Chap I Smooth mfds & maps

1.1 Definitions

Def • $U \subset \mathbb{R}^n$ open. A map $f: U \to \mathbb{R}^m$ is smooth iff it has (continuous) partial derivatives of all orders ($= C^\infty$).

• $X \subset \mathbb{R}^n$ any subset. $f: X \to \mathbb{R}^m$ is smooth iff $\forall x \in X$, $\exists$ open (in $\mathbb{R}^n$) ngbd, $x \in U \subset \mathbb{R}^n$, and smooth $F: U \to \mathbb{R}^m$

s.t. $f|_{X\cap U} = F|_{X \cap U}$

Smoothness is a "local property" (vs "global property")

Def A smooth map $f: X^{\subset \mathbb{R}^n} \to Y^{\subset \mathbb{R}^m}$ is a diffeomorphism iff it's bijective & the inverse $f^{-1}: Y \to X$ is smooth.

X & Y are called diffeomorphic

Def $X \subset \mathbb{R}^N$ is a k-dimensional manifold if each point has an open (in X) ngbd $U \in X$ which is diffeomorphic to an open subset of $\mathbb{R}^k$.

A diffeo^m $\varphi: U \to V$, U open in $\mathbb{R}^k$, is called a parametris^n of $V \subset X$. The inverse diffeo^m $\varphi^{-1}: V \to U$ is called a chart, or a coordinate system, on V.

Can write $\varphi^{-1} = (\underbrace{x_1, \dots, x_k}_{\substack{f^n\text{s on } V, \\ \text{"coordinate" } f^n\text{s}}})$

Dimension written $k = \dim X$.

Ex Unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, smooth mfd of dim 2.

• The diffeo $(x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$ for $x^2+y^2<1$ parametrises the upper half sphere

• Similarly can parametrise the other hemispheres (southern, eastern, ...) whence S^2 is a smooth mfd.

* $S^k = \{ x \in \mathbb{R}^{k+1} \mid \|x\| = 1 \}$ is a smooth k-dim mfd

Exercise (ES1) If $\underset{\subset\,\mathbb{R}^m}{X}, \underset{\subset\,\mathbb{R}^n}{Y}$ smooth mfds, so is $X \times Y \subset \mathbb{R}^{m+n}$.
And $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Def • If X, Z are both manifolds in $\mathbb{R}^N$, and $Z \subset X$.
Then Z is called a submanifold of X.

• X is itself a submanifold of $\mathbb{R}^N$

• any open subset $U \subset X$ is a submanifold of X

The codimension of Z in X is $\dim X - \dim Z$

1.2 Tangent spaces & derivatives

Recall (A&T): for $U \subset \mathbb{R}^n$ open, $f\ (= (f_1, \dots, f_m)) : U \to \mathbb{R}^m$ smooth, the derivative of f at $x \in U$ is a linear map

$Df_x : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(x+h) = f(x) + Df_x(h) + \varepsilon(h)$,

where $\frac{\|\varepsilon(h)\|}{\|h\|} \to 0$ as $h \to 0$.

⚠ Paternain uses df_x

$$Df_x = \left(\frac{\partial f_i}{\partial x_j}\right)_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_x$$

← the Jacobian matrix

(Note $Df_x(e_i) = \frac{\partial f}{\partial x_i}$)

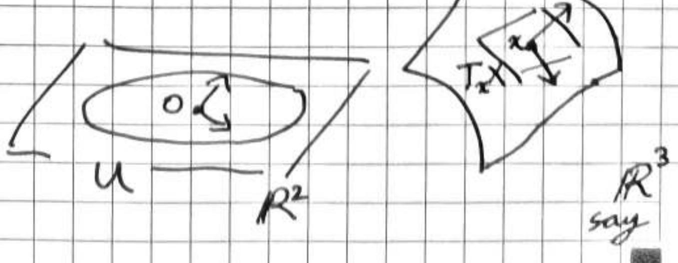
Def • For $x \in U \overset{\text{open}}{\subset} \mathbb{R}^n$, the tangent space at x is just $\mathbb{R}^n$.

• For an arbitrary smooth mfd $X \in \mathbb{R}^N$, we define the tangent (tgt) space to X at x as follows:

Local parametrisation $\varphi : \underset{\subset\,\mathbb{R}^k}{U} \to X$, with $\varphi(0) = x$.

Define $T_x X = D\varphi_0(\mathbb{R}^k)$

linear map $\mathbb{R}^k \to \mathbb{R}^N$



<u>Lemma</u> $T_x X$ is independent of the choice of param^n φ, and $\dim T_x X = k$.

<u>Pf</u> Suppose $\psi: \cancel{U}\, V \to X$ another param^n with $\psi(0) = x$.
$V \subset \mathbb{R}^{k'}$

Shrink U, V if necessary, assume $\varphi(U) = \psi(V) = W \subset X$.

h is a diffeo

$$\begin{tikzcd} & W & \\ U \arrow[ur, "\varphi"] \arrow[rr, "h:=\psi^{-1}\circ\varphi"'] & & V \arrow[ul, "\psi"'] \end{tikzcd}$$

By the chain rule,

$$D\varphi_0 = D\psi_0 \circ Dh_0$$

Dh_0 is invertible, so $D\varphi_0$ & $D\psi_0$ have the same image in $\mathbb{R}^N$ and $k = k'$

$\varphi^{-1}: W \to U$ smooth, so we can choose $\tilde{W}$ open in $\mathbb{R}^N$, containing x, s.t there's a smooth map $\Phi: \tilde{W} \to \mathbb{R}^k$ extending φ^{-1}.

Then $\Phi \circ \varphi$ is the identity on a ngbd of $\varphi^{-1}(x) = 0$.

So by the chain rule $D\Phi_x \circ D\varphi_0 = id_{\mathbb{R}^k}$

In particular $D\varphi_0$ has full rank and we are done. $\square$

<u>Rmk</u> can think of coset $\underbrace{x + T_x X}_{\text{did this in picture}} \subset \mathbb{R}^N$ as being a linear approximation to $X \subset \mathbb{R}^N$ at x.

Suppose $f: X^{\subset \mathbb{R}^n} \longrightarrow Y^{\subset \mathbb{R}^m}$ is a smooth map between mfds.

i.e. given $x \in X$, $\exists$ open $U \subset \mathbb{R}^n$, $x \in U$ and smooth $F: U \to \mathbb{R}^m$ with $F|_{U \cap X} = f|_{U \cap X}$

We <u>define</u> $Df_x : T_x X \to T_{f(x)} Y$ to be the restriction of $DF|_x$ to the tangent space $T_x X \subset \mathbb{R}^n$.

<u>Lemma</u> The restriction $DF_x|_{T_xX} : T_xX \longrightarrow \mathbb{R}^m$ does not depend on the choice of F, and its image lies in $T_{f(x)}Y$.

(<u>Csq</u>: Df_x well-defined)

<u>Proof</u> Let $y = f(x) \in Y$, and take parametrizations

$$\begin{array}{ccc} \varphi : U \to X \subseteq \mathbb{R}^n, & & \psi : V \longrightarrow Y \subseteq \mathbb{R}^m \\ \cap\mid & & \cap\mid \\ \mathbb{R}^k \leftarrow \dim X & & \mathbb{R}^l \leftarrow \dim Y \end{array}$$

s.t. $\varphi(0) = x$, $\psi(0) = y$ and wlog $f(\varphi(U)) \subset \psi(V)$.

Set $\varphi(U) = W \cap X$, W open in $\mathbb{R}^n$, and say $F : W \to \mathbb{R}^m$ extends $f|_{W\cap X}$

$$\begin{tikzcd} \varphi(U) = W \cap X \arrow[r, "f"] & \psi(V) \\ U \arrow[u, "\varphi", "\text{(diffeo)}"'] \arrow[r, "{h := \psi^{-1}\circ f \circ \varphi = \psi^{-1}\circ F\circ\varphi}"'] & V \arrow[u, "\psi"', "\text{(diffeo)}"] \end{tikzcd}$$

h only depends on f and the param^{n}s

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "DF_x"] & \mathbb{R}^m \\ T_xX \arrow[u, phantom, "\cup"] & T_yY \arrow[u, phantom, "\cup"] \\ \mathbb{R}^k \arrow[u, "D\varphi_0"] \arrow[r, "Dh_0"'] & \mathbb{R}^l \arrow[u, "D\psi_0"'] \end{tikzcd}$$

(diagram above for derivatives)

$$DF_x \circ D\varphi_0 = D\psi_0 \circ Dh_0 \qquad (*)$$

so the image lies in T_yY

⌜ $D\varphi_0$ surj so $\operatorname{Im} DF_x|_{T_xX} = \operatorname{Im}(DF_x \circ D\varphi_0)$
$= \operatorname{Im}(D\psi_0 \circ Dh_0) \subset T_yY$ ⌟

$(*) \Rightarrow \underbrace{Dh_0}_{\text{indep of } F} = (D\psi_0)^{-1} \circ DF_x \circ D\varphi_0$

□

Cor Chain rule If $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $Z \subset \mathbb{R}^p$

are smooth maps of manifolds, then

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x .$$

Pf Immediate from chain rule for maps $\mathbb{R}^n \to \mathbb{R}^m \to \mathbb{R}^p$. $\square$

$\boxed{1.2}$ Inverse function theorem

(Version I assume from IB:)

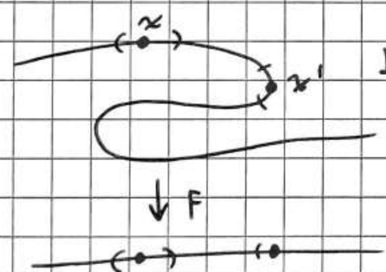
If $U \subset \mathbb{R}^n$ open, and $f : U \ni x \to \mathbb{R}^n$ smooth s.t. $Df_x : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then f is a local diffeo^m at x.

Def $f : X \to Y$ smooth map between mfds

We say f is a local diffeo^m at $x \in X$ if f maps an open nghd of x in X diffeomorphically to its image, an open nghd of $f(x)$ in Y.

Thm (Inverse function thm) Suppose $f : X \to Y$ is smooth, and $Df_x : T_x X \to T_{f(x)} Y$ is an isomorphism. Then f is a local diffeo^m at x.

e.g.



$Df_{x'}$ not an isom

Pf In the notation of the previous proof, apply the IB IFT with $n = k = l$ to the smooth map $h : U \to V$.

1.3 Regular values & Sard's thm

<u>Def</u> A smooth map $f: X \to Y$ is called a <u>submersion</u> or <u>regular</u> at $x \in X$ if $Df_x : T_x X \to T_{f(x)} Y$ is onto.

Let $C \subset X$ be the collection of non-regular points.

<u>Def</u> A point of C is called a <u>critical point</u>.

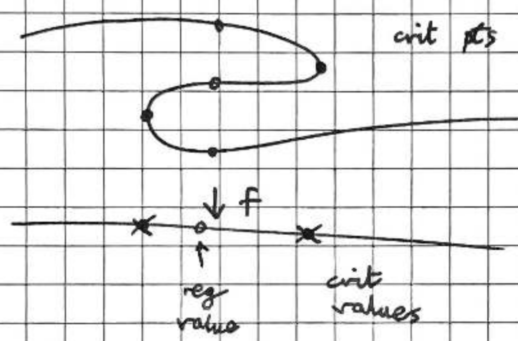
A point in $f(C)$ is a <u>critical value</u>.

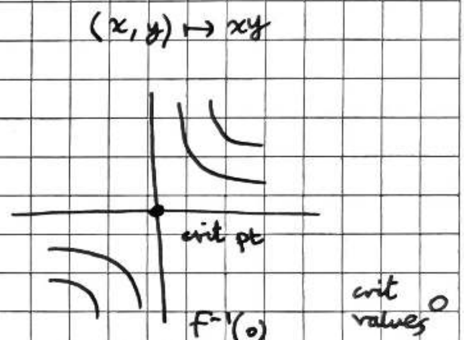
A point in $Y \setminus f(C)$ is a <u>regular value</u>.

all preimages are regular points

<u>Rk</u> If $\dim X < \dim Y$, then Df_x is never surjective, so $C = X$ and $y \in Y$ is regular $\iff y \notin f(X)$.

<u>e.g.</u>





<u>Thm</u> (pre-image thm)

Let y be a regular value of $f: X \to Y$, w/ $\dim X \geq \dim Y$.

Then $f^{-1}(y)$ is a submanifold of X.

Assuming $y \in f(X)$, we have $\dim f^{-1}(Y) = \dim X - \dim Y$.

<u>Pf</u> Assume $y \in f(X)$, and let $x \in f^{-1}(y)$.

y is a regular value, so $Df_x : T_x X \to T_y Y$ is surj.

Let $K = \ker(Df_x)$, $\dim K = \dim X - \dim Y =: p$.

Say $X \subset \mathbb{R}^n$. Choose $T: \mathbb{R}^n \to \mathbb{R}^p$ to be any linear map with $\ker T \cap K = \{0\}$

Consider map
$$F: X \to Y \times \mathbb{R}^p$$
$$z \mapsto (f(z), T(z))$$

$DF_x(v) = (Df_x(v), T(v))$ for $v \in T_x X$

non-singular by our choice of T (!)

IFT for $F \Rightarrow F$ is a local diffeo at x, so F maps some open ngbd U of x diffeo^m^ly onto a ngbd V of $(y, T(x))$

Hence F maps $f^{-1}(y) \cap U$ diffeo^m^ly onto $(\{y\} \times \mathbb{R}^p) \cap V$.

Hence we have a local chart around x, and $f^{-1}(y)$ is a manifold of dim p. $\square$

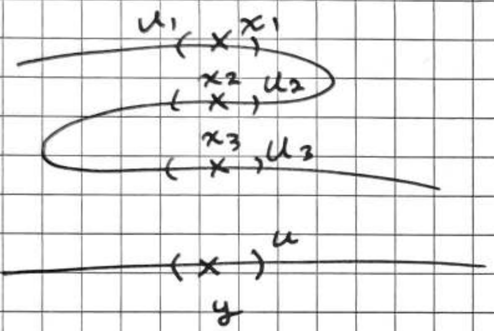
Cor Let $f: X \to Y$ smooth map, assume $\dim X = \dim Y$.

Then if X is cpct and y is a regular value of f, then $f^{-1}(y)$ is a finite set of points.

Pf $f^{-1}(y)$ is a 0-dim^l^ mfd by IFT; there can't be an accumulation point, as at each regular pt, f is a local diffeo^m^. $\square$

Rmk on preimage theorem: pf shows that $T_x f^{-1}(y) = \ker Df_x$

Thm (Stack of records theorem) Let $f: X \to Y$ smooth map b/w manifolds of the same dimension and with X compact.
Let y be a regular value, and $f^{-1}(y) = \{x_1, \dots, x_n\}$.
Then there exists an open ngbd of $y \in \mathcal{U}$ in Y s.t. $f^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_1 \sqcup \mathcal{U}_2 \sqcup \cdots \sqcup \mathcal{U}_k$ where each $\mathcal{U}_i$ is an open ngbd of x_i in X & f maps each $\mathcal{U}_i$ diffeomorphically onto $\mathcal{U}$.



"stack of USB sticks"

Pf By IFT we can pick disjoint ngbds W_i of x_i in X s.t. f maps W_i diffeo^{ly} onto $f(W_i) \ni y$.
Observe $f(X \setminus \cup_i W_i)$ is compact, hence closed in Y.
Then $\mathcal{U} := \bigcap_i f(W_i) \setminus f(X \setminus \cup_i W_i)$ works. $\square$

So if $\# f^{-1}(y)$ denotes the number of pts in $f^{-1}(y)$, stack of records thm $\Rightarrow$ the f^n $y \mapsto \# f^{-1}(y)$ is locally constant as y ranges over regular values.

Applications of preimage thm

Ex 1 $f: \mathbb{R}^{n+1} \to \mathbb{R}$, $\underline{x} \mapsto \sum x_i^2$, then

$$Df_{(a_1,\dots,a_{n+1})}(h_1,\dots,h_{n+1}) = 2\sum a_i h_i$$

zero is the only critical value

So $f^{-1}(1) = S^n$ is a smooth n-manifold.

Ex2 (Orthogonal gp)

$$O(n) = \{ A \in \underbrace{M(n)}_{\substack{\text{real } n\times n \\ \simeq \mathbb{R}^{n^2}}} \mid AA^T = I \}$$

Claim $O(n)$ is a smooth submfd of dim $\frac{n(n-1)}{2}$

Pf Let $S(n) \subset M(n)$ be the subspace of symmetric matrices which is $(\simeq \mathbb{R}^{\frac{n(n+1)}{2}})$

$T_S M(n) \simeq M(n)$, $T_S S(n) \simeq S(n)$

$f: M(n) \longrightarrow S(n)$ given by $f(A) = AA^T$ $(\Rightarrow f^{-1}(I) = O(n))$

STP I is a regular value of f

Compute (A & T): $Df_A(H) = HA^T + AH^T$

Fix $A \in O(n)$. Subclaim Given $C \in T_{f(A)} S(n)$ $(\simeq S(n))$,

$\exists H \in T_A M(n)$ $(\simeq M(n))$

s.t. $Df_A(H) = HA^T + AH^T = C$

Pf of subclaim Can solve $HA^T = \frac{1}{2} C$ for H

Then $AH^T = \frac{1}{2} C^T = \frac{1}{2} C$,

$\therefore Df_A(H) = C$. $\square$

So Df_A is surjective, and I is a regular value. $\square$

1.3.1 Sard's Theorem

Def (P&M) A set $A \subset \mathbb{R}^n$ has measure zero if it can be covered by a countable collection of rectangular solids with arbitrarily small total volume. (i.e. given $\varepsilon > 0$, $\exists$ ctble collection $\{R_1, R_2, \dots\}$ of rectangles $R_i = [\alpha_1^{(i)}, \beta_1^{(i)}) \times \cdots \times [\alpha_n^{(i)}, \beta_n^{(i)})$ with $A \subset \bigcup_i R_i$ and $\sum_i \operatorname{vol} R_i < \varepsilon$.)

If X is a manifold, $A \subset X$ has measure zero if for every (check a few) local parametrisation φ of X, $\varphi^{-1}(A)$ has measure zero in $\mathbb{R}^n$.

Rmk A measure zero set cannot contain a non-empty open subset

Thm (Sard, 1942) The set of critical values of a smooth map $f: X \to Y$ has measure zero.

Pf Non-eximanable, see Milnor §3. $\square$

1.4 Transversality

$f: X \to Y$, $Z \subseteq Y$ submfd

What can we say about $f^{-1}(Z)$?

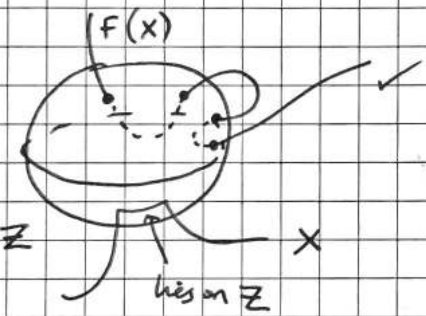
Def A smooth map $f: X \to Y$, ~~is~~ is transversal (or transverse) to a submfd $Z \subset Y$ if $\forall x \in f^{-1}(Z)$,

$$\mathrm{Im}\, Df_x + T_{f(x)} Z = T_{f(x)} Y.$$

not necessarily direct

Write $f \pitchfork Z$.

$Y = \mathbb{R}^3$,



Def $g: Z \to Y$ is an immersion at $y \in Z$ if $Dg_y : T_y Z \to T_{g(y)} Y$ is an injection (immersion $\leftrightarrow$ injective D $g: \hookrightarrow$)

submersion $\leftrightarrow$ surjective D e.g. $S^1 \to \mathbb{R}^2$

Local immersion theorem If $g: Z \hookrightarrow Y$, then we can choose local coords on Z, Y s.t. g is locally of the form $g(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$

Pf (ES1) Hint: $\underset{Z}{\mathcal{U} \subseteq}$, $\underset{Y}{V \subseteq}$, build $G: \mathcal{U} \times \mathbb{R}^{\mathrm{codim}} \to V$ & apply IFT. $\square$

Thm (Generalised preimage thm)

If a smooth $f: X \to Y$ is transversal to a submanifold $Z \subset Y$, then $f^{-1}(Z)$ is a submanifold of X, and

$$\operatorname{codim} f^{-1}(Z) = \operatorname{codim}(Z)$$

Pf By the local immersion theorem, Z can be written in a ngbd of a pt $y = f(x) \in Z$ as the zero set of smooth functions $\underbrace{h_1, \ldots, h_r}_{\substack{\text{coords} \\ k+1, \ldots, k+r \text{ which} \\ \text{zero on } Z}}$ where $r = \operatorname{codim} Z$.

Let $H = (h_1, \ldots, h_r)$. Then near x, $f^{-1}(Z)$ is the zero set of $H \circ f$.

Thus if $\underline{0} \in \mathbb{R}^r$ is a regular value of $H \circ f$, done by preimage theorem.

$D(H \circ f)_x = DH_y \circ Df_x$ is surjective

iff $\operatorname{Im}(Df_x) + T_{f(x)} Z = T_{f(x)} Y$

because $DH_y : T_y Y \to \mathbb{R}^r$ is surjective with kernel $T_{f(x)} Z$.

by construction / def of H.

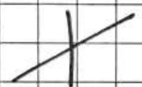
$\square$

<u>Rk</u> In previous thm : if $Z = \{pt\}$, recover the preimage theorem

If $f : X \hookrightarrow Y$ inclusion of a submanifold, then X is transversal to another submanifold iff $T_xX + T_xZ = T_xY \quad \forall x \in X$

e.g. in $\mathbb{R}^2$

$G^{\underline{\text{nled}}}$ preimage thm $\Rightarrow$ the intersection of two transversal submflds X and Z of $\mathbb{R}^n$ is also a submanifold with $\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z$

<u>Fact</u> • Transversality of a map is a <u>stable</u> condition, i.e. the condition survives under small perturbations (reasonably clear using continuity )

• Arbitrary smooth maps may be deformed by an arbitrarily small amount to maps which are transverse to Z. ("tranverse maps are generic") Pf uses Sard's theorem.

(e.g. Guillemin & Pollack, Ch 2. § 3)

<u>1.5 Manifolds w/ boundary</u>

<u>Def</u> Half space $\mathbb{H}^k := \{ (x_1, \dots, x_k) \in \mathbb{R}^k \mid x_k \geq 0 \}$

The boundary $\partial \mathbb{H}^k$ is defined to be the hyperplane $\{x_k = 0\}$.

<u>Def</u> • A subset $X \subset \mathbb{R}^n$ is called a smooth manifold with boundary if each $x \in X$ has an open nbhd $U \subset X$ diffeo$^{\underline{c}}$ to an open subset of $\mathbb{H}^k$.

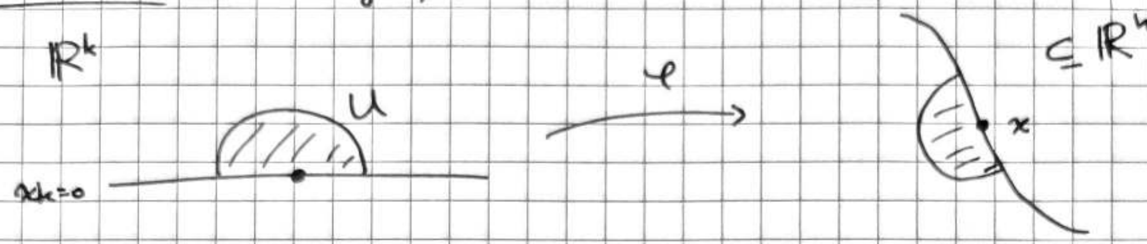
Such a diffeo^m is (no boundary $\leadsto \mathbb{R}^k$) called a chart / parametrisation as before.

• The boundary of X, denoted ∂X, consists of points which belong to the image of $\partial \mathbb{H}^k$ under <u>some</u> local param^n

ES1 $\Rightarrow$ any local param^n

- The interior of X, $\mathrm{Int}\, X$, is defined to be $X \setminus \partial X$.

<u>New feature</u>: boundary points



<u>Derivatives</u>: parametrisation φ, pick an extension $\tilde{\varphi}$ to a smooth map on an open ngbd of 0 in $\mathbb{R}^k$ (if needed, shrink $\mathcal{U}$)

$$D\tilde{\varphi}_0 = \lim_{\substack{a \to 0 \\ a \in \mathrm{Int}\, \mathbb{H}^k \cap \mathcal{U}}} D\varphi_a \qquad (\varphi, \tilde{\varphi} \text{ have cts partials})$$

So $D\tilde{\varphi}_0$ depends only on φ, not on the choice of extension $\tilde{\varphi}$

Thus can define $D\varphi_0 = D\tilde{\varphi}_0$.

<u>Tangent spaces</u>: φ^{-1} extends to a smooth map Φ on an open ngbd W of x in $\mathbb{R}^n$. $\Phi \circ \varphi = \mathrm{id}$ on $\mathrm{Int}\, \mathbb{H}^k \cap \mathcal{U}$

$\Rightarrow D\Phi \circ D\varphi = \mathrm{id}_{\mathbb{R}^k}$ on $\mathrm{Int}\, \mathbb{H}^k \cap \mathcal{U}$

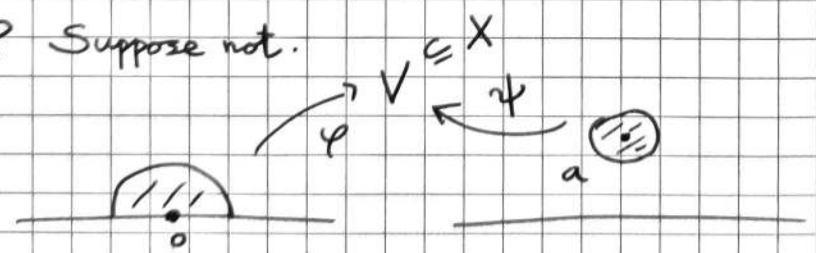
So by continuity, $D\Phi_x \circ D\varphi_0 = \mathrm{id}_{\mathbb{R}^k}$.

In particular, $D\varphi_0$ is injective $\mathbb{R}^k \to \mathbb{R}^n$, and we can define the tangent space (as before) as $D\varphi_0(\mathbb{R}^k)$ whenever $x \in \partial X$.

Idea for ES Q^n (above)

If $x \in \partial X$, then for any parametrisation ψ, $\psi^{-1}(x) \in \partial \mathbb{H}^k$.

Why? Suppose not.



$g = \varphi^{-1} \circ \psi$ is a diffeo

$Dg_a = (D\varphi_0)^{-1} \circ D\psi_a$

iso: $\mathbb{R}^k \to \mathbb{R}^k$

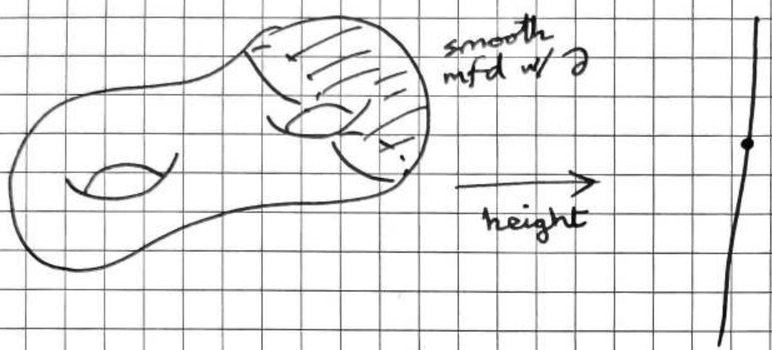
IFT $\Rightarrow$ g local diffeo from open ngbd of a in $\mathbb{R}^k$ to an open ngbd of 0 in $\mathbb{R}^k$ #

<u>Lemma</u> If X is a mfd w/o boundary, and $f: X \to \mathbb{R}$ is smooth w/ 0 a regular value, then the subset

$\{x \in X \mid f(x) \geq 0\}$ is a smooth manifold w/ boundary $f^{-1}(0)$.

<u>E.g.</u> • Unit ball B^k is given by $f(x) \geq 0$ in $\mathbb{R}^k$ where $f(x) = 1 - |x|^2$, so B^k is smooth mfd w/ boundary

•



<u>Pf</u> The set where $f > 0$ is open in X so automatically a mfd of the same dimension.

For $x \in X$ with $f(x) = 0$, the same proof as for the preimage theorem shows that (w/ $p = \dim X - 1$)

$\exists$ open ngbd U of x s.t.

$$\{x : f(x) \geq 0\} \cap U \underset{\text{diffeo}}{\simeq} \left(\{y \geq 0\} \times \mathbb{R}^{k-1}\right) \cap V$$

for V some ngbd of $(0, T(x))$ in $\mathbb{R}^k$

same notation as in preimage theorem, i.e. $\ker T \cap \ker Df_x = \{0\}$

($k = \dim X$)

Trading the roles of x_1 & x_k, done. $\square$

<u>Thm</u> Let $f: X \to Y$ be a smooth map w/:

- X : m-mfd w/ boundary
- Y : n-mfd w/o boundary, $m > n$

Assume $y \in Y$ is a regular value for both f and $f|_{\partial X}$.

Then $f^{-1}(y)$ is a smooth $(m-n)$ mfd w/ boundary $\cap\, \partial X$.

<u>Proof</u> By taking charts, wlog $f : \underset{\mathbb{H}^m}{\overset{\cap}{V}} \to \mathbb{R}^n$,
with $\partial V = \mathbb{R}^{m-1} \cap V$ and y
a regular value for f and $f|_{\partial V}$.

Consider $z \in f^{-1}(y)$.

- If $z \in \operatorname{Int} \mathbb{H}^m$ then use the preimage theorem
- If $z \in \partial \mathbb{H}^m$, see you next time (!)

Pf (cont.)

If $z \in \partial \mathbb{H}^m$, extend f to an open subset $\mathcal{U} \subseteq \mathbb{R}^m$ containing x, say $F : \mathcal{U} \to \mathbb{R}^n$.

Wlog F has no critical points in $\mathcal{U}$ (just shrink, $DF_z = Df_z$)

So $F^{-1}(y) \cap \mathcal{U}$ is a smooth manifold of dim $m-n$.

Consider $\pi : F^{-1}(y) \to \mathbb{R}$, $(x_1, \dots, x_m) \longmapsto x_m$.

Claim 0 is a regular value of π

Pf of claim Let $g = f|_{\mathbb{R}^{m-1} \cap V = \partial V} = f \circ i$
where $i : V \cap \mathbb{R}^{m-1} \hookrightarrow V \subset \mathbb{H}^m$.

So $dg_x = df_x|_{\mathbb{R}^{m-1}}$ (chain rule).

If 0 is a critical value of π, then $D\pi_x = 0$ for some $x \in \pi^{-1}(0)$.

$T_x F^{-1}(y) = \ker Df_x \subset \mathbb{R}^{m-1}$ since $D\pi_x = 0$

This implies that $Dg_x : \mathbb{R}^{m-1} \to \mathbb{R}^n$ has rank $n-1$.

So x is a critical point for $f|_{\partial X}$ $(= g)$.

This contradicts y a regular value of $f|_{\partial X}$. $\square$

Notice $F^{-1}(y) \cap \mathbb{H}^m = f^{-1}(y) \cap \mathcal{U}$

$$= \{ x \in F^{-1}(y) \mid \pi(x) \geq 0 \}$$

By the previous lemma (L4), this is a smooth mfd with boundary $\pi^{-1}(0) = \mathbb{R}^{m-1} \cap F^{-1}(y)$. $\square$

Rmk There's a common generalisation of this theorem & generalised preimage theorem, for $f : X \to Y$, X mfd w/ bdry Y mfd w/o boundary, $Z \subset Y$, $f \pitchfork Z$. See Paternain's notes.

1.6 Degree mod 2

X smooth mfd w/o boundary. Then $X \times [0,1]$ is a smooth mfd w/ boundary ($X \times \{0,1\}$).

<u>Def</u> Two maps $f, g : X \to Y$ are <u>smoothly homotopic</u> if $\exists$ a smooth map $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x),$$
$$F(x,1) = g(x), \quad \forall x \in X.$$

The map F is a smooth homotopy between f, g.

<u>Check</u> (!) The relation $f \sim g$ iff f, g smoothly homotopic is an equivalence relation.

The equivalence class to which a map belongs is its (smooth) homotopy class.

Denote $f_t : X \to Y$, $x \mapsto F(x,t)$ a one-parameter family of maps $\forall\, t \in [0,1]$.

<u>Def</u> Two diffeo^{m}s $f, g : X \to Y$ are <u>smoothly isotopic</u> if $\exists$ a smooth homotopy $F : X \times [0,1] \to Y$ from f to g s.t. f_t is a diffeo^m for all t.

<u>Classification of compact 1-mfds</u> (cf Milnor, App A)

Every compact 1-mfd is diffeo^c to a finite union of S^1s and $[0,1]$s. Proof is funny.

<u>Cor</u> The boundary of a compact 1-mfd consists of an even number of points.

<u>Lemma</u> (Homotopy lemma) Let $f, g : X \to Y$ smoothly commutative maps. Suppose $\partial X = \emptyset$, X is cpct, and $\dim X$ equals $\dim Y$. Let $y \in Y$ be a regular value for both f, g.

Then $\# f^{-1}(y) = \# g^{-1}(y) \pmod 2$.

finite from L2

Pf Let $F: X \times [0,1] \to Y$ be a smooth homotopy between f, g.
Assume that $y \in Y$ is a regular value for F.
Then $F^{-1}(y)$ is a compact 1-mfd, w/ boundary
$$F^{-1}(y) \cap (X \times \{0,1\})$$
by our first theorem today.
So $\# \partial F^{-1}(y) = \# f^{-1}(y) + \# g^{-1}(y)$
is even by classification of 1-mfds.
$\therefore \# f^{-1}(y) = \# g^{-1}(y) \pmod 2$


$F^{-1}(y) \subset X \times [0,1]$

Now assume y is not a regular value for F.
By the stack-of-records theorem, there is $U \subseteq Y$ open containing y s.t. every point in U is regular value for both f, g. Moreover $\forall z \in U$, $\# f^{-1}(y) = \# f^{-1}(z)$,
$$\# g^{-1}(y) = \# g^{-1}(z).$$
By Sard's theorem, there is $z \in U$ regular value for F.
Then by the above
$$\begin{aligned} \# f^{-1}(y) &= \# f^{-1}(z) \\ &= \# g^{-1}(z) \mod 2 \\ &= \# g^{-1}(y) \end{aligned}$$
□

Lem (Homogeneity lemma) Let X be smooth connected mfd, possibly with boundary. Let $y, z \in \operatorname{Int} X$.
Then $\exists$ diffeo^m $h: X \to X$ smoothly isotopic to id_X, such that $h(y) = z$.

Will use Slightly more general theorem about existence of solutions to ODEs. "IB version of Picard-Lindelöf plus smooth dep^{n}dce on params"

Pf of homogeneity lemma

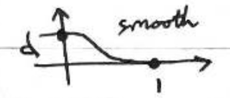
Claim Let $B \subset \mathbb{R}^n$ be the closed unit ball, $z \in \mathrm{Int}(B)$.
Then $\exists$ diffeo $\mathbb{R}^n \to \mathbb{R}^n$, smoothly isotopic to id, s.t.

- all points outside B fixed
- $0 \mapsto z$

Pf Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be C^∞ with

- $\psi(x) > 0 \quad \forall\, x \in \mathrm{Int}(B)$
- $\psi(x) = 0 \quad \forall\, x$ with $\|x\| \geq 1$

⌜How? $\mathbb{R}^n \xrightarrow{\|\cdot\|} \mathbb{R} \longrightarrow \mathbb{R}$



⌟

Set $c := \frac{z}{\|z\|} \in S^{n-1} \subseteq \mathbb{R}^n$.

w/ $F(0,x) = x$

ODE: $$\frac{d}{dt} F(t,x) = c \cdot \psi(F(t,x)) \qquad (*)$$

Picard-Lindelöf $\Rightarrow$ $(*)$ has unique smooth solution

$\|\psi\| = K$ (fixed const.)

$\Rightarrow$ solution for all initial conditions & all time

indeed ε in Picard Lindelöf uniform in t, x

(Note for $\|x\| \geq 1$, $F(t,x) = x \quad \forall\, t \in \mathbb{R}$)

Properties $\quad F(t,x) := F_t(x)$

$\{F_t\}$ is a flow, of the vector field $c \cdot \psi(\cdot)$

- $F_0(x) = x$
- $F_{s+t}(x) = F_s \circ F_t(x)$ (b/c of uniqueness of sol^{n}s)
- $F_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeo ($F_t \circ F_{-t} = \mathrm{id}$)
- Each F_t is smoothly isotopic to id via F_s, $s \in [0,t]$
  w/ $F_s|_{\mathbb{R}^n \setminus B} = \mathrm{id}_{\mathbb{R}^n \setminus B}$
- $F_{t_*}(0) = z$ for some suitable time t_*

⌜Why? Take a closed ball $B_r(z) \subset \mathrm{Int}\, B$.
Then ψ is bounded below on $B_{|z|+r}(0)$ by a positive no.⌟

=

Now, say $x, y \in \mathrm{Int}\, X$ are isotopic if $\exists$ diffeo h, isotopic to Id, s.t. $h(x) = y$.

Check This is an equivalence relation

Moreover, the claim implies the equivalence classes are open.
As X is connected, we're done. $\square$

<u>Thm</u> (degree mod 2) X compact, $\partial X = \emptyset$, Y connected,
$\dim X = \dim Y$, $f: X \to Y$ smooth
If y, z are regular values of f then
$$\# f^{-1}(y) = \# f^{-1}(z) \mod 2$$
<u>Pf</u> By homogeneity lemma, $\exists$ diffeo h of Y isotopic to id, s.t. $h(y) = z$.
y regular for $f \Rightarrow z$ regular for $h \circ f$
$h \circ f$ smoothly isotopic to f (compose the isotopy b/w h & id with f)
So by the homotopy lemma,
$$\# (h \circ f)^{-1}(z) = \# f^{-1}(z) \mod 2$$
But LHS $= \# f^{-1}(y)$ so done. $\square$

<u>Def</u> The number $\# f^{-1}(y)$ is called the degree mod 2 of f, denoted $\deg_2 f$
<u>Lemma</u> If f, g are homotopic then $\deg_2 f = \deg_2 g$
<u>Pf</u> By Sard's theorem, f, g have a common regular value.
So result follows from homotopy lemma. $\square$

<u>Ex</u> X compact mfd, $\partial X = \emptyset$ $\cdot \deg_2(\mathrm{Id}) = 1$
$\cdot \deg_2(\text{const.}) = 0$
$\Rightarrow$ the identity cannot be homotopic to a constant map

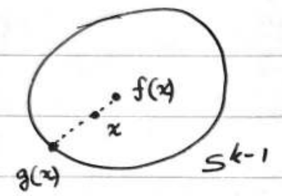
<u>Cor</u> $\nexists$ smooth $f: B^{k+1} \to S^k$ s.t. $f|_{S^k} = id_{S^k}$
$B^{k+1} = \{\|x\| < 1\} \subseteq \mathbb{R}^{k+1}$
⌈ "$\nexists$ a smooth retraction of B^{k+1} to S^k" ⌋

<u>Pf</u> Assume f exists. Consider the smooth map
$$F: S^k \times [0,1] \to S^k, \ (x,t) \mapsto f(tx)$$

This is a homotopy b/w the constant map (when $t=0$) and the identity (when $t=1$). ✗ $\square$

<u>Thm</u> (Smooth Brouwer fixed point theorem)
Any smooth map $f: B^k \to B^k$ has a fixed point.

<u>Pf</u> Suppose $f: B^k \to B^k$ has no fixed point.
We can construct a smooth map $g: B^k \to S^{k-1}$ as follows:



"Clearly" g is smooth and we contradict our corollary. $\square$

<u>Rmk</u> Can prove the continuous Brouwer f.p. theorem from the smooth version via Stone-Weierstrass.

1.6.1 Intersection numbers mod 2

Want: Generalization of degree mod 2 for smooth maps $f: X \to Y$ and $Z \subseteq Y$ submfd.

Suppose: (1) X compact, $\partial X = \emptyset$

(2) Z closed submfd $\subseteq Y$, $\partial Z = \emptyset$

(3) $f \pitchfork Z$

(4) $\dim X + \dim Z = \dim Y$

Given these, $f^{-1}(Z) \subseteq X$ closed submfd of dim 0 (generalized preimage thm) so a finite # of points.

Def. Mod 2 intersection of f with Z,

$$I_2(f, Z) := \# f^{-1}(Z) \bmod 2$$

Analogue of homotopy lemma: If f_0, f_1 are transversal to Z and are homotopic, then $I_2(f_0, Z) = I_2(f_1, Z)$.

Pf (sketch) $F: X \times [0,1] \to Z$ s.t. $F(x,0) = f_0(x)$
$F(x,1) = f_1(x)$

If $F \pitchfork Z$, $F^{-1}(Z)$ is a 1-mfd with boundary
$F^{-1}(Z) \cap (X \times \{0,1\})$

In general, if NOT$(F \pitchfork Z)$, perturb F & use the fact that transversality is generic. $\square$

Rmk For any smooth f, define $I_2(f, Z)$ to be $I_2(g, Z)$ where g is homotopic to f and $g \pitchfork Z$.

[· g exists because transversality is generic

· choice of doesn't matter b/c of homotopy lemma]

See Guillemin-Pollack p70-72

Special case: $X \xhookrightarrow{i} Y$ compact submanifold, Z closed submfd of Y, complementary dimension

If $X \pitchfork Z$, then $I_2(X, Z) := I_2(i, Z) = \#\{X \cap Z\} \bmod 2$

Note: By the above remark, we can drop the assumption that $X \pitchfork Z$.

If $I_2(X,Z) \neq 0$ (i.e. 1), deduce that you can't move (homotope) X to be disjoint from Z.

<u>Note</u> If $X \cap Z = \emptyset$, intersec^n is transv^l

forgive me father

<u>Ex</u> $Y = 2$ torus $S^1 \times S^1$

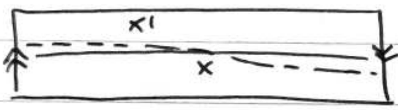
$X = S^1 \times \{pt\}$

$Z = \{pt\} \times S^1$ $\qquad I_2(X,Z) = 1$

"can't pull them apart"

When $\dim X = \frac{1}{2} \dim Y$, we can consider $I_2(X,X)$, the self-intersection mod 2.

<u>Ex</u> X = equator on a Mobius strip



X' = perturbation

$I_2(X,X) = 1$

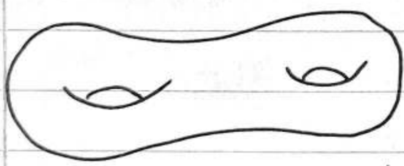
<u>Harder application of mod 2 degree</u>:

<u>Jordan-Brouwer separation theorem</u> (sketch)

$X \subseteq \mathbb{R}^n$ compact, connected, hypersurface (i.e. codim$^n = 1$)

Then $\mathbb{R}^n \setminus X$ has two connected components D_0, D_1, where $\overline{D_1}$ is compact, $\partial \overline{D_1} = X$ (the "inside"),

D_0 is unbounded (the "outside")



$X \subseteq \mathbb{R}^3$ $\qquad$ (<u>Csq</u>: X is orientable)

<u>Idea</u>: Take $z \notin X$, $u_z : X \to S^{n-1}$ to be the map

$$u_z(x) = \frac{x-z}{|x-z|}, \quad W_2(z) = \deg_2 u_z$$

(cf winding # (mod 2) in complex analysis or Gauss map)

$D_0 = \{ z \in \mathbb{R}^n \setminus X : W_2(z) = 0 \}$

$D_1 = \{ z \in \mathbb{R}^n \setminus X : W_2(z) = 1 \}$

(see Guillemin & Pollack for more)

1.7 Abstract mfds & Whitney's embedding thm

Generalis^n of IB abstract surface

Def X abstract mfd is a top space, Hausdorff, 2nd ctble, with an atlas of charts:

- chart $U \overset{open}{\subseteq} X$ w/ $\varphi_U : U \to V \subseteq \mathbb{R}^n$ homeo onto V open
- X covered by these
- $\varphi_{U_1} \circ \varphi_{U_2}^{-1} |_{\varphi_2(U_1 \cap U_2)}$ smooth

(n fixed)

Whitney's embedding thm An abstract n-dim mfd X can be embedded in $\mathbb{R}^{2n}$ (really hard, $\mathbb{R}^{2n+1}$ less so)

NB embedded in smooth sense

$\rightsquigarrow$ X is a manifold in the sense of this course (+)

- $\mathbb{R}^N$ greedily with bump functions 
- project $\mathbb{R}^N \to \mathbb{R}^{2n}$ generically, get immersion  w/ transverse self $\cap$ pts
- use $(2n+1)^{st}$ direction, bump functions, to push off intersection points

Harder fact Any compact orientable smooth n-manifold can be embedded in $\mathbb{R}^{2n-1}$ $(n > 1)$

(+) This is me at 10PM working through this.

- Actually not too bad, have injective df where you look at injection of open in $\mathbb{R}^n$ into $\mathbb{R}^N$. Take coordinates to give canonical immersion, now can produce smooth inverse via projection.
- The point is smooth structure matches the one given by us at the start, i.e. that where charts are smooth understood via extensions to $\mathbb{R}^N$

Chapter 2 Length, area, curvature

2.1 Arc-length, curvature & torsion of curves

Def · Let $I \subset \mathbb{R}$ be an interval, X a mfd. A curve in X is a smooth map $\alpha : I \to X$.

· The velocity of α is $\dot\alpha(t) := D\alpha_t(1) \in T_{\alpha(t)}X$.

· α is regular if α is an immersion (i.e. velocity non-zero)

Rmk $X \subset \mathbb{R}^N$, $\dot\alpha(t)$ has the usual meaning

Def $\alpha : I \to \mathbb{R}^N$ regular curve, $t_0, t \in I$.

The arc-length from t_0 to t is

$$s(t) = \int_{t_0}^{t} |\dot\alpha(t)|\, dt .$$

If $I = [a,b]$, $a<b$, the length of α is

$$\ell(\alpha) = \int_a^b |\dot\alpha(t)|\, dt .$$

Def A curve is parametrised by arc-length (i.e. is of unit speed) if $|\dot\alpha(t)| = 1 \ \forall t \in I$.

Note If α regular, the f^n $t \mapsto s(t)$ has strictly positive derivative so has a smooth inverse $\ell = \ell(s)$.

Consider $\beta : s \mapsto \alpha(\ell(s))$:

$$\beta'(s) = \dot\alpha(\ell(s)) \frac{d\ell}{ds}(s) , \quad \frac{d\ell}{ds}(s) = \frac{1}{\frac{ds}{dL}(\ell(s))} = \frac{1}{|\dot\alpha(\ell(s))|}$$

so $|\beta'(s)| = 1$,

i.e. β is parametrised by arc-length,

same image & length (if $\exists$) as α.

Def Let $\alpha : I \to \mathbb{R}^3$ a regular curve of unit speed.

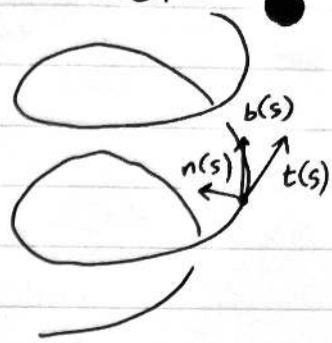
· The curvature of α at $s \in I$ is $\kappa(s) := |\ddot\alpha(s)|$

· If $\kappa(s) \neq 0$, define unit vector $n(s)$ by taking

$$\ddot\alpha(s) = \kappa(s)\, n(s).$$

Notation $\dot\alpha(s) := t(s)$ is the tangent vector at s

$n(s)$ is the normal vector at s



· $\|\dot\alpha\|^2 = 1 \Rightarrow \langle \ddot\alpha, \dot\alpha \rangle = 0$

Def The plane spanned by $t(s)$, $n(s)$ is called the osculating plane.
(if $\alpha : I \to \mathbb{R}^2 \subseteq \mathbb{R}^3$, this is just $\mathbb{R}^2$)

Def The unit $b(s) := t(s) \times n(s)$ is called the binormal vector at s.

Note $\dot b(s) = \underbrace{\dot t \times n}_{\substack{\text{zero} \\ \text{as } \dot t = \kappa n}} + t \times \dot n$

$\|b\| = 1 \Rightarrow b \cdot \dot b = 0$

$\dot b \perp b$ and t
so $\dot b(s) = \tau(s) n(s)$

for some $\tau(s) \in \mathbb{R}$, the torsion of α at s.

Def At any point s s.t. $\kappa(s) \neq 0$, the o.n. triple
$t(s), n(s), b(s)$ is the Frenet trihedron at s

$\dot n = ?$ $\quad n = b \times t \Rightarrow \dot n = \dot b \times t + b \times \dot t$

$$= \tau n \times t + b \times \kappa n$$
$$= -\tau b - \kappa t$$

Frenet formulae:

$$\begin{cases} \dot t = \kappa n \\ \dot n = -\tau b - \kappa t \qquad (*) \\ \dot b = \tau n \end{cases}$$

Thm (Fundmental local structure of curves in $\mathbb{R}^3$)
I finite interval
Given smooth functions $\kappa(s) > 0$ and $\tau(s)$ for $s \in I$, there exists a regular curve $\alpha : I \to \mathbb{R}^3$ s.t. s is the arc-length, $\kappa(s)$ is its curvature, and $\tau(s)$ is its torsion.
Moreover, any other curve β regular and satisfying the above, differs from α by an isometry of $\mathbb{R}^3$.

Pf $(*)$: ODE for $(t, n, b) \in M_{3\times3}(\mathbb{R})$
Given initial condition (t_0, n_0, b_0) get a unique solution $(t(s), n(s), b(s))$ by the existence & uniqueness of solutions to ODEs.

If $(t_0, n_0, b_0) \in O_3(\mathbb{R}) \subset M_{3\times 3}(\mathbb{R})$, claim that $(t(s), n(s), b(s))$ stay in $O_3(\mathbb{R})$ for all s.

$$\begin{pmatrix} \dot t \\ \dot n \\ \dot b \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\substack{\text{derivative!} \\ \text{antisymmetric}}} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

We saw $TO(n) \simeq$ antisymm matrices

$\therefore (t(s), n(s), b(s))$ is orthonormal too!

To conclude: pick $\alpha(0) \in \mathbb{R}^3$ & integrate $t(s)$.

Different choices for $\alpha(0)$ give translations of $\mathbb{R}^3$, different choices of $(t_0, n_0, b_0) \in SO(3)$ give action of $SO(3)$ on $\mathbb{R}^3$. $\square$

<u>Rmk</u> For plane curve $\alpha: I \to \mathbb{R}^2$ (i.e. $\tau = 0$), can give a sign to the curvature.

Let $\{e_1, e_2\}$ the std basis for $\mathbb{R}^2$. Define $n(s)$ by requiring that $\{t(s), n(s)\}$ have the same orientation as $\{e_1, e_2\}$.

Curvature defined by $\dot t(s) = \kappa(s) n(s)$, $\kappa \in \mathbb{R}$ (any sign)

absolute value gives previous def

2.2 The isoperimetric inequality in the plane

Let $\Omega \subseteq \mathbb{R}^2$ be a domain (connected open) with boundary $\partial\Omega$, a connected C^1 mfd.

Let $A(\Omega)$ be the area of Ω.

<u>Thm</u> $\ell^2(\partial\Omega) \geq 4\pi A(\Omega)$ with equality iff Ω disk

<u>Lemma</u> (Wirtinger's inequality)

Let $f: \mathbb{R} \to \mathbb{R}$ a C^1 f^n, periodic with period L.

Suppose $\int_0^L f(t)\, dt = 0$.

Then $\int_0^L |f'(t)|^2\, dt \geq \frac{4\pi^2}{L^2} \int_0^L |f(t)|^2\, dt$

with equality iff $\exists$ constants a_{-1} & $a_1 \in \mathbb{C}$ with $a_{-1} = \overline{a_1}$ and
$f(t) = a_{-1} e^{-2\pi i t/L} + a_1 e^{2\pi i t/L}$.

Recall Wirtinger's inequality :

$$\int_0^L |f'(t)|^2\,dt \geq \frac{4\pi^2}{L^2}\int_0^L |f(t)|^2\,dt$$

for f with period L, $\int_0^L f(t) = 0$.

<u>Pf</u> Consider the Fourier series for f, f'.

f C^1 means they converge absolutely uniformly.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k t/L}, \quad f'(t) = \sum_{k=-\infty}^{\infty} b_k e^{2\pi i k t/L}$$

where $$a_k = \frac{1}{L}\int_0^L f(t)\,e^{-2\pi i k t/L}\,dt,$$

$$b_k = \frac{1}{L}\int_0^L f'(t)\,e^{-2\pi i k t/L}\,dt.$$

- $a_0 = 0$ (hypothesis)
- $b_0 = 0$ (direct computation)
- integration by parts gives

$$b_k = \frac{2\pi i k}{L} a_k$$

- Parseval's identity

$$\int_0^L |f'(t)|^2\,dt = \sum_{k=-\infty}^{\infty} L|b_k|^2 \geq \frac{4\pi^2}{L^2}\sum_{k=-\infty}^{\infty} L|a_k|^2$$

$$= \int_0^L |f(t)|^2\,dt \cdot \frac{4\pi^2}{L^2}$$

Used $k=0$ terms absent.

Equality iff $a_k = 0$ for $k \notin \{-1, 1\}$. $\square$

Isoperimetric Inequality

$\Omega \subseteq \mathbb{R}^2$ domain, $\partial\Omega$ connected, C^1 1-mfd

also Ω cpct

Then $\ell^2(\partial\Omega) \geq 4\pi A(\Omega)$ w/ equality iff Ω a disc.

Pf Parametrise $\partial\Omega$ by arc-length.

Define $X(x,y) = (x,y)$ on $\mathbb{R}^2$.

By translating, assume $\int_{\partial\Omega} X \, ds = 0$. ("centering")

n: outward unit normal on $\partial\Omega$

By the 2D divergence theorem

$$\underbrace{\int_{\Omega} \nabla \cdot X \, dA}_{= 2A(\Omega)} = \int_{\partial\Omega} \langle X, n \rangle \, ds$$

By Cauchy-Schwartz, $\langle X, n \rangle \leq |X|$

$$\therefore 2A(\Omega) \leq \int_{\partial\Omega} |X| \, ds$$

$$\leq \left(\int_{\partial\Omega} |X|^2 ds \right)^{1/2} \left(\int_{\partial\Omega} ds \right)^{1/2}$$

equality iff $|X|$ const.

By Wirtinger,

$$\int_{\partial\Omega} x^2 \, ds \geq \frac{\ell(\partial\Omega)^2}{4\pi^2} \int_{\partial\Omega} |x'|^2 ds$$

since by assumption $\int_{\partial\Omega} x \, ds = 0$.

Thus
$$2A(\Omega) \leq \left(\cancel{*} \frac{\ell(\partial\Omega)^3}{4\pi^2} \right)^{1/2} \ell(\partial\Omega)^{1/2}$$

$$= \frac{1}{2\pi} \ell(\partial\Omega)^2.$$

$\square$

Rmk $\exists$ generalization to arbitrary smooth surfaces S involving curvature.

E.g. S orientable, cst curvature K, $\Omega \subset S$ a domain with compact closure $\overline{\Omega}$, $\partial\Omega$ piecewise C^2. Then

$$4\pi A(\Omega) \chi(\Omega) \leq \ell^2(\partial\Omega) + K A(\Omega)^2$$

2.3 First fundamental form & area

A surface is a smooth 2-manifold. This section: $S \subset \mathbb{R}^3$, $\partial S = \emptyset$

Def The quadratic form I_p on $T_p S$ given by $I_p(w) = \langle w, w \rangle = |w|^2$ (comes from $\mathbb{R}^3$) is the first fundamental form (1FF) of S.

Def If X is a smooth mfd, a Riemannian metric on X is a smoothly varying family of inner products on $T_p X$.
(with p)

Ex 1FF

If $f : X \to Y$ an immersion and Y has a Riemannian metric g, get a natural pullback metric on X.

$$h_p(v,w) = g_{f(p)}(df_p(v), df_p(w))$$

1FF for $S \subset \mathbb{R}^3$ comes from the standard i.p. on $\mathbb{R}^3$ via inclusion.

<u>Def</u> S_1, S_2 are <u>isometric</u> if $\exists$ diffeo^{m} $f: S_1 \to S_2$ s.t. $\forall p \in S_1$, Df_p is a linear isometry b/w T_pS_1 & $T_{f(p)}S_2$ (equipped w/ their inner products).

<u>1FF in coordinates</u>

Let $\varphi: \underset{\mathbb{R}^2}{\underset{\cap}{U}} \longrightarrow \underset{\mathbb{R}^3}{\underset{\cap}{S}}$ be a parametrisation of an open ngbd of $p \in S$.

$$\left.\begin{aligned} \varphi_u(u,v) &= \frac{\partial \varphi}{\partial u}(u,v) = D\varphi_{(u,v)}(e_1) \\ \varphi_v(u,v) &= \frac{\partial \varphi}{\partial v}(u,v) = D\varphi_{(u,v)}(e_2) \end{aligned}\right\} \text{basis for } T_{\varphi(u,v)}S$$

<u>Def</u> Let $p = \varphi(u,v) \in S$. Set
$$\begin{aligned} E(u,v) &= \langle \varphi_u, \varphi_u \rangle_p \\ F(u,v) &= \langle \varphi_u, \varphi_v \rangle_p \\ G(u,v) &= \langle \varphi_v, \varphi_v \rangle_p \end{aligned}$$

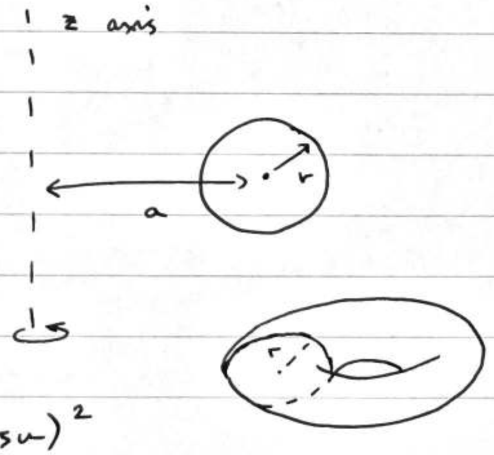
Given $w \in T_pS$, pick a curve α, $\alpha(0) = p$, $\alpha'(0) = D\alpha_0(1) = w$ with $\alpha(t) = \varphi(u(t), v(t))$, $t \in (-\varepsilon, \varepsilon)$.

<u>Note</u> $\dot\alpha = \varphi_u \dot u + \varphi_v \dot v$

Then
$$\begin{aligned} I_p(\underbrace{\dot\alpha(0)}_{=w}) &= \langle \dot\alpha(0), \dot\alpha(0) \rangle \\ &= E(\dot u)^2 + 2F\dot u \dot v + G(\dot v)^2 \end{aligned}$$

Equiv 1FF is $E\,du^2 + 2F\,du\,dv + G\,dv^2$ ⌜cf 79–80 "Curved spaces"⌟

<u>Ex</u> S = torus of revolution



$$\varphi(u,v) = \begin{pmatrix} (a + r\cos u)\cos v \\ (a + r\cos u)\sin v \\ r \sin u \end{pmatrix}$$

where $(u,v) \in (0, 2\pi)$

[& obv. other charts]

IB: $E = r^2$, $F = 0$, $G = (a + r\cos u)^2$

<u>Def</u> Given a curve $\alpha: I = [0,1] \longrightarrow S \subseteq \mathbb{R}^3$, recall its length is
$$\ell(\alpha) = \int_0^1 \langle \dot\alpha(t), \dot\alpha(t) \rangle^{1/2}\, dt$$

In local coordinates (where $\alpha(t) = \varphi(u(t), v(t))$):

$$\ell(\alpha) = \int_0^1 (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\, dt$$

Check $\|\varphi_u \times \varphi_v\| = \sqrt{\underbrace{EG - F^2}_{>0 \text{ as } I_p \text{ +ve def}}}$

$\ulcorner$ i.e. $\langle \varphi_u, \varphi_v \rangle^2 + \|\varphi_u \times \varphi_v\|^2 = \|\varphi_u\|^2 \|\varphi_v\|^2$ $\lrcorner$

Suppose Ω is contained in the image of a local parametrisation $\varphi: U \to S$.

Lemma The integral $\displaystyle\int_{\varphi^{-1}(\Omega)} \|\varphi_u \times \varphi_v\|\, du\, dv$

doesn't depend on the choice of parametrisation.

Proof Consider another parametrisation $\psi: \tilde{U} \to S$ with $\Omega \subset \psi(\tilde{U})$. Wlog $\psi(\tilde{U}) = \varphi(U)$.

$$\begin{tikzcd} & S \\ U \arrow[ur, "\varphi"] & \tilde{U} \arrow[l, "h := \varphi^{-1}\circ\psi"] \arrow[u, "\psi"'] \end{tikzcd}$$

Set $\underbrace{J(\tilde{u}, \tilde{v})}_{\substack{\text{Jacobian for} \\ \text{change of variables} \\ \text{map } h}} = \det \underbrace{\left(\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} \right)}_{\substack{\text{Jacobian} \\ \text{matrix}}}$

$$\psi = \varphi \circ h \Rightarrow D\psi_a(e_i) = D\varphi_{h(a)} \circ Dh_a(e_i), \quad i = 1, 2$$

$$\Rightarrow \|\psi_{\tilde{u}} \times \psi_{\tilde{v}}\|_a = \|\varphi_u \times \varphi_v\|_{h(a)} \cdot |J(a)| \qquad \left(\text{det of a product}\right)$$

By change-of-variables formula (from VC)

$$\int_{\varphi^{-1}(\Omega)} \|\varphi_u \times \varphi_v\|\, du\, dv = \int_{\psi^{-1}(\Omega)} \|\varphi_u \times \varphi_v\| \circ h \cdot |J|\, d\tilde{u}\, d\tilde{v}$$

$$= \int_{\psi^{-1}(\Omega)} \|\psi_{\tilde{u}} \times \psi_{\tilde{v}}\|\, d\tilde{u}\, d\tilde{v}$$

$\square$

Def Let $\Omega \subset S$ be a bounded domain contained in the image of a parametrisation $\varphi: U \to S$.

Then the <u>area</u> of Ω is

$$A(\Omega) = \int_{\varphi^{-1}(\Omega)} \|\varphi_u \times \varphi_v\| \, du \, dv$$

$$= \int_{\varphi^{-1}(\Omega)} (EG - F^2)^{1/2} \, du \, dv .$$

<u>Rmk</u> We can still define areas of bounded domains which are not contained in the image of a parametrisation, by using a "partition of unity".

<u>But</u>: almost never do this — <u>Fact</u>: We can always find a parametrisation for which the image contains Ω, except perhaps for a finite number of curves (which don't contribute to area) — see exponential map in Chap 4

<u>Aside</u> This gives us a "Riemannian measure" dA on S. If $f : S \to \mathbb{R}$ is cts, and $\varphi : U \to S$ a parametrisation of S up to a finite number of curves,

$$\int_S f \, dA := \int_U f(u,v) \sqrt{EG - F^2} \, du \, dv$$

<u>Ex</u> Torus of rev, ctd. Here $\sqrt{EG - F^2} = r(a + r\cos u)$

$\therefore A = 4\pi^2 r a$

2.4 The Gauss map

Given a param^n, $\varphi : U \to S$ around $p \in S$, we can choose a unit normal vector at each point $q \in \varphi(U)$ via

$$N(q) := \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}(q)$$

$$N : \varphi(U) \longrightarrow \underbrace{S^2}_{\text{unit sphere}} \subseteq \mathbb{R}^3$$

Def A smooth field of unit normal vectors on a surface S is a smooth map $N : S \longrightarrow S^2 \subseteq \mathbb{R}^3$ s.t. $\forall p \in S$, $N(p) \perp T_pS$

A surface $S \subset \mathbb{R}^3$ is orientable if it admits a choice of smooth field of unit normal vectors.

an orientation (two possible)

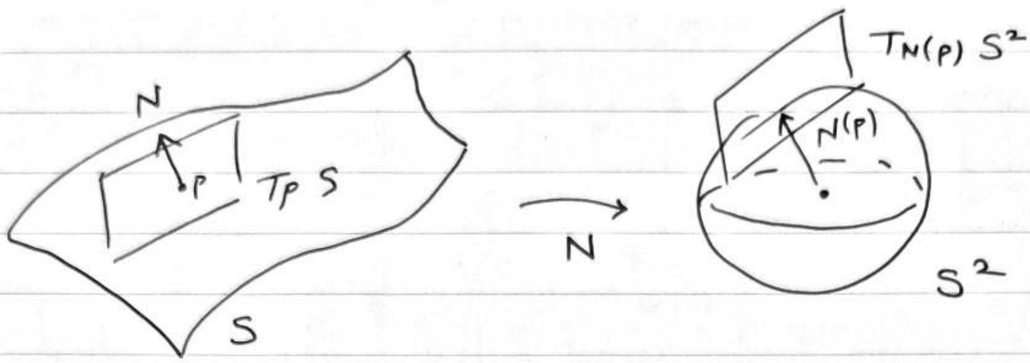
Orientability is a global property of S.

Ex The Möbius band is a non-orientable surface (ES 2)

Rmk An orientation defines an orientation on every tangent plane: basis (u, v) for T_pS is positively oriented if (u, v, N) is positively oriented basis for $\mathbb{R}^3$.

This is def of orientation which generalises

Def Let S be an oriented surface and $N: S \to S^2$ the smooth field of unit normals giving the orientation. N is called the Gauss map.



Note $T_pS = T_{N(p)}S^2 = \langle N(p)\rangle^{\perp} \subseteq \mathbb{R}^3$

$\rightsquigarrow$ Get $DN_p : T_pS \to T_{N(p)}S^2 = T_pS$

Key property of the Gauss map DN_p is self-adjoint

Pf Let $\varphi : U \to S$ be a param^n around p, and suppose $\alpha(t) = \varphi(u(t), v(t))$ is a curve in $\varphi(U)$ with $\alpha(0) = p$.

Then
$$\begin{aligned} DN_p(\dot\alpha(0)) &= DN_p(\dot u(0)\varphi_u + \dot v(0)\varphi_v) \\ &\parallel \\ \left.\frac{d}{dt}\right|_{t=0} & N(u(t), v(t)) \\ &\parallel \\ \dot u(0) N_u &+ \dot v(0) N_v \end{aligned}$$

$\therefore DN_p(\varphi_u) = N_u, \quad DN_p(\varphi_v) = N_v$

Since $\{\varphi_u, \varphi_v\}$ is a basis for T_pS, enough to show that
$$\langle N_u, \varphi_v\rangle = \langle \varphi_u, N_v\rangle$$
(since $\langle N_u, \varphi_u\rangle = \langle \varphi_u, N_u\rangle$ & similarly for v)

Note that $\langle N, \varphi_u\rangle = 0 = \langle N, \varphi_v\rangle$.

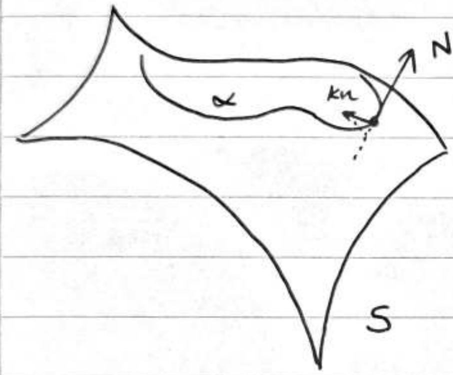
Taking partial derivatives wrt v, u get
$$\langle N_v, \varphi_u\rangle = -\langle N, \varphi_{uv}\rangle = -\langle N, \varphi_{vu}\rangle = \langle N_u, \varphi_v\rangle. \quad \square$$

Def The quadratic form given by
$$\mathrm{II}_p(w) = -\langle DN_p(w), w\rangle$$
is called the second fundamental form (2FF) of S at p.

Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be parametrised by arc-length with $\alpha(0) = p$.
Let $N(s) := N(\alpha(s))$.

$$\langle N(s), \dot\alpha(s) \rangle = 0$$
$$\Rightarrow \langle \dot N(s), \dot\alpha(s) \rangle + \langle N(s), \ddot\alpha(s) \rangle = 0$$
$$\Rightarrow \langle N(s), \ddot\alpha(s) \rangle = \mathrm{II}_p(\dot\alpha(s)) \qquad \text{(no minus sign)}$$

$$\therefore \mathrm{II}_p(\dot\alpha(0)) = \underset{\substack{\uparrow \\ \text{curvature of } \alpha}}{k} \langle \underset{\substack{\uparrow \\ \text{unit normal to } \alpha \text{ at } p}}{n}, N \rangle$$



<u>Def</u> $k_n(p) = \langle N, kn \rangle (p)$ is the <u>normal curvature</u> of α at p.
It may be +ve or −ve depending on the choice of normal N.

<u>Note</u> $k_n(p)$ only depends on the tangent vector $\dot\alpha(0)$, not on the choice of specific α

<u>Remark</u> Conversely, given $\overset{\text{non-zero}}{v} \in T_pS$, consider $V \subseteq \mathbb{R}^3$ plane through p containing v and N.

$C := V \cap S$ is called the <u>normal section</u> of S at p, along the direction v.

Generalised preimage theorem $\Rightarrow$ C is locally a smooth 1-mfd by transversality.

We may parametrise this locally by arc-length, say $\alpha : (-\varepsilon, \varepsilon) \to S$, so that $\dot\alpha(0) = \frac{v}{\|v\|}$ and $\ddot\alpha(0) = \pm k N(p)$ so $k_n(p) = \underset{\substack{\uparrow \\ \text{dep on choice of } N}}{\pm} k$

<u>Recall</u> A self-adjoint map has an o.n. basis of eigenvectors.

<u>Cor</u> $\exists$ o.n. basis $\{e_1, e_2\}$ of T_pS, real numbers k_1, k_2, say $k_1 \geq k_2$, such that $DN_p(e_i) = -k_i e_i$, $i = 1, 2$.
Moreover, k_1 & k_2 are the max, resp min values of II_p, restricted to unit vectors.

<u>Def</u> k_1, k_2 are called the <u>principal curavetrs</u> at p

e_1, e_2 " <u>principal directions</u> at p.

<u>Note</u> The k_i are the extremal values for curvature of normal sections.

<u>Def</u> • The <u>Gauss curvature</u> $K(p)$ of S at p is $\det(DN_p) = k_1 k_2$

• The <u>mean curvature</u> $H(p)$ of S at p is $-\frac{1}{2}\operatorname{tr}(DN_p) = \frac{k_1+k_2}{2}$

<u>Rmk</u> $K(p)$ is independent of orientation, but $H(p)$ is not

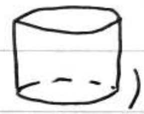
<u>Def</u> $p \in S$ is <u>elliptic</u> if $K(p) > 0$ (e.g. all points on S^2

$z = x^2 + y^2$ 

"elliptic paraboloid")

<u>hyperbolic</u> if $K(p) < 0$ (e.g. $z = x^2 - y^2$ 

"hyperbolic paraboloid")

<u>parabolic</u> if $K(p) = 0$, $DN_p \neq 0$ (e.g. cylinder )

<u>planar</u> if $DN_p = 0$

<u>Def</u> A point is <u>umbilic</u> if $k_1 = k_2$

<u>Fact</u> (ES2) If all points on a connected surface are umbilic, then it is contained in a sphere or in a plane

<u>Example</u> "Monkey saddle" $\{z = x^3 - 3y^2 x\} = S \subset \mathbb{R}^3$

$T_{\underline{0}} S = \{z = 0\} \Rightarrow N_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Setting $x = r\cos\theta$, $y = r\sin\theta$ have $\underbrace{z = r^3 \cos 3\theta}_{\text{normal section in direction } \theta}$

$\Rightarrow$ all normal sections have $k = 0$

<u>Upshot</u> $\underline{0} \in S$ is planar

2.5 Second fundamental form in local coords

Let φ be a param^n about $p \in S$.

Def $e := \langle N, \varphi_{uu} \rangle = - \langle N_u, \varphi_u \rangle$ $\quad \because \langle N, \varphi_u \rangle = 0$

$f := \langle N, \varphi_{uv} \rangle = - \langle N_u, \varphi_v \rangle = - \langle N_v, \varphi_u \rangle$

$g := \langle N, \varphi_{vv} \rangle = - \langle N_v, \varphi_v \rangle$ [IB: they were called L, M, N]

Lem In coord system (u,v), the 2FF is given by $e\,du^2 + 2f\,du\,dv + g\,dv^2$

Pf If α is a curve passing throug p at $t=0$, then

$$\begin{aligned} II_p(\dot\alpha(0)) &= -\langle DN_p(\dot\alpha(0)), \dot\alpha(0) \rangle \\ &= -\langle DN_p(\dot u \varphi_u + \dot v \varphi_v), \dot u \varphi_u + \dot v \varphi_v \rangle \\ &= -\langle \dot u N_u + \dot v N_v, \dot u \varphi_u + \dot v \varphi_v \rangle \\ &= e\dot u^2 + 2f \dot u \dot v + g \dot v^2. \end{aligned}$$ $\square$

Lem $K = \dfrac{eg - f^2}{EG - F^2}$, $\quad H = \dfrac{1}{2} \dfrac{eG - 2fF + gE}{EG - F^2}$

The principal curvatures k_1, k_2 are the roots of the poly $k^2 - 2Hk + K = 0$.

Pf Write $DN_p(\varphi_u) = N_u = a_{11}\varphi_u + a_{21}\varphi_v$ at p,

$DN_p(\varphi_v) = N_v = a_{12}\varphi_u + a_{22}\varphi_v$.

Taking inner products with φ_u & φ_v, get:

$$\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (*)$$

- $K = \det DN_p = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underset{(*)}{=} \dfrac{eg - f^2}{EG - F^2}$
- $H = -\frac{1}{2} \operatorname{tr} DN_p = -\frac{1}{2} \operatorname{tr} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \right]$ $\square$

Ex Torus of revl^n, ctd (2)

$\varphi(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r \sin u)$

Check
- $E = r^2$, $F = 0$, $G = (a + r\cos u)^2$
- $e = r$, $f = 0$, $g = \cos u(u + v\cos u)$

$$K = \frac{\cos u}{r(a + r\cos u)}$$

Taylor series interpretation of 2FF

Param^n $\varphi: U \to S$ around p, $\varphi(0) = p$

Expand $\varphi(h,k)$ as a Taylor series.

$$\varphi(h,k) = \varphi(0,0) + h\varphi_u + k\varphi_v + \tfrac{1}{2}\left(\varphi_{uu}h^2 + 2\varphi_{uv}hk + \varphi_{vv}k^2\right)$$
$$+ R(h,k) \leftarrow \text{'remainder'}$$

$$\langle N, \varphi(h,k) - \varphi(0,0)\rangle = \tfrac{1}{2}\left(eh^2 + 2fhk + gk^2\right) + \langle N, R(h,k)\rangle$$

i.e. $\mathrm{II}_p(h,k)$ gives the normal displacement from the tangent plane in the direction (h,k)

(cf 1FF : $\|\varphi(h,k) - \varphi(0,0)\|^2 = Eh^2 + 2Fhk + Gk^2 + S(h,k)$)

- $K > 0 \iff$ 2FF either +ve def or −ve def

  geometrically, S stays locally on one side of T_pS

  ⚠ not an equivalence

  $\iff$ principal curvatures have the same sign

  $\iff$ all normal curvatures have the same sign

- $K < 0 \iff$ 2FF non-deg, indef

  $\iff$ ppal curvatures have opposite signs

  geometrically, S has points on either side of T_pS locally

2.6 Theorema Egregium

<u>Thm</u> (Gauss, 1827) The Gauss curvature of a surface is invariant under local isometries

<u>Pf</u> Isometries preserve 1FF. We'll show that in local coords, K can be expressed purely in terms of E, F, G and their derivatives. (Hence the result follows)

Let $\varphi: U \to S$ be a parametrisation. At each point of $\varphi(U)$, have a basis of $\mathbb{R}^3$ given by $\{\varphi_u, \varphi_v, N\}$.

<u>Step 1</u> Express the derivatives of φ_u, φ_v in this basis

$$\varphi_{uu} = \Gamma_{11}^1\varphi_u + \Gamma_{11}^2\varphi_v + eN \quad \text{-- } \langle \varphi_{uu}, N\rangle \qquad ~~(1234)~~$$
$$\left.\begin{aligned}\varphi_{uv} &= \Gamma_{12}^1\varphi_u + \Gamma_{12}^2\varphi_v + fN \\ \varphi_{vu} &= \Gamma_{21}^1\varphi_u + \Gamma_{21}^2\varphi_v + fN\end{aligned}\right\}\text{ equal}$$
$$\varphi_{vv} = \Gamma_{22}^1\varphi_u + \Gamma_{22}^2\varphi_v + gN$$

Γ_{ij}^k : Christoffel symbols

Step 2 Show that Γ_{ij}^k can be expressed in terms of E, F, G and their first derivatives.

Take inner product of (1) with φ_u & φ_v, to get:

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle \varphi_{uu}, \varphi_u \rangle = \frac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle \varphi_{uu}, \varphi_v \rangle = F_u - \frac{1}{2} E_v \end{cases}$$

Since $EG - F^2 \neq 0$, solve for Γ_{11}^i.

Similarly for the other Christoffel symbols, using (2)–(4).

Step 3 Friday

Recall: Thm (Gauss) The Gauss curvature is invariant under isometry

Pf (cont.) Step 1 $\varphi_{uu} = \Gamma_{11}^1 \varphi_u + \Gamma_{11}^2 \varphi_v + eN$ etc

Step 2 Show that Γ_{ij}^k can be expressed via E, F, G and partials

Step 3 Express K in terms of Christoffel & derivatives

Using $\varphi_{uuv} = \varphi_{uvu}$ and (1) & (2) get

$$(\Gamma_{11}^1)_v \varphi_u + \Gamma_{11}^1 \varphi_{uv} + (\Gamma_{11}^2)_v \varphi_v + \Gamma_{11}^2 \varphi_{vv} + e_v N + e N_v$$
$$= (\Gamma_{12}^1)_u \varphi_u + \Gamma_{12}^1 \varphi_{uu} + (\Gamma_{12}^2)_u \varphi_v + \Gamma_{12}^2 \varphi_{vu} + f_u N + f N_u$$

Notation from L12: $N_u = a_{11}\varphi_u + a_{21}\varphi_v$
$N_v = a_{12}\varphi_u + a_{22}\varphi_v$

Use (1-4) to substitute for φ_{uu} etc, to get two expressions in terms of φ_u, φ_v, N. Equate coeffs of φ_v and get

$$(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + (\Gamma_{12}^2)^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2 \qquad (\dagger)$$
$$= -f a_{21} + e a_{22}$$

Also recall $-\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

Rearranging, $-f a_{21} + e a_{22} = -\dfrac{eg - f^2}{EG - F^2} \cdot E = -KE$

Finally, note $E \neq 0$ so done. $\square$

Rmk The expression $(\dagger)$ for $-KE$ is called the Gauss formula.

Rmk If φ is an orthogonal param^n, i.e. $F = 0$, the Gauss formula gives:

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

Def A parametrisation is called isothermal if $E = G = \lambda(u,v)^2$ and $F = 0$.

If φ is isothermal, then

$$K = -\frac{1}{2\lambda^2}\left[\left(\frac{2\lambda_v}{\lambda}\right)_v + \left(\frac{2\lambda_u}{\lambda}\right)_u\right] = -\frac{1}{\lambda^2}\Delta(\log \lambda).$$

⚠ H is not invariant under isometry, e.g.  $\mathbb{R}^2$ and  $S^1 \times \mathbb{R}$

Chapter 3: Critical points of length & area

Theme: Extremals of the length & area functionals.

Functional	Extrema	Var problem
Length	geodesic	Euler-Lagrange eqs ← ODEs ☺
area	minimal sfcs	Plateau problem ← PDE ☹ existence & uniqueness pb v. delicate

3.1 Geodesics

$S \subseteq \mathbb{R}^3$ a surface, $p, q \in S$. Let $\Omega(p,q)$ be the set of all curves $\alpha: [0,1] \to S$ with $\alpha(0) = p$, $\alpha(1) = q$.

Let ℓ be the functional $\Omega(p,q) \to \mathbb{R}$ given by

$$\ell(\alpha) = \int_0^1 |\dot{\alpha}(t)|\, dt \qquad \text{(length)}$$

Closely related & more convenient for variational purposes: energy functional

$$E(\alpha) = \underbrace{\tfrac{1}{2}}_{\text{often omitted}} \int_0^1 |\dot{\alpha}(t)|^2\, dt$$

⚠ $E(\alpha)$, unlike $\ell(\alpha)$, is not invariant under reparametrisation

Lemma $\alpha \in \Omega(p,q)$. $\ell(\alpha)^2 \leq 2E(\alpha)$ w/ equality iff α has constant speed.

($\Rightarrow$ To minimise E, need to minimise ℓ and parametrize at constant speed)

Pf Cauchy-Schwartz: $\left(\int_0^1 fg\right)^2 \leq \left(\int_0^1 f^2\right)\left(\int_0^1 g^2\right)$

Equality iff f, g linearly dependent

Put $f = 1$, $g = |\dot{\alpha}(t)|$ and done. □

Extremals for E: Consider a 1-parameter family of curves $\alpha_s \in \Omega(p,q)$, $s \in (-\varepsilon, \varepsilon)$, $\alpha_0 = \alpha$.

Let $E(s) := E(\alpha_s)$.

We have $$\frac{dE}{ds} = \int_0^1 \left\langle \frac{\partial^2 \alpha_s}{\partial s \partial t}, \frac{\partial \alpha_s}{\partial t} \right\rangle dt$$ [all functions are nice so $\frac{d}{ds}\int = \int \frac{d}{ds}$]

$$\Rightarrow \left.\frac{dE}{ds}\right|_{s=0} = \int_0^1 \left\langle \frac{\partial}{\partial t}\frac{\partial \alpha_s}{\partial s}, \frac{\partial \alpha_s}{\partial t} \right\rangle_{s=0} dt$$

$$= \int_0^1 \left\langle \frac{d}{dt} W(t), \dot{\alpha} \right\rangle dt \quad (*)$$

where $W(t) = \left.\frac{\partial \alpha_s(t)}{\partial s}\right|_{s=0}$

· If $\beta(s) = \alpha_s(t)$ for fixed t, then $W(t) = \beta'(0) \in T_{\beta(0)} S$

· Integrating $(*)$ by parts:

$$\left.\frac{dE}{ds}\right|_{s=0} = \int_0^1 \frac{d}{dt}\langle W(t), \dot{\alpha}\rangle dt - \int_0^1 \langle W(t), \ddot{\alpha}\rangle dt$$

$$= \underbrace{\langle W(1), \dot{\alpha}(1)\rangle}_{=0} - \underbrace{\langle W(0), \dot{\alpha}(0)\rangle}_{=0} - \int_0^1 \langle W(t), \ddot{\alpha}\rangle dt$$

because $\alpha_s \in \Omega(p,q)$
$\therefore \alpha_s(0) = p, \ \alpha_s(1) = q \ \forall s$

$$\therefore \left.\frac{dE}{ds}\right|_{s=0} = -\int_0^1 \langle W(t), \ddot{\alpha}\rangle dt$$

Upshot If $\ddot{\alpha} \perp T_{\alpha(t)} S$, then α is an extremum for E

Fact This is an iff condition (need more calc of var^{n}s ; see Wilson)

Def A curve $\alpha : I \to S$ is called a geodesic if $\forall t \in I$,
$\ddot{\alpha}(t) \perp T_{\alpha(t)} S$

<u>Rmk</u> This is also the "correct" def^n for geodesics on any embedded smooth manifold $X \subset \mathbb{R}^N$.

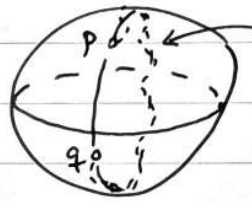
<u>Rmk 2</u> Geodesics have constant speed:

$$\frac{d}{dt}\langle \dot\alpha(t), \dot\alpha(t)\rangle = 2\langle \underbrace{\dot\alpha(t)}_{\in T_{\alpha(t)}S}, \underbrace{\ddot\alpha(t)}_{\perp \text{ to } T_{\alpha(t)}S}\rangle = 0$$

<u>Ex 1</u> For α a curve in the plane $\mathbb{R}^2 \subseteq \mathbb{R}^3$, α geodesic iff $\ddot\alpha = 0$ iff α a segment of a straight line w/ constant speed.

<u>Ex 2</u> The great circles on $S^2 \subset \mathbb{R}^3$ are geodesics
(Normal vector to a curve proportional to normal to S^2.)

<u>Rmk 3</u> Geodesics are stationary points of length functional, need not be max or min.



geod, neither max nor min

<u>3.2 Covariant derivative & parallel transport</u>

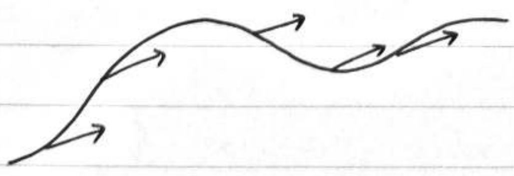
<u>Def</u> Let $\alpha: I \to S$ be a curve. A <u>vector field</u> V along α is a smooth map $V: I \to \mathbb{R}^3$ s.t. $V(t) \in T_{\alpha(t)}S \ \forall t$.

<u>Def</u> The vector obtained by the normal projection of $\frac{dV}{dt}$ onto the tangent plane $T_{\alpha(t)}S$ is called the <u>covariant derivative</u> of V at t, denoted $\frac{DV}{dt}(t)$.

<u>Rmk</u> α geodesic $\iff \frac{D\dot\alpha}{dt}(t) = 0$.

<u>Def</u> A vector field V along α is called <u>parallel</u> if $\frac{DV}{dt} = 0$

<u>Ex</u> $\alpha: I \to \mathbb{R}^2 \subset \mathbb{R}^3$ plane curve
V is parallel iff it is constant



<u>Prop</u> Let V, W be parallel vector fields along $\alpha : I \to S$. Then $\langle V(t), W(t) \rangle$ is const.

<u>Pf</u> $$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle \underset{\perp \text{ to } T_{\alpha(t)}S}{\dot V(t)}, \underset{\in T_{\alpha(t)}S}{W(t)} \rangle + \langle \underset{\in T_{\alpha(t)}S}{V(t)}, \underset{\perp \text{ to } T_{\alpha(t)}S}{\dot W(t)} \rangle = 0. \qquad \square$$

<u>Lemma</u> The covariant derivative only depends on the 1FF [not the embedding]

<u>Proof</u> Enough to work in a coordinate ptch. Let $\varphi : U \to S$ be a param^n, and $\alpha : I \to S$ a curve s.t. $\alpha(I) \subset \varphi(U)$.
Write $\alpha(t) = \varphi(u(t), v(t))$, some u, v.
Let V be a vector field along α : have
$$V(t) = a(t)\varphi_u + b(t)\varphi_v$$
for some smooth functions a, b.
Then $$\frac{dV}{dt} = a(\underbrace{\varphi_{uu}}_{=\Gamma^1_{11}\varphi_u + \Gamma^2_{11}\varphi_v + eN}\dot u + \varphi_{uv}\dot v) + b(\varphi_{vu}\dot u + \varphi_{vv}\dot v) + \dot a\varphi_u + \dot b\varphi_v$$
[pf of Thm Egregium]

Using eq^{n}s (1) – (4) in the pf of Thm Egregium & def^n of covariant derivative, get:
$$\left.\begin{aligned}\frac{DV}{dt} = {} & (\dot a + a\Gamma^1_{11}\dot u + b\Gamma^1_{21}\dot u + a\Gamma^1_{12}\dot v + b\Gamma^1_{22}\dot v)\varphi_u \\ & + (\dot b + a\Gamma^2_{11}\dot u + b\Gamma^2_{21}\dot u + a\Gamma^2_{12}\dot v + b\Gamma^2_{22}\dot v)\varphi_v\end{aligned}\right\} (\dagger)$$
The Γ^k_{ij} only depend on 1FF, so done. $\square$

<u>Cor</u> (of proof) Equations for geodesics in terms of local coords:
$$\dot\alpha(t) = \underset{a}{\dot u}\varphi_u + \underset{b}{\dot v}\varphi_v \qquad (*)$$
$$\left\{\begin{aligned} \ddot u + \Gamma^1_{11}\dot u\dot u + \Gamma^1_{21}\dot u\dot v + \Gamma^1_{12}\dot v\dot u + \Gamma^1_{22}\dot v\dot v = 0 \\ \ddot v + \Gamma^2_{11}\dot u\dot u + \Gamma^2_{21}\dot u\dot v + \Gamma^2_{12}\dot v\dot u + \Gamma^2_{22}\dot v\dot v = 0\end{aligned}\right. \qquad (**)$$
If we regard $(\dagger)$ as a linear ODE in a & b, then given $v_0 \in T_{\alpha(t_0)}S$, $t_0 \in I$, then there exists a unique parallel vector field $V(t)$ along α with $V(t_0) = v_0$.

<u>Def</u> The vector $V(t_1)$, $t_1 \in I$, is the parallel transport of v_0 along α at the point t_1.

Let $\alpha \in \Omega(p,q)$. Denote by $P: T_pS \to T_qS$ the map which assigns to $v_0 \in T_pS$ the result of parallel transport at T_qS.

<u>Properties</u> • P is a linear map, ⌈(†) is a linear ODE⌋
• P is an isometry. ⌈earlier prop about $\langle V(t), W(t)\rangle$⌋

<u>Prop</u> Given a point $p \in S$ and a vector $v \in T_pS$, $\exists\, \varepsilon > 0$ and a unique geodesic (with unit speed) $\gamma: (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

<u>Pf</u> Existence & uniqueness of sol^{n}s to ODEs, fixed initial conditions, applied to $(**)$. Integrate using $(*)$. □

3.3 Minimal surfaces

<u>Def</u> A surface $S \subseteq \mathbb{R}^3$ is <u>minimal</u> if its mean curvature H vanishes everywhere.

<u>Rk</u> $k_1 + k_2 = 0$ implies S can't be compact (ES3)

Let $\varphi: \mathcal{U} \to S$ be a param^n, and let $D \subseteq \mathcal{U}$ be a bdd domain with $\overline{D} \subseteq \mathcal{U}$.
($\mathcal{U} \subseteq \mathbb{R}^2$)

Let $h: \overline{D} \to \mathbb{R}$ be a smooth function.

<u>Def</u> The <u>normal variation</u> of $\varphi(\overline{D})$ determined by h is the map:

$$\rho: \overline{D} \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^3$$
$$(u, v, \cancel{\lambda}\, t) \longmapsto \varphi(u,v) + \cancel{\lambda}\, t\, h(u,v)\, N(u,v)$$

<u>Rk</u> If ∂D is a nice curve, e.g. piecewise smooth, we often demand that $h(\partial \overline{D}) \equiv 0$.

For fixed $t \in (-\varepsilon, \varepsilon)$, define $\rho^t(u,v) = \rho(u,v,t) : D \to \mathbb{R}^3$
$(\rho^0 = \varphi|_{\bar{D}})$.

$$\left.\begin{aligned} \rho^t_u &= \varphi_u + thN_u + th_u N \\ \rho^t_v &= \varphi_v + thN_v + th_v N \end{aligned}\right\} \text{lin indep for small } t\text{, on } \bar{D}$$

$\therefore$ for small t, $\rho^t(D)$ smooth embedded surface in $\mathbb{R}^3$

$\therefore$ wlog for all $t \in (-\varepsilon, \varepsilon)$

Let E^t, F^t, G^t be the coeffs of the 1FF of ρ^t

$$E^t = E + 2th\langle \varphi_u, N_u\rangle + t^2h^2\langle N_u, N_u\rangle + t^2 h_u h_u$$

$$F^t = F + th(\langle \varphi_u, N_v\rangle + \langle \varphi_v, N_u\rangle) + t^2h^2\langle N_u, N_v\rangle + t^2 h_u h_v$$

$$G^t = G + 2th\langle \varphi_v, N_v\rangle + t^2h^2\langle N_v, N_v\rangle + t^2 h_v h_v$$

Recall: $$\left.\begin{aligned} e &= -\langle \varphi_u, N_u\rangle \\ f &= -\langle \varphi_v, N_u\rangle = -\langle \varphi_u, N_v\rangle \\ g &= -\langle \varphi_v, N_v\rangle \end{aligned}\right\} \text{coeffs of 2FF of } \varphi$$

$$E^tG^t - (F^t)^2 = EG - F^2 - 2th(eG - 2fF + gE) + O(t^2)$$

Recall: $H = \dfrac{eG - 2fF + gE}{2(EG - F^2)}$ (mean curvature for φ)

So $E^tG^t - (F^t)^2 = (EG - F^2)(1 - 4thH) + r$,

unif on $\bar{D}$ → where $r/t \to 0$ as $t \to 0$.

Let $A(t)$ denote the area of $\rho^t(D)$. ($t \mapsto A(t)$ smooth)

$$A(t) = \int_D \sqrt{EG - F^2}\sqrt{1 - 4thH + \bar{r}}\; du\, dv \quad ; \quad \bar{r} = \frac{r}{\sqrt{EG - F^2}}$$

$$A'(0) = \lim_{t \to 0} \frac{A(t) - A(0)}{t}$$

$$= -\int_D 2hH\sqrt{EG - F^2}\; du\, dv \qquad (*)$$

<u>Prop</u> $\varphi(\mathcal{U})$ is minimal iff $A'(0) = 0$ for all bounded domains D with $\bar{D} \subset \mathcal{U}$ and all normal variations of $\varphi(\bar{D})$.

<u>Pf</u> If $H \equiv 0$ then $(*) \Rightarrow A'(0) = 0$.

actually $h = H$ works → Conversely, suppose $H(q) \neq 0$ for some $q \in \mathcal{U}$. Then set h non-zero in a small ngbd of q so that $A'(0) < 0$. □

Rmk They're called "minimal" surfaces, but they're just an extremum of area – needn't be minimum.

Def The mean value vector $\underline{H} := HN$

A normal variation in the direction of $\underline{H}$ always has $A'(0) < 0$ if H doesn't vanish. ($\underline{H}$ indep of orientation choice)

Soap films At regular points of soap films, the mean curvature vanishes; they minimise area for physical reasons. But a soap film might have self-intersections.

[pics or it didn't happen]

3.4 The Weierstrass representation

Prop Assume $\varphi: U \to \mathbb{R}^3$ is an isothermal param^n ($F=0$, $E=G=\lambda^2$)

Then $\varphi_{uu} + \varphi_{vv} = 2\lambda^2 \underline{H}$.

Proof Isothermal $\Rightarrow \langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle$

Differentiate wrt u: $\langle \varphi_{uu}, \varphi_u \rangle = \langle \varphi_{uv}, \varphi_v \rangle$

OTOH, $\langle \varphi_u, \varphi_v \rangle = 0 \Rightarrow \langle \varphi_{uv}, \varphi_v \rangle = -\langle \varphi_u, \varphi_{vv} \rangle$

$\therefore \langle \varphi_{uu} + \varphi_{vv}, \varphi_u \rangle = 0$

Similarly $\langle \varphi_{uu} + \varphi_{vv}, \varphi_v \rangle = 0$.

$\therefore \varphi_{uu} + \varphi_{vv} \sim N$

From local coord formulae,

$$H = \frac{g+e}{2\lambda^2} \qquad \text{Thus } 2\lambda^2 H = e + g = \langle \varphi_{uu} + \varphi_{vv}, N \rangle$$

$$\text{So } \varphi_{uu} + \varphi_{vv} = 2\lambda^2 HN = 2\lambda^2 \underset{\sim}{H}. \qquad \square$$

Cor Let $\varphi: U \to \mathbb{R}^3$ be an isothermal param^n.

Then $\varphi(U)$ is minimal iff $\Delta\varphi = 0$, where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Rmk Isothermal coords always exist for any surface (hard thm)

For minimal surfaces without planar points, see ES3.

Rmk The corollary says the coord fcts of such a φ are harmonic.

Fact: A non-constant harmonic function has no local maxima (locally the real part of a holomorphic function)

($\rightsquigarrow$ min surfaces never compact [overkill])

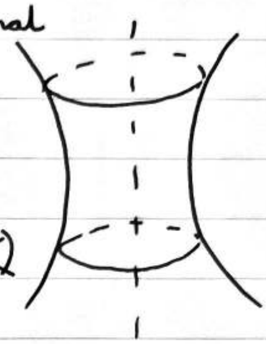
<u>Ex</u> <u>Catenoid</u>: $\varphi(u,v) = (a\cosh v \cos u,\ a\cosh v \sin u,\ av)$

$u \in [0, 2\pi],\ v \in \mathbb{R}$

<u>Check</u>: $E = G = a^2\cosh^2 v,\quad F = 0$ so isothermal

$\Delta\varphi = 0$ so minimal

<u>Fact</u>: only complete minimal surface of rev^{n}



"I swear that's not a hyperboloid"

<u>Helicoid</u>: $\varphi(u,v) = (a\sinh v \cos u,\ a\sinh v \sin u,\ \cancel{av}\ au)$

$u \in [0, 2\pi],\ v \in \mathbb{R}$

<u>Check</u>: $E = G = a^2\cosh^2 v,\quad F = 0$

$\rightsquigarrow$ locally isometric to catenoid

(& $\Delta\varphi = 0$ so minimal)

Prob of finding minimal surfaces : Lagrange 1760s
Catenoid & helicoid : Meusnier 1776 . Next one : ~1830s (Scherk)

Lots of further examples via holomorphic functions !

Write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, param^n of a surface $S \subseteq \mathbb{R}^3$.

<u>Prop^n</u> Consider the cx valued functions

$$\theta_1 = x_u - i x_v ,$$
$$\theta_2 = y_u - i y_v ,$$
$$\theta_3 = z_u - i z_v .$$

Then • φ is isothermal iff $\theta_1^2 + \theta_2^2 + \theta_3^2 = 0$,

• if φ is isothermal, then S is minimal iff $\theta_1, \theta_2, \theta_3$ are holomorphic (domain $\mathcal{U} \subseteq \mathbb{C}$)

<u>Pf</u> • Calculate $\theta_1^2 + \theta_2^2 + \theta_3^2 = (E-G) - 2iF$, whence the first claim.

• For the second claim, consider the Cauchy-Riemann equations for θ_1 :

$x_{uv} = x_{vu}$ ← always holds since φ smooth

$x_{uu} = -x_{vv}$ ← iff $\Delta x = 0$

$\left[(Re)_u = (Im)_v \quad (Re)_v = -(Im)_u \right]$

Similarly for θ_2 & θ_3. □

<u>Goal</u> : solutions to $\theta_1^2 + \theta_2^2 + \theta_3^2 = 0$ (*)

<u>Lemma</u> Let $D \subseteq \mathbb{C}$ be a domain, g meromorphic on D, and f a holomorphic f^n on D such that at each point where g has a pole of order k, f has a zero of order at least $2k$.

Then $\theta_1 = \frac{1}{2}\underset{f}{(}\cancel{1} - g^2)$, $\theta_2 = \frac{i}{2} f (1+g^2)$, $\theta_3 = fg$ (†)

are holomorphic on D & satisfy (*).

Conversely, every triple of holo^c f^n's on D, $\theta_1, \theta_2, \theta_3$ satisfying (*) may be represented in the form (†), except for the case $\theta_1 = i\theta_2$, $\theta_3 = 0$.

<u>Pf</u> • Direct calculation ; the condition on zeros, poles ensures that $\theta_1, \theta_2, \theta_3$ defined by $(\dagger)$ are holomorphic on D.

• Conversely, suppose $\theta_1, \theta_2, \theta_3$ holomorphic satisfy $(*)$. Set

$$f = \theta_1 - i\theta_2 \ , \quad g = \frac{\theta_3}{f} \qquad [\text{ok unless } \theta_1 \equiv i\theta_2]$$

$(*)$ becomes $\quad (\theta_1 - i\theta_2)(\theta_1 + i\theta_2) = -\theta_3^2$

$$\therefore \quad \theta_1 + i\theta_2 = -\frac{\theta_3^2}{\theta_1 - i\theta_2}$$

$$= -fg^2$$

Solve for the θ_i to obtain $(\dagger)$.

By definition, f is holomorphic, g is meromorphic, and by the above fg^2 is holomorphic which implies the relation between zeros and poles of f, g.

This rep^n can only fail if $\theta_1 \equiv i\theta_2$ in which case $\theta_3 \equiv 0$ and this is the exception remarked above. $\square$

Assume D is a simply connected domain, $\zeta_0 \in D$.

θ_i, $i = 1, 2, 3$ holomorphic, $\theta_1^2 + \theta_2^2 + \theta_3^2 = 0$, f, g associated to them as above.

<u>Prop</u> The functions $x(u,v)$, $y(u,v)$, $z(u,v)$ given by

$$x(u,v) = \mathrm{Re}\int_{\zeta_0}^{u+iv} \theta_1(\zeta)\, d\zeta = \tfrac{1}{2}\mathrm{Re}\int_{\zeta_0}^{u+iv} \cancel{i} f(\zeta)\left(1 - g(\zeta)^2\right) d\zeta \ ,$$

$$y(u,v) = \mathrm{Re}\int_{\zeta_0}^{u+iv} \theta_2(\zeta)\, d\zeta = \tfrac{1}{2}\mathrm{Re}\int_{\zeta_0}^{u+iv} f(\zeta)\left(1 - g(\zeta)^2\right) d\zeta \ ,$$

$$z(u,v) = \mathrm{Re}\int_{\zeta_0}^{u+iv} \theta_3(\zeta)\, d\zeta = \mathrm{Re}\int_{\zeta_0}^{u+iv} f(\zeta)\, g(\zeta)\, d\zeta \ ,$$

satisfy $x_u - ix_v = \theta_1$, $y_u - iy_v = \theta_2$, $z_u - iz_v = \theta_3$.

<u>Corollary</u> $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ is a minimal surface when embedded. This is called the Weierstrass parametrisation.

Pf For h holo^c, $\int_{\zeta_0}^{u+iv} h(\zeta)d\zeta$ is independent of the chosen path [D simply connected],

and $\frac{d}{d\zeta}\left(\int_{\zeta_0}^{\zeta} h(\xi)d\xi\right) = h(\zeta)$ [Cauchy].

Say $\chi(\omega) = \int_{\zeta_0}^{\omega} \theta_1(\zeta)d\zeta = \alpha(\omega) + i\beta(\omega)$.

$$\underset{x(u,v)}{\uparrow}$$

$$\theta_1(\omega) = \frac{d}{d\omega}\chi(\omega) = \alpha_u + i\beta_u \underset{\text{[Cauchy - Riemann]}}{=} \alpha_u - i\alpha_v$$

Ditto for θ_2 & θ_3. $\square$

ES 3: With φ as above, φ is an immersion iff f vanishes only at the poles of g, and it does so at order $2k$ where k is the order of the pole of g.

Rk Immersions are local embeddings – global is harder to check

Rk Given the existence of isothermal coordinates, the prop^n means any minimal surfaces has a local Weierstrass rep^n, for suitable D, f, g (& ζ_0).

Example Enneper's surface (1864)

$D = \mathbb{C}$, $f(\zeta) = 1$, $g(\zeta) = \zeta$ gives a minimal immersion $\varphi: \mathbb{C} \to \mathbb{R}^3$; $(\zeta_0 = 0)$ embedding on ngbd of origin.

$$\varphi(u,v) = \frac{1}{2}\left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - u^2v, u^2 - v^2\right)$$

Gaussian curvature $K = \frac{-16}{(1+|\zeta|^2)^4}$, $(\zeta = u+iv)$

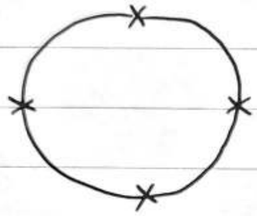
<u>Example</u> Scherk's doubly periodic surface (1835)

D = open unit disc in $\mathbb{C}$ $\qquad f(\zeta) = \dfrac{4}{1-\zeta^4}$

$\zeta_0 = 0$ $\qquad g(\zeta) = \zeta$

This gives an immersion $\phi : D \to \mathbb{R}^3$, $\zeta = u+iv$ gives

$$\phi(u,v) = \left(-\arg \frac{\zeta + i}{\zeta - i} - \pi,\ -\arg \frac{\zeta+1}{\zeta-1} - \pi,\ \log \left| \frac{\zeta^2+1}{\zeta^2-1} \right| \right)$$

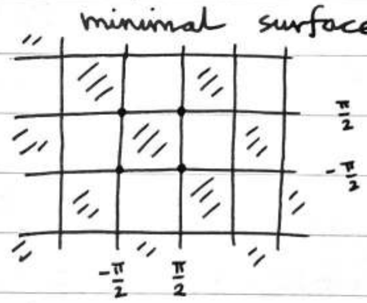


"weird behaviour" at 4^{th} roots of unity

$\varphi(D)$ lies over open square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq (x,y)$-plane with $z = \log\left(\dfrac{\cos y}{\cos x}\right)$

$\Rightarrow$ $\underline{\cos y = e^z \cos x}$ $\quad$ complete, embedded, doubly periodic, minimal surface S

Domain in x-y plane:

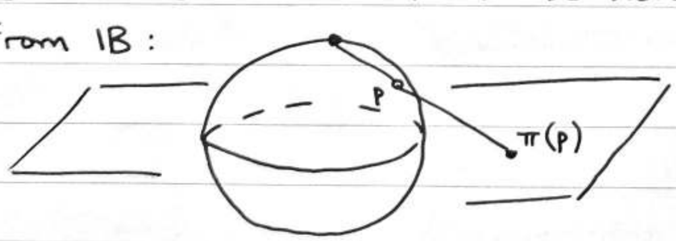


vertical lines at the vertices

3.5 Meaning of g in the Weierstrass representation

Let $\pi : S^2 \to \mathbb{C} \cup \{\infty\}$ be stereographic projection

From IB:



$$\pi^{-1}(z) = \left(\frac{2\,\mathrm{Re}\, z}{1+|z|^2},\ \frac{2\,\mathrm{Im}\, z}{1+|z|^2},\ \frac{|z|^2-1}{|z|^2+1} \right)$$

Suppose φ is a Weierstrass param^n for an immersed minimal surface. Then $\quad \varphi_u - i\varphi_v = (\theta_1, \theta_2, \theta_3)$

& hence $\quad \lambda^2 = E = G = \frac{1}{2} \| \varphi_u - i\varphi_v \|^2 = \frac{1}{2} \sum |\theta_j|^2$

i.e. $\lambda^2 = \left(\frac{|f|(1+|g|^2)}{2}\right)^2$ & $F = 0$

Check: $\varphi_u \times \varphi_v = \operatorname{Im}(\theta_2\overline{\theta_3}, \theta_3\overline{\theta_1}, \theta_1\overline{\theta_2})$

Use expressions for θ_i in terms of f & g,

$$\varphi_u \times \varphi_v = \frac{|f|^2(1+|g|^2)}{4}\left(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2-1\right)$$

$$\Rightarrow |\varphi_u \times \varphi_v| = \lambda^2 = \left(\frac{|f|^2(1+|g|^2)^2}{4}\right)$$

$$\therefore N = \left(\frac{2\operatorname{Re} g}{1+|g|^2}, \frac{2\operatorname{Im} g}{1+|g|^2}, \frac{|g|^2-1}{|g|^2+1}\right)$$ is the Gauss map

So $\pi \circ N = g$ as a map $D \to \mathbb{C} \cup \{\infty\}$.

RK $\exists$ non-orientable min surfaces (!)

ES3 As a function on D, the Gaussian curvature is

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2$$

Cor Either $K \equiv 0$ or zeros of K are isolated

Thm If $D = \mathbb{C}$, then either the minimal surface (immersed) $\varphi(D)$ lies in the plane, or the Gauss map hits all but two values.

E.g.: • Enneper surface, $g(\zeta) = \zeta$

• Catenoid, has an immersed param^n from $\mathbb{C}$, and the Gauss map misses 2 directions

Proof • $g(\zeta)$ fails to be defined iff $\theta_1 \equiv i\theta_2$, $\theta_3 \equiv 0$.
In this case z is constant so the surface lies on a plane.

• If g is defined, it's a meromorphic function on $\mathbb{C}$.

- If g is constant, again lie on a plane.
- If not, then by Picard's theorem, g takes all values in $\mathbb{C} \cup \{\infty\}$ except for at most 2 of them; so the Gauss map hits all but at most 2 directions. $\square$

<u>Fact</u> If S is a complete minimal surface in $\mathbb{R}^3$ and non-planar, then the image of the Gauss map misses at most 4 points (Xavier 1981, Fujimoto 1988)

Chap IV Global Riemannian Geometry

4.1 The exponential map & geodesic polar coordinates

Let $S \subset \mathbb{R}^3$ be an embedded surface.

Recall: given $p \in S$, and $v \in T_pS$, $\exists\, \varepsilon > 0$ and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$, $\dot\gamma(0) = v$. Write $\gamma_v := \gamma$.

Note: The geodesic $\gamma_{\lambda v}(t)$, $t \in (-\varepsilon/|\lambda|, \varepsilon/|\lambda|)$
is $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$ via uniqueness.

Def If $v \in T_pS$, $v \neq 0$ is such that $\gamma_{\frac{v}{|v|}}(|v|) = \gamma_v(1)$ is defined, then we set $\exp_p(v) = \gamma_v(1)$; and $\exp_p(0) = p$.

Prop/fact: $\exp_p$ is always defined, and smooth, in some ball B_ε w/ centre at $0 \in T_pS$.

Pf Existence & uniqueness of solutions to ODEs, including the smooth dependence on initial conditions.

Def $\exp_p$ is called the exponential map

Prop $\exp_p : B_\varepsilon \to S$ is a diffeo^m from a ngbd $U \subset B_\varepsilon$ of 0 in T_pS onto its image in S.

Pf By IFT, enough to show that $D(\exp_p)_0$ is non-singular.
Let $\alpha(t) := tv$, $v \in T_pS$.

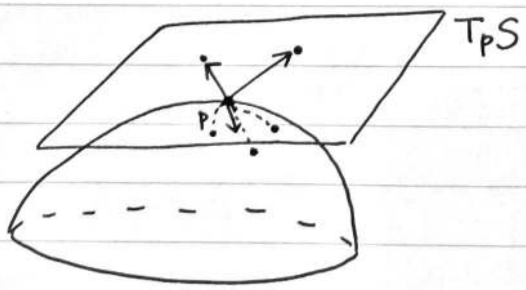
The curve
$$(\exp_p \circ \alpha)(t) = \exp_p(tv) = \gamma_v(t)$$
has tangent vector v at $t = 0$.

Then $D(\exp_p)_0(v) = v$ by the chain rule.

$\therefore D(\exp_p)_0 : T_pS \to T_pS$ is the identity ; done. $\square$

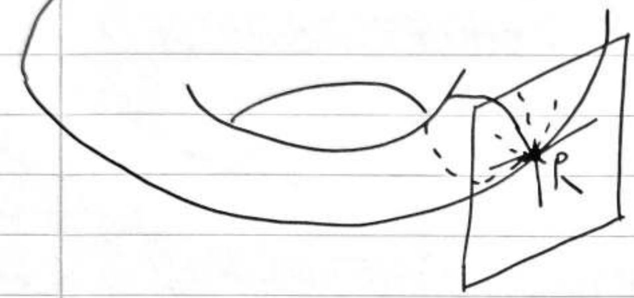
Cor When restricted to a sufficiently small ball U, $\exp_p$ is a parametrisation of S around p.

Def $V : \exp_p(U)$ is called a normal ngbd of p.



Further csq of ODE theory

- If $S \subset \mathbb{R}^3$ is closed (as a subset of $\mathbb{R}^3$), then $\exp_p$ is defined on all of T_pS; geodesics are defined for all $t \in \mathbb{R}$.
- The exponential map depends smoothly on p.



geodesics meet for large t
$\therefore \exp_p$ will generally not be an embedding

Rmk All of this generalises to arbitrary embedded manifolds $X \subset \mathbb{R}^N$.

Aside (cf ES4) The terminology comes from matrix groups, i.e. groups X which are closed submfds of $M_n(\mathbb{R})$.
E.g. $X = O(n) \subset M_n(\mathbb{R})$
$M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, inner product $\langle L_1, L_2 \rangle = \mathrm{tr}(L_1 L_2^T)$
Given $A \in T_I X$, define $\alpha : \mathbb{R} \to M_n\mathbb{R}$ by:

$$\alpha(t) = \exp(At) = I + At + \tfrac{1}{2}A^2t^2 + \cdots$$

Note: $\alpha'(t) = A\alpha(t)$,
$\alpha''(t) = A^2\alpha(t)$, etc.

You can show that: $\left.\begin{array}{l} \cdot\ \alpha(t) \in X \ \ \forall t \\ \cdot\ \alpha''(t) \perp T_{\alpha(t)}X \end{array}\right\} \Rightarrow$ α is a geodesic on X — must be the unique one through I with $\alpha'(0) = A$ ☺

(r, θ) polar coords on $\mathbb{R}^2$
⌜ Formally: delete the half-line $\theta = 0$
i.e. $0 < r$, $0 < \theta < 2\pi$ ⌟

Assume $p \in S$, $\exp_p : B_\varepsilon(0) \to V \subseteq S$ is a diffeo^{m}

Def The geodesic polar coordinates about p are the images of the polar coords on $B_\varepsilon(0) \subseteq \mathbb{R}^2$ under $\exp_p$.
Formally: $0 < r < \varepsilon$, $0 < \theta < 2\pi$, but we can still define a smooth map

$$\varphi : \tilde{U} := (-\varepsilon, \varepsilon) \times \mathbb{R} \longrightarrow V \quad (\ni p)$$
$$(r, \theta) \longmapsto \exp_p(r(\cos\theta e_1 + \sin\theta e_2))$$
$$= \exp_p(r v(\theta))$$
$$= \gamma_{v(\theta)}(r)$$

unit vector "at angle θ"

<u>Def</u> The images under $\exp_p : U \to V$ of circles in U centred at the origin are called <u>geodesic circles</u>.

wrt 1FF, gives metric on T_pS

The images of lines through the origin under $\exp_p$ are called <u>radial geodesics</u>.

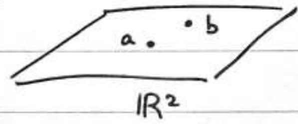
<u>Prop</u> The coefficients for 1FF in geodesic polar coords, say $E(r,\theta)$, $F(r,\theta)$, $G(r,\theta)$ satisfy:

- $E \equiv 1$
- $F \equiv 0$
- $G(0,\theta) = 0$
- $(\sqrt{G})_r(0,\theta) = 1$

Moreover, the Gauss curvature K can be written as

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$$

<u>QQs</u>



Recall: $\varphi(r,\theta) = \exp_p(rv(\theta)) = \gamma_{v(\theta)}(r)$

Pf of Prop • From the def of φ, $\varphi_r = \dot\gamma_{v(\theta)}(r)$, and $v(\theta) = \dot\gamma_{v(\theta)}(0)$ has norm 1. Geodesics travel at constant speed, so

$$E(r,\theta) = \langle \varphi_r, \varphi_r \rangle \equiv 1.$$

• Let $w = \frac{dv}{d\theta}$. Also from the def of φ, we see that

$$\begin{aligned}\varphi_\theta &= D(\exp_p)_{rv}(rw)\\ &= r\,D(\exp_p)_{rv}(w)\end{aligned}$$

Hence
$$F(r,\theta) = r\langle \dot\gamma_{v(\theta)}(r), D(\exp_p)_{rv}(w)\rangle,$$
$$G(r,\theta) = r^2 \langle D(\exp_p)_{rv}(w), D(\exp_p)_{rv}(w)\rangle.$$

• $F(0,\theta) = 0$

• Formula for $G(r,\theta)$ implies $(\sqrt{G})_r(0,\theta) = \| D(\exp_p)_0(w)\|$
$$= \|w\| = 1$$
$\ulcorner D(\exp)_0 = id \lrcorner$

• Compute $F_r(r,\theta) = \underbrace{\langle \varphi_{rr}, \varphi_\theta\rangle}_{=0} + \langle \varphi_r, \varphi_{\theta r}\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}\langle \varphi_r, \varphi_r\rangle$
$\equiv 0$ by above.

since $\gamma_{v(\theta)}(r)$ is geodesic, so $\varphi_{rr} \perp T_q S$

So since $F(0,\theta) = 0$, get that $F \equiv 0$.

Finally, the expression for K follows immediately from the simplificated formula for an orthogonal param^n.

$$\ulcorner\, 2K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right] \lrcorner$$

$\square$

Rmk The fact that $F(r,\theta) = 0$ $\forall r > 0$, i.e. $\langle \varphi_r, \varphi_\theta\rangle = 0$, says that the geodesic circles are everywhere orthogonal to the radial geodesics.

Rmk This system of coords on S shows that a geodesic on S minimises length / energy. (ES4)

4.2 Gauss - Bonnet theorem

4.2.1 Geodesic curvature

Def Let w be a smooth field of unit vectors along a curve $\alpha : I \to S \subset \mathbb{R}^3$, on an oriented surface S.

Since $|w(t)| = 1$, $\frac{dw}{dt}$ is orthogonal to w $\forall t$, and so

$$\frac{Dw}{dt} = \lambda N \wedge w$$

The real number $\lambda = \lambda(t)$, denoted $\left[\frac{Dw}{dt}\right]$, is called the algebraic value of the covariant derivative.

Rmk The sign of $\left[\frac{Dw}{dt}\right]$ depends on the choice of orientation.

$$\left[\frac{Dw}{dt}\right] = \left\langle \frac{Dw}{dt}, N \wedge w \right\rangle = \left\langle \frac{dw}{dt}, N \wedge w \right\rangle$$

Def Let $\alpha : I \to S$ be a regular curve param^d by arc length. The algebraic value of the covariant derivative of $\dot{\alpha}(s)$

$$k_g(s) := \left[\frac{D\dot{\alpha}}{dt}\right] = \langle \ddot{\alpha}, N \wedge \dot{\alpha} \rangle$$

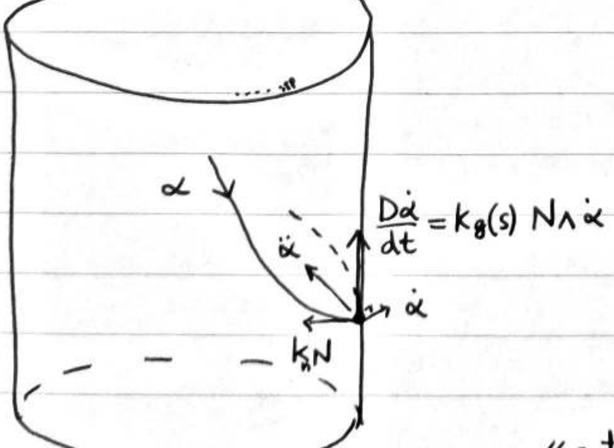
is called the geodesic curvature of α at $\alpha(s)$.

So $k(s)^2 = |\ddot{\alpha}(s)|^2 = k_g(s)^2 + k_n(s)^2$

(total) curvature — normal curvature $= \langle N, k\underline{n} \rangle = \langle N, \ddot{\alpha} \rangle$

Rmk Sign of geodesic curvature depends on orientation.

α is a geodesic iff $k_g(s) = 0$ $\forall s$



« artist's impression »

Let V, W be two smooth unit vector fields along a curve $\alpha : I \to S$

Let $\underbrace{iV}_{\text{notation!}}$ be the unique vector field along α s.t. $\forall t$,

$(V(t), iV(t))$ is a +vely oriented basis for $T_{\alpha(t)}S$.

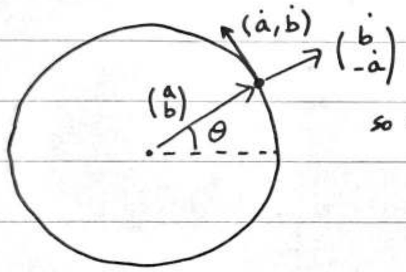
Write $W(t) := a(t)V(t) + b(t)iV(t)$

where a, b smooth & $a^2 + b^2 = 1$.

Ex If a, b smooth fns on I s.t. $a^2+b^2=1$, and φ_0 s.t. at $t_0 \in I$, $a(t_0) = \cos\varphi_0$, $b(t_0) = \sin\varphi_0$, then

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} (a\dot{b} - b\dot{a})\, d\tau$$

is a smooth fn on I s.t. $\varphi(t_0) = \varphi_0$, $\begin{cases} \cos(\varphi(t)) = a(t), \\ \sin(\varphi(t)) = b(t). \end{cases}$



so $\dot{\theta} = a\dot{b} - b\dot{a}$ etc

Def The smooth function $\varphi(t)$ obtained from V, W as above is called a smooth determination of the angle from V to W along α.

Prop Let V, W be smooth unit vector fields along a curve α.

Then $\left[\dfrac{DW}{dt}\right] - \left[\dfrac{DV}{dt}\right] = \dfrac{d\varphi}{dt}$ where φ is a smooth determination of the angle from V to W.

Recall: Prop Let V, W be smooth vector fields along $\alpha : I \to S$.
Then $\left[\frac{DW}{dt}\right] - \left[\frac{DV}{dt}\right] = \frac{d\varphi}{dt}$

where $\varphi(t)$ is a smooth determination of the angle from V to W.

Proof By def^n $\left[\frac{DW}{dt}\right] = \langle \dot{W}, N \times W \rangle$

$$\left[\frac{DV}{dt}\right] = \langle \dot{V}, N \times V \rangle = \langle \dot{V}, iV \rangle$$

$W = \cos\varphi(t)\, V + \sin\varphi(t)\, iV$
$\Rightarrow \dot{W} = \dot{\varphi}\left(-\sin\varphi\, V + \cos\varphi\, iV\right) + \cos\varphi\, \dot{V} + \sin\varphi\, \dot{(iV)}$
$N \times W = \cos\varphi\, iV - \sin\varphi\, V$ $\quad\ulcorner V, iV, N$ o.n. triple$\lrcorner$
$\Rightarrow \left[\frac{DW}{dt}\right] = \dot{\varphi} + \langle -\sin\varphi\, V + \cos\varphi\, iV, \cos\varphi\, \dot{V} + \sin\varphi\, \dot{(iV)} \rangle$

$$= \dot{\varphi} + \cos^2\varphi \langle iV, \dot{V} \rangle - \sin^2\varphi \langle V, \dot{(iV)} \rangle$$

Note: $\langle V, \dot{V} \rangle = \langle iV, \dot{(iV)} \rangle = 0$ b/c unit vector fields.
$\langle V, iV \rangle = 0 \Rightarrow \langle iV, \dot{V} \rangle + \langle \dot{(iV)}, V \rangle = 0$

So: $\left[\frac{DW}{dt}\right] = \dot{\varphi} + \cos^2\varphi \langle iV, \dot{V} \rangle + \sin^2\varphi \langle iV, \dot{V} \rangle$

$$= \dot{\varphi} + \langle iV, \dot{V} \rangle = \dot{\varphi} + \left[\frac{DV}{dt}\right]. \qquad \square$$

Rmks • Let $\alpha : I \to S$ regular param^d by arc-length.
Let $V(s)$ be a parallel (unit) vector field along α.
Let ψ be a smooth determination of the angle from V to $\dot{\alpha}$.
Then by the prop, $k_g(s) = \left[\frac{D\dot{\alpha}}{Dt}\right] = \frac{d\psi}{dt}$

• If α is plane curve, we can take $V(t)$ to be a fixed direction in the plane, and k_g reduces to the usual curvature.

S oriented. $\varphi(u,v)$ param^n. Say that φ is compatible w/ the orientation if (φ_u, φ_v) is a positively oriented basis of T_pS.

<u>Prop</u> Let $\varphi(u,v)$ be an orthogonal param^{n}, compatible w/ orientation. Let $W(t)$ be a smooth unit vector field along a curve $\alpha(t) = \varphi(u(t), v(t))$. Then:

$$\left[\frac{DW}{dt}\right] = \frac{1}{2\sqrt{EG}}\left[\frac{G_u \dot v - E_v \dot u}{\text{,}\smile\text{,}}\right] + \frac{d\psi}{dt}$$

where ψ is the angle from φ_u to W (in given orientation).

If $\alpha : I \to \varphi(U)$ param^{d} by arc-length,

$$K_g(s) = \frac{1}{2\sqrt{EG}}\left[G_u \dot v - E_v \dot u\right] + \frac{d\psi}{dt}$$

where $\psi(s)$ is the angle from φ_u to $\dot\alpha(s)$ (in given orientation)

<u>Pf</u> Let $e_1 = \dfrac{\varphi_u}{\sqrt E}$, $e_2 = \dfrac{\varphi_v}{\sqrt G}$.

Then (e_1, e_2) is a +vely oriented o.n. basis of tangent spaces.

· By previous result, $\left[\dfrac{DW}{dt}\right] = \left[\dfrac{De_1}{dt}\right] + \dfrac{d\psi}{dt}$ ⌜cf V, iV⌟

· $$\left[\frac{De_1}{dt}\right] = \langle \dot e_1, N \times e_1 \rangle = \langle \dot e_1, e_2\rangle$$
$$= \langle (e_1)_u \dot u + (e_1)_v \dot v, e_2\rangle$$
$$= \langle (e_1)_u \dot u, e_2\rangle + \langle (e_1)_v \dot v, e_2\rangle$$

· $$\langle (e_1)_u, e_2\rangle = \left\langle \left(\frac{\varphi_u}{\sqrt E}\right)_u, \frac{\varphi_v}{\sqrt G}\right\rangle$$
$$= \frac{1}{\sqrt{EG}}\langle \varphi_{uu}, \varphi_v\rangle$$

$\langle \varphi_u, \varphi_v\rangle = 0$ so other term drops

$$\langle \varphi_{uu}, \varphi_v\rangle = \underbrace{\langle \varphi_u, \varphi_v\rangle_u}_{\text{zero}} - \langle \varphi_u, \varphi_{uv}\rangle = -\tfrac12 E_v$$

$$\Rightarrow \langle (e_1)_u, e_2\rangle = -\frac{E_v}{2\sqrt{EG}}$$

· Similarly $$\langle (e_1)_v, e_2\rangle = \frac{G_u}{2\sqrt{EG}}$$

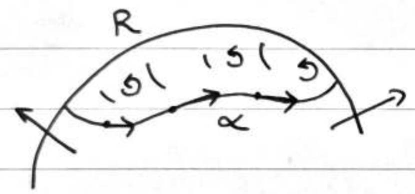
□

4.2.2 Local Gauss-Bonnet theorem

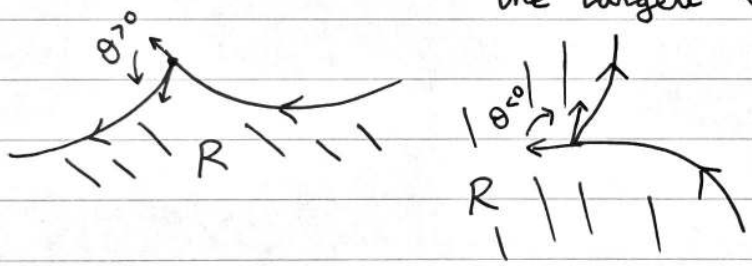
Let S be an oriented surface. A region $\overline{R} \subset S$ (closure of a domain R) is called a simple region if

- $\overline{R}$ is homeo^c to a closed disc
- $\partial\overline{R}$ can be param^d by a simple, closed, $\underbrace{\text{piecewise}}_{\text{"may have corners"}}$ regular curve $\alpha: I \to S$ param^d by arc length

Say that α is positively oriented if at all regular points, $N(\alpha(s)) \wedge \dot{\alpha}(s)$ points into the region R.

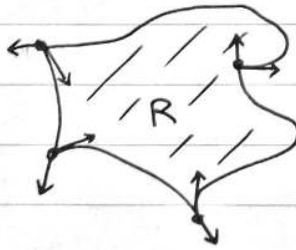


The exterior (signed) angles at the vertices of α (i.e. points where not reg) are given by the discontinuity of the tangent vectors.



⌜$\theta \in (-\pi, \pi)$⌟ ⊗∼

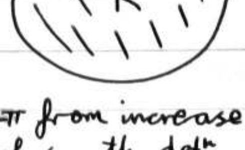
Observe: Suppose $\overline{R} \subseteq \varphi(U)$, some $\varphi: U \to S$ param^n, U disc. Then there exists a piecewise diff'ble fn $\psi: I \to \mathbb{R}$ which is given by a smooth determination of the angle from φ_u to $\dot{\alpha}$ on the smooth parts of α, and at each vertex $\alpha(t_j)$ jumps by the exterior angle θ_j.



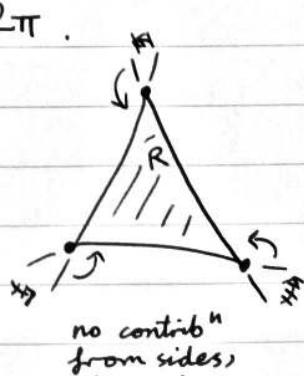
$\vec{\varphi_u}$, say

Our assumptions imply
$$\psi(L) - \psi(0) = 2\pi.$$

(No proof)



2π from increase of smooth det^n



no contrib^n from sides, but angles

Thm (local Gauss-Bonnet)

Let $\varphi : U \to S$ be an orthogonal parametrisation, S oriented surface, $U \subset \mathbb{R}^2$ disc, φ compatible with orientation. Let $\alpha : \underset{[0,L]}{I} \to \varphi(U)$ be a piecewise regular, simple closed curve, enclosing a domain R, parametrised by arc-length, positively oriented.

Then
$$\int_I k_g(s)\,ds + \int_R K\,dA = 2\pi - \sum_{i=1}^{N} \theta_i$$
where θ_i are the signed exterior angles at the vertices.

Pf From L20, have the piecewise diff'ble & piecewise cts function $\psi : I \to \mathbb{R}$, a smooth determination of the angle from φ_u to $\dot\alpha$ in given orientation on the regular arcs, jumps by θ_i at vertices.

By the previous proposition, $k_g(s) = \frac{1}{2\sqrt{EG}}\left[G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right] + \frac{d\psi}{ds}$

away from vertices, where $\alpha(s) = \varphi(u(s), v(s))$.

$$\Rightarrow \int_I k_g(s)\,ds = \int_I \frac{1}{2\sqrt{EG}}\left[G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right] ds + \int_I \frac{d\psi}{ds}\,ds$$

By L20 discussion, $\int_I \frac{d\psi}{ds}\,ds + \sum_{i=1}^{N} \theta_i = 2\pi$.

Green's thm:
$$\int_I k_g(s)\,ds = \int_{\varphi^{-1}(R)} \left[\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right] du\,dv + 2\pi - \sum_{i=1}^{N} \theta_i$$

Using Gauss formula for o.g. param^n, we have:
$$\int_I k_g(s)\,ds = -\int_{\varphi^{-1}(R)} K\sqrt{EG}\,du\,dv + 2\pi - \sum_{i=1}^{N} \theta_i$$
$$\underset{F=0}{=} -\int_R K\,dA + 2\pi - \sum_{i=1}^{N} \theta_i .$$
$\square$

Rmk If T is a geodesic triangle w/ interior angles $\alpha_1, \alpha_2, \alpha_3$, get:
$$\int_T K\,dA = 2\pi - \sum_{i=1}^{3} (\pi - \alpha_i) = \sum_{i=1}^{3} \alpha_i - \pi .$$

(For now, $T \subseteq \varphi(U)$ for o.g. param^n, general T later)

4.2.3 Triangulations & Euler characteristic

Def Let S be a compact surface, perhaps w/ boundary.
A triangulation of S consists of a finite number of topological triangles $\{T_1, \dots, T_N\}$ (i.e. each T_i closed & homeo to a Euclidean triangle), which cover S s.t. two distinct triangles are either disjoint, or have a single vertex in common, or have an entire edge in common.


✓


×

Def The Euler characteristic of the triangle is defined as

$$\chi := V - E + F$$

where $F = \#\text{faces}$, $E = \#\text{edges}$, $V = \#\text{vertices}$.

Fact · Every compact surface has a triangulation — moreover, the triangles can be taken to be smooth (i.e. edges are regular curves)

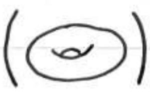
· Any triangulation of S can be approximated by a smooth one, with the same E, V, F. ⚠ Not true in higher dim

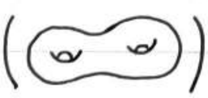
· χ is independent of the choice of triangulation, (L22) by global Gauss-Bonnet, so we write
$\chi = \chi(S)$, the Euler characteristic of the surface.
(also seen in alg top)

· Compact surfaces w/o boundary who are orientable are completely classified by $\chi(S)$. Any such surface is diffeo^c to a sphere with g handles, where $\chi(S) = 2 - 2g$ and g is called the genus of S.

e.g. $g=0$: $\chi($  $) = 2$

cf IB geom, alg top, Riem sfc, alg geom

$g=1$: $\chi($  $) = 0$

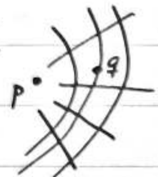
$g=2$: $\chi($  $) = -2$

⋮

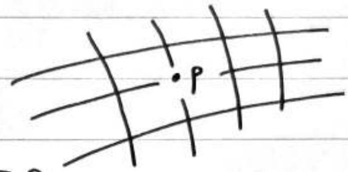
4.2.4 The global Gauss-Bonnet theorem

Prop Given $p \in S$, $\exists$ param^n $\varphi : \mathcal{U} \to S$ of a ngbd of p which is orthogonal.

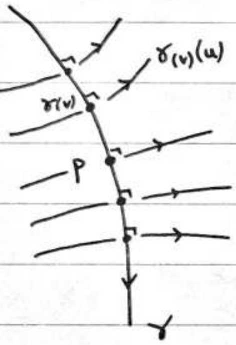
Pf



- Either geodesic polars + smooth dependence of $\exp_p$ on p + local compactness
- Or : "normal coordinates"



Start w/ unit speed geodesic through p, say $\gamma : (-\varepsilon, \varepsilon) \to S$, $\gamma(0) = p$.



For any point $\gamma(v)$, we take a unit speed geodesic $\gamma_{(v)}(u)$ with
- $\gamma_{(v)}(0) = \gamma(v)$
- $\frac{d}{du}\gamma_{(v)}(0) \perp \frac{d\gamma}{dv}(\gamma(v))$ & +vely oriented

Existence & uniqueness of geodesics & smooth dependence on ICs imply we get a smooth map $\varphi(u,v) = \gamma_{(v)}(u)$ from a ngbd of $\underline{0} \in \mathbb{R}^2$ to S.

Moreover: $$\left.\begin{aligned} (D\varphi)_0(e_1) &= \frac{\partial \gamma_{(0)}}{\partial u}(0) \\ (D\varphi)_0(e_2) &= \frac{\partial \gamma}{\partial v}(0) \end{aligned}\right\} \text{o.n. so } D\varphi_0 \text{ iso}$$

$\therefore \varphi$ is a local diffeo by IFT ; get a param^n

- Exercise : check φ is orthogonal $\ulcorner\ E \equiv 1$

$$F(0,v) = 0$$

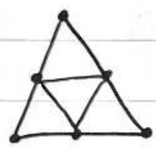
$$F_u = 0 \ \lrcorner$$

$\square$

Suppose S is a <u>compact surface.</u> We may cover S by a finite number of geodesic balls, of the form $B(x_j, \varepsilon_j)$, $j = 1, \dots, N$, s.t each $B(x_j, \varepsilon_j)$ has an orthogonal parametrisation.

Set $\varepsilon = \min \varepsilon_j$.

Start w/ a smooth triangulation of S. We can subdivide it by dividing each triangle into 4 smaller triangles.



$$\begin{aligned} \chi_{\text{new triangulation}} &= F' - E' + V' \\ &= 4F - (2E + 3F) + (V + E) \\ &= F - E + V = \chi_{\text{old triangulation}} \end{aligned}$$

By repeating this process enough times, get a new triangulation in which every triangle has geodesic diameter $< \varepsilon$ (& w/ same χ).

not sure this works → use Lebesgue no. Lemma

And hence (!) each is contained in some $B(x_j, \varepsilon_j 2)$, some j.

Thus local G-B applies to each triangle in the subdivided smooth triangulation.

<u>Thm</u> (Global Gauss-Bonnet)

Let S be a smooth compact surface, $\partial S = \emptyset$.

Given any triangulation of S w/ Euler characteristic χ, we have:

$$\int_S K \, dA = 2\pi \chi.$$

<u>Cor</u> χ does not depend on the choice of triangulation.

Write it as $\chi = \chi(S)$.

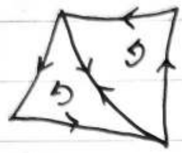
<u>Pf</u> By discussion above, assume wlog that the triangulation is smooth, and that local Gauss-Bonnet may be applied to each of its triangles, T_i $i = 1, \dots, m$.

If T_i has interior angles $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ at its vertices, then:

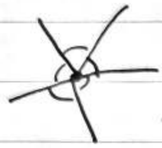
$$\int_{\underset{\text{+ve orientation}}{\partial T_i}} k_g \, dl + \int_{T_i} K \, dA = \sum_{j=1}^{3} \alpha_{ij} - \pi$$

Now sum over the T_i.

The boundary terms cancel, e.g. look at each edge.



So $$\int_S K\,dA = \sum_{i,j} \alpha_{ij} - \pi F$$



at a vertex

$$= 2\pi V - \pi F$$

$$= 2\pi V - 2\pi E + 2\pi F$$

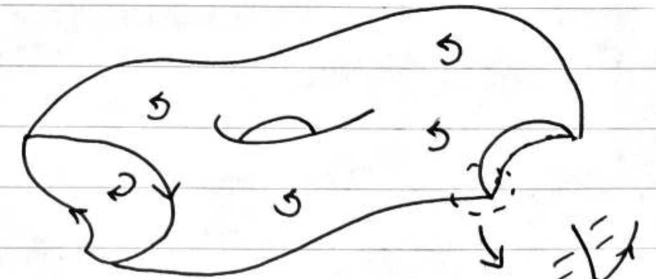
3F=2E

$$= 2\pi \chi. \qquad \square$$

Thm (Global G-B w/ boundary)
Let $R \subset S$ be a domain in an oriented surface, and suppose R has compact closure, and its boundary consists of n piecewise regular simple closed curves: $\alpha_i : I_i \to S$, $i = 1, \dots, n$ where the images do not intersect (!), the α_i are param'd by arc-length & positively oriented. Let θ_j be the exterior angles of the vertices of the curves α.
Then $$\sum_{i=1}^{n} \int_{I_i} k_g(s)\,ds + \int_R K\,dA = 2\pi \chi(R) - \sum_{j=1}^{p} \theta_j.$$

Pf We generalise the argument in the previous proof.
Assume that we have a smooth triangulation as before (note: all vertices of the α_i are vertices of the triangulation)
Modification to previous proof:

$\int_{\text{boundary edge}} k_g\,ds$ don't cancel



Adding them up get the sum over the α_i of $k_g\,ds$.

negative external angle

$$\sum_{T_i} \left(\sum_j \alpha_{ij} - \pi \right) = 2\pi V - \sum_j \theta_j - \pi F$$
$$- \pi(\#\text{smooth boundary vertices})$$
$$- \pi(\#\text{vertices on boundary})$$

$$= 2\pi V - \pi V_\partial - \sum_j \theta_j - \pi F$$

Now, $3F = 2E - E_\partial$, $V_\partial = E_\partial$

so we obtain

$$2\pi\chi(R) - \sum_j \theta_j \quad \text{and done.}$$

$\square$

4.3 Application of Gauss-Bonnet

<u>Thm</u> A connected compact oriented surface S w/ everywhere positive Gaussian curvature is diffeomorphic to S^2.

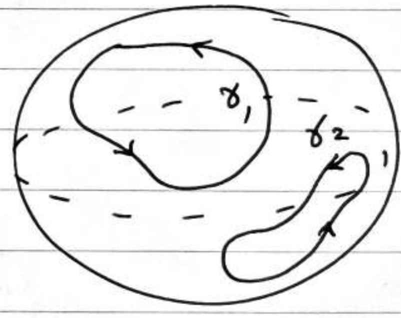
Moreover, if $\exists$ two simple closed geodesics on S, then they must intersect.

<u>Pf</u> If $K>0$ everywhere, then every point of S^2 is a regular value for the Gauss map $N: S \to S^2$.

Stack of records theorem $\Rightarrow$ N is a covering map c.f. Alg Top

Since S^2 is simply connected, $\nexists$ non-trivial topological covering, so N is a homeomorphism. Hence a diffeo.

Now let γ_1, γ_2 be simple closed geodesics and suppose $\gamma_1 \cap \gamma_2 = \phi$.



Then the γ_i bound a domain $R \subset S^2$ homeo^c to an annulus.

Now $\chi(R) = 0$, so by G-B

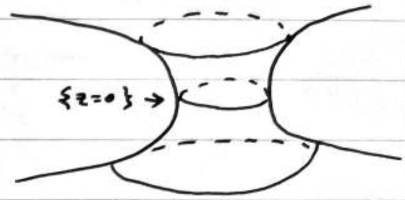
$$\int_R K \, dA = 0 \quad ※$$

$\square$

Prop Let S be a surface homeo^c to a cylinder. Suppose S has everywhere negative Gaussian curvature. Then S has at most one simple closed geodesic.

Pf ExSh4. $\square$

Ex Catenoid has precisely one simple closed geodesic.



§4.4 Fenchel's theorem

Def Let $\alpha : [0,L] \to \mathbb{R}^2$ or $\mathbb{R}^3$ param^d by arc-length. The total curvature of α is $\int_0^L |k(s)|\, ds$.

Note If α is a simple closed plane curve (so contained in a normal ngbd), then «by» local G–B:

$$\int_0^L k(s)\, ds = 2\pi \qquad \ulcorner k(s) = k_g(s) = \text{signed curvature}\lrcorner$$

$\Rightarrow$ The total curvature is $\geqslant 2\pi$, with equality iff $k(s)$ never changes sign.

Thm (Fenchel)

If $\alpha : [0,L] \to \mathbb{R}^3$ is a simple regular closed curve with nowhere vanishing curvature, then the total curvature is $\geqslant 2\pi$, with equality iff α is a plane curve.

could drop w/ more work
⚠ Do Carmo uses it secretly

Fact Let S be a compact surface in $\mathbb{R}^3$ (possibly with boundary), $N : S \to S^2$ the Gauss map. Then

$$\int_S |K|\, dA = \int_{S^2} \# N^{-1}(y)\, dy \qquad \ulcorner\text{cf ES4}\lrcorner$$

Pf (sketch) The critical values of N have measure 0 by Sard, so they don't affect the RHS. They don't affect the LHS either, as critical points have $K=0$.

Around any regular value $y \in S^2$, by the Stack-of-Records-thm, $\exists$ open ngbd of y s.t. $N^{-1}(\mathcal{U}) = \bigsqcup_{i=1}^{k} V_i$ where $N : V_i \to \mathcal{U}$ is a diffeo^m.

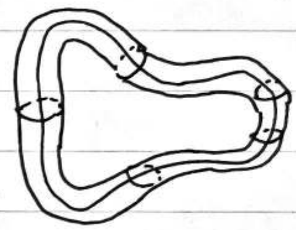
Since $K(x) = \det DN_x$ $(\forall x \in S)$, the change of variables formula implies $\displaystyle\int_{V_i} |K|\, dA = \int_{\mathcal{U}} dy$ = area of $\mathcal{U}$ $\forall i$.

Thus $\displaystyle\sum_{i=1}^{k} \int_{V_i} |K|\, dA = \int_{\mathcal{U}} \# N^{-1}(y)\, dy$.

this, works! → To conclude, either define a sufficiently fine grid on S^2, or use partitions of unity (☺). (cf ES1 *, Guillemin & Pollack) □

<u>Pf of Fenchel</u> Recall (!) from ES2 that the <u>tube T</u> around α is given by $\varphi(s,v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v)$ where $s \in [0,L]$, $v \in [0,2\pi]$ and $r > 0$ is fixed, sufficiently small. (normal) (binormal)

This is where we use $k \neq 0$.



a bit of a bitch to prove look at diag in $[0,L]\times D$ → WLOG r small enough s.t. T doesn't have self-intersections ($[0,L]$ compact)

- Check: $|\varphi_s \times \varphi_v|^2 = EG - F^2 = r^2(1 - rk\cos v)^2$.

- WLOG also assume $r \cdot \max_{s\in[0,L]} |k(s)| < 1$

(This implies φ is regular, & $N = \overset{\forall v}{-}(n\cos v + b \sin v)$.

$\varphi_s \times \varphi_v = r(1 - rk\cos v)N$

Using Frenet formulae: $\quad\left[N_s = DN_p(\varphi_s),\ N_v = DN_p(\varphi_v) \right]$

$$\begin{aligned} N_s \times N_v &= k\cos v(n\cos v + b\sin v) \\ &= -k\cos v\, N \\ &= -\frac{k\cos v}{r(1-rk\cos v)}\, \varphi_s \times \varphi_v \end{aligned}$$

$$\therefore \boxed{K = -\frac{k\cos v}{r(1-rk\cos v)}}$$

Thus the points of T where $K = 0$ correspond to the points where $\underline{b}$ and $-\underline{b}$ intersect T.

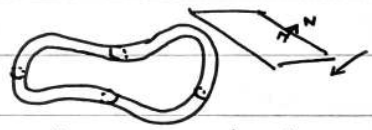
Now let $R := \{ p \in T : K(p) \geqslant 0 \}$,
$R^+ := \{ p \in T : K(p) > 0 \}$.

Then
$$\int_R K\, dA = \int_0^L \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} K \sqrt{EG-F^2}\, d\overset{s}{\cancel{k}}\, dv$$
$$= -\int_0^L \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k \cos v \, ds\, dv$$
$$= \left(-\int_0^L k(s)\, ds \right) \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos v \, dv \right)$$
$$= 2 \int_0^L k(s)\, ds .$$

i.e. twice the total curvature.

By argument on ES2, each normal direction is achieved at a point of T at which the Gauss curvature is $\geqslant 0$.

⌈ using T cpct, sliding plane $\perp N$



Thus the Gauss map $N : R \to S^2$ is surjective.

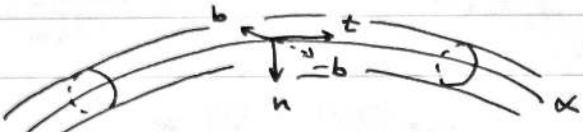
Using the formula from earlier,
$$2 \int_0^L k(s)\, ds = \int_R K \, dA = \int_{S^2} \# (N|_R)^{-1}(y)\, dy \underset{\uparrow \text{ surj}}{\geqslant} 4\pi .$$

<u>RTP</u>: If $\int_R K\, dA = 4\pi$,

then α is a plane curve.

<u>Claim 1</u> Equality $\Rightarrow$ for every point $p \in R^+$, T lies on one side of the tangent plane at p (?!) not just locally

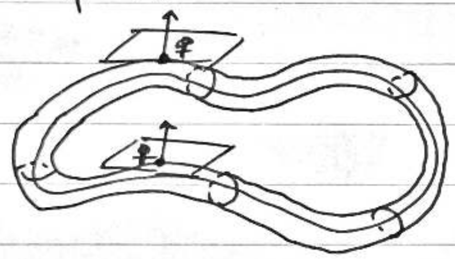
<u>Pf of Fenchel</u> (cont.)



R.T.P If $\int_R K\,dA = 4\pi$, then α is a plane curve

<u>Claim 1</u> Equality $\Rightarrow$ for every point $p \in R_+$, T lies on one side of the tangent plane at p.

<u>Pf of claim 1</u> Suppose not. The usual "sliding planes" argument implies that you can find two different points $q \neq p$ with $N(p) = N(q)$ and $q \in R$.



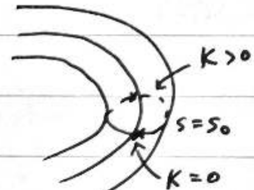
But we really want $q \in R_+$. Using the fact that N is a local diffeo^m around p, we can find $p', q' \in R_+$ (p' near p) s.t. $N(p') = N(q')$.

something
something →
regular value

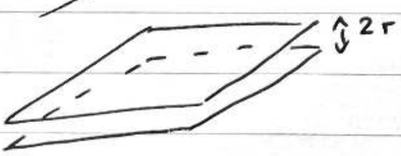
This implies (by the area formula) that we have strict inequality $\int_R K\,dA > 4\pi$ ※ □

<u>Crucial observation</u> Taking limits in claim 1, we deduce that if $K(p) \geq 0$ then T lies on one side of the tangent plane at p.

· Now fix a circle $s = s_0$ on T; have corresponding semi-circle on which $K \geq 0$. At the two endpoints of this semi-circle, the two tangent planes P_1, P_2 are parallel ($\perp$ to $+b$, resp $-b$) & at distance $2r$ apart.

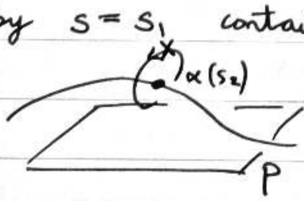


OTOH, T must lie between P_1 & P_2



<u>Claim 2</u> α lies on the parallel plane between P_1 & P_2 i.e. $\frac{1}{2}(P_1 + P_2) =: P$.

<u>Pf of claim 2</u> Suppose not. Consider $\alpha(s_1)$ furthest from P. The tangent vector $t(s_1)$ is parallel to P, so the circle of radius r on T given by $s = s_1$ contains a perpendicular vector to P.



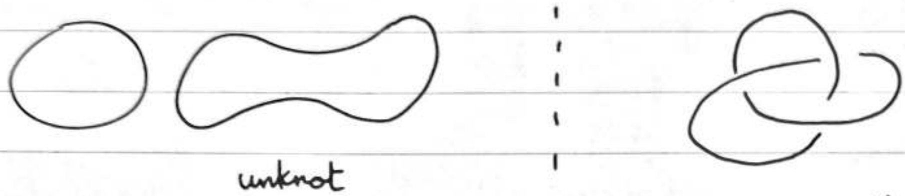
So some point of T (x on pic) lies outside the region between P_1, P_2. ※ □ □

4.5 The Fáry-Milnor theorem

This is about knotted curves $\alpha : [0, L] \to \mathbb{R}^3$.

Def A curve α is unknotted if $\alpha([0,L])$ is the boundary of an embeded disc (i.e. homeo^c to disc $D \subset \mathbb{R}^2$)

E.g.



unknot

is knotted ("trefoil knot")

— but this isn't easy to prove !

Thm (Fáry-Milnor) Let $\alpha : [0,L] \to \mathbb{R}^3$ be a regular knotted simple closed curve with nowhere vanishing curvature. Then its total curvature is $\geq 4\pi$.

Rmk Milnor showed $> 4\pi$ and that 4π was the best lower bound (!)

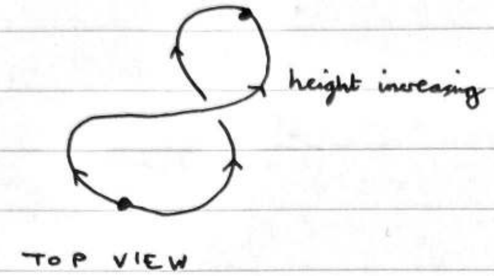
Pf As for Fenchel, consider the tube T. Use same notation.

$$\int_T |K|\, dA = 4 \int_0^L |k(s)|\, ds$$

$$\parallel$$

$$\int_{S^2} \# N^{-1}(y)\, dy \qquad \text{by area formula}$$

Suppose now that the total curvature of α is $< 4\pi$.
Then $\int_T |K|\, dA < 16\pi$, so some point $v \in S^2$ (wlog regular) is the image under N of ≤ 3 points of T.
This means $\alpha'(s) \perp v$ for at most 3 values of s.
Wlog $v = (0,0,1)$. Since $\langle v, \alpha'(s)\rangle = \frac{d}{ds}\langle v, \alpha(s)\rangle$, the function $s \mapsto \underbrace{\langle v, \alpha(s)\rangle}_{\text{"height of } \alpha\text{"}}$ has derivative 0 for at most 3 values of s.
Since the number of local max/min of the height function is even, it must be 2: one max and one min.



So α consists of two arcs, joining the highest and lowest points on the curve, each with monotonically increasing height. Each plane $\{z = \text{const}\}$ between max and min cuts α at 2 points.

Joining each pair by the line segment between them, we get an embedded topological disc with boundary α. $\square$

~ FIN ~