II Classical Dynamics

§1 Review of Newtonian Mechanics

1.1 Newton's 2^{nd} law

Particle mass m, position $\underline{r}(t)$. Then in an inertial frame

$$\boxed{m\ddot{\underline{r}} = \underline{F}}$$ i.e. the usual "F = ma"

- ▷ System of 2^{nd} order ODEs for components of $\underline{r}$. Solution determined by $\underline{r}(0)$ and $\dot{\underline{r}}(0)$.
- ▷ Any frame moving at const velocity wrt inertial frame is also inertial.
- ▷ In a non-iternial frame, eq^n of motion modified and can be defined including 'fictitious' forces.
- ▷ N2 also applies to extended bodies if m is the total mass, $\underline{r}$ is the position of the CoM, $\underline{F}$ is the net force.

The (lin) momentum is $\underline{p} = m\dot{\underline{r}}$

The angular momentum (about origin) is $\underline{L} = \underline{r} \times \underline{p} = m\underline{r} \times \dot{\underline{r}}$

The kinetic energy is $T = \frac{1}{2} m |\dot{\underline{r}}|^2$

~ m const

Their rates of change are
$$\dot{\underline{p}} = m\ddot{\underline{r}} = \underline{F}$$
$$\dot{\underline{L}} = m\dot{\underline{r}} \times \dot{\underline{r}} + m\underline{r} \times \ddot{\underline{r}} = \underline{r} \times \underline{F} = \underline{G}$$
$$\dot{T} = m\ddot{\underline{r}} \cdot \dot{\underline{r}} = \underline{F} \cdot \dot{\underline{r}}$$

torque on particle

Change in T along a path is
$$\int dT = \int_{t_1}^{t_2} \dot{T}\, dt = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}}\, dt = \int_C \underline{F} \cdot d\underline{r}$$

Call this line integral the work done by the force along C.

1.2 Systems of particles

Consider N particles of masses m_i, positions $\underline{r}_i(t)$.

The i^{th} eq^n of motion is $m_i \ddot{\underline{r}}_i = \underline{F}_i$

If $\underline{F}_i$ given as a f^n of the $\underline{r}_j$, $\dot{\underline{r}}_j$, t then we have a system of coupled 2^{nd} order ODEs.

Sol^n depends on initial pos^{n}s, vel.

Assume $\underline{F}_i$ decomposed as $\sum_{j=1}^{N} \underline{F}_{ij} + \underline{F}_i^{ext}$

inter-particle

external force,

zero for an isolated system

Newton's Third Law states $\underline{F}_{ji} = -\underline{F}_{ij}$, so $\underline{F}_{ii} = 0$ for example

Then we get $m_i \ddot{\underline{r}}_i = \sum_{j=1}^{N} \underline{F}_{ij} + \underline{F}_i^{ext}$

Centre of mass given by $M\underline{R} = \sum_{i=1}^{N} m_i \underline{r}_i$, $M = \sum_{i=1}^{N} m_i$

Summing eq^{n}s of motion, find

$$M\ddot{\underline{R}} = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} \underline{F}_{ij}}_{\text{zero}} + \sum_{i=1}^{N} \underline{F}_i^{ext} = \underline{F}^{ext}, \text{ net ext force}$$

$\therefore$ N2 applies to multiparticle systems, via internal forces cancelling out

The total momentum is $\underline{P} = \sum_{i=1}^{N} \underline{p}_i = \sum_{i=1}^{N} m_i \dot{\underline{r}}_i = M\dot{\underline{R}}$

and satisfies $\dot{\underline{P}} = M\ddot{\underline{R}} = \underline{F}^{ext}$

Total angular momentum $\underline{L} = \sum_{i=1}^{N} m_i \underline{r}_i \times \dot{\underline{r}}_i$

and so $\dot{\underline{L}} = \sum_{i=1}^{N} m_i \underline{r}_i \times \ddot{\underline{r}}_i = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i$

If strong N3 applies, i.e. $\underline{F}_{ij}$ parallel to $\underline{r}_i - \underline{r}_j$, then internal torque zero

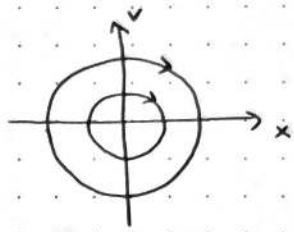
$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N} \underline{r}_i \times \underline{F}_{ij} &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \underline{r}_i \times (\underline{F}_{ij} - \underline{F}_{ji}) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ij} \\ &= 0 \end{aligned}$$

And then $\dot{\underline{L}} = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{ext} = \sum_{i=1}^{N} \underline{G}_i^{ext} = \underline{G}^{ext}$, net ext torque.

For an isolated system, both $\underline{P}$, $\underline{L}$ are conserved.

1.3 Configuration space & phase space

- For a system of N particles moving in 3D without constraints
  The physical space in which the particles move is $\mathbb{R}^3$.
- The instantaneous config^n described by the vectors $\underline{r}_1, \dots, \underline{r}_N$
  No constraints $\Rightarrow$ config^n space is $\mathbb{R}^{3N}$ — 3N degrees of freedom
- The future evolution also depends on velocities.
  Hence define phase space as $\mathbb{R}^{6N}$ with points $(\underline{r}_1, \dots, \underline{r}_N; \dot{\underline{r}}_1, \dots, \dot{\underline{r}}_N)^T$
- The equations of motion define a flow in phase space
  Ex SHM in 1D has $\ddot{x} = -x$
  Physical space, config space are $\mathbb{R}$
  Phase space is $\mathbb{R}^2$, flow given by $\dot{x} = v$, $\dot{v} = -x$



1.4 Conservative systems

- For a single particle, a conservative force satisfies $\underline{F} = -\underline{\nabla} V$
  Call $V(\underline{r})$ the potential energy
- The work done simplifies as $\int_C (-\underline{\nabla} V) \cdot d\underline{r} = V(\underline{a}) - V(\underline{b})$
  Depends only on the endpoints, and $T + V$ is conserved
- For a system of N particles, $\underline{F}_i = -\underline{\nabla}_i V$ for $V(\underline{r}_1, \dots, \underline{r}_n)$
  Then total energy $T + V = \sum_{i=1}^{n} \frac{1}{2} m_i |\dot{\underline{r}}_i|^2 + V(\underline{r}_1, \dots, \underline{r}_n)$ conserved
- If $\underline{F}_i = -\underline{\nabla}_i V$ for some $V(\underline{r}_1, \dots, \underline{r}_N, t)$ with explicit time-dep
  Then total energy need not be conserved
- The gravitational & electrostatic forces between particles are of the form
  $$F_{ij} = C_{ij} \frac{r_{ij}}{|\underline{r}_{ij}|^3} \quad \text{where } \underline{r}_{ij} = \underline{r}_i - \underline{r}_j,$$
  with $$C_{ij} = -Gm_i m_j \text{ or } \frac{q_i q_j}{4\pi\varepsilon_0}$$
  Corresponding potential energy is $V_{ij} = V_{ji} = \frac{C_{ij}}{r_{ij}}$ (self-energy zero)
  Then note $-\underline{\nabla}_i V_{ij} = \underline{F}_{ij}$, $-\underline{\nabla}_j V_{ij} = \underline{F}_{ji} = -\underline{F}_{ij}$ (N3 satisfied)

- More generally, could have $V_{ij} = V_{ij}(r_{ij})$ and

$$\underline{F}_{ij} = -\underline{\nabla}_i V_{ij} = -\frac{1}{r_{ij}} \frac{dV_{ij}}{dr_{ij}} \underline{r}_{ij}$$

[Note that $r_{ij}^2 = (\underline{r}_i - \underline{r}_j)\cdot(\underline{r}_i - \underline{r}_j)$

$$2 r_{ij} \underline{\nabla}_i r_{ij} = 2(\underline{r}_i - \underline{r}_j)$$

$$\therefore \underline{\nabla}_i r_{ij} = \frac{\underline{r}_{ij}}{r_{ij}}$$]

- Total potential energy is then $V = \sum_{i<j} V_{ij} = \frac{1}{2}\sum_{i\neq j} V_{ij}$
  If not isolated, add terms $\sum_i V_i^{ext}$

- The gravitational N-body problem $m_i \ddot{\underline{r}}_i = \sum_{j=1}^{N} \underline{F}_{ij}$, $\underline{F}_{ij} = -\frac{G m_i m_j \underline{r}_{ij}}{r_{ij}^3}$
  e.g. ($N \approx 10$ Solar System, $N \approx 10^5$ star cluster, $N \approx 10^{11}$ galaxy)

1.5 Euclidean & Galilean symmetries

> Translations in space
> Rotations in space
> Translations in time
> Galilean transformations

general form

$$\underline{r} \mapsto R\underline{r} + \underline{v}_0 t + \underline{r}_0 \, , \quad t \mapsto t + t_0$$

where $R \in SO(3)$, $\underline{v}_0, \underline{r}_0, t_0$ constant

$$\begin{pmatrix} \underline{r} \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R & \underline{v}_0 & \underline{r}_0 \\ 0 & 1 & t_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{r} \\ t \\ 1 \end{pmatrix}$$

Under this transformation, distances and time intervals are invariant, as is the potential energy of an isolated system. Relative positions, forces, are all rotated by R.

These operations constitute the Galilean group.

There is a relation between continuous symmetries and conservation laws:

> Translation in space ~ lin momentum
> Rotations in space ~ ang momentum
> Translation in time ~ energy

Discrete symmetries include
> Point reflection
> Time reversal

Sol^n to exercise Verify conservation law $T+V = \sum \frac{1}{2} m_i |\dot{\underline{r}}_i|^2 + V(\underline{r}_1, \dots, \underline{r}_n)$

$$\frac{d}{dt}(T+V) = \sum m_i \ddot{\underline{r}}_i \cdot \dot{\underline{r}}_i + \sum (\nabla_i V) \cdot \dot{\underline{r}}_i$$

$$= \sum \underbrace{(m_i \ddot{\underline{r}}_i + \nabla_i V)}_{\text{zero}} \cdot \dot{\underline{r}}_i$$

§2 Lagrange's Equations

2.1 Generalized coordinates

While Newtonian mechanics uses vectors, the Lagrangian approach is more flexible.

A system with n degrees of freedom requires n independent generalized coordinates $q_i(t)$, $i = 1, \dots, n$ to specify its config^n.

The generalized velocities are $\dot{q}_i = dq_i/dt$

Why do this?

- Non-Cartesian coordinates for systems with circular, spherical symmetry
- Systems with constraints e.g. confining surfaces, rods, etc

Lagrangian mechanics deals mostly with conservative systems.

Their dynamics can be derived from a scalar $L(\underline{q}, \dot{\underline{q}}, t)$.

Notation: ▷ $\underline{q}$ is $(q_1, \dots, q_n)$, $\dot{\underline{q}}$ is $(\dot{q}_1, \dots, \dot{q}_n)$

$\frac{\partial L}{\partial \underline{q}}$ is $\left(\frac{\partial L}{\partial q_1}, \dots, \frac{\partial L}{\partial q_n}\right)$, $\underline{q} \cdot \underline{p}$ is $\sum p_i q_i$

2.2 Hamilton's Principle

If a system evolves from $\underline{q}_1 = \underline{q}(t_1)$ to $\underline{q}_2 = \underline{q}(t_2)$ then the *action* is defined as $S[\underline{q}] = \int_{t_1}^{t_2} L \, dt$.

This is a functional of the path $\underline{q}(t)$ taken in configuration space.

Hamilton's principle states that the physical path is such that the action has a stationary value, i.e.

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0$$

subject to $\underline{q}(t_1)$, $\underline{q}(t_2)$ being fixed.

According to the calculus of variations, this means that the functional derivative vanishes $\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ for $i = 1, \dots, n$

This is E-L eq^n.

In analytical mechanics we call these <u>Lagrange's equations</u> and interpret them as the <u>equations of motion</u> of the system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad \text{or} \quad \frac{d}{dt}\frac{\partial L}{\partial \underline{\dot{q}}} = \frac{\partial L}{\partial \underline{q}}$$

Since L depends on $\underline{q}$ and $\underline{\dot{q}}$, these are n second-order ODEs.

The LHS of Lagrange's equations can be written using p_i,

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}, \quad \text{where} \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is the generalized / conjugate momentum to coordinate q_i

Derivation of E-L eq^n (wh dh):

Note $\delta \int L\, dt = \int \delta L\, dt$, $\delta(\dot{q}_i) = \frac{d}{dt}(\delta q_i)$

$$\delta S = \int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) dt$$

$$= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i\, dt + \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i \right]_{t_1}^{t_2}$$

If δS is to vanish for all variations δq_i that vanish at the endpoints, then we require $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ for $i = 1, \dots, n$

Is the Lagrangian unique? No, if we add to L the expression

$$\frac{d}{dt} f(\underline{q}, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \underline{q}} \cdot \underline{\dot{q}}$$

then we add to $S = \int L\, dt$ the quantity

$$f(\underline{q}_2, t_2) - f(\underline{q}_1, t_1)$$

which is independent of the path.

Indeed, extra terms in Lagrange's equations cancel out.

<u>Ex</u> N particles with potential forces

The standard form of L is the <u>difference</u> of T and V,

$$L = \sum \tfrac{1}{2} m_i |\dot{\underline{r}}_i|^2 - V(\underline{r}_1, \dots, \underline{r}_N)$$

Identify $\underline{q}$ with $(\underline{r}_1, \dots, \underline{r}_N)$,

$$\frac{\partial L}{\partial \dot{\underline{r}}_i} = m_i \dot{\underline{r}}_i \,, \quad \frac{\partial L}{\partial \underline{r}_i} = -\underline{\nabla}_i V$$

So Lagrange's equations just say $m_i \ddot{\underline{r}}_i = -\underline{\nabla}_i V$; Newton 2.

<u>Ex</u> Particle in a rotating frame

If $\underline{r}(t)$ is the position vector measured in a frame rotating with angular velocity $\omega(t)$ about an axis through the origin, then

$$L = \tfrac{1}{2} m |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2 - V,$$

because $\dot{\underline{r}} + \underline{\omega} \times \underline{r}$ is the absolute velocity, including that due to rotation.

$$\begin{aligned} L &= \tfrac{1}{2} m |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2 - V \\ &= \tfrac{1}{2} m \dot{x}_i \dot{x}_i + m \varepsilon_{ijk} \dot{x}_i \omega_j x_k + \tfrac{1}{2} m \varepsilon_{ijk} \varepsilon_{ilm} x_k \omega_j x_m \omega_l - V \end{aligned}$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i},$$

$$\frac{d}{dt}\left(m \dot{x}_i + m \varepsilon_{ijk} \omega_j x_k\right) = m \cancel{\varepsilon_{ijk}}\, \varepsilon_{kji} \dot{x}_k \omega_j + m \varepsilon_{kji} \varepsilon_{klm} \omega_j \omega_l x_m - \frac{\partial V}{\partial x_i}$$

$$\Rightarrow \quad m\left(\ddot{x}_i + 2 \varepsilon_{ijk} \omega_j \dot{x}_k + \varepsilon_{ijk} \dot{\omega}_j x_k + \varepsilon_{ijk} \varepsilon_{klm} \omega_j \omega_l x_m\right) = -\frac{\partial V}{\partial x_i}$$

$$m\left(\ddot{\underline{r}} + 2 \underline{\omega} \times \dot{\underline{r}} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})\right) = -\underline{\nabla} V$$

which is the equation of motion in a rotating frame, obtained without vectors.

The extra terms are the <u>ficticious</u> forces arising in a non-inertial frame of reference

$$m \ddot{\underline{r}} = -\underline{\nabla} V - 2m \underset{\substack{\uparrow \\ \text{Coriolis}}}{\underline{\omega}} \times \dot{\underline{r}} - m \underset{\substack{\uparrow \\ \text{Euler}}}{\dot{\underline{\omega}}} \times \underline{r} - m \underline{\omega} \underset{\substack{\uparrow \\ \text{centrifugal}}}{\times} (\underline{\omega} \times \underline{r})$$

The Lagrangian in the rotating frame,

$$L = \tfrac{1}{2} m |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2 - V$$

can be thought of as

$$L = \tfrac{1}{2} m |\dot{\underline{r}}|^2 - \left(V - \tfrac{1}{2} m |\underline{\omega} \times \underline{r}|^2\right) + m \dot{\underline{r}} \cdot (\underline{\omega} \times \underline{r})$$

which is like $T - V$, plus an extra term for the velocity dependent force.

<u>Example</u> Particle moving in a plane (r,θ), with $ds^2 = dr^2 + r^2 d\theta^2$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r,\theta)$$

Lagrange's equations are

$$\frac{d}{dt}\left(m\dot{r}\right) = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \Rightarrow m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial V}{\partial r}, \quad (*)$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -\frac{\partial V}{\partial \theta} \Rightarrow \text{\textcircled{}}$$

Eq^n (†) relates the rate of change of angular momentum to the torque. If $V = V(r)$ then we have a central force, $p_\theta = mr^2\dot{\theta}$ conserved.

<u>Exercise</u> Show that adding $\frac{d}{dt}f(\underline{q},t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \underline{q}} \cdot \dot{\underline{q}}$

then the extra terms in Lagrange's equations cancel out.

Solution to exercise

Extra terms $\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q}_j \frac{\partial f}{\partial q_j}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$ leads to $\frac{d}{dt}\frac{\partial f}{\partial q_i} = \frac{\partial^2 f}{\partial t \partial q_i} + \dot{q}_j \frac{\partial^2 f}{\partial q_j \partial q_i}$

$\frac{\partial L}{\partial q_i}$ leads to $\frac{\partial^2 f}{\partial q_i \partial t} + \dot{q}_j \frac{\partial^2 f}{\partial q_j \partial q_i}$ which cancel out

2.3 Point transformations

- Lagrange's equations are invariant under a change of generalized coordinates.
- This is obvious from the variational principle, because L and $S = \int L\, dt$ are scalars, invariant under a transformation of coordinates.
- Verify explicitly by considering a change from $\underline{q}$ to $\underline{Q}(\underline{q}, t)$. This is known as a point transformation
- Recall $\delta S = \int \frac{\delta S}{\delta q_i} \delta q_i \, dt = \int \frac{\delta S}{\delta Q_j} \delta Q_j \, dt$
- From the chain rule, $\delta Q_j = \frac{\partial Q_j}{\partial q_i} \delta q_i$

So we should expect the following chain rule $\frac{\delta S}{\delta q_i} = \frac{\delta S}{\delta Q_j} \frac{\partial Q_j}{\partial q_i}$

- To verify, use the chain rule

$$\dot{Q}_j = \frac{\partial Q_j}{\partial t} + \frac{\partial Q_j}{\partial q_k} \dot{q}_k$$

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial q_i \partial t} + \frac{\partial^2 Q_j}{\partial q_i \partial q_k} \dot{q}_k \right)$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial Q_j}{\partial q_i}$$

Thus
$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} - \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial q_i \partial t} + \frac{\partial^2 Q_j}{\partial_k \partial q_i}\dot{q}_k\right)$$
$$= \left(\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i}$$
$$= \frac{\delta S}{\delta Q_j}\frac{\partial Q_j}{\partial q_i}$$

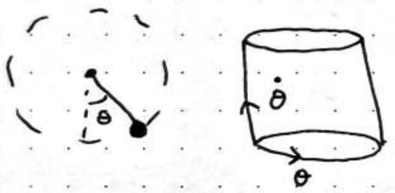
- So for a non-singular transformation, $\frac{\delta S}{\delta \underline{q}} = 0 \Rightarrow \frac{\delta S}{\delta \underline{Q}} = 0$ $\square$

L4.2

2.4 Holonomic constraints

- Often deal with systems of particles whose motion is constrained. In a rigid body, $|\underline{r}_i - \underline{r}_j| = \text{const}$ for any pair of particles. In a simple pendulum, the particle can move only on a circle.
- Forces of constraint are required to enforce these conditions. In the Newtonian approach we introduce these forces explicitly and eliminate them algebraically. Much easier in the Lagrangian approach.
- Consider a system of N particles moving in d dimensions, subject to M independent constraints, holonomic, $f_m(\underline{r}_1, \dots, \underline{r}_N, t) = 0$, $m = 1, \dots, M$
- The constraints restrict the possible configuration to a manifold in $\mathbb{R}^{dN}$ of dimension $n = dN - M$, called the configuration manifold.
- Scleronomic constraints do not depend on time, while rheonomic do.

Ex Simple pendulum, configuration space is a circle; phase space is cylinder



Ex Particle in 3D confined to move along a moving wire, so subject to two rheonomic constraints, and has one degree of freedom.

- It is possible to handle constraints using Lagrange multipliers. To enforce $f_m = 0$ at every $t \in (t_1, t_2)$, require time dependent $\lambda_m(t)$. Variation of the action $\int L\, dt$ subject to $f_m = 0$ is equivalent to the unconstrained variation of $\int (L - \sum \lambda_m f_m)\, dt$
- Get modified Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{r}}_i}\right) = \frac{\partial L}{\partial \underline{r}_i} - \sum \lambda_m \frac{\partial f_m}{\partial \underline{r}_i}$$

The λ terms are the forces of constraint, which can be eliminated algebraically as in the Newtonian approach.

- Usually a better method is to solve the equations of constraint by introducing n independent generalized coordinates $q_1, \dots, q_n$. We can write

$$\underline{r}_i = \begin{cases} \underline{r}_i(\underline{q}) & \text{for scleronomic,} \\ \underline{r}_i(\underline{q}, t) & \text{for rheonomic.} \end{cases}$$

L4.3

- For each $q_1, \dots, q_n$ obtain the unmodified Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

because the f_m do not depend on the q_i.

<u>Ex</u> Simple pendulum of length ℓ, in 2D. The constraint $f = |\underline{r}| - \ell = 0$, produces a force of constraint $-\lambda \underline{\nabla} f = -\lambda \frac{\underline{r}}{|\underline{r}|}$.

Here λ is the tension in the string, or the normal reaction from a surface, required to constrain the particle to the circle.

If instead we use generalized coordinate θ, just solve $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$

- To rewrite the kinetic energy in terms of generalized coordinates,

$$\dot{r}_k = \frac{\partial r_k}{\partial q_i} \dot{q}_i + \frac{\partial r_k}{\partial t}$$

If the constraints are scleronomic, last term zero and

$$T = \frac{1}{2} m_k |\dot{\underline{r}}_k|^2$$
$$= \sum_k \frac{1}{2} m_k \frac{\partial \underline{r}_k}{\partial q_i} \frac{\partial \underline{r}_k}{\partial q_j} \dot{q}_i \dot{q}_j$$

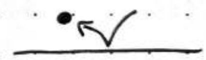
- This is a positive definite, homogeneous quadratic function

$$T = \sum_{i,j} \frac{1}{2} T_{ij}(\underline{q}) \dot{q}_i \dot{q}_j$$

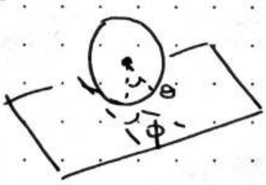
where $T_{ij} = \sum_k m_k \frac{\partial \underline{r}_k}{\partial q_i} \cdot \frac{\partial \underline{r}_k}{\partial q_j}$ is a positive definite, symmetric matrix.

<u>Ex</u> (Non-holonomic constraints)

- Inequalities, e.g. a particle bouncing on a table, $z \geq 0$



- Non-integrable constraints involving velocities, e.g. sphere or disc (but not cylinder) rolling on a surface



L5.1

2.5 Symmetries and conservation laws

Conservation of momentum

If L does not depend on q_k, say that coordinate is <u>ignorable</u> (or cyclic).

Lagrange's equation for q_k reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \quad \text{and hence} \quad \frac{\partial L}{\partial \dot{q}_k} = p_k = \text{const}$$

Conservation of energy

The total time derivative, via chain rule

$$\frac{dL}{dt} = \sum \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) + \frac{\partial L}{\partial t}$$

If L does not depend explicitly on time, using Lagrange's equations get

$$\frac{dL}{dt} = \sum \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right)\dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$
$$= \frac{d}{dt}\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

Therefore we have the conservation law

$$\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad ,$$ sometimes called the <u>Beltrami identity</u>

If $L = T(\underline{q}, \dot{\underline{q}}) - V(\underline{q})$ where T is a homogeneous quadratic in $\dot{\underline{q}}$, then Euler's theorem on homogeneous functions gives

$$\sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

In this case the conserved quantity is $2T - (T - V) = T + V = E = \text{const}$

2.6 Noether's Theorem

If the Lagrangian is invariant under a continuous symmetry transformation, then there is a conserved quantity associated.

<u>Proof</u> Let s be the parameter of the continuous transf^{n} $\underline{q}(t) \mapsto \underline{Q}(s,t)$, where $s=0$ corresponds to $\underline{q}(t) = \underline{Q}(0,t)$.

By Hypothesis, $L(\underline{Q}, \dot{\underline{Q}}, t) = L(\underline{q}, \dot{\underline{q}}, t)$ where $\dot{\underline{Q}} = \frac{\partial \underline{Q}}{\partial t}$

Differentiate wrt s at constant t,

$$0 = \frac{\partial L}{\partial \underline{Q}} \cdot \frac{\partial \underline{Q}}{\partial s} + \frac{\partial L}{\partial \dot{\underline{Q}}} \cdot \frac{\partial \dot{\underline{Q}}}{\partial s}$$

and evaluate at $s=0$, where $\underline{Q} = \underline{q}$, $\dot{\underline{Q}} = \dot{\underline{q}}$

$$\frac{\partial L}{\partial \underline{q}} \cdot \left.\frac{\partial \underline{Q}}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \dot{\underline{q}}} \cdot \left.\frac{\partial \dot{\underline{Q}}}{\partial s}\right|_{s=0} = 0$$

Using Lagrange's equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}}\right)\cdot\frac{\partial \underline{Q}}{\partial s}\bigg|_{s=0} + \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{\partial \dot{\underline{Q}}}{\partial s}\bigg|_{s=0} = 0$$

The conserved quantity is $\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{\partial \underline{Q}}{\partial s}\Big|_{s=0}$ $\square$

<u>Example</u> Isolated N-body system

$$L = \sum \tfrac{1}{2} m_i |\dot{\underline{r}}_i|^2 - V(\underline{r}_1, \dots, \underline{r}_N)$$

If V depends only on the relative positions, $\underline{r}_{ij} = \underline{r}_j - \underline{r}_i$, then L is invariant under a translation $\underline{r}_i \longmapsto \underline{r}_i + s\underline{n}$

where s is a continuous parameter and $\underline{n}$ is any unit vector.

Noether's Theorem gives us the conserved

$$\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{\partial \underline{Q}}{\partial s}\bigg|_{s=0} = \sum m_i \dot{\underline{r}}_i \cdot \underline{n} = \underline{P}\cdot\underline{n}$$

which is the component of total linear momentum in the direction $\underline{n}$.

If V depends only on the magnitudes $|\underline{r}_{ij}|$, then L is invariant under a rotation of the system. Under an infinitesimal rotation through angle s about the unit vector $\underline{n}$, $\underline{r}_i \mapsto \underline{r}_i + s\,\underline{n}\times\underline{r}_i$

Noether's Theorem gives

$$\begin{aligned}\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{\partial \underline{Q}}{\partial s}\bigg|_{s=0} &= \sum m_i \dot{\underline{r}}_i \cdot (\underline{n}\times\underline{r}_i) \\ &= \sum m_i (\underline{r}_i \times \dot{\underline{r}}_i)\cdot n \\ &= \underline{L}\cdot\underline{n}\end{aligned}$$

which is the component of the total angular momentum in direction $\underline{n}$.

Thus translational symmetry, associated with <u>homogeneity</u> of space, is related to the conservation of linear momentum.

Rotational symmetry, associated with the <u>isotropy</u> of space, is related to the conservation of angular momentum.

L5.3

<u>Autonomous</u> systems have Lagrangians that do not depend explicitly on time, $\frac{\partial L}{\partial t} = 0$. As we have seen, implies conservation of energy.

Energy is to time, as momentum is to space. Also see this in relativity.

L6 I

2.7 Examples

Ex Simple pendulum (planar)

- Mass m, gravity g, length l

Use the angle θ to describe the configuration

One degree of freedom

$$L = T - V = \tfrac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

so Lagrange is $\quad ml^2\ddot{\theta} = -mgl\sin\theta$

This is the well-known pendulum equation $\quad \ddot{\theta} = -\frac{g}{l}\sin\theta$

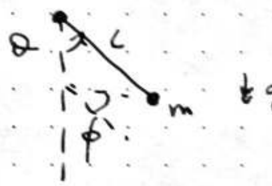
- We could have noted that $\frac{\partial L}{\partial t} = 0$ so energy is conserved:

$$E = \tfrac{1}{2} ml^2\dot{\theta}^2 - mgl\cos\theta = \text{const}$$

This is the first integral of Lagrange's equation wrt θ.

Ex Spherical pendulum

This can be thought of as a particle moving without friction on the surface of a sphere.



Use spherical polar angles (θ, ϕ) to describe the configuration. Two degrees of freedom.

- $$L = T - V = \tfrac{1}{2} ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta$$

(θ, ϕ are orthogonal coords with scale factors $l, l\sin\theta$)

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta$$

$$\frac{\partial L}{\partial \theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta, \quad \frac{\partial L}{\partial \phi} = 0$$

so Lagrange's equations are

$$ml^2\ddot{\theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta, \qquad ml^2\dot{\phi}\sin^2\theta = \text{const.}$$

- Recognize the conserved angular momentum about the vertical axis (because of rotational symmetry.

L6.2

Simplify the θ equation of motion

$$ml^2\ddot{\theta} = \frac{L_z^2 \cos\theta}{ml^2 \sin^3\theta} - mgl\sin\theta$$

Can write as

$$ml^2\ddot{\theta} = -\frac{dV_{eff}}{d\theta} \quad \text{with} \quad V_{eff} = \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

Alternatively, note that energy is conserved because $\frac{\partial L}{\partial t} = 0$

$$\begin{aligned} E &= T + V \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + V_{eff} \\ &= \text{const} \end{aligned}$$

If $L_z = 0$, the problem is identical to the planar pendulum

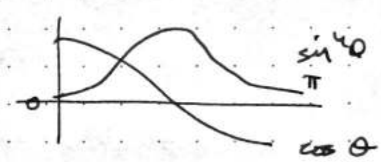
If $L_z \neq 0$, the pendulum cannot approach $\theta = 0$ or $\theta = \pi$

Equilibrium requires

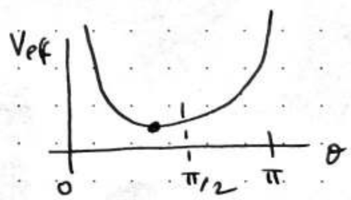
$$\sin^4\theta = \frac{L_z^2}{m^2 l^3 g}\cos\theta$$

which has one solution in $0 < \theta < \frac{\pi}{2}$

and none in $\frac{\pi}{2} < \theta < \pi$



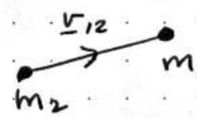
V_{eff} has a minimum here and the equilibrium is stable



The general motion consists of an oscillation in θ about the equilibrium value (which depends on the value of L_z) combined with a non-uniform rotation about the axis.

2.8 The two-body problem

For two particles with inverse-square force



$$L = \tfrac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\underline{r}}_2|^2 + \frac{k}{r_{12}}$$

Instead of $\underline{r}_1$ and $\underline{r}_2$ we work with generalized coords

$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_1}{m_1 + m_2}$, pos^n of COM and $\underline{r}_{12}$, separation vector

L6.3

We find $\underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}$, $\underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}$

and so (exercise)

$$L = \tfrac{1}{2}M|\dot{\underline{R}}|^2 + \tfrac{1}{2}\mu|\dot{\underline{r}}|^2 + \frac{k}{r}$$ for $\mu = \frac{m_1 m_2}{M}$, the <u>reduced mass</u>

Lagrange's equation for $\underline{R}$ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{R}}} = \frac{d}{dt}(M\dot{\underline{R}}) = M\ddot{\underline{R}} = 0$$

so the centre of mass moves with constant velocity

Lagrange's equation for $\underline{r}$ is

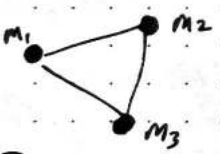
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}} = \frac{\partial L}{\partial \underline{r}} \qquad \text{i.e.} \quad \mu\ddot{\underline{r}} = -\frac{k\underline{r}}{r^3}$$

The general solution, making use of angular momentum conservation, is a conic section (IA Dyn Rel)

2.9 The restricted 3-body problem

For 3 gravitating particles

$$L = \tfrac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\underline{r}}_2|^2 + \tfrac{1}{2}m_3|\dot{\underline{r}}_3|^2 + \frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}} + \frac{Gm_3m_1}{r_{31}}$$



In the limit $m_3 \ll m_1, m_2$, we treat m_3 as a <u>test particle</u>

$$\begin{aligned} L &= L_{12} + L_3 \\ &= \left(\tfrac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\underline{r}}_2|^2 + \frac{Gm_1m_2}{r_{12}}\right) \\ &\quad + \left(\tfrac{1}{2}m_3|\dot{\underline{r}}_3|^2 + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right) \end{aligned}$$

We first solve for the motion of m_1 and m_2, neglecting m_3, using L_{12}. This is just the two-body problem and we obtain a conic section

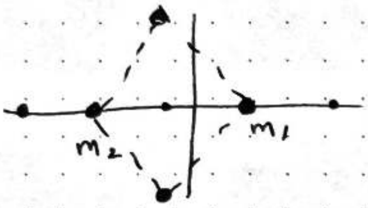
We then solve for the motion of m_3 using L_3. Here $\underline{r}_1, \underline{r}_2$ have a prescribed motion given by their two body orbit. We have a non-autonomous system for the third body.

L6.4

The simplest case is when m_1, m_2 have a circular orbit of radius $r_{12} = a$ and angular velocity $\omega = \left(\frac{GM}{a^3}\right)^{1/2}$

Take the orbit to be in the xy-plane with the centre of mass at the origin. In a frame that rotates with the orbit, at angular velocity $(0,0,\omega)$ the separation vector $\underline{r}_{12} = \text{const} = (a, 0, 0)$ WLOG, so

$$\underline{r}_1 = \left(\frac{m_2 a}{M}, 0, 0\right), \quad \underline{r}_2 = \left(-\frac{m_1 a}{M}, 0, 0\right)$$



In this frame, the third body experiences the Coriolis force and the autonomous potential

$$-\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} - \frac{1}{2}m_3\omega^2(x^2+y^2)$$

This has five equilibrium points, the Lagrange points $L_1, \dots, L_5$

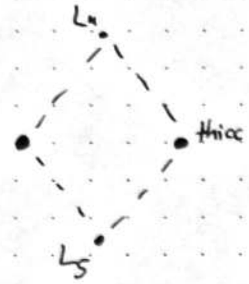
The first three (saddle points) are on the x-axis, and are unstable

L_4 and L_5 (maxima) are at the equilateral triangular points in the plane of the orbit.

<u>Exercise</u> (hard)

Show that, despite being potentially max, the equilibria L_4, L_5 are stabilized by the Coriolis force if ($m_1 \geq m_2$ WLOG) $\frac{m_1}{m_2} > \frac{25+3\sqrt{69}}{2} \simeq 24.96$

L7.1



Solution to Lagrange things

$$\ddot{x} - 2\omega\dot{y} = -\frac{\partial\phi}{\partial x} \qquad \phi = -\frac{Gm_1}{r_{13}} - \frac{Gm_2}{r_{23}} - \frac{1}{2}\omega^2(x^2+y^2)$$

$$\ddot{y} = -2\omega\dot{x} - \frac{\partial\phi}{\partial y}$$

Expand to first order near equilibrium

$$(x,y) = (x_e, y_e) + (\xi, \eta)$$

$$\ddot{\xi} - 2\omega\dot{\eta} = -\phi_{xx}\xi - \phi_{xy}\eta$$

$$\ddot{\eta} + 2\omega\dot{\xi} = -\phi_{yx}\xi - \phi_{yy}\eta$$

Solutions are $\propto e^{\lambda t}$ with

$$\lambda^2\xi - 2\omega\lambda\eta = -\phi_{xx}\xi - \phi_{yx}\eta$$

$$\lambda^2\eta + 2\omega\lambda\xi = -\phi_{yx}\xi - \phi_{yy}\eta$$

Compatible when

$$(\lambda^2+\phi_{xx})(\lambda^2+\phi_{yy}) + (2\omega\lambda - \phi_{xy})(2\omega\lambda + \phi_{yx}) = 0$$

$$\lambda^4 + \lambda^2(\phi_{xx} + \phi_{yy} + 4\omega^2) + (\phi_{xx}\phi_{yy} - \phi_{xy}^2) = 0$$

For our potential, at L_4 and L_5

$$\phi_{xx} = -\frac{3}{4}\omega^2, \quad \phi_{yy} = -\frac{9}{4}\omega^2, \quad \phi_{xy} = \frac{3\sqrt{3}}{4}\left(\frac{m_1-m_2}{m_1+m_2}\right)\omega^2$$

$$\therefore \lambda^4 + \lambda^2\omega^2 + \frac{27}{4}\frac{m_1m_2}{(m_1+m_2)^2}\omega^4 = 0$$

$$\lambda^2/\omega^2 = -\frac{1}{2}\left[1 \pm \sqrt{1 - \frac{27m_1m_2}{(m_1+m_2)^2}}\right]$$

For $\frac{27m_1m_2}{(m_1+m_2)^2} < 1$, i.e. $\frac{m_1}{m_2} > \frac{25+3\sqrt{69}}{2} \approx 24.96$

both roots λ^2 are negative real numbers, so λ is imaginary, leading to oscillatory solutions (stable equilibrium)

Otherwise we have complex conjugate pair of λ^2, giving a solution with $\mathrm{Re}(\lambda) > 0$, unstable.

2.10 Charged particles in EM field

The eq^n of motion for a charged particle of mass m, charge q in a field $\underline{E}(\underline{r},t)$, $\underline{B}(\underline{r},t)$ is $\quad m\ddot{\underline{r}} = q(\underline{E} + \dot{\underline{r}} \times \underline{B})$

Can we obtain from a Lagrangian?

$\underline{E}, \underline{B}$ can be derived from scalar and vector potentials

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}, \quad \underline{B} = \underline{\nabla} \times \underline{A}$$

This ensures that they satisfy

$$\frac{\partial \underline{B}}{\partial t} = -\underline{\nabla} \times \underline{E} \quad \text{and} \quad \underline{\nabla} \cdot \underline{B} = 0$$

In terms of the potentials, the eq^n of motion is

$$\begin{aligned} m\ddot{x}_i &= q\left(-\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} + \varepsilon_{ijk}\dot{x}_j \varepsilon_{klm}\frac{\partial A_m}{\partial x_l}\right) \\ &= q\left(-\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} + \dot{x}_j\frac{\partial A_j}{\partial x_i} - \dot{x}_j \frac{\partial A_i}{\partial x_j}\right) \qquad (*) \end{aligned}$$

Consider the Lagrangian

$$L = \tfrac{1}{2}m|\dot{\underline{r}}|^2 + q(-\phi + \dot{\underline{r}}\cdot\underline{A})$$

This looks like $T - V$ (with $V = q\phi$) but with an extra term due to the magnetic field.

We have $\displaystyle p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i$

$$\frac{\partial L}{\partial x_i} = q\left(-\frac{\partial \phi}{\partial x_i} + \dot{x}_j \frac{\partial A_j}{\partial x_i}\right)$$

Lagrange's equation for x_i is

$$\frac{d}{dt}(m\dot{x}_i + qA_i) = q\left(-\frac{\partial \phi}{\partial x_i} + \dot{x}_j\frac{\partial A_j}{\partial x_i}\right)$$

$$\parallel$$

$$m\ddot{x}_i + q\frac{\partial A_i}{\partial x_j}\dot{x}_j + q\frac{\partial A_i}{\partial t} \qquad \text{rearrange, get } (*)$$

Note that, in the presence of a magnetic vector potential, the conjugate momentum $\underline{p} = m\dot{\underline{r}} - q\underline{A} \neq m\underline{v}$

Note also that $L \neq T - V$. This is typical in the case of velocity-dep forces.

2.11 Purely Kinetic Lagrangians and geodesics

As previously discussed, often possible to write kenetic term as

$$T = \frac{1}{2} T_{ij}(\underline{q})\, \dot{q}_i \dot{q}_j$$

where T_{ij} is a posdef, symmetric matrix and we use summation convention.

e.g. for N particles subject to scleronomic constraints

$$T_{ij} = \sum m_k \frac{\partial \underline{r}_k}{\partial q_i} \cdot \frac{\partial \underline{r}_k}{\partial q_j}$$

If no forces other than those of contraint, we have a purely kinetic Lagrangian, $L = T = \frac{1}{2} T_{ij}(\underline{q})\, \dot{q}_i \dot{q}_j$

As this is an autonomous system, the energy T is conserved.

The action integral $S = \int L\, dt = \int T\, dt = T \int dt$

is proportional to the time taken between the initial and final configuration, and should be minimized.

An important example is a single particle constrained to move on a surface. Here $n=2$ and (q_1, q_2) are general curvilinear coordinates on the surface.

$T = \text{constant} \Rightarrow$ constant $|\underline{v}|$, so the action is also proportional to the distance travelled.

Hamilton's principle implies that the particle follows a <u>geodesic</u> on the surface: a curve that minimizes the distance between two points

<u>Examples</u> A free particle in a plane follows a straight line

On a sphere, follow arc of great circle

In this case $\frac{T_{ij}}{m} = \frac{\partial \underline{r}}{\partial q_i} \cdot \frac{\partial \underline{r}}{\partial q_j} = g_{ij}$ is the <u>metric</u> (tensor) of surface

Lagrange's equations for such are

$$L = \frac{1}{2} T_{ij} \dot{q}_i \dot{q}_j \quad \Rightarrow \quad \frac{d}{dt}(T_{ij} \dot{q}_j) = \frac{1}{2} \frac{\partial T_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k$$

$$T_{ij} \ddot{q}_j = \left(\frac{1}{2} \frac{\partial T_{jk}}{\partial q_i} - \frac{\partial T_{ij}}{\partial q_k} \right) \dot{q}_j \dot{q}_k$$

In the case $T_{ij}/m = g_{ij}$ this can be rearranged into geodesic eq^n.

2.12 Small oscillations & stability

Consider $L = \frac{1}{2} T_{ij}(\underline{q})\, \dot{q}_i \dot{q}_j - V(\underline{q})$

Lagrange's eq^{n}s $\quad \frac{d}{dt}\left(T_{ij}\dot{q}_j\right) = \frac{1}{2}\frac{\partial T_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k - \frac{\partial V}{\partial q_i}$

An equilibrium config is a static solution $\underline{q} = \underline{q}_e = \text{const}$

It is therefore a stationary point of the potential energy $\frac{\partial V}{\partial \underline{q}} = 0$

For small departures from equilibrium, expand L to 2^{nd} order in $\underline{q}, \dot{\underline{q}}$, taking $\underline{q}_e = 0$ WLOG: $\quad L \approx L_2 = \frac{1}{2} T_{ij}(0)\dot{q}_i\dot{q}_j - V(0) - \frac{1}{2}V_{ij}(0)\, q_i q_j$

where $V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$

The constant term $V(0)$ may be omitted.

Now write $T_{ij}(0) = T_{ij}$, constant sym matrix. Likewise for V_{ij}

The quadratic approx to L gives linear approx to Lagrange

$$\frac{d}{dt}\left(T_{ij}\dot{q}_j\right) = -V_{ij}q_j \quad \text{or} \quad T_{ij}\ddot{q}_j + V_{ij}q_j = 0$$

These are n coupled linear homogeneous 2^{nd} order ODEs with const coeffs

The conserved energy is $E = \frac{1}{2}T_{ij}\dot{q}_j\dot{q}_i + \frac{1}{2}V_{ij}q_jq_i = \text{const}$

In matrix-vector notation $T\ddot{\underline{q}} + V\underline{q} = 0$

Seek solutions of the form $\underline{q}(t) = \underline{e}\, e^{-i\omega t}$, $\underline{e} \neq 0$; ω angular frequency

Satisfied if $-\omega^2 T\underline{e} + V\underline{e} = 0$,

generalized eigenvalue problem $V\underline{e} = \lambda T\underline{e}$

with eigenvalue $\lambda = \omega^2$ and eigenvector $\underline{e}$.

The eigenvalues λ are the roots of $\det(V - \lambda T) = 0$, poly deg n.

Claim The eigenvalues are real, eigenvectors are orthogonal in that

$\underline{e}^T T \underline{f} = 0$ for $\underline{e}, \underline{f}$ having distinct evals λ, μ

Proof $V\underline{e} = \lambda T \underline{e} \Rightarrow \underline{e}^\dagger V \underline{e} = \lambda \underline{e}^\dagger T \underline{e}$

Since V real symmetric, LHS is real.

Similarly $\underline{e}^\dagger T \underline{e}$ is real, and also positive from positive definite $\underline{e} \neq 0$

Hence $\lambda = \dfrac{\underline{e}^\dagger V \underline{e}}{\underline{e}^\dagger T \underline{e}}$ is real as desired.

The eigenvectors can also be taken to be real.

Now consider $V\underline{e} = \lambda T \underline{e}$, $V\underline{f} = \mu T \underline{f}$

$\Rightarrow \underline{e}^T V = \lambda \underline{e}^T T$

$\Rightarrow \underline{e}^T V \underline{f} = \lambda \underline{e}^T T \underline{f} = \mu \underline{e}^T T \underline{f}$

$\Rightarrow (\lambda - \mu) \underline{e}^T T \underline{f} = 0$ $\square$

Solutions $\underline{q} = \underline{e}\, e^{-i\omega t}$ are known as <u>normal modes</u>

If $\lambda = \omega^2 > 0$ then ω is real and the physical solution ~~solution~~ $\mathrm{Re}[\underline{e} e^{-i\omega t}]$

Represents <u>simple-harmonic motion</u> with angular freq $\omega > 0$ (WLOG)

If $\lambda < 0$, then $\omega = \pm i \sqrt{-\lambda}$ is imaginary. The + sol^n grows exponentially while the − sol^n decays. $\exists$ of growing solution $\Rightarrow$ equilibrium <u>unstable</u>

In the stable case (all $\lambda > 0$) the general sol^n involves a lin combi of normal modes with diff frequencies. If the frequencies are incommesurate then the motion is not strictly periodic but <u>quasiperiodic</u>

$$\begin{cases} T\ddot{\underline{q}} + V\underline{q} = 0 \\ \underline{q} = \underline{e}\, e^{-i\omega t} \end{cases}$$

Assume eigenvectors normalized $\underline{e}_i^T T \underline{e}_j = \delta_{ij}$

Then $\underline{e}_i^T V \underline{e}_j = \lambda_j \delta_{ij} = \omega_j^2 \delta_{ij}$ (no sum)

Since eigenvectors form a basis, general sol^n

$$\underline{q} = \sum Q_i(t)\, \underline{e}_i$$

with mode amplitudes $Q_i = \underline{e}_i^T T \underline{q}$ known as <u>normal coordinates</u>.

The normal coordinates satisfy $\ddot{Q}_i + \omega_i^2 Q_i = 0$

The kinetic, potential energies are

$$T = \tfrac{1}{2} \dot{\underline{q}}\, T \dot{\underline{q}} = \tfrac{1}{2} \sum_i \sum_j \dot{Q}_i \dot{Q}_j\, \underline{e}_i^T T \underline{e}_j = \tfrac{1}{2} \sum_i \dot{Q}_i^2$$

$$V = \tfrac{1}{2} \underline{q}^T V \underline{q} = \tfrac{1}{2} \sum_i \sum_j Q_i Q_j\, \underline{e}_i^T V \underline{e}_j = \tfrac{1}{2} \sum_i \omega_i^2 Q_i^2$$

L8.3

So Lagrangian reduces to

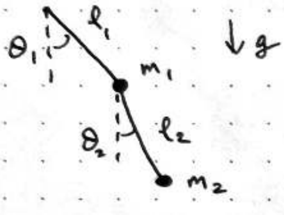
$$L = T - V = \sum \tfrac{1}{2}\left(\dot{Q}_i^2 - \omega_i^2 Q_i^2\right)$$

and the conserved total energy is

$$E = T + V = \sum \tfrac{1}{2}\left(\dot{Q}_i^2 + \omega_i^2 Q_i^2\right) = \sum E_i$$

In fact each mode oscillates independently and each E_i is separately conserved

<u>Ex</u> Double pendulum



Exact Lagrangian – see ExSh 1.4

For small oscillations,

$$L \approx L_2 = \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2$$
$$- \tfrac{1}{2}(m_1 + m_2) g l_1 \theta_1^2 - \tfrac{1}{2} m_2 g_2 l_2 \theta_2^2$$

Thus
$$T = \begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$$

$$V = \begin{pmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}$$

In the case $m_1 = m_2 = m$, $l_1 = l_2 = l$, the evals satisfy

$$\det(V - \lambda T) = \begin{vmatrix} 2mgl - 2ml^2\lambda & -ml^2\lambda \\ -ml^2\lambda & mgl - ml^2\lambda \end{vmatrix}$$

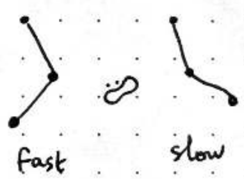
$$= 2m^2 l^2 (g - l\lambda)^2 - m^2 l^4 \lambda^2 = 0$$

$$\omega^2 = \lambda = (2 \pm \sqrt{2})\, g/l$$

Both roots are +ve, so equilibrium is stable

The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}$

Upper and lower signs : fast and slow modes



The general solution will be a linear combi ; the freq ratio is $1 + \sqrt{2}$

So motion is quasiperiodic

Solution of exercise

Section 3 Motion of a Rigid Body

3.1 Rotating frames of reference

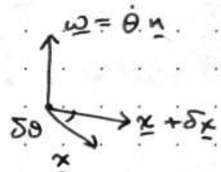
The rotation of a vector through an angle θ about an axis parallel to $\underline{n}$ described by the linear map

$$\underline{x} \mapsto \underline{x} + \sin\theta\, \underline{n} \times \underline{x} + (1-\cos\theta)\, \underline{n} \times (\underline{n} \times \underline{x})$$

For an infinitesimal rotation by $\delta\theta$, we have $\delta \underline{x} = \delta\theta\, \underline{n} \times \underline{x}$.

The rate of change of a vector due to a time-dep rotation is hence

$$\dot{\underline{x}} = \lim_{\delta t \to 0} \frac{\delta \underline{x}}{\delta t} = \underline{\omega} \times \underline{x} \qquad \text{where } \underline{\omega} = \dot{\theta}\, \underline{n}$$



is the angular velocity vector.

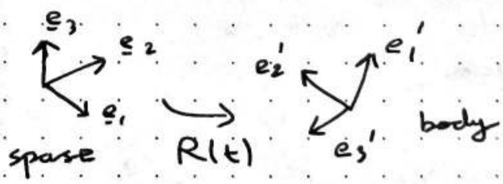
Consider two right-handed orthonormal bases in $\mathbb{R}^3$

- The space frame, an inertial frame of reference with time-independent basis vectors $\underline{e}_i$ $(i=1,2,3)$
- The body frame, a generally non-inertial frame with time-dependant basis vectors $\underline{e}_a'$ $(a=1,2,3)$. It is related to the space frame by a time-dependent rotation.

We have $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, $[\underline{e}_i, \underline{e}_j, \underline{e}_k] = \varepsilon_{ijk}$

The two bases are related by a time-dependent proper orthogonal matrix $R(t)$ with $R^{-1} = R^T$ and $\det(R) = 1$:

$$\underline{e}_a' = R_{ai}\, \underline{e}_i \quad , \quad \underline{e}_i = R_{ai}\, \underline{e}_a'$$



Orthonormality is preserved because

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \Rightarrow \underline{e}_a' \cdot \underline{e}_b' = R_{ai} R_{bj}\, \underline{e}_i \cdot \underline{e}_j = R_{ai} R_{bi} = \delta_{ab}$$

Right-handedness is preserved because

$$[\underline{e}_i, \underline{e}_j, \underline{e}_k] = \varepsilon_{ijk} \Rightarrow [\underline{e}_a', \underline{e}_b', \underline{e}_c'] = R_{ai} R_{bj} R_{ck}\, [\underline{e}_i, \underline{e}_j, \underline{e}_k]$$

$$= \varepsilon_{ijk} R_{ai} R_{bj} R_{ck} = \varepsilon_{abc} \underbrace{\det(R)}_{1}$$

L9.2

Consider the time derivatives of the body basis vectors in the space frame

$$\dot{\underline{e}}'_a = \dot{R}_{ai}\,\underline{e}_i = \dot{R}_{ai} R_{bi}\,\underline{e}'_b = A_{ab}\,\underline{e}'_b$$

where $A_{ab} = \dot{R}_{ai} R_{bi}$ or $A = \dot{R}R^T$ is antisymmetric

$$A + A^T = \dot{R}R^T + R\dot{R}^T = \frac{d}{dt}(RR^T) = 0$$

We would like to interpret the equation

$$\dot{\underline{e}}'_a = A_{ab}\,\underline{e}'_b \quad \text{as} \quad \dot{\underline{e}}'_a = \underline{\omega} \times \underline{e}'_a$$

where $\underline{\omega}(t) = \omega_i \underline{e}_i = \omega'_a \underline{e}'_a$ is the angular velocity of the body frame with respect to the space frame.

Associated with 3×3 anti-sym matrix have a 3-vector

$$A_{ab} = \varepsilon_{abc}\,\omega'_c \quad , \quad \omega'_a = \tfrac{1}{2}\varepsilon_{abc} A_{bc}$$

So we have $\dot{\underline{e}}'_a = A_{ab}\,\underline{e}'_b$

$$= \varepsilon_{abc}\,\omega'_c\,\underline{e}'_b$$

$$= \omega'_c\,\underline{e}'_c \times \underline{e}'_a$$

$$= \underline{\omega} \times \underline{e}'_a$$

with angular velocity (body frame components)

$$\omega'_c = \tfrac{1}{2}\varepsilon_{abc}\,\dot{R}_{ai} R_{bi}$$

Let $\underline{v} = v_i \underline{e}_i = v'_a \underline{e}_a$ be a time dep vector with components $v_i(t)$ or $v'_a(t)$.

In space frame,

$$\dot{\underline{v}} = \dot{v}_i \underline{e}_i = \dot{v}'_a \underline{e}'_a + v'_a\,\underline{\omega} \times \underline{e}'_a$$

$$= \dot{v}'_a \underline{e}'_a + \underline{\omega} \times \underline{v}$$

Therefore time-derivatives of vectors in the space, body frames are related by

$$\left(\frac{d\underline{v}}{dt}\right)_{space} = \left(\frac{d\underline{v}}{dt}\right)_{body} + \underline{\omega} \times \underline{v}$$

<u>Exercise</u> Show that $\omega_k = \tfrac{1}{2}\varepsilon_{ijk} R_{ai}\dot{R}_{aj}$, which is the vector (index not^n) HARD! associated with the antisymm matrix $R^T\dot{R}$ rather than $\dot{R}R^T$.

In the previous episode

$$\omega_c' = \frac{1}{2}\varepsilon_{abc}\dot{R}_{ai}R_{bi}$$

To ensure $\omega_k \underline{e}_k = \omega_c' \underline{e}_c' = \underline{\omega}$, we have

$$\omega_k = R_{ck}\omega_c' = \frac{1}{2}\varepsilon_{abc}\dot{R}_{aj}R_{bj}R_{ck}$$

Now $\varepsilon_{abc}R_{ai}R_{bj}R_{ck} = \varepsilon_{ijk}$ since $\det(R) = 1$.

Multiply by R_{di} and use $R_{ai}R_{di} = \delta_{ad}$:

$$\varepsilon_{dbc}R_{bj}R_{ck} = \varepsilon_{ijk}R_{di}$$

Relabel $d \mapsto a$, substitute

$$\omega_k = \frac{1}{2}\varepsilon_{ijk}R_{ai}\dot{R}_{aj} = \frac{1}{2}\varepsilon_{ijk}(R^T\dot{R})_{ij} \qquad \square$$

3.2 The inertia tensor

The general motion of a rigid body consists of translation and rotation. Here we consider the rotational part.

If a rigid body is instantaneously rotating with angular velocity $\underline{\omega}$ about an axis through the origin (wlog), then particles in the body have velocities

$$\dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i$$

This ensures that the distance between any pair of particles remains fixed.

The total angular momentum about the origin is

$$\begin{aligned}\underline{L} &= \sum_{i=1}^{N} m_i \underline{r}_i \times \dot{\underline{r}}_i \\ &= \sum_{i=1}^{N} m_i \underline{r}_i \times (\underline{\omega}\times\underline{r}_i) \\ &= \sum_{i=1}^{N} m_i \left(|\underline{r}_i|^2\underline{\omega} - (\underline{r}_i\cdot\underline{\omega})\,\underline{r}_i\right) \\ &= \underline{\underline{I}}\,\underline{\omega}\end{aligned}$$

where $\underline{\underline{I}}$ is the inertia tensor with components

$$\begin{aligned}I_{ab} &= \sum_{i=1}^{N} m_i\left(|\underline{r}_i|^2\delta_{ab} - x_{ia}x_{ib}\right) \\ &= \sum_{i=1}^{N} m_i \begin{pmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -x_iy_i & x_i^2+z_i^2 & -y_iz_i \\ -x_iz_i & -y_iz_i & x_i^2+y_i^2 \end{pmatrix}\end{aligned}$$

For a continuous body, replace mass-weighted sum with a mass weighted integral $I_{ab} = \int\left(|\underline{r}|^2\delta_{ab} - x_ax_b\right)dm$ where dm is mass element

The rotational kinetic energy also involves $\underline{\underline{I}}$

$$
\begin{aligned}
T &= \sum_{i=1}^{N} \tfrac{1}{2} m_i |\dot{\underline{r}}_i|^2 \\
&= \sum_{i=1}^{N} \tfrac{1}{2} m_i (\underline{\omega} \times \underline{r}_i)\cdot(\underline{\omega} \times \underline{r}_i) \\
&= \sum_{i=1}^{N} \tfrac{1}{2} m_i\, \underline{\omega} \cdot (\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)) \\
&= \tfrac{1}{2} \underline{\omega} \cdot (\underline{\underline{I}}\, \underline{\omega}) \\
&= \tfrac{1}{2} I_{ab}\, \omega_a \omega_b
\end{aligned}
$$

I is symmetric, and positive definite, so it has three orthogonal eigenvectors and three positive eigenvalues

These define the <u>principal axes</u> and the principal moments of inertia

Wrt the principal axes, $\underline{\underline{I}} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$

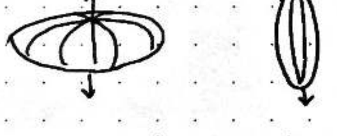
Natural to identify body frame with the principal axes.

Differently shaped bodies with the same inertia tensors have identical rotational dynamics and are said to be inertially similar.

Rigid bodies can be classified based on their inertia tensors:

- Spherical top $I_1 = I_2 = I_3$
- Symmetric top $I_1 = I_2 \neq I_3$

  Either oblate ($I_3 > I_1$) or prolate ($I_3 < I_1$)



- Asymmetric top $I_3 > I_2 > I_1$

A degenerate limit of a rigid body is a <u>linear rotor</u>, which is a linear configuration of particles along a single axis. In this case the moment of inertia about this axis vanishes and the inertia tensor is only positive semi-definite.

<u>Ex</u> Uniform solid cuboid of sides a, b, c about its centre

$$
I_{xx} = \frac{M}{abc} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\, dx\, dy\, dz
$$

$$
= \frac{M}{abc} \left(\frac{2}{3}\left(\frac{b}{2}\right)^3 ac + \frac{2}{3}\left(\frac{c}{2}\right)^3 ab \right) = \frac{M}{12}(b^2 + c^2)
$$

$$
I_{xy} = \frac{M}{abc} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (-xy)\, dx\, dy\, dz = 0 \quad \text{, etc.}
$$

L10.3

$$I = \frac{M}{12}\begin{pmatrix} b^2+c^2 & & \\ & a^2+c^2 & \\ & & a^2+b^2 \end{pmatrix}$$

Since a real symm matrix can be diagonalized by a rotation, even a completely asymmetric body is inertially similar to a cuboid

<u>Ex</u> Spherically symmetric body of density $\rho(r)$

$$I_{ab} = \int \left(|\underline{r}|^2 \delta_{ab} - x_a x_b \right) \rho \, dV$$

Because of rotational symmetry, I_{ab} must be isotropic:

$$I_{ab} = I \, \delta_{ab}$$

Find I from the trace

$$3I = I_{aa} = \int (3r^2 - r^2) \rho \, dV = \int_0^R 2r^2 \rho \, 4\pi r^2 dr$$

$$I_{ab} = \delta_{ab} \frac{8\pi}{3} \int_0^R \rho r^4 dr$$

e.g. for uniform sphere $M = \rho \frac{4\pi R^3}{3}$,

$$I_{ab} = \frac{2}{5} M R^2 \delta_{ab}$$

<u>Natural examples</u>

<u>Sun</u> Nearly spherical but very centrally concentrated:

$$I_1 \approx I_2 \approx I_3 \approx 0.0735 \, MR^2$$

small

<u>Earth</u> Nearly an oblate spheroid (rot flattening) and not uniform:

$$I_1 \approx I_2 \approx 0.3296 \, MR^2$$

$$I_3 \approx 0.3307 \, MR^2$$

<u>Moon</u> Nearly uniform but measurably asymmetric (frozen deformation)

$$I_1 \approx 0.3926 MR^2, \quad I_2 \approx 0.3927 MR^2, \quad I_3 \approx 0.3929 MR^2$$

L10.4

Parallel axis theorem

The inertia tensor depends on the choice of origin (which is on the rotation axis).

The inertia tensor I^P_{ab} wrt any point P is related to the inertia tensor I^C_{ab} wrt the centre of mass C by

$$I^P_{ab} = I^C_{ab} + M(|\underline{c}|^2\delta_{ab} - c_a c_b)$$

where M is the total mass and $\underline{c}$ is the position vector of P wrt C.

Proof (sketch) Taking the origin at C, we have

$$I^P_{ab} = \int (|\underline{r}-\underline{c}|^2\delta_{ab} - (x_a - c_a)(x_b - c_b))\, dm$$

When this is expanded out, the terms linear in $\underline{r}$ integrate to zero because of the definition of C. $\square$

Note that the added term $M(|\underline{c}|^2\delta_{ab} - c_a c_b)$ is just the inertia tensor about P of a point mass M located at C.

3.3 Kinematics of a rigid body

A rigid body is a system of particles constrained such that

$$|\underline{r}_i - \underline{r}_j| = \text{const}$$

The possible motions of a rigid body are the continuous isometries of Euclidean space: translations & rotations

It has six degrees of freedom of freedom and its config space is

$$\mathbb{R}^3 \times SO(3)$$

The general motion can be considered as a translation of the centre of mass plus a rotation about the centre of mass:

$$\dot{\underline{r}}_i = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_i^C \quad \text{where } \underline{r}_i^C = \underline{r}_i - \underline{R}$$

L10.5

The total angular momentum about the origin decomposes into translational plus rotational parts

$$
\begin{aligned}
\underline{L} &= \sum_{i=1}^{N} m_i \underline{r}_i \times \dot{\underline{r}}_i \\
&= \sum_{i=1}^{N} m_i (\underline{R} + \underline{r}_i^c) \times (\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i^c) \\
&= \sum_{i=1}^{N} m_i \underline{R} \times \dot{\underline{R}} + \sum_{i=1}^{N} m_i \underline{r}_i^c \times (\underline{\omega} \times \underline{r}_i^c) \\
&= M \underline{R} \times \dot{\underline{R}} + \underline{\underline{I}}^c \underline{\omega}
\end{aligned}
$$

Terms linear in $\underline{r}_i^c$ drop out because of def^n of $\underline{R}$, $\sum_{i=1}^{N} m_i \underline{r}_i^c = 0$

Similarly, total kinetic energy is

$$
\begin{aligned}
T &= \sum_{i=1}^{N} \tfrac{1}{2} m_i |\dot{\underline{r}}_i|^2 \\
&= \sum_{i=1}^{N} \tfrac{1}{2} m_i |\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i^c|^2 \\
&= \sum_{i=1}^{N} \tfrac{1}{2} m_i |\dot{\underline{R}}|^2 + \sum_{i=1}^{N} \tfrac{1}{2} m_i |\underline{\omega} \times \underline{r}_i^c|^2 \\
&= \tfrac{1}{2} M |\dot{\underline{R}}|^2 + \tfrac{1}{2} \underline{\omega}^T \underline{\underline{I}}^c \underline{\omega}
\end{aligned}
$$

L11.1

3.4 The Euler top

The Euler-top refers to the torque-free motion of a general rigid body.

- In a uniform gravitational field $\underline{g}$, the next-ernal force on the body is $\underline{F} = \sum m_i \underline{g} = M\underline{g}$

and the net torque about the centre of mass is

$$\underline{G}^c = \sum \underline{r}_i^c \times (m_i \underline{g}) = 0$$

The motion of a rigid body in a uniform gravitational field therefore consists of an accelerated motion of the centre of mass, $\ddot{\underline{R}} = \underline{g}$ and a rotation about the centre of mass with constant angular momentum

$\underline{L} = \text{const}$

- For a spherical top, $I_{ab} = I\delta_{ab}$, we have $\underline{L} = \underline{\underline{I}}\,\underline{\omega} = I\underline{\omega}$,

so $\underline{L} = \text{const} \Rightarrow \underline{\omega} = \text{const}$

In other cases the motion is generally much more interesting

We now abandon the unprimed/primed component notation.

The space axes will be called xyz and the body axes will be 123

When $\underline{\underline{I}}$ is not isotropic, $\underline{L}$ is not generally parallel to $\underline{\omega}$.

- In the space frame (inertial), $\underline{L}$ is constant, but $\underline{\underline{I}}$ need not be, because the body is spinning, so $\underline{\omega}$ need not be constant.

The time-derivatives are related by

$$0 = \left(\frac{dL}{dt}\right)_{space} = \left(\frac{\partial L}{\partial t}\right)_{body} + \underline{\omega} \times \underline{L}$$

Both sides vanish because $\underline{L}$ is constant in an inertial frame.

In the body frame, components of $\underline{L}$ time dep. But $\underline{\underline{I}}$ is diagonal and constant whence $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$

We obtain $I_1 \dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2 = 0$ etc.

which are Euler's equations

$$\begin{cases} I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 \\ I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 \end{cases}$$

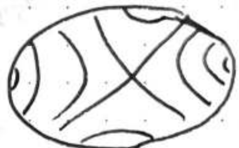
Two quantities are conserved: the squared ang mom

$$L^2 = L_1^2 + L_2^2 + L_3^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2$$

and the (purely kinetic) energy

$$E = T = \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2$$
$$= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

The point with position vector $\underline{L}(t)$ therefore lies on the intersection of a sphere (of radius $|\underline{L}|$) and an ellipsoid (of semi-axes $\sqrt{2EI_i}$)


Sussman & Wisdom

The asymmetric top

The non-trivial equilibria of Euler's equations correspond to rotation about a principal axis (e.g. $\omega_1 \neq 0$, $\omega_2 = \omega_3 = 0$)

Take $I_3 > I_2 > I_1$ wlog. Then

$$I_1 \leq \frac{L^2}{2E} \leq I_3$$

with equalities corresponding to rotation about the long and short axes

$$L^2 = L_1^2 + L_2^2 + L_3^2$$
$$E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

Rotation about the long axis (smallest I_1) maximises E at fixed $|\underline{L}|$, so is stable (huh)

Rotation about the short axis (largest I_3) maximises $|\underline{L}|$ at fixed E, so is stable (woah)

L 11.3

Rotation about the intermediate axis is <u>unstable</u>.

The curves of intersection near this equilibrium are hyperbolic.

A small initial departure from equilibrium leads to a large-amplitude periodic excursion.

Sometimes known as the Tennis Racket Theorem!

To analyse the stability about $\underline{e}_3$, treat ω_1, ω_2 as small quantities

To first order, Euler's eq^{n}s give

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$

$$I_3 \dot{\omega}_3 = 0 \quad \Rightarrow \omega_3 = \text{const.}$$

and so $\ddot{\omega}_1 = C_3\omega_1$, $\ddot{\omega}_2 = C_3\omega_2$

with
$$C_3 = \frac{(I_2 - I_3)(I_3 - I_1)\omega_3^2}{I_1 I_2}$$

With $I_3 > I_2 > I_1$, have $C_3 < 0$ and so stable oscillations

Permuting indices to consider other axes, we have

$$C_1 < 0 \text{ (stable)} \quad \text{and} \quad C_2 > 0 \text{ (unstable)}$$

As a result of dissipation, astronomical bodies generally rotate about the short axis, which minimizes energy at fixed $|\underline{L}|$.

But many smaller asteroids are tumbling.

e.g. Oumuamua, the first interstellar asteroid discovered

The symmetric top

If $I_1 = I_2 \neq I_3$, Euler's equations reduce to

$$I_1\dot{\omega}_1 = (I_1 - I_3)\,\omega_2\omega_3$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\,\omega_3\omega_1$$

$$I_3\dot{\omega}_3 = 0$$

which imply $\omega_3 = \text{const}$ and

$$\dot{\omega}_1 = -\Omega_p\,\omega_2$$

$$\dot{\omega}_2 = +\Omega_p\,\omega_1 \qquad \text{where } \Omega_p = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3$$

The general solution

$$\omega_1 = A\cos(\Omega_p t + B), \quad \omega_2 = A\sin(\Omega_p t + B)$$

represents uniform precession of the angular velocity vector around the symmetry axis $\underline{e}_3$

Precession means rotation of the spin axis about another axis

This precession is prograde (in the same sense as the spin) if the body is oblate $(I_3 > I_1)$ and retrograde if the body is prolate

EX For the Earth, $\left(\frac{I_3 - I_1}{I_1}\right) \approx \frac{1}{300}$ so the free precession of the Earth's spin axis should have a period of about 300 days.

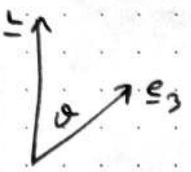
The observed Chandler wobble has a period of 433 days because the Earth is not a rigid body.

L11.5

For a symmetric top

$$2E = \frac{L_1^2 + L_2^2}{I_1} + \frac{L_3^2}{I_3} = |\underline{L}|^2\left(\frac{\sin^2\vartheta}{I_1} + \frac{\cos^2\vartheta}{I_3}\right)$$

where ϑ is the angle between $\underline{L}$ and $\underline{e}_3$



So $E = \text{const} \Rightarrow \vartheta = \text{const}$

Hence precession is the only possibility, and precession must be uniform because of axial symmetry.

3.5 Poinsot's construction

Euler's equations for a free asymmetric top can be solved analytically in terms of Jacobian elliptic functions (see Landau & Lifshitz)

Here we examine a geometrical ~~gn~~ interpretation of the motion due to Poinsot.

The evolution of the angular velocity vector corresponds to the motion of a point in $\mathbb{R}^3$ with position vector $\underline{\omega}(t)$. The curve traced by this point is constrained by conservation of angular momentum & energy:

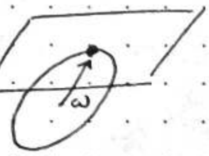
The kinetic energy is quadratic in $\underline{\omega}$, $T = \frac{1}{2}\underline{\omega}\cdot(\underline{\underline{I}}\,\underline{\omega})$

with gradient $\underline{\nabla}_\omega T = \underline{\underline{I}}\,\underline{\omega}$

Conservation of energy means that $\underline{\omega}$ is constrained to lie on the ellipsoid $T = E = \text{const}$, known as the <u>inertia ellipsoid</u>. It is centred on the origin, is aligned with the body axes, and has semi-axes $\sqrt{2E/I_i}$

We also have $\underline{\omega}\cdot\underline{L} = 2E$,

which means that $\underline{\omega}$ is constrained to lie in a plane. It is $\perp \underline{L}$ and located at a distance $2E/|\underline{L}|$ from the origin.



Viewed in the space frame, the vector $\underline{L}$ is constant (because no torque acts and frame is inertial), so the plane is stationary; known as the <u>invariable plane</u>

The ellipsoid, however, rotates in the same way as the body (it can be thought of as being physically attached to the body).

The ellipsoid is tangent to the plane at the point $\underline{\omega}$ because the normal to the ellipsoid is parallel to $\underline{\nabla}_\omega T = \underline{\underline{I}}\,\underline{\omega} = \underline{L}$, which is also normal to the plane.

The point of contact traces out a curve called the <u>herpolhode</u> on the invariable plane and a curve called the <u>polhode</u> on the inertia ellipsoid.

L12.2

Since the centre of the centre of the ellipsoid is fixed, the velocity of the point of contact is $\underline{\omega} \times \underline{\omega} = 0$; the first $\underline{\omega}$ here is the angular velocity of the ellipsoid and the second is the position vector of the point of contact. Therefore the ellipsoid rolls on the plane w/o slipping.

In the case of a symmetric top the ellipsoid is a spheroid and the polhode and herpolhode are circular, corresponding to uniform precession of $\underline{\omega}$ around $\underline{L}$.

For an asymmetric top the rolling motion of the inertia ellipsoid is non-uniform.

3.6 Euler angles

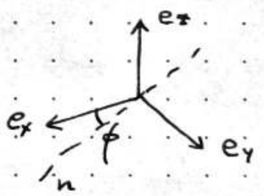
An explicit representation of the rotation group $SO(3)$, providing a generalized coordinate system for a rigid body that is fixed at one point.

Euler showed that an arbitrary rotation can be composed from three elementary rotations (i.e. about coord axes)

This is one of the conventional definitions:

- First rotate through an angle ϕ about the z-axis of the space frame

$$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$



- Now rotate through an angle θ about the new x-axis (called line of nodes)

$$R_n(\theta) = \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix}$$

- Then rotate through an angle ψ about the new z-axis (which becomes the 3-axis of the body frame):

$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1 \end{pmatrix}$$

L12.3

The result is $R = R_3(\psi)\, R_n(\theta)\, R_z(\phi)$

$$= \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$\frac{\pi}{2} - \theta$ and $\phi - \frac{\pi}{2}$ give the latitude and longitude of the 3-axis, while ψ is a rotational angle measured around that axis

ψ and ϕ are defined modulo 2π, while $\theta \in [0, \pi]$

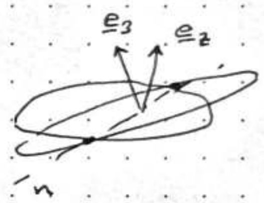
Intuitively, thinking about the meanings of the three elemental ~~rotates~~ rotations, the angular velocity vector is

$$\underline{\omega} = \dot{\psi}\, \underline{e}_3 + \dot{\theta}\, \underline{e}_n + \dot{\phi}\, \underline{e}_z$$

where the unit vector parallel to the line of nodes is

$$\underline{e}_n = \underline{e}_x \cos\phi + \underline{e}_y \sin\phi = \underline{e}_1 \cos\psi - \underline{e}_2 \sin\psi$$

From the diagram,



$$\underline{e}_3 = \underline{e}_x \sin\theta\sin\phi - \underline{e}_y \sin\theta\cos\phi + \underline{e}_z \cos\theta$$

and conversely

$$\underline{e}_z = \underline{e}_1 \sin\theta\sin\psi + \underline{e}_2 \sin\theta\cos\psi + \underline{e}_3 \cos\theta$$

And so the angular velocity is given in the space frame by

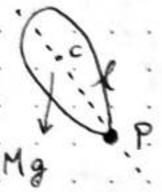
$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix}$$

and in the body frame by

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{pmatrix}$$

§3.7 The Lagrange top

The **Lagrange top** refers to a symmetric top fixed at a point on the axis of symmetry (not the c.o.m.) and subject to a gravitational torque



Let M be the mass, ℓ the distance of the fixed point P from the centre of mass C along the axis of symmetry

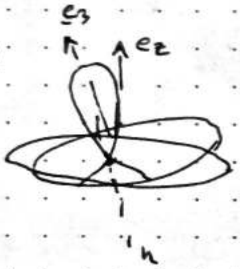
We have $I_1^P = I_2^P = I_1^C + M\ell^2$ by the Parallel Axis Theorem

The kinetic energy is

$$T = \tfrac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \tfrac{1}{2} I_3 \omega_3^2$$

Using our expressions for $(\omega_1, \omega_2, \omega_3)$ we have

$$T = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3(\dot\psi + \dot\phi\cos\theta)^2$$



ϕ rotates n horizontally
θ rotates e_3 about n
ψ rotates about e_3.

The potential energy is given by the height of the c.o.m.

$$V = MgZ = Mg\ell\cos\theta$$

So the energy of the Lagrange top is

$$E = T + V = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2} I_3(\dot\psi + \dot\phi\cos\theta)^2 + Mg\ell\cos\theta$$

and its Lagrangian is $L = T - V$.

There are three degrees of freedom, but both ψ and ϕ are ignorable.

The corresponding conserved quantities are

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3 = L_3$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\,\dot\phi + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta = L_z$$

L_z conserved because system invariant under a rotation about vertical

L_3 is conserved because of the axial symmetry of the top

E is also conserved

What does it mean for L_3 to be conserved when the 3 axis changes in time?

In vector terms, $\dot{\underline{L}} = \underline{G}$ and $L_3 = \underline{L}\cdot\underline{e}_3(t)$, so

$$\dot{L}_3 = \underline{G}\cdot\underline{e}_3 + \underline{L}\cdot\dot{\underline{e}}_2$$

The gravitational torque is

$$\underline{G} = l\, \underline{e}_3 \times (-Mg\underline{e}_z)$$

so $\underline{G}\cdot\underline{e}_3 = 0$. The other term vanishes because $\dot{\underline{e}}_3 = \underline{\omega}\times\underline{e}_3$ and $\underline{L}, \underline{\omega}, \underline{e}_3$ are coplanar for a symmetric top (exercise). nice ✓

The possible motions of the top are

- spin ($\omega_3 = \dot\psi + \dot\phi\cos\theta$)
- precession ($\dot\phi$)
- nutation ($\dot\theta$) "nodding XD"

Nutation and precession

$$p_\psi = I_3(\dot\psi + \dot\phi\cos\theta)$$

$$p_\phi = I_1\dot\phi\sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta$$

Define constants a and b by $p_\psi = I_1 a$, $p_\phi = I_1 b$.

Then $\dot\phi = \dfrac{b - a\cos\theta}{\sin^2\theta}$, $\dot\psi = \dfrac{I_1}{I_3}a - \dot\phi\cos\theta$

and we aim to solve for $\theta(t)$.

The energy equation simplifies to

$$E - \tfrac{1}{2}I_3\omega_3^2 = E' = \tfrac{1}{2}I_1\dot\theta^2 + \tfrac{1}{2}I_1\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

which defines a new constant E', the reduced energy

The dynamics reduces to 1D motion in the effective potential

$$V_{\text{eff}}(\theta) = \frac{1}{2}I_1\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

To analyse this, let $u = \cos\theta$ and define constants

$$\alpha = \frac{2E'}{I_1}, \quad \beta = \frac{2Mgl}{I_1} > 0$$

Then the energy equation becomes

$$\dot u^2 = f(u)$$

where $f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$ is a cubic. We also have

$$\dot\phi = \frac{b - au}{1 - u^2}, \quad \dot\psi = \frac{I_1}{I_3}a - \dot\phi u$$

The physical system exists where $f(u) \geq 0$ and $-1 \leq u \leq 1$.

(If the top is spinning on a table, however, the top will hit the table when u falls below some positive value.)

For large $|u|$ the βu^3 dominates, and since $\beta > 0$; $f(u) \to \pm\infty$ as $u \to \pm\infty$

At this stage the cubic could have either one real root or three.

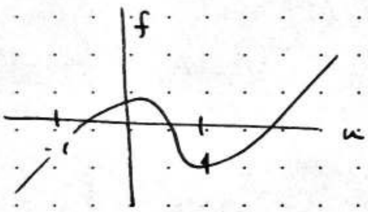
$\ulcorner f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2 \lrcorner$

We have $f(\pm 1) = -(b \mp a)^2 \leq 0$.

On the other hand, $f > 0$ for some range of u within $-1 < u < 1$, where the physical system moves and $\dot{u}^2 > 0$ (except in the equilibrium case $u = \text{const}$ seen later)

We also know that $f > 0$ for large u

Thus f must have 3 real roots, one of which is in $u > 1$, and two in $-1 \leq u \leq 1$.

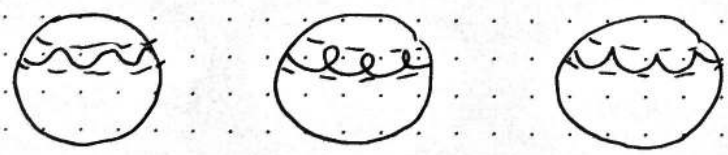


Let $u_1 < u_2$ be the two roots in $-1 \leq u \leq 1$.
Then we have a periodic oscillation of u (and of θ) in the range $u_1 \leq u \leq u_2$
This is the <u>nutation</u> of the top.

As u oscilliates, so does the precession rate $\dot{\phi} = \dfrac{b - au}{1-u^2}$

There are three types of possible motion:

- 'Wavy': nutation with consistent procession ($\dot{\phi} > 0$ or < 0 throughout)
- 'Loopy': nutation with mixed prograde and retrograde precession ($\dot{\phi}$ changes sign)
- 'Cuspy': nutation with halting precession



The last one is what naturally occurs if the spinning top is released with $\dot{\theta} = \dot{\phi} = 0$ from some intermediate θ.

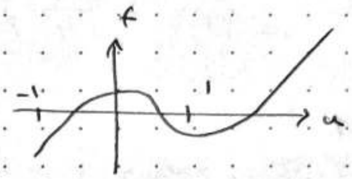
(Recap of the Lagrange top)

$$L = \tfrac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta$$

$$p_\psi = I_1 a = I_3 (\dot\psi + \dot\phi \cos\theta)$$

$$p_\phi = I_1 b = I_1 \dot\phi \sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta$$

$$V_{eff}(\theta) = \tfrac{1}{2} I_1 \left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$



$$\dot u^2 = f(u) \;, \quad u = \cos\theta$$

$$f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$$

Uniform precession without nutation

$\theta = \text{const}$ possible if the roots coincide, $u_1 = u_2$

Easier to analyse by setting $V'_{eff}(\theta) = 0$

$$I_1 \left(\frac{b - a\cos\theta}{\sin\theta}\right)\left[a - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta}\right] - Mgl\sin\theta = 0$$

Assuming $\sin\theta \neq 0$ (see below), this implies

$$\dot\phi\,(p_\psi - I_1 \dot\phi \cos\theta) - Mgl = 0 ,$$

a quadratic for $\dot\phi$. This has two real solutions (fast and slow precession) if

$$p_\psi^2 > 4\,Mgl\,I_1 \cos\theta \qquad (*)$$

To find a solution with $0 < \theta < \frac{\pi}{2}$ ($\cos\theta > 0$), we need the top to have sufficient spin. It can then process uniformly.

To set this up, we need to push it in the azimuthal direction at the right rate

(Condition $(*)$ is trivially satisfied if $\theta > \frac{\pi}{2}$, i.e. if the top hangs below the pivot, but may not be physically allowed)

$$\dot\phi\,(p_\psi - I_1 \dot\phi \cos\theta) - Mgl = 0$$

The fast and slow precession rates are

$$\dot\phi_\pm = \frac{p_\psi \pm \sqrt{p_\psi^2 - 4Mgl I_1 \cos\theta}}{2 I_1 \cos\theta}$$

and are both prograde (same sign as spin).

In the limit of a <u>fast</u> top, $p_\psi^2 \gg 4 Mgl I_1 \cos\theta$, we have

$$\dot\phi_+ \approx \frac{p_\psi}{I_1 \cos\theta} = \frac{I_3}{I_1\cos\theta}\,\omega_3\,, \qquad \dot\phi_- \approx \frac{Mgl}{p_\psi}$$

The slow solution is easier to observe in practice

<u>The sleeping top</u>

$$V_{\text{eff}}(\theta) = \frac{1}{2} I_1 \left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

To reach $\theta = 0$, we require $p_\psi = p_\phi$, so $a = b$

Expanding V_{eff} to second order in θ in this case gives

$$V_{\text{eff}} \approx \frac{1}{8} I_1 a^2 \theta^2 + Mgl\left(1 - \frac{1}{2}\theta^2\right)$$

This is a minimum (stable equilibrium) if

$$p_\psi^2 > 4 Mgl I_1$$

In practice, friction slows the top until this condition is no longer satisfied, then the top falls over.

<u>The free symmetric top revisited</u>

Return to the Euler top in the symmetric case $I_1 = I_2 \neq I_3$

Since gravity is irrelevant, choose the z-axis of the space frame to be aligned with the conserved angular momentum vector $\underline{L}$: $\underline{L} = |\underline{L}|\,\underline{e}_z$

In terms of the Euler angles, we have

$$\underline{\omega} = \dot\psi\,\underline{e}_3 + \dot\theta\,\underline{e}_n + \dot\phi\,\underline{e}_z$$

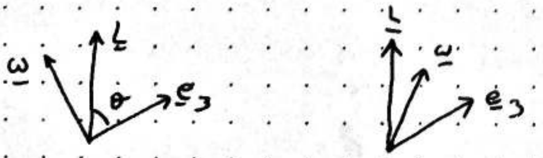
and so $\underline{L} = \underline{\underline{I}}\,\underline{\omega} = I_3\dot\psi\,\underline{e}_3 + I_1\dot\theta\,\underline{e}_n + \dot\phi\left(I_1\underline{e}_z + (I_3 - I_1)\cos\theta\,\underline{e}_3\right)$

[See this from $I_{ab} = I_1\delta_{ab} + (I_3 - I_1)\delta_{a3}\delta_{b3}$] $\quad \| \; |\underline{L}|\,\underline{e}_z$

Therefore $\dot\theta = 0$ (as noted previously), $\dot\phi = \dfrac{|\underline{L}|}{I_1}$ $\quad \cos\theta\,\underline{e}_3 - \underline{e}_z \in \langle \underline{e}_1, \underline{e}_2\rangle$

and $I_3\dot\psi + (I_3 - I_1)\dot\phi\cos\theta = 0$

So $\underline{L}$, $\underline{\omega}$ and $\underline{e}_3$ are coplanar and have a fixed relative orientation.

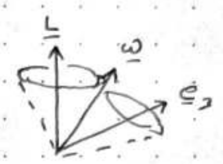


oblate ($I_3 > I_1$) prolate ($I_3 < I_1$)

L14.3

We have $I_3\omega_3 = I_3(\dot\psi + \dot\phi\cos\theta) = I_1\dot\phi\cos\theta = |\underline{L}|\cos\theta$

This solution corresponds to the uniform precession of a free symmetric top, seen in the space frame.



The symmetry axis of the body sweeps out a cone of angle θ about $\underline{L}$ at angular velocity $\dot\phi$

$\underline{\omega}$ also sweeps out a cone (space cone)

Relative to the body frame, $\underline{e}_z$ and $\underline{\omega}$ rotate about $\underline{e}_3$ at the rate

$$-\dot\psi = \dot\phi\cos\theta - \omega_3 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3$$

which agrees with Ω_p from the Euler top analysis.

So $\underline{\omega}$ also sweeps out a cone (the body cone) around $\underline{e}_3$ at rate $-\dot\psi$.

L15.1

§4 Hamilton's Equations

Sir William Rowan Hamilton

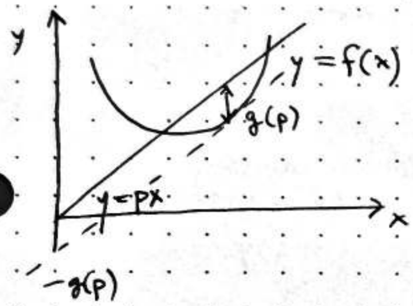
The Hamiltonian approach to dynamics uses phase-space variables (often denoted $\underline{q}$ and $\underline{p}$) rather than the generalized coordinates and velocities $\underline{q}$ and $\dot{\underline{q}}$ of the Lagrangian approach.

The Lagrangian approach $L(\underline{q}, \dot{\underline{q}}, t)$ is replaced by the Hamiltonian $H(\underline{q}, \underline{p}, t)$

Lagrange's equations are replaced by Hamilton's equations

4.1 The Legendre transformation

Let $f(x)$ be a convex function of x



Let $g(p)$ be the max value of $px - f(x)$ wrt x

This defines the Legendre transform $g(p)$ of $f(x)$.

The maximum occurs at $x = x^*(p)$ defined by

$$p = f'(x^*)$$

This equation is invertible because convexity of f implies that f' is increasing.

Ex $f(x) = \frac{1}{2}ax^2$, $p = ax \Rightarrow x = \frac{p}{a}$

$$\Rightarrow g(p) = px^* - f(x^*) = p\left(\frac{p}{a}\right) - \frac{1}{2}a\left(\frac{p}{a}\right)^2 = \frac{1}{2}p^2/a$$

Ex $f(x) = x^4$, $p = 4x^3 \Rightarrow x = (p/4)^{1/3}$

$$\Rightarrow g(p) = px^* - f(x^*) = p(p/4)^{1/3} - (p/4)^{4/3} = 3\,(p/4)^{4/3}$$

Ex $f(x) = e^x$, $p = e^x \Rightarrow x = \log p$

$$\Rightarrow g(p) = px^* - f(x^*) = p\log p - p$$

In terms of differentials, $f(x)$ has the differential $df = p\,dx$

where $p = \frac{df}{dx}$. Invert the relation $p = \frac{df}{dx}$ to express x in terms of p.

Let $g = px - f$, so that $dg = p\,dx + x\,dp - df = x\,dp$

which means that $x = \frac{dg}{dp}$.

The two functions' first derivatives are inverse functions of each other

$$p(x) = f'(x), \quad x(p) = g'(p)$$

The LT is an <u>involution</u>: if the LT of f is g, then the LT of g is f.

For functions of several variables $\underline{x}$, the LT generalizes to

$$g(\underline{p}) = \underline{p}\cdot\underline{x} - f(\underline{x}) \qquad \text{where } \underline{x} \text{ is determined by } \underline{p} = \frac{\partial f}{\partial \underline{x}}.$$

Then $dg = \underline{p} \cdot d\underline{x} + \underline{x} \cdot d\underline{p} - df = \underline{x} \cdot d\underline{p}$

§4.2 Hamilton's equations

The Hamiltonian $H(\underline{q}, \underline{p}, t)$ is the Legendre transform of the Lagrangian $L(\underline{q}, \dot{\underline{q}}; t)$ with respect to $\dot{\underline{q}}$.

The variables $\underline{q}$ and t are passive in the LT.

Thus $H = \underline{p} \cdot \dot{\underline{q}} - L$ with $\underline{p} = \frac{\partial L}{\partial \dot{\underline{q}}}$

It is important that $\dot{\underline{q}}$ is to be eliminated in favour of $\underline{p}$.

The total differential of H is

$$dH = \underline{p} \cdot d\dot{\underline{q}} + \dot{\underline{q}} \cdot d\underline{p} - \left(\frac{\partial L}{\partial \underline{q}} \cdot d\underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}} \cdot d\dot{\underline{q}} + \frac{\partial L}{\partial t} dt \right)$$

$$= \dot{\underline{q}} \cdot d\underline{p} - \dot{\underline{p}} \cdot d\underline{q} - \frac{\partial L}{\partial t} dt$$

where we have used the definition $\underline{p} = \frac{\partial L}{\partial \dot{\underline{q}}}$ and Lagrange's equations $\dot{\underline{p}} = \frac{\partial L}{\partial \underline{q}}$. Compare this with

$$dH = \frac{\partial H}{\partial \underline{q}} \cdot d\underline{q} + \frac{\partial H}{\partial \underline{p}} \cdot d\underline{p} + \frac{\partial H}{\partial t} dt,$$

to deduce Hamilton's equations

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}, \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

For a system with n degrees of freedom, we obtain 2n first-order ODEs, rather than n second order ODEs as in Lagrangian dynamics.

$\underline{p}$ is thought of as independent variable from $\underline{q}$ on an equal footing (they are treated almost symmetrically), whereas $\dot{\underline{q}}$ is the derivative of $\underline{q}$

In components, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

Hamilton's equations define a <u>flow</u> in phase space. The flow is orthogonal to the gradient of the Hamiltonian.

For an autonomous system for which L and therefore H do not depend explicitly on t, we have $H = \text{const.}$ along the path in phase space. H can often be identified with the energy of the system.

L15.3

Ex Particle in 1D potential, $L = \frac{1}{2} m\dot{x}^2 - V(x)$

$$q = x, \; p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\begin{aligned} H &= p\dot{q} - L \\ &= m\dot{x}^2 - \left(\frac{1}{2} m\dot{x}^2 - V(x)\right) \\ &= \frac{1}{2} m\dot{x}^2 + V(x) \\ &= p^2/2m + V(x) \end{aligned}$$

which is the total energy.

Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} = -V'(q)$$

Ex Charged particle in an electromagnetic field. As found previously,

$$L = \tfrac{1}{2}m|\dot{\underline{r}}|^2 + q(-\phi + \dot{\underline{r}}\cdot\underline{A})$$

So $\underline{q} = \underline{r}$; $\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} + q\underline{A} \Rightarrow \dot{\underline{r}} = \frac{\underline{p} - q\underline{A}}{m}$ Note $\underline{p} \neq m\dot{\underline{r}}$ (!)

$$\begin{aligned}\therefore H &= \underline{p}\cdot\dot{\underline{q}} - L \\ &= m|\dot{\underline{r}}|^2 + q\underline{A}\cdot\dot{\underline{q}} - L \\ &= \tfrac{1}{2}m|\dot{\underline{r}}|^2 + q\phi \\ &= \frac{|\underline{p} - q\underline{A}|^2}{2m} + q\phi\end{aligned}$$

This is the total energy of the particle, to which $\underline{B}$ does not contribute.

Eliminate $\dot{\underline{r}}$ to obtain

$$H = \frac{|\underline{p} - q\underline{A}|^2}{2m} + q\phi \quad \text{(lol sorry)}$$

Hamilton's equations are

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}} \Rightarrow \dot{\underline{r}} = \frac{\underline{p} - q\underline{A}}{m}$$

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}} \Rightarrow \text{in components } \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{p}_i = q\frac{(p_j - qA_j)}{m}\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

$$m\ddot{x}_i + q\frac{\partial A_i}{\partial t} + q\dot{x}_j\frac{\partial A_i}{\partial x_j} = q\dot{x}_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

$$m\ddot{x}_i = q\left[\left(-\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t}\right) + \dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\right]$$

$$m\ddot{x}_i = q(E_i + \varepsilon_{ijk}\dot{x}_j B_k)$$

which is the Newtonian form of the equation of motion.

Ex Motion in a constant unif $\underline{B} = (0,0,B)$ with $\underline{E} = 0$

Choosing the gauge $\underline{A} = (0, Bx, 0)$ (non-unique, recall $\underline{B} = \nabla\times\underline{A}$), we have

$$H = \frac{p_x^2}{2m} + \frac{(p_y - qBx)^2}{2m} + \frac{p_z^2}{2m}$$

Hamilton's equations are

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y - qBx}{m}, \quad \dot{z} = \frac{p_z}{m}$$

$$\dot{p}_x = \frac{qB(p_y - qBx)}{m}, \quad \dot{p}_y = 0, \quad \dot{p}_z = 0$$

So p_y, p_z are const.

Let $\omega = qB/m$ (the gyrofrequency). Then

$$\dot{p}_x = \omega m \dot{y} \Rightarrow p_x - \omega m y = \text{const.}$$

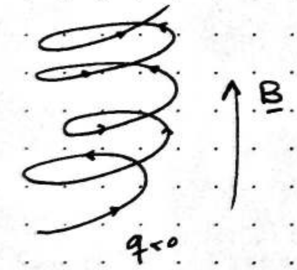
Define constants x_0, y_0 by

$$p_y = qBx_0, \quad p_x - \omega m y = -qBy_0$$

Then $\dot{x} = \omega(y - y_0)$, $\quad \dot{y} = -\omega(x - x_0)$

which describes clockwise circular motion around (x_0, y_0) with angular frequency ω.

Meanwhile $\dot{z} = \text{const}$ implies uniform motion along $\underline{B}$. The result is helical motion with angular frequency ω.



Ex Particle in a rotating frame. As found previously,

$$L = \tfrac{1}{2} m |\dot{\underline{r}} + \underline{\omega} \times \underline{r}|^2 - V(\underline{r})$$

So $\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m(\dot{\underline{r}} + \underline{\omega} \times \underline{r}) \Rightarrow \dot{\underline{r}} = \frac{\underline{p}}{m} - \underline{\omega} \times \underline{r}$

$$\begin{aligned} H &= \underline{p} \cdot \dot{\underline{r}} - L \\ &= \underline{p} \cdot \left(\frac{\underline{p}}{m} - \underline{\omega} \times \underline{r} \right) - \frac{|\underline{p}|^2}{2m} + V(\underline{r}) \\ &= \frac{|\underline{p}|^2}{2m} - \underline{p} \cdot (\underline{\omega} \times \underline{r}) + V(\underline{r}) \end{aligned}$$

This is equal to the energy in the rotating frame, i.e.

$$\tfrac{1}{2} m |\dot{\underline{r}}|^2 + V(\underline{r}) - \tfrac{1}{2} m |\underline{\omega} \times \underline{r}|^2$$

Hamilton's equations are

$$\dot{\underline{r}} = \frac{\partial H}{\partial \underline{p}} = \frac{\underline{p}}{m} - \underline{\omega} \times \underline{r}$$

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{r}} = \underline{p} \times \underline{\omega} - \underline{\nabla} V$$

The eq^n can also be read as

$$\dot{\underline{p}} + \underline{\omega} \times \underline{p} = -\underline{\nabla} V$$

The LHS is the time derivative of $\underline{p}$ in the non-rotating frame, and $\underline{p}$ is in fact the linear momentum in the non-rotating frame.

Check that this agrees with the Newtonian form:

$$\frac{d}{dt}\left(m(\dot{\underline{r}} + \underline{\omega} \times \underline{r})\right) + \underline{\omega} \times \left(m(\dot{\underline{r}} + \underline{\omega} \times \underline{r})\right) = -\underline{\nabla} V$$

$$m\Big(\ddot{\underline{r}} + \underbrace{2\underline{\omega} \times \dot{\underline{r}}}_{\text{Coriolis}} + \underbrace{\dot{\underline{\omega}} \times \underline{r}}_{\text{Euler}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r})}_{\text{centrifugal}}\Big) = -\underline{\nabla} V \quad \checkmark$$

e.g. taking $\underline{\omega} = (0, 0, \omega)$ and $\underline{\nabla} V = 0$, we have

$$\dot{p}_x = \omega p_y, \quad \dot{p}_y = -\omega p_x$$

so $\underline{p}$ rotates clockwise with angular frequency ω

<u>Exe</u> Solve explicitly for $x(t), y(t)$

4.3 The Principle of least action

We obtained the Hamiltonian from the Lagrangian via the Legendre transform

$$H = \underline{p} \cdot \dot{\underline{q}} - L$$

The action integral is therefore

$$S = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} (\underline{p} \cdot \dot{\underline{q}} - H) \, dt$$

Consider the variation of the action under independent variations of $\underline{q}$ and $\underline{p}$

$$\delta S = \int_{t_1}^{t_2} \left(\underline{p} \cdot \delta \dot{\underline{q}} + \dot{\underline{q}} \cdot \delta \underline{p} - \frac{\partial H}{\partial \underline{q}} \cdot \delta \underline{q} - \frac{\partial H}{\partial \underline{p}} \cdot \delta \underline{p} \right) dt$$

$$= \left[\underline{p} \cdot \delta \underline{q} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(-\left(\dot{\underline{p}} + \frac{\partial H}{\partial \underline{q}}\right) \cdot \delta \underline{q} + \left(\dot{\underline{q}} - \frac{\partial H}{\partial \underline{p}}\right) \cdot \delta \underline{p} \right) dt$$

The action is therefore stationary wrt variations in which $\underline{q}$ is fixed at the endpoints if and only if Hamilton's equations

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}} \quad \text{and} \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}} \quad \text{are satisfied.}$$

L16.4

$$S = \int_{t_1}^{t_2} (\underline{p} \cdot \dot{\underline{q}} - H)\, dt$$

If H does not depend explicitly on time, then it is conserved and we can write

$$S = \int \underline{p} \cdot d\underline{q} - (t_2 - t_1) H$$

The second part can be ignored because it is indep of path.

The first part, the <u>abbreviated action</u> $\int \underline{p} \cdot d\underline{q}$ depends strictly on the path taken in phase space (and not on the rate at which it is traversed).

The actual path makes this line integral stationary.

Solution of exercise

$$\dot{\underline{r}} = \frac{\underline{p}}{m} - \underline{\omega} \times \underline{r}, \quad \dot{\underline{p}} = \underline{p} \times \underline{\omega}$$

$$\dot{p}_x = \omega p_y, \quad \dot{p}_y = -\omega p_x$$

Combine to give

$$\ddot{p}_x + \omega^2 p_x = 0$$

Choose origin of time such that

$$p_x = mU \sin\omega t, \quad p_y = mU \cos\omega t,$$

where $U = \text{const}$. Then

$$\dot{x} = U\sin\omega t + \omega y, \quad \dot{y} = U\cos\omega t - \omega x$$

Combine to give

$$\ddot{x} + \omega^2 x = 2U\omega\cos\omega t$$

General solution

$$x = Ut\sin\omega t + A\sin\omega t + B\cos\omega t$$

$$y = Ut\cos\omega t + A\cos\omega t - B\sin\omega t$$

4.4 The Poisson bracket

The Poisson bracket of two functions $f(\underline{p}, \underline{q}; t)$ and $g(\underline{q}, \underline{p}, t)$ is defined as

$$\{f, g\} = \frac{\partial f}{\partial \underline{q}} \cdot \frac{\partial g}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}} \cdot \frac{\partial g}{\partial \underline{q}}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

This is an antisymmetric product of the partial derivatives of f, g

In the $n=1$ case it corresponds to the Jacobian determinant

$$\left| \frac{\partial(f,g)}{\partial(q,p)} \right| = \begin{vmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{vmatrix}$$

Warning!
- ▷ Landau & Lifschitz define with relative minus sign, use square brackets.
- ▷ Goldstein uses square brackets for Poisson, braces for Lagrange (!)
- ▷ Arnold uses round brackets

$$\{f,g\} = \frac{\partial f}{\partial \underline{q}} \cdot \frac{\partial g}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}} \cdot \frac{\partial g}{\partial \underline{q}}$$

The Poisson bracket has the following properties, where f, g, h are functions, α, β are constants:

- Antisymmetry: $\{f,g\} = -\{g,f\}$
- Linearity: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$
- Leibniz rule: $\{fg, h\} = f\{g,h\} + \{f,h\}g$
- Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Clearly $\{f,f\} = 0$ and $\{f,c\} = 0$ where c = const.

The Poisson brackets of the <u>canonical variables</u> are

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$$

These are called the <u>canonical commutation relations</u>

Two functions whose bracket vanishes are said to <u>Poisson commute</u>

The time evolution of any function $f(\underline{q}, \underline{p}, t)$ is given by

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial \underline{q}} \cdot \dot{\underline{q}} + \frac{\partial f}{\partial \underline{p}} \cdot \dot{\underline{p}} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial \underline{q}} \cdot \frac{\partial H}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}} \cdot \frac{\partial H}{\partial \underline{q}} + \frac{\partial f}{\partial t} \\ &= \{f, H\} + \frac{\partial f}{\partial t}\end{aligned}$$

(There is a close analogy between the Poisson bracket and the commutator of operators in QM — see later)

Any function $f(\underline{q}, \underline{p}, t)$ s.t. $\frac{df}{dt} = 0$ is a <u>conserved quantity</u> or <u>constant of motion</u>.

So any function $f(\underline{q}, \underline{p})$ that Poisson commutes with the Hamiltonian is a conserved quantity; called an <u>integral of motion</u>

Most commonly (c.f. Noether's theorem)

- H is conserved if $\frac{\partial H}{\partial t} = 0$
- p_i is conserved if $\frac{\partial H}{\partial q_i} = 0$

Poisson's Theorem

The Poisson bracket of any two conserved quantities is also a conserved quantity

This follows quickly from the Jacobi identity if the functions do not depend explicitly on time:

$$\frac{d}{dt}\{f,g\} = \{\{f,g\},H\}$$
$$= -\{H,\{f,g\}\}$$
$$= \{f,\{g,H\}\} + \{g,\{H,f\}\}$$
$$= \{f,\{g,H\}\} + \{\{f,H\},g\}$$
$$= \left\{f, \frac{dg}{dt}\right\} + \left\{\frac{df}{dt}, g\right\}$$

General proof:

$$\frac{d}{dt}\{f,g\} = \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\}$$
$$= \{f,\{g,H\}\} + \{\{f,H\},g\} + \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}$$
$$= \left\{f, \{g,H\} + \frac{\partial g}{\partial t}\right\} + \left\{\{f,H\} + \frac{\partial f}{\partial t}, g\right\}$$
$$= \left\{f, \frac{dg}{dt}\right\} + \left\{\frac{df}{dt}, g\right\}. \quad \square$$

Angular momentum and the Laplace-Runge-Lenz vector

The angular momentum of a particle is

$$\underline{L} = \underline{r} \times \underline{p}$$

In components, $L_1 = x_2 p_3 - x_3 p_2$

$$L_2 = x_3 p_1 - x_1 p_3$$
$$L_3 = x_1 p_2 - x_2 p_1$$

Consider the Poisson bracket (identity $\underline{q} = \underline{r}$)

$$\{L_1, L_2\} = \{x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3\}$$

Recall that

$$\{L_1, L_2\} = \sum_{i=1}^{3} \left(\frac{\partial L_1}{\partial x_i}\frac{\partial L_2}{\partial p_i} - \frac{\partial L_1}{\partial p_i}\frac{\partial L_2}{\partial x_i} \right)$$

The non-zero terms are $\frac{\partial}{\partial x_3}(-x_3 p_2)\frac{\partial}{\partial p_3}(-x_1 p_3)$

$$- \frac{\partial}{\partial p_3}(x_2 p_3)\frac{\partial}{\partial x_3}(x_3 p_1) = p_2 x_1 - x_2 p_1$$

By extension, $\{L_i, L_j\} = \varepsilon_{ijk} L_k$

Also,

$$\{L_i, |\underline{L}|^2\} = \{L_i, L_j L_j\}$$

$$= 2\{L_i, L_j\} L_j \quad \text{(Leibniz)}$$

$$= 2\varepsilon_{ijk} L_k L_j$$

$$= 0$$

The Hamiltonian for the Kepler problem; a particle of mass m in a potential $V(r) = -\frac{k}{r}$, where $k = \text{const}$ and $r = |\underline{r}|$, is

$$H = \frac{|\underline{p}|^2}{2m} - \frac{k}{r}$$

The Laplace-Runge-Lenz vector is defined as

$$\underline{A} = \underline{p} \times \underline{L} - mk\hat{\underline{r}} \quad , \quad \text{where } \hat{\underline{r}} = \frac{\underline{r}}{r}$$

It can be shown that

$$\{L_i, H\} = 0 \,, \quad \{A_i, H\} = 0 \,,$$

$$\{L_i, A_j\} = \varepsilon_{ijk} A_k \quad , \quad \{A_i, A_j\} = -2mH\varepsilon_{ijk} L_k \quad \text{(ex 4.2)}$$

So both $\underline{L}$ and $\underline{A}$, as well as H, are integrals of motion

Their Poisson brackets do not generate any further integrals of motion

Rescale $\underline{A}$ to form the dimensionless eccentricity vector

$$\underline{e} = \frac{\underline{A}}{mk} = \frac{\underline{p} \times \underline{L}}{mk} - \hat{\underline{r}}$$

Now

$$\underline{e} \cdot \underline{r} = \frac{\underline{r} \cdot (\underline{p} \times \underline{L})}{mk} - r$$

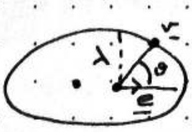
$$er\cos\theta = \frac{(\underline{r} \times \underline{p}) \cdot \underline{L}}{mk} - r = \frac{|\underline{L}|^2}{mk} - r$$

where θ is the angle between $\underline{e}$ and $\underline{r}$

Rearrange to give the polar equation for a conic section of eccentricity e and semilatus rectum $\lambda = |\underline{L}|^2 / mk$

$$r = \frac{\lambda}{1 + e\cos\theta}$$

Alternative method of solution for the Kepler problem.



Conservation of $\underline{A}$, in addition to $\underline{L}$, is related to a hidden $SO(4)$ symmetry of Kepler problem

Exercise

Show that $\{L_i, L_j\} = \epsilon_{ijk} L_k$

directly from the canonical commutation relations using suffix notation and the Leibniz rule.

Sol^n of exercise

Using the def^n $L_i = \varepsilon_{ijk} x_j p_k$ and the canonical commutation relations

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}$$

we have

$$\begin{aligned}
\{L_i, L_j\} &= \{\varepsilon_{ikl} x_k p_l, \varepsilon_{jmn} x_m p_n\} \\
&= \varepsilon_{ikl}\varepsilon_{jmn} \{x_k p_l, x_m p_n\} \\
&= \varepsilon_{ikl}\varepsilon_{jmn} \left(x_k \{p_l, x_m p_n\} + \{x_k, x_m p_n\} p_l \right) \\
&= \varepsilon_{ikl}\varepsilon_{jmn} \left(\{p_l, p_n\} x_k x_m + x_k \{p_l, x_m\} p_n \right. \\
&\qquad \left. + x_m \{x_k, p_n\} p_l + \{x_k, x_m\} p_l p_n \right) \\
&= \varepsilon_{ikl}\varepsilon_{jmn} \left(- x_k p_n \delta_{lm} + x_m p_l \delta_{kn} \right) \\
&= - \varepsilon_{ikl}\varepsilon_{jln} x_k p_n + \varepsilon_{ikl}\varepsilon_{jmk} x_m p_l \\
&= - x_j p_i + \cancel{x_k p_k \delta_{ij}} + x_i p_j - \cancel{x_l p_l \delta_{ij}} \\
&= \varepsilon_{ijk}\varepsilon_{klm} x_l p_m \\
&= \varepsilon_{ijk} L_k \quad \square
\end{aligned}$$

4.5 Canonical transformations

Consider a change of variables in phase space from $(\underline{q}, \underline{p})$ to $(\underline{Q}, \underline{P})$, where $\underline{Q}, \underline{P}$ are functions of $(\underline{q}, \underline{p}, t)$.

The transformation is said to be <u>canonical</u> if it preserves the form of Hamilton's equations, i.e.

$$\dot{\underline{Q}} = \frac{\partial K}{\partial \underline{P}}, \quad \dot{\underline{P}} = -\frac{\partial K}{\partial \underline{Q}},$$

where $K(\underline{Q}, \underline{P}, t)$ is a new Hamiltonian (sometimes called the Kamiltonian!)

We can combine Hamilton's equations

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}, \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}$$

into the form $\dot{\underline{x}} = \Omega \dfrac{\partial H}{\partial \underline{x}}$ or $\dot{x}_a = \Omega_{ab} \dfrac{\partial H}{\partial x_b}$ $(a, b = 1, \ldots, 2n)$

where $\underline{x} = (\underline{q}, \underline{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ combines all $2n$ canonical variables and

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is an antisymmetric block matrix, where I_n is the $n \times n$ identity matrix.

The Poisson bracket can then be written as

$$\{f, g\} = \Omega_{ab} \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$$

Let J be the Jacobian matrix of the transformation $\underline{x} \mapsto \underline{X}(\underline{x}, t)$:

$$J_{ab} = \frac{\partial X_a}{\partial x_b}$$

Then
$$\dot{X}_a = \frac{\partial X_a}{\partial x_b} \dot{x}_b + \frac{\partial X_a}{\partial t}$$
$$= \frac{\partial X_a}{\partial x_b} \Omega_{bc} \frac{\partial H}{\partial x_c} + \frac{\partial X_a}{\partial t}$$
$$= \frac{\partial X_a}{\partial x_b} \Omega_{bc} \frac{\partial X_d}{\partial x_c} \frac{\partial H}{\partial X_d} + \frac{\partial X_a}{\partial t}$$

For a <u>restricted transformation</u> that does not involve time explicitly, the last term vanishes and we have

$$\underline{\dot{X}} = J \Omega J^T \frac{\partial H}{\partial \underline{X}}$$

We recover the canonical form of Hamilton's equations, (w/ same H)

$$\underline{\dot{X}} = \Omega \frac{\partial H}{\partial \underline{X}} \quad \text{iff} \quad J \Omega J^T = \Omega$$

If J satisfies this condition, it is called a <u>symplectic matrix</u>

We have then found a canonical transformation to an alternative set of canonical variables $(\underline{Q}, \underline{P})$. In block form,

$$J = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}, \quad J^T = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial P_j}{\partial q_i} \\ \frac{\partial Q_j}{\partial p_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix}$$

$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, so the symplectic condition $J \Omega J^T = \Omega$ is

$$\begin{pmatrix} \{Q_i, Q_j\} & \{Q_i, P_j\} \\ \{P_i, Q_j\} & \{P_i, P_j\} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The transformation is therefore canonical iff the new variables satisfy the canonical commutation relations, $\{Q_i, Q_j\} = 0$, $\{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$

Alternatively, using $J^{-1} = \begin{pmatrix} \frac{\partial q_i}{\partial Q_j} & \frac{\partial q_i}{\partial P_j} \\ \frac{\partial p_i}{\partial Q_j} & \frac{\partial p_i}{\partial P_j} \end{pmatrix}$, $\Omega^{-1} = -\Omega$,

we can write the symplectic condition as $J = -\Omega (J^{-1})^T \Omega$, i.e.

$$\begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_j}{\partial P_i} & -\frac{\partial q_j}{\partial P_i} \\ -\frac{\partial p_j}{\partial Q_i} & \frac{\partial q_j}{\partial Q_i} \end{pmatrix}$$

These are known as the <u>direct conditions</u> for the transformation to be canonical.

<u>Invariance of Poisson brackets under canonical transformations</u>

Let $(\underline{q}, \underline{p}) \mapsto (\underline{Q}, \underline{P})$ be a canonical transformation

Denote the Poisson bracket wrt the original canonical variables $(\underline{q}, \underline{p})$ by:

$$\{f, g\}_{(\underline{q}, \underline{p})} = \Omega_{ab} \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$$

and the Poisson bracket wrt the new canonical variables $(\underline{Q}, \underline{P})$ by

$$\{f, g\}_{(\underline{Q}, \underline{P})} = \Omega_{ab} \frac{\partial f}{\partial X_a} \frac{\partial g}{\partial X_b}$$

By the chain rule,

$$\begin{aligned} \{f, g\}_{(\underline{q}, \underline{p})} &= \Omega_{ab} \frac{\partial f}{\partial X_c} \frac{\partial X_c}{\partial x_a} \frac{\partial g}{\partial X_d} \frac{\partial X_d}{\partial x_b} \\ &= \Omega_{ab} \frac{\partial f}{\partial X_c} J_{ca} \frac{\partial g}{\partial X_d} J_{db} \\ &= J_{ca} \Omega_{ab} J_{db} \frac{\partial f}{\partial X_c} \frac{\partial g}{\partial X_d} \\ &= (J \Omega J^T)_{cd} \frac{\partial f}{\partial X_c} \frac{\partial g}{\partial X_d} \\ &= \Omega_{cd} \frac{\partial f}{\partial X_c} \frac{\partial g}{\partial X_d} \quad \text{(symplectic)} \\ &= \{f, g\}_{(\underline{Q}, \underline{P})} \end{aligned}$$

$\therefore$ The Poisson bracket is invariant under a canonical transformation, and no subscripts are needed.

This invariance is related to the 'symplectic structure' of Hamiltonian systems, often described using differential forms (see Arnold)

4.6 Generdating functions

These are a practical way of finding canonical transformations.

Let $(\underline{q}, \underline{p})$ be canonical variables with Hamiltonian H. Then Hamilton's Principle takes the form

$$\delta \int (\underline{p} \cdot d\underline{q} - H\,dt) = 0$$

If the new variables $(\underline{Q}, \underline{P})$ are also canonical with Hamiltonian K, then we must have

$$\delta \int (\underline{P} \cdot d\underline{Q} - K\,dt) = 0$$

The two forms are equivalent if the integrands differ by the total differential of some function F, i.e.

$$\underline{p} \cdot d\underline{q} - H\,dt = \underline{P} \cdot d\underline{Q} - K\,dt + dF$$

In this case F is called the <u>generating function</u> of the canonical transform^{n}

If F is given as a function $F = F_1(\underline{q}, \underline{Q}, t)$, which is called a <u>generating function of type 1</u>, then we write

$$dF_1 = \underline{p} \cdot d\underline{q} - \underline{P} \cdot d\underline{Q} + (K - H)\,dt$$

and deduce that

$$\underline{p} = \frac{\partial F_1}{\partial \underline{q}}, \quad \underline{P} = -\frac{\partial F_1}{\partial \underline{Q}}, \quad K = H + \frac{\partial F_1}{\partial t}$$

giving us the old and new momenta and the new Hamiltonian.

If, instead, the generdating function is expressed in terms of $(\underline{q}, \underline{P}, t)$, called a <u>generating function of type 2</u>, then we make a Legendre transformation:

Let $F = F_2(\underline{q}, \underline{P}, t) - \underline{Q} \cdot \underline{P}$

Then $dF_2 = \underline{p} \cdot d\underline{q} + \underline{Q} \cdot d\underline{P} + (K - H)\,dt$

from which

$$\underline{p} = \frac{\partial F_2}{\partial \underline{q}}, \quad \underline{Q} = \frac{\partial F_2}{\partial \underline{P}}, \quad K = H + \frac{\partial F_2}{\partial t}$$

Similarly for other two types:

$F = F_3(\underline{p}, \underline{Q}, t) + \underline{q}\cdot\underline{p}$ gives

$$\underline{q} = -\frac{\partial F_3}{\partial \underline{p}}, \quad \underline{P} = -\frac{\partial F_3}{\partial \underline{Q}}, \quad K = H + \frac{\partial F_3}{\partial t}$$

and $F = F_4(\underline{p}, \underline{P}, t) + \underline{q}\cdot\underline{p} - \underline{Q}\cdot\underline{P}$ gives

$$\underline{q} = -\frac{\partial F_4}{\partial \underline{p}}, \quad \underline{Q} = \frac{\partial F_4}{\partial \underline{P}}, \quad K = H + \frac{\partial F_4}{\partial t}$$

In each case the two Hamiltonians differ by the partial time-derivative of the generelating function.

Canonical transf^{n}s generally mix coordinates $\underline{q}$ and momenta $\underline{p}$, so these lose their individual meanings.

Ex ▷ If $F_2 = \underline{q}\cdot\underline{P}$, then $\underline{Q} = \underline{q}$ and $\underline{P} = \underline{p}$, which is the trivial <u>identity transformation</u>

▷ If $F_1 = \underline{q}\cdot\underline{Q}$, then $\underline{Q} = \underline{p}$ and $\underline{P} = -\underline{q}$, so coordinates and momenta are interchanged (with some sign changes).

▷ If $F_2 = \sum_{i=1}^{n} f_i(\underline{q}) P_i$, then

$$Q_i = f_i(\underline{q}), \quad p_i = \sum_{j=1}^{n} \frac{\partial f_j}{\partial q_i} P_j$$

This is a point transformation of the coordinates, $\underline{q} \mapsto \underline{Q}(\underline{q})$.

The transformation of momenta is consistent with

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial \dot{Q}_j}{\partial \dot{q}_i}\frac{\partial L}{\partial \dot{Q}_j} = \frac{\partial Q_j}{\partial q_i} P_j$$

▷ Consider the generating function $F_1 = \frac{1}{2}q^2 \cot Q$

We obtain $\quad p = q\cot Q, \quad P = \frac{1}{2} q^2 \operatorname{cosec}^2 Q$

So $\quad P = \frac{1}{2}(p^2 + q^2), \quad Q = \arctan\left(\frac{q}{p}\right)$

The inverse relation is

$$q = \sqrt{2P}\sin Q, \quad p = \sqrt{2P}\cos Q$$

This transformation mixes coordinates and momenta like the transf^n between rectangular and polar coordinates:

L19.3

4.7 Infinitesimal canonical transformations

Consider the near identity transformation

$$Q = q + \delta q, \qquad P = p + \delta p$$

where δq and δp are small.

This is canonical iff

$$\{Q, P\} = \left(1 + \frac{\partial(\delta q)}{\partial q}\right)\left(1 + \frac{\partial(\delta p)}{\partial p}\right) - \frac{\partial(\delta q)}{\partial p}\frac{\partial(\delta p)}{\partial q} = 1$$

To first order in the small quantities, this is a condition of vanishing 'divergence': $\dfrac{\partial(\delta q)}{\partial q} + \dfrac{\partial(\delta p)}{\partial p} = 0$

More generally, consider the generating function

$$F_2 = \underline{q} \cdot \underline{P} + \varepsilon G(\underline{q}, \underline{P}, t)$$

where ε is a small parameter that tends to zero.

This generates the near identity transformation

$$\underline{p} = \frac{\partial F_2}{\partial \underline{q}} = \underline{P} + \varepsilon \frac{\partial G}{\partial \underline{q}}$$

$$\underline{Q} = \frac{\partial F_2}{\partial \underline{P}} = \underline{q} + \varepsilon \frac{\partial G}{\partial \underline{P}}$$

To first order in ε this gives

$$\delta \underline{q} = \varepsilon \frac{\partial G}{\partial \underline{P}} = \varepsilon \frac{\partial G}{\partial \underline{p}}, \qquad \delta \underline{p} = -\varepsilon \frac{\partial G}{\partial \underline{q}}$$

since we replace $\dfrac{\partial G}{\partial \underline{P}}$ with $\dfrac{\partial G}{\partial \underline{p}}$ to this level of approximation and consider G as a function of $(\underline{q}, \underline{p}, t)$.

This is known as an <u>infinitesimal canonical transformation</u> with generating function G.

Time-evolution as a canonical transformation

Consider the changes in $\underline{q}$ and $\underline{p}$ due to time evolution in a Hamiltonian system during an infinitesimal time interval ε:

$$\delta \underline{q} = \varepsilon \dot{\underline{q}} = \varepsilon \frac{\partial H}{\partial \underline{p}} , \quad \delta \underline{p} = \varepsilon \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}} \varepsilon$$

This corresponds to an infinitesimal canonical transformation generated by the Hamiltonian.

Integrating this result, we conclude that the time-evolution of a Hamiltonian system in phase space over a finite time-interval corresponds to a canonical transformation.

This corresponds to an <u>active</u> view of canonical transformation rather than the <u>passive</u> view with which we started.

In the passive view, the transformation is a change of coordinates labelling the same point in phase space.

In the active view, the transformation takes us to another point of the phase space.

Noether's Theorem revisited

The condition for $G(\underline{q}, \underline{p})$ to be an integral of motion is $\{G, H\} = 0$.

This is satisfied iff the Hamiltonian is invariant under the infinitesimal canonical transformation generetad by G, i.e.

$$\delta H = \frac{\partial H}{\partial \underline{q}} \cdot \delta \underline{p} + \frac{\partial H}{\partial \underline{p}} \cdot \delta \underline{p}$$

$$= \frac{\partial H}{\partial \underline{q}} \cdot \left(\varepsilon \frac{\partial G}{\partial \underline{p}} \right) + \frac{\partial H}{\partial \underline{p}} \cdot \left(-\varepsilon \frac{\partial G}{\partial \underline{q}} \right)$$

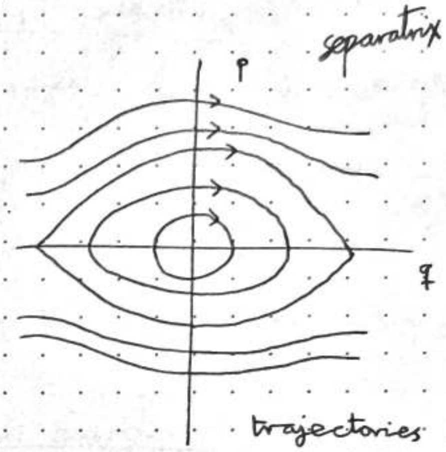
$$= \varepsilon \{H, G\} = 0$$

<u>Ex</u> If $\frac{\partial H}{\partial q_k} = 0$ for some k, then $\delta H = 0$ for the perturbation $\delta q_k = \varepsilon$ generated by $G = p_k$, in which case p_k is an integral of motion.

4.8 Liouville's Theorem

The time-evolution of a Hamiltonian system corresponds to the flow of an incompressible fluid in $2n$-dimensional phase space.

For $n=1$,
$$\dot{q} = +\frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$
is an incompressible 2D flow, with streamfunction H.



trajectories are curves $H=$const.

Ex Simple pendulum

In dimensionless variables (here $q=\theta$),
$$H = \frac{p^2}{2} - \cos q,$$
Leading to $\dot{q} = p$, $\dot{p} = -\sin q$,

implying the pendulum equation $\ddot{q} = -\sin q$

More generally, the Hamiltonian velocity field is
$$v_a = \dot{x}_a = \Omega_{ab}\frac{\partial H}{\partial x_b}$$
with divergence
$$\underline{\nabla}\cdot\underline{v} = \frac{\partial v_a}{\partial x_a} = \Omega_{ab}\frac{\partial^2 H}{\partial x_a \partial x_b} = 0$$
because Ω_{ab} is antisymmetric while $\frac{\partial^2 H}{\partial x_a \partial x_b}$ is symmetric.

More explicitly, the divergence is
$$\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

The Hamiltonian flow can also seen to be volume-preserving because it corresponds to a canonical transformation, and the symplectic condition $J\Omega J^T = \Omega \Rightarrow |\det(J)| = 1$

Do note $\det(\Omega) = 1$.

Consider an ensemble of systems with the same Hamiltonian function but diff initial conditions.

Each is represented by a 'particle' following the flow in phase space.

If they are closely spaced, we can discuss their phase-space density $\rho(\underline{q},\underline{p},t)$

Since the 'particles' are conserved, the density satisfies the conservation eq^n

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{v}) = 0$$

But the Hamiltonian flow satisfies $\underline{\nabla}\cdot\underline{v} = 0$, so the Lagrangian / material derivative vanishes $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{v}\cdot\underline{\nabla}\rho = 0$

This is <u>Liouville's Theorem</u> : the phase-space density is conserved following the flow in phase space.

Equilibrium (time-indep) solutions for the phase-space density are possible if ρ is a f^n of the integrals of motion.

The most important example is the <u>Boltzmann dist^n</u> $\rho \propto \exp(-H/kT)$

<u>Poincaré recurrence theorem</u>

Let P be any point in the phase space of an autonomous Hamiltonian, and let D_0 be any ngbd of P. If the accessible phase space is bounded, then $\exists$ point in D_0 that returns to D_0 within a finite time.

<u>Proof</u> Evol^n in the Hamiltonian flow for any +ve time step T corresponds to an invertible, volume preserving map g of phase space onto itself.

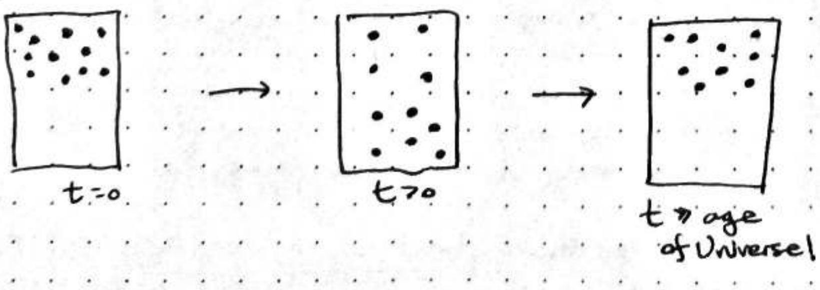
The image of D_0 after n time-steps is $D_n = g^n D_0$.

The images must intersect, because they all have the same volume and the accessible volume of phase space is finite.

Spose $g^n D_0 \cap g^m D_0 \neq \phi$ for some $n > m$.

Then $g^{n-m} D_0 \cap D_0 \neq \phi$ i.e. D_k intersects D_0 for $k = n-m$. $\square$

<u>Ex</u> Particles in a box. The accessible phase space is bdd in $\underline{r}$ by the walls and in $\dot{\underline{r}}$ by the conservation of total energy.



4.9 Angle-action variables

For an autonomous system with one degree of freedom,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},$$

we aim to make a canonical transformation $(q, p) \mapsto (Q, P)$ such that $H(Q,P) = H(P)$ depends only on P.

In this case Hamilton's equations will take the simple form

$$\dot{P} = 0 \Rightarrow P = \text{const.}$$

$$\dot{Q} = \frac{dH}{dP} = \text{const.}$$

In fact we know $H = \text{const.}$, so P could be any function of H.

Assuming that the motion is bounded, we choose (Q, P) s.t. Q increases by 2π around each orbit, like an angular variable.

The area in phase space enclosed by a periodic orbit is

$$A = \iint dq\, dp = \iint dQ\, dP$$

and is invariant under a canonical transf^n:

$$\left| \frac{\partial(Q,P)}{\partial(q,p)} \right| = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \{Q, P\} = 1$$

The area can also be written as the line integral

$$A = \oint p\, dq = \oint P\, dQ$$ by Green's theorem, provided that the orbit is traversed <u>clockwise</u>.

$$\oint (f\, dx + g\, dy) = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy$$

Since $P = \text{const}$ on the orbit, and Q is to increase by 2π around it, we choose

$$P = \frac{A}{2\pi}$$

Relabelling (Q, P) as (θ, I) we have identified <u>angle-action variables</u> where θ is the <u>angle</u> and $I = \frac{1}{2\pi}\oint p\, dq$ is the <u>action</u>.

(The generetating function of the canonical transformation is the indefinite integral $F_2(q, I) = \int p\, dq$)

Hamilton's equations reduce to

$$\dot{\theta} = \frac{dH}{dI}, \quad \dot{I} = 0$$

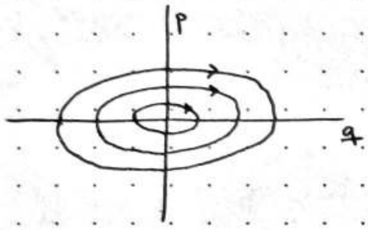
with general solution

$$I = \text{const}; \quad \theta = \theta_0 + \omega t$$

where $\omega(I) = \frac{dH}{dI}$ is the angular frequency of the orbit.

<u>Ex</u> SHO $H = \frac{p^2}{2m} + \frac{kq^2}{2}$

The orbits are ellipses $H = \text{const}$ in phase space



The area is $A = \pi \sqrt{\frac{2H}{k}} \sqrt{2mH}$

so the action is $I = \frac{A}{2\pi} = \sqrt{\frac{m}{k}}\, H$

and the angular frequency is $\omega = \frac{dH}{dI} = \sqrt{\frac{k}{m}} = \text{const}$

independent of energy. Thus $I = \frac{H}{\omega}$.

The angle can be found from

$$\dot{\theta} = \omega = \text{const}$$

$$\theta = \omega \int dt = \omega \int \frac{dq}{\dot{q}} = \omega \int \frac{dq}{p/m}$$

$$= \omega \int \frac{dq}{\sqrt{\frac{2H}{m} - \frac{kq^2}{m}}} = \arcsin\left(q\sqrt{\frac{k}{2H}} \right)$$

from which $q = \sqrt{\frac{2H}{k}} \sin\theta$, $p = \sqrt{2mH - mkq^2} = \sqrt{2mH} \cos\theta$

$$\theta = \arctan\left(\sqrt{mk}\, \frac{q}{p} \right)$$

<u>Integrable systems</u> The evolution of systems with $n > 1$ degrees of freedom may or may not be regular.

An autonomous Hamiltonian $H(\underline{q}, \underline{p})$ with n degrees of freedom is said to be <u>integrable</u> if $\exists$ n integrals of motion $f_i(\underline{q}, \underline{p})$ that are independent and <u>in involution</u> i.e. $\{f_i, f_j\} = 0$, $i,j = 1, \dots, n$

The motion takes place on an n-dim level surface S_n of the functions $f_1, \dots, f_n$ in phase space.

L21.3

We can always take $f_1 = H$.

The case $n=1$ is always integrable.

The case $n=2$ is integrable if there exists an integral of motion independat of H.

In general it is difficult to determine whether a Hamiltonian system is integrable.

The <u>Arnold-Liouville Theorem</u> states that if S_n is compact and connected, then $\exists$ a canonical transformation to angle-action variables $(\underline{\theta}, \underline{I})$ s.t. the ~~functions~~ actions are functions of $f_1, \ldots, f_n$ the angles are 2π-periodic and $H = H(\underline{I})$.

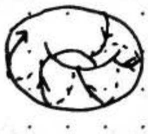
Hamilton's equations in angle-action variables reduce to

$$\dot{\underline{\theta}} = \frac{\partial H}{\partial \underline{I}}, \quad \dot{\underline{I}} = 0$$

with general solution $\underline{I} = \text{const.}$, $\underline{\theta} = \underline{\theta}_0 + \underline{\omega} t$

where $\underline{\omega}(\underline{I}) = \frac{\partial H}{\partial \underline{I}}$

The general solution is quasiperiodic and winds on an n-dimensional torus in phase space.



The action variables are

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} \underline{p} \cdot d\underline{q}$$

where γ_i is a cycle in which θ_j increases by 2π if $j=i$ and by 0 otherwise.

<u>Ex</u> The Lagrange top $(n=3)$ is an integrable system, with

$f_1 = H, \ f_2 = L_z, \ f_3 = L_3$

<u>Exe</u> Show that $\{L_z; L_3\} = 0$

Systems that are not integrable can have irregular, chaotic motion, e.g. the double pendulum for large amplitudes

In the planar Kepler problem we have (with $k = GMm$)

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} = E = -\frac{k}{2a} = \text{const.}$$

$$p_\theta = mr^2\dot{\theta} = L_z = \sqrt{mka(1-e^2)} = \text{const.}$$

so the radial action for a bound orbit is

$$I_r = \frac{1}{2\pi}\oint p_r\,dr = \frac{2}{2\pi}\int_{r_{min}}^{r_{max}} \sqrt{2mE - \frac{L_z^2}{r^2} + \frac{2mk}{r}}\,dr$$

where $r_{min} = a(1-e)$ and $r_{max} = a(1+e)$.

This evaluates to (see EX 4.11)

$$I_r = \sqrt{\frac{mk^2}{-2E}} - L_z = \sqrt{mka}\left(1 - \sqrt{1-e^2}\right)$$

The azimuthal action is just

$$I_\theta = \frac{1}{2\pi}\oint p_\theta\,d\theta = p_\theta = L_z$$

The Hamiltonian in action variables is

$$H = -\frac{mk^2}{2(I_r + I_\theta)^2}$$

The Hamiltonian in ; We see from this that the radial and azimuthal frequencies,

$$\omega_r = \frac{\partial H}{\partial I_r}, \quad \omega_\theta = \frac{\partial H}{\partial I_\theta}$$

are degenerate, which is why the orbits are closed ellipses.

Sol^n of exercise

We have $L = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2 + Mgl\cos\theta$,

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = I_1 \dot\theta \ , \quad p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + I_3 (\dot\psi + \dot\phi \cos\theta)\cos\theta = L_z ,$$

$$p_\psi = \frac{\partial L}{\partial \psi} = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\phi \cos\theta) = L_3 ,$$

so $$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} - Mgl\cos\theta ,$$

$$\{L_z, H\} = -\frac{\partial p_\phi}{\partial p_\phi} \frac{\partial H}{\partial \phi} = 0 ,$$

$$\{L_3 ; H\} = -\frac{\partial p_\psi}{\partial p_\psi} \frac{\partial H}{\partial \psi} = 0 ,$$

$$\{L_z ; L_3\} = \frac{\partial p_\phi}{\partial \psi} \frac{\partial p_\psi}{\partial p_\psi} - \frac{\partial p_\phi}{\partial p_\phi} \frac{\partial p_\psi}{\partial \phi} = 0$$

4.10 Adiabatic Invariant

Consider a Hamiltonian system that depends on a parameter λ that changes in time. The system is non-autonomous and there may be no integrals of motion (even the energy is not conserved).

Suppose λ changes adiabatically; i.e. on a time-scale much longer than the typical oscillation period.

Certain quantities known as adiabatic invariants may then be approximately conserved.

Consider $H(q, p; \lambda)$ where $\dot\lambda$ is small.

For any fixed λ, there is a canonical transf^n to angle-action variables $(q ; p) \mapsto (\theta ; I)$ with generating function F, giving Hamilton's eq^{n}s in the form
$$\dot\theta = \frac{\partial H}{\partial I} = \omega(I) , \quad \dot I = 0$$

Applying the same transf^n when λ varies in time, we obtain a new Hamiltonian $K = H + \frac{\partial F}{\partial t} = H + G\dot\lambda$

where $G(\theta, I; \lambda) = \frac{\partial F}{\partial \lambda}$

n sure what fixed? ∂

$$K = H + G\dot{\lambda}$$

Hamilton's equations are then

$$\dot{\theta} = \omega(I) + \frac{\partial G}{\partial I}\dot{\lambda}$$

$$\dot{I} = -\frac{\partial G}{\partial \theta}\dot{\lambda}$$

Over one oscillation, only θ changes significantly.

Clearly $\dot{I}$ is of first order in the small quantity $\dot{\lambda}$.

To this order we can evaluate the change in I over one oscillation by approximating $\lambda \approx \lambda_0 = \text{const.}$, $\dot{\lambda} \approx (\dot{\lambda})_0 = \text{const.}$, $I \approx I_0 = \text{const.}$ and $\theta \approx \theta_0 + \omega_0 t$ where $\theta_0 = \text{const.}$ and $\omega_0 = \omega(I_0)$.

$$\Delta I = \int \dot{I}\, dt \approx \int_0^{2\pi} -\frac{\partial G}{\partial \theta}(\dot{\lambda})_0 \frac{d\theta}{\omega_0}$$

$$\approx -\frac{(\dot{\lambda})_0}{\omega_0}\int_0^{2\pi} \frac{d}{d\theta}\left(G(\theta, I_0, \lambda_0)\right) d\theta$$

$\leftarrow$ G single valued

$$\approx 0$$

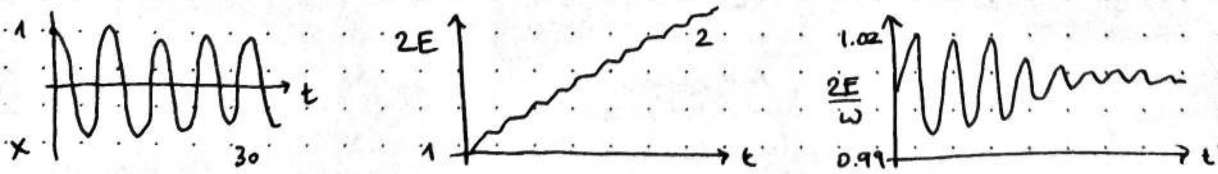
The variations in I are purely oscillatory and do not accumulate, to first order in $\dot{\lambda}$. So I is an adiabatic invariant.

More generally, if a Hamiltonian system is integrable when $\lambda = \text{const.}$, then a similar argument shows that the <u>action variables are adiabatic invariants</u>.

<u>Ex</u> If the natural frequency ω of a harmonic oscillator is varied slowly, the energy E changes, but the action $\frac{E}{\omega}$ is approximately conserved.

(This is reassuring because, in quantum mechanics, the energy of the harmonic oscillator is $E = (n + \frac{1}{2})\hbar\omega$, where n is an integer)

Numerical solution of $\ddot{x} + \omega^2 x = 0$ with $\omega^2 = 1 + 0.1t$:



small range!

Ex: If the central mass of the Kepler problem is varied slowly, the orbital eccentricity is approximately conserved, because the actions I_r and I_θ are adiabatic invariants and

$$1-e^2=\left(\frac{I_\theta}{I_r+I_\theta}\right)^2$$

4.11 Point vortices

The motion of an incompressible fluid in 2D is described by a streamfunction $\psi(x,y,t)$. The velocity field is

$$\underline{v} = \underline{\nabla} \times (\psi \underline{e}_z) = \underline{\nabla}\psi \times \underline{e}_z$$

or

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}$$

satisfying $\underline{\nabla} \cdot \underline{v} = 0$ (incompressible flow)

and $\underline{v} \cdot \underline{\nabla}\psi = 0$ (flow parallel to contours of ψ).

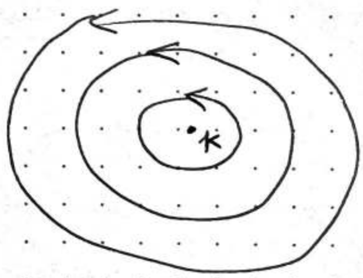
The vorticity is

$$\omega = \underline{e}_z \cdot (\underline{\nabla} \times \underline{v}) = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\nabla^2 \psi$$

The motion of an unbounded fluid is determined by its vorticity distribution. In an ideal fluid, the vortices move with the fluid velocity.

A point vortex of strength κ at the origin corresponds to the point source streamfunction

$$\psi = -\frac{\kappa}{2\pi} \log r = -\frac{\kappa}{4\pi} \log(x^2+y^2)$$



Note that $\omega = -\nabla^2 \psi = 0$ for $r \neq 0$

In fact $\omega = \kappa\, \delta(\underline{r})$

In polar coordinates

$$v_r = 0, \quad v_\theta = \frac{\kappa}{2\pi r},$$

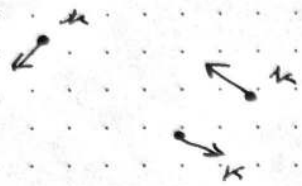
and in Cartesian components

$$v_x = -\frac{\kappa y}{2\pi(x^2+y^2)}, \quad v_y = \frac{\kappa x}{2\pi(x^2+y^2)}$$

Note that $\underline{v} = \underline{\nabla}\left(\frac{\kappa\theta}{2\pi}\right)$. The strength κ is also called the circulation, because $\oint \underline{v} \cdot d\underline{r} = \kappa$

around any closed curve that encloses the vortex once in a positive sense.

Suppose we have n point vortices with strengths κ_i at locations (x_i, y_i).
The combined velocity field is the sum of those generated by each vortex.



Therefore the vortices move according to

$$\dot{x}_i = -\frac{1}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\kappa_j (y_i - y_j)}{r_{ij}^2}$$

$$\dot{y}_i = \frac{1}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\kappa_j (x_i - x_j)}{r_{ij}^2}$$

where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$.

This can be written as

$$\dot{x}_i = \frac{1}{\kappa_i} \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{1}{\kappa_i} \frac{\partial H}{\partial x_i}$$

where

$$H = -\frac{1}{4\pi} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \kappa_i \kappa_j \log r_{ij}$$

H is the part of the kinetic energy of the fluid (divided by its mass per unit area) that depends on the interaction of the vortices. Note that

$$\frac{\partial \log r_{ij}}{\partial x_i} = \frac{x_i - x_j}{r_{ij}^2}$$

This looks like a Hamiltonian system but with coordinates x and y playing the roles of both coordinates and momenta!

To deal with the factors of κ, we can either identify the canonical variables as $q_i = \sqrt{\kappa_i}\, x_i$, $p_i = \sqrt{\kappa_i}\, y_i$, or we can use a non-standard Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \frac{1}{\kappa_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right),$$

which still satisfies antisymmetry, linearity, Leibniz and Jacobi.

Then we have $\dot{x}_i = \{x_i, H\}$, $\dot{y}_i = \{y_i, H\}$.

Since H is invariant under translations in time, translations in space and rotations, we have conservation of energy (H), linear momentum,

$$P_x = \sum_{i=1}^{n} \kappa_i y_i, \qquad P_y = -\sum_{i=1}^{n} \kappa_i x_i$$

and angular momentum

$$L = -\sum_{i=1}^{n} \frac{1}{2}\kappa_i (x_i^2 + y_i^2),$$

e.g. the infinitesimal canonical transf^n generated by P_x produces the perturbation $\delta x_i = \varepsilon \{x_i, P_x\} = \frac{\varepsilon}{\kappa_i}\frac{\partial P_x}{\partial y_i} = \varepsilon$

$$\delta y_i = \varepsilon \{y_i, P_x\} = -\frac{\varepsilon}{\kappa_i}\frac{\partial P_x}{\partial x_i} = 0$$

The infinitesimal canonical transf^n generated by L produces the perturbation $\delta x_i = \varepsilon \{x_i, L\} = \frac{\varepsilon}{\kappa_i}\frac{\partial L}{\partial y_i} = -\varepsilon y_i$

$$\delta y_i = \varepsilon \{y_i, L\} = -\frac{\varepsilon}{\kappa_i}\frac{\partial L}{\partial x_i} = \varepsilon x_i$$

and H is invariant under this rotation of all the vortices about the origin.

Their Poisson brackets are

$$\{\underline{P}, H\} = 0, \quad \{L, H\} = 0, \quad \{P_x, P_y\} = \sum \kappa_i = \text{const.}$$

$$\{P_x, L\} = -P_y, \quad \{P_y, L\} = P_x$$

By the Leibniz rule, $|P|^2 = P_x^2 + P_y^2$ Poisson-commutes with both H and L.

We have three integrals of motion, H, L and $|P|^2$ that are independent and in involution. So the system is integrable for $n \leq 3$, but not otherwise. Indeed, chaos can occur with four vortices.

If $K = \sum \kappa_i \neq 0$, the centre of vorticity (X, Y) and the dispersion D can be defined by

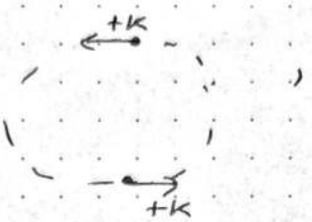
$$X = \frac{1}{K}\sum \kappa_i x_i, \qquad Y = \frac{1}{K}\sum \kappa_i y_i$$

$$D = \frac{1}{K}\sum \kappa_i \left((x_i - X)^2 + (y_i - Y)^2\right)$$

These are conserved because $\underline{P}$ and L are

L23.4

The motion of two vortices is very simple: their separation d is constant and they rotate in circular motion about the centre of vorticity with angular velocity $\dfrac{\kappa_1+\kappa_2}{2\pi d^2}$



unless $\kappa_1+\kappa_2=0$; in which case they move in parallel straight lines, perpendicular to their separation, with speed $\frac{\kappa_1}{2\pi d}$

If n vortices of equal strength are placed at the vortices of a regular polygon, the system will rotate uniformly. But the configuration is unstable for $n>7$.

4.12 Classical spin in a magnetic field

Elementary particles have two types of angular momentum: orbital angular momentum $\underline{r} \times \underline{p}$ and spin, which is an intrinsic property of particles. Both forms of angular momentum are associated with a magnetic (dipole) moment.

Spin is properly described by quantum mechanics (e.g. the Dirac equation) but here we discuss a useful classical model.

Consider a rotating body with mass m and charge q, centred on the origin. Within the body, the velocity is $\underline{v}(\underline{r})$, the mass density is $\rho_m(\underline{r})$ and the charge density is $\rho_q(\underline{r})$.

The angular momentum is

$$\underline{L} = \int \underline{r} \times \underline{v}\, \rho_m \, dV = \int \underline{r} \times \underline{v}\, dm$$

and the magnetic dipole moment is

$$\underline{m} = \int \tfrac{1}{2} \underline{r} \times \underline{J} \, dV = \tfrac{1}{2} \int \underline{r} \times \underline{v} \, dq$$

where $\underline{J} = \rho_q \underline{v}$ is the electric current density.

If the charge-to-mass ratio is uniform;

$$\frac{\rho_q}{\rho_m} = \frac{dq}{dm} = \frac{q}{m}$$

then $\underline{m} = \dfrac{q}{2m} \underline{L}$

In quantum mechanics, magnetic moment and angular momentum are related by $\underline{m} = \gamma \underline{L}$ where $\gamma = \dfrac{gq}{2m}$ is the gyromagnetic ratio and g is a correction factor (the 'g-factor' or dimensionless magnetic moment), which is slightly larger than 2 for electron spin (but 1 for electron orbits).

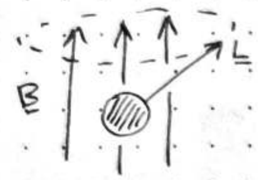
From electromagnetism, the potential energy of a magnetic dipole in an external magnetic field $\underline{B}$ is $V = -\underline{m} \cdot \underline{B}$ and the torque on the dipole is $\underline{G} = \underline{m} \times \underline{B}$.

So the spin evolves according to

$$\dot{\underline{L}} = \underline{G} = \gamma \underline{L} \times \underline{B} = (-\gamma \underline{B}) \times \underline{L}$$

$|\underline{L}|$ is conserved, but the spin axis precesses around $\underline{B}$ at angular frequency $\omega = -\gamma B = -\frac{gqB}{2m}$

This Larmor precession is the basis for MRI (nuclear magnetic resonance imaging).



Can we derive this equation of motion from a Hamiltonian?

For a charged spherical top in a magnetic field, the energy is $H = \frac{|\underline{L}|^2}{2I} - \gamma \underline{L} \cdot \underline{B}$ where I is the moment of inertia

The components of $\underline{L}$ satisfy

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k$$

So $\dot{L}_i = \{L_i, H\} = \{L_i, L_j\} \frac{L_j}{I} - \gamma \{L_i, L_j\} B_j$

$$= 0 - \gamma \varepsilon_{ijk} L_k B_j$$

giving $\dot{\underline{L}} = -\gamma \underline{B} \times \underline{L}$ as required.

4.13 * Classical dynamics and quantum mechanics

Commutators and Poisson brackets

QM uses Hermitian operators (or matrices) acting on cx vector spaces, to represent observables (physical quantities); e.g. position $\underline{r}$, momentum $\underline{p}$ and (orbital) angular momentum $\underline{L} = \underline{r} \times \underline{p}$, as well as spin.

The commutator of two operators is defined as

$$[A, B] = AB - BA$$

The position and momentum operators satisfy the canonical commutation rel^{n}s

$$[x_i, p_j] = i\hbar \delta_{ij}$$

We also have, e.g.

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$

$$[L_i, |\underline{L}|^2] = 0$$

and similarly for spin angular momentum.

This is also highly reminiscent of Poisson brackets in classical dynamics with $[A, B] \sim i\hbar \{A, B\}$.

The expectation value of an observable is given by the inner product $\langle \psi | A | \psi \rangle$ where $|\psi\rangle$ is the <u>state vector</u>.

- In the <u>Schrödinger picture</u>, A is time-independent, e.g. $\underline{p} = -i\hbar \underline{\nabla}$ but $|\psi\rangle$ evolves.

In the <u>Heisenberg picture</u>, $|\psi\rangle$ is time-independent, but A evolves according to
$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H] + \frac{\partial A}{\partial t}$$
which corresponds to
$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$
in classical dynamics.

<u>The Hamilton-Jacobi and Schrödinger equations</u>

Suppose that $F_2(\underline{q}, \underline{P}, t)$ is the generating function of a canonical transformation $(\underline{q}, \underline{p}) \mapsto (\underline{Q}, \underline{P})$ such that the Hamiltonian vanishes:
$$K = H + \frac{\partial F_2}{\partial t} = 0$$
Then
$$\underline{p} = \frac{\partial F_2}{\partial \underline{q}}, \quad \underline{Q} = \frac{\partial F_2}{\partial \underline{P}}$$
and Hamilton's equations become trivial:
$$\dot{\underline{Q}} = \frac{\partial K}{\partial \underline{P}} = 0, \quad \dot{\underline{P}} = -\frac{\partial K}{\partial \underline{Q}} = 0$$

- The total-time derivative of F_2 is
$$\frac{dF_2}{dt} = \frac{\partial F_2}{\partial \underline{q}} \cdot \dot{\underline{q}} + \frac{\partial F_2}{\partial \underline{P}} \cdot \dot{\underline{P}} + \frac{\partial F_2}{\partial t}$$
$$= \underline{p} \cdot \dot{\underline{q}} - H = L$$
so
$$F_2 = S(\underline{q}, t) = \int L \, dt$$
is the action, considered as an indefinite integral from some initial configuration to the current configuration $(\underline{q}, t)$ along the physical path.

$S(\underline{q}, t)$ is known as

- <u>Hamilton's characteristic function</u>

Given a Hamiltonian function $H(\underline{q},\underline{p},t)$, S satisfies the <u>Hamilton-Jacobi equation</u>

$$-\frac{\partial S}{\partial t} = H\left(\underline{q}, \frac{\partial S}{\partial \underline{q}}, t\right)$$

This is a first-order non-linear PDE for $S(\underline{q},t)$.

An important example is

$$H = \frac{|\underline{p}|^2}{2m} + V(\underline{r})$$

in which case

$$-\frac{\partial S}{\partial t} = \frac{|\nabla S|^2}{2m} + V(\underline{r})$$

This is closely related to the Schrödinger eq^n

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\underline{r})\psi$$

for the wavefunction $\psi(\underline{r},t)$ in QM.

Let $\psi = R e^{iS/\hbar}$ where $R(\underline{r},t)$, $S(\underline{r},t)$ are real.

The real part of the Schrödinger eq^n, divided by ψ, is

$$-\frac{\partial S}{\partial t} = \frac{|\nabla S|^2}{2m} + V(\underline{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

which differs from the Hamilton-Jacobi equation only by a quantum correction to the potential.

<u>Actions and path integrals</u>

The old quantum theory of Bohr and Sommerfeld is based on a quantization of the classical action variables:

$$\frac{1}{2\pi}\oint p_i \, dq_i = n\hbar, \quad n \in \mathbb{Z}$$

Later, Feynman showed that QM can be derived from an extension of Hamilton's principle. The wavefunction $\psi(\underline{q},t)$ can be written as

$$\psi = C\int e^{iS[\underline{q}]/\hbar} \, D\underline{q} \quad (*)$$

where $C = \text{const}$ and $S[\underline{q}] = \int L \, dt$

is the action integral over <u>any</u> path $\underline{q}(t)$ connecting the current state to a fixed initial state.

Eq^n (*) is a functional integral over all possible paths, known as a path integral or sum over histories. Different paths have equal weighting but different phases.

In the classical limit $S \gg \hbar$, appropriate for a macroscopic system or when the quantum numbers are very large, there is substantial cancellation because the phase varies rapidly with the path.

However, constructive interference occurs in the nghd of the classical path $q_c(t)$ for which S has a stationary value, so that ψ is strongly peaked near the classical path.