24 lecture course, modified schedule Dr Claude Warnick
4 problem sheets, 4 long exam QQs cwm50@

Aims of the course: Develop ideas from linear anlysis and measure theory, with an eye to applications in PDE theory etc

Review of integration + Measure Theory

Def^n Given a set E, a σ-algebra on E is a collection $\mathcal{E}$ of subsets of E such that:
i) $\emptyset \in \mathcal{E}$
ii) $A \in \mathcal{E} \Rightarrow A^c = \{x \in E : x \notin A\} \in \mathcal{E}$
iii) $A_n \in \mathcal{E} \ \forall n \in \mathbb{N} \Rightarrow \bigcup_n A_n \in \mathcal{E}$

The pair $(E, \mathcal{E})$ is called a measurable space, and elements of $\mathcal{E}$ are measurable sets.

A measure on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ s.t.
i) $\mu(\emptyset) = 0$
ii) if $A_n \in \mathcal{E} \ \forall n \in \mathbb{N}$ are disjoint, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$

The triple $(E, \mathcal{E}, \mu)$ is a measure space.

Example E is any set, $\mathcal{E} = 2^E$
$\mu(A) = \#(A)$ called the counting measure

Given any collection $\mathcal{A}$ of subsets of E, we define $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$ to be the intersection of all the σ-algebras that contain $\mathcal{A}$. (is a σ-algebra)

If E carries a topology, τ, then $\sigma(\tau)$ is the Borel algebra, written $\mathcal{B}(E)$.

Example $E = \mathbb{R}^n$, we can define a σ-algebra $\mathcal{M}$, and a measure λ with the following properties:
i) $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}$
ii) If $A = (a_1, b_1] \times \cdots \times (a_n, b_n]$ is a rectangle

then $\hat{\lambda}(A) = (b_1 - a_1)\cdots(b_n - a_n)$.

iii) $A \in \mathcal{M}$ iff for any $\varepsilon > 0$ there exists an open set O and a closed set C s.t. $C \subset A \subset O$ and

$$\hat{\lambda}(O \setminus C) < \varepsilon$$

As any open set in $\mathbb{R}^n$ is a countable union of disjoint rectangles, this uniquely determines $\mathcal{M}$, λ.

We call λ the <u>Lebesgue measure</u> and $\mathcal{M}$ the <u>Lebesgue measurable sets</u>.

We often write $dx = \lambda$ and $|A| = \lambda(A)$.

<u>Def^n</u> A function $f : E \to G$ between two measurable spaces $(E, \mathcal{E})$, $(G, \mathcal{G})$ is <u>measurable</u> if $f^{-1}(A) \in \mathcal{E}$ $\forall A \in \mathcal{G}$.

Special cases: a) If $(G, \mathcal{G}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we simply say f is measurable on $(E, \mathcal{E})$.

b) If $(G, \mathcal{G}) = ([0,\infty], \mathcal{B}([0,\infty]))$ we say f is a <u>non-negative</u> measurable f^n

c) If E, G are topological spaces, with their Borel algebras, we say f is <u>Borel</u>.

The class of measurable functions is closed under vector space operations ($+$, mult by $a \in \mathbb{R}$), products and limits.

A <u>simple function</u> is a function of the form

$$f = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$$

for $A_k \in \mathcal{E}$, a_k constant (typically in $[0,\infty]$, $\mathbb{R}$ or $\mathbb{C}$)

For a non-negative simple measurable f^n, define

$$\mu(f) = \int_E f \, d\mu = \sum_{k=1}^{N} a_k \mu(A_k) \qquad \left(\begin{array}{c}\text{convention} \\ 0 \cdot \infty = 0\end{array}\right)$$

For a non-negative measurable function, define

$$\mu(f) = \int_E f \, d\mu = \sup\{\mu(g) : g \text{ simple}, 0 \leq g \leq f\}$$

A measurable function is <u>integrable</u> if $\mu(|f|) < \infty$, in this case we can write $f = f^+ - f^-$ with $f^\pm$ non-negative and $\mu(f^\pm) < \infty$. Then

$$\mu(f) = \int_E f \, d\mu := \mu(f^+) - \mu(f^-)$$

The integral satisfies all properties expected (linearity, order preserving etc.)

⌜If f takes values in $\mathbb{C}$, split into real, im parts to define $\int_E f d\mu$ ⌟

<u>Thm</u> (Monotone convergence)

If $(f_n)_{n=1}^{\infty}$ is an increasing sequence of non-negative measurable functions converging to f (on $(E, \mathcal{E}, \mu)$), then

$$\int_E f_n \, d\mu \longrightarrow \int_E f \, d\mu \quad \text{as } n \to \infty.$$

<u>Thm</u> (Dominated convergence)

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions on $(E, \mathcal{E}, \mu)$ such that

i) $f_n \to f$ pointwise almost everywhere

ii) $|f_n| \leq g$ a.e. for some integrable g

Then $\int_E f_n \, d\mu \longrightarrow \int_E f \, d\mu$ as $n \to \infty$.

L^p - spaces

Associated to a measure space $(E, \mathcal{E}, \mu)$ is a scale of Banach spaces:

Def For $1 \leq p < \infty$ and $f : E \to \mathbb{C}$ measurable, we define $\|f\|_{L^p} := \left(\int_E |f|^p \, d\mu \right)^{1/p}$

while for $p = \infty$

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_E |f| = \inf \{ K : |f| \leq K \text{ a.e.} \}$$

Then the space $L^p(E, \mu)$ is defined to be

$$L^p(E, \mu) = \{ f : E \to \mathbb{C} \text{ meas} \mid \|f\|_{L^p} < \infty \} / \sim$$

Where $f \sim g$ if $f = g$ a.e.

Thm (Riesz - Fischer)

The space $L^p(E, \mu)$ with the norm $\|\cdot\|_{L^p}$ is a Banach space for $1 \leq p \leq \infty$.

When $(E, \mathcal{E}, \mu) = (\mathbb{R}^n, \mathcal{M}, \lambda)$, we write $L^p(\mathbb{R}^n, \lambda) = L^p(\mathbb{R}^n)$

Mollification and dense subspaces in $L^p(\mathbb{R}^n)$

It's often useful to identify dense subsets of a particular space consisting of 'nicer' objects (e.g. $\mathbb{Q} \subset \mathbb{R}$)

From the definitions, we can show that the set of simple functions $s : E \to \mathbb{C}$ such that $\mu(\{x : s(x) \neq 0\}) < \infty$ (†) is dense in $L^p(E, \mu)$ for $1 \leq p < \infty$. ($p = \infty$ drop (†))

We will establish the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

For $p = \infty$, result is not true. If $f \in C^0(\mathbb{R})$, then $\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}} |f(x)|$. So any sequence $(f_n)_{n=1}^\infty$ of cts f's which converges in L^∞ must converge in the sup-norm.

So the limit is cts.

$\therefore$ There exists no sequence in $C^0(\mathbb{R})$ which converges to $g(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$

But $g(x) \in L^\infty(\mathbb{R})$.

We shall prove the result for $p < \infty$ using Mollification by convolution

Def If $f, g : \mathbb{R}^n \to \mathbb{C}$, define their convolution

$$f * g : \mathbb{R}^n \longrightarrow \mathbb{C}$$

$$x \longmapsto \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

provided the integral exists.

[e.g. $f \in L^1(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n)$]

We have the following basic properties:

Lemma If $f, g, h \in C_c^\infty(\mathbb{R}^n)$, then

$$f * g = g * f,$$

$$(f*g)*h = f*(g*h),$$

$$\int_{\mathbb{R}^n} (f*g)(x)\, dx = \left(\int_{\mathbb{R}^n} f(x)\, dx\right)\left(\int_{\mathbb{R}^n} g(x)\, dx\right)$$

Pf Change of variables formula and Fubini.

Note $C_c^\infty(\mathbb{R}^n)$ is overkill.

A more substantial result concerns the differentiability of $f*g$, for which we require some notation.

A multi-index α is an element of $\mathbb{Z}_{\geq 0}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$

We define $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$

If $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ $(x = (x_1, \dots, x_n))$

If $f \in C^k(\mathbb{R}^n)$ for $k \geq |\alpha|$ then

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

~~Theorem~~ (nope) Final def^n before theorem:

We say $f \in L^p_{loc}(\mathbb{R}^n)$ if $f \mathbb{1}_K \in L^p(\mathbb{R}^n)$ for all compact K.

<u>Theorem</u> Suppose $f \in L^1_{loc}(\mathbb{R}^n)$, $g \in C^k_c(\mathbb{R}^n)$ for some $k \geq 0$.
Then $f * g \in C^k(\mathbb{R}^n)$ and

$$D^\alpha(f*g) = f * D^\alpha g \quad \text{for all } |\alpha| \leq k.$$

<u>Pf</u> • First suppose $k=0$. Let $\tau_z f(x) = f(x-z)$ for any $z \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{C}$.

Note that $\tau_z(f*g) = f * \tau_z g$ from def^n.

Further $\tau_z g(x) \longrightarrow g(x)$ as $|z| \to 0$ (since $g \in C^0$)

Moreover $|\tau_z g(x)| \leq \sup_y |g(y)| \, \mathbb{1}_{B_R(0)}(x)$

if $|z| < 1$ for some $R > 0$.

Thus

$$|f(y) \tau_z g(x-y)| \leq \underbrace{C |f(y)| \, \mathbb{1}_{B_R(0)}(x-y)}_{\text{integrable (in } y)}$$

So by DCT $\tau_z(f*g) = f * \tau_z(g) = \int_{\mathbb{R}^n} f(y) \tau_z g(x-y)\, dy$

$$\longrightarrow f*g \quad \text{as } |z| \to 0$$

$\therefore$ $f*g$ is continuous

• Now suppose $k=1$, let $\Delta_i^h g(x) := \dfrac{g(x+he_i) - g(x)}{h}$

for $h \in \mathbb{R}$, $i = 1, \dots, n$

Then $\Delta_i^h g(x) \longrightarrow D_i g(x)$ as $h \to 0$ and

$\cancel{|\Delta_i^h g(x)| \leq}$ by mean value theorem,

$$\exists\, t \in (-|h|, |h|) \text{ s.t. } \frac{g(x+he_i) - g(x)}{h} = D_i g(x + te_i)$$

$$\therefore |\Delta_i^h g(x)| \leq \sup_y |D_i g(y)| \, \mathbb{1}_{B_R(0)}(x) \quad \forall\, |h| < 1$$

for R sufficiently large

Now again from def^n of $f*g$ we have

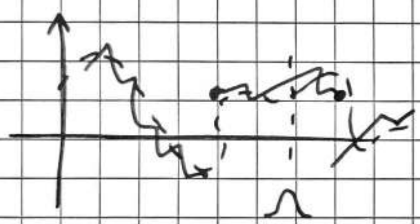
$$\Delta_i^h (f*g) = f * \Delta_i^h g \longrightarrow f * D_i g \quad \text{by DCT}$$

(identical argument to $k=0$ case)

$\therefore$ $f*g$ has its first partial derivs, so $f*g \in C^1(\mathbb{R}^n)$

For $k>1$ we use induction.

<u>Idea</u> If we replace a L^p function f by ~~x~~ the function whose value at x is a smoothly weighted average of f in a ngbd of x, then this new function should approximate f and be smooth.



Before proving that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we require two results.

<u>Lemma</u> (Minkowski's integral inequality)

If $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is measurable, and $1 \leq p < \infty$,

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)\, dx \right|^p dy \right)^{1/p}$$

$$\leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p\, dy \right]^{1/p} dx$$

<u>Rmk</u> $F \geq 0$ or F integrable

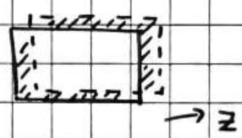
<u>Proof</u> Problem 1.4

<u>Lemma</u> Suppose $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^n)$. Then

$$\|\tau_z g - g\|_{L^p} \to 0 \quad \text{as} \quad z \to 0.$$

(Recall $\tau_z g(x) = g(x-z)$)

<u>Pf</u> • consider $g = \mathbb{1}_R$, R a rectangle



result is fairly straightforward,

hence true for a finite disjoint union of rectangles

$$\|\tau_z \mathbb{1}_R - \mathbb{1}_R\|_{L^p} = (\text{Area of } /\!/\!/)^{1/p} \longrightarrow 0 \text{ as } z \to 0$$

• If B is measurable, $|B| < \infty$, then $\forall \varepsilon > 0$, $\exists A$ finite disjoint union of rectangles with $\|\mathbb{1}_A - \mathbb{1}_B\|_{L^p} = |A \Delta B|^{1/p} < \varepsilon$

Thus
$$\begin{aligned} \|\tau_z \mathbb{1}_B - \mathbb{1}_B\|_{L^p} \leq\ & \|\tau_z \mathbb{1}_B - \tau_z \mathbb{1}_A\|_{L^p} \\ & + \|\tau_z \mathbb{1}_A - \mathbb{1}_A\|_{L^p} \\ & + \|\mathbb{1}_A - \mathbb{1}_B\|_{L^p} \end{aligned}$$

classic 3ε-proof

$$\|\tau_z(\mathbb{1}_B - \mathbb{1}_A)\|_{L^p} = \|\mathbb{1}_B - \mathbb{1}_A\|_{L^p}$$

$$\therefore \|\tau_z \mathbb{1}_B - \mathbb{1}_B\|_{L^p} \leq 2\varepsilon + \cancel{X} \|\overset{\tau_z}{}\mathbb{1}_{\cancel{X}\,A} - \mathbb{1}_A\|_{L^p} < 3\varepsilon$$

if z is small

$\therefore$ result holds for $g = \mathbb{1}_B$, B Lebesgue measurable set

$\therefore$ result holds for any simple function

And if $\tilde{g} \in L^p$, $\exists\ \tilde{g}$ simple (+...) such that

$$\|g - \tilde{g}\|_{L^p} < \varepsilon$$

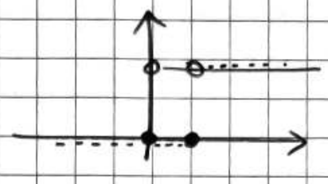
Using $\|\tau_z g - g\|_{L^p} \leq \|\tau_z g - \tau_z \tilde{g}\|_{L^p} + \|\tau_z \tilde{g} - \tilde{g}\|_{L^p} + \|\tilde{g} - g\|_{L^p}$ obtain our final result. $\square$

$$\boxed{\text{Translation is continuous on } L^p \text{ for } 1 \leq p < \infty.}$$

Note result is false for $p = \infty$.

Let $f : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



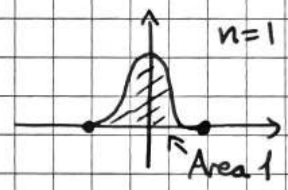
then $\|\tau_z f - f\|_{L^\infty} = 1$ for all $z \neq 0$

<u>Theorem</u> Suppose $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfies

(i) $\phi \geq 0$

(ii) $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$

$\left(\begin{array}{l} \text{may also} \\ \text{require} \end{array} \quad \operatorname{supp} \phi \subset B_1(0) \right)$



Define $\phi_\varepsilon(y) = \frac{1}{\varepsilon^n} \phi\left(\frac{y}{\varepsilon}\right)$ $\quad \left(\text{so } \int_{\mathbb{R}^n} \phi_\varepsilon(y)\, dy = 1\right)$

Then if $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $\phi_\varepsilon * g$ is smooth and

$$\phi_\varepsilon * g \longrightarrow g \text{ in } L^p \text{ as } \varepsilon \to 0.$$

<u>Pf</u> Smooth by previous lemma and the fact that if $f \in L^p(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} |f| \mathbb{1}_K\, dx \leq \|f\|_{L^p} \|\mathbb{1}_K\|_{L^q}, \quad \underset{\text{Hölder}}{\frac{1}{p} + \frac{1}{q} = 1}$$

$$\Rightarrow f \in L^1_{loc}(\mathbb{R}^n)$$

Need to show $\|\phi_\varepsilon * g - g\|_{L^p} \to 0$ as $\varepsilon \to 0$.

We estimate:

$$\|\phi_\varepsilon * g(x) - g(x)\| = \left| \int_{\mathbb{R}^n} \phi_\varepsilon(y)\, g(x-y)\, dy - g(x) \right| \quad (y = \varepsilon z)$$

$$= \left| \int_{\mathbb{R}^n} \phi(z)\, g(x - \varepsilon z)\, dz - \int_{\mathbb{R}^n} \phi(z)\, g(x)\, dz \right|$$

$$\leq \int_{\mathbb{R}^n} \phi(z) \left| g(x - \varepsilon z) - g(x) \right| dz$$

$$|\phi_\varepsilon * g(x) - g(x)| \leq \int_{\mathbb{R}^n} \phi(z)\, |\tau_{\varepsilon z} g(x) - g(x)|\, dz$$

$$\therefore \|\phi_\varepsilon * g - g\|_{L^p} = \left(\int_{\mathbb{R}^n} |\phi_\varepsilon * g(x) - g(x)|^p\, dx \right)^{1/p}$$

$$\leq \left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi(z)\, |\tau_{\varepsilon z} g(x) - g(x)|\, dz \right|^p dx \right]^{1/p}$$

$$\overset{\text{Mink}}{\leq} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \phi(z)^p\, |\tau_{\varepsilon z} g(x) - g(x)|^p\, dx \right]^{1/p} dz$$

$$= \int_{\mathbb{R}^n} \phi(z)\, \|\tau_{\varepsilon z} g - g\|_{L^p}\, dz$$

$$\to 0 \text{ as } \varepsilon \to 0 \text{ by DCT}$$

since $\phi(z) \|\tau_{\varepsilon z} g - g\|_{L^p} \to 0 \quad \forall z$

and $|\phi(z)| \|\tau_{\varepsilon z} g - g\|_{L^p} \leq 2\phi(z) \|g\|_{L^p}$ $\square$

This immediately implies that $C^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

If $f \in L^p(\mathbb{R}^n)$, then $|f| \mathbb{1}_{B_R(0)} \nearrow |f|$ p.w.,

so $f \mathbb{1}_{B_R(0)} \to f$ in L^p,

so by considering $\phi_\varepsilon * (f \mathbb{1}_{B_R(0)})$ we can see that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Rmk $\mathrm{supp}(f*g) \subset \mathrm{supp} f + \mathrm{supp}\, g$

Lebesgue Differentiation Theorem

Recall that if $f: \mathbb{R} \to \mathbb{R}$ is continuous, then

$$F(x) = \int_0^x f(t)\, dt$$

is diff'ble, and $F'(x) = f(x)$.

i.e. $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$

$$\Rightarrow \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(x)$$

$$\Leftrightarrow \lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = f(x)$$

$$\Leftrightarrow \lim_{h\to 0} \frac{1}{|B_h(x)|}\int_{B_h(x)} \big(f(t)-f(x)\big)\,dt = 0$$

We shall show that if $f:\mathbb{R}^n \to \mathbb{C}$ is integrable,

$$\lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|\,dy = 0 \quad \text{a.e. } x.$$

Def Given $f: \mathbb{R}^n \to \mathbb{C}$ integrable, the Hardy-Littlewood maximal function $Mf: \mathbb{R}^n \longrightarrow [0,\infty]$ is defined via

$$Mf(x) =: \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy$$

#cricket

Lemma Suppose $f \in L^1(\mathbb{R}^n)$ for $n \geq 1$. Then Mf is measurable, finite a.e. and there exists a constant $C_n > 0$ such that

$$|\{Mf > \lambda\}| \leq \frac{C_n}{\lambda} \|f\|_{L^1} \quad \forall \lambda > 0.$$

Pf Let $A_\lambda = \{Mf > \lambda\}$.

For each $x \in A_\lambda$, $\exists r_x > 0$ s.t.

$$\frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x)} |f(y)| \, dy > \lambda.$$

We claim A_λ is open, which implies Mf is measurable.

[Since $(\lambda, \infty]$ for $\lambda \in [0,\infty)$ generate $\mathcal{B}([0,\infty])$]

We show A_λ^c is closed. Suppose $(x_m)_{m=1}^\infty$ is a sequence with $x_m \in A_\lambda^c$ and $x_m \to x$. Suppose for contradiction that $x \in A_\lambda$, and let r_x be as above. Consider

$$\frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x_m)} |f(y)| \, dy \longrightarrow \frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x)} |f(y)| \, dy$$

as $m \to \infty$, by DCT.

Since $x_m \notin A_\lambda$, LHS $\leq \lambda$ for all m.

But by assumption RHS $> \lambda$. $\times$

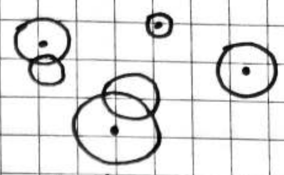
So $x \in A_\lambda^c$ and A_λ is open.

Let $K \subset A_\lambda$ be compact. Since $\{B_{r_x}(x), x \in A_\lambda\}$ is an open cover of K, pick a finite subcover, say

$$K \subset \bigcup_{i=1}^N B_i \qquad \text{with } B_i = B_{r_x}(x) \text{ for some } x.$$

By Wiener's covering lemma, there is a disjoint subcollection $B_{i_1}, \dots, B_{i_k}$ such that

$$|K| \leq \left| \bigcup_{i=1}^{N} B_i \right| \leq 3^n \left| \sum_{j=1}^{k} |B_{i_j}| \right|$$



"ball expansion"

Each B_i satisfies

$$\frac{1}{|B_i|} \int_{B_i} |f(y)|\, dy > \lambda .$$

So

$$|K| \leq 3^n \sum_{j=1}^{k} |B_{i_j}| \leq \frac{3^n}{\lambda} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)|\, dy$$

$$\leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy \qquad \text{since } B_{i_j} \text{ disjoint} \;\smile$$

True for all compact $K \subset A_\lambda$.

So by inner regularity of Lebesgue measure,

$$|A_\lambda| \leq \frac{3^n}{\lambda} \|f\|_{L^1} \qquad (C_n = 3^n \text{ works then})$$

Since $\{M_f = \infty\} \subset A_\lambda$ for all λ, the bound above gives $|\{M_f = \infty\}| < C/\lambda \;\; \forall \lambda$

$\therefore \; |\{M_f = \infty\}| = 0.$ $\square$

<u>Thm</u> (Lebesgue differentiation)

Let $f : \mathbb{R}^n \to \mathbb{C}$ be integrable. Then for almost every $x \in \mathbb{R}^n$, we have

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0 .$$

Such an x is called a <u>Lebesgue point</u> of f.

<u>Note</u> that if f is continuous, this is straightforward.

<u>Pf</u> For $\lambda > 0$, define

$$A_\lambda = \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy > \lambda 2 \right\}$$

If we can show $|A_\lambda| = 0$, we are done, as $\bigcup_n A_{\frac{1}{n}}$ is the set of non-Lebesgue points of f.

Fix $\varepsilon > 0$. We can find $g \in C_c^\infty(\mathbb{R}^n)$ such that
$\|g - f\|_{L^1} < \varepsilon$.

We estimate

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f(x)|\,dy$$

$$\leq \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - g(y)|\,dy \quad \text{Ⓐ}$$

$$+ \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y) - g(x)|\,dy \quad \text{Ⓑ}$$

$$+ \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(x) - f(x)|\,dy \quad \text{Ⓒ}$$

Ⓐ $\leq \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - g(y)|\,dy = M[f-g](x)$

Ⓑ $\to 0$ as $r \to 0$ by continuity of g

Ⓒ $= |g(x) - f(x)|$

$$\therefore \limsup_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f(x)|\,dy \leq M[f-g](x) + |g(x) - f(x)|$$

Now if $x \in A_\lambda$, then either $M[f-g](x) > \lambda$,
or $|g(x) - f(x)| > \lambda$.

By previous lemma,

$$|\{M[f-g](x) > \lambda\}| \leq \frac{C_n}{\lambda}\|f-g\|_{L^1} \leq \frac{C_n \varepsilon}{\lambda}.$$

By Markov's inequality,

$$|\{|g(x) - f(x)| > \lambda\}| \leq \frac{\|f-g\|_{L^1}}{\lambda} \leq \frac{\varepsilon}{\lambda}.$$

Thus

$|A_\lambda| \leq \frac{(1+C_n)}{\lambda}\varepsilon$, but $\varepsilon > 0$ was arbitrary.

So $|A_\lambda| = 0$. $\square$

Corollary If ϕ_ε, f are as in the mollification theorem from last lecture, then

$$\phi_\varepsilon * f \longrightarrow f \quad \text{pointwise a.e. as } \varepsilon \to 0$$

(in addition to convergence in L^p)

Corollary If $g \in L^1(\mathbb{R})$ and $G(x) = \int_{-\infty}^{x} g(t)\,dt$, then G is differentiable at a.e. x, with $G'(x) = g(x)$.

Littlewood's Principles

"Every (measurable) set is nearly a finite sum of intervals, every function (integrable) is nearly continuous, every (pointwise a.e.) convergent sequence is nearly uniformly convergent" Littlewood 1944

Theorem, the first of these principles we can rephrase as:

Suppose $|A| < \infty$. Then for any $\varepsilon > 0$ there exists B, a finite union of rectangles such that $|A \Delta B| < \varepsilon$.

Theorem (Egorov)

Suppose $(f_k)_{k=1}^{\infty}$ is a sequence of measurable f^ns $f : E \to \mathbb{C}$ where $E \subset \mathbb{R}^n$, $|E| < \infty$, and suppose $f_k \to f$ a.e. in E.

Then $\forall \varepsilon > 0$ we can find a closed set $A_\varepsilon \subset E$ such that

$|E \setminus A_\varepsilon| \leq \varepsilon$ and $f_k \to f$ uniformly on A_ε

Pf Wlog, assume $f_k \to f$ pointwise (else discard measure zero)

Let $E_k^n = \{ x \in E : |f_j(x) - f(x)| < \frac{1}{n} \ \forall j > k \}$.

Fixing n, have $E_k^n \subset E_{k+1}^n$ and $\bigcup_{k=1}^{\infty} E_k^n = E$.

Thus $|E_k^n| \nearrow |E|$.

Pick k_n such that $|E \setminus E_{k_n}^n| < 2^{-n}$.

By construction,

$$|f_j(x) - f(x)| < \frac{1}{n} \quad \forall j > k_n, \ x \in E_{k_n}^n .$$

Pick N s.t. $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon/2$ and set $A_\varepsilon' = \bigcap_{n=N}^{\infty} E_{k_n}^n$.

Note $|E \setminus A_\varepsilon'| \leq \sum_{n=N}^{\infty} |E \setminus E_{k_n}^n| < \varepsilon/2$

Now suppose $\delta > 0$. Pick $n \geq N$ s.t. $\frac{1}{n} < \delta$ and suppose $x \in A_\varepsilon'$.

Then $x \in E_{k_n}^n$. Thus $|f_j(x) - f(x)| < \delta \ \forall j > k_n$

$\therefore f_j \to f$ uniformly on A_ε'

Finally, pick a closed $A_\varepsilon \subset A_\varepsilon'$ with $|A_\varepsilon' \setminus A_\varepsilon| < \varepsilon/2$.

Then we're done, as $|E \setminus A_\varepsilon| < \varepsilon$. $\square$

Theorem (Lusin)

Suppose f is measurable and finite valued on E, where $E \subset \mathbb{R}^n$, $|E| < \infty$. Then given $\varepsilon > 0$ we can find a closed set $F_\varepsilon \subset E$ with $|E \setminus F_\varepsilon| < \varepsilon$ such that $f|_{F_\varepsilon}$ is continuous.

Pf Suppose first f is a simple function

$$f = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k}$$

where $|A_k| < \infty$ and the A_k are disjoint, with $E = \bigcup_{k=1}^m A_k$. (if necessary include $0 \mathbb{1}_{f^{-1}(0)}$)

For any $\varepsilon > 0$, we can pick compact sets $K_k \subset A_k$ with $|A_k \setminus K_k| < \varepsilon/m$.

Let $B = \bigcup_{k=1}^m K_k$. Then $|E \setminus B| < \varepsilon$.

Since the sets K_k are compact, disjoint (hence distance between $K_i, K_j > 0$ if $i \neq j$) have $f|_B$ is continuous.

Now let f be measurable, finite valued and let f_n be a sequence of simple functions with $f_n \to f$ a.e.

Then we can find C_n s.t. $|C_n| < 2^{-n}$ and $f_n|_{E \setminus C_n}$ is continuous. Let $\varepsilon > 0$. By Egorov, we can find a set $A_{\varepsilon/3}$ such that $f_n \to f$ unif. on $A_{\varepsilon/3}$ and $|E \setminus A_{\varepsilon/3}| \leq \frac{\varepsilon}{3}$.

Let N be s.t. $\sum_{n=N}^\infty 2^{-n} < \varepsilon/3$.

Set $F_\varepsilon' = A_{\varepsilon/3} \setminus \bigcup_{n=N}^\infty C_n$.

Then $|E \setminus F_\varepsilon'| < 2\varepsilon/3$ and for $n > N$, $f_n|_{F_\varepsilon'}$ is cts.

So since $f_n \to f$ unif on F_ε', $f|_{F_\varepsilon'}$ is cts.

Finally let $F_\varepsilon \subset F_\varepsilon'$ be closed with $|F_\varepsilon' \setminus F_\varepsilon| < \varepsilon/3$.

Then $|E \setminus F_\varepsilon| < \varepsilon$ and we are done. $\square$

Warning $f|_{F_\varepsilon}$ is cts means f is continuous if viewed as being defined only on points of F_ε NOT $f: E \to \mathbb{C}$ cts at F_ε

Banach and Hilbert Space Analysis

We move on to consider the linear structures that $L^p(\mathbb{R}^n)$ has as a result of being a Banach space. The case $p=2$ is especially relevant as this is moreover a Hilbert space.

In fact: given any measure space $(E, \mathcal{E}, \mu)$ the space $L^2(E,\mu)$ is naturally a Hilbert space, with inner product

$$(f,g)_{L^2} =: \int_E \bar{f}\, g \, d\mu$$

Orthogonal systems of functions

If $S = \{u_j\}_{j\in J}$ is a subset of a Hilbert space H, we say it is orthogonal if $(u_i, u_j) = 0 \ \forall i \neq j$.

We say it is orthonormal if $(u_j, u_j) = 1 \ \forall j$.

We say S is complete if $\overline{\text{span}(S)} = H$

A complete orthonormal set is an orthonormal basis.

A Hilbert space is separable if (and only if) it admits a countable orthonormal basis.

(separable $\longleftrightarrow$ $\exists$ countable dense set)

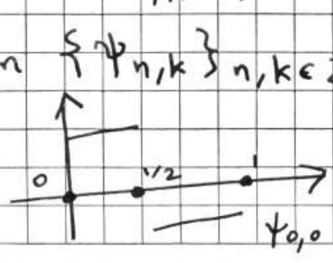
Examples 1. Consider $L^2([0,1])$ and let $S = \{e^{-2\pi i n x}\}$.

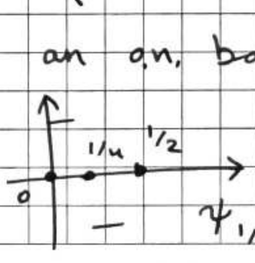
S is an orthonormal basis: completeness follows from Stone-Weierstrass plus density of C^∞ in L^2.

2. Let $\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$

Define $\psi_{n,k}(x) := 2^{n/2}\, \psi(2^n x - k)$

Then $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an o.n. basis for $L^2(\mathbb{R})$





Haar System

3. Consider $L^2(\mathbb{R}, e^{-x^2}dx)$,

the Hilbert space with Gaussian-weighted inner product

$$(f,g) = \int_{\mathbb{R}} \bar{f}(x)\, g(x)\, e^{-x^2}\, dx.$$

Applying Gram-Schmidt to the lin indep set $\{1, x, x^2, \dots\}$ we construct a sequence of polynomials $H_k(x)$ of degree k s.t. $\int_{\mathbb{R}} \overline{H_k(x)}\, H_l(x)\, e^{-x^2}\, dx = 0, \quad k<l$

By convention, we normalise H_k s.t. the coeff of x^k is 2^k.

The set $\{H_k\}_{k=1}^{\infty}$ are the Hermite polynomials and are a o.n. basis for $L^2(\mathbb{R}, e^{-x^2}dx)$.

An important result in the theory of Hilbert spaces is

<u>Thm</u> (Riesz representation thm)
Suppose $\Lambda : H \to \mathbb{C}$ is a bounded linear operator.
Then $\exists!\, w \in H$ s.t. $\Lambda u = (w, u)$ $\forall u \in H$.

We will apply this to prove an important result in measure theory.

<u>Def</u> Suppose $(E, \mathcal{E})$ is a measurable space, and μ, ν are measures defined on it. We say ν is <u>absolutely continuous with respect to μ</u>, written $\nu \ll \mu$ if for any measurable set $\mu(A) = 0 \Rightarrow \nu(A) = 0$.
We say μ, ν are <u>mutually singular</u>, written $\mu \perp \nu$ if there is some measurable set B s.t.
$$\mu(B) = 0, \quad \nu(B^c) = 0.$$

<u>Theorem</u> (Radon-Nikodym)
Suppose $(E, \mathcal{E})$ is measurable space with finite measures μ, ν s.t. $\nu \ll \mu$. Then there exists a $w \in L^1(E, \mu)$ s.t.
$$\nu(A) = \int_A w(x)\, d\mu(x) \qquad \forall A \in \mathcal{E}$$

This implies $$\int_E F(x)\, d\nu(x) = \int_E F(x)\, w(x)\, d\mu(x)$$
for all non-negative measurable $F : E \to \mathbb{C}$.

<u>Pf</u> Let $\alpha = \mu + 2\nu$, $\beta = 2\mu + \nu$. These are finite measures on $(E, \mathcal{E})$.
On the space $$H = L^2(E, \alpha) = L^2(E, \mu) \cap L^2(E, \nu) = L^2(E, \beta)$$
we define $$\Lambda(f) = \int_E f\, d\beta.$$
Now $$|\Lambda(f)| \leq \int_E |f|\, d\beta \leq 2\int_E |f|\, d\alpha \leq 2\sqrt{\alpha(E)}\, \|f\|_{L^2(\alpha)}$$

So $\Lambda : H \to \mathbb{C}$ is bounded and linear.

Thus by Riesz, $\exists\, g \in H$ s.t. $\forall f \in H$,

$$\int_E f \, d\beta = \int_E f g \, d\alpha$$

$$\Rightarrow \int_E f(2g-1)\, d\nu = \int_E f(2-g)\, d\mu \quad \forall f \in H \quad (*)$$

NEEDS FIXING

via averages show $g \in [\tfrac{1}{2}, 2]$ α-a.e.

Set $f = \mathbb{1}_{A_j}$ for $A_j = \{x \in E : g(x) < \tfrac{1}{2} - \tfrac{1}{j}\}$

$$\int_E f(2g-1)\, d\nu \leq -\frac{1}{j} |A_j|_\nu$$

$$\int_E f(2-g)\, d\mu \geq \frac{3}{2} |A_j|_\mu$$

$\therefore\ \nu(A_j) = \mu(A_j) = 0$ and $g \geq \frac{1}{2}$ μ-a.e. and ν-a.e.

Similarly, setting $A_j = \{x \in E : g(x) > 2 + \tfrac{1}{j}\}$ we can show $g \leq 2$ μ-a.e. and ν-a.e.

By MCT, $(*)$ holds for all non-negative measurable f.

Now take $Z = \{g(z) = \tfrac{1}{2}\}$. Setting $f = \mathbb{1}_Z$ in $(*)$ we deduce $\mu(Z) = 0$. Since $\nu \ll \mu$ we have $\nu(Z) = 0$.

Given F non-negative, measurable, set

$$f(x) = \frac{F(x)}{2g(x)-1}, \quad w(x) = \frac{2-g(x)}{2g(x)-1} \quad \forall x \notin Z$$

$f(x) = w(x) = 0$ for $x \in Z$.

Now
$$\int_E F\, d\nu = \int_{E\setminus Z} F\, d\nu = \int_E f(2g-1)\, d\nu$$
$$= \int_E f(2-g)\, d\mu = \int_{E\setminus Z} F w\, d\mu = \int_E F w\, d\mu.$$

Setting $F = 1$, deduce $w \in L^1(E, \mu)$, set $f = \mathbb{1}_A$, $A \in \mathcal{E}$ to obtain the result. $\square$

Dual Spaces

Given a topological vector space (TVS) X, the continuous dual (or simply dual), X' is the set of continuous linear maps $\Lambda : X \to \mathbb{C}$. X' is a vector space in a natural way.

If X is normed, we can equip X' with a norm

$$\|\Lambda\|_{X'} := \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\Lambda(x)|$$

With this norm X' is Banach. We will often assume X is Banach.

The dual of a Banach space separates points

Lemma Let X be Banach. Suppose $x, y \in X$, $x \neq y$. Then $\exists\, \Lambda \in X'$ such that $\Lambda(x) \neq \Lambda(y)$.

Pf/ Postpone until we look at Hahn-Banach.

If X is Banach, then for each $x \in X$ there is a natural map
$$f_x : X' \to \mathbb{C}$$
$$\Lambda \mapsto \Lambda(x)$$

This is bounded, linear, hence $f_x \in X''$. If $f_x(\Lambda) = f_y(\Lambda)$ for all Λ, then by the lemma $x = y$.

Thus we have a natural injection $X \hookrightarrow X''$.

If this map is surjective, we say X is reflexive, write $X = X''$ (canonical).

Special case

If $X = H$ is a Hilbert space, then by Riesz H' is canonically identified with H. Thus $H'' = H$.

The dual of $L^p(\mathbb{R}^n)$

Suppose $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$ and let $p^{-1} + q^{-1} = 1$. Then by Hölder's inequality, if $g \in L^q(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} f(x)\, g(x)\, dx \right| \leq \|g\|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

Thus the map $\Lambda_g : L^p(\mathbb{R}^n) \longrightarrow \mathbb{C}$ given by

$$\Lambda_g(f) = \int_{\mathbb{R}^n} g(x) f(x)\, dx$$

is bounded and linear.

And by Exercise 1.4

$$\|\Lambda_g\|_{(L^p)'} = \|g\|_{L^q}$$

Thus the map $K : L^q(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)'$

$$g \longmapsto \Lambda_g$$

is linear, isometric (hence injective).

We deduce $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)'$.

If $p = q = 2$, we know $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)'$.

What if $p \neq 2$?

<u>Theorem</u> (Dual of $L^p(\mathbb{R}^n)$)

Let $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$. Then $L^q(\mathbb{R}^n) = L^p(\mathbb{R}^n)'$, where we identify functions in $L^q(\mathbb{R}^n)$ with linear maps using K.

<u>Remarks</u> • We assert $L^1(\mathbb{R}^n)' = L^\infty(\mathbb{R}^n)$, however it is the case that $L^\infty(\mathbb{R}^n)' \neq L^1(\mathbb{R}^n)$.

Thus $L^p(\mathbb{R}^n)$ is reflexive for $1 < p < \infty$ and not otherwise.

• The result holds in more general measure spaces.

• To prove result, we will reduce to the case where Λ is positive and real.

<u>Thm</u> If $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$ then $L^p(\mathbb{R}^n)' = L^q(\mathbb{R}^n)$, where we identify $g \in L^q(\mathbb{R}^n)$ with the linear map $\Lambda_g \in L^p(\mathbb{R}^n)'$ given by $\Lambda_g(f) = \int_{\mathbb{R}^n} g(x) f(x)\, dx$.

Let $L^p(\mathbb{R}^n; \mathbb{R})$ be the (real) subspace of $L^p(\mathbb{R}^n)$ consisting of functions which take real values a.e.

If $f \in L^p(\mathbb{R}^n)$ then $\exists!\ f_r, f_i \in L^p(\mathbb{R}^n; \mathbb{R})$ such that $f = f_r + i f_i$.

Given a bounded linear map $\Lambda : L^p(\mathbb{R}^n) \longrightarrow \mathbb{C}$, we define two real-linear maps $\Lambda_r, \Lambda_i : L^p(\mathbb{R}^n; \mathbb{R}) \longrightarrow \mathbb{R}$ by

$$\Lambda_r(f) = \mathrm{Re}(\Lambda f), \quad \Lambda_i(f) = \mathrm{Im}(\Lambda f).$$

Λ is determined uniquely by Λ_i, Λ_r:

$$\Lambda(f_r + i f_i) = \Lambda_r(f_r) - \Lambda_i(f_i) + i \Lambda_r(f_i) + i \Lambda_i(f_r)$$

A real-linear map $u : L^p(\mathbb{R}^n; \mathbb{R}) \longrightarrow \mathbb{R}$ is positive if $u(f) \geq 0$ for all $f \geq 0$. Any real-linear map u may be written $u = u_+ - u_-$ for positive $u_\pm : L^p(\mathbb{R}^n; \mathbb{R}) \longrightarrow \mathbb{R}$.

This means that to prove our theorem, it suffices to show:

<u>Lemma</u> Let $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$. Suppose $u : L^p(\mathbb{R}; \mathbb{R}) \longrightarrow \mathbb{R}$ is a bounded, positive, linear map. Then there exists a non-negative $g \in L^q(\mathbb{R}^n; \mathbb{R})$ with $\|g\|_{L^q} = \|u\|_{L^{p\prime}}$ such that

$$u(f) = \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

<u>Pf</u> Let $\mu = e^{-|x|^2} dx$ be Gaussian measure on $\mathbb{R}^n$, note $\mu(\mathbb{R}^n) < \infty$. For a Lebesgue measurable A, let

$$\nu(A) := u\left(e^{-|x|^2/p} \mathbb{1}_A\right)$$

Clearly $\nu(A) \in [0, \infty]$ and $\nu(\emptyset) = 0$, $\nu(\mathbb{R}^n) < \infty$.

Further if $B = \bigcup_{n=1}^\infty A_n$ for disjoint meas sets A_n, then setting $B_k = \bigcup_{n=1}^k A_n$ we have

$$\| e^{-|x|^2/p} \mathbb{1}_B - e^{-|x|^2/p} \mathbb{1}_{B_k} \|_{L^p} = \left| \int_{B \setminus B_k} e^{-|x|^2} dx \right|^{1/p}$$

$$= \left[\mu(B \setminus B_k) \right]^{1/p} \longrightarrow 0$$

$$\nu(B \setminus B_k) = u\big(\underbrace{e^{-|x|^2/p} \mathbb{1}_B - e^{-|x|^2/p} \mathbb{1}_{B_k}}_{\to 0 \text{ in } L^p} \big) \longrightarrow 0 \quad \text{as } k \to \infty$$

$$= \nu(B) - \nu(B_k) \qquad \therefore \nu(B_k) \to \nu(B)$$

$\therefore \nu$ is a finite measure on Lebesgue sets

Further $\nu \ll \mu$ since if $\mu(A) = 0$ then

$$\| e^{-|x|^2/p} \mathbb{1}_A \|_{L^p} = [\mu(A)]^{1/p} = 0$$

$\therefore \nu(A) = 0$

Hence by Radon-Nikodym, there exists a non-negative $G \in L^1(\mathbb{R}^n, \mu)$ such that

$$\nu(A) = \int_A G(x) \, d\mu(x) = \int_A G(x) e^{-|x|^2} dx$$

By linearity of u, if $f = e^{-|x|^2/p} F$ for some simple F, we have

$$u(f) = \int_{\mathbb{R}^n} f(x) g(x) \, dx \qquad \text{where } g(x) = e^{-|x|^2/q} G(x)$$

Functions of the form $e^{-|x|^2/p} F$ for F simple are dense in $L^p(\mathbb{R}^n; \mathbb{R})$ and moreover we have

$$|u(f)| \leq \|u\|_{L^{p\prime}} \|f\|_{L^p}$$

Thus $\sup \left\{ \left| \int_{\mathbb{R}^n} f(x) g(x) \, dx \right| : f \in L^p(\mathbb{R}^n; \mathbb{R}), \|f\|_{L^p} \leq 1 \right\} \leq \|u\|_{L^{p\prime}}$

By example sheet question, $g \in L^q(\mathbb{R}^n; \mathbb{R})$ with

$$\|g\|_{L^q} \leq \|u\|_{L^{p\prime}}$$

On the other hand, $\|u\|_{L^{p\prime}} \leq \|g\|_{L^q}$ is just Hölder. $\square$

<u>Theorem</u> (Riesz representation for spaces of cts functions)
~ a.k.a. Riesz - Markov

Another space whose dual can be conveniently characterised is $C_c^0(\mathbb{R}^n)$, the space of cts, compactly supported functions on $\mathbb{R}^n$ with the sup norm.

We shall argue ä that any positive functional on $C_c^0(\mathbb{R}^n, \mathbb{R})$ can be represented by integration against a suitable measure.

Suppose $\mathcal{M}$ is a σ-algebra on $\mathbb{R}^n$, μ a measure with $\mu(\mathbb{R}^n) < \infty$, we require μ is regular:

<u>Def^n</u> Suppose E is a topological space and $\mathcal{E}$ is a σ-algebra containing the Borel algebra. Then a measure on $(E, \mathcal{E})$ is <u>regular</u> if for any $A \in \mathcal{E}$ and any $\varepsilon > 0$ we can find a closed set C and an open set O s.t. $C \subset A \subset O$ and $\mu(O \setminus C) < \varepsilon$.

<u>Example</u> Lebesgue measure

~~Since~~ Suppose μ is a regular measure on $\mathbb{R}^n$, σ-algebra $\mathcal{M}$. Since $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}$, any $f : C_c^0(\mathbb{R}^n; \mathbb{R})$ is measurable. Then the map

$$\Lambda : C_c^0(\mathbb{R}^n; \mathbb{R}) \longrightarrow \mathbb{R}$$

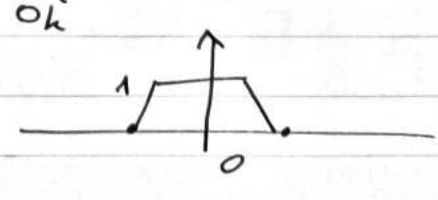
$$f \longmapsto \int_{\mathbb{R}^n} f \, d\mu \qquad (\mu(\mathbb{R}^n) < \infty)$$

is a positive, bounded linear map.

Suppose we are given Λ, can we recover μ? We'd like to set $f = \mathbb{1}_A$ for $A \in \mathcal{M}$, but $\mathbb{1}_A \notin C_c^0(\mathbb{R}^n; \mathbb{R})$.

Suppose O is open, for $k \in \mathbb{N}$ let $O_k = O \cap \{|x| < k\}$.

Define
$$\chi_k(x) := \begin{cases} 1, & x \in O_k,\ d(x, O_k^c) \geq k^{-1} \\ k\, d(x, O_k^c), & x \in O_k,\ d(x, O_k^c) < k^{-1} \\ 0, & x \in O_k^c \end{cases}$$



Then $\chi_k(x) \in C_c^0(\mathbb{R}^n; \mathbb{R})$, $\chi_k \nearrow \mathbb{1}_O$, so by MCT

$$\mu(O) = \lim_{k\to\infty} \int_{\mathbb{R}^n} \chi_k \, d\mu = \lim_{k\to\infty} \Lambda(\chi_k)$$

In particular,

$$\mu(O) = \sup \{ \Lambda(g) : g \in C_c^0(\mathbb{R}^n, \mathbb{R}), \ 0 \le g \le \mathbb{1}_O \}.$$

Now since μ is regular, if we know μ on open sets, we know μ on all measurable sets. We've shown

<u>Lemma</u> Given a σ-algebra $\mathcal{M}$ on $\mathbb{R}^n$ containing $\mathcal{B}(\mathbb{R}^n)$ and a regular measure μ s.t. $\mu(\mathbb{R}^n) < \infty$, the map

$$\Lambda : f \mapsto \int f \, d\mu$$

defines a positive, bounded, linear map on $C_c^0(\mathbb{R}^n; \mathbb{R})$. Furthermore, Λ uniquely determines μ.

In fact more is true:

<u>Theorem</u> (Riesz - Markov)

Given a positive, bounded, linear operator $\Lambda : C_c^0(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ there exists a σ-algebra $\mathcal{M}$ containing $\mathcal{B}(\mathbb{R}^n)$ and a unique regular measure μ s.t. $\mu(\mathbb{R}^n) < \infty$ and

$$\Lambda(f) = \int_{\mathbb{R}^n} f \, d\mu$$

Pf/ See "Functional Analysis" by Rudin.

Strong, weak and weak-* topologies

The spaces we have mostly considered so far, being Banach spaces, have a natural topology associated to the norm which is adequate for many purposes. For some purposes they are not as convenient.

Recall:

Thm Let $(x_n)_{n=1}^{\infty}$ with $x_k \in \mathbb{R}^n$ be a bounded sequence. Then $(x_k)_{k=1}^{\infty}$ has a convergent subsequence.

This is a very important result, but fails if $\mathbb{R}^n$ is replaced by a Banach space X with $\dim X = \infty$.

To recover (something like) compactness, we can consider different topologies.

Def A semi-norm p on a vector space X over a field $\Phi = \mathbb{C}$ or $\mathbb{R}$ is a map $p: X \to \mathbb{R}$ such that

(i) $p(x+y) \leq p(x) + p(y) \quad \forall\, x, y \in X$

(ii) $p(\lambda x) = |\lambda|\, p(x) \quad \forall\, x \in X,\ \lambda \in \Phi$

(iii) $p(x) \geq 0 \quad \forall\, x \in X$ (Follows from (i), (ii))

A family $\mathcal{P}$ of seminorms is separating if for every $x \in X$ with $x \neq 0$, there exists $p \in \mathcal{P}$ with $p(x) \neq 0$.

If $\mathcal{P}$ is a separating family of seminorms, we define a topology $\tau_{\mathcal{P}}$ as follows:

For $p \in \mathcal{P}$, $n \in \mathbb{N}$, let

$$V(p,n) = \{ x \in X : p(x) < \tfrac{1}{n} \}.$$

Let $\mathring{\beta}$ be the collection of finite intersections of $V(p,n)$'s,

$$\beta = \{ x + B : B \in \mathring{\beta},\ x \in X \}.$$

Theorem β as described above is a base for a Hausdorff topology $\tau_{\mathcal{P}}$ on X such that vector space operations are continuous.

That is, $+ : X \times X \to X$, $(x,y) \mapsto x+y$ and $\cdot : \Phi \times X \to X$, $(\lambda, x) \mapsto \lambda x$ are cts

$$\left.\begin{aligned} + &: X \times X \to X \\ &(x,y) \mapsto x+y \\ \cdot &: \Phi \times X \to X \\ &(\lambda, x) \mapsto \lambda x \end{aligned}\right\} \text{ are cts}$$

and moreover each $p \in \mathcal{P}$ is continuous.

We say the space $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space (LCTVS). If $\mathcal{P}$ is countable, $\{p_i\}_{i=1}^{\infty}$ say, the topology is a metric topology with metric

$$d(x,y) = \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x-y)}{1+p_i(x-y)}$$

If this metric is complete, we say $(X, \tau_{\mathcal{P}})$ is <u>Fréchet</u>.

If X is Banach, we can take $\mathcal{P}_S = \{\|\cdot\|\}$, the corresponding topology $\tau_S := \tau_{\mathcal{P}_S}$ is the norm topology.

$(x_k)_{k=1}^{\infty}$ with $x_k \in X$ converges to x in τ_S iff

$$\|x - x_k\| \longrightarrow 0 \text{ as } k \to \infty.$$

τ_S is called the <u>strong</u> topology on X.

Suppose $\Lambda \in X'$. Then $p_\Lambda : x \mapsto |\Lambda(x)|$ is a seminorm, and the family $\mathcal{P}_W = \{p_\Lambda : \Lambda \in X'\}$ is separating (Hahn-Banach). Thus $\tau_W := \tau_{\mathcal{P}_W}$ makes X into a LCTVS. This topology is called the <u>weak</u> topology.

$(x_k)_{k=1}^{\infty}$ with $x_k \in X$ converges to x in τ_W if

$$\Lambda(x_k) \longrightarrow \Lambda(x) \quad \forall\ \Lambda \in X'.$$

We say x_k converges weakly to x and write

$$x_k \rightharpoonup x \text{ in } X.$$

($x_k \to x$ implies $x_k \rightharpoonup x$, not the converse)

Now X' is also a Banach space, so has associated strong and weak topologies. It also carries another, the <u>weak-* topology</u>.
For any $x \in X$, we define $p_x : \Lambda' \mapsto |\Lambda'(x)|$.
This is a seminorm on X'.
Setting $\mathcal{P}_{w*} := \{ p_x : x \in X \}$ we have a separating family of seminorms on X'; the associated topology $\tau_{w*} := \tau_{\mathcal{P}_{w*}}$ is called the <u>weak-* topology</u>.
$(\Lambda_k)_{k=1}^{\infty}$ with $\Lambda_k \in X'$ converges weakly-* to Λ if

$$\Lambda_k(x) \longrightarrow \Lambda(x) \quad \forall\, x \in X$$

We write $\Lambda_k \xrightarrow{*} \Lambda$.

Note, if X is reflexive then $X'' = X$ and the weak and weak-* topologies on X' coincide. ($\Lambda_k \rightharpoonup \Lambda$ implies $\Lambda_k \xrightarrow{*} \Lambda$)

<u>Example</u>

Let $1 \leq p < \infty$, $f_i \in L^p(\mathbb{R}^n)$, $i = 1, \ldots$

$$f_i \longrightarrow f \text{ in } L^p \iff \| f_i - f \|_{L^p} \longrightarrow 0$$

$$f_i \rightharpoonup f \text{ in } L^p \iff \int_{\mathbb{R}^n} g f_i \, dx \to \int_{\mathbb{R}^n} g f \, dx \quad \forall\, g \in L^q(\mathbb{R}^n)$$

$$\underset{1<p<\infty}{\iff} f_i \xrightarrow{*} f$$

If $f_i \in L^\infty(\mathbb{R}^n)$,

$$f_i \longrightarrow f \text{ in } L^\infty \iff \| f_i - f \|_{L^\infty} \longrightarrow 0$$

$$f_i \xrightarrow{*} f \text{ in } L^\infty \iff \int_{\mathbb{R}^n} g f_i \, dx \to \int_{\mathbb{R}^n} g f \, dx \quad \forall\, g \in L^1(\mathbb{R}^n)$$

$$\not\Longleftrightarrow f_i \rightharpoonup f$$

<u>Compactness</u>

The prototypical compactness result for functions is the <u>Arzelà-Ascoli</u>

<u>Thm</u> Let $I = [0,1]$ and suppose $(f_i)_{i=1}^{\infty}$ is a sequence of functions $f_i : I \to \mathbb{C}$ which are (uniformly) bounded:

$$\sup_{x \in I} | f_i(x) | \leq M$$

and equicontinuous: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, k$ and $|x - y| < \delta$,

$$| f_k(x) - f_k(y) | < \varepsilon$$

Then (f_i) admits a subsequence converging uniformly to a uniformly continuous function.

A nice corollary is the following. We recall that f is γ-Hölder continuous for some $0 < \gamma \leq 1$ if

$$\|f\|_{C^{0,\gamma}} := \sup_{x \in I} |f(x)| + \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty$$

The space $C^{0,\gamma}(I)$ of γ-Hölder cts functions with this norm is Banach.

Corollary The closed unit ball in the $C^{0,\gamma}$ topology is compact in the C^0 topology.

Note that the compactness is in a different topology to the boundedness.

"weird & wacky"

For general normed spaces, we have an important compactness result concerning the dual space.

<u>Thm</u> (Banach-Alaoglu 1)
Let X be a normed space, and let $\overline{B}' = \{ \Lambda \in X' \mid \|\Lambda\|_{X'} \leq 1 \}$ be the closed unit ball in X'. Then $\overline{B}'$ is compact in the weak-* topology on X'.

The proof in full generality requires Tychonoff's theorem (and hence choice). We will prove the weaker (but still powerful)

<u>Thm</u> (Banach-Alaoglu 2)
Let X be a separable Banach space and let $(\Lambda_j)_{j=1}^{\infty}$ be a sequence with $\Lambda_j \in \overline{B}'$. Then there exists a subsequence $(\Lambda_{j_k})_{k=1}^{\infty}$ and $\Lambda \in \overline{B}'$ such that $\Lambda_{j_k} \xrightarrow{*} \Lambda$.

<u>Pf</u> Let $D = \{x_k\}_{k=1}^{\infty}$ be a dense countable subset of X.
Consider $(\Lambda_j(x_1))_{j=1}^{\infty}$. This is a bounded sequence in $\mathbb{C}$ since $|\Lambda_j(x_1)| \leq \|\Lambda_j\|_{X'} \|x_1\|_X \leq \|x_1\|_X$.
By Bolzano-Weierstrass, $\exists\, \Lambda(x_1) \in \mathbb{C}$ and a subsequence $\Lambda_{j_k}(x_1) \longrightarrow \Lambda(x_1)$ as $k \to \infty$; moreover $|\Lambda(x_1)| \leq \|x_1\|$.
Write $\Lambda_{1,k} = \Lambda_{j_k}$. $(\Lambda_{1,k})_{k=1}^{\infty}$ is a subsequence of $(\Lambda_j)_{j=1}^{\infty}$.
By a similar argument, applied to $\Lambda_{1,k}$, we can find a subsequence Λ_{1,k_i} and $\Lambda(x_2) \in \mathbb{C}$ s.t.

$$\Lambda_{1,k_i}(x_2) \longrightarrow \Lambda(x_2) \text{ as } i \to \infty \quad ; \quad |\Lambda(x_2)| \leq \|x_2\|.$$

Since $\Lambda_{1,k}(x_1) \longrightarrow \Lambda(x_1)$, we also have $\Lambda_{1,k_i}(x_1) \longrightarrow \Lambda(x_1)$.
Write $\Lambda_{2,i} = \Lambda_{1,k_i}$
Proceeding iteratively, we produce $(\Lambda_{l,k})_{k=1}^{\infty}$ a subsequence of $(\Lambda_{l-1,k})_{k=1}^{\infty}$ s.t. $\Lambda_{l,k}(x_j) \longrightarrow \Lambda(x_j)$ as $k \to \infty$, for $j \leq l$,
where $\Lambda(x_j) \in \mathbb{C}$ satisfy $|\Lambda(x_j)| \leq \|x_j\|$.
Now, consider $(\Lambda_{j,j})_{j=1}^{\infty}$. This is a subsequence of $(\Lambda_i)_{i=1}^{\infty}$ such that

$$\Lambda_{j,j}(x) \longrightarrow \Lambda(x) \quad \text{for all } x \in D.$$

Claim 1 $\Lambda : D \to \mathbb{C}$ is uniformly cts

Fix $\varepsilon > 0$. Suppose $x, y \in D$, $\|x - y\| < \varepsilon/3$.

Pick k suff large that $|\Lambda_{k,k}(x) - \Lambda(x)| < \varepsilon/3$,

and $|\Lambda_{k,k}(y) - \Lambda(y)| < \varepsilon/3$.

Then
$$
\begin{aligned}
|\Lambda(x) - \Lambda(y)| &\le |\Lambda(x) - \Lambda_{k,k}(x)| \\
&\quad + |\Lambda_{k,k}(x) - \Lambda_{k,k}(y)| \\
&\quad + |\Lambda_{k,k}(y) - \Lambda(y)| \\
&< \varepsilon,
\end{aligned}
$$
noting that $\|\Lambda_{k,k}\|_{X'} \le 1$.

Thus Λ extends uniquely to $\bar{D} = X$ as a continuous function.

Claim 2 $\Lambda : X \to \mathbb{C}$ is linear

Suppose $x, y \in X$, $a \in \mathbb{C}$ and $z = x + ay$. Fix $\varepsilon > 0$.
$$
\begin{aligned}
|\Lambda(z) - \Lambda(x) - a\Lambda(y)| &\le |\Lambda(z) - \Lambda(z')| + |\Lambda(x) - \Lambda(x')| \\
&\quad + |a|\,|\Lambda(y) - \Lambda(y')| \\
&\quad + |\Lambda(z') - \Lambda_{j,j}(z')| + |\Lambda(x') - \Lambda_{j,j}(z')| \\
&\quad + |a|\,|\Lambda(y') - \Lambda_{j,j}(y')| \\
&\quad + |\Lambda_{jj}(z' - x' - ay')|
\end{aligned}
$$

in D

Pick x', y', z' s.t. 1st and 3rd lines are $< \varepsilon/3$ for all j.

Then pick j suff large that 2nd line is $< \varepsilon/3$.

Thus $|\Lambda(z) - \Lambda(x) - a\Lambda(y)| < \varepsilon$ $\forall \varepsilon > 0$. $\therefore \Lambda$ linear

Claim 3 $\Lambda_{j,j} \xrightarrow{*} \Lambda$ in X'

If $x \in X$, estimate
$$
\begin{aligned}
|\Lambda_{j,j}(x) - \Lambda(x)| &\le |\Lambda_{j,j}(x) - \Lambda_{j,j}(x')| \\
&\quad + |\Lambda_{j,j}(x') - \Lambda(x')| \\
&\quad + |\Lambda(x') - \Lambda(x)|
\end{aligned}
$$
Pick x' in D close to x s.t. 1st, 3rd terms $< \varepsilon/3$ for all j.

Then pick j suff large that $|\Lambda_{j,j}(x') - \Lambda(x')| < \varepsilon/3$.

$\therefore$ for j suff large $|\Lambda_{j,j}(x) - \Lambda(x)| < \varepsilon$.

Claim 4 $\|\Lambda\| \leq 1$

$$\|\Lambda\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\Lambda(x)| = \sup_{\substack{x \in D \\ \|x\| \leq 1}} |\Lambda(x)| \quad \text{by density}$$

By construction $|\Lambda(x)| \leq \|x\|$ for $x \in D$.

$\therefore \|\Lambda\| \leq 1$. $\square$

Example Suppose $1 < p \leq \infty$ and let $(f_j)_{j=1}^{\infty}$ be a sequence in $L^p(\mathbb{R}^n)$ satisfying $\|f_j\|_{L^p} \leq K \ \forall j$

Then there exists a subsequence $(f_{j_k})_{k=1}^{\infty}$ and $f \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f_{j_k} g \, dx \longrightarrow \int_{\mathbb{R}^n} f g \, dx$$

holds for all $g \in L^q(\mathbb{R}^n)$. Further $\|f\|_{L^p} \leq K$.

Pf/ Note $L^p(\mathbb{R}^n) = L^q(\mathbb{R}^n)'$ where $p^{-1} + q^{-1} = 1$ and $L^q(\mathbb{R}^n)$ is separable for $1 \leq q < \infty$.

(f_j) is a sequence, uniformly bounded in $L^q(\mathbb{R}^n)'$.

Apply Banach-Alaoglu to $(f_j / K)_{j=1}^{\infty}$.

Hahn-Banach

Hahn-Banach is one of the cornerstones of Functional Analysis.

It concerns the problem of extending a linear map $\Lambda : M \to \mathbb{C}$ defined on a subspace $M \leq X$ which is bdd, to a bdd linear map $\tilde{\Lambda} : X \to \mathbb{C}$.

For finite dimensional X this is relatively straightforward.

If X is ∞-dim, we have to invoke AoC.

Suppose X is a vector space over $\mathbb{C}$. Then X is also a vector space over $\mathbb{R}$ in a natural way. A real-linear map $\ell : X \to \mathbb{R}$ is a map satisfying

$$\ell(x+ay) = \ell(x) + a\ell(y), \quad x,y \in X, a \in \mathbb{R}$$

Suppose $\Lambda : X \to \mathbb{C}$ is a (complex) linear, then $\ell(x) = \operatorname{Re}\Lambda(x)$ is a real linear map. Conversely, given a real-linear map ℓ, we have $\Lambda(x) = \ell(x) - i\ell(ix)$, $x \in X$, defines a complex linear map $\Lambda : X \to \mathbb{C}$ with $\operatorname{Re}\Lambda(x) = \ell(x)$. If we can extend a real-linear map then we can transfer the result to the complex setting.

In the context of Hahn-Banach thm, it's useful to consider a more general notion of boundedness.

Def A sublinear functional on a real vector space X is a map $p : X \to \mathbb{R}$ satisfying

$$\begin{cases} p(x+y) \le p(x) + p(y) & \forall x,y \in X \\ p(tx) = tx & \forall x \in X, t \geq 0 \end{cases}$$

E.g. If $\ell : X \to \mathbb{R}$ is linear, then $p(x) = |\ell(x)|$ is sublinear. All semi-norms are sublinear, but not all sublinear f^ns are semi-norms.

For a linear map $\ell : X \to \mathbb{R}$, we can work with one-sided bounds of the form $\ell(x) \le p(x)$ but this implies

$$-p(-x) \le \ell(x) \le p(x).$$

Lemma (Bounded extension lemma)

Let X be a real vector space, $p : X \to \mathbb{R}$ sublinear and $M \le X$. Suppose $\ell : M \to \mathbb{R}$ is linear and satisfies $\ell(y) \le p(y)$ $\forall y \in M$. Fix $x \in X \setminus M$; then letting $\tilde{M} = \operatorname{span}\{M, x\}$, $\exists \tilde{\ell} : \tilde{M} \to \mathbb{R}$ linear such that $\tilde{\ell}(w) \le p(w)$ $\forall w \in \tilde{M}$ and $\tilde{\ell}|_M = \ell$.

Pf If $w \in \tilde{M}$, we may uniquely write w as

$$w = \lambda x + y, \quad y \in M, \lambda \in \mathbb{R}.$$

$\therefore$ to define $\tilde{\ell}$ it suffices to define $\tilde{\ell}(x) = a$ as then

$$\tilde{\ell}(w) = \lambda a + \ell(y).$$

Suppose $y, z \in M$. Then for all

$$\ell(y) + \ell(z) = \ell(y+z) \leq p(y+z) \leq p(y-x) + p(z+x)$$

hence $\ell(y) - p(y-x) \leq p(z+x) - \ell(z)$. $\quad (+)$

Let $a = \sup_{y \in M} \left(\ell(y) - p(y-x)\right)$.

By $(+)$, a is finite and further

$$\ell(y) - a \leq p(y-x) \quad \forall y \in M \quad (*_1)$$

$$\ell(z) + a \leq p(z+x) \quad \forall z \in M \quad (*_2)$$

If $\lambda > 0$, let $z = \lambda^{-1} y$ and multiply $(*_2)$ by λ.

If $\lambda < 0$, let $y = -\lambda^{-1} y$ and multiply $(*_1)$ by $-\lambda$

to deduce $\ell(y) + a\lambda \leq p(y + \lambda x) \quad \forall y \in M, \lambda \in \mathbb{R}$

$\Rightarrow \tilde{\ell}(y) \leq p(y) \ \forall w \in \tilde{M}$. $\square$

If $\dim(X/M)$ is finite, we can extend $\ell : M \to \mathbb{R}$ to a functional $\tilde{\ell} : X \to \mathbb{R}$ in a finite no. of steps. If is separable, we can use induction to construct an extension. But in general we need to invoke axiom of choice.

Zorn's Lemma

Zorn's lemma concerns sets equipped with a partial ordering

Def Let S be a set. A partial order on S is a binary relation $\leq$ satisfying: (i) $a \leq a \quad \forall a \in S$ (reflexivity)

(ii) If $a \leq b$ and $b \leq a$ then $a = b$ (anti-symmetry)

(iii) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

$\forall a, b, c \in S$

A set with a partial ordering is called a partially ordered set or poset. Note it need not hold that for $a, b \in S$, either

$a \leq b$ or $b \leq a$. If this holds for all $a, b \in S$ we say $\leq$ is a total order.

A subset T of a poset S which is totally ordered is called a chain.
An element $u \in S$ is an upper bound for $T \subset S$ if $a \leq u$ for all $a \in T$. A maximal element of S is an element $m \in S$ such that $m \leq x$ implies $x = m$.

Examples a) If A is any set, $S = 2^A$ is a poset, ordered by set inclusion.

b) $\mathbb{R}$ with its standard ordering is a totally ordered set, with no maximal element.

c) The collection of open balls in $\mathbb{R}^n$ is a poset ordered by inclusion. The subset $T = \{B_r(0) : 0 < r \leq 1\}$ is a chain. $B_1(0)$ is a maximal element of T, $B_2(0)$ is an upper bound.

Proposition (Zorn's lemma)
Let $(S, \leq)$ be a poset in which every totally ordered subset has an upper bound. Then $(S, \leq)$ contains at least one maximal element.

This is equivalent to A.o.C. We'll take it for granted. (an axiom even)
With Zorn's lemma, we are ready to prove:

Theorem (Hahn Banach)
Let X be a real vector space, $p : X \to \mathbb{R}$ sublinear and $M \subset X$ a subspace. Suppose $\ell : M \to \mathbb{R}$ is linear and satisfies $\ell(x) \leq p(x)$ for all $x \in M$. Then there exists a linear operator $\tilde{\ell} : X \to \mathbb{R}$ such that:
- $\tilde{\ell}(x) \leq p(x) \quad \forall x \in X$
- $\tilde{\ell}|_M = \ell$

Pf Consider the set S of extensions of ℓ to a subspace of X.
That is a pair $(N, \ell^*) \in S$ if:

(i) $N \leq X$ and $M \leq N$,

(ii) $\ell^* : N \to \mathbb{R}$ is linear map,

(iii) $\ell^*(x) \leq p(x) \; \forall x \in N$,

(iv) , $\ell^*|_M = \ell$.

S is a poset with partial ordering

$(N_1, \ell_1^*) \leq (N_2, \ell_2^*)$ if $\quad \bullet \; N_1 \leq N_2$

$\bullet \; \ell_2^*|_{N_1} = \ell_1^*$

Suppose T is a totally ordered subset of S.

Define $(\mathcal{N}, L)$ by $\mathcal{N} = \bigcup_{(N,\ell^*) \in T} N$

and for $x \in \mathcal{N}$, $L(x) = \ell^*(x)$ for some $(N, \ell^*) \in T$ s.t. $x \in N$

This is well-defined since T totally ordered.

Further for all $(N, \ell^*) \in T$, $(N, \ell^*) \leq (\mathcal{N}, L)$.

Thus T has an upper bound.

By Zorn, S has a maximal element $(\tilde{N}, \tilde{\ell})$.

We claim $\tilde{N} = X$.

If not $\exists \; x \in X \setminus \tilde{N}$ and by our extension lemma we can extend $\tilde{\ell}$ to some $\tilde{\ell}^*$ on $\text{span}(\tilde{N}, x) \supsetneq \tilde{N}$.

This contradicts maximality of $(\tilde{N}, \tilde{\ell})$.

So $\tilde{N} = X$ and we are done. $\square$

Note this proof is <u>non-constructive</u>.

This is typical of proofs invoking AoC.

<u>Corollary</u> Let X be a Banach space over Φ, where Φ is $\mathbb{R}$ or $\mathbb{C}$, and $M \leq X$. Let $\Lambda : M \to \Phi$ be a bounded linear operator. Then there exists $\tilde{\Lambda} : X \to \Phi$ with $\|\tilde{\Lambda}\|_{X'} = \|\Lambda\|_{M'}$ such that $\Lambda = \tilde{\Lambda}|_M$.

<u>Pf</u> If $\Phi = \mathbb{R}$, apply previous result with $p(x) = \|\Lambda\| \, \|x\|$ ($\Lambda = 0$ is trivial). If $\Phi = \mathbb{C}$, write $\Lambda(x) = \ell(x) - i\ell(ix)$ for $\ell : X \to \mathbb{R}$, $\ell(x) = \operatorname{Re}\Lambda(x)$ a real linear functional.
Noting that $|\Lambda(x)| = \ell(e^{i\theta}x)$ for suitable θ, we have

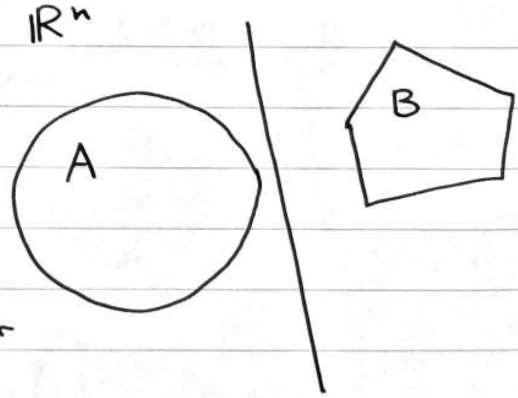
$$\sup_{\substack{x \in N, \\ \|x\| \leq 1}} |\Lambda(x)| = \sup_{\substack{x \in N \\ \|x\| \leq 1}} |\ell(x)| \quad , \quad N \leq X$$

Apply last result to ℓ and reconstruct Λ. $\square$

We move onto some 'geometric' consequences of Hahn-Banach, known as the <u>Hahn-Banach separation theorems</u>.
Recall a subset A of a vector space V is convex if

$$\forall x, y \in A, \quad tx + (1-t)y \in A \quad \forall t \in [0,1]$$

The prototypical result is the <u>supporting hyperplane theorem</u>.
A, B are convex, disjoint compact subsets of $\mathbb{R}^n$
We can find a codimension 1 surface
such that A, B are on opposite sides.



<u>Theorem</u> Suppose A, B are disjoint, non-empty, convex sets in a (real or complex) Banach space X.

a) If A is open, there exists $\Lambda \in X'$ and $\gamma \in \mathbb{R}$ s.t.

$$\operatorname{Re}\Lambda x < \gamma \leq \operatorname{Re}\Lambda y \qquad \forall x \in A, y \in B$$

If B is ~~closed~~ open too, second inequality may be taken to be strict.

b) If A is compact, B closed, then $\exists\, \Lambda \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\operatorname{Re}\Lambda x < \gamma_1 < \gamma_2 < \operatorname{Re}\Lambda y \quad \forall x \in A, y \in B$

Pf Wlog, assume $\Phi = \mathbb{R}$, else treat X as a real v.s. to find $\ell : X \to \mathbb{R}$ linear with properties required. Set $\Lambda(x) = \ell(x) - i\ell(ix)$, this will have the required properties.

a) Pick $a_0 \in A$, $b_0 \in B$ and let $x_0 = b_0 - a_0$.

Let $C = A - B + x_0$. This is a convex neighbourhood of 0 in X (Check!)

Since A, B are disjoint, $x_0 \notin C$.

Let $p(x) = \inf\{t > 0 : t^{-1}x \in C\}$.

By exercise 2.10, p is sublinear, $p(x) \leq k\|x\|$ for some $k > 0$, and $p(y) < 1$ for $y \in C$.

Since $x_0 \notin C$, $p(x_0) \geq 1$.

Let $M = \{t x_0, t \in \mathbb{R}\}$, define $f : M \longrightarrow \mathbb{R}$ by

$f(t x_0) = t$. (obviously linear)

If $t > 0$ then $f(t x_0) = t \leq t p(x_0) = p(t x_0)$.

If $t \leq 0$ then $f(t x_0) = t \leq 0 \leq p(t x_0)$.

So $f \leq p$ on M. By Hahn-Banach, extend f to a functional Λ satisfying

$$-k\|x\| \leq -p(-x) \leq \Lambda(x) \leq p(x) \leq k\|x\|$$

so $\Lambda \in X'$.

Suppose $a \in A$, $b \in B$. Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + x_0) \leq p(\underbrace{a - b + x_0}_{\in C}) < 1$$

$\therefore \Lambda a < \Lambda b$

Thus $\Lambda(A)$ and $\Lambda(B)$ are disjoint convex subsets of $\mathbb{R}$ with $\Lambda(A)$ to the left of $\Lambda(B)$.

By Ex 2.2, Λ is an open mapping, so $\Lambda(A)$ is open, we can take γ to the right endpoint of $\Lambda(A)$.

If B is also open, $\Lambda(B)$ is open and we will have that

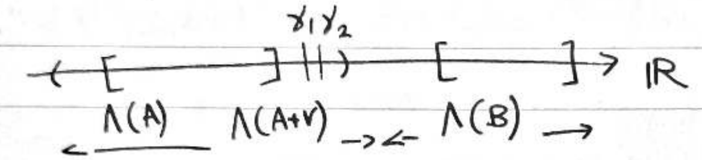
$\gamma < \Lambda(b) \quad \forall b \in B$.

b) Since A is compact, B closed, disjoint

$$d = \inf \{ \|a-b\| \mid a \in A, b \in B \} > 0$$

Let $V = B_{d/2}(0)$ and consider $A+V$. This is open, convex and disjoint from B. By part a) there exists $\Lambda \in X'$ such that $\Lambda(A+V)$ and $\Lambda(B)$ are disjoint convex subsets of $\mathbb{R}$, with $\Lambda(A+V)$ to the left of $\Lambda(B)$.

Since $\Lambda(A+V)$ open and $\Lambda(A)$ is a compact subset of $\Lambda(A+V)$, the result follows. $\square$



Setting $A = \{x\}$, $B = \{y\}$ and applying b) gives immediately that if $x \neq y$ $\exists \Lambda \in X'$ such that $\Lambda(x) \neq \Lambda(y)$. (Justifies a lemma claimed earlier)

<u>Corollary</u> Suppose M is a subspace of a Banach space X and $x_0 \in X$. Then if x_0 is not in the closure of M, there exists $\Lambda \in X'$ such that $\Lambda(x_0) = 1$ and $\Lambda|_M = 0$.

<u>Pf</u> Apply b) with $A = \{x_0\}$, $B = \overline{M}$; find $\Lambda \in X'$ s.t. $\{\Lambda x_0\}$ and $\Lambda(\overline{M})$ are disjoint.

$\Lambda(\overline{M})$ must be a proper subspace of the scalar field, so must be $\{0\}$. Normalise by dividing by Λx_0. $\square$

Distributions

For many applications, it's useful to have available a space of objects that are 'rougher' than functions.

E.g. Consider the statement:

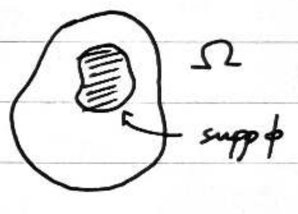
$G : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\Delta G = \delta(x)$

$x \mapsto \frac{1}{4\pi|x|}$

The theory of distributions (or generalized functions) provides a setting in which we can make sense of such statements. The idea is to consider the dual space of a space of 'test functions'.

The space $\mathcal{D}(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be open. The set $C_c^\infty(\Omega)$ consists of smooth functions $\phi : \Omega \to \mathbb{C}$ s.t. $\operatorname{supp}\phi = \overline{\{x : \phi(x) \neq 0\}}$ is compact. $C_c^\infty(\Omega)$ is a vector space in the obvious way.



We can equip this space with a topology such that it is a LCTVS. The details are in Appendix of online notes. We summarize:

<u>Theorem</u> $C_c^\infty(\Omega)$ may be equipped with a topology τ such that (i) vector space operations are continuous

(ii) a sequence $(\phi_j)_{j=1}^\infty$ with $\phi_j \in C_c^\infty(\Omega)$ converges to 0 if there exists $K \subset \Omega$ compact such that $\forall j$, $\operatorname{supp}\phi_j \subset K$, and for each multi-index α we have $|D^\alpha \phi_j(x)| \to 0$ unif in K

$(\phi_j \to \phi \iff \phi_j - \phi \to 0)$

(iii) if Y is a LCTVS and $\Lambda : C_c^\infty(\Omega) \to Y$ is linear, then Λ is continuous if and only if it is sequentially continuous.

(i.e. $\Lambda\phi_j \to \Lambda\phi$ $\forall$ sqces $\phi_j \to \phi$)

$C_c^\infty(\Omega)$ equipped with τ is denoted $\mathcal{D}(\Omega)$, the <u>space of test functions</u>.

<u>Examples</u> Fix $\phi \in C_c^\infty(\mathbb{R})$, $\phi \neq 0$

(a) If $\phi_j(x) = e^{-j}\phi(jx)$ then $\phi_j \to 0$ in $\mathcal{D}(\Omega)$

(b) If $\phi_j(x) = \frac{1}{j^{100}}\phi(jx)$ then $\phi_j \not\to 0$ in $\mathcal{D}(\Omega)$ ⌜in general⌟ nvm

(c) If $\phi_j(x) = e^{-j}\phi(x-j)$ then $\phi_j \not\to 0$ in $\mathcal{D}(\Omega)$

<u>The space $\mathcal{E}(\Omega)$</u>

If $\Omega \subset \mathbb{R}^n$ is open, $C^\infty(\Omega)$ is the vector space of smooth functions defined on Ω. We can equip $C^\infty(\Omega)$ with a Fréchet topology.

First we recall Ω admits an exhaustion by compact sets, i.e. sequence of compact sets $(K_i)_{i=1}^\infty$, $K_i \subset \Omega$ with $K_i \subset K_{i+1}^\circ$ and $\Omega = \bigcup_{i=1}^\infty K_i$

We let, for $\phi \in C^\infty(\Omega)$,

$$p_N(\phi) := \sup_{\substack{x \in K_N \\ |\alpha| \le N}} |D^\alpha(x)|$$

Then $\mathcal{P} = \{p_N : N \in \mathbb{N}\}$ is a separating family of seminorms which induces a topology τ such that $(C^\infty(\Omega), \tau)$ is a LCTVS. We call this space $\mathcal{E}(\Omega)$. This is a metric topology which is complete, hence Fréchet.

We note $(\phi_j)_{j=1}^\infty$ with $\phi_j \in C^\infty(\Omega)$ converges to zero in $\mathcal{E}(\Omega)$ if for each compact $K \subset \Omega$ and multi-index α,

$$\sup_{x \in K} |D^\alpha \phi_j(x)| \longrightarrow 0 \qquad (\phi_j \to \phi \iff \phi_j - \phi \to 0)$$

<u>Example</u> If $\phi \in C_c^\infty(\mathbb{R})$ is fixed, then $\phi_j(x) = e^{-j}\phi(x-j)$ converges to zero as $j \to \infty$.

Indeed, $e^{-j}\phi(x-j) \to 0$.

We note $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ and the inclusion is continuous.

The space $\mathcal{S}(\mathbb{R}^n)$

<u>Def</u> A function $\phi \in C^\infty(\mathbb{R}^n)$ is <u>rapidly decreasing</u> if

$$\sup_{x \in \mathbb{R}^n} |(1+|x|)^N D^\alpha \phi(x)| < \infty$$

for all $N \in \mathbb{N}$ and multi-indices α.

<u>Example</u> $\phi(x) = e^{-|x|^2}$ is rapidly decreasing

$\phi(x) = \frac{1}{(1+|x|^2)^{100}}$ is <u>not</u> rapidly decreasing

For ϕ rapidly decreasing, we define

$$p_N(x) = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq N}} |(1+|x|)^N D^\alpha \phi(x)|$$

$\mathcal{P} = \{p_N\}_{N=1}^\infty$ is a separating family of seminorms, making the set of rapidly decreasing functions into a LCTVS, $\mathcal{S}(\mathbb{R}^n)$, known as the <u>Schwartz space</u>.

(Note we could use equivalent seminorms, e.g.

$$\sup_{x \in \mathbb{R}^n, |\alpha| \leq N} |(1+|x|^2)^N D^\alpha \phi(x)|$$

or $$\sup_{x \in \mathbb{R}^n, |\alpha| \leq N} |D^\alpha((1+|x|^2)^N \phi(x))|$$

If $(\phi_j)_{j=1}^\infty$ is a sequence in $\mathcal{S}(\mathbb{R}^n)$ then $\phi_j \to 0$ iff $\forall N \in \mathbb{N}$ and all α multi-indices, we have

$$\sup_{x \in \mathbb{R}^n} |(1+|x|)^N D^\alpha \phi(x)| \to 0$$

$\mathcal{S}(\mathbb{R}^n)$ is Frèchet. We have $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$,

$$\begin{array}{ccc} \mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n) \\ & \cap & \cup \\ & \mathcal{S}(\mathbb{R}^n) & \end{array}$$

with the inclusion map(s) continuous.

Distributions

For $\Omega \subset \mathbb{R}^n$ open, the space of distributions on Ω, $\mathcal{D}'(\Omega)$, is the continuous dual space of $\mathcal{D}(\Omega)$.

A linear map $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is continuous if and only if $u(\phi_j) \to 0$ for all sequences $\phi_j \to 0$ in $\mathcal{D}(\Omega)$.

$\mathcal{D}'(\Omega)$ is equipped with the weak-$*$ topology induced by the family of semi-norms $\mathcal{P} = \{ p_\phi : \phi \in \mathcal{D}(\Omega) \}$ where

$$p_\phi(u) = |u(\phi)|$$

$$u_j \to u \text{ in } \mathcal{D}'(\Omega) \quad \text{iff} \quad u_j[\phi] \to u[\phi] \ \forall \phi \in \mathcal{D}(\Omega)$$

Example (a) For $x \in \Omega$, define

$$\begin{aligned} \delta_x : \mathcal{D}(\Omega) &\to \mathbb{C} \\ \phi &\mapsto \phi(x) =: \delta_x[\phi] \end{aligned}$$

This is a distribution, the Dirac delta.

(b) If $f \in L^1_{loc}(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ meas} \mid f \mathbb{1}_K \in L^1 \ \forall K \text{ cpct} \}$ then

$$\begin{aligned} T_f : \mathcal{D}(\Omega) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_\Omega f \phi \, dx \end{aligned}$$

is also a distribution.

<u>Lemma</u> Suppose $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear. Then u is continuous iff for each compact $K \subset \Omega$ there exists $N_K \in \mathbb{N}$ and $c_K > 0$ such that

$$|u[\phi]| \le c_K \sup_{x \in K} \sum_{|\alpha| \le N_K} |D^\alpha \phi(x)|$$

for all $\phi \in C_c^\infty(K)$.

<u>Pf</u> "$\Leftarrow$" If $\phi_j \to 0$ in $\mathcal{D}(\Omega)$ then $\exists\, K \subset \Omega$ cpct s.t. $\forall j$, $\operatorname{supp} \phi_j \subset K$ and $D^\alpha \phi_j \to 0$ unif on K for each α. Apply the criterion above to the set K to deduce

$$|u[\phi_j]| \to 0 \quad \text{so } u \text{ is cts.}$$

"$\Rightarrow$" Suppose not. ~~Fix K. For all j, $\exists\, \phi_j \in C_c^\infty(K)$ with~~ $\exists\, K$ and sqce $\phi_j \in C_c^\infty(K)$

s.t. $\forall j$, $|u[\phi_j]| \ge j \sup_{x \in K} \sum_{|\alpha| \le j} |D^\alpha \phi_j(x)|$

Let $\psi_j = \dfrac{\phi_j}{|u[\phi_j]|}$.

Then $\psi_j \to 0$ in $\mathcal{D}(\Omega)$:

For any β multi-index,

$$|D^\beta \psi_j(x)| = |D^\beta \phi_j(x)| \,/\, |u[\phi_j]|$$
$$\le \frac{|D^\beta \phi_j(x)|}{j \sup_{x \in K} \sum_{|\alpha| \le j} |D^\alpha \phi_j(x)|}$$

If $j \ge |\beta|$ then $|D^\beta \psi_j(x)| \le \frac{1}{j} \to 0$.

On the other hand, $|u[\psi_j]| = 1$ ⨳ $\square$

If we can choose $N_K = N$ <u>independently of K</u>, we say u has <u>finite order</u>. The least such N is called the <u>order of the dist^n</u>.

Examples • δ_x and T_f have order 0.

• $u: \varphi \mapsto \varphi'(x)$ has order 1.

• $u: \mathcal{D}(\mathbb{R}) \to \mathbb{C}$
$\varphi \mapsto \sum_{i=1}^{\infty} \varphi^{(i)}(i)$ does not have finite order

Properties of Distributions

Recall that for $f \in L^1_{loc}(\Omega)$ we introduced

$$T_f[\phi] = \int_\Omega f\phi \, dx$$

Note that if $T_f = T_g$, then $\int_\Omega (f-g)\phi \, dx = 0$ for all $\phi \in C_c^\infty(\Omega)$. This implies $f = g$ a.e. (recall mollification by convolution)

Thus the map

$$T: L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega)$$
$$f \mapsto T_f$$

is an injection.

We will use this to generalise operations defined on functions to act on distributions.

Operations on distributions

$\mathcal{D}'(\Omega)$ is naturally a vector space over $\mathbb{C}$, so we can add distributions and multiply by elements of $\mathbb{C}$:

If $u_1, u_2 \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{C}$, then

$$(u_1 + \alpha u_2)[\phi] := u_1[\phi] + \alpha u_2[\phi] \quad \forall \phi \in \mathcal{D}(\Omega)$$

Suppose $a \in C^\infty(\Omega)$. Then for $\phi \in \mathcal{D}(\Omega)$ we have

$$T_{af}[\phi] = \int_\Omega af\phi \, dx = T_f[a\phi]$$

for all $f \in L^1_{loc}(\Omega)$ since $a\phi \in \mathcal{D}(\Omega)$.

This suggests that we define au for $a \in C^\infty(\Omega)$, $u \in \mathcal{D}'(\Omega)$ by

$$(au)[\phi] = u[a\phi] \qquad \forall \phi \in \mathcal{D}(\Omega)$$

Similarly, if $f \in C^1(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ then

$$T_{D_i f}[\phi] = \int_\Omega D_i f \phi \, dx = -\int_\Omega f D_i \phi \, dx$$

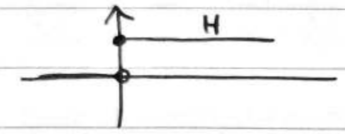
$$= T_f[-D_i \phi]$$

This suggests that for $u \in \mathcal{D}'(\Omega)$ and α a multi-index, we <u>define</u>

$$D^\alpha u[\phi] = (-1)^{|\alpha|} u[D^\alpha \phi] \qquad \forall \phi \in \mathcal{D}(\Omega)$$

This is <u>always</u> well-defined : we can differentiate any distribution.

<u>Example</u> Let $H : \mathbb{R} \to \mathbb{R}$ be given by $H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$



Consider $T_H \in \mathcal{D}'(\mathbb{R})$.
We compute, for $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned} DT_H[\phi] &= - T_H[D\phi] = - \int_{\mathbb{R}} H \, D\phi \, dx \\ &= - \int_{-\infty}^{\infty} H(x) \phi'(x) \, dx \\ &= - \int_0^\infty \phi'(x) \, dx \\ &= \phi(0) = \delta_0[\phi] \end{aligned}$$

Thus $DT_H = \delta_0$.
Often abuse notation to write $H'(x) = \delta(x)$.

We now have enough operations to make sense of distributional PDEs:
e.g. solve $\displaystyle\sum_{|\alpha| \leq k} a^\alpha D^\alpha u = \omega$

where $\omega \in \mathcal{D}'(\Omega)$ is given, $a^\alpha \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ is the unknown. Separates out problems of existence and regularity.

<u>The space $\mathcal{E}'(\Omega)$</u>

$\mathcal{E}'(\Omega)$ is the continuous dual space of $\mathcal{E}(\Omega)$. Since $\mathcal{E}(\Omega)$ is a metric space, a linear map $u : \mathcal{E}(\Omega) \to \mathbb{C}$ is cts iff $u[\phi_j] \to 0$ for all sqces $\phi_j \to 0$ in $\mathcal{E}(\Omega)$.

Lemma Suppose $u : \mathcal{E}(\Omega) \to \mathbb{C}$ is linear.
Then u is continuous iff there exists $K \subset \Omega$ cpct, $N \in \mathbb{N}$, $C > 0$ such that $\forall \phi$

$$|u[\phi]| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |D^\alpha \phi(x)| \qquad (+)$$

Pf Suppose $(+)$ holds and let $\phi_j \in \mathcal{E}(\Omega)$ converge to 0.
Then $\forall$ compact $\tilde{K} \subset \Omega$ and $\tilde{N} \in \mathbb{N}$ we have

$$\sup_{x \in \tilde{K}, |\alpha| \leq \tilde{N}} |D^\alpha \phi_j(x)| \longrightarrow 0$$

Taking $\tilde{K} = K$, $\tilde{N} = N$ applying $(+)$ implies $|u[\phi_j]| \to 0$
So u is cts.

Now suppose $(+)$ is not true: Then let (K_j) be an exhaustion of Ω by compact sets, i.e. $K_j \subset \Omega$ cpct, $K_j \subset \mathring{K}_{j+1}$, $\bigcup_j K_j = \Omega$.
In particular, since $(+)$ is false, for each j we can find $\phi_j \in \mathcal{E}(\Omega)$ s.t.

$$|u[\phi_j]| \geq j \sup_{\substack{x \in K_j \\ |\alpha| \leq j}} |D^\alpha \phi_j(x)|$$

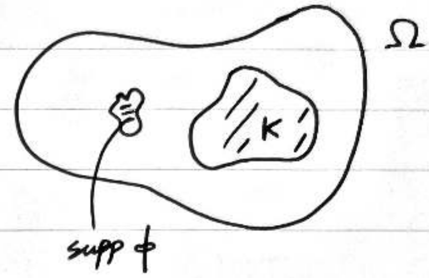
Define $\psi_j = \dfrac{\phi_j}{|u[\phi_j]|}$. Then $\psi_j \to 0$ in $\mathcal{E}(\Omega)$ since for any compact $\tilde{K}$, any $\tilde{N} \in \mathbb{N}$, $\exists J > \tilde{N}$ with $\tilde{K} \subset K_j$ $\forall j \geq J$, hence

$$\sup_{x \in \tilde{K}, |\alpha| \leq \tilde{N}} |D^\alpha \psi_j(x)| \leq \frac{1}{j} \qquad \forall j \geq J.$$

However $|u[\psi_j]| = 1$ so $u[\psi_j] \not\to 0$ and u not cts. $\square$

Since $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ and the inclusion is continuous, it then follows that $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. We can identify $\mathcal{E}'(\Omega)$ with the distributions of compact support: we say $u \in \mathcal{D}'(\Omega)$ has compact support if $\exists$ compact $K \subset \Omega$ such that

$$u[\phi] = 0 \quad \forall \; \phi \in C_c^\infty(\Omega \setminus K)$$



By the previous lemma, if $u \in \mathcal{E}'(\Omega)$, then u has compact support.
Conversely, if $u \in \mathcal{D}'(\Omega)$ has compact support inside some compact K, then we can extend u uniquely to $\tilde{u} \in \mathcal{E}'(\Omega)$ by setting $\tilde{u}[\phi] = u[\chi\phi] \; \forall \, \phi \in \mathcal{E}(\Omega)$, where $\chi \in C_c^\infty(\Omega)$ satisfies $\chi = 1$ on K.

<u>Examples</u> a) If $f \in L^1(\Omega)$ vanishes a.e. in $\Omega \setminus K$ for some compact $K \subset \Omega$, then $T_f \in \mathcal{E}'(\Omega)$.

b) For any $x \in \Omega$, $\delta_x \in \mathcal{E}'(\Omega)$.

c) $u \in \mathcal{D}'(\mathbb{R})$ defined by $u[\phi] = \sum_{m=-\infty}^{\infty} \phi(m)$ does not belong to $\mathcal{E}'(\Omega)$.

<u>Tempered Distributions</u>

The space $\mathcal{S}'(\mathbb{R}^n)$, the continuous dual of $\mathcal{S}(\mathbb{R}^n)$, is called the space of <u>tempered distributions</u>. Since we have cts inclusions

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$$

we deduce

$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

~~Examples~~ [<u>Recall</u> A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is cts iff $\forall \, \phi_j \in \mathcal{S}(\mathbb{R}^n)$, $\phi_j \to 0$ we have $u[\phi_j] \to 0$]

<u>Examples</u> a) If $f \in L^1_{loc}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} (1+|x|)^{-N} |f(x)| \, dx < \infty$$

for some $N \in \mathbb{Z}$, then $Tf \in \mathcal{S}'(\mathbb{R}^n)$.

Note that if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$|T_f[\phi]| = \left| \int_{\mathbb{R}^n} f(x)\, \phi(x)\, dx \right|$$
$$\leq \int_{\mathbb{R}^n} (1+|x|)^{-N} |f(x)| dx \cdot \sup_{x \in \mathbb{R}^n} |(1+|x|)^N \phi(x)|$$

If $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $\sup_{x \in \mathbb{R}^n} |(1+|x|^N)\phi_j(x)| \to 0$

$\Rightarrow |T_f[\phi_j]| \to 0$

b) If $f = e^{|x|^2}$, then $T_f \in \mathcal{D}'(\mathbb{R}^n)$, but $T_f \notin \mathcal{S}'(\mathbb{R}^n)$

c) The map $u[\phi] = \sum_{m=-\infty}^{\infty} m^{50}\phi(m)$ belongs to $\mathcal{S}'(\mathbb{R})$ but not to $\mathcal{E}'(\mathbb{R}^n)$

Convolutions

Recall that if $\phi : \mathbb{R}^n \to \mathbb{C}$ then $\tau_x \phi(y) = \phi(y-x)$ for $x \in \mathbb{R}^n$.
We introduce the involute of ϕ: $\check{\phi}(y) = \phi(-y)$, and take the convention $\tau_x \check{\phi}(y) = \phi(x-y)$.
Suppose $f \in L^1_{loc}(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\underset{\text{at } (x)}{f * \phi} = \int_{\mathbb{R}^n} f(y) \underbrace{\phi(x-y)}_{\tau_x \check{\phi}(y)}\, dy = T_f[\tau_x \check{\phi}]$$

This suggests that if $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, we define

$$u * \phi\,(x) = u[\tau_x \check{\phi}]$$

Note that for $a \in \mathbb{C}$,

$$\left.\begin{aligned} (u_1 + a u_2) * \phi &= u_1 * \phi + a u_2 * \phi \\ u * (\phi_1 + a\phi_2) &= u * \phi_1 + a u * \phi_2 \end{aligned}\right\} \begin{aligned} &u, u_i \in \mathcal{D}'(\mathbb{R}^n) \\ &\phi, \phi_i \in \mathcal{D}(\mathbb{R}^n) \end{aligned}$$

Notice that

$$u * \check{\phi}\,(0) = u[\phi],$$

so that if we know $u * \phi$ for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, we can get back u.

Example $\quad \delta_0 * \phi(x) = \delta_0[\tau_x \check{\phi}] = \phi(x-y)\big|_{y=0} = \phi(x)$

$\therefore \ \delta_0 * \phi = \phi$

Lemma Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

i) $u * \phi \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(u * \phi) = D^\alpha u * \phi = u * D^\alpha \phi$

ii) If $u \in \mathcal{E}'(\mathbb{R}^n)$ then $u * \phi$ has compact support, hence $u * \phi \in \mathcal{D}(\mathbb{R}^n)$.

Proof i) $\dfrac{u*\phi(x+he_i) - u*\phi(x)}{h} = u\left[\frac{1}{h}\left\{\tau_{\{x+he_i\}}\check{\phi} - \tau_x\check{\phi}\right\}\right]$

but $\frac{1}{h}\left\{\tau_{(x+he_i)}\check{\phi} - \tau_x\check{\phi}\right\}(y) = \dfrac{\phi(x+he_i-y) - \phi(x-y)}{h}$

$$\xrightarrow[\text{as } h\to 0]{\mathcal{D}(\mathbb{R}^n)} D_i\phi(x-y) = \tau_x(\check{D_i\phi})(y)$$

by Exercise (3.2??)

$\therefore \lim_{h\to 0} \dfrac{u*\phi(h+e_i) - u*\phi(x)}{h} = u[\tau_x(\check{D_i\phi})] = u * D_i\phi$

Iterate to find $u*\phi \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha(u*\phi) = u * D^\alpha\phi \quad \forall \alpha.$$

Further, $\left(D^\alpha \tau_x\check{\phi}\right)(y) = \dfrac{\partial^{|\alpha|}}{\partial y^\alpha}\phi(x-y) = (-1)^{|\alpha|} D^\alpha\phi(x-y)$

$$= (-1)^{|\alpha|}\left(\tau_x(\check{D^\alpha\phi})\right)(y)$$

Hence $D^\alpha u * \phi \overset{\ast\ast}{=} D^\alpha u[\tau_x \check{\phi}]$
$= (-1)^{|\alpha|} u[D^\alpha(\tau_x\check{\phi})]$
$= (-1)^{|\alpha|} u[(-1)^{|\alpha|}\tau_x(\check{D^\alpha\phi})]$
$= u * D^\alpha\phi$

ii) Suppose $u[\phi] = 0 \ \forall \phi \in C_c^\infty(\mathbb{R}^n \setminus K)$. Then for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp}\tau_x\check{\phi} \cap K = \emptyset$ for $|x|$ large enough. Hence $u*\phi(x) = 0$ for $|x|$ large. So $u*\phi \in \mathcal{D}(\mathbb{R}^n)$. $\square$

Since $u * \phi \in \mathcal{D}(\mathbb{R}^n)$ for $u \in \mathcal{E}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, we can define

Def Suppose $u_1 \in \mathcal{D}'(\mathbb{R}^n)$, $u^2 \in \mathcal{E}'(\mathbb{R}^n)$. Then $u_1 * u_2$ is the unique distribution satisfying

$$(u_1 * u_2) * \phi = u_1 * (u_2 * \phi) \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$$

Example If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, recall $\delta_0 \in \mathcal{E}'(\mathbb{R}^n)$.
Then

$$(u * \delta_0) * \phi = u * (\delta_0 * \phi) = u * \phi$$

So $u * \delta_0 = u$.

<u>Recall</u> $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\tau_x \check{\phi}(y) = \phi(x-y)$

$u \in \mathcal{D}'(\mathbb{R}^n)$: $u * \phi(x) := u[\tau_x \check{\phi}]$

$u \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow u * \phi \in \mathcal{D}(\mathbb{R}^n)$, $D^\alpha(u*\phi) = D^\alpha u * \phi$
$= u * D^\alpha \phi$

If $u_1 \in \mathcal{D}'(\mathbb{R}^n)$, $u_2 \in \mathcal{E}'(\mathbb{R}^n)$, $u_1 * u_2$ is defined by

$$(u_1 * u_2) * \phi = u_1 * (u_2 * \phi) \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$$

<u>Lemma</u> If $u_1 \in \mathcal{D}'(\mathbb{R}^n)$, $u_2 \in \mathcal{E}'(\mathbb{R}^n)$ then

$$D^\alpha(u_1 * u_2) = u_1 * D^\alpha u_2 = D^\alpha u_1 * u_2$$

<u>Pf</u> Pick $\phi \in \mathcal{D}(\mathbb{R}^n)$ and compute

$$\begin{aligned} D^\alpha(u_1 * u_2) * \phi &= (u_1 * u_2) * D^\alpha \phi \\ &= u_1 * (u_2 * D^\alpha \phi) && \text{def}^n \\ &= u_1 * (D^\alpha u_2 * \phi) \\ &= (u_1 * D^\alpha u_2) * \phi && \text{def}^n \end{aligned}$$

$\therefore D^\alpha(u_1 * u_2) = u_1 * D^\alpha u_2$

Other case similarly. $\square$

We can apply these methods to solve PDE problems.

Suppose $L = \sum_{|\alpha| \leq k} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$

is a constant coefficient differential operator of order k.

A fundamental solution of L is a distribution G satisfying

$$LG = \delta_0$$

<u>Theorem</u> If $G \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of L and $u_0 \in \mathcal{E}'(\mathbb{R}^n)$, then $u := G * u_0$ solves

$$Lu = \sum_{|\alpha| \leq k} a_\alpha D^\alpha u = u_0$$

<u>Pf</u> $$\begin{aligned} L(G * u_0) &= \sum_{|\alpha| \leq k} a_k D^\alpha (G * u_0) \\ &= \sum_{|\alpha| \leq k} a_\alpha (D^\alpha G * u_0) \\ &= \Big(\sum_{|\alpha| \leq k} a_\alpha D^\alpha G \Big) * u_0 = \delta_0 * u_0 = u_0. \end{aligned}$$ $\square$

The same argument shows that if $f \in \mathcal{D}(\mathbb{R}^n)$, then $G * f \in C^\infty(\mathbb{R}^n)$ solves $Lu = f$, where derivatives are understood classically.

<u>Example</u> $L = -\Delta = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ operator on $\mathbb{R}^3$

Define $g : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto \frac{1}{4\pi|x|}$. Then $G = T_g$ is a fundamental solution for L.

Thus if $f \in C_c^\infty(\mathbb{R}^n)$, then $u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, d^3y$

is a smooth sol^n of $-\Delta u = f$.

The Fourier Transform

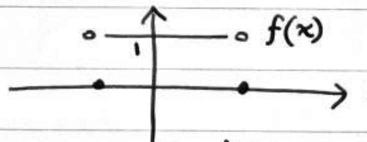
Given $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform of f $\mathcal{F}[f] = \hat{f} : \mathbb{R}^n \to \mathbb{C}$ given by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx$$

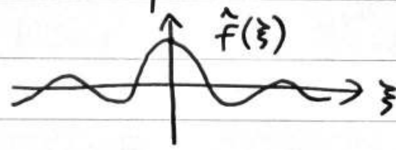
Since $|f(x) e^{-i\xi\cdot x}| = |f(x)| \in L^1(\mathbb{R}^n)$, the integral converges absolutely for all $\xi \in \mathbb{R}^n$.

<u>Examples</u> $(n=1)$

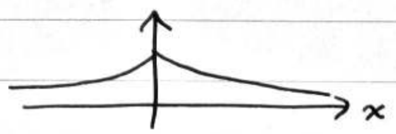
i) $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{o/w} \end{cases}$



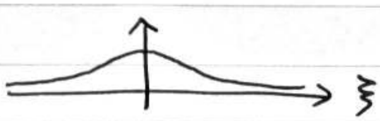
$\hat{f}(\xi) = \frac{2 \sin \xi}{\xi}$



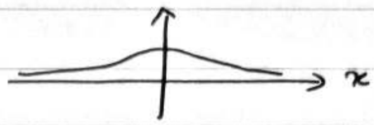
ii) $f(x) = \begin{cases} e^{x}, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$



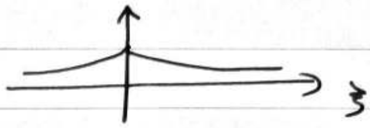
$\hat{f}(\xi) = \frac{2}{(1+\xi^2)}$



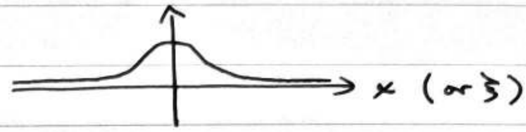
iii) $f(x) = \dfrac{1}{1+x^2}$



$$\hat{f}(\xi) = \begin{cases} \pi e^{\xi}, & \xi < 0 \\ \pi e^{-\xi}, & \xi \geq 0 \end{cases}$$



iv) $f(x) = e^{-x^2/2}$



$$\hat{f}(\xi) = \sqrt{2\pi}\, e^{-\xi^2/2}$$

Note: in all examples,

f regular $\longleftrightarrow$ $\hat{f}$ decays near ∞

f decays near ∞ $\longleftrightarrow$ $\hat{f}$ regular

<u>Lemma</u> (Riemann-Lebesgue Lemma)

Suppose $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^0(\mathbb{R}^n)$ with the estimate

$$\sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \leq \|f\|_{L^1}$$

and moreover $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

<u>Pf</u> If $\xi_k \to \xi$ in $\mathbb{R}^n$, then for all $x \in \mathbb{R}^n$,

$$f(x) e^{-ix\cdot\xi_k} \longrightarrow f(x) e^{-ix\cdot\xi}$$

and $|f(x) e^{-ix\cdot\xi_k}| \leq |f(x)| \in L^1(\mathbb{R}^n)$.

So by DCT, $\hat{f}(\xi_k) \to \hat{f}(\xi)$. Hence $\hat{f} \in C^0(\mathbb{R}^n)$.

Further $|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x}\, dx \right| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1}$.

Now, given $f \in L^1(\mathbb{R}^n)$ and $\varepsilon > 0$, we can find $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ with $\|f - f_\varepsilon\|_{L^1} < \varepsilon$.

We compute

$$\hat{f_\varepsilon}(\xi) = \int_{\mathbb{R}^n} f_\varepsilon(x) e^{-i\xi\cdot x}\, dx = \int_{\mathbb{R}^n} f_\varepsilon(x)\, \mathrm{div}_x\left(\frac{\xi}{-i|\xi|^2} e^{-ix\cdot\xi} \right) dx$$

$$= \int_{\mathbb{R}^n} \frac{\xi}{-i|\xi|^2} \cdot Df_\varepsilon(x)\, e^{-ix\cdot\xi}\, dx \quad \leftarrow \text{using } f_\varepsilon \text{ has cpct support to drop boundary terms}$$

Thus $|\hat{f}_\varepsilon(\xi)| \leq \int_{\mathbb{R}^n} \frac{|\xi \cdot Df_\varepsilon(x)|}{|\xi|^2}\,dx$

$$\leq \frac{1}{|\xi|} \|Df_\varepsilon\|_{L^1} < \varepsilon$$

for $|\xi|$ large enough. For such a $|\xi|$,

$$|\hat{f}(\xi)| = |\hat{f}(\xi) - \hat{f}_\varepsilon(\xi) + \hat{f}_\varepsilon(\xi)|$$

$$\leq |\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)| + |\hat{f}_\varepsilon(\xi)|$$

$$\leq \|f - f_\varepsilon\|_{L^1} + \varepsilon < 2\varepsilon .$$

$\square$

Properties of the Fourier Tr

Recall that if $f: \mathbb{R}^n \to \mathbb{C}$, $y \in \mathbb{R}^n$, then $\tau_y f(x) = f(x-y)$, and $e_y(x) = e^{ix\cdot y}$.

Lemma (i) Suppose $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$ and $f_\lambda(x) = \lambda^{-n} f(\lambda^{-1} x)$.

Then $\widehat{f_\lambda}(\xi) = \hat{f}(\lambda \xi)$,

$\widehat{e_y f}(\xi) = \tau_y \hat{f}(\xi)$,

$\widehat{\tau_y f}(\xi) = e_{-y}(\xi)\hat{f}(\xi)$. important L^1, only defined a.e.

(ii) Suppose $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Pf Change of variables + Fubini. $\square$

Theorem (i) Suppose $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^1(\mathbb{R}^n)$ $\forall j$

Then $\widehat{D_j f}(\xi) = i\xi_j \hat{f}(\xi)$.

(ii) Suppose $(1+|x|)f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^1(\mathbb{R}^n)$ and

$$D_j \hat{f}(\xi) = -i\widehat{x_j f}(\xi).$$

Pf For $\varepsilon > 0$ we may find $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|f - f_\varepsilon\|_{L^1} + \sum_j \|D_j f - D_j f_\varepsilon\|_{L^1} < \varepsilon$$

somewhat subtle: want $D_j(f\chi)$ to be close to $D_j(f)$ in L^1, need nice χ?

⌈ Consider mollification by convolution of $f\chi_R$, where $\chi_R(x) = \chi(\frac{x}{R})$ with $\chi \in C_c^\infty(B_2(0))$ s.t. $\chi(x) = 1$ for $|x| < 1$. ⌋

By integration by parts we have

$$\widehat{D_j f_\varepsilon}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D_j f_\varepsilon(x)\, dx$$

$$= \int_{\mathbb{R}} i\xi_j e^{-ix\cdot\xi} f_\varepsilon(x)\, dx$$

$$= i\xi_j \hat{f}_\varepsilon(\xi).$$

Now $\left|\widehat{D_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| \leq \left|\widehat{D_j f}(\xi) - \widehat{D_j f_\varepsilon}(\xi)\right| + \left|i\xi_j \hat{f_\varepsilon}(\xi) - i\xi_j \hat{f}(\xi)\right|$

$$\leq \|D_j f - D_j f_\varepsilon\|_{L^1} + |\xi|\,\|f - f_\varepsilon\|_{L^1}$$

$$< (1+|\xi|)\,\varepsilon$$

So $\widehat{D_j f}(\xi) = i\xi_j \hat{f}(\xi) \quad \forall \xi$.

(ii) By assumption $x_j f \in L^1(\mathbb{R}^n)$ so $-i\widehat{x_j f} \in C^0(\mathbb{R}^n)$.
STP that $D_j \hat{f}$ exists and $D_j \hat{f} = -i\widehat{x_j f} \quad \forall \xi$.
We compute

$$\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} = \frac{1}{h}\int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot(\xi + he_j)}\,dx - \frac{1}{h}\int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\,dx$$

$$= \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h} - 1}{h}\right) dx$$

$$f(x)e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h}-1}{h}\right) \longrightarrow -ix_j f(x) e^{-ix\cdot\xi} \quad \text{pointwise}$$

and since $|e^{i\theta} - 1| = 2\left|\sin\frac{\theta}{2}\right| \leq |\theta|$

$$\left|f(x)e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h}-1}{h}\right)\right| \leq |f(x)|\,|x_j| \quad \in L^1(\mathbb{R}^n)$$

$\therefore$ by DCT, $\dfrac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} \longrightarrow -i\widehat{x_j f}(\xi)$. $\square$

Corollary The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ continuously.

Pf We note that for any $f : \mathbb{R}^n \to \mathbb{C}$

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} |f|\,dx \leq \sup_{\mathbb{R}^n}\left((1+|x|)^{n+1}|f(x)|\right) \underbrace{\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{n+1}}}_{\text{finite}}$$

$\therefore$ if $f \in \mathcal{S}(\mathbb{R}^n)$, we have that

$$D^\alpha(x^\beta f(x)) \in L^1(\mathbb{R}^n)$$

for any multi-indices α, β

By iteratively applying previous result,

$$|\widehat{D^\alpha(x^\beta f)}(\xi)| = |\xi^\alpha D^\beta \hat{f}(\xi)|$$

$$\therefore \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha D^\beta \hat{f}(\xi)| \leq C \sup_{\substack{x\in\mathbb{R}^n \\ |\delta|\leq\alpha}} \left((1+|x|)^{|\beta|+n+1} |D^\delta f(x)| \right)$$

Now if $f_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, we see that

$$\sup_\xi |\xi^\alpha D^\beta \hat{f}_j(\xi)| \to 0$$

$\therefore \hat{f}_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$.

We deduce that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ continuously. $\square$

<u>Theorem</u> (Fourier Inversion)

Suppose $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$. Then

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi \qquad \text{for a.e. } x$$

(recalling $\check{f}(x) = f(-x)$, then $\mathcal{F}^2[f] = (2\pi)^n \check{f}$).

<u>Pf</u> We consider the limit $\varepsilon \to 0$ of

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\cdot\xi} d\xi$$

By DCT, (use $\hat{f}\in L^1$) $I_\varepsilon(x) \to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$ as $\varepsilon \to 0$

On the other hand,

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) e^{-i\xi\cdot y} dy \right) e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\cdot\xi} d\xi$$

Fubini

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{-i\xi\cdot(y-x)} d\xi \right) dy$$

$$= \int_{\mathbb{R}^n} f(y) \frac{1}{\varepsilon^n (2\pi)^{n/2}} e^{-\frac{|y-x|^2}{2\varepsilon^2}} dy$$

$$= f * \psi_\varepsilon(x)$$

where $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(\varepsilon^{-1}x)$, $\psi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}|x|^2}$.

Note ψ is smooth, $\psi \geq 0$, $\int \psi = 1$.

By our mollification theorem from Lecture 3, we have

$$f * \psi_\varepsilon \to f \quad \text{in } L^1(\mathbb{R}^n).$$

$$\therefore \; I_\varepsilon(x) \longrightarrow \frac{1}{\cdots}\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$

and $I_\varepsilon \to f$ in L^1

$$\therefore \; f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi \qquad \text{a.e. } x.$$

$\square$

(If f is cts, then this holds for every x)

Extensions of the Fourier Transform

The Fourier transform on $L^2(\mathbb{R}^n)$

In order to extend $\mathcal{F}$ to act on L^2, we require the following result.

Theorem (Plancherel - Parseval)

Suppose $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.
Then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$, and further

$$(f, g)_{L^2} = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})_{L^2} .$$

Pf Suppose $f, g \in \mathcal{S}(\mathbb{R}^n)$ (hence $\hat{f}, \hat{g} \in \mathcal{S}(\mathbb{R}^n)$).
We calculate

$$\begin{aligned}
(f,g)_{L^2} &= \int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx \\
&\overset{\text{F.I.T.}}{=} \int_{\mathbb{R}^n} \overline{f(x)} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{i\xi\cdot x} d\xi \right) dx \\
&\overset{\text{Fubini}}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \overline{f(x)}\, e^{ix\cdot\xi} dx \right) \hat{g}(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\left(\int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx \right)}\, \hat{g}(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\, \hat{g}(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})_{L^2} .
\end{aligned}$$

Now suppose $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Pick $f_j, g_j \in \mathcal{S}(\mathbb{R}^n)$ with $\|f_j - f\|_{L^1} + \|f_j - f\|_{L^2} \to 0$ (sim for g_j).
By Riemann-Lebesgue,

$$\sup_{\xi \in \mathbb{R}^n} |\hat{f}_j(\xi) - \hat{f}(\xi)| \leq \|f_j - f\|_{L^1} \to 0$$

so $\hat{f}_j \to \hat{f}$ uniformly.
Moreover $\|\hat{f}_j - \hat{f}_k\|_{L^2}^2 = (2\pi)^n \|f_j - f_k\|_{L^2}^2$ by the above.
But (f_j) is Cauchy in L^2, hence $(\hat{f}_j)$ is Cauchy in L^2.

So $\hat{f} \in L^2(\mathbb{R}^n)$ and $\hat{f}_j \to \hat{f}$ in $L^2(\mathbb{R}^n)$.
Similarly $\hat{g} \in L^2(\mathbb{R}^n)$ and $\hat{g}_j \to \hat{g}$ in $L^2(\mathbb{R}^n)$.
Then
$$\begin{aligned}(f,g) &= \lim_{j\to\infty} (f_j, g_j) \\ &= \frac{1}{(2\pi)^n} \lim_{j\to\infty} (\hat{f}_j, \hat{g}_j) \\ &= \frac{1}{(2\pi)^n} (\hat{f}, \hat{g}).\end{aligned}$$
$\square$

We note that if $f \in L^1 \cap L^2(\mathbb{R}^n)$, then
$$\|\hat{f}\|_{L^2} = (2\pi)^{n/2} \|f\|_{L^2}.$$

Thus $\mathcal{F} : L^1 \cap L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, and $\mathcal{F}$ is isometric when we think of $L^1 \cap L^2(\mathbb{R}^n)$ as a dense subset of $L^2(\mathbb{R}^n)$ with the L^2 inner product. Hence $\mathcal{F} : L^1 \cap L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is a uniformly continuous map.

$\mathcal{F}$ extends uniquely to a linear map $\overline{\mathcal{F}}$ from $\overline{L^1 \cap L^2(\mathbb{R}^n)}^{L^2}$ i.e. just $L^2(\mathbb{R}^n)$, into $L^2(\mathbb{R}^n)$.

The extension $\overline{\mathcal{F}}$ is sometimes called the <u>Fourier-Plancherel</u> transform. All the results of last lecture carry over, mutatis mutandis.

We can given an explicit formula for $\overline{\mathcal{F}}$ as follows:
If $f \in L^2(\mathbb{R}^n)$ then $f_R = f \mathbb{1}_{B_R(0)} \in L^2 \cap L^1(\mathbb{R}^n)$, and as $R \to \infty$, $f_R \to f$ in $L^2(\mathbb{R}^n)$. Hence
$$\hat{f}_R \to \overline{\mathcal{F}}[f] \text{ in } L^2(\mathbb{R}^n), \text{ as } R \to \infty.$$
$$\text{i.e.} \left(\xi \mapsto \int_{B_{R(0)}} f(x) e^{-ix\cdot\xi}\, dx\right) \xrightarrow{L^2} \hat{f} = \overline{\mathcal{F}}[f]$$

Typically one writes $\mathcal{F} = \overline{\mathcal{F}}$ and ignore the distinction.

The Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$

Suppose $f \in L^1(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then we compute

$$T_{\hat{f}}[\phi] = \int_{\mathbb{R}^n} \hat{f}(x)\,\phi(x)\,dx$$ * $\hat{f} \in L^1_{loc}$? (yes)

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)\, e^{-ix\cdot y}\, dy \right) \phi(x)\, dx$$

↓ Fubini

$$= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-ix\cdot y}\, \phi(x)\, dx \right) dx$$

$$= \int_{\mathbb{R}^n} f(y)\, \hat{\phi}(y)\, dy$$

$$= T_f[\hat{\phi}].$$

Noting that $\phi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$, this computation suggests we define,

Defn For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the Fourier transform of u, $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by $\hat{u}[\phi] = u[\hat{\phi}] \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.

This makes sense since $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, linear, so $u \circ \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous, linear.

Rmk We couldn't do this for $u \in \mathcal{D}'(\mathbb{R}^n)$, because

$$\phi \in \mathcal{D}(\mathbb{R}^n) \not\Rightarrow \hat{\phi} \in \mathcal{D}(\mathbb{R}^n)$$ ⌈very much so⌋

Examples (a) Fix $\xi \in \mathbb{R}^n$ and consider δ_ξ.

$$\hat{\delta}_\xi[\phi] = \delta_\xi[\hat{\phi}] = \hat{\phi}(\xi)$$

$$= \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, \phi(x)\, dx$$

ink refill

$$= T_{e_{-\xi}}[\phi] \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

$$\therefore \hat{\delta}_\xi = T_{e_{-\xi}}$$

⌈ $f(x) = \delta(x-\xi)$, $\hat{f}(\eta) = e^{-i\eta\cdot\xi}$ ⌋

(b) For $x \in \mathbb{R}^n$ consider T_{e_x}

$$\widehat{T_{e_x}}[\phi] = T_{e_x}[\hat{\phi}] = \int_{\mathbb{R}^n} e^{ix\cdot y}\hat{\phi}(y)\,dy$$

$\downarrow$ F.I.T.

$$= (2\pi)^n \phi(x)$$

$$= (2\pi)^n \delta_x[\phi] \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

$$\therefore \widehat{T_{e_x}} = (2\pi)^n \delta_x$$

For α multi-index, define $X^\alpha : \mathbb{R}^n \to \mathbb{C}$

$$x \mapsto x^\alpha.$$

Recall that if u is a distribution, $x \in \mathbb{R}^n$, then

$$\tau_x u[\phi] = u[\tau_{-x}\phi] \qquad \text{by def}^n$$

<u>Lemma</u> Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$\widehat{e_x u} = \tau_x \hat{u}, \quad \widehat{\tau_x u} = e_{-x}\hat{u}$$

$$\widehat{D^\alpha u} = i^{|\alpha|} X^\alpha \hat{u}, \quad D^\alpha \hat{u} = (-1)^{|\alpha|}\widehat{X^\alpha u}.$$

Moreover $\hat{\hat{u}} = (2\pi)^n \check{u}$.

<u>Pf</u> Pick $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\widehat{e_x u}[\phi] = (e_x u)[\hat{\phi}] = u[e_x\hat{\phi}]$$

$$= u[\widehat{\tau_{-x}\phi}] = \hat{u}[\tau_{-x}\phi]$$

$$= \tau_x \hat{u}[\phi]$$

Since ϕ arbitrary, $\widehat{e_x u} = \tau_x \hat{u}$.

$$\widehat{D^\alpha u}[\phi] = D^\alpha u[\hat{\phi}] = (-1)^{|\alpha|} u[D^\alpha \hat{\phi}]$$

$$= (-1)^{|\alpha|} u[(-i)^{|\alpha|}\widehat{X^\alpha \phi}]$$

$$= i^{|\alpha|} u[\widehat{X^\alpha\phi}] = i^{|\alpha|}\hat{u}[X^\alpha\phi]$$

$$= i^{|\alpha|} X^\alpha \hat{u}[\phi]$$

$\therefore \widehat{D^\alpha u} = i^{|\alpha|} X^\alpha \hat{u}$. *Rest similar*

$$\hat{\hat{u}}[\phi] = \hat{u}[\hat{\phi}] = u[\hat{\hat{\phi}}] = u[(2\pi)^n \check{\phi}]$$
$$= (2\pi)^n \check{u}[\phi]$$

$\therefore \mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is invertible

$$\frac{1}{(2\pi)^n} \check{\hat{\hat{u}}} = u. \qquad \square$$

Lemma $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is linear homeomorphism.

Pf Linearity is straightforward.
We call $u_j \to u$ in the weak-$*$ topology of $\mathcal{S}'(\mathbb{R}^n)$
if $u_j[\phi] \to u[\phi]$ $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
Suppose $u_j \to u$. Then

$$\hat{u}_j[\phi] = u_j[\hat{\phi}] \longrightarrow u[\hat{\phi}] = \hat{u}[\phi]$$

$\therefore \hat{u}_j[\phi] \longrightarrow \hat{u}[\phi]$ $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$

$\therefore \mathcal{F}[u_j] \longrightarrow \mathcal{F}[u_j]$ in $\mathcal{S}'(\mathbb{R}^n)$

Thus $\mathcal{F}$ is (sequentially) continuous.
Since $\mathcal{F}^4 = (2\pi)^{2n} \iota$, we have that $\mathcal{F}$ is invertible
with (sequentially) cts inverse. $\square$

Periodic Distributions

We've already said that if $f \in L^2(0,1)$, we may write

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{+2\pi i n x} \qquad \text{converging in } L^2.$$

$$f_n = \int_0^1 e^{-2\pi i n x} f(x)\, dx$$

For $u \in \mathcal{D}'(\mathbb{R}^n)$, recall the translate of u by $z \in \mathbb{R}^n$, $\tau_z u$ is defined by $\tau_z u[\phi] = u[\tau_{-z}\phi] \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$

<u>Definition</u> We say $u \in \mathcal{D}'(\mathbb{R}^n)$ is <u>periodic</u> if for all $g \in \mathbb{Z}^n$ we have $\tau_g u = u$.

<u>Example</u> (a) For $g \in \mathbb{Z}^n$, the distribution $T_{e_{2\pi g}}$ is periodic:

$$\begin{aligned}
\tau_{g'} T_{e_{2\pi g}}[\phi] &= T_{e_{2\pi g}}[\tau_{-g'}\phi] \\
&= \int_{\mathbb{R}^n} e^{2\pi i g\cdot y} \phi(y+g')\, dy \\
&= \int_{\mathbb{R}^n} e^{2\pi i g\cdot(x-g')} \phi(x)\, dx \\
&= \underbrace{e^{-2\pi i g\cdot g'}}_{1} \int_{\mathbb{R}^n} e^{2\pi i g\cdot x} \phi(x)\, dx \\
&= T_{e_{2\pi g}}[\phi]
\end{aligned}$$

(Exercise: If $f \in L^1_{loc}(\mathbb{R}^n)$ and $f(x+g) = f(x) \;\forall g \in \mathbb{Z}^n$, then T_f is periodic)

(b) Suppose $v \in \mathcal{E}'(\mathbb{R}^n)$. Then

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v \qquad \text{is periodic.}$$

u defines a distribution since $u[\phi]$ is always a finite sum

$$\tau_{g'} u[\phi] = u[\tau_{-g'}\phi] = \sum_{g\in\mathbb{Z}^n} \tau_g v[\tau_{-g'}\phi]$$

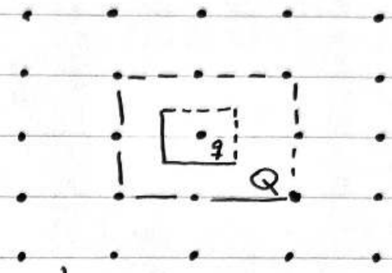
$$= \sum_{g\in\mathbb{Z}^n} v[\tau_{-g-g'}\phi] = \sum_{g\in\mathbb{Z}^n} \tau_{g+g'} v[\phi] = u[\phi]$$

for $g' \in \mathbb{Z}^n$, $\phi \in \mathcal{D}(\mathbb{R}^n)$.

For periodic functions $f \in C^\infty(\mathbb{R}^n)$, often wish to compute averages over a fundamental cell of the lattice

$$q = \{ x \in \mathbb{R}^n : -\tfrac{1}{2} \le x_i < \tfrac{1}{2},\ i = 1, \dots, n \}$$

E.g. $M[f] = \int_q f(x)\,dx \overset{!}{=} T_f[\mathbb{1}_q]$



Would like to extend to periodic distributions, but $\mathbb{1}_q \notin \mathcal{D}(\mathbb{R}^n)$.

<u>Lemma</u> Let $Q = \{ x \in \mathbb{R}^n : -1 \le x_i < 1,\ i = 1, \dots, n \}$

Then there exists $\psi \in C^\infty(\mathbb{R}^n)$ s.t.

(i) $\psi \geq 0$

(ii) $\operatorname{supp} \psi \subset Q^\circ$

(iii) $\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1$

Call such a ψ a <u>periodic partition of unity</u>.

Suppose ψ, ψ' are both P.P.U. Then if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic, $u[\psi] = u[\psi']$.

We define $M[u] = u[\psi]$.

<u>Proof</u> Pick $\psi_0 \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp} \psi_0 \subset Q^\circ$ and $\psi_0(x) = 1$ on q, and $\psi_0 \geq 0$.

Let $S(x) = \sum_{g \in \mathbb{Z}^n} \psi_0(x-g)$. This is a locally finite sum,

so S is smooth. For each $x \in \mathbb{R}^n$, $\exists\, g \in \mathbb{Z}^n$ s.t.

$x - g \in \bar{q}$ $\therefore$ $S(x) \geq 1$

S is manifestly periodic

Thus $\psi(x) = \dfrac{\psi_0(x)}{S(x)}$ satisfies (i) – (iii) ☺

Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic and ψ, ψ' are P.P.U.

Then $$u[\psi] = u\Big[\psi \sum_{g \in \mathbb{Z}^n} \tau_g \psi'\Big] = \sum_{g \in \mathbb{Z}^n} u[\psi\, \tau_g \psi']$$

$$= \sum_{g \in \mathbb{Z}^n} \tau_{-g} u[\tau_{-g}\psi \cdot \psi'] = u\Big[\psi' \sum_{g \in \mathbb{Z}^n} \tau_{-g} \psi\Big]$$

$$= u[\psi']$$ o.k. nice □

If $u = T_f$ for $f \in L^1_{loc}(\mathbb{R}^n)$, f periodic, then by choosing a sequence of ψ_n P.P.U.s such that $\psi_n \to \mathbb{1}_q$ pointwise and ψ_n bounded, we have

$$M[T_f] = \int_q f(x)\,dx$$

This justifies our notation.

We now ~~see~~ show example (b) above is typical.

<u>Lemma</u> Suppose $v \in \mathcal{E}'(\mathbb{R}^n)$. Then $u = \sum_{g \in \mathbb{Z}^n} \tau_g v$ (+)

converges weakly-* in $\mathcal{S}'(\mathbb{R}^n)$. Conversely, if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic, then there exists $v \in \mathcal{E}'(\mathbb{R}^n)$ s.t. (+) holds.

In particular every periodic distribution is tempered.

<u>Pf</u> Let $K = \operatorname{supp} v$. By Lemma from Lecture 14, $\exists\ \mathbb{N} \ni N$, $C > 0$ s.t. $|v[\phi]| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |D^\alpha \phi(x)| \quad \forall \phi \in \mathcal{E}(\mathbb{R}^n)$

Suppose $\phi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$. Then

$$|\tau_g v[\phi]| = |v[\tau_{-g}\phi]| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |D^\alpha \phi(x+g)|$$

K is bounded, so $K \subset B_R(0)$ for some $R > 0$.

For $x \in K$,

$$1 + |g| = 1 + |x + g - x| \leq 1 + R + |x+g|$$
$$\leq (1+R)(1+|x+g|)$$

$$\therefore\ 1 \leq \frac{(1+R)(1+|x+g|)}{1+|g|} \quad \forall x \in K$$

$$\therefore \text{ For } M \geq 1,\ |\tau_g v[\phi]| \leq C \frac{(1+R)^M}{(1+|g|)^M} \sup_{\substack{x \in K \\ |\alpha| \leq N}} \left((1+|x+g|)^M \cdot |D^\alpha \phi(x+g)| \right)$$

$$\leq C \frac{(1+R)^M}{(1+|g|)^M} \underbrace{\sup_{\substack{y \in \mathbb{R}^n \\ |\alpha| \leq N}} (1+|y|)^M |D^\alpha \phi(y)|}_{<\infty}$$

$\therefore\ M = n+1$ gives $|\tau_g v[\phi]| \leq \dfrac{\tilde{C}}{(1+|g|)^{N+1}}$

Since
$$\sum_{g\in\mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty$$
we conclude that
$$\sum_{g\in\mathbb{Z}^n} \tau_g v[\phi] \longrightarrow \text{converges}$$
Hence $\sum_{g\in\mathbb{Z}^n} \tau_g v$ converges weakly-$*$ in $\mathcal{S}'(\mathbb{R}^n)$.

Now, suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic, take ψ a P.P.U.
For $\phi \in \mathcal{D}(\mathbb{R}^n)$:
$$u[\phi] = \Big(\sum_{g\in\mathbb{Z}^n} \tau_g \psi\Big) u[\phi]$$
$$= \sum_{g\in\mathbb{Z}^n} u[\tau_g \psi \phi]$$
Since u is periodic
$$u[\tau_g\psi\cdot\phi] = \tau_g u[\tau_g\psi\cdot\phi]$$
$$= u[\psi\cdot\tau_{-g}\phi]$$
$$= (\psi u)[\tau_{-g}\phi]$$
$$= \tau_g(\psi u)[\phi]$$
Let $v = \psi u$. Then v has compact support since if $\operatorname{supp}\phi \cap \operatorname{supp}\psi = 0$, $v[\phi] = u[\phi\psi] = 0$.
So v extends uniquely to an element of $\mathcal{E}'(\mathbb{R}^n)$.
We're shown
$$u = \sum_{g\in\mathbb{Z}^n} \tau_g v$$
By first part of the proof, $u \in \mathcal{S}'(\mathbb{R}^n)$. $\square$

In particular, this means that every periodic distribution may be Fourier transformed.

Today we shall establish a result analogous to the usual result concerning Fourier series of a periodic function on $\mathbb{R}$. First we require a technical lemma.

Lemma Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $(e_{-g'}-1)u = 0$ for all $g' \in \mathbb{Z}^n$. Then
$$u = \sum_{g \in \mathbb{Z}^n} c_g \, \delta_{2\pi g}$$
where $c_g \in \mathbb{C}$ satisfy $|c_g| \leq C(1+|g|)^N$ for some $C>0$, $N \in \mathbb{N}$ and the sum converges in $\mathcal{S}'(\mathbb{R}^n)$.

Pf Let $\Lambda^* = \{2\pi g : g \in \mathbb{Z}^n\}$. Suppose $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \cap \Lambda^* = \emptyset$. Then for $g' \in \mathbb{Z}^n$, $(e_{-g'}-1)$ is non-zero on $\operatorname{supp}\phi$, so $(e_{-g'}-1)^{-1}\phi \in \mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, hence
$$\begin{aligned} 0 &= (e_{-g'}-1)u\left[(e_{-g'}-1)^{-1}\phi\right] \\ &= u[\phi] \end{aligned}$$
$\therefore \operatorname{supp} u \subset \Lambda^*$

Pick ψ a periodic partition of unity. Let $\tilde{\psi}(x) = \psi\left(\frac{x}{2\pi}\right)$.
Then $\sum_{g \in \mathbb{Z}^n} \tau_{2\pi g}\tilde{\psi}(x) = 1$, $\tilde{\psi} \geq 0$, $\operatorname{supp}\tilde{\psi} \subset \{|x_i| < 2\pi\}$

Consider $v_g = (\tau_{2\pi g}\tilde{\psi})u$.
$\operatorname{supp} v_g \subset \{2\pi g\}$
$$\sum_{g \in \mathbb{Z}^n} v_g = u, \quad \text{and} \quad (e_{-g'}-1)v_g = 0$$
Setting $g' = \ell_j$, $j = 1, \dots, n$, $\{\ell_j\}$ canonical basis for $\mathbb{R}^n$
We have
$$\left(e^{-i(x_j - 2\pi g_j)} - 1\right)v_g = 0 \qquad \text{(cheeky!)}$$

Note that $\left(e^{-i(x_j-2\pi g_j)} - 1\right) = (x_j - 2\pi g_j)K(x_j)$ for some smooth $K(x_j)$ which is non-zero near $2\pi g_j$.
Thus $(x_j - 2\pi g_j)v_g = 0$.
Suppose $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then
$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^n (x_j - 2\pi g_j)\phi_j(x)$$
for $\phi_j \in C^\infty(\mathbb{R}^n)$. v_g has compact support, so extends to

act on $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, hence

$$v_g[\phi] = v_g[\phi(2\pi g)] + \sum_{j=1}^{n} (x_j - 2\pi g_j)\, v_g[\phi_j(x)]$$
$$= v_g[\phi(2\pi g)]$$

$$\therefore \ (\tau_{2\pi g}\tilde{\psi})u[\phi] = (\tau_{2\pi g}\tilde{\psi})u[\phi(2\pi g)]$$
$$= u[\tau_{2\pi g}\tilde{\psi}\,\phi(2\pi g)]$$
$$= u[\tau_{2\pi g}\tilde{\psi}]\cdot \delta_{2\pi g}[\phi]$$

$$\therefore \ v_g = u[\tau_{2\pi g}\tilde{\psi}]\cdot \delta_{2\pi g}$$

$$\therefore \ u = \sum_{g\in\mathbb{Z}^n} v_g = \sum_{g\in\mathbb{Z}^n} c_g\,\delta_{2\pi g}$$

with $c_g = u[\tau_{2\pi g}\tilde{\psi}]$

Now by Exercise 3.6. $\exists\, N, k \in \mathbb{N}$, $C > 0$ s.t.

$$|u[\phi]| \leq C \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq k}} (1+|x|)^N |D^\alpha \phi(x)| \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

$$\Rightarrow |c_g| \leq C \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq k}} (1+|x|)^N |D^\alpha \tilde{\psi}(x-2\pi g)|$$

$$\leq C \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq k}} (1+|x+2\pi g|)^N |D^\alpha \tilde{\psi}(x)|$$

$$\leq C' \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq k}} (1+|x|)^N |D^\alpha \tilde{\psi}(x)|\ (1+|g|)^N$$

(using $1+|x+y| \leq c(1+|x|)(1+|y|)$)

$$\therefore \ |c_g| \leq C(1+|g|)^N$$

Convergence in $\mathcal{S}'(\mathbb{R}^n)$ follows straightforwardly. $\square$

<u>Theorem</u> Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic.
Then there exist constants $c_g \in \mathbb{C}$ s.t.

$$u = \sum_{g \in \mathbb{Z}^n} c_g T e_{2\pi g} \qquad (*)$$

Here $c_g = M(e_{-2\pi g} u)$ satisfy the bound

$$|c_g| \leq C(1+|g|)^N \quad \text{for some } C \geq 0,\ N \in \mathbb{N}$$

and $(*)$ converges in $\mathcal{S}'(\mathbb{R}^n)$. [Here c_g are the Fourier coeffs.]

<u>Pf</u> u is periodic $\therefore u \in \mathcal{S}'(\mathbb{R}^n)$ so we may consider the Fourier transform
We note
$$\tau_{g'} u = u \quad \forall g' \in \mathbb{Z}^n$$

$$\Rightarrow \quad e_{-g'} \hat{u} = \hat{u}$$
$$\Rightarrow \quad (e_{-g'} - 1)\hat{u} = 0 \qquad \forall g' \in \mathbb{Z}^n$$

By our previous result,
$$\hat{u} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}, \qquad [|c_g| \leq C(1+|g|)^N]$$

sum converging in $\mathcal{S}'(\mathbb{R}^n)$. We can apply inverse Fourier transform, (noting that $\mathcal{F}$ is cts on $\mathcal{S}'(\mathbb{R}^n)$) so
$$u = \sum_{g \in \mathbb{Z}^n} c_g T e_{2\pi g} \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n) \qquad (+)$$

We note that since $e_{2\pi g} \in L^1_{loc}$ we have that
$$M(e_{-2\pi g'} T e_{2\pi g}) = \int_{\mathcal{Q}} e^{2\pi i (g-g')\cdot x}\, dx = \delta_{gg'}$$

The map $u \mapsto M(e_{-2\pi g'} u)$ is a cts map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathbb{C}$,
so deduce acting on $(+)$,
$$M(e_{-2\pi g} u) = \sum_{g' \in \mathbb{Z}^n} c_{g'} M(e_{-2\pi g} T e_{2\pi g'})$$
$$= c_g.$$

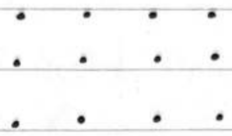
$\square$

We often abuse notation to write

$$u = \sum_{g\in\mathbb{Z}^n} c_g e_{2\pi g} \quad \left(\text{or worse } u(x) = \sum_{g\in\mathbb{Z}^n} c_g e^{2\pi i g\cdot x}\right)$$

Example

Consider $u = \sum_{g\in\mathbb{Z}^n} \delta_g$



Then $c_g = M(e_{-2\pi g} u)$ ← ψ PPU

$$= u[\psi e_{-2\pi g}]$$

$$= \sum_{g'\in\mathbb{Z}^n} \psi(g')\, \underbrace{e^{-2\pi g\cdot g' i}}_{1} = \sum_{g'\in\mathbb{Z}^n} \tau_{-g'}\psi(0) = 1$$

We have established Poisson's formula

$$\sum_{g\in\mathbb{Z}^n} \delta_g = \sum_{g\in\mathbb{Z}^n} T e_{2\pi g} \qquad (\text{converging in } \mathcal{S}'(\mathbb{R}^n))$$

Sometimes abuse notation to write

$$\sum_{g\in\mathbb{Z}^n} \delta(x-g) = \sum_{g\in\mathbb{Z}^n} e^{2\pi i g\cdot x}$$

Theorem Suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic, with Fourier coefficients c_g. Then

(i) $D^\alpha u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic and

$$D^\alpha u = (2\pi i)^{|\alpha|} \sum_{g\in\mathbb{Z}^n} g^\alpha c_g T e_{2\pi g}$$

(ii) If $f \in L^1_{loc}(\mathbb{R}^n)$ is periodic, and $u = T_f$, then

$$|c_g| \leq \|f\|_{L^1(q)} \qquad (\sim\text{Riemann-Lebesgue})$$

And moreover, $c_g \to 0$ as $|g| \to \infty$.

(iii) If $f \in C^{n+1}(\mathbb{R}^n)$ is periodic, $u = T_f$, then

$$f(x) = \sum_{g\in\mathbb{Z}^n} c_g e^{2\pi i g\cdot x}$$

with uniform convergence.

(iv) If $f, h \in L^2_{loc}$ are periodic with Fourier coeffs f_g, h_g resp then
$$\int_q \bar{f}(x) h(x)\, dx = \sum_{g \in \mathbb{Z}^n} \bar{f}_g h_g \qquad (\sim \text{Parseval})$$

Furthermore
$$f(x) = \sum_{g \in \mathbb{Z}^n} f_g e^{2\pi i g \cdot x}$$
converges in $L^2(q)$.

<u>Pf</u> See online notes.

Sobolev Spaces

An important class of spaces for many applications (e.g. PDEs) are the Sobolev spaces. These consist of L^p functions whose dist^{n}l derivatives are also in L^p.

<u>Def</u> Suppose $\Omega \subset \mathbb{R}^n$ is open, let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p \leq \infty$. We say $f \in L^p(\Omega)$ belongs to the <u>Sobolev space</u> $W^{k,p}(\Omega)$ if $\forall\, |\alpha| \leq k$ there exists $f^\alpha \in L^p(\Omega)$ such that

$$D^\alpha T_f = T_{f^\alpha} \quad \text{in } \mathcal{D}'(\Omega) \qquad (*)$$

We say f^α is the α-th weak derivative of f, and write $D^\alpha f := f^\alpha$. $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{1/p}, \quad p < \infty$$

$$\|f\|_{W^{k,\infty}} = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty}$$

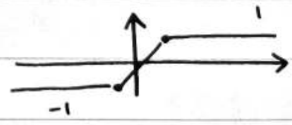
Note that $(*)$ means:

$$\int_\Omega f^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega f \, D^\alpha \phi \, dx$$

for all $\phi \in \mathcal{D}(\Omega)$.

<u>Example</u>

- Let $f(x) = \begin{cases} -1, & x < -1 \\ x, & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$



$f \in W^{1,\infty}(\mathbb{R})$ with $Df(x) = \begin{cases} 0, & |x| > 1, \\ 1, & |x| \leq 1. \end{cases}$

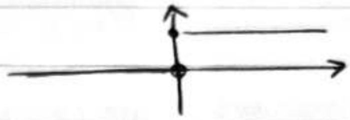
Suppose $\phi \in \mathcal{D}(\mathbb{R})$. Then

$$-\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = +\int_{-\infty}^{-1} \phi'(x)\,dx - \int_{-1}^{1} x\phi'(x)\,dx - \int_1^\infty \phi'(x)\,dx$$

$$= \phi(-1) - \big[x\phi(x)\big]_{-1}^{1} + \int_{-1}^{1} \phi(x)\,dx + \phi(1)$$

$$= \int_{-\infty}^{\infty} Df(x)\,\phi(x)\,dx \quad \checkmark$$

· $H(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}$



$H \notin W^{1,p}(\mathbb{R})$ for any p, since
$DH = \delta_0 \neq T_f$ for any $f \in L^1_{loc}$

We specialise to the case $p=2$, $\Omega = \mathbb{R}^n$, well-adapted to Fourier analysis.

From now on we drop the distinction between a function $f \in L^1_{loc}(\mathbb{R}^n)$ and the distribution $T_f \in \mathcal{D}'(\mathbb{R}^n)$.

In particular, we'll write (for example) $D^\alpha f \in L^1(\mathbb{R}^n)$ to mean $\exists\, f^\alpha \in L^1(\mathbb{R}^n)$ such that $D^\alpha T_f = T_{f^\alpha}$.

Suppose $f \in L^2(\mathbb{R}^n)$; $D^\alpha f \in L^2(\mathbb{R}^n) \iff \xi^\alpha \hat{f}(\xi) \in L^2(\mathbb{R}^n)$, by our earlier results. This motivates:

<u>Def^n</u> For $s \in \mathbb{R}$, we say that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^s(\mathbb{R}^n)$ if $\hat{f} \in L^2_{loc}(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 \, d\xi < \infty .$$

$H^s(\mathbb{R}^n)$ is a Hilbert space with inner product

$$(f,g)_{H^s} = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \, \hat{g}(\xi) (1+|\xi|^2)^s \, d\xi$$

If $s = k \in \mathbb{Z}_{\geq 0}$, then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

An important property of Sobolev spaces is that for k (or s) high enough, they embed into the Hölder spaces.

<u>Theorem</u> (Sobolev embedding)

Suppose $f \in H^s(\mathbb{R}^n)$ for some $s > \frac{n}{2} + k$. Then there exists $f^* \in C^k(\mathbb{R}^n)$ with $f = f^*$ a.e. We write $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

<u>Pf</u> First, suppose $f \in \mathcal{S}(\mathbb{R}^n)$. Then by Fourier inversion

$$D^\alpha f(x) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \xi^\alpha \hat{f}(\xi) \, d\xi \qquad (|\alpha| \leq k)$$

We estimate

$$|D^\alpha f(x)| = \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi)\, d\xi \right|$$

$$\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)|\, d\xi$$

$\downarrow$ C-S

$$\leq \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \cdot \left(\int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|^2)^s}\, d\xi \right)^{1/2}$$

Now, $|\alpha| \leq k \Rightarrow |\xi^\alpha|^2 \leq C_k (1+|\xi|^2)^k$ for some $C_k > 0$. #1

Hence $$\int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|^2)^s}\, d\xi \leq \underbrace{\int_{\mathbb{R}^n} \frac{C_n}{(1+|\xi|^2)^{s-k}}\, d\xi}_{\substack{s > \frac{n}{2}+k \\ \therefore \text{ integrable}}} =: C_{k,n,s}^2 < \infty.$$

We deduce that

$$\sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} |D^\alpha f(x)| \leq C_{n,k,s} \|f\|_{H^s} \qquad (\dagger)$$

Now suppose $f \in H^s(\mathbb{R}^n)$. We can approximate f by a sequence $(f_n)_{n=1}^\infty$ with $f_n \in \mathcal{S}(\mathbb{R}^n)$, $f_n \to f$ in $H^s(\mathbb{R}^n)$. (Exercise 4.?), and $f_n \to f$ pointwise a.e. In particular (f_n) is Cauchy in $H^s(\mathbb{R}^n)$, hence by $(\dagger)$, (f_n) is Cauchy in $C^k(\mathbb{R}^n)$, so there exists $f^* \in C^k(\mathbb{R}^n)$ s.t. $f_n \to f^*$ uniformly. But $f_n \to f$ pointwise a.e.

Hence $f = f^*$ a.e.

Traces of Sobolev functions

We saw last time that if $s > n/2$, then $H^s(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$. As a result, we can make sense of $f|_\Sigma$ for $f \in H^s(\mathbb{R}^n)$ and Σ some surface if $s > n/2$. If $s \geq 0$, $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, so $f \in H^s(\mathbb{R}^n)$ is at best defined pointwise a.e. in general. If $\Sigma = \{x^n = 0\}$ for example, we cannot obviously consider $f|_\Sigma$. To resolve this issue we have:

Theorem (Trace theorem)

Let $s > 1/2$. Then there is a bounded linear map
$$T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$$
such that $Tf = f|_{\{x^n=0\}} \quad \forall \; f \in \mathcal{S}(\mathbb{R}^n)$.

Tf is the trace of f on $\{x^n = 0\}$.

Pf Exercise 4.?? $\square$

By combining this result with coordinate transformations, it is possible to establish a similar result with $\{x^n = 0\}$ replaced by any sufficiently regular 'surface' Σ^{n-1}.

The space $H^1_0(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be open, and suppose $f \in C^\infty_c(\Omega)$. Extending f by zero outside Ω, we have $f \in H^1(\mathbb{R}^n)$, so naturally we have $C^\infty_c(\Omega) \subset H^1(\mathbb{R}^n)$. We denote by $H^1_0(\Omega)$ the closure of this $C^\infty_c(\Omega)$ with respect to the H^1-norm:

$$\|f\|^2_{H^1} = \int_{\mathbb{R}^n} (1 + |\xi|^2)\,|\hat{f}(\xi)|^2 \, d\xi$$

REMEMBER: WEAK DERIVATIVE !

$$= (2\pi)^n \int_\Omega \left(|Df(x)|^2 + |f(x)|^2 \right) dx$$

$H^1_0(\Omega)$ is a Hilbert space with inner product
$$(u, v)_{H^1} = \int_\Omega \left(\overline{Du} \cdot Dv + \bar{u} v \right) dx$$

Suppose that $\phi \in C^\infty_c((\Omega^c)^\circ)$. Then
$$\int \phi u \, dx = 0 \qquad \forall u \in C^\infty_c(\Omega)$$

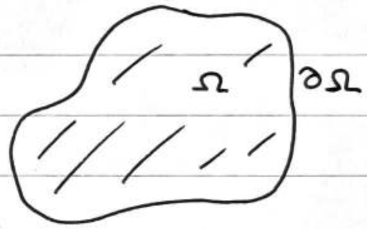
and $|\Lambda_\phi(u)| \leq \|\phi\|_{L^2}\|u\|_{L^2} \leq \|\phi\|_{L^2}\|u\|_{H^1}\cdot C$

$\therefore$ If $u_n \to u$ in $H^1(\mathbb{R}^n)$ with $u_n \in C_c^\infty(\Omega)$
then $0 = \Lambda_\phi(u_n) \longrightarrow \Lambda_\phi(u)$
$\therefore$ If $u \in H_0^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \phi u \, dx = 0 \quad \forall \phi \in C_c^\infty((\Omega^c)^\circ)$

$\therefore$ u must vanish a.e. outside Ω
Further, since $T: H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is bounded, and $Tu = 0 \; \forall u \in C_c^\infty(\Omega)$, we see that if $\partial\Omega$ is sufficiently regular, if $u \in H_0^1(\Omega)$, then u vanishes on $\partial\Omega$ in the trace sense.

$H_0^1(\Omega)$: "H^1 functions with vanish on $\partial\Omega$, outside Ω"

<u>Elliptic Boundary Value Problems</u>



Suppose $\Omega \subset \mathbb{R}^n$ is open. Consider

$$\left.\begin{aligned} -\Delta u + u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right\} \; (\dagger).$$

We will seek a solution to $(\dagger)$ with $u \in H_0^1(\Omega)$ which satisfies the equation in a distributional sense. This encodes the boundary condition in the function space.
Suppose $v \in C_c^\infty(\Omega)$.
Then, assuming $f \in L^2(\Omega)$, we deduce

$$\int_\Omega \left(\overline{Du}\cdot Dv + \bar{u} v\right) dx = \int_\Omega \bar{f} v \, dx \quad (*)$$

This holds because if $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$, then
$\int_\Omega -\bar{u}\Delta v + \bar{u} v = \int_\Omega \bar{f} v \, dx \quad \forall v \in C_c^\infty(\Omega)$
$(*)$ follows since $u \in H^1$, so $\int_\Omega \bar{u} D_i \tilde{v} = -\int_\Omega D_i \bar{u} \tilde{v}$
$\forall v \in C_c^\infty(\mathbb{R}^n)$. Let $\tilde{v} = D_i v$

By continuity, $(*)$ must hold for any $v \in H_0^1(\Omega)$.

Conversely, if ~~=~~ (*) holds for all $v \in H^1_0(\Omega)$, then taking $v \in C^\infty_c(\Omega)$ and integrating by parts we see that u solves

$$-\Delta u + u = f \quad \text{in } \mathcal{D}'(\Omega).$$

This motivates

Def We say $u \in H^1_0(\Omega)$ is a weak solution of (†) for some $f \in L^2(\Omega)$ if $(u,v)_{H^1} = (f,v)_{L^2} \quad \forall\, v \in H^1_0(\Omega)$.

Note that $v \mapsto (f,v)_{L^2}$ is a bounded linear map $H^1_0(\Omega) \to \mathbb{C}$ (since $|(f,v)|_{L^2} \leq \|f\|_{L^2}\|v\|_{L^2} \leq C\|f\|_{L^2}\|v\|_{H^1}$).
By Riesz, we deduce that there exists a unique $u \in H^1_0(\Omega)$ s.t. $(u,v)_{H^1} = (f,v)_{L^2} \quad \forall\, v \in H^1_0(\Omega)$

We've shown

Lemma Given $f \in L^2(\Omega)$, there exists a unique $u \in H^1_0(\Omega)$ solving (†) in the weak sense, and further $\|u\|_{H^1} \leq C\|f\|_{L^2}$.
If we let $Af := u$, we claim A is a linear, bounded map $L^2(\Omega) \to H^1_0(\Omega)$.
Suppose $f, g \in L^2(\Omega)$, $a \in \mathbb{C}$. Then if $u = Af$, $w = Ag$:

$$\begin{aligned}(u + aw, v)_{H^1} &= (u,v)_{H^1} + \bar{a}(w,v)_{H^1} \\ &= (f,v)_{L^2} + \bar{a}(g,v)_{L^2} \\ &= (f + ag, v)_{L^2} \qquad \forall\, v \in H^1_0(\Omega)\end{aligned}$$

Hence $A(f+ag) = Af + aA(g)$.
A is bounded:

$$\underbrace{(Af, Af)_{H^1}}_{\|Af\|^2_{H^1}} = (f, Af)_{L^2} \leq \|f\|_{L^2}\|Af\|_{L^2} \leq \|f\|_{L^2} C\|Af\|_{H^1}$$

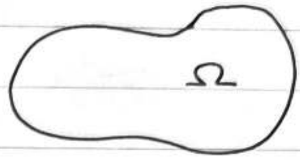
$$\therefore \|Af\|_{H^1} \leq C\|f\|_{L^2}$$

Finally, A is symmetric: $(Af, g)_{L^2} = (f, Ag)_{L^2}$

$$\begin{aligned}(f, Ag)_{L^2} &= (f, w)_{L^2} = (u, w)_{H^1} = \overline{(w,u)_{H^1}} \\ &= \overline{(g,u)_{L^2}} = (Af, g)_{L^2}\end{aligned}$$

Ideally, we'd like to be able to say that if f is sufficiently nice, then a solution to $(\dagger)$ exists where the derivatives are understood classically.

This is the problem of <u>regularity</u>.



$$\left.\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} (+)$$

$$\overset{\text{ω}}{\Longleftrightarrow} \quad u \in H_0^1(\Omega) : \int_\Omega \overline{Du} \cdot Dv + \bar{u} v \, dx = \int_\Omega \bar{f} v \, dx \quad \forall v \in H_0^1(\Omega)$$

« weak solution »

Given $f \in L^2(\Omega)$, $\exists!$ weak solution $u \in H_0^1(\Omega)$

If $Af := u$ then A is linear, bounded, <u>Hermitian</u>

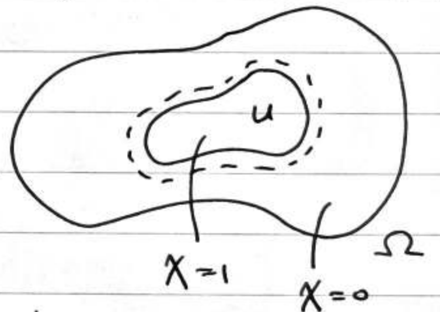
Is u more regular?

<u>Def^n</u> For $s > 0$, we define

$$H^s_{loc}(\Omega) = \{ u \in L^2_{loc}(\Omega) \mid \chi u \in H^s(\mathbb{R}^n) \text{ for all } \chi \in C_c^\infty(\Omega) \}$$

Note that if U is open, $\overline{U} \subset \Omega$, then we can find $\chi \in C_c^\infty(\Omega)$ such that $\chi(x) = 1$ for $x \in U$.

need $\bar{U}$ cpct !



We deduce that if $s > \frac{n}{2} + k$, $u \in H^s_{loc}(\Omega)$, then $\chi u \in C^k(U)$. Since U is arbitrary, this implies $u \in C^k(U)$. ω

Returning to our elliptic boundary value problem, fix $K \subset \Omega$ compact, let $\chi_K \in C_c^\infty(\Omega)$, $\chi_K = 1$ on K.

Suppose $u \in H_0^1(\Omega)$ is a weak solution of $(+)$ i.e.

$$\int_\Omega \overline{Du} \cdot Dv + \bar{u} v \, dx = \int_\Omega \bar{f} v \, dx \quad \forall v \in H_0^1(\Omega)$$

Pick $\phi \in \mathcal{S}(\mathbb{R}^n)$, let $v = \chi_K \cdot \phi \in H_0^1(\Omega)$. After some algebra, we deduce

$$\int_\Omega \bar{w} (-\Delta\phi + \phi) \, dx = \int_\Omega \bar{g} \phi \, dx$$

where $w = \chi_K u$ and $g = -2 Du \cdot D\chi_K - u \Delta \chi_K + f \chi_K$ $\quad (\Delta)$

Now $w \in H_0^1(\Omega) \subset H^1(\mathbb{R}^n)$ solves $-\Delta w + w = g$ in $\mathcal{S}'(\mathbb{R}^n)$ with $g \in L^2(\mathbb{R}^n)$.

From lecture 20 that $-\Delta w + w \in L^2(\mathbb{R}^n) \Rightarrow w \in H^2(\mathbb{R}^n)$.

Thus $w \in H^2(\mathbb{R}^n)$. Let $\psi \in C_c^\infty(\Omega)$ be arbitrary.
Then taking $K = \operatorname{supp}\psi$, we deduce
$$\psi u = \psi(\chi_K u) \in H^2(\mathbb{R}^n)$$
Thus $u \in H^2_{loc}(\Omega)$.

Now, suppose $f \in L^2 \cap H^1_{loc}(\Omega)$. Then returning to (Δ), we see $g \in H^1(\mathbb{R}^n)$. Hence (by lecture 20) $w \in H^3(\mathbb{R}^n)$ and by a similar argument this implies that $u \in H^1_0 \cap H^3_{loc}(\Omega)$.
Iterating, we see that if $f \in L^2 \cap H^k_{loc}(\Omega)$, then $u \in H^1_0 \cap H^{k+2}_{loc}(\Omega)$. In particular, if $f \in L^2 \cap C^\infty(\Omega)$ then $u \in H^1_0 \cap C^\infty(\Omega)$, and
$$-\Delta u + u = f \quad \text{holds classically in } \Omega.$$

This property of the operator $-\Delta + 1$ is called elliptic regularity.
It doesn't hold for all PDE, e.g.
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{has sol}^{\text{n}}\text{s} \quad u(x,t) = u_+(x+t) + u_-(x-t)$$
Can have u singular even though $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = 0$

Rellich-Kondrachov

Our final major result concerns compactness for Sobolev spaces.
Suppose $\Omega \subset \mathbb{R}^n$ is open & bounded, and let $(u_n)_{n=1}^\infty$ be a sequence in $H^1_0(\Omega)$ satisfying $\forall n$, $\|u_n\|_{H^1} \leq K$.
By Banach-Alaoglu, after extracting a subsequence (and relabelling) we may assume $u_n \xrightarrow{H^1} u$ for some $u \in H^1_0(\Omega)$, $\|u\|_{H^1} \leq K$. Moreover, if $w \in L^2(\Omega)$ then since
$$|(w,v)_{L^2}| \leq \|w\|_{L^2}\|v\|_{L^2} \leq \|w\|_{L^2}\|v\|_{H^1}$$
we deduce that $v \mapsto (w,v)_{L^2}$ is a bounded linear map $H^1_0(\Omega) \to \mathbb{C}$, so $(w,u_n)_{L^2} \to (w,u)_{L^2}$ i.e. $u_n \xrightarrow{L^2} u$.

In fact, we can say more than this.

Theorem (Rellich–Kondrachov)

Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded. Let $(u_n)_{n=1}^{\infty}$, $u_n \in H_0^1(\Omega)$ satisfy $u_n \overset{L^2}{\rightharpoonup} u$ for some $u \in H_0^1(\Omega)$ and $\|u_n\|_{H^1} \leq K$.
Then $u_n \overset{L^2}{\to} u$.

Pf Fix $\varepsilon > 0$. By Perceval:

$$\|u_n - u\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|\hat{u}_n - \hat{u}\|_{L^2}^2$$

$$= \frac{1}{(2\pi)^n} \int_{|\xi| < R} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \, d\xi + \frac{1}{(2\pi)^n} \int_{|\xi| > R} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \, d\xi .$$

We deal separately with the two integrals.

rmk 2 comes from AM-GM

$$\frac{1}{(2\pi)^n} \int_{|\xi| \geq R} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \, d\xi \leq \frac{2}{(2\pi)^n R^2} \int_{|\xi| > R} |\xi|^2 \left[|\hat{u}_n(\xi)|^2 + |\hat{u}(\xi)|^2 \right] d\xi$$

$$\leq \frac{2K^2}{R^2} \quad \text{from the bound on } \|u_n\|_{H^1}$$

rmk bound on $\|u\|_{H^1}$ not exactly same as before, but subseq $u_n \overset{H^1}{\rightharpoonup} u$

$$< \varepsilon \quad \text{if } R > 0 \text{ is suff large}$$

Having fixed such an R, we return to

$$\frac{1}{(2\pi)^n} \int_{|\xi| < R} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \, d\xi .$$

We recall that

$$\hat{u}_n(\xi) = \int_{\Omega} u_n(x) e^{-ix\cdot\xi} \, dx = (e_{+\xi}, u_n)_{L^2(\Omega)}$$

$\ulcorner e_y(x) = e^{ix\cdot y} \lrcorner$ Since $|\Omega| < \infty$, we deduce $e_\xi \in L^2(\Omega)$.
We deduce, using $u_n \overset{L^2}{\rightharpoonup} u$, that

$$\hat{u}_n(\xi) \to \hat{u}(\xi) .$$

Further, for $|\xi| < R$:

$$|\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \leq 2\left(|\hat{u}_n(\xi)|^2 + |\hat{u}(\xi)|^2\right)$$
$$\leq 2\left(\|u_n\|_{L^1}^2 + \|u\|_{L^1}^2\right) \quad \text{Riemann Lebesgue}$$
$$\leq 2|\Omega|\left(\|u_n\|_{L^2}^2 + \|u\|_{L^2}^2\right)$$
$$\leq 2|\Omega|\left(K^2 + \|u\|_{L^2}^2\right) \qquad \in L^1(B_R(0))$$

So by DCT, we deduce

$$\frac{1}{(2\pi)^n}\int_{|\xi|<R} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \, d\xi \longrightarrow 0 \quad \text{as } n \to \infty.$$

So picking n sufficiently large, we have

$$\|u_n - u\|_{L^2}^2 < 2\varepsilon.$$

$\square$

Corollary 1

If $\Omega \subset \mathbb{R}^n$ is open, bounded, and $(u_n)_{n=1}^\infty$ is a bounded sequence in $H_0^1(\Omega)$, then (u_n) admits a subsequence which converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. $\square$

Corollary 2

Suppose $\Omega \subset \mathbb{R}^n$ is open + bounded, and that

$$A : L^2(\Omega) \to H_0^1(\Omega) \text{ is a bounded linear map.}$$

Then $A : L^2(\Omega) \to L^2(\Omega)$ is compact. $\square$

⌈ $B : H \to H$ is compact if $\forall$ seq $(x_n) \subset H$ with x_n bdd, (Bx_n) has a convergent subsequence ⌋

We can use Rellich-Kondrachov to help understand more general PDE problems.

Consider for Ω open + bounded:

$$\left.\begin{aligned} -\Delta u + Vu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} (\dagger)$$

$V: \Omega \to \mathbb{R}$ is smooth and bounded (+ bdd derivative ?)

In a similar way to our previous problem, we say $u \in H_0^1(\Omega)$ is a weak solution to $(\dagger)$ if

$$(*) \int_\Omega \left(\overline{Du} \cdot Dv + V \bar{u} v \right) dx = \int_\Omega \bar{f} v \, dx \qquad \forall v \in H_0^1(\Omega)$$

Our previous argument fails: the LHS need not be an inner product. We rewrite $(*)$ as

$$\int_\Omega \left(\overline{Du} \cdot Dv + \bar{u} v \right) dx = \int_\Omega \left((1-V) \bar{u} v + \bar{f} v \right) dx \qquad (\ddagger)$$

Recalling $A: L^2(\Omega) \to H_0^1(\Omega)$ is the map taking $g \in L^2(\Omega)$ to the unique weak sol^n of

$$\begin{aligned} -\Delta u + u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

We note that $(\ddagger)$ holds $\forall v \in H_0^1(\Omega)$ iff

$$u = A\left(f + (1-V) u \right)$$
$$\Longleftrightarrow \quad (I - K) u = Af$$

where $Ku = A\left((1-V) u \right)$.

Now $K: L^2(\Omega) \to H_0^1(\Omega)$ is bounded and linear, so $K: L^2(\Omega) \to L^2(\Omega)$ is compact.

Thus either

a) $\exists\, w \in L^2(\Omega)$ s.t. $(I-K) w = 0$,

or b) $\exists!\, u \in L^2(\Omega)$ s.t. $(I-K) u = Af$.

In case a), the space of such w is finite dim^l. Further

$$(I-K) w = 0 \Rightarrow w = A\left((1-V) w \right) \Rightarrow w \in H_0^1(\Omega)$$

Can use regularity theory to show $w \in H_0^1(\Omega) \cap C^\infty(\Omega)$.

In case b), $u = A(f-(1-V)u) \Rightarrow u \in H_0^1(\Omega)$, so u is a weak sol^n to $(\dagger)$. We've shown:

- Either $\exists\, w \in H_0^1 \cap C^\infty(\Omega)$ solving
$$-\Delta w + Vw = 0$$
- or $\forall f \in L^2(\Omega)$, $\exists\, u \in H_0^1(\Omega)$ solving $(\dagger)$ in the weak sense.

Now let's consider $A : L^2(\Omega) \to H_0^1(\Omega)$ in some more detail. We showed previously that A is bounded, linear, Hermitian. By Rellich-Kondrachov $A : L^2(\Omega) \to L^2(\Omega)$ is compact. By the spectral theorem, we know
$$\sigma(A) = \{0, \mu_1, \mu_2, \dots\}$$
where $\mu_k \in \mathbb{R}$, $\mu_k \to 0$*. Further there is an o/n basis for $L^2(\Omega)$ consisting of e-vectors of A.

[*: if $\exists\, \infty$ many of them]

Suppose $Aw = \mu w$ for $\mu \in \mathbb{R}$, $w \in L^2(\Omega)$ $(w \neq 0)$. Then $w \in H_0^1(\Omega)$ and for any $v \in H_0^1(\Omega)$,
$$(w, v)_{L^2} \underset{\text{def of } A}{=} (Aw, v)_{H^1} = \mu\, (w, v)_{H^1}$$

Setting $w = v$, see $\mu \neq 0$, and hence $w \in H_0^1(\Omega)$ is a weak solution to
$$\left.\begin{aligned} -\Delta w + w &= \frac{1}{\mu} w \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \quad (\Delta)$$

Now $w \in H_0^1(\Omega) \subset H^1(\mathbb{R}^n)$ $\therefore w \in H^1_{loc}(\Omega)$

$\therefore$ By regularity theory $w \in H_0^1 \cap H^3_{loc}(\Omega)$. Iterating, we deduce that $w \in H_0^1 \cap C^\infty(\Omega)$

Now (Δ) holds iff w is a weak soln to
$$-\Delta w = \left(-1 + \frac{1}{\mu}\right) w \quad \text{in } \Omega$$
$$w = 0 \quad \text{on } \partial\Omega$$

We deduce

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open + bounded. Then there exists an o/n basis $\{w_k\}_{k=1}^{\infty}$ for $L^2(\Omega)$ s.t. $w_k \in H_0^1 \cap C^\infty(\Omega)$ satisfy $-\Delta w_k = \lambda_k w_k$ in Ω,
where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_k \to \infty$ ☺

Corollary $(\sin nx)_{n=1}^{\infty}$ is a basis for $L^2((0,\pi))$.

The Direct Method of the Calculus of Variations

Given $u \in H_0^1(\Omega)$, $f \in L^2(\Omega)$, consider

$$S[u] = \int_\Omega \left(|Du|^2 + |u|^2 - \bar{f}u - f\bar{u} \right) dx$$

Clearly $S[u] \in \mathbb{R}$. Further

$$\begin{aligned} S[u] &= \|u\|_{H^1}^2 - 2\,\mathrm{Re}\,(f,u)_{L^2} \\ &\overset{\text{C-S}}{\geq} \|u\|_{H^1}^2 - 2\|f\|_{L^2}\|u\|_{L^2} \\ &\overset{\text{AMGM}}{\geq} \|u\|_{H^1}^2 - \tfrac{1}{2}\|u\|_{L^2}^2 - 2\|f\|_{L^2}^2 \\ &\geq \tfrac{1}{2}\|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 \end{aligned}$$

Thus $\{ S[u] \mid u \in H_0^1(\Omega) \}$ is bounded below.
We may consider
$\sigma = \inf \{ S[u] \mid u \in H_0^1(\Omega) \} > -\infty$.
We claim that this infimum is attained.
To see this, let $u_n \in H_0^1(\Omega)$ be s.t. $S[u_n] \searrow \sigma$.
$(S[u_n])_{n=1}^{\infty}$ is convergent, hence bounded, so since

$$\|u_n\|_{H^1}^2 \leq 2S[u_n] + 4\|f\|_{L^2}^2 ,$$

$(u_n)_{n=1}^{\infty}$ is bounded in $H_0^1(\Omega)$.

By Banach-Alaoglu, there exists $w \in H^1_0(\Omega)$ such that (after extracting a subsequence and relabelling)
$u_n \rightharpoonup w$ in $H^1_0(\Omega)$.

Now by exercise 2.8 $\|w\|_{H^1} \leq \liminf_{n\to\infty} \|u_n\|_{H^1}$.

Since $u_n \rightharpoonup w$ in H^1,
$$\mathrm{Re}(f,w)_{L^2} = \lim_{n\to\infty} \mathrm{Re}(f,u_n) = \liminf_{n\to\infty} \mathrm{Re}(f,u_n)$$

$$\begin{aligned} S[w] &= \|w\|^2_{H^1} - 2\,\mathrm{Re}(f,w)_{L^2} \\ &\leq \liminf_{n\to\infty}\left(\|u_n\|^2_{H^1} - 2\,\mathrm{Re}(f,u_n)_{L^2} \right) \\ &\leq \liminf_{n\to\infty}\left(S[u_n]\right) = \sigma \quad \ddot\smile \end{aligned}$$

$\therefore S[w] \leq \sigma$. But $\sigma = \inf_{u\in H^1_0} S[u]$, hence $S[w] = \sigma$.

We've shown that $S[u] \geq S[w]$ $\forall\, u \in H^1_0(\Omega)$.
w is a minimisar of S.
Consider for $v \in H^1_0(\Omega)$, $t \in \mathbb{R}$
$$S[w+tv] = S[w] + 2t\,\mathrm{Re}\left(\int_\Omega (\overline{Dw}\cdot Dv + \bar{w}v - \bar{f}v)\,dx\right) + t^2\|v\|^2_{H^1}$$

In order that $S[w+tv] \geq S[w]$ $\forall t$, coeff of t must vanish. Since this is true for all v (replacing v with $e^{i\theta}v$) we see that
$$\int_\Omega (\overline{Dw}\cdot Dv + \bar{w}v)\,dx = \int_\Omega \bar{f}v\,dx \quad \forall\, v\in H^1_0(\Omega)$$

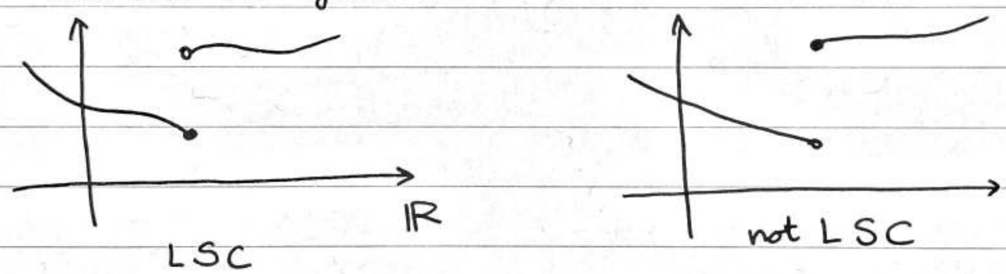
i.e. if w is a weak solution to
$$\left.\begin{aligned} -\Delta w + w &= f \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}$$

We also see that such an w must be unique from $(*)$.

* Direct method for Calculus of Variations

We will generalise the approach we used in an example last lecture.
Let X a topological space (e.g. Banach space or subset thereof, with strong, weak or weak-* topology).
We say a functional $S : X \to \mathbb{R}$ is
coercive if $\{ u \mid S[u] \leq K \}$ is sequentially relatively compact $\forall K$.
i.e. if $(u_n)_{n=1}^{\infty}$ satisfies $S[u_n] \leq K$ then (u_n) has a convergent subsequence.
lower semi-continuous (LSC) if for any sequence $(u_n)_{n=1}^{\infty}$ with $u_n \to u^*$ in X, we have $S[u^*] \leq \liminf_{n\to\infty} S[u_n]$.

Note that lower semi-continuity is implied by continuity, but the converse is false.



Theorem Suppose $S : X \to \mathbb{R}$ is coercive and LSC. Then S achieves its minimum.

Pf Let $\alpha = \inf \{ S[u] \mid u \in X \}$ (possibly $= -\infty$)
Pick (u_n) s.t. $S[u_n] \to \alpha$. $(S[u_n])_{n=1}^{\infty}$ is bounded above, hence by coercivity, (u_n) has a convergent subsequence.
Relabel so that wlog $u_n \to u^*$ in X.

$$S[u^*] \leq \liminf_{n\to\infty} S[u_n] = \alpha$$

But $S[u] \geq \alpha \ \forall u$. $\therefore S[u^*] = \alpha$, and S achieves its minimum at u^* (hence $\alpha > -\infty$). $\square$

We've worked here with sequential def^{n}s. Good enough for most purposes. Non-sequential versions of this theorem also exist.

From now on, X is a <u>reflexive</u> <u>separable</u> Banach space!

Typically, the functions we've interested in are <u>not</u> coercive in the strong topology. However, by Banach-Alaoglu, it is easier to show coercivity in the weak (= weak-$*$) topology.

<u>Lemma</u> Suppose $S: X \to \mathbb{R}$ satisfies
$$S[u] \leq K \Rightarrow \|u\| \leq \tilde{K}$$
Then S is coercive on X with the <u>weak topology</u>.

<u>Pf</u> $\{S[u] \leq K\} \subseteq \{\|u\| \leq \tilde{K}\}$

So done by Banach-Alaoglu. $\square$

Thus working in the weak topology makes coercivity easier to prove. On the other hand, since there are more weakly convergent sequences than strongly convergent sequences, showing LSC is harder in the weak topology. Typically, we require some <u>convexity</u>.

<u>Lemma</u> If $\mathcal{U} \subset X$ is closed (in the strong topology) & convex, then $\mathcal{U}$ is closed in the weak topology.

<u>Pf</u> Pick $x \in \mathcal{U}^c$. By Hahn-Banach separation theorem applied to $\{x\}$ (convex + compact) and $\mathcal{U}$ (convex + closed) $\exists \Lambda \in X'$ s.t. $\quad \mathrm{Re}(\Lambda x) < \gamma_1 < \gamma_2 < \mathrm{Re}(\Lambda y) \quad \forall y \in \mathcal{U}$

Now $\{\mathrm{Re}(\Lambda z) < \gamma_1\}$ is open in the weak topology, contains x and is disjoint from $\mathcal{U}$. Thus $\mathcal{U}^c$ is weakly open and thus $\mathcal{U}$ is weakly closed. $\square$

Using this, we can show:

<u>Theorem</u> If $S: X \to \mathbb{R}$ is convex and LSC in the strong topology, then it is LSC in the weak topology.

<u>Pf</u> Consider $\{S[u] \leq K\}$. This is convex and sequentially closed in the strong topology on X, hence strongly closed. So by Lemma $\{S[u] \leq K\}$ is weakly closed.

Now suppose $u_n \rightharpoonup u^*$,

Choosing a subsequence (u_{n_j}), we may assume wlog that

$$S[u_{n_j}] \longrightarrow \liminf S[u_n]$$

For any $\varepsilon > 0$, $\exists J$ s.t. $\forall j \geq J$,

$$S[u_{n_j}] \leq \underbrace{\liminf S[u_n] + \varepsilon}_{= K}$$

But $\{S[u] \leq K\}$ is weakly closed, so since $u_{n_j} \rightharpoonup u^*$, we deduce

$$S[u^*] \leq K = \liminf_{n \to \infty} S[u_n] + \varepsilon$$

ε arbitrary $\therefore S[u^*] \leq \liminf_{n\to\infty} S[u_n]$. $\square$

<u>Example</u> Let $\Omega \subset \mathbb{R}^n$ be bounded open and suppose $L : \mathbb{C}^n \to \mathbb{R}$ is convex, and satisfies $L[z] \geq \gamma |z|^2 - C$ for some $C, \gamma > 0$ all $z \in \mathbb{C}^n$

Then $S : H_0^1(\Omega) \longrightarrow \mathbb{R}$

$$u \longmapsto \int_\Omega L(Du)\, dx$$

has a minimisor.

<u>Pf</u> <u>Coercivity</u>: $S[u] \geq \gamma \int_\Omega |Du|^2 dx - C \geq \tilde{\gamma} \|u\|_{H^1}^2 - C$ (not same const)

by Exercise 4.3 $\left(\int_\Omega |u|^2 dx \leq \beta \int_\Omega |Du|^2 dx \;\; \forall u \in H_0^1(\Omega) \right)$

$$\therefore S[u] \leq K \Rightarrow \|u\|_{H^1}^2 \leq \frac{C+K}{\gamma} \quad \therefore S \text{ is coercive}$$

(weak topology!)

<u>LSC</u>: Suppose $u_n \rightharpoonup u$ in $H_0^1(\Omega)$; by taking a subsequence (u_{n_j}) we may assume

$$S[u_{n_j}] \longrightarrow \liminf S[u_n],$$

and that $Du_{n_j} \longrightarrow Du$ pointwise a.e.

Now $L(Du_{n_j}) + C \geq 0$, so by Fatou

$$\liminf_{n\to\infty} \left(S[u_n] + C|\Omega| \right) = \liminf_{j \to \infty} \left(\int_\Omega [L(Du_{n_j}) + C]\, dx \right)$$

$$\geq \int_\Omega [L(Du) + C]\, dx$$

$$= S[u] + C|\Omega|$$

$\Rightarrow \quad S[u] \leq \liminf_{n\to\infty} S[u_n]$

So S is LSC in the strong topology.
S is convex, since L is convex (check), so by previous theorem S is weakly LSC.
$\therefore$ achieves its minimum :) $\square$

<u>Example</u> $\quad S[u] = \int_\Omega \left(1 + |Du|^4\right)^{1/2} dx$

satisfies conditions

~ FIN ~