' \in A, \ 1 \in S^1$

e.g. if $\# A = 3$,



$=(S^1 \vee S^1) \vee S^1$

$$\pi_1(\vee_A S^1, x_0) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{\# A \text{ copies}} = F(A)$$

<u>Theorem</u> (Seifert-van Kampen theorem)

Suppose $Y_1, Y_2 \subseteq X$ are open subsets,

$Z = Y_1 \cap Y_2$ non-empty,

$X = Y_1 \cup Y_2$

and suppose Y_1, Y_2, Z are all path connected.

Let $x_0 \in Z$, $i_k : Z \hookrightarrow Y_k$, $j_k : Y_k \hookrightarrow X$ inclusions.

Then the diagram

$$\begin{tikzcd} \pi_1(X, x_0) & \pi_1(Y_2, x_0) \arrow[l, "j_{2*}"'] \\ \pi_1(Y_1, x_0) \arrow[u, "j_{1*}"'] & \pi_1(Z, x_0) \arrow[l, "i_{1*}"] \arrow[u, "i_{2*}"'] \end{tikzcd}$$

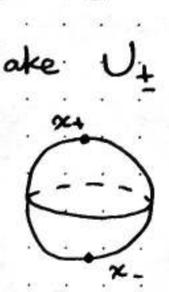
is a pushout.

<u>Pf</u> <u>Omitted</u>. □

<u>Example</u> $X = S^n \subseteq \mathbb{R}^{n+1}$ the unit sphere

Let $x_\pm = (\pm 1, 0, \ldots, 0)$

Take $U_\pm = S^n \setminus \{x_\pm\}$, $V = U_+ \cap U_- = S^n \setminus \{x_+, x_-\}$



Note $V \cong S^{n-1} \times (-1, 1)$ via

$$(x_1, \ldots, x_{n+1}) \longmapsto \left(\frac{(x_2, \ldots, x_{n+1})}{\|(x_2, \ldots, x_{n+1})\|}, x_1 \right)$$

Note that this is path-connected provided $n \geqslant 2$.

Also, U_+, U_- are both homeomorphic to $\mathbb{R}^n$ via setereographic projection.

$X = U_+ \cup U_-$

S-vK! get pushout

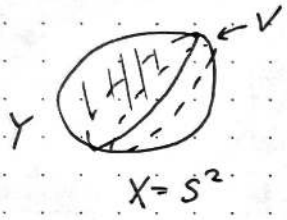
$$\begin{tikzcd} \pi_1(S^n) & \pi_1(U_+) = \{e\} \arrow[l] \\ \{e\} = \pi_1(U_-) \arrow[u] & \pi_1(V) \arrow[l] \arrow[u] \end{tikzcd}$$

Thus $\pi_1(S^n) = \{e\}$ for $n \geqslant 2$.

Thus S^n is simply connected for $n \geqslant 2$.

Seifert-van Kampen for closed sets

<u>Def</u> A subset $Y \subseteq X$ is called a <u>ngbd retract</u> if $\exists V \subseteq X$ open, $Y \subseteq V$ such that Y is a deformation retract of V.



Seifert van-Kampen for closed sets

Suppose $Y_1, Y_2 \subseteq X$ are closed subsets, $X = Y_1 \cup Y_2$, $Z = Y_1 \cap Y_2$ and Y_1, Y_2, Z all path connected as before.

Suppose Z is a ngbd retract in both Y_1 and Y_2.

Then

$$\begin{tikzcd} \pi_1(X, x_0) & \pi_1(Y_1, x_0) \arrow[l] \\ \pi_1(Y_2, x_0) \arrow[u] & \pi_1(Z, x_0) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout.

§ Attaching cells

<u>Def</u> Let X be a space; $\alpha : S^{n-1} \to X$ a map. Define

$$X \cup_\alpha D^n = (X \amalg D^n)/\sim$$

where $\sim$ is the smallest equivalence relation that identifies

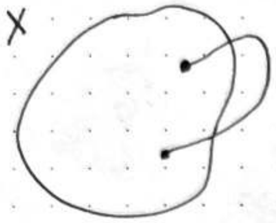
$$x \in S^{n-1} = \partial D^n \text{ with } \alpha(x) \in X$$

The resulting space is called the <u>attaching of an n-cell to X</u>

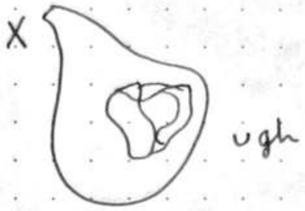
(D^n is an "n-cell")

L11.4

Ex $n=1$, $D^1 =$ •——• , $S^0 =$ • •



$n=2$, D^2 ▨ S^1 ○



L12.1

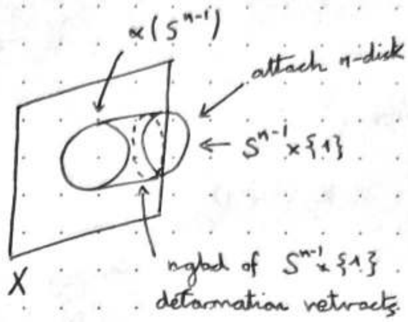
How does this operation affect π_1?

<u>lemma</u> If $n \geqslant 3$, $i : X \longrightarrow X \cup_\alpha D^n$ induces an isomorphism on fundamental groups

<u>proof</u> The <u>mapping cylinder</u> of α is the space

$$M_\alpha =: \left(X \amalg (S^{n-1} \times I) \right) / \sim$$

where $(\theta, 0) \sim \alpha(\theta)$ for all $\theta \in S^{n-1}$.

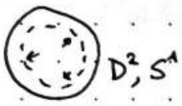


Identify S^{n-1} with $S^{n-1} \times \{1\} \subseteq M_\alpha$

Note S^{n-1} is now a ngbd retract in M_α,

and note $X \cup_\alpha D^n \cong M_\alpha \cup_{id_{S^{n-1}}} D^n$

Also, S^{n-1} is a ngbd retract in D^n



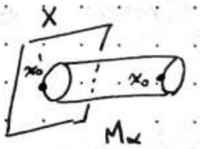
Now apply the closed S-vK theorem with basepoint in copy of $S^{n-1} \subseteq M_\alpha \cup_{id} D^n$, and take

$$Y_1 = M_\alpha, \; Y_2 = D^n, \; Z = Y_1 \cap Y_2 = S^{n-1}$$

Thus get a pushout

$$\begin{tikzcd}
\pi_1(X \cup_\alpha D^n, x_0) & \pi_1(M_\alpha, x_0) \arrow[l] \\
\pi_1(D^n; x_0) \arrow[u] & \pi_1(S^{n-1}, x_0) = \{e\} \arrow[l] \arrow[u] \\
\{e\} \arrow[u, equal] &
\end{tikzcd}$$

Thus $\pi_1(X \cup_\alpha D^n, x_0) \cong \pi_1(M_\alpha, x_0)$.



Note X is a deformation retract of M_α, thus

$$\pi_1(M_\alpha, x_0) \cong \pi_1(X, x_0')$$

□

<u>lemma</u> Let $\alpha : S^1 \longrightarrow X$ be a map, $x_0 = \alpha(\theta_0)$ for some $\theta_0 \in S^1$.

Then $\pi_1(X \cup_\alpha D^2; x_0) \cong \pi_1(X, x_0) / \langle\langle [\alpha] \rangle\rangle$,

viewing α as a loop based at x_0.

The inclusion map $i : X \to X \cup_\alpha D^2$ induces the quotient map

$$i_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0) / \langle\langle [\alpha] \rangle\rangle$$

<u>proof</u> Same setup as in the previous lemma, and get

$$\begin{tikzcd}
\pi_1(X \cup_\alpha D^2, x_0) & \pi_1(X, x_0) \arrow[l] & {[\alpha]} \\
\{e\} = \pi_1(D^2, \theta_0) \arrow[u] & \pi_1(S^1, \theta_0) \cong \mathbb{Z} \arrow[l] \arrow[u, "\alpha_*"] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

We know then that

$$\pi_1(X \cup_\alpha D^2, x_0) \cong \pi_1(X, x_0) / \langle\langle [\alpha] \rangle\rangle \qquad \square$$

<u>Theorem</u> For any finitely generated group G, ~~nay~~ finitely presented, i.e. $G = \langle A | R \rangle$ with A, R both finite, there exists a compact space X with $\pi_1(X, x_0) = G$.

<u>Proof</u> Let $Y = \bigvee_A S^1 \left[= \underset{(a,1)\sim(a',1)}{(A \times S^1)/\sim} \right]$

Then $\pi_1(Y, y_0) = F(A)$.

For each relation $r \in R$, we get a loop

$\alpha_r : S^1 \longrightarrow Y$ representing $r \in \pi_1(Y, y_0)$

Attach a D^2 to Y via α_r. Do this for each $r \in R$.

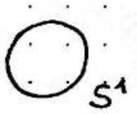
Then we get a space X with $\pi_1(X, x_0) = F(A) / \langle\langle R \rangle\rangle$. $\square$

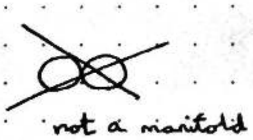
§ Classification of surfaces

<u>Def</u> An <u>n-dimensional (topological) manifold</u> is a Hausdorff topological space M such that every point $x \in M$ has a ngbd $\mathcal{U}$ homeomorphic to an open ~~ngbd~~ set of $\mathbb{R}^n$.

<u>Ex</u> ① S^n is a manifold, e.g. $U_\pm = S^n \setminus \{x_\pm\} \cong \mathbb{R}^n$ via stereographic proj^n

②





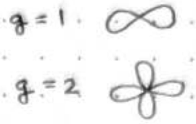
③ $\alpha : S^1 \longrightarrow X = \{*\}$ 1 point space

$Y = X \cup_\alpha D^2 \cong S^2$

L12.3

④ g a positive integer,

$\Gamma_{2g} = \bigvee_{i=1}^{2g} S^1_i$ a wedge of $2g$ circles



$S^1_1, \dots, S^1_{2g}$ $2g$ distinct circles

Let $\alpha_i : I \to S^1_i$ be a simple loop with basepoint the wedge point, $1 \le i \le g$, and $\beta_i : I \to S^1_{i+g}$ again a simple loop based at the wedge point, $1 \le i \le g$.

Let $\rho_g := \alpha_1 \beta_1 \overline{\alpha_1} \overline{\beta_1} \alpha_2 \beta_2 \overline{\alpha_2} \overline{\beta_2} \cdots \alpha_g \beta_g \overline{\alpha_g} \overline{\beta_g}$

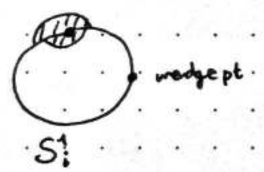
Think of this loop as a map $\rho_g : S^1 \to \Gamma_{2g}$

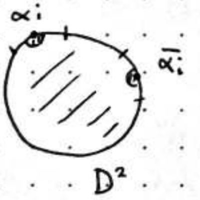
and define $\Sigma_g := \Gamma_{2g} \cup_{\rho_g} D^2$

Claim Σ is a compact surface

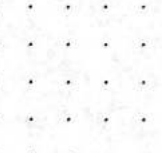
Need to check each point of Σ has a ngbd homeo to an open subset of $\mathbb{R}^2$.

- The interior of D^2 is an open ngbd of any point in the interior of D^2, homeo^c to an open subset of $\mathbb{R}^2$.
- At a point in S^1_i, not the wedge point; the corresponding path α_i or β_{i-g} (if $i > g$) appears twice in the loop

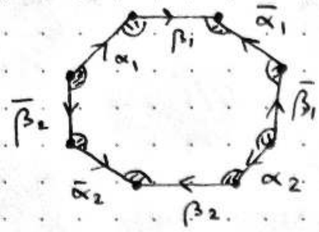


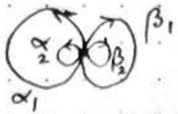


- At wedge point, need to be more careful

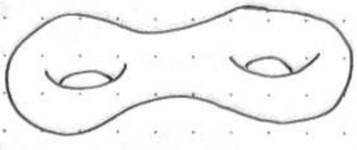


$g=2$





Σ_2

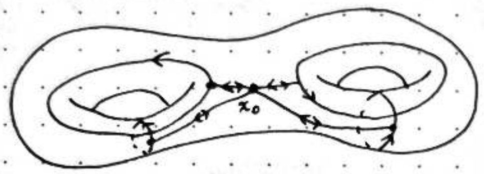


see video on the moodle

Note $\pi_1(\Sigma_g) = \langle a_1,\dots,a_g, b_1,\dots,b_g \mid a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1} \rangle$

$\pi_1(\Sigma_1) = \langle a_1, b_1 \mid a_1b_1a_1^{-1}b_1^{-1} \rangle \cong \mathbb{Z}^2$

For $g \geq 2$, $\pi_1(\Sigma_g)$ is non-abelian.



Take $\Gamma_{g+1} = \bigvee_{i=0}^{g} S^1_i$ the wedge of $g+1$ circles.

Let $\alpha_i : I \to S^1_i$ be the loop around the i^{th} circle, and take

$$\sigma_g := \alpha_0\alpha_0\alpha_1\alpha_1\cdots\alpha_g\alpha_g$$

which we may view as a map $\sigma_g : \partial D^2 \longrightarrow \Gamma_{g+1}$

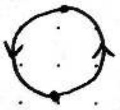
e.g. $g=0$, Γ_1 σ_g is twice the generator of $\pi_1(\Gamma_1)$

Take

$$S_g := \Gamma_{g+1} \cup_{\sigma_g} D^2$$

S_g is the non-orientable surface of genus g

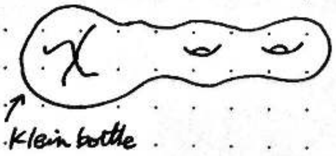
$g=0$



i.e. we get D^2 with opposite points on boundary identified giving $\mathbb{R}P^2$, the real projective plane

S_1 is the Klein bottle

S_g for $g > 1$



$\pi_1(S_g) \cong \langle a_0,\dots,a_g \mid a_0^2a_1^2\cdots a_g^2 \rangle$

e.g. $g=0$; $\langle a_0 \mid a_0^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$

Theorem Any compact surface X is homeomorphic to S_g or Σ_g for some g

How do we know these are distinct surfaces?

Lemma Let $g \in \mathbb{N} = \{0,1,2,\dots\}$.

Then $\pi_1(\Sigma_g)$ surjects onto $\mathbb{Z}^{2g}$ but not $\mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z})$

$\pi_1(S_g)$ surjects onto $\mathbb{Z}^{g} \oplus (\mathbb{Z}/2\mathbb{Z})$ but not $\mathbb{Z}^{g+1}$

Proof Let $\{\bar{a}_i, \bar{b}_i\}$ be a basis for $\mathbb{Z}^{2g}$.

The map $a_i \mapsto \bar{a}_i$; $b_i \mapsto \bar{b}_i$ takes $\rho_g = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

to $\bar{a}_1 + \bar{b}_1 - \bar{a}_1 - \bar{b}_1 + \cdots + \bar{a}_g + \bar{b}_g - \bar{a}_g - \bar{b}_g = 0$.

Thus there is a surjective homomorphism $\pi_1(\Sigma_g) \to \mathbb{Z}^{2g}$.

Suppose $f : \pi_1(\Sigma_g) \to \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z})$ is a surjection

Compose with reduction mod 2

$$f' \cancel{g} : \pi_1(\Sigma_g) \xrightarrow[f]{} \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z}) \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^{2g+1}$$

Thus $\underset{f'}{\cancel{g}}(a_1), \dots, \underset{f'}{\cancel{g}}(a_g); \dots, f'(b_1), \dots, f'(b_g)$ generate $(\mathbb{Z}/2\mathbb{Z})^{2g+1}$ ⋇

For $\pi_1(S_g)$, let $\{\bar{a}_i\}$ be a basis for the $\mathbb{Z}^g$ part of $\mathbb{Z}^g \oplus (\mathbb{Z}/2\mathbb{Z})$ and let $\bar{c}_i$ generate the $\mathbb{Z}/2\mathbb{Z}$ part.

Define $a_0 \mapsto \bar{c}_0 - \sum_{i=1}^{g} \bar{a}_i$

$a_i \mapsto \bar{a}_i$, $1 \le i \le g$

$$\sigma_g = a_0^2 \cdots a_g^2 \mapsto \underset{\substack{\uparrow \\ \text{in } \mathbb{Z}/2\mathbb{Z}}}{2\bar{c}_0} - \sum_{i=1}^{g} 2\bar{a}_i + 2\bar{a}_1 + \cdots + 2\bar{a}_g = 0$$

This gives a surjective homomorphism $\pi_1(S_g) \to \mathbb{Z}^g \oplus (\mathbb{Z}/2\mathbb{Z})$

If $f : \pi_1(S_g) \to \mathbb{Z}^{g+1}$ is surjective; then $\mathbb{Z}^{g+1}$ is generated by $f(a_0), \dots, f(a_g)$. But $0 = f(\sigma_g) = 2f(a_0) + \cdots + 2f(a_g)$

This is a contradiction (see GRM) □

Corollary Σ_g, $S_{g'}$ are never homeomorphic,

$\Sigma_g \cong \Sigma_{g'}$ iff $g = g'$ $\quad$ $S_g \cong S_{g'}$ iff $g = g'$

L13.3

<u>Pf</u> If $\Sigma_g \cong \Sigma_{g'}$ and $g < g'$, there is a surjection

$$\pi_1(\Sigma_g) \cong \pi_1(\Sigma_{g'}) \longrightarrow \mathbb{Z}^{2g'} \longrightarrow \mathbb{Z}^{2g} \oplus \mathbb{Z}/2\mathbb{Z} \quad \#$$

The other cases are similar. $\square$

<u>Remark</u> There exists a higher dim'l analogue of π_1;

$\pi_n(X, x_0)$ is the set of homotopy class of map from $(\underset{\underset{p}{\cup}}{S^n}, p)$ to (X, x_0) (i.e. $f: S^n \to X$ takes p to x_0)

$\pi_n(X, x_0)$ is a group, but is always abelian for $n > 1$.

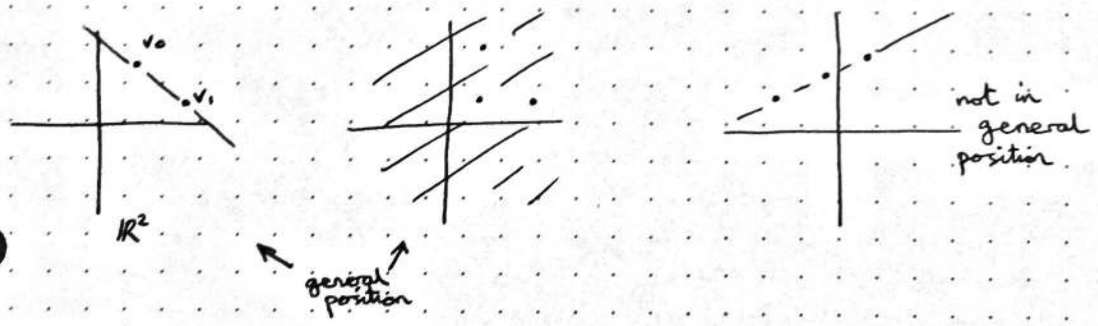
<u>Hard to calculate</u> $\pi_n(S^m)$ is still, in general, unknown

Different approach to understanding higher dim'l aspects of topological spaces:

<u>homology</u>

§ Simplicial complexes

<u>Def</u> A finite set $V = \{v_0, \dots, v_n\} \subseteq \mathbb{R}^m$ (not necessarily $n=m$) is said to be <u>in general position</u> if the smallest affine linear subspace of $\mathbb{R}^n$ containing V is of dimension n.



Equivalently

① $\{v_1 - v_0, \dots, v_n - v_0\}$ are linearly independent

or ② $\forall\, s_1, \dots, s_n \in \mathbb{R}$ if $\sum_{i=1}^{n} s_i(v_i - v_0) = 0$ then $s_1 = \dots = s_n = 0$

or ③ $\forall\, t_0, \dots, t_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} t_0 = 0$,

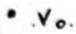
if $\sum_{i=0}^{n} t_i v_i = 0$ then $t_0 = \dots = t_n = 0$

We write the <u>span</u> or <u>convex hull</u> of a set $V = \{v_0, \dots, v_n\} \subseteq \mathbb{R}^m$ as

$$\langle V \rangle = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \right\}$$

If V is in general position, then $\langle V \rangle$ is said to be an <u>n-simplex</u>.

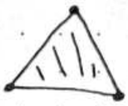
0 - simplex



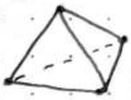
1 - simplex

$tv_0 + (1-t)v_1$,
$0 \leq t \leq 1$

2 - simplex



3 - simplex



solid tetrahedron

<u>Def</u> If $V \subseteq \mathbb{R}^n$ is in general position, and $U \subseteq V$, then $\langle U \rangle \subseteq \langle V \rangle$ is said to be a <u>face</u> of $\langle V \rangle$, and we write $\langle U \rangle \leq \langle V \rangle$.

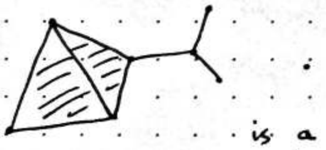
If $\langle U \rangle \leq \langle V \rangle$ but $\langle U \rangle \neq \langle V \rangle$ we call $\langle U \rangle$ a <u>proper face</u> of $\langle V \rangle$.

[$\emptyset$ is viewed as a face]

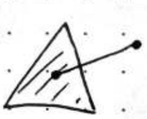
<u>Def</u> A <u>simplicial complex</u> is a finite set K of simplices in some $\mathbb{R}^m$ such that ① If $\sigma \in K$ and $\tau \leq \sigma$, then $\tau \in K$

② If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is a face of both σ and τ

<u>E.g.</u>



is a simplicial complex in $\mathbb{R}^2$,



not a simplicial complex

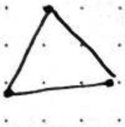
consisting of 2 two - simplices,
8 one - simplices,
8 zero - simplices.

<u>Def</u> We write $\dim K$ (the <u>dimension</u> of K) to be the largest n such that K contains an n-simplex.

We write $K_{(d)} := \{\sigma \in K \mid \dim \sigma \leq d\}$, the <u>$d$-skeleton</u> of K, also a simplicial complex if K is.

<u>Example</u> The set of faces of a simplex σ is a simplicial complex

The set of <u>proper</u> faces of a simplex is also a simplicial complex (if the simplex is n-dim^l, this is just the $(n-1)$ skeleton of the previous example.



This called the boundary of σ, written $\partial\sigma$

The set of points of σ not contained in a simplex of $\partial\sigma$ is called the interior of σ, written as σ°.

If $\dim\sigma = 0$, $\partial\sigma = \emptyset$, $\sigma^\circ = \sigma$.

Def The realisation of K is

$$|K| := \bigcup_{\sigma\in K} \sigma \subseteq \mathbb{R}^m,$$ viewed as a topological space with topology induced from $\mathbb{R}^m$.

If X is a space, a triangulation of X is a simplicial complex K and a homeomorphism $|K| \xrightarrow{\sim} X$.

Examples ① If σ_n is the n-simplex, then $\sigma_n \cong D^n$, so D^n is triangleable

② $\partial\sigma_n$ is a simplicial complex with $|\partial\sigma_n| \cong S^{n-1}$, so S^{n-1} is too

Remark There may be many different triangulations of the same space, e.g. D^2 is homeo to  & 

Def Let K, L be simplicial complexes. A simplicial map is a map $f: K \to L$ such that ① each 0-simplex $\langle v\rangle \in K$ is sent to a 0-simplex $\langle f(v)\rangle \in L$

② $f(\langle v_0, \dots, v_n\rangle) = \langle f(v_0), \dots, f(v_n)\rangle$ for any $\langle v_0, \dots, v_n\rangle \in K$

The realisation of f is the continuous map

$|f| : |K| \longrightarrow |L|$ given by,

on $\sigma = \underset{\hat{K}}{\langle v_0, \dots, v_n\rangle} \subseteq |K|$,

$$f_\sigma : \langle v_0, \dots, v_n\rangle \longrightarrow \langle f(v_0), \dots, f(v_n)\rangle$$
$$\sum_{i=0}^n t_i v_i \longmapsto \sum_{i=0}^n t_i f(v_i)$$

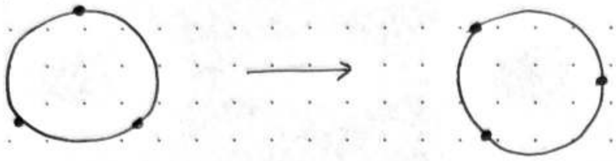
It's clear that f_σ is continuous and that if $\tau \leq \sigma$, then $f_\sigma|_\tau = f_\tau$

Thus the f_σ's glue to give the continuous map $|f| : |K| \to |L|$.

L14.3

<u>Basic problem</u> There aren't enough simplicial maps to capture all possible maps $|K| \longrightarrow |L|$ up to homotopy,

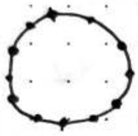
<u>e.g.</u>



consider $\underset{\text{3 vertices}}{\partial\sigma_2} \longrightarrow \underset{\text{3 vertices}}{\partial\sigma_2}$

There are at most 3^3 simplicial maps here, since a simplicial map is determined by what it does on the level of vertices.

<u>Solution</u> Barycentric subdivision ; refine triangulations



<u>Def</u> Let $V = \{v_0, \dots, v_n\}$ be in general position.

Then $\hat{\sigma} = \frac{1}{n+1} \sum_{i=0}^{n} v_i$ is the <u>barycentre</u> of $\sigma = \langle V \rangle$.

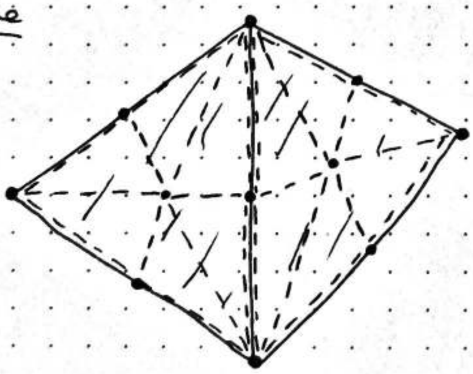
Note $\hat{\sigma} \in \mathrm{Int}(\sigma) = \sigma^\circ$

<u>Def</u> Let K be a simplicial complex. The <u>barycentric subdivision</u> K' of K is the complex such that ① The vertices of K' are the barycentres of simplices of K

② Vertices $\hat{\sigma}_0, \dots, \hat{\sigma}_n$ span a simplex of K' iff

$$\sigma_0 \lneq \sigma_1 \lneq \cdots \lneq \sigma_n$$

<u>Example</u>



<u>Lemma</u> Let K be a simplicial complex. Then K' is a simplicial complex, and

$$|K'| = |K|$$

<u>Proof</u> Each simplex in K' is really a simplex ; i.e. $\hat{\sigma}_0, \dots, \hat{\sigma}_n$ are in general position

Suppose given $\sigma_0 \lneq \cdots \lneq \sigma_n \in K$ and suppose $\sum_{i=0}^{n} t_i = 0$

and $\sum_{i=0}^{n} t_i \hat{\sigma}_i = 0$

Take j to be the largest index such that $t_j \neq 0$.

Then $\hat{\sigma}_j = -\sum_{i=0}^{j-1} \frac{t_i}{t_j} \hat{\sigma}_i$

is then contained in a <u>proper</u> face of σ_j, namely σ_{j-1}

This is a contradiction. Thus $\hat{\sigma}_0, \dots, \hat{\sigma}_n$ are in general position.

K' is a simplicial complex

① Clearly closed under passing to faces:

if $\langle \hat\sigma_0, \dots, \hat\sigma_k \rangle \in K'$, then so is any face $\langle \hat\sigma_{i_0}, \dots, \hat\sigma_{i_\ell} \rangle$

$0 \le i_0 < i_1 < \dots < i_\ell \le k$

② We need to show that if $\sigma', \tau' \in K'$, then $\sigma' \cap \tau'$ is a face of both σ', τ'.

Let $\sigma' = \langle \hat\sigma_0, \dots, \hat\sigma_m \rangle \in K'$, $\sigma_0 \underset{\neq}{\leq} \sigma_1 \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} \sigma_m \in K$

$\tau' = \langle \hat\tau_0, \dots, \hat\tau_n \rangle \in K'$, $\tau_0 \underset{\neq}{\leq} \tau_1 \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} \tau_n \in K$

So if $\sigma' \cap \tau' \neq \phi$, the intersection must be contained in $\sigma_m \cap \tau_n$, which is a face of both σ_m, τ_n.

So we can assume that σ', τ' are contained in a common simplex $\delta \in K$, namely by intersection with $\sigma_m \cap \tau_n$.

Induction on dim K

If one of σ', τ' doesn't contain $\hat\delta$, then the intersection $\sigma' \cap \tau'$ is contained in $\partial\delta$; or both σ', τ' contain $\hat\delta$, in which case $\sigma' \cap \tau'$ is the convex hull of $\hat\delta$ and $(\sigma' \cap \partial\delta) \cap (\tau' \cap \partial\delta)$.

In either case, $\sigma' \cap \tau'$ or $(\sigma' \cap \partial\delta) \cap (\tau' \cap \delta\delta)$ are intersections of simplices in K' contained in the skeleton $K^{(n-1)}$ where $n = \dim K$.

Hence by induction, $\sigma' \cap \tau'$ or $(\sigma' \cap \partial\delta) \cap (\tau' \cap \partial\delta)$ are faces of either simplex. Thus, in any event, $\sigma' \cap \tau'$ is a simplex which is a face of σ' and τ'.

$\therefore$ K' is a simplicial complex

$|K'| = |K|$ Induction in dim K

$[\dim K = 0 \Rightarrow K' = K]$

Certainly any $\langle \hat\sigma_0, \dots, \hat\sigma_n \rangle \in K'$ is contained in σ_n

So $|K'| \subseteq |K|$.

Let $\sigma = \langle v_0, \dots, v_m \rangle \in K$, $x \in \sigma$.

If $x = \hat\sigma$ then $x \in |K'|$.

Otherwise, project $\pi : \sigma \setminus \{\hat\sigma\} \longrightarrow |\partial\sigma|$

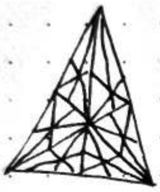
By induction, $\pi(x) \in \langle \hat\sigma_1, \dots, \hat\sigma_n \rangle \in K'$

and $x \in \langle \hat{\sigma}_1, \dots, \hat{\sigma}_n, \hat{\sigma} \rangle \in K'$. ▨

Def Set $K^{(0)} = K$,

$$K^{(r)} := \left(K^{(r-1)}\right)'$$

the *r^{th} barycentric subdivision of K*



Def For K a simplicial complex, we set

$$\operatorname{mesh}(K) := \max_{\langle u,v \rangle \in K} \|u - v\|,$$

the *mesh of K*.

Lemma If $\dim K = n$, then

$$\operatorname{mesh}(K^{(r)}) \leq \left(\frac{n}{n+1}\right)^r \operatorname{mesh}(K)$$

Pf Sufficient to check for $r = 1$.

If $\langle u, v \rangle \in K' = K^{(1)}$,

then $u = \hat{\tau}$, $v = \hat{\sigma}$ for $\tau \lneq \sigma$.

We can assume that τ is a 0-simplex, as this maximizes the distance between $\hat{\sigma}$ and $\partial\sigma$.

Let $\sigma = \langle v_0, \dots, v_m \rangle$, $\hat{\tau} = \tau = \langle v_0 \rangle$.

Then

$$\begin{aligned} \|\hat{\tau} - \hat{\sigma}\| &= \left\| v_0 - \sum_{i=0}^{m} \frac{1}{m+1} v_i \right\| \\ &= \left\| \frac{m}{m+1} v_0 - \frac{1}{m+1} \sum_{i=1}^{m} v_i \right\| \\ &= \left\| \frac{1}{m+1} \sum_{i=1}^{m} (v_i - v_0) \right\| \\ &\leq \frac{1}{m+1} \left\| \sum_{i=1}^{m} (v_i - v_0) \right\| \\ &\leq \frac{1}{m+1} \sum_{i=1}^{m} \|v_i - v_0\| \\ &\leq \frac{m}{m+1} \operatorname{mesh}(K). \end{aligned}$$

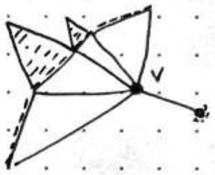
▨

Simplicial approximation

Idea Any map $|K| \longrightarrow |L|$ is homotopic to a simplicial map provided we pass to some barycentric subdivisions of K, L.

Def Let K be a simplicial complex. The (open) star of a vertex v of K is the union of interiors of simplices of K containing V, i.e:

$$St_K(v) := \bigcup_{v \in \sigma \in K} \sigma^\circ$$



complement of the ⫽ stuff

Def Let $\phi : |K| \to |L|$ be a map. A simplicial map $f : K \to L$ is said to be a simplicial approximation to ϕ if

$$\phi(St_K(v)) \subseteq St_L(f(v))$$

Note If $\phi = |f|$, then $\phi(St_K(v)) = St_L(f(v))$

Lemma If $f : K \to L$ is a simplicial approximation to $\phi : |K| \to |L|$, then $|f| \simeq \phi$ are homotopic maps

L16.1

Pf Suppose $|L| \subseteq \mathbb{R}^m$.

Will show straight line homotopy

$$H(x,t) = t\,\phi(x) + (1-t)\,|f|(x)$$

has image contained in $|L|$.

Let $x \in |K|$ lie in the interior of some simplex σ (necessarily unique)

Then $\phi(x)$ lies in the interior of some unique $\tau \in L$.

Claim: $f(\sigma) \leq \tau$

Pf Let v_i be a vertex of σ.

Then $x \in St_K(v_i)$, so

$$\phi(x) \in \phi(St_K(v_i)) \subseteq St_L(f(v_i))$$

since f is a simplicial approximation of ϕ.

Thus $\tau^\circ \subseteq St_L(f(v_i))$, so $f(v_i)$ is a vertex of τ.

Thus $f(\sigma) \leq \tau$. ▨

Note we didn't use the fact that f is a simplicial map $\swarrow$ only used f sends vertices to vertices

Now since τ is convex in $\mathbb{R}^m$, the straight line between $|f|(x) \in \tau$ and $\phi(x) \in \tau^\circ$ is contained in τ.

Hence H has image in L. ▨

The simplicial approximation theorem

Let K, L be simplicial complexes and $\phi : |K| \to |L|$ be a map.

Then there exists a positive integer r and a simplicial approximation

$$f : K^{(r)} \longrightarrow L \quad \text{to} \quad \phi : |K^{(r)}| \longrightarrow |L|$$

Pf Note $\mathcal{U} = \{ \phi^{-1}(St_L(v)) \mid v \in L \text{ vertex} \}$ is an open cover of $|K|$.

We will show $\exists \delta > 0$ s.t. $\forall x \in |K| \subseteq \mathbb{R}^m$, the ball $B(x, \delta) \subseteq U$ for some $U \in \mathcal{U}$. [Here $B(x,\delta)$ is the δ-ball in $|K|$ with distance coming from Euclidean distance on $\mathbb{R}^m$]

(This is the Lebesgue number Lemma)

If not, for each n, $\exists\, x_n \in |K|$ s.t. $B(x_n, 1/n)$ is not contained in any open set of $\mathcal{U}$.

Using compactness of $|K|$, we may pass to a convergent subsequence and assume $x_n \longrightarrow x \in |K|$.

But $x \in U \in \mathcal{U}$ for some U, so $\exists\, \varepsilon > 0$ such that

$B(x; \varepsilon) \subseteq U$.

Taking n sufficiently large so that $\|x - x_n\| < \varepsilon/2$, we then get

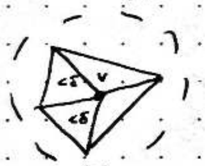
$B(x_n, \varepsilon/2) \subseteq U$

When $\frac{1}{n} < \varepsilon/2$, this contradicts $B(x_n, 1/n) \notin U$ #

Now choose r sufficiently large so that $\text{mesh}(K^{(r)}) < \delta$.

For each vertex v of $K^{(r)}$, we have

$$St_{K^{(r)}}(v) \subseteq B(v, \delta) \subseteq \phi^{-1}(St_L(u)) \quad ^{(*)} \text{ for some vertex } u \in L.$$



We define $f(v) = u$.

We need to show f defines a simplicial map approximating ϕ.

But if $\sigma \in K$ and $x \in \sigma^\circ$, then $f(\sigma)$ is a face of τ, the unique simplex of L containing $\phi(x)$ in its interior. This follows exactly as in the proof of the claim in the previous lemma. So $f(\sigma)$ is a simplex of L.

f is a simplicial approximation by $(*)$, which implies

$$\phi(St_{K^{(r)}}(v)) \subseteq St_L(f(v)) \quad \square$$

§5 Homology

Modify the def^n of simplex.

Consider points $v_0, \dots, v_n$ in general position giving a simplex $\sigma = \langle v_0, \dots, v_n \rangle$

S_{n+1} the symmetric group on $n+1$ letters acts on $v_0, \dots, v_n$

$A_{n+1} \leq S_{n+1}$ the alternating group, i.e. the subgroup of even permutations also acts on $v_0, \dots, v_n$, and there are two orbits of this action.

An <u>orientation</u> on a simplex $\sigma = \langle v_0, \dots, v_n \rangle$ is a choice of one of these orbits, i.e. an ordering of $v_0, \dots, v_n$ defined up to an element of A_{n+1}.

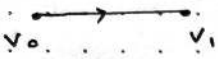
L16.3

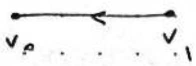
Now view $\langle v_0, \dots, v_n \rangle$ as an oriented simplex, with orientation given by the stated ordering.

<u>Ex</u> 0-simplex: we don't have such a notion $\langle v_0 \rangle$

1-simplex $\langle v_0, v_1 \rangle$ and $\langle v_1, v_0 \rangle$

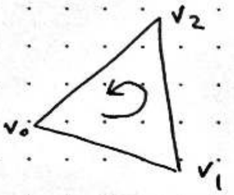
are now viewed as distinct simplices with opposite orientation

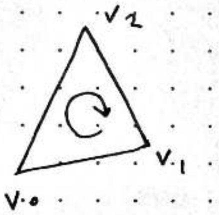




2-simplex $\langle v_0, v_1, v_2 \rangle = \langle v_1, v_2, v_0 \rangle = \langle v_2, v_0, v_1 \rangle$

$\langle v_1, v_0, v_2 \rangle = \langle v_0, v_2, v_1 \rangle = \langle v_2, v_1, v_0 \rangle$





<u>Def</u> Let K be a simplicial complex.

We define the <u>group of n-chains</u> of K to be the free abelian group generated by the simplices of dimension n in K;

i.e. $C_n(K) = \bigoplus_{\substack{\sigma \in K \\ \dim \sigma = n}} \langle \sigma \rangle$

where $\langle \sigma \rangle$ is the free group generated by σ.

We will use the convention that we make an arbitrary choice of orientation for each $\sigma \in K$ once and for all, and write $\bar{\sigma}$ for the opposite choice of orientation, and identify $\bar{\sigma}$ with $-\sigma$ in $C_n(K)$.

We define the <u>boundary homomorphism</u>

$$\partial := \partial_n : C_n(K) \to C_{n-1}(K)$$

defined by, for $\sigma = \langle v_0, \dots, v_n \rangle \in K$

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \underset{\nwarrow \text{omit } v_i}{\hat{v}_i}, \dots, v_n \rangle$$

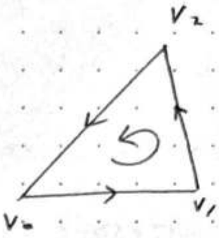
Note $\langle v_0, \dots, \hat{v}_i, v_n \rangle$ may not be oriented as the chosen orientation of this simplex τ, in which case use the convention $\bar{\tau} = -\tau$.

L16.4

Example $\partial\langle v_0, v_1\rangle = \langle v_1\rangle - \langle v_0\rangle$

$$\partial\langle v_0, v_1, v_2\rangle = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle$$

$$= \langle v_1, v_2\rangle + \langle v_2, v_0\rangle + \langle v_0, v_1\rangle$$



L17.1

Note if $\sigma = \langle v_0, \dots, v_n \rangle$ (oriented) we may write

$$\bar{\sigma} = \langle v_0, \dots, v_{j-1}, v_{j+1}, v_j, v_{j+2}, \dots, v_n \rangle \quad \text{(interchange vertices } v_j, v_{j+1} \text{)}$$

$$\begin{aligned}
\partial \bar{\sigma} &= \sum_{i=0}^{j-1} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_{j-1}, v_{j+1}, v_j, \dots \rangle \\
&\quad + (-1)^j \langle v_0, \dots, v_{j-1}, v_j, v_{j+2}, \dots, v_n \rangle \\
&\quad + (-1)^{j+1} \langle v_0, \dots, v_{j-1}, v_{j+1}, v_{j+2}, \dots, v_n \rangle \\
&\quad + \sum_{i=j+2}^{n} \langle v_0, \dots, v_{j+1}, v_j, v_{j+2}, \dots, \hat{v}_i, \dots, v_n \rangle \\
&= \sum_{i=0}^{j-1} (-1)^i \overline{\langle v_0, \dots, \hat{v}_i, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_n \rangle} \\
&\quad + (-1)^j \overline{\langle v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n \rangle} \\
&\quad + (-1)^{j+1} \overline{\langle v_0, \dots, v_j, v_{j+2}, \dots, v_n \rangle} \\
&\quad + \sum_{i=j+2}^{n} (-1)^i \overline{\langle v_0, \dots, v_j, v_{j+1}, \dots, \hat{v}_i, \dots, v_n \rangle} \\
&= \sum_{i=0}^{n} (-1)^n \overline{\langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle} \\
&= -\partial \sigma
\end{aligned}$$

$$\therefore \boxed{\partial \bar{\sigma} = -\partial \sigma}$$

So convention that $\bar{\sigma} = -\sigma$ is compatible with ∂.

Def Let $n \in \mathbb{Z}$. The group of n-cycles of K is

$$Z_n(K) := \operatorname{Ker}(\partial_n : C_n(K) \longrightarrow C_{n-1}(K))$$

The group of n-boundaries is

$$B_n(K) := \operatorname{im}(\partial_{n+1} : C_{n+1}(K) \longrightarrow C_n(K))$$

Lemma $\partial \circ \partial = 0$ and hence $B_n(K) \subseteq Z_n(K)$

Pf Need to calculate

$$\begin{aligned}
&\partial_{n-1} \circ \partial_n (\langle v_0, \dots, v_n \rangle) \\
&= \partial_{n-1} \Big(\sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle \Big) \\
&= \sum_{i=0}^{n} (-1)^i \partial_{n-1} (\langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle) \\
&= \sum_{j<i} (-1)^j (-1)^i \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n \rangle \\
&\quad + \sum_{j>i} (-1)^{j-1} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots \hat{v}_j, \dots, v_n \rangle \\
&= \sum_{j<i} (-1)^{i+j} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n \rangle \\
&\quad + \sum_{j<i} (-1)^{i+j+1} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n \rangle = 0
\end{aligned}$$

not $(-1)^j$!

☺

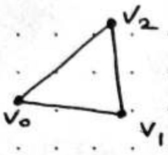
☒

Since $\partial_n \circ \partial_{n+1} = 0$; $\underset{B_n(K)}{\underset{\shortparallel}{\operatorname{im} \partial_{n+1}}} \subseteq \underset{Z_n(K)}{\underset{\shortparallel}{\operatorname{Ker} \partial_n}}$

<u>Def</u> The n^{th} <u>(simplicial) homology group</u> of K is

$$H_n(K) := Z_n(K) / B_n(K)$$

<u>Example</u> S^1



0 simplices $\langle v_0 \rangle$ $\langle v_1 \rangle$ $\langle v_2 \rangle$

1 simplices $\langle v_0, v_1 \rangle$, $\langle v_1, v_2 \rangle$, $\langle v_2, v_1 \rangle$

$$C_0(K) \cong C_1(K) \cong \mathbb{Z}^3, \quad C_n(K) = 0 \text{ for } n \notin \{0,1\}$$

Only need to understand $\partial_1 : C_1(K) \to C_0(K)$

$$\partial_1(\langle v_0; v_1 \rangle) = \langle v_1 \rangle - \langle v_0 \rangle$$

$$\partial(\langle v_1; v_2 \rangle) = \langle v_2 \rangle - \langle v_1 \rangle$$

$$\partial_1(\langle v_2, v_0 \rangle) = \langle v_0 \rangle - \langle v_2 \rangle$$

∂_1 has matrix $\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

$$\operatorname{Ker} \partial_1 = \langle (1,1,1)^T \rangle = \operatorname{span}(\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle)$$

$$\operatorname{Im} \partial_2 = 0$$

$$\therefore H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}$$

$$Z_0(K) = \operatorname{Ker} \partial_0 = C_0(K) \cong \mathbb{Z}^3$$

$$B_0(K) = \operatorname{im} \partial_1 = \frac{\mathbb{Z}^3}{\langle (-1,1,0), (0,-1,1) \rangle} \cong \mathbb{Z}$$

gives $H_0(K) \cong \mathbb{Z}$

$\langle (-1,1,0), (0,-1,1) \rangle \leq \mathbb{Z}^3$

$$H_n(K) = 0 \quad \forall n \notin \{0,1\}$$

Example $L =$  with K as before,

$$C_0(L) = C_0(K), \quad C_1(L) = C_1(K)$$
$$C_2(L) = \langle \langle v_0, v_1, v_2 \rangle \rangle$$

$$\partial_2(\langle v_0, v_1, v_2 \rangle) = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle$$

Note this is the generator of $\operatorname{Ker} \partial_1$, so $Z_1(L) = B_1(L)$ and hence

$$H_1(L) = 0 \quad (!)$$

Note $H_0(L) \cong H_0(K) \cong \mathbb{Z}$

$$Z_2(L) = \operatorname{Ker} \partial_2 = 0$$
$$H_2(L) = 0$$

TL;DR: $\boxed{\begin{array}{l} H_0(L) = \mathbb{Z} \\ H_n(L) = 0 \quad \forall n \neq 0 \end{array}}$

Lemma Let K be a simplicial complex. If d is no. of path components of $|K|$, then $H_0(K) \cong \mathbb{Z}^d$

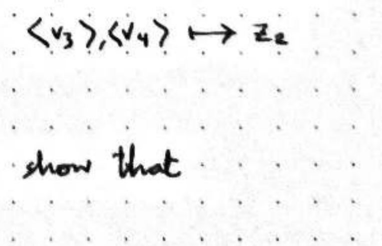
Proof Denote by $\pi_0(K)$ the set of path components of $|K|$.

Let $\mathbb{Z}[\pi_0(K)]$ be the free abelian group generated by $\pi_0(K)$.

Note $\mathbb{Z}[\pi_0(K)] \cong \mathbb{Z}^d$.

Have a map $q : C_0(K) \longrightarrow \mathbb{Z}[\pi_0(K)]$

$\langle v \rangle \longmapsto$ path component containing v

e.g. 

$\langle v_0 \rangle, \langle v_1 \rangle, \langle v_2 \rangle \mapsto z_1$

$\langle v_3 \rangle, \langle v_4 \rangle \mapsto z_2$

As every path component of $|K|$ has a vertex, this map is surjective.

Note $\partial_0 = 0$, so $Z_0(K) = C_0(K)$ so

$$H_0(K) \cong C_0(K) / B_0(K)$$

By Noether's first isomorphism theorem (!) it is enough to show that

$$B_0(K) = \operatorname{Ker} q$$

If $\sigma = \langle v_0, v_1 \rangle \in C_1(K)$

$$(q \circ \partial_1)(\langle v_0, v_1\rangle) = q(\langle v_1\rangle - \langle v_0\rangle)$$

$$= 0 \quad \text{since } v_1, v_0 \text{ lie in same path component of } |K|$$

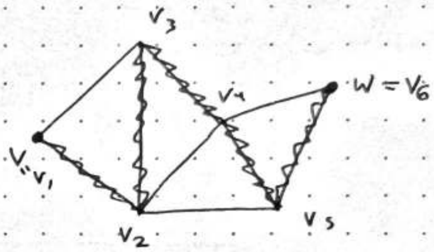
Thus $B_0(K) \leq \operatorname{Ker} q$

Conversely, $\operatorname{Ker}(q)$ is generated by elements of $C_0(K)$ of the form $\langle v\rangle - \langle w\rangle$ with v, w in the same path component of $|K|$.

But then there is a sequence of vertices

$v = v_1, v_2, \dots, v_n = w$ in K with $\langle v_i, v_{i+1}\rangle \in K$

Then
$$\langle w\rangle - \langle v\rangle = (\langle v_n\rangle - \langle v_{n-1}\rangle) + \dots + (\langle v_2\rangle - v_1\rangle)$$
$$= \partial(\langle v_{n-1}, v_n\rangle) + \dots + \partial(\langle v_1, v_2\rangle)$$
$$\in B_0(K)$$



$\therefore \operatorname{Ker} q \leq B_0(K)$ and we are done □

Remark Analogy with π_1

$Z_1(K) \leftrightsquigarrow$ loops

$B_1(K) \leftrightsquigarrow$ homotopies

§ Chain maps and homotopies

Goal Understand maps on homology induced by maps of simplicial complexes

Def A chain complex $C_\bullet$ is a sequence of abelian groups C_n, $n \in \mathbb{Z}$, with homomorphisms $\partial_n : C_n \to C_{n-1}$ $\forall n$ such that $\partial_{n-1} \circ \partial_n = 0$.

A chain map $f_\bullet : C_\bullet \to D_\bullet$ between chain complexes is a collection of homomorphisms $f_n : C_n \to D_n$ $\forall n$ such that

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"'] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_n"'] & D_{n-1} \end{tikzcd}$$

is commutative $\forall n$, i.e. $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

Given a chain complex, can define groups

$$Z_n(C_\bullet) = \operatorname{Ker} \partial_n$$
$$B_n(C_\bullet) = \operatorname{Im} \partial_{n+1}$$
$$H_n(C_\bullet) = Z_n(C_\bullet) / B_n(C_\bullet)$$

[$\partial_n \circ \partial_{n+1} = 0$ implies $B_n(C_\bullet) \leq Z_n(C_\bullet)$]

Lemma If $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, then we obtain a map

$$f_* : H_n(C_\bullet) \to H_n(D_\bullet) \text{ defined by}$$
$$[c] \longmapsto [f_n(c)]$$
$$(c \in Z_n(C_\bullet))$$

Pf Let $c \in Z_n(C_\bullet)$. Then $\partial_n \circ f_n(c) = f_{n-1} \circ \partial_n(c) = 0$.
So $f_n(c) \in Z_n(D_\bullet)$. Thus we get a well-defined composed map

$$Z_n(C_\bullet) \to Z_n(D_\bullet) \to H_n(D_\bullet)$$

Further, if $c \in B_n(C_\bullet)$, $\exists\, c' \in C_{n+1}$ s.t. $c = \partial_{n+1}(c')$.
Then $f_n(c) = f_n \circ \partial_{n+1}(c') = \partial_{n+1} \circ f_{n+1}(c')$
So $f_n(c) \in B_n(D_\bullet)$.

Thus we obtain a well-defined map $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$. ▨

Lemma A simplicial map $f: K \to L$ induces a chain map

$$f_\bullet : C_\bullet(K) \to C_\bullet(L)$$

via $f_n : \underset{n \text{ simplex}}{\sigma \in K} \longmapsto \begin{cases} f(\sigma) & \text{if } \dim^{f}\sigma = n, \\ 0 & \text{otherwise.} \end{cases}$

Hence $f_\bullet$ induces a homomorphism $f_*: H_n(K) \longrightarrow H_n(L)$.

Pf We need to show $\partial_n \circ f_n = f_{n-1} \circ \partial_n$

Let $\sigma = \langle v_0, \dots, v_n \rangle$.

If $\dim f(\sigma) = n$, then $f(\sigma) = \langle f(v_0), \dots, f(v_n) \rangle$.

Then it is clear that, since we have a one-to-one correspondence between faces of σ and faces of $f(\sigma)$, that $f_{n-1} \circ \partial_n = \partial_n \circ f_n$.

If $\dim f(\sigma) \leq n-2$, then every face of σ is mapped via f to a simplex of dimension $\leq n-2$.

Thus $f_n(\sigma) = 0$, so $\partial_n \circ f_n(\sigma) = 0$

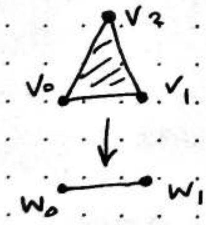
and $f_{n-1}(\tau) = 0$ for any $(n-1)$-dim face of σ,

so $f_{n-1} \circ \partial_n(\sigma) = 0$.

If $\dim f(\sigma) = n-1$, we may assume $f(v_0) = f(v_1)$ and

$$f(\langle v_0, \dots, v_n \rangle) = f(\langle v_1, \dots, v_n \rangle) = f(\langle v_0, v_2, \dots, v_n \rangle) = f(\sigma)$$

e.g.



$v_0, v_1 \mapsto w_0$
$v_2 \mapsto w_1$

Then $f_n(\sigma) = 0$, so $\partial_n \circ f_n(\sigma) = 0$

Now $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle$

Because $f(v_0) = f(v_1)$, if $i \notin \{0,1\}$

we have $f_{n-1}(\langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle) = 0$

Thus $f_{n-1} \circ \partial_n(\sigma) = f_{n-1}(\langle v_1, \dots, v_n \rangle) - f_{n-1}(\langle v_0, v_2, \dots, v_n \rangle)$

$= 0$. ▨

Remarks ① If $f: K \to L$, $g: L \to M$ simplicial maps, then

$$(g \circ f)_* = g_* \circ f_* \quad ✓$$

② If K is a simplicial complex, then $(id_K)_* = id_{H_n(K)}$

Question When do chain maps induce the same maps on homology?

Def Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be chain maps.

A chain homotopy $h_\bullet$ between $f_\bullet$ and $g_\bullet$ is a collection of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n(c) - f_n(c) = \partial_{n+1} \circ h_n(c) + h_{n-1} \circ \partial_n(c).$$

We say $f_\bullet$ and $g_\bullet$ are chain homotopic and write $f_\bullet \simeq g_\bullet$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1} - g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n - g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[d, "f_{n-1} - g_{n-1}"] \arrow[dl, "h_{n-1}"'] \\
D_{n+1} \arrow[r, "\partial_{n+1}"'] & D_n \arrow[r, "\partial_n"'] & D_{n-1}
\end{tikzcd}$$

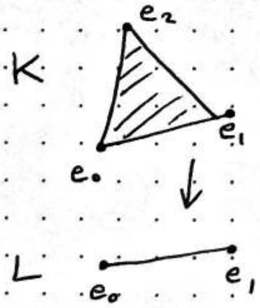
Lemma If $f_\bullet \simeq g_\bullet : C_\bullet \to D_\bullet$, then $f_* = g_* : H_n(C_\bullet) \to H_n(D_\bullet)$

Pf Let $c \in Z_n(C_\bullet)$. Then

$$\begin{aligned}(g_n - f_n)(c) &= \partial_{n+1} \circ h_n(c) + h_{n-1} \circ \partial_n(c) \\ &= \partial_{n+1} \circ h_n(c) \in B_n(D_\bullet)\end{aligned}$$

So $[g_n(c)] = [f_n(c)]$. ▨

Example



$e_0, e_2 \mapsto e_0$
$e_1 \mapsto e_1$

$i : L \to K$ the natural inclusion
$r : K \to L$ the "simplicial retraction"
$e_0 \mapsto e_0, e_1 \mapsto e_1, e_2 \mapsto e_0$

$r \circ i = id_L$, but $i \circ r \neq id_K$

We'll define a chain homotopy between $(i \circ r)_\bullet : C_\bullet(K) \to C_\bullet(K)$
$id_{C_\bullet(K)} : C_\bullet(K) \to C_\bullet(K)$

Then we conclude that $(i \circ r)_* = i_* \circ r_* = id_{H_n(K)}$

Thus $r_* : H_n(K) \to H_n(L)$ is an isomorphism.

Define $h_n : C_n(K) \to C_{n+1}(K)$ which sends every simplex to 0, except $h_0(\langle e_2 \rangle) = \langle e_2, e_0 \rangle$

$h_1(\langle e_1, e_2 \rangle) = -\langle e_0, e_1, e_2 \rangle$

Easy to check that $h_\bullet$ is the desired chain homotopy.

The only non-zero checks are

$$(\partial_1 \circ h_0 + h_{-1} \circ \partial_0)(\langle e_2 \rangle) = \partial_1(\langle e_2, e_0 \rangle) = \langle e_0 \rangle - \langle e_2 \rangle$$

$$= (i_0 \circ r_0 - id_{C_0(K)})(\langle e_2 \rangle)$$

Degree 1 :

$$(\partial_2 \circ h_1 + h_0 \circ \partial_1)(\langle e_1, e_2 \rangle)$$

$$= \partial_2(-\langle e_0, e_1, e_2 \rangle) + h_0(\langle e_2 \rangle - \langle e_1 \rangle)$$

$$= -\langle e_0, e_1 \rangle - \langle e_1, e_2 \rangle - \langle e_2, \cancel{e_0} \rangle + \langle e_2, \cancel{e_0} \rangle$$

another?

$$= (i_1 \circ r_1 - id_{C_1(K)})(\langle e_1, e_2 \rangle)$$

Degree 2

$$(\partial_3 \circ h_{\cancel{1}2} + h_1 \circ \partial_2)(\langle e_0, e_1, e_2 \rangle)$$

$$= h_1(\langle e_1, e_2 \rangle + \langle e_2, e_0 \rangle + \langle e_0, e_1 \rangle)$$

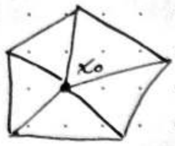
$$= -\langle e_0, e_1, e_2 \rangle$$

$$= (i_2 \circ r_2 - id_{C_2(K)})(\langle e_0, e_1, e_2 \rangle)$$

Thus $h_\bullet$ is the desired chain homotopy.

L19.1

Def A simplicial complex K is a cone if there exists a vertex x_0 such that $\forall \tau \in K$, $\exists \sigma \in K$ such that $x_0 \in \sigma$ and $\tau \leq \sigma$



Lemma If K is a cone then

$$H_n(K) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0 \end{cases}$$

Pf Let $i = \{\langle x_0 \rangle\} \longrightarrow K$ be the inclusion,

$r : K \longrightarrow \{\langle x_0 \rangle\}$ be the constant simplicial map.

Certainly $r \circ i = id_{\{\langle x_0 \rangle\}}$, so $r_* \circ i_* = id_{H_n(\{x_0\})}$

Want to show $i_* \circ r_* = id_{H_n(K)}$, showing $H_n(K) \cong H_n(\{x_0\})$ and the latter is easily calculated to be as desired.

Will build a chain homotopy between $id_{C_\bullet(K)}$ and $i_\bullet \circ r_\bullet$

⌈ $i_\bullet : C_\bullet(\{\langle x_0 \rangle\}) \longrightarrow C_\bullet(K)$

$r_\bullet : C_\bullet(K) \longrightarrow C_\bullet(\{\langle x_0 \rangle\})$ are induced chain maps ⌋

Let σ be a simplex $\langle v_0, \ldots, v_n \rangle \in K$. Then

$$h_n(\sigma) = \begin{cases} 0 & \text{if } x_0 \in \sigma \\ \langle x_0, v_0, \ldots, v_n \rangle & \text{otherwise} \end{cases}$$

$\in K$ since K is a cone

Need to show

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = id_{C_n(K)} - i_n \circ r_n$$

Suppose $n > 0$, $x_0 \notin \sigma$

Then $(\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n)(\sigma)$

$$= \partial_{n+1}(\langle x_0, v_0, \ldots, v_n \rangle) + h_{n-1}\left(\sum_{i=0}^{n} (-1)^i \langle v_0, \ldots, \hat{v}_i, \ldots, v_n \rangle\right)$$

$$= \langle v_0, \ldots, v_n \rangle + \sum_{i=0}^{n} (-1)^{i+1} \langle x_0, v_0, \ldots, \hat{v}_i, \ldots, v_n \rangle$$

$$+ \sum_{i=0}^{n} (-1)^i \langle x_0, v_0, \ldots, \hat{v}_i, \ldots, v_n \rangle$$

$$= \langle v_0, \ldots, v_n \rangle = \sigma$$

$$= id_{C^n(K)}(\sigma) - \underset{\substack{\uparrow \\ \text{zero } \because n>0}}{i_n \circ r_n(\sigma)}$$

Suppose $n>0$, $x_0 \in \sigma$. Say $x_0 = v_j$.

$$(\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n)(\sigma)$$

$$= 0 + h_{n-1}\left(\sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle \right)$$

$$= (-1)^j \langle x_0 = v_j, v_0, \dots, \hat{v}_j, \dots, v_n \rangle$$

$$= \langle v_0, \dots, v_n \rangle = \sigma$$

$$= \mathrm{id}_{C^n(K)}(\sigma) - i_n \circ r_n(\sigma)$$

If $n=0$; $\sigma = \langle v_0 \rangle$, $v_0 \neq x_0$, get

$$(\partial_1 \circ h_0 + h_{-1} \circ \partial_0)(\sigma) = \partial_1(\langle x_0, v_0 \rangle)$$

$$= \langle v_0 \rangle - \langle x_0 \rangle$$

$$= \mathrm{id}_{C_0(K)}(\sigma) - i_0 \circ r_0(\sigma)$$

If $\sigma = \langle x_0 \rangle$, get

$$(\partial_1 \circ h_0 + h_{-1} \circ \partial_0)(\sigma) = 0$$

$$= \mathrm{id}_{C_0(K)}(\sigma) - i_0 \circ r_0(\sigma)$$

$\therefore$ $h_\cdot$ is the desired chain homotopy. ▨

<u>Example</u> Let K be an n-simplex along with all its faces

K is a cone (any vertex will play the role of x_0)

Thus $H_n(K) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{o/w} \end{cases}$

<u>Example</u> Take $L = \partial\sigma_n$, $|L| = S^{n-1}$, $n \geq 2$

Note the obvious inclusion $L \hookrightarrow K$ of simplicial complexes (K as in previous example) induces a chain map

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & C_{n-1}(L) \arrow[r, "\partial_{n-1}"] \arrow[d, "="] & \cdots \arrow[r, "\partial_1"] & C_0(L) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C_n(K) \arrow[r] & C_{n-1}(K) \arrow[r, "\partial_{n-1}"'] & \cdots \arrow[r] & C_0(K) \arrow[r] & 0
\end{tikzcd}$$

Vertical equalities; set of d-simplices of L and K are the same for $d<n$. Also every square is commutative. Note then that $H_d(L) \cong H_d(K)$ for $d \leq n-2$ (same kernels & images)

Note since $H_{n-1}(K) = 0$, we must have $Z_{n-1}(K) = B_{n-1}(K)$.

But $Z_{n-1}(L) = Z_{n-1}(K)$,

but $B_{n-1}(L) = 0$.

Thus $H_{n-1}(L) \cong Z_{n-1}(L) \cong B_{n-1}(K)$

But note K has one n-simplex σ, so $C_n(K) = \mathbb{Z}\sigma$ and then

∂_n is injective, and hence $B_{n-1}(K) \cong \mathbb{Z}$.

Thus $H_d(L) = \begin{cases} \mathbb{Z} & \text{if } d=0 \text{ or } n-1 \\ 0 & \text{otherwise} \end{cases}$

<u>Moral</u> Homology can detect "higher dimensional holes"

§ <u>Continuous maps and homotopies</u>

We would like, for any map $\phi : |K| \to |L|$ a map

$$\phi_* : H_n(K) \longrightarrow H_n(L).$$

<u>Idea</u> Use a simplicial approximation $f : K^{(r)} \longrightarrow L$ and then define

$$\phi_* = f_* : H_n(K^{(r)}) \longrightarrow H_n(L)$$

We would then need to show $H_n(K^{(r)}) = H_n(K)$.

<u>Eventually</u> We'll show $H_n(K)$ only depends on $|K|$

First, will define a simplicial version of homotopy.

<u>Def</u> Two simplicial maps $f, g : K \to L$ are <u>contiguous</u> if, for every $\sigma \in K$, there is $\tau \in L$ such that $f(\sigma)$ and $g(\sigma)$ are both faces of τ.

<u>Remark</u> Suppose given $\phi : |K| \to |L|$ with $f, g : K \to L$ two different simplicial approximations to ϕ.

If $x \in \sigma^\circ$ and $\phi(x) \in \tau^\circ$, we showed that $f(\sigma) \leq \tau$ & $g(\sigma) \leq \tau$ (in the proof that $|f|, |g| \simeq \phi$) and hence f and g are contiguous

<u>Lemma</u> $f, g : K \to L$ contiguous, then

$$f_* = g_* : H_n(K) \longrightarrow H_n(L) \quad \forall n$$

<u>Proof</u> Will construct a chain homotopy.

Fix a total order $<$ on vertices of K and use the convention that each $\sigma \in K$ is oriented $\sigma = \langle v_0, \dots, v_n \rangle$ with $v_0 < v_1 < \dots < v_n$.

L19.4

Notationally, will write

$$\langle f(v_0), \dots, f(v_i), g(v_i), \dots, g(v_n) \rangle = 0$$

if these points are not in general position.

Define $h_n : C_n(K) \longrightarrow C_{n+1}(L)$

by $h_n(\langle v_0, \dots, v_n \rangle) = \sum_{j=0}^{n} (-1)^j \langle f(v_0), \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle$

Note that the i^{th} term, if non-zero, is a simplex of L, because $f(v_0), \dots, f(v_i), g(v_i), \dots, g(v_n)$ are all vertices of some simplex of L, by contiguity of f, g.

$$\begin{aligned}
&(\partial \circ h + h \circ \partial)(\sigma) \\
&= \partial \Big(\sum_{j=0}^{n} (-1)^j \langle f(v_0), \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle \Big) \\
&\quad + h \Big(\sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle \Big) \\
&= \sum_{i \le j} (-1)^{i+j} \langle f(v_0), \dots, \widehat{f(v_i)}, \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle \\
&\quad - \sum_{i \geq j} (-1)^{i+j} \langle f(v_0), \dots, f(v_j), g(v_j), \dots, \widehat{g(v_i)}, \dots, g(v_n) \rangle \\
&\quad + \sum_{j < i} (-1)^{i+j} \langle f(v_0), \dots, f(v_j), g(v_j); \dots, \widehat{g(v_i)}, \dots, g(v_n) \rangle \\
&\quad - \sum_{j > i} (-1)^{i+j} \langle f(v_0), \dots, \widehat{f(v_i)}, \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle \\
&= \sum_{i=0}^{n} \langle f(v_0), \dots, f(v_{i-1}), g(v_i); \dots, g(v_n) \rangle \\
&\quad - \sum_{i=0}^{n} \langle f(v_0), \dots, f(v_i); g(v_{i+1}); \dots, g(v_n) \rangle \\
&= \langle g(v_0), \dots, g(v_n) \rangle - \langle f(v_0), \dots, f(v_n) \rangle \quad \text{as desired.} \quad ▨
\end{aligned}$$

L20.1

Lemma Let K be a simplicial complex, K' be its first barycentric subdivision. A simplicial map $s : K' \to K$ is a simplicial approximation to the identity $id_{|K|}$ if and only if for every $\sigma \in K$, ~~$\hat{\sigma}(s)$~~ $s(\hat{\sigma})$ is a vertex of ~~t~~ σ.

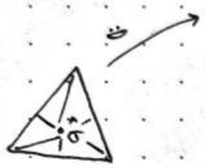
Also such an s exists.

Proof Let $s : K' \to K$ be a simplicial approximation to $id_{|K|}$.

Then $id_{|K|}(St_{K'}(\hat{\sigma})) \subseteq St_K(s(\hat{\sigma}))$

$\cup|$

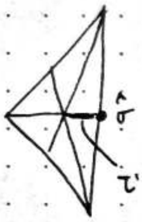
σ°



So $s(\hat{\sigma})$ is a vertex of σ.

Conversely, suppose $s(\hat{\sigma})$ is a vertex of σ $\forall \sigma \in K$.

Let $\tau' \in K'$ with $\mathring{\tau}' \subseteq St_{K'}(\hat{\sigma})$ i.e. $\hat{\sigma}$ a vertex of τ'.



Then $\mathring{\tau}'$ is contained in the entirior of a simplex $\tau \in K$ such that $\sigma \leq \tau$.

Thus $s(\hat{\sigma})$ is also a vertex of τ.

$$\mathring{\tau}' \subseteq \tau^\circ \subseteq St_K(s(\hat{\sigma}))$$

Since the $\mathring{\tau}'$ cover $St_{K'}(\hat{\sigma})$, we get

$$Id_{|K|}(St_{K'}(\hat{\sigma})) \subseteq St_K(s(\hat{\sigma}))$$

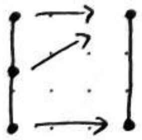
Thus s is a simplicial approximation to $id_{|K|}$.

Constructing s For each $\hat{\sigma}$, choose $s(\hat{\sigma})$ to be a vertex of σ.

A simplex of K' is of the form $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$ with $\sigma_0 \leq \dots \leq \sigma_n$.

So all vertices of σ_i are vertices of σ_n and hence

$\langle s(\hat{\sigma}_0), \dots, s(\hat{\sigma}_n) \rangle$ is a face of σ_n; in particular a simplex. ▨



Prop If $s : K' \to K$ is a simplicial approximation to $id_{|K|}$, then $s_* : H_n(K') \to H_n(K)$ is an isomorphism for all n.

Proof Postponed until more machinery is developed. ▨

Cor Let K be a simplicial complex. We may canonically identity $H_n(K^{(r)})$ with $H_n(K)$.

Pf Enough to check for $r = 1$.

We choose a simplicial approximation to $id_{|K|}$, $s : K' \to K$, which then induces an isomorphism $H_n(K') \to H_n(K)$ by prop^n.

But any two choices $s, s' : K' \to K$ are contiguous, and contiguous simplicial maps induce the same maps $s_* = s'_*$ on homology. ▨

Notation We write $\nu_{K,r} : H_n(K)^{(r)} \xrightarrow{\sim} H_n(K)$

$$\nu_{K,r,s} : H_n(K^{(r)}) \xrightarrow{\sim} H_n(K^{(s)}) \quad , r \geqslant s$$

Note $\nu_{K,r_1,r_2} \circ \nu_{K,r_3,r_1} = \nu_{K,r_3,r_2}$ for $r_3 \geqslant r_1 \geqslant r_2$

Prop To each continuous map $f : |K| \to |L|$ there is an associated homomorphism $f_* : H_n(K) \to H_n(L)$ given by $f_* = s_* \circ \nu_{K,r}^{-1}$ where $s : K^{(r)} \to L$ is a simplicial approximation to f. Then

① f_* does not depend on the choice of r or s

② If $g : |M| \to |K|$ is a map then $(f \circ g)_* = f_* \circ g_*$

Proof For ①, let $s : K^{(r)} \to L$, $t : K^{(q)} \to L$ both simplicial approximations to f, $r \geqslant q$. Let $a : K^{(r)} \to K^{(q)}$ be a simplicial approx^n to the identity.

Then $s, t \circ a : K^{(r)} \to L$ are both simplicial approximations to f.

⌈Using that if $f : |K| \to |L|$, $g : |M| \to |K|$ have simplicial approx^{n}s $s : K \to L$, $t : M \to K$ then $s \circ t$ is a simp approx^n to $f \circ g$.⌋ ✓

Thus $s, t \circ a$ are contiguous, hence induce the same homomorphisms

$$s_* = (t \circ a)_* = t_* \circ a_* : H_n(K^{(r)}) \to H_n(L)$$

Note $a_* = \nu_{K,r,q}$, so

$$\underbrace{s_* \circ \nu_{K,r}^{-1}}_{f_* \text{ via } s} = t_* \circ \nu_{K,r,q} \circ \nu_{K,r}^{-1} = \underbrace{t_* \circ \nu_{K,q}}_{f_* \text{ via } t}$$

Hence f_* is well-defined.

For ②, let $s: K^{(r)} \to L$, $t: M^{(q)} \to \cancel{M} \, K^{(r)}$ be simplicial approx^{n}s to f, g respectively. Then $s \circ t$ is a simplicial approx^n to $f \circ g$, so

$$
\begin{aligned}
(f \circ g)_* &= (s \circ t)_* \circ \nu_{M,q}^{-1} \\
&= s_* \circ t_* \circ \nu_{M,q}^{-1} \\
&= (s_* \circ \nu_{K,r}^{-1}) \circ (\nu_{K,r} \circ t_* \circ \nu_{M,q}^{-1}) \\
&= f_* \circ g_* . \quad ▨
\end{aligned}
$$

Cor If $|K| \cong |L|$ then $H_n(K) \cong H_n(L)$

One remaining question; what is the relation between f_* and g_* if $f \simeq g$?

Lemma For L a simplicial complex in $\mathbb{R}^m$, there exists $\varepsilon = \varepsilon(L) > 0$ such that if $f, g : |K| \to |L|$ satisfy $|f(x) - g(x)| < \varepsilon \ \forall x \in |K|$, then $f_* = g_* : H_n(K) \to H_n(L)$.

Pf The set $\{ St_L(w) \mid w \in L \}$ forms an open cover of $|L|$.

So by the Lebesgue number Lemma, $\exists \varepsilon > 0$ s.t. each ball of radius 2ε in $|L|$ lies in some $St_L(w)$. We take $\varepsilon(L) = \varepsilon$.

Let $f, g : |K| \to |L|$ as in the statement.

Consider the open cover of $|K|$ given by

$$\{ f^{-1}(B_\varepsilon(y)) \mid y \in |L| \}$$

By the Lebesgue number lemma, $\exists \delta > 0$ s.t. each $B_\delta(x)$ lies in some $f^{-1}(B_\varepsilon(y))$, or equivalently, $f(B_\delta(x)) \subseteq B_\varepsilon(y)$.

So $g(B_\delta(x)) \subseteq B_{2\varepsilon}(y)$.

Choose r so that $\mathrm{Mesh}(K^{(r)}) < \frac{1}{2}\delta$. Then for each vertex $v \in K^{(r)}$, the diameter of $St_{K^{(r)}}(v) < \delta$, so $f(St_{K^{(r)}}(v))$ and $g(St_{K^{(r)}}(v))$ both lie in some $St_L(w)$.

(since $B_{2\varepsilon}(y) \subseteq St_L(w)$ for some w.)

Set $s(v) = w$. Then s is a simplicial approx^n to both f and g (see the construction of simplicial approximations)

Thus $f_* = s_* \circ \nu_{K,r}^{-1} = g_*$. ▨

Theorem If $f \simeq g : |K| \longrightarrow |L|$ then $f_* = g_* : H_n(K) \longrightarrow H_n(L)$

Pf Let $H : |K| \times I \rightarrow |L|$ be the homotopy between f and g.

As $|K| \times I$ is compact, H is uniformly continuous.

Thus for $\varepsilon = \varepsilon(L)$ from lemma, $\exists \delta > 0$ s.t. $\forall x \in |K|$

$$|s - t| < \delta \Rightarrow |H(x,s) - H(x,t)| < \varepsilon$$

Now choose $0 = t_0 < t_1 < \cdots < t_k = 1$ s.t. $t_i - t_{i-1} < \delta \ \forall i$, and let $f_i(x) := H(x, t_i)$, $f_i : |K| \rightarrow |L|$.

Then $|f_i(x) - f_{i-1}(x)| < \varepsilon \ \forall x \in |K|$, and thus by the Lemma,

$$(f_i)_* = (f_{i-1})_*$$

Thus $f_* = (f_0)_* = \cdots = (f_k)_* = g_*$. $\square$

Cor If $|K|$ and $|L|$ are homotopy equivalent, then $H_n(K) \cong H_n(L) \ \forall n$.

Def [Still dependant on proof that $H_n(K') \cong H_n(K)$] We have seen that $H_n(K)$ only depends on $|K|$, so if $X \cong |K|$, we write

$$H_n(X) := H_n(K)$$

§6 Homology calculations and applications

Ex If $S^n \simeq S^m$ then $n = m$

[Follows immediately from fact that $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n, \\ 0 & \text{o/w.} \end{cases}$]

Theorem If $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$

Proof Suppose given a homeomorphism $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Compose with translation and can then assume $\phi(0) = 0$, and then we get a homeomorphism $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$.

But each are homotopic to S^{m-1} and S^{n-1} respectively, so $m = n$ by the example. $\square$

Exercise Any map $\phi : D^n \rightarrow D^n$ has a fixed point (Exactly as in D^2)

§ Mayer–Vietoris theorem

(Analog of Seifert-van Kampen theorem)

Def A sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{i+1} \xrightarrow{f_i} A_i \xrightarrow{f_{i-1}} A_{i-1} \longrightarrow \cdots$$

is exact at A_i if $\mathrm{im}(f_i) = \mathrm{Ker}(f_{i-1})$.

The sequence is exact if it is exact at every A_i.

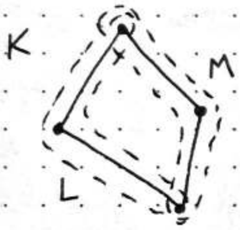
A short exact sequence is an exact sequence

$$0 \to A \to B \to C \to 0$$

[i.e. $A \to B$ is injective, $B \to C$ is surjective;

$C \cong B/A$ by iso thm]

Setup for Mayer-Vietoris



Let K be a simplicial complex with

$K = L \cup M$ with $L, M \subseteq K$ subcomplexes

$N = L \cap M$ is also a subcomplex

We write $K = L \cup_N M$.

Let $i : N \to L$, $j : N \to M$, $l : L \to K$, $m : M \to K$ be the inclusion maps

Theorem (Mayer-Vietoris) In the above setup, $\exists$ a map $\forall n$

$\delta_* : H_n(K) \longrightarrow H_{n-1}(N)$ making

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}(K) \arrow[dll, "\delta_*"'] & \\ H_n(N) \arrow[r, "i_* \oplus j_*"'] & H_n(L) \oplus H_n(M) \arrow[r, "l_* - m_*"'] & H_n(K) \arrow[dl, "\delta_*"'] & \\ & H_{n-1}(N) \arrow[r] & \cdots & \end{tikzcd}$$

an exact sequence.

Proof will follow from more <u>homological algebra</u>

<u>Def</u> We say a sequence of chain maps

$$\begin{tikzcd} A_\bullet \arrow[r, "f_\bullet"'] & B_\bullet \arrow[r, "g_\bullet"'] & C_\bullet \end{tikzcd}$$

is <u>exact at $B_\bullet$</u> if

$$\begin{tikzcd} A_n \arrow[r, "f_n"'] & B_n \arrow[r, "g_n"'] & C_n \end{tikzcd} \quad \text{is exact } \forall n.$$

<u>Lemma</u> (Snake lemma) Let

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] & C_\bullet \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence of chain complexes. Then for every $n \in \mathbb{Z}$, $\exists$ a homomorphism $\delta_* : H_{n+1}(C_\bullet) \to H_n(A_\bullet)$ making

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(C_\bullet) \arrow[dll, rounded corners, "\delta_*"'] & \\ H_n(A_\bullet) \arrow[r, "f_*"'] & H_n(B_\bullet) \arrow[r, "g_*"'] & H_n(C_\bullet) \arrow[dll, rounded corners, "\delta_*"'] \\ H_{n-1}(A_\bullet) \arrow[r] & \cdots & \end{tikzcd}$$

exact.

<u>Pf</u> <u>'Trivial' diagram chase</u>

$$\begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & {} \arrow[d, "\partial_{n+2}"] & \\ 0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \ni y \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"] & C_{n+1} \ni x \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\ 0 \arrow[r] & z \in A_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\ & {} & {} & {} & \end{tikzcd}$$

<u>Construction of $\delta_* : H_{n+1}(C_\bullet) \to H_n(A_\bullet)$</u>

Take $[x] \in H_{n+1}(C_\bullet)$, $x \in Z_{n+1}(C_\bullet)$

As g_{n+1} is surjective, $\exists\, y \in B_{n+1}$ s.t. $g_{n+1}(y) = x$.

Now $g_n \circ \partial_{n+1}(y) = \partial_{n+1} \circ g_{n+1}(y) = \partial_{n+1}(x) = 0$ since $x \in Z_{n+1}(C_\bullet)$

Hence by exactness, $\exists\, z \in A_n$ s.t. $f_n(z) = \partial_{n+1}(y)$. (unique)

Note that $f_{n-1} \circ \partial_n(z) = \partial_n \circ f_n(z) = \partial_n \circ \partial_{n+1}(y) = 0$

so since f_{n-1} is injective, $z \in Z_n(A_\bullet)$ and we define $[z] \in H_n(A_\bullet)$

L21.4

We define $\delta_*([x]) = [z]$, but we need to check this is well-defined.

Two issues: ① May replace x with $x + \partial_{n+2}(x')$.

Let $y' \in B_{n+2}$ be s.t. $g_{n+2}(y') = x'$

Then replace y with $y + \partial_{n+2}(y')$, indeed

$$g_{n+1}(y + \partial_{n+2}(y')) = g_{n+1}(y) + \partial_{n+2} \circ g_{n+2}(y')$$
$$= x + \partial_{n+2}(x')$$

Now $\partial_{n+1}(y + \partial_{n+2}(y')) = \partial_{n+1}(y) + \cancelto{\text{zero}}{\partial_{n+1} \circ \partial_{n+2}(y')}$

$$= \partial_{n+1}(y),$$

so need not change z.

② We may choose a different lift y' of x, i.e.

$$g_{n+1}(y') = g_{n+1}(y) = x$$

Hence $g_{n+1}(y' - y) = 0$

so $\exists\, z' \in A_{n+1}$ with $f_{n+1}(z') = y' - y$

Thus $y' = y + f_{n+1}(z')$.

So $\partial_{n+1}(y') = \partial_{n+1}(y) + \partial_{n+1} \circ f_{n+1}(z')$

$$= \partial_{n+1}(y) + f_n \circ \partial_{n+1}(z')$$

Thus, if we replace z by $z + \partial_{n+1}(z')$ we see that

$$f_n(z + \partial_{n+1}(z')) = \partial_{n+1}(y) + f_n \circ \partial_{n+1}(z')$$
$$= \partial_{n+1}(y')$$

Thus $[z] = [z + \partial_{n+1}(z')]$.

Thus δ_* is well-defined.

Exactness of the sequence

Exactness at $H_n(B_\cdot)$

If $[a] \in H_n(A_\cdot)$ then

$$g_* \circ f_* ([a]) = [g_n \circ f_n(a)] = 0$$

since $0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ is exact, so $g_n \circ f_n = 0$.

Thus $\operatorname{Im} f_* \subseteq \operatorname{Ker} g_*$.

With $[b] \in H_n(B_\cdot)$ s.t. $0 = g_*([b]) = [g_n(b)]$

there exists $c \in C_{n+1}$ with $g_n(b) = \partial_{n+1}(c)$.

But then there exists $b' \in B_{n+1}$ with $g_{n+1}(b') = c^{\ddagger}$. Then

$$\begin{aligned} g_n(b - \partial_{n+1}(b')) &= g_n(b) - \partial_{n+1} \circ g_{n+1}(b') \\ &= g_n(b) - \partial_{n+1}(c) \\ &= 0 \end{aligned}$$

Thus $\exists\, a \in A_n$ s.t. $f_n(a) = b - \partial_{n+1}(b')$.

In particular $f_*[a] = [f_n(a)] = [b - \partial_{n+1}(b')] = [b]$.

check $a \in Z_n(A_\cdot)$

Thus $\operatorname{Ker} g_* \subseteq \operatorname{Im} f_*$ and hence $\operatorname{Ker} g_* = \operatorname{Im} f_*$.

Exactness at $H_n(A_\cdot)$

Suppose $[z] = \delta_*([x])$. Then

$$f_*([z]) = [f_n(z)] = [\partial_{n+1}(y)]$$

where y as in the construction of $\delta_*([x])$, so $g_{n+1}(y) = x$.

But $[\partial_{n+1}(y)] = 0$, so $f_*([z]) = 0$ so

$\operatorname{im} \delta_* \subseteq \ker f_*$.

Suppose now that $f_*([z]) = 0$.

Then $f_n(z) = \partial_{n+1}(y)$ for some $y \in B_{n+1}$.

Take $x = g_{n+1}(y)$.

Then $\partial_{n+1}(x) = \partial_{n+1} \circ g_{n+1}(y) = g_n \circ \partial_{n+1}(y) = g_n \circ f_n(z) = 0$.

Thus x is a cycle, so x represents a homology class $[x]$.

But $\delta_*([x]) = [z]$ by construction of δ_*.

Thus $\operatorname{Ker} f_* \subseteq \operatorname{im} \delta_*$.

Exactness at $H_n(C_\cdot)$

If $[x] \in \mathrm{Im}\, g_*$; i.e. after replacing x with another representative ; we can assume that $\exists\, y \in B_n$ s.t. $g_n(y) = x$, and $y \in Z_n(B_\cdot)$, i.e. $\partial_n(y) = 0$.

Then $\delta_*([x]) = 0$ by construction, so $\mathrm{Im}\, g_* \subseteq \mathrm{Ker}\,\delta_*$

Next suppose $\delta_*([x]) = 0$, i.e. in the construction of $\delta_*([x])$, $z \in A_{n-1}$ is a boundary i.e. $\exists\, a \in A_n$ s.t. $\partial_n(a) = z$

Then $\partial_n \circ f_n(a) = f_{n-1} \circ \partial_n(a) = f_{n-1}(z) = \partial_n(y)$ with $g_n(y) = x$ by construction of δ_*

Thus $\partial_n(y - f_n(a)) = 0$, so $y - f_n(a)$ is a cycle.

Also
$$\begin{aligned} g_n(y - f_n(a)) &= g_n(y) - g_n \circ f_n(a) \\ &= g_n(y) = x \end{aligned}$$

So $[x] = g_*[y - f_n(a)]$ i.e. x is the image of a cycle in $Z_n(B_\cdot)$.

Thus $[x] \in \mathrm{Im}\, g_*$ so $\mathrm{Ker}\,\delta_* \subseteq \mathrm{Im}\, g_*$. □

Proof of Mayer – Vietoris

It's easy to check that

$$H_n(\underset{\uparrow}{C_\cdot \oplus D_\cdot}) = H_n(C_\cdot) \oplus H_n(D_\cdot)$$

chain complex whose
n^{th} term is $C_n \oplus D_n$,
$\partial_n(c,d) = (\partial_n c, \partial_n d)$

It then suffices so thow that

$$0 \to C_\cdot(N) \xrightarrow[i_\cdot \oplus j_\cdot]{} C_\cdot(L) \oplus C_\cdot(M) \xrightarrow[l_\cdot - m_\cdot]{} C_\cdot(K) = 0$$

is a short exact sequence of chain complexes:

Note $C_\cdot(N)$ is a 'subgroup' of both $C_\cdot(L)$ and $C_\cdot(M)$, so $i_\cdot \oplus j_\cdot$ is injective.

Since $K = L \cup M$, if $c \in C_n(K)$, we can write $c = c_L + c_M$ where c_L is a linear combination of simplices in L and c_M a lin combi at M.

Thus $\exists\, b_L \in C_n(L)$, $b_M \in C_n(M)$ with $l_n(b_L) = c_L$, $m_n(b_M) = c_M$

Thus $c = l_n(b_L) - m_n(-b_M)$

$= (l_n - m_n)(b_L, -b_M)$

So $l_\cdot - m_\cdot$ is surjective.

Exactness in the middle

For $(b_L, b_M) \in C_n(L) \oplus C_n(M)$ note

$l_n(b_L) - m_n(b_M) = 0$

iff every simplex that occurs in b_L also occurs in b_M with the same coeff.

Thus both b_L, b_M are linear combinations of simplices in $L \cap M = N$.

Thus $\exists\, a \in C_n(M)$ with $b_L = i_n(a)$, $b_M = j_n(a)$

$\therefore (b_L, b_M) = i_n \oplus j_n(a)$

Thus $\operatorname{Ker} l_\cdot - m_\cdot \subseteq \operatorname{im} i_\cdot \oplus j_\cdot$

$(l_\cdot - m_\cdot)(i_\cdot \oplus j_\cdot)(a) = (l_\cdot - m_\cdot)(i_\cdot(a), j_\cdot(a))$

$= l_\cdot \circ i_\cdot(a) - m_\cdot \circ j_\cdot(a)$

$= 0$ since $l \circ i = m \circ j$

Thus $\operatorname{im} i_\cdot \oplus j_\cdot \subseteq \operatorname{Ker} l_\cdot - m_\cdot$ $\square$

5 - Lemma Given a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

of abelian groups with exact rows.

If $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, so is γ

Pf Exercise, c.f. Example Sheet 4. ▨

Can now prove $s_*: H^n(K') \to H^n(K)$ is an isomorphism

[Recall $s: K' \to K$ was constructed to be a simplicial approx^n to $id_{|K|}$.

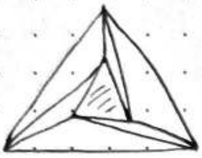
Pf Induction on the number of simplices of K.

If $\# K = 1$, i.e. $K = \{pt\} = \{\cdot\}$ then since $K' = K$, the statement is obvious.

L22.4

Induction step Let $\sigma \in K$ be of maximal dimension, in particular not the proper face of another simplex in K.

Let $L = K \setminus \{\sigma\}$, M = the simplicial complex consisting of σ and its faces



$N = L \cap M = \partial\sigma$

We also write s for the restriction of s to the barycentric subdivisions of L, M and N, so we get

$s_* : H_n(L') \to H_n(L)$

$s_* : H_n(M') \to H_n(M)$

$s_* : H_n(N') \to H_n(N)$

By induction hypothesis, these are all isomorphisms.

Apply Mayer-Vietoris to get a diagram

$$\begin{tikzcd}
H_n(N') \arrow[r] \arrow[d, "s_*"] & H_n(L') \oplus H_n(M') \arrow[r] \arrow[d, "s_*"] & H_n(K') \arrow[r] \arrow[d, "s_*"] & H_{n-1}(N') \arrow[r] \arrow[d, "s_*"] & H_{n-1}(L') \oplus H_{n-1}(M') \arrow[d, "s_*"] \\
H_n(N) \arrow[r] & H_n(L) \oplus H_n(M) \arrow[r] & H_n(K) \arrow[r] & H_{n-1}(N) \arrow[r] & H_{n-1}(L) \oplus H_{n-1}(M)
\end{tikzcd}$$

commutes!

Thus by 5-Lemma, $s_* : H_n(K') \to H_n(K)$ is an isomorphism $\square$

§ Homology of compact surfaces

We constructed the oriented surface of genus g,

$$\Sigma_g = \underset{\substack{\uparrow \\ \text{bouquet} \\ \text{of } 2g\ S^1\text{'s}}}{\Gamma_{2g}} \cup_{\rho_g} D^2 \quad , \quad \rho_g = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}$$

<u>Example</u> $\Gamma_r = \bigvee_{i=1}^{r} S^1$

Then Γ_r is triangleable, e.g.

Γ_2 

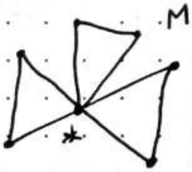
Have $\Gamma_1 \cong S^1$, so

$$H_0(\Gamma_1) = \mathbb{Z}$$
$$H_1(\Gamma_1) = \mathbb{Z}$$
$$H_i(\Gamma_1) = 0 \quad \forall i > 1$$

Suppose we have shown that

$$H_i(\Gamma_{r-1}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^{r-1} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Then we can use Mayer-Vietoris to calculate $H_i(\Gamma_r)$



M a triangulation of S^1

L a triangulation of Γ_{r-1}

$K = L \cup M$ is a traigulation of Γ_r

$L \cap M = N = \{*\}$

Mayer-Vietoris gives a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \underset{\text{zero}}{\underset{\shortparallel}{H_1(N)}} \arrow[r] & \underset{\mathbb{Z}^{r-1}}{\underset{\shortparallel}{H_1(L)}} \oplus \underset{\mathbb{Z}}{\underset{\shortparallel}{H_1(S^1)}} \arrow[r, "\text{surj}"] & H_1(\Gamma_r) \arrow[dll, "\leftarrow \text{zero}"] & \\
& \underset{\mathbb{Z}}{\underset{\shortparallel}{H_0(N)}} \arrow[r, "\underset{\text{inj}}{\uparrow}"'] & \underset{\mathbb{Z}}{\underset{\shortparallel}{H_0(\Gamma^{r-1})}} \oplus \underset{\substack{\uparrow \\ \text{since } H_0 \\ = \mathbb{Z}^{\#\text{ connected components}}}}{\underset{\mathbb{Z}}{\underset{\shortparallel}{H_0(S^1)}}} \arrow[r] & \underset{\mathbb{Z}}{\underset{\shortparallel}{H_0(\Gamma_r)}} \arrow[r] & 0
\end{tikzcd}$$

Note that the bottom row gives an exact sequence

$$\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

which implies the <u>first</u> map is necessarily injective.

Note also one can argue that whenever N is connected, the maps

$$H_0(N) \to H_0(L) \ , \quad H_0(M) \quad \text{are injective.}$$

L23.2

$\therefore$ we get a short exact sequence

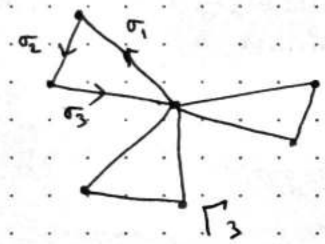
$$0 \to \mathbb{Z}^{r-1} \oplus \mathbb{Z} \longrightarrow H_1(\Gamma^r) \longrightarrow 0$$

Thus $H_1(\Gamma^r) \cong \mathbb{Z}^r$ as desired.

Conclusion

$$H_i(\Gamma_r) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}^r, & i=1 \\ 0, & i>1 \end{cases}$$

Note if $\alpha_1, \dots, \alpha_r$ represent the homology classes of the r circles, then $H_1(\Gamma_r)$ is generated by $\alpha_1, \dots, \alpha_r$.



$\alpha_1 = \sigma_1 + \sigma_2 + \sigma_3$

Remark Whenever ~~K, L, M~~, N ~~are~~ is connected, the map

$H_0(N) \longrightarrow H_0(L) \oplus H_0(M)$ is injective

hence $H_1(L) \oplus H_1(M) \longrightarrow H_1(K)$ is surjective.

Calculate homology of Σ_g We attack a disk in 2-steps

First attach a cylinder

$\Sigma_g^* := \Gamma_{2g} \cup_{\rho_g} (S^1 \times I)$, $\rho_g : S^1 \times \{0\} \to \Gamma_{2g}$ given by $\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}$

By shrinking I to a point, we obtain a deformation retraction of Σ_g^* onto Γ_{2g}, hence $\Sigma_g^* \simeq \Gamma_{2g}$.

Thus $H_i(\Sigma_g^*) \cong H_i(\Gamma_{2g})$.

Then $\Sigma_g = \Sigma_g^* \cup_\alpha D^2$ where $\alpha : \partial D^2 \to S^1 \times \{1\}$ the identity

Choose triangulations of Σ_g^* and D^2 so that they are compatible via gluing map α

Let L be the triangulation of ~~X~~ Σ_g^*

M the triangulation of D^2

$K = L \cup M$ a triangulation of Σ_g

$N = L \cap M$ a triangulation of $S^1 \times \{1\}$

$$\begin{tikzcd}
0 \arrow[r] & \underset{\underset{0 = H_2(\Gamma_{2g})}{\parallel}}{H_2(\Sigma_g^*)} \oplus \overset{\overset{0}{\parallel}}{H_2(D^2)} \arrow[r] & H_2(\Sigma_g) \arrow[dll, rounded corners] & \\
\underset{\underset{\mathbb{Z}}{\parallel}}{H_1(S^1)} \arrow[r] & \underset{\underset{\mathbb{Z}^{2g} \oplus 0}{\parallel}}{H_1(\Sigma_g^*) \oplus H_1(D^2)} \arrow[r] & H_1(\Sigma_g) \arrow[r, "\underset{\text{by remark}}{\uparrow}"'] & 0
\end{tikzcd}$$

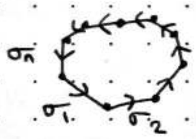
$$\begin{tikzcd}
0 \arrow[r] & H_2(\Sigma_g) \arrow[r] & \overset{\overset{\mathbb{Z}}{\parallel}}{H_1(S^1)} \arrow[r, "i_*"'] & \overset{\overset{\mathbb{Z}^{2g}}{\parallel}}{H_1(\Sigma_g^*)} \arrow[r] & H_1(\Sigma_g) \arrow[r] & 0
\end{tikzcd}$$

So $H_2(\Sigma_g) \cong \operatorname{Ker} i_*$

$H_1(\Sigma_g) \cong \operatorname{coker} i_* := \dfrac{H_1(\Sigma_g^*)}{\operatorname{Im} i_*}$ (Noether's first isom theorem)

Have to understand i_*. Note that

$1 \in \mathbb{Z} \cong H_1(S^1)$ is represented by the circle:



the generator of $H_1(S^1)$ is $\sigma_1 + \cdots + \sigma_n$

The deformation retract of Σ_{2g}^* to Γ_{2g} identifies S^1 with image of $\rho_g : S^1 \longrightarrow \Gamma_{2g}$, thus

$$\rho_{g*}(\sigma_1 + \cdots + \sigma_n) = [\alpha_1] + [\beta_1] - [\alpha_1] - [\beta_1] + \cdots$$
$$= 0$$

Thus $i_* = 0$, whence

$H_2(\Sigma_g) \cong H_1(S^1) \cong \mathbb{Z}$

$H_1(\Sigma_g) \cong H_1(\Sigma_g^*) \cong \mathbb{Z}^{2g}$

$H_0(\Sigma_g) \cong \mathbb{Z}$

<u>Note</u> We can detect g via the homology of Σ_g

Calculate $H_1(S_g)$, S_g the non-orientable surface of genus g

$$S_g = \Gamma_{g+1} \cup_\alpha D^2, \quad \alpha = \alpha_0^2 \alpha_1^2 \cdots \alpha_g^2$$

Repeating the previous argument, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_2(S_g) \arrow[r] & \underset{\underset{\mathbb{Z}}{\parallel}}{H_1(S_1)} \arrow[r, "i_*"'] & \underset{\underset{\mathbb{Z}^{g+1}}{\parallel}}{H_1(\Gamma_{g+1})} \arrow[r] & H_1(S_g) \arrow[r] & 0
\end{tikzcd}$$

Now $i_*(\text{generator}) = 2[\alpha_0] + \cdots + 2[\alpha_g]$, and i_* is injective.

$$\operatorname{coker} i_* = \frac{\mathbb{Z}^{g+1}}{\mathbb{Z}(2,2,\dots,2)} \cong \mathbb{Z}^g \oplus \mathbb{Z}/2\mathbb{Z}$$

So
$$\begin{cases} H_0(S_g) = \mathbb{Z} \\ H_1(S_g) = \mathbb{Z}^g \oplus \mathbb{Z}/2\mathbb{Z} \\ H_i(S_g) = 0 \quad \text{for } i>1 \end{cases}$$

§ Rational homology and Euler characteristic

Def Let K be a simplicial complex

Define the $\mathbb{Q}$-vector space of rational n-chains to be the $\mathbb{Q}$-vector space with basis the set of n-simplices of K. We denote this space by $C_n(K;\mathbb{Q})$.

Define $\partial_n : C_n(K;\mathbb{Q}) \to C_{n-1}(K;\mathbb{Q})$

exactly as before, $Z_n(K;\mathbb{Q}) = \ker \partial_n$

$$B_n(K;\mathbb{Q}) = \operatorname{im} \partial_{n+1}$$

$$H_n(K;\mathbb{Q}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

This is a $\mathbb{Q}$-vector spaces.

This contains less information than $H_n(K;\mathbb{Z})$

Lemma If $H_n;\mathbb{Z}(K) \cong \mathbb{Z}^b \oplus F$ where F is a finite abelian group, then $H_n(K;\mathbb{Q}) \cong \mathbb{Q}^b$

Pf Have a natural map

$$C_n(K;\mathbb{Z}) \longrightarrow C_n(K;\mathbb{Q})$$

$$\Sigma a_i \sigma_i \longmapsto \Sigma a_i \sigma_i$$

This defines a chain map

$$C_\bullet(K;\mathbb{Z}) \longrightarrow C_\bullet(K;\mathbb{Q})$$

and hence obtain a natural homomorphism

$$H_n(K;\mathbb{Z}) \longrightarrow H_n(K;\mathbb{Q})$$

If $c \in Z_n(K,\mathbb{Q})$, can multiply by m the product of the denominators occuring in c, so that $m \cdot c$ has integer coefficients.

Hence $m \cdot c$ is in the image of the map

$$Z_n(K;\mathbb{Z}) \longrightarrow Z_n(K;\mathbb{Q})$$

Now $H_n(K;\mathbb{Q}) \cong \mathbb{Q}^{b'}$ for some b',

and the above shows that $b' \leqslant b$.

Let $[c_1], \dots, [c_b] \in H_n(K;\mathbb{Z})$ generating the $\mathbb{Z}^b$ factor in $H_n(K;\mathbb{Z})$, and use the same notation for images in $H_n(K;\mathbb{Q})$.

Suppose $\exists \lambda_1, \dots, \lambda_b \in \mathbb{Q}$ such that $\sum_{i=1}^{b} \lambda_i [c_i] = 0$ in $H_n(K;\mathbb{Q})$.

Thus $\exists m$ such that $m\lambda_i \in \mathbb{Z}$ $\forall i$ and $\sum_{i=1}^{b} (\lambda_i m)[c_i] = 0$ $\ddot{\frown}$

Thus $\exists c \in C_{n+1}(K;\mathbb{Q})$ s.t. $\partial_{n+1} c = \sum_{i=1}^{b} \lambda_i c_i$.

So $\exists m \in \mathbb{Z}$ s.t. $mc \in C_{n+1}(K,\mathbb{Z})$ and then

$$\partial_{n+1}(mc) = \sum_{i=1}^{b} m\lambda_i c_i$$

But then $\sum_{i=1}^{n} m\lambda_i [c_i] = 0$ in $H_n(K;\mathbb{Z})$, so $\lambda_i = 0$ $\forall i$ since the $[c_i]$ are assumed to be linearly independent in $\mathbb{Z}^b$.

Thus the $[c_i]$ are linearly independent in $H_n(K;\mathbb{Q})$.

So $b' \geqslant b$ and $b' = b$. ▨

<u>Remark</u> Using rational homology, can't distinguish between $\mathbb{RP}^2$ and a point.

<u>Def</u> Let K be a simplicial complex. The <u>Euler characteristic</u> of K is $\chi(K) = \sum_{n\in\mathbb{Z}} (-1)^n \dim_{\mathbb{Q}} H_n(K;\mathbb{Q})$

If $X \cong |K|$, we write $\chi(X) = \chi(K)$.

<u>Lemma</u> $\chi(K) = \sum_{n\in\mathbb{Z}} (-1)^n \,\#\{n\text{-simplices of } K\}$

<u>Pf</u> As $\#\{n\text{-simplices in } K\} = \dim_{\mathbb{Q}} C_n(K;\mathbb{Q})$, it's enough to show that $\chi(K) = \sum_{n\in\mathbb{Z}} (-1)^n \dim_{\mathbb{Q}} C_n(K;\mathbb{Q})$.

Note $\dim H_n(K;\mathbb{Q}) = \dim Z_n(K;\mathbb{Q}) - \dim B_n(K;\mathbb{Q})$

We may apply rank-nullity to ∂_n, getting

$$\begin{array}{ccccc} \operatorname{rank}\partial_n & + & \dim\ker\partial_n & = & \dim C_n(K;\mathbb{Q}) \\ \parallel & & & & \\ \dim B_{n-1}(K;\mathbb{Q}) & + & \dim Z_n(K;\mathbb{Q}) & & \end{array}$$

Thus $\sum_{n\in\mathbb{Z}} (-1)^n \dim C_n(K;\mathbb{Q})$

$$= \sum_{n\in\mathbb{Z}} (-1)^n \dim Z_n(K;\mathbb{Q}) + \sum_{n\in\mathbb{Z}} (-1)^n \dim B_{n-1}(K;\mathbb{Q})$$

$$= \sum_{n\in\mathbb{Z}} (-1)^n \dim Z_n(K;\mathbb{Q}) - \sum_{n\in\mathbb{Z}} (-1)^n \dim B_n(K;\mathbb{Q})$$

$$= \sum_{n\in\mathbb{Z}} (-1)^n \dim H_n(K;\mathbb{Q}).$$ ▨

<u>Example</u> If $\dim K = 2$, then

$$\chi(K) = \underset{\substack{\nearrow \\ \#\text{vertices}}}{V} - \underset{\substack{\uparrow \\ \#\text{edges}}}{E} + \underset{\substack{\nwarrow \\ \#\text{faces}}}{F}$$

$$\chi(S^2) = \underset{\substack{\uparrow \\ \dim H_0}}{1} - \underset{\substack{\uparrow \\ \dim H_1}}{0} + \overset{\nwarrow \dim H_2}{1}$$

$\therefore$ Any triangulation of the two-sphere satisfies $\boxed{2 = V - E + F}$

§ The Lefschetz fixed point theorem

Let X be triangulable.

Def Let $\phi : X \to X$ be a map.

The Lefschetz number of ϕ is

$$L(\phi) := \sum_{n \in \mathbb{Z}} (-1)^n \underset{\substack{\uparrow \\ \text{trace}}}{\operatorname{tr}} \left(\phi_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}) \right)$$

Notice that if $\phi = id$, then $L(\phi) = \chi(X)$.

Lemma If $f : K \to K$ is a simplicial map, then

$$L(|f|) = \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{tr}\left(f_n : C_n(K;\mathbb{Q}) \to C_n(K;\mathbb{Q}) \right)$$

Pf sketch Given a diagram (comm)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

use twice ☺

with exact rows, it's easy to check that $\operatorname{Tr}(\beta) = \operatorname{Tr}(\alpha) + \operatorname{Tr}(\gamma)$.

Using this, the proof of the similar statement for $\chi(K)$ goes through. □

Theorem (Lefschetz fixed point theorem)

Let $\phi : X \to X$ be a map, X triangulable.

If $L(\phi) \neq 0$, then ϕ has a fixed point.

Pf (sketch) We will show that if ϕ has no fixed point then $L(\phi) = 0$.

By compactness of X, if ϕ has no fixed points, then $\exists\, \delta > 0$ such that

$\|x - \phi(x)\| > \delta \quad \forall x \in X$

Now choose K such that $X \cong |K|$ and $\operatorname{Mesh}(K) < \delta/2$. $\therefore$

Thus if $x \in \sigma \in K$ then $\phi(x) \notin \sigma$

Now let $f : K^{(r)} \to K$ be a simplicial approximation to ϕ.

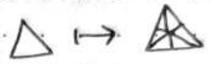
If $r \in K^{(r)}$ a vertex with $v \in \sigma \in K$ then

$\phi(r) \in St_K(f(r))$ (since simplicial approx^{n})

So $\| \phi(v) - f(v) \| < \delta/2$

But $\| \phi(v) - v \| > \delta$ so $\| v - f(v) \| > \delta/2$

So $f(v) \notin \sigma$.



Let $i_\bullet : C_\bullet(K ; \mathbb{Q}) \to C_\bullet(K^{(r)}, \mathbb{Q})$

be the chain map inducing the canonical isomorphism on homology.

For each n-simplex $\sigma \in K$, $i_n(\sigma)$ is supported on simplices of $K^{(r)}$ supported in σ.

Since f takes vertices ~~of~~ in σ out of σ, it follows that

$f_n \circ i_n(\sigma)$ is supported on simplices disjoint from σ.

Since ϕ_* is induced at the level of chains by $f_n \circ i_n$, we now have

$L(\phi) = \sum_{n \in \mathbb{Z}} (-1)^n (\operatorname{Tr} f_n \circ i_n)$ by the lemma

But $f_n \circ i_n$ maps each n-simplex of K off of itself.

Thus the matrix of $f_n \circ i_n$ using the basis of n-simplices of K, has no diagonal entries.

Thus $\operatorname{Tr} f_n \circ i_n = 0$ so $L(\phi) = 0$. $\square$

<u>Cor</u> If X is a triangulable and contractible, then any map $\phi : X \to X$ has a fixed point.

<u>Pf</u> As $H_n(X ; \mathbb{Q}) = 0$ for $n > 0$

$H_0(X ; \mathbb{Q}) = \mathbb{Q}$,

so only non-zero map ϕ_* is $\phi_* : \underset{\mathbb{Q}}{\underset{\shortparallel}{H_0(X ; \mathbb{Q})}} \to \underset{\mathbb{Q}}{\underset{\shortparallel}{H_0(X ; \mathbb{Q})}}$,

the identity. Thus $L(\phi) = 1$. $\square$