SI Units · metres, sec, kg

adapted to humans → numbers $O(1)$

Natural units : takes into account consts of Nature

1) unifies physical dimensions (treat space & time on = footing)

2) dimensional analysis → determine which effects are important

3) indicates possible breakdown of this

SPEED OF LIGHT

$c = 299792458\ m/s \approx 3\times10^8\ m/s = \text{const.} = 1$

$1s = 3\times10^8\ m$ lightyear $1\ yr = 9.4607\times10^{15}\ m$

$v \ll c = 1 \Rightarrow$ Galilean transf, Newton kin accurate

$v \lesssim 1 \Rightarrow$ need SR

GRAVITATIONAL CONSTANT

$G = 6.67408\times10^{-11}\ m^3 kg^{-1} s^{-2} = \text{const.}$

$c=1, G=1 \Rightarrow 1m = 1.3466\times10^{27}\ kg$

$1s = 4.0370\times10^{35}\ kg$

$M_\odot = 1.477\ \cancel{kg}\ km = 4.927\ \mu s \rightarrow \sim$ Schwarzchild radius of sun

$\frac{M}{R} \ll \frac{c^2}{G} \Rightarrow$ Newtonian gravity accurate

$\frac{M}{R} \approx 1 \Rightarrow$ NG breaks down! Need GR

If velocities are themselves determined by gravity,

$v \approx 1 \leftrightarrow M/R \approx 1$

Spherical orbit around mass M: $\frac{v^2}{c^2} = \frac{G}{c^2}\frac{M}{R} \Rightarrow v^2 = M/R$

Escape velocity $\frac{v_e^2}{c^2} = \frac{2G}{c^2}\frac{M}{R}$

Planck's constant

$$\hbar \equiv \frac{h}{2\pi} = 1.0545718 \times 10^{-34}\ \mathrm{kg\,m^2 s^{-1}}$$

Set $c = \hbar = 1 \Rightarrow 1\ \mathrm{kg} = 8.5223 \times 10^{50}\ \mathrm{Hz}$
(QFT)
or $1\mathrm{m} = 1/3.51767282 \times 10^{-48}\ \mathrm{kg}$

Compton wavelength $\lambda\!\!\!^- = \frac{\hbar}{mc} = \frac{1}{m}$

Compare w/ size or available volume

Sun $\lambda\!\!\!^-_\odot = \frac{\hbar}{M_\odot c} = 0.177 \times 10^{-72}\ \mathrm{m} \Rightarrow \frac{\lambda\!\!\!^-_\odot}{R_\odot} \ll 1$

Proton $\lambda\!\!\!^-_\odot = \frac{\hbar}{M_p c} = 0.210268\ \mathrm{fm} \sim$ radius of atomic nuclei
proton!
$\sim 1 - 10\ \mathrm{fm}$

$\frac{\lambda\!\!\!^-}{R} \ll 1 \rightarrow$ classical physics is accurate

$\frac{\lambda\!\!\!^-}{R} \approx 1 \Rightarrow$ need Quantum physics

Consider a system with

$$\frac{G}{c^2}\frac{M}{R} = 1 \quad (GR) \quad \text{and} \quad \frac{\hbar}{McR} = 1 \quad (QM)$$

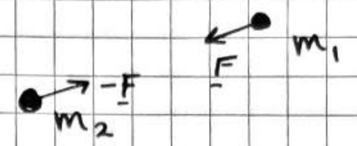
$$M_{Pl} = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8}\ \mathrm{kg} = 1.22 \times 10^{19}\ \mathrm{GeV}$$

$$L_{Pl} = \frac{G}{c^2} M_{PL} = 1.61 \times 10^{-35}\ \mathrm{m}$$

$$T_{Pl} = \frac{1}{c} L_{Pl} = 5.37 \times 10^{-44}\ \mathrm{s}$$

Theory of Quantum Gravity $\longrightarrow$ speculative, unknown

Newtonian 2 body problem



$$\underline{F}_{1 \text{ on } 2} = G m_{1a} m_{2p} \frac{\underline{r}_1 - \underline{r}_2}{|\underline{r}_1 - \underline{r}_2|^3}$$

$$= m_{2i} \ddot{\underline{r}}_2$$

m_a = active mass : source of gravity

m_p = passive mass : sensitivity of GRAVITY

m_i = inertial mass : resistance to change of motion

"Action = Reaction" $\Rightarrow \vec{F}_{2 \text{ on } 1} = -\vec{F}_{1 \text{ on } 2}$

$\therefore m_{1a} m_{2p} = m_{2a} m_{1p} \quad \forall$ bodies m_1, m_2

$\therefore \frac{m_{1p}}{m_{1a}} = \frac{m_{2p}}{m_{2a}}$ active/passive ratio same for all bodies

WLOG $m_a = m_p$ [electric charge $q_a = q_p$]

What about inertial mass ?

~1590 : Galileo rolls balls of different mass down slope
↪ needs same time

1922 : Eötvös : torque from Sun's gravity on torsion balance is $< 5 \times 10^{-9}$

all compatible w/ $m_a = m_p = m_i$

Equivalence Principles

Weak equivalence principle (WEP). Freely falling bodies with negligible gravitational self-interaction follow the same paths if they have the same initial velocity & position.

Def^n "local inertial frame" = coordinate frame (t, x, y, z) defined by a freely falling observer similar to Minkowski spacetime
↑ inertial frames

"local" means ≪ length scale of variations of $\underline{g}$

Einstein Equiv Principle (EEP)

In a local inertial frame, the results of all <u>non</u>-gravitational experiments are indistinguishable from the same experiment performed in an inertial frame in Mink

Strong Equivalence Principle (SEP)

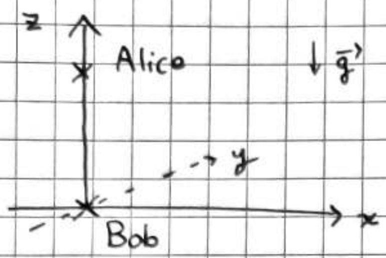
The grav motion of a small test body (that may have gravitational self-interactions) deps only on its initial v & x

- SEP $\Rightarrow$ WEP, WEP $\not\Rightarrow$ SEP

Ex of SEP-violating but WEP-satisfying theory models where gravity has an extra <u>scalar field component</u> in addition to spacetime metric $g_{\mu\nu}$

- SEP $\Rightarrow$ G = const. everywhere
- GR satisfies all 3 EP's

GRAVITATIONAL REDSHIFT

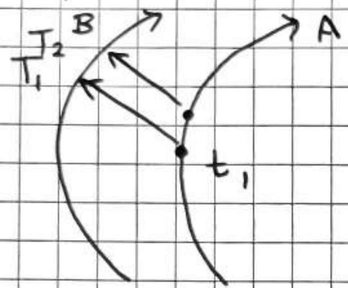


$\vec{g} = (0, 0, -g)$

Alice at $z = h$ } sends light to
Bob at $z = 0$

EEP $\Rightarrow$ equivalent to frame accelerated w/ $(0, 0, +g)$ in Mink

Assumption $v_{A,B} \ll c \Rightarrow$ ignore $\frac{v^2}{c^2}$, higher order SR terms



$$z_A = h + \tfrac{1}{2}gt^2, \quad z_B = \tfrac{1}{2}gt^2$$

Alice emits 1st signal at t_1

$$\Rightarrow z_1(t) = z_A(t_1) - c(t - t_1)$$
$$= h + \tfrac{1}{2}gt_1^2 - c(t - t_1)$$

$$h + \tfrac{1}{2}gt_1^2 - c(T_1 - t_1) = \tfrac{1}{2}gT_1^2 \quad (**)$$

Alice emits 2nd signal at $t_2 = t_1 + \Delta\tau_A$

reaches Bob at $T_2 = T_1 + \Delta\tau_B$ ← approx $v \ll c$

$$\Rightarrow h + \tfrac{1}{2}g(t_1 + \Delta_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \tfrac{1}{2}g(T_1 + \Delta\tau_B)^2$$

Subtract $(**)$ to get

$$c(\Delta\tau_A - \Delta\tau_B) + \tfrac{1}{2}g\Delta\tau_A(2t_1 + \Delta\tau_A) = \tfrac{1}{2}g\Delta\tau_B(2T_1 + \Delta\tau_B)$$

Assumption $\Delta\tau_A \ll t_1$, $\Delta\tau_B \ll T_1$ (more subtle)

$$\Rightarrow c(\Delta\tau_A - \tau_{B\Delta}) + g\Delta_{\tau A}t_1 = g\Delta_{\tau B}T_1$$

$$\Rightarrow \Delta\tau_B(gT_1 + c) = \Delta\tau_A(gt_1 + c)$$

$$\Delta\tau_B = \Delta\tau_A \cdot \left(1 + \frac{gt_1}{c}\right)\left(1 + \frac{gT_1}{c}\right)^{-1}$$

$$\simeq \Delta\tau_A\left(1 - g\frac{(T_1 - t_1)}{c}\right) \quad \frac{gt}{c} \ll 1$$

$$(**) \Rightarrow \frac{h}{c} - (T_1 - t_1) = \frac{1}{2}\frac{g}{c}(T_1 + t_1)(T_1 - t_1) \approx 0$$

$$\therefore \Delta\tau_B \simeq \Delta\tau_A\left(1 - \frac{gh}{c^2}\right) < \Delta\tau_A$$

$\boxed{\text{BLUE SHIFT}}$

Confirmed in Pound-Rebka expt (1960)

[light falling in a tower]

Invariant interval in SR $c^2\Delta\tau^2 = c^2(\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2)$

For weak, static gravitational field, $\phi/c^2 \ll 1$

$$c^2 d\tau^2 = \left[1 + \frac{2\phi(x,y,z)}{c^2}\right] c^2 dt^2 - \left[1 - \frac{2\phi(x,y,z)}{c^2}\right](dx^2 + dy^2 + dz^2)$$

Alice $\vec{x}_A$, Bob $\vec{x}_B$ at fixed positions

Alice emits signals at t_A, $t_A + \Delta t$

Bob receives at t_B, $t_B + \Delta t$

$$\Delta\tau_A^2 = \left(1 + \frac{2\phi(\underline{x}_A)}{c^2}\right)\Delta t^2$$

$$\Delta\tau_B^2 = \left(1 + \frac{2\phi(\underline{x}_B)}{c^2}\right)\Delta t^2$$

$$\therefore \Delta\tau_B \approx \left(1 + \frac{\phi(\underline{x}_B)}{c^2}\right)\Delta t, \quad \Delta\tau_A \approx \left(1 + \frac{\phi(\underline{x}_A)}{c^2}\right)\Delta t$$

$$\therefore \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\left(1 + \frac{\phi_A}{c^2}\right)^{-1}\Delta\tau_A$$

$$\approx \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A$$

Newtonian gravity for matter fields

[INDEX NOTATION]

- write vectors, matrices as components $x_i = (x_1, x_2, x_3) \equiv (x, y, z)$
- repeated indices get summed over
- no index can apper > twice

  # Eisentein Summation Convention
- rename 'dummy' indices
- in an equation or a sum, 'free' indices must match on all sides, terms
- denote partial derivatives by

  $\partial_i \equiv \dfrac{\partial}{\partial x_i}$ or a comma $v_{k,i} \equiv \dfrac{\partial v_k}{\partial x_i}$

Ex Motion of a pt particle in gravity

$$m\ddot{\vec{x}} = m\vec{g}(\vec{x},t) \quad \text{vs} \quad m\ddot{x}_i = mg_i(x_k,t)$$

In a non-inertial coordinate system

$$\tilde{x}_i = x_i - b_i(t)$$

$$\ddot{\tilde{x}}_i = \tilde{g}_i(\tilde{x}_k,t) = g_i(\tilde{x}_k,t) - \ddot{b}_i(t)$$

1) If g_i is uniform (x_k indep) $\Rightarrow \exists\, b_i$ s.t. $\tilde{g}_i = 0$

2) If g_i not uniform, can only get $\tilde{g}_i = 0$ locally

« freely falling frame »

Index version of Newton

Tidal forces on two particles at $x_i, x_i + \delta x_i$

$$\frac{d^2}{dt^2} x_i = g_i(x_k,t) \;, \quad \frac{d^2}{dt^2}(x_i + \delta x_i) = g_i(x_k + \delta x_k, t)$$

$$\therefore \frac{d^2}{dt^2}\delta x_i = \delta x_k\, \partial_k g_i + O(\delta x_k^2)$$

$$\Rightarrow \frac{d^2}{dt^2}\delta x_i + E_{ij}\delta x_j = 0$$

$E_{ij} = -\partial_j g_i$

$\Rightarrow \sim$

$\vec{g}$ is curl free $\Rightarrow \vec{g} = -\vec{\nabla}\phi$

so $E_{ij} = \partial_i\partial_j\phi = E_{ji}$

Poisson equation

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$$

mass density

$$\Rightarrow \nabla^2\phi = \partial_i\partial_i\phi = 4\pi G\rho$$

$$\partial_k E_{ij} = -\partial_k\partial_j g_i = \partial_j E_{ik}$$

$$\therefore E_{i[j,k]} = \frac{1}{2}(E_{ij,k} - E_{ik,j}) = 0$$

GR motivated from incompatibality of SR w/ Newtonian spacetime

Extend index notation

- Distinguish "upstairs" and "downstairs" indices $v^i \neq v_i$
- Summation only over one up & one down index [sic.]

$$v^j u_j = \sum_{j=1}^{3} v^j u_j$$

- Latin indices i, j range from 1, 2, 3 (space)

  Greek indices $\alpha, \beta, \ldots$ range from $0 \ldots 3$ (spacetime)

Metric of Euclidean 3-space

Pythagoras as matrix eq $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

$$= \underbrace{\delta_{ij}}_{\mathrm{diag}(1,1,1)} \Delta x^i \Delta x^j$$

in Cartesian coords

Polar coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$= g_{ij} \, d\tilde{x}^i d\tilde{x}^j \qquad g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

Lorentz Transformations ($c=1$)

"Proper distance" between spacetime events (t, x, y, z)
& $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

In SR, no inertial frame is preferred to another

$\Rightarrow$ same proper distance

$\Delta s = 0$ light ray

$$\Delta s^2 = \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta \quad , \quad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

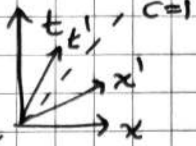
$$= \eta_{\tilde{\alpha}\tilde{\beta}} \Delta \tilde{x}^{\tilde{\alpha}} \Delta \tilde{x}^{\tilde{\beta}} \leftarrow \text{if } \tilde{x} \text{ related to } x \text{ by a Lorentz boost}$$

$$\eta^{\alpha\beta} = \eta^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

inverse matrix

$$\tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_{\mu} x^{\mu} + x_0^{\mu}$$

const.



wlog $x_0^{\mu} = 0$

want $\eta_{\tilde{\alpha}\tilde{\beta}} \Delta \tilde{x}^{\tilde{\alpha}} \Delta \tilde{x}^{\tilde{\beta}} = \eta_{\tilde{\alpha}\tilde{\beta}} \Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\tilde{\beta}}{}_{\nu} \Delta x^{\mu} \Delta x^{\nu} \stackrel{?}{=} \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}$

$$\eta_{\mu\nu} = \Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\tilde{\beta}}{}_{\nu} \eta_{\tilde{\alpha}\tilde{\beta}}$$

e.g. $\Lambda^{\tilde{\alpha}}{}_{\mu} = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

satisfied by Lorentz transformations

you can show $\Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\mu}{}_{\tilde{\beta}} = \delta^{\tilde{\alpha}}_{\tilde{\beta}}$

$\Lambda^{\mu}{}_{\tilde{\alpha}} \Lambda^{\tilde{\alpha}}{}_{\nu} = \delta^{\mu}_{\nu}$

$\gamma = \frac{1}{\sqrt{1-v^2}}$

inverse $\Lambda^{\mu}{}_{\tilde{\alpha}} = \begin{pmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

WORLDLINES & 4-VELOCITY

Def The interval between 2 spacetime events

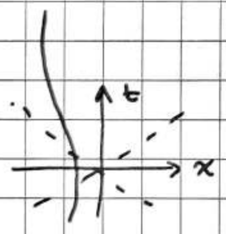
x^{α} & $x^{\alpha} + \Delta x^{\alpha}$ is called timelike if $\eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} < 0$

null if $\eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} = 0$

spacelike if $\eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} > 0$

Def Proper time $\Delta \tau^2 = -\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

Clock Postulate A clock moving on a (timelike) worldline $x^{\alpha}(\lambda)$, $\lambda \in \mathbb{R}$



measures the proper time

$$\tau \equiv \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}}$$

$$= \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\nu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\mu+1}}{d\lambda}}\, d\lambda$$

comments:
- τ is invariant under reparametrizing $\lambda \mapsto f(\lambda)$ (monotonic)
- often convenient to choose $\lambda = \tau$
- $d\tau = \sqrt{-\eta_{\mu\nu} dx^{\mu} dx^{\nu}} \Rightarrow \eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \equiv \eta_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = -1$

Def The 4-velocity along a time-like curve is

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau} \Rightarrow \eta_{\mu\nu} u^{\mu} u^{\nu} = -1$$

Geodesics Consider the action

$$S[x^\alpha(\lambda)] = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}\, d\lambda = \int \mathcal{L}\, d\lambda$$

geodesic def'd as extremising this action

$$\frac{\delta S}{\delta x^\alpha(\lambda)} = 0$$

keep endpoints fixed or sth

E-L equations $\displaystyle \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu}$

$$\Rightarrow \cdots \Rightarrow \frac{d^2x^\alpha}{d\tau^2} = 0$$

"not accelerating"

Can also derive same eq^n for null & spacelike geodesics.

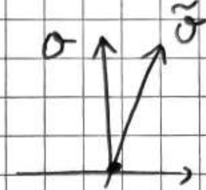
Postulate free massive (massless) particles in SR move on geodesic timelike (null) curves

Time dilation

Let $\mathcal{O}, \tilde{\mathcal{O}}$ be 2 observers w/ coordinates x^μ, $\tilde{x}^{\tilde{\alpha}}$ related by a Lorentz boost.

Let $\tilde{\mathcal{O}}$ move with vel. v in the x-direction relative to $\mathcal{O}$.

Recall $U^\alpha = \dfrac{dx^\alpha}{d\tau}$.



Clock at rest $\tilde{U}^{\tilde{\alpha}} = \left(\dfrac{d\tilde{t}}{d\tau}, 0, 0, 0\right) = (1,0,0,0)$

View from $\mathcal{O}$: $U^\mu = \left(\dfrac{dt}{d\tau}, \dfrac{dx^i}{d\tau}\right)$

$$= \Lambda^\mu{}_{\tilde{\alpha}} \tilde{U}^{\tilde{\alpha}} = \left(\gamma \frac{d\tilde{t}}{d\tau}, \gamma v^i \frac{d\tilde{t}}{d\tau}\right)$$

U^t component: $\dfrac{dt \rightarrow \text{coord time of } \mathcal{O}}{d\tau \rightarrow \text{proper time of } \tilde{\mathcal{O}}} = \gamma$ by above

$\therefore dt = \dfrac{d\tau}{\sqrt{1-v^2}}$, $\mathcal{O}$ "sees" $\tilde{\mathcal{O}}$ aging more slowly

Lorentz contraction

Define length of a rod in $\mathcal{O}$'s frame $\equiv$ proper distance Δs between 2 events A, B where x^i_A, x^i_B give position of two ends of the rod at some time t_0

$$\Delta s = \sqrt{\eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta} = \sqrt{\delta_{ij} \Delta x^i \Delta x^j} \quad \text{if no } \Delta t$$

Let the rod be at rest wrt $\mathcal{O}$ frame.

Worldlines of the two ends of the rod

$$x^\mu = (t_{tail}, x^i_0), \quad y^\mu = (t_{head}, x^i_0 + \underbrace{\Delta x^i}_{\text{"proper" Length}}) \quad \text{in } \mathcal{O}$$

In $\tilde{\mathcal{O}}$, have

$$(\tilde{t}_{tail}, \tilde{x}^i) = \tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu x^\mu$$

$$(\tilde{t}_{head}, \tilde{y}^i) = \tilde{y}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu y^\mu = \Lambda^{\tilde{\alpha}}{}_\mu (x^\mu + \Delta x^\mu)$$

We pick $\tilde{A}, \tilde{B}$ with $\tilde{t}_{tail} = \tilde{t}_{head}$.

$$\therefore t_{tail} = t_{head} + \frac{\Lambda^0{}_i \Delta x^i}{\Lambda^0{}_0} = t_{head} - v_i \Delta x^i$$

Proper distance

$$\Delta s^2_{\tilde{A}\tilde{B}} \text{ via } x^{\mu}_{\tilde{A}} = (t_{head} - v_i \Delta x^i, x^i_0)$$
$$x^{\mu}_{\tilde{B}} = (t_{head}, x^i_0 + \Delta x^i)$$

$$\therefore \tilde{l}^2 = \Delta s^2_{\tilde{A}\tilde{B}} = -(v_i \Delta x^i)^2 + \delta_{ij} \Delta x^i \Delta x^j$$

Orient the rod along x-axis, so

$$\tilde{l} = \sqrt{1 - v_x^2}\, \Delta x \leq l$$

Comments

- The sign of v doesn't matter
- Velocity $\perp$ to the rod causes no contraction

4-momentum & Doppler shift

Def: 4-momentum of particle with rest mass $m > 0$, and 4 velocity U^α : $p^\alpha = m U^\alpha$

Recall $\eta_{\mu\nu} U^\mu U^\nu = -1$

$$\therefore \eta_{\mu\nu} U^\mu m U^\nu m = \eta_{\mu\nu} P^\mu P^\nu = -m^2$$

Let $\tilde{\mathcal{O}}$ move with vel v in x-dir^n rel to $\mathcal{O}$.

Consider a particle at rest in $\tilde{\mathcal{O}}$.

$$\tilde{p}^{\tilde{\alpha}} = (m, 0, 0, 0)$$

$$p^\mu = (\underbrace{\gamma m}_{\text{relativistic energy}}, \underbrace{\gamma m v}_{\text{relativistic 3-momentum}}, 0, 0) \quad \text{via Lorentz}$$

$$p_\mu p^\mu = -m^2 = -E^2 + p^2 \quad \text{so} \quad E^2 = p^2 + m^2 \quad \left(\text{restore } c \text{ via units}\right)$$

Comment Null curves do not have a 4-velocity U^μ but they do have 4-momentum.

Recall that for massless particles $E = \hbar\omega$, $p = \hbar k$

$$p^\alpha = \hbar\omega(1, 1, 0, 0) \quad \text{for e.g. a photon moving in } x\text{-dir}^n$$

Let $\tilde{\mathcal{O}}$ move with v in x-dir^n rel to $\mathcal{O}$

$$\therefore \tilde{p}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_{\mu} P^\mu = (\gamma E - \gamma v E, -\gamma v E + \gamma E, 0, 0)$$
$$= (\tilde{E}, \tilde{E})$$

So get redshift

$$\frac{\tilde{\nu}}{\nu} = \frac{\tilde{E}}{E} = \gamma(1-v) = \sqrt{\frac{1-v}{1+v}}$$

redshift if $\tilde{\mathcal{I}}$ moves in same dir^n as photon

blueshift if $v < 0$

Transverse Doppler shift if $v \perp$ dir^n of propagation of photon

V^μ vector

W_μ covector b/c acts on vectors $V^\mu \mapsto W_\mu V^\mu$

Isomorphism with dual via

$V^\mu \mapsto V_\mu = \eta_{\mu\nu} V^\nu$

#visualthinker

Differential geometry

extend to curved spacetime

motivation : GR generalises SR like

Riemannian geometry generalises Euclidean geometry

Conventions : an upstairs index in a denominator counts as a downstairs index

e.g. $\partial_i = \frac{\partial}{\partial x^i}$

contravariant : upstairs , covariant : downstairs

transform differently under a coordinate transformation

Manifolds & Tensors

Strategy : start w/ manifold M (rep^n space or spacetime)

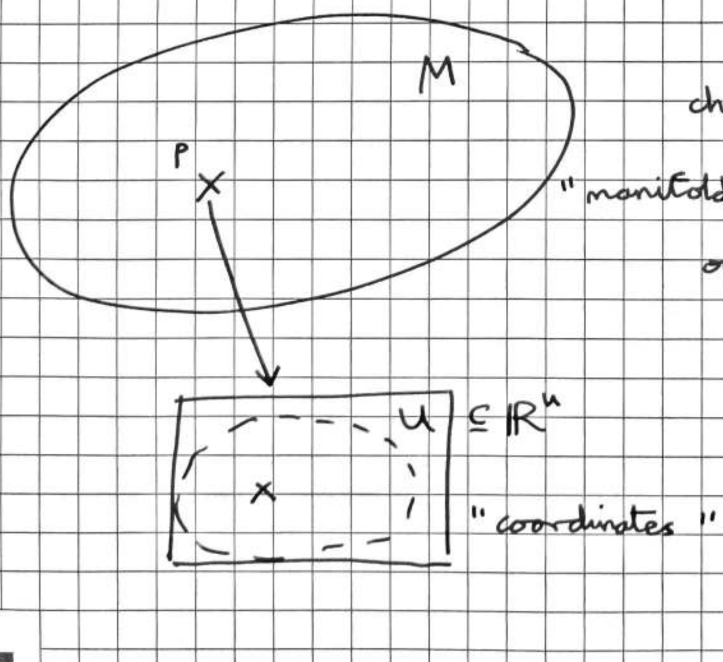
Establish structure step-by-step

Def n-dim manifold M = topological space of points that locally resembles Euclidean space $\mathbb{R}^n$ at every point

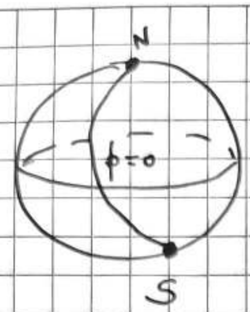
An example of a manifold is M for which there exists a 1-to-1 and onto map $\phi : M \to \mathcal{U} \subseteq \mathbb{R}^n$, for $\mathcal{U}$ open

$$p \mapsto x^\alpha \quad , \quad \alpha = 0, 1, \ldots, n-1$$

More generally, it suffices to chop M up into open subsets "manifold" which each maps to an open subset of $\mathbb{R}^n$.



S^2



usual polar coordinates (θ, φ)
$0<\varphi<2\pi,\ 0<\theta<\pi$
parametrise open subset

Coordinates are like house no.s on a street.

Look for objects that are independent of coordinates.

Curves, vectors etc live on M, not $\mathbb{R}^n$

Since ϕ is 1-to-one, this distinction is often blurred.

<u>Def</u> Function on M $\quad f: M \to \mathbb{R}$

To proceed further we want a notion of differentiation $\frac{\partial}{\partial x_i} f$ and a notion of smooth (C^∞) functions.

In this course we restrict attention to manifolds & coordinate systems for which these notions are defined in the usual way on coordinates $\mathbb{R}^n$.

f is smooth $\leftrightarrow$ $f(x^\alpha)$ smooth f^n on $\mathbb{R}^n$

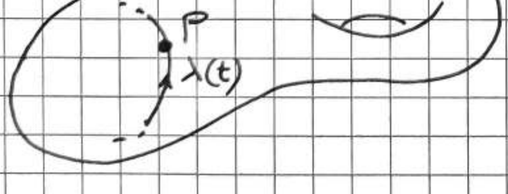
If f at a given point $P \in M$ is invariant under a change of coordinates, it is a scalar.

If you change coordinates $y^\beta(x^\alpha)$

$$f(y^\alpha) = f(y^\alpha(x^\alpha)) \qquad \text{WTF} \because$$

<u>Vectors</u> <u>Def</u> Let C^∞ be the space of all smooth f^ns on M.
Let λ be a smooth curve parametrised by $t \in \mathbb{R}$ s.t.

$$P = \lambda(0)$$



The <u>tangent vector</u> to the curve λ at P is the map $V: C^\infty \longrightarrow \mathbb{R}$

$$f \longmapsto \left.\frac{d f(\lambda(t))}{dt}\right|_{t=0}$$

Call $T_P(M)$ the space of all vectors at P.

i.e. look at arbitrary curves through P

A vector is a differential operator, in particular obeys

(i) $V(f+g) = V(f) + V(g)$, $V(\alpha f) = \alpha V(f)$ — linearity

(ii) $V(fg) = V(f)\,g(P) + f(P)\,V(g)$ — Leibniz

Consider coordinate system x^α near P.

Then
$$V(f) = \frac{d}{dt} f(x^\alpha(\lambda(t)))$$
$$= \underbrace{\left.\frac{dx^\mu}{dt}\right|_{t=0}}_{\text{components}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\substack{\text{basis}\\ \text{vectors}}} f(x^\alpha)$$

This shows $T_P(M)$ is an n-dimensional real vector space with basis $e_\mu := \partial_\mu = \frac{\partial}{\partial x^\mu}$

Components $V^\mu := \frac{dx^\mu}{dt} \Rightarrow V = V^\mu e_\mu$

V itself doesn't care about a coordinate change

Suppose change $x^\mu \to \tilde{x}^\alpha$. Then

$$e_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\alpha} = \left(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right)\tilde{e}_\alpha$$

& likewise $\tilde{e}_\alpha = \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right) e_\mu$

What about the vector components?

$$V^\mu = \frac{dx^\mu}{dt}, \quad \tilde{V}^\alpha = \frac{d\tilde{x}^\alpha}{dt} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{dx^\mu}{dt} = \left(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right) V^\mu$$

& likewise $V^\mu = \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right)\tilde{V}^\alpha$

Note how in particular

$$V = V^\mu e_\mu = \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right)\tilde{V}^\alpha \left(\frac{\partial \tilde{x}^\beta}{\partial x^\mu}\right)\tilde{e}_\beta = \tilde{V}^\alpha \tilde{e}_\alpha$$

Covectors / 1-forms

Def a covector / 1-form is a linear map

$$\eta : T_p(M) \longrightarrow \mathbb{R}$$
$$V \longmapsto \eta(V)$$

$T_p^*(M) =:$ cotangent space at p

$=$ space of covectors at p

is an n-dimensional vector space

Let e_μ be basis of $T_p(M)$. Then the components of a covector η are $\eta_\mu \equiv \eta(e_\mu)$

Properties

Linearity : $\alpha, \beta \in \mathbb{R}$, $V, W \in T_p(M)$

$$\eta(\alpha V + \beta W) = \alpha \eta(V) + \beta \eta(W)$$

Components : $\eta(V) = \eta(V^\mu e_\mu)$

$= V^\mu \eta(e_\mu)$ by lin

$= V^\mu \eta_\mu$

Transformation rule : 'require' $\eta(V)$ to be a scalar

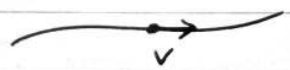
$$\Rightarrow \eta(V) = \eta_\mu V^\mu = \tilde{\eta}_\alpha \tilde{V}^\alpha = \tilde{\eta}_\alpha \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu$$

$$\therefore \tilde{\eta}_\alpha \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \eta_\mu \quad \text{or} \quad \tilde{\eta}_\beta = \frac{\partial x^\mu}{\partial \tilde{x}^\beta} \eta_\mu$$

Def Gradient df of a smooth f^n f

$$df : T_p(M) \longrightarrow \mathbb{R}$$
$$V \longmapsto V(f)$$

recall $V = \frac{d}{dt}$ along some γ



Basis: Let $f = x^\alpha$ for some α

$$dx^\alpha(e_\beta) = e_\beta(x^\alpha)$$
$$= \frac{\partial}{\partial x^\beta}(x^\alpha) = \delta^\alpha_\beta$$

$$\Rightarrow \eta_\alpha dx^\alpha(V) = \eta_\alpha dx^\alpha(V^\beta \partial_\beta) = \eta_\alpha V^\alpha = \eta(V)$$

i.e. dx^α is basis of $T_p^*(M)$ dual to e_α, $\boxed{\eta = \eta_\alpha dx^\alpha}$

Tensors

Def A tensor T at $p \in M$ of rank $\binom{r}{s}$ is a multilinear map $T: \underbrace{T_p^*(M) \times \cdots \times T_p^*(M)}_{r \text{ factors}} \times \underbrace{T_p(M) \times \cdots \times T_p(M)}_{s \text{ factors}} \longrightarrow \mathbb{R}$

"plug in r covectors, s vectors, get a real numbers"

Ex • covector η is a tensor of rank $\binom{0}{1}$ by definition

• vector V can be viewed as $V: T_p^*(M) \longrightarrow \mathbb{R}$

$$\eta \longmapsto \eta(V)$$

i.e. V is a $\binom{1}{0}$ tensor

components of V: $\eta(V) = \eta_\alpha dx^\alpha(V) = \eta_\alpha V^\alpha$

$$\therefore V^\alpha = dx^\alpha(V) = V(\cancel{d}x^\alpha)$$

for any tensor define components

$$T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} = T(dx^{\alpha_1}, \dots, dx^{\alpha_r}, e_{\beta_1}, \dots, e_{\beta_s})$$

• $\delta: T_p^*(M) \times T_p(M) \longrightarrow \mathbb{R}$

$$(\eta, V) \longmapsto \eta(V)$$

is a $\binom{1}{1}$ tensor, with components $\delta^\alpha_\beta = \delta(dx^\alpha, e_\beta) = dx^\alpha(e_\beta)$

"special tensor"

same components in all coordinate systems

Can show that tensors of rank $\binom{r}{s}$ form a vector space of dimension n^{r+s} and transform according to

$$\tilde{T}^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} = \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \tilde{x}^{\alpha_r}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_s}}{\partial \tilde{x}^{\beta_s}} T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$$

Tensor operations

(1) addition & scalar mult^n, e.g. if S, T are $\binom{1}{1}$ tensors, $c_1, c_2 \in \mathbb{R}$, $c_1 S + c_2 T : T_p^*(M) \times T_p(M) \longrightarrow \mathbb{R}$

defined via $(\eta, V) \longmapsto c_1 S(\eta, V) + c_2 T(\eta, V)$

(2) symmetrization, anti-symm, e.g. if T is $\binom{0}{2}$ tensor

symmetric part $S_{\alpha\beta} \equiv \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) \equiv T_{(\alpha\beta)}$

antisymm part $A_{\alpha\beta} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \equiv T_{[\alpha\beta]}$

Index subset $T^{(\alpha\beta)\gamma}{}_{\delta} \equiv \frac{1}{2}\left(T^{\alpha\beta\gamma}{}_{\delta} + T^{\beta\alpha\gamma}{}_{\delta}\right)$

Non-adjacent $T_{(\alpha|\beta\gamma|\delta)} \equiv \frac{1}{2}\left(T_{\alpha\beta\gamma\delta} + T_{\delta\beta\gamma\alpha}\right)$

Over $n>2$ indices

- sum over all permutations
- apply sign of permn for anti-sym
- factor of $\frac{1}{n!}$

e.g. $$T^{\alpha}{}_{[\beta\gamma\delta]} = \frac{1}{6}\begin{pmatrix} T^{\alpha}{}_{\beta\gamma\delta} + T^{\alpha}{}_{\gamma\delta\beta} + T^{\alpha}{}_{\delta\beta\gamma} \\ -T^{\alpha}{}_{\beta\delta\gamma} - T^{\alpha}{}_{\gamma\beta\delta} - T^{\alpha}{}_{\delta\gamma\beta} \end{pmatrix}$$

(3) contraction of $\binom{r}{s}$ tensor into $\binom{r-1}{s-1}$

summing over 1 upper & 1 lower index

e.g. if T is $\binom{3}{2}$ tensor, can contract to get

$$S : T_p^*(M) \times T_p^*(M) \times T_p(M) \to \mathbb{R}$$
$$(\omega, \eta, V) \mapsto T(dx^{\mu}, \omega, \eta; \partial_{\mu}, V)$$

sum over μ

this is basis independent, in that

$$T(d\tilde{x}^{\alpha}, \omega, \eta; \tilde{\partial}_{\alpha}, V) = T(dx^{\mu}, \omega, \eta; \partial_{\mu}, V)$$

$$\parallel$$

$$\underbrace{\left(\frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\mu}}\right)}_{\delta^{\alpha}_{\beta}} T(dx^{\alpha}, \omega, \eta; \partial_{\beta}, V)$$

in components $S^{\mu\nu}{}_{\rho} = T^{\alpha\mu\nu}{}_{\alpha\rho}$

4) Outer Product : Let S be a rank $\binom{p}{q}$ tensor,
T be a rank $\binom{r}{s}$ tensor.

Then $S \otimes T$ is a rank $\binom{p+r}{q+s}$ tensor defined via

$$(S\otimes T)(\eta_1,\dots,\eta_p,\eta_{p+1},\dots,\eta_{p+r};V_1,\dots,V_q,V_{q+1},\dots,V_{q+s})$$
$$= S(\eta_1,\dots,\eta_p;V_1,\dots,V_q)\,T(\eta_{p+1},\dots,\eta_{p+r};V_{q+1},\dots,V_{q+s})$$

(i) In terms of components,

$$(S\otimes T)^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_r}{}_{\mu_1\cdots\mu_q\nu_1\cdots\nu_s} = S^{\alpha_1\cdots\alpha_p}{}_{\mu_1\cdots\mu_q}\,T^{\beta_1\cdots\beta_r}{}_{\nu_1\cdots\nu_s}$$

(ii) Given a coord basis, a e.g. $\binom{2}{1}$ tensor can be written

$$T = T^{\mu\nu}{}_{\rho}(e_\mu \otimes e_\nu \otimes dx^\rho)$$

So far, have looked at tensors at P

A tensor field of rank $\binom{r}{s}$ is defined as the assignment to each point $P\in M$ of a tensor of rank $\binom{r}{s}$ at P, X_P

Ex Vector field $p \mapsto X_p$

For a f^n f, $X(f) : M \to \mathbb{R}$, $p \mapsto X_p(f)$

A tensor field is smooth iff its components in any coordinate basis are smooth functions on $\mathbb{R}^n$.

Integral Curves

Def An integral curve λ of a vector field through $P\in\mathcal{M}$ is the curve through P whose tangent vector at Q is V_Q

In coords, $\left.\frac{d}{dt}\right|_\lambda = V \Rightarrow \left.\frac{dx^\mu}{dt}\right|_\lambda = V^\mu$

$$\lambda(t_0) = P$$

They exist and are unique, at least locally.

Metric tensor (field) used to measure distances, volumes etc.

Def A metric at $P\in\mathcal{M}$ is a $\binom{0}{2}$ tensor s.t.

(i) symmetric $g(V,W) = g(W,V) \quad \forall\, V,W \in T_p(\mathcal{M})$

equivalently $g_{\alpha\beta} = g_{\beta\alpha}$

(ii) non-degenerate, i.e. if V is s.t. $\forall\, W \in T_p(M)$

$$g(V,W) = 0 \quad , \quad \text{then} \quad V = 0$$

in components, $\quad g = g_{\alpha\beta}\, dx^\alpha dx^\beta$, (omit $\otimes$)

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

A metric induces an isomorphism between the space of vectors and the space of 1-forms, $\quad V \longmapsto g(V,\cdot) = \underline{V}$

i.e. $\underline{V}(W) = g(V,W) = \underline{V}_\mu W^\mu = g_{\mu\nu} V^\mu W^\nu$

define $\quad V_\mu = g_{\mu\nu} V^\nu$

Non-degeneracy implies g is "invertible", i.e. we can define the inverse metric g^{-1}, a symm $\binom{2}{0}$ tensor with components $g^{\alpha\beta}$ s.t. $\quad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$

Ex line element on unit sphere

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} , \quad g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

g^{-1} maps 1-forms to vectors, $\quad \eta \longmapsto g^{-1}(\eta,\cdot)$

This is inverse to the isomorphism given by g,

$$g^{-1}(g(V)) = V , \quad g(g^{-1}(\eta)) = \eta$$

Signature g symmetric $\Rightarrow$ comp. of $g \in M$ form a symm matrix

$\therefore \exists$ basis where $g_{\mu\nu}$ is diagonal

g non-degenerate $\Rightarrow$ all diagonal elements are nonzero

$\therefore$ can rescale the basis s.t. the diagonal elements $= \pm 1$

such a basis is called an "orthonormal" basis, not unique

Sylvester's law of inertia: no. of $+1$, -1 is indep of o.n. basis

Def Signature $= \sum +1, -1$ over diagonal

Riemannian metric $(+, + \cdots +)$ sig $= +n$, no. of dim

Lorentzian metric $(-, + \cdots +)$ sig $n-2$

Others use $(+, -, \ldots, -)$ sig $2-n$

Note: Equivalence principle $\Rightarrow$ local frame of ref, SR holds

$\therefore \exists$ local coords s.t. $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$

at a point p In general at $q \neq p$, $g_{\mu\nu} \neq \eta_{\mu\nu}$

Def A Riemannian (Lorentzian) manifold is the pair (M, g), where M is a manifold & g is a Riemannian (Lorentzian) metric on it.

Spacetime = Lorentzian manifold $n=4$ to match our universe

Ex Minkowski metric on $\mathbb{R}^4$ with Cartesian coords x^0, x^1, x^2, x^3

$$\eta = -\underbrace{(dx^0)^2}_{(dx^0 \otimes dx^0)} + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

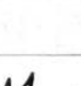
Def Let (M, g) be a Lor. M & $V \in T_p(M)$, $V \neq 0$

V is timelike if $g(V,V) < 0$

null if $g(V,V) = 0$

spacelike if $g(V,V) > 0$

at any local inertial frame locally have a lightcone structure just like SR



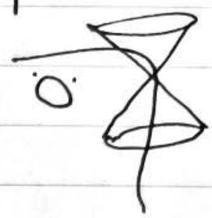
Def Norm of spacelike $V \in T_p(M)$

defined as $\|V\| = \sqrt{g(V,V)}$

Angle between spacelike V, W

$$\cos\theta = \frac{g(V,W)}{\|V\|\,\|W\|}$$

Geodesics **Def** Say a curve is timelike/null/spacelike at p if its tangent vector $\frac{d}{dt} = \dot{x}^\alpha \partial_\alpha$ is ___



Def Length of a spacelike curve

$$S = \int_{t_0}^{t_1} \sqrt{g(\dot{x}^\alpha \partial_\alpha, \dot{x}^\alpha \partial_\alpha)}\, dt$$

$$= \int_{t_0}^{t_1} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, dt$$

Rmk param^n indep

Similarly for a timelike curve,

$$T = \int_{t_0}^{t_1} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, dt$$

Def 4 velocity along timelike curve

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

Then $\quad g_{\mu\nu} u^\mu u^\nu = -1$

Action $\quad S = \int \mathcal{L}(q_k, \dot{q}_k, \lambda)\, d\lambda \quad , \quad k = 1, \dots, n$

is extremised if $q_k(\lambda)$ solve the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\right) = \frac{\partial \mathcal{L}}{\partial q^k} \quad , k = 1, \dots, n$$

Nother's Theorem: symmetry of $\mathcal{L} \longleftrightarrow$ conserved quantity

(i) $\mathcal{L}$ indep of q^k, i.e. $\frac{\partial \mathcal{L}}{\partial q^k} = 0 \Rightarrow p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k}$ conserved along integral curve

(ii) $\mathcal{L}$ indep of t, i.e. $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \dot{q}^k p_k - \mathcal{L}$ conserved along integral curve

Geodesics, var^n 1 $\quad$ timelike curves $A \to B$

$$S = \int_0^1 \mathcal{L}\, d\lambda, \quad \mathcal{L} = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

Indep of param^n, in that if $\kappa : [0,1] \to [a,b]$ monotone

$$S = \int_0^1 \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\lambda = \int_a^b \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, \frac{d\lambda}{d\kappa}\, d\kappa = \int_a^b \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\kappa} \frac{dx^\nu}{d\kappa}}\, d\kappa$$

E-L eq^{n}s $$\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} = \frac{1}{2\mathcal{L}}\left(-g_{\mu\nu}\delta^\mu_\alpha \dot{x}^\nu - g_{\mu\nu}\dot{x}^\mu \delta^\nu_\alpha\right)$$
$$= -\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} = \frac{1}{2\mathcal{L}}\left(-\dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\right)$$

$$\therefore \frac{d}{d\lambda}\left(-\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}\right) + \frac{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu,\alpha}}{2\mathcal{L}} = 0 \quad , \alpha = 0,1,2,3$$

Change param to $\tau(\lambda)$ proper time, $\frac{d\tau}{d\lambda} = \mathcal{L}$

$$\therefore -\mathcal{L}\frac{d}{d\tau}\left(g_{\mu\alpha}\frac{dx^\mu}{d\tau}\right) + \frac{\mathcal{L}}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu,\alpha} = 0$$

$$\therefore \frac{d^2x^\mu}{d\tau} g_{\mu\alpha} + g_{\mu\alpha,\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - \frac{1}{2}g_{\mu\nu,\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\boxed{\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$ "geodesic equations"

where $\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\rho}\left(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}\right)$ ← Christoffel symbols

is symmetric in μ,ν.

Spacelike geodesics $\tilde{\mathcal{L}} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$, $\frac{ds}{d\lambda} = \tilde{\mathcal{L}}$

Var ※1: $$S = \int_0^1 \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda$$

$$\Rightarrow \frac{d^2x^\beta}{d\tau^2} + \left\{ {\beta \atop \mu\nu} \right\} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (*)$$

Christoffel symbol $\left\{ {\beta \atop \mu\nu} \right\} = \frac{1}{2} g^{\beta\alpha}(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})$

Var ※2: $$S = \int_0^1 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\lambda$$

Advantage: no restriction to timelike geodesics

Disadvantage: not inv under reparametrisation

E-L give $\ddot{x}^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\} \dot{x}^\beta \dot{x}^\gamma = 0$ where dot denotes $\frac{d}{d\lambda}$ $\qquad (+)$

Let $\tau = \tau(\lambda)$, $\frac{d\tau}{d\lambda} > 0$

Then $\frac{d}{d\tau} = \frac{d\lambda}{d\tau}\frac{d}{d\lambda}$, $\frac{d^2}{d\tau^2} = \frac{d^2\lambda}{d\tau^2}\frac{d}{d\lambda} + \left(\frac{d\lambda}{d\tau}\right)^2 \frac{d^2}{d\lambda^2}$

$\therefore$ (*) gives $\left(\frac{d\lambda}{d\tau}\right)^2 \ddot{x}^\beta + \frac{d^2\lambda}{d\tau^2}\dot{x}^\beta + \left(\frac{d\lambda}{d\tau}\right)^2 \dot{x}^\mu \dot{x}^\nu \left\{ {\beta \atop \mu\nu} \right\} = 0$

$$\therefore \ddot{x}^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\} \dot{x}^\beta \dot{x}^\gamma = -\left(\frac{d\lambda}{d\tau}\right)^{-2} \frac{d^2\lambda}{d\tau^2}\dot{x}^\alpha \propto \dot{x}^\alpha$$

Agree only if $\frac{d^2\lambda}{d\tau^2} = 0$ i.e. $\lambda = C_1\tau + C_2$ is affine parameter

Makes sense even for lightrays

Def If a curve $C: I \to M$, $\lambda \mapsto x^\alpha(\lambda)$ satisfies (+) then it is geodesic and affinely parametrised.

If RHS is $f(\lambda)\dot{x}^\alpha$ then it is geodesic but not affinely param^d

Geodesic Postulate Particles with positive (zero) mass move along timelike (null) geodesics. At least test particles do.

Var ※ 2 gives an easy way to compute $\left\{ {\alpha \atop \beta\gamma} \right\}$

Ex $ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$, $\quad f = 1 - \frac{2M}{r}$

$\therefore\ \mathcal{L} = f\dot{t}^2 - f^{-1}\dot{r}^2 + -r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$

$\therefore\ \frac{d}{d\tau}(2f\dot{t}) = 0$ by E-L for t

$\therefore\ \ddot{t} + f^{-1}\frac{df}{dr}\dot{t}\dot{r} = 0$ so $\left\{ {t \atop tr} \right\} = \left\{ {t \atop rt} \right\} = \frac{df/dr}{2f}$

& other $\left\{ {t \atop \mu\nu} \right\} = 0$

Covariant derivative ∇ on Manifold $\mathcal{M}$

Def For scalar field f,

$$\nabla f : T_p(M) \longrightarrow \mathbb{R},\ V \mapsto \nabla_V(f) = V(f)$$

∇f is a $\binom{0}{1}$ tensor, $\nabla_\alpha f \equiv (\nabla f)_\alpha = \partial_\alpha f$

Def For vector field V,

$$\nabla V : T_p(M) \longrightarrow T_p(M),\ W \mapsto \nabla_W V$$

satisfying the following (α_1, β_1, f scalar fields, V vector)

bilinear $\quad \nabla_{\alpha_1 W_1 + \beta_1 W_2} V = \alpha_1 \nabla_{W_1} V + \beta_1 \nabla_{W_2} V$

$\quad \nabla_W(V_1 + V_2) = \nabla_W V_1 + \nabla_W V_2$

Leibniz $\quad \nabla_W(fV) = \nabla_W(f)V + f\nabla_W(V)$

Equivalently $\quad \nabla V : T_p^*(M) \times T_p(M) \longrightarrow \mathbb{R}$

$$(\eta, X) \longmapsto \eta(\nabla_X V)$$

so ∇V is a $\binom{1}{1}$ tensor

$$V^\alpha{}_{;\beta} \equiv \nabla_\beta V^\alpha \equiv (\nabla V)^\alpha{}_\beta$$

Def Let $\{e^\mu\}$ be a basis for $T_p(M)$

The connection coefficients are $\Gamma^\rho_{\mu\nu}$ defined via

$$\nabla_{e_\nu} e_\mu = \Gamma^\rho_{\mu\nu} e_\rho$$

For vectors V, W get $\quad (\nabla_V W)^\rho = V^\nu(\partial_\nu W^\rho) + V^\nu \Gamma^\rho_{\mu\nu} W^\mu$

$$\therefore W^\rho{}_{;\nu} \equiv \nabla_\nu W^\rho \equiv (\nabla W)^\rho{}_\nu = \partial_\nu W^\rho + \Gamma^\rho_{\mu\nu} W^\mu$$

<u>Recall</u> Let $\{e^\mu\}$ be basis of T_pM

Connection coeff $\Gamma_{\mu\nu}^{\ \ \rho}$ given by $\nabla_{e_\nu} e_\mu = \Gamma_{\mu\nu}^{\ \ \rho} e_\rho$

For $V = V^\mu e_\mu$, $W = W^\mu e_\mu$,

$$\begin{aligned}
\nabla_V W &= \nabla_{V^\mu e_\mu}(W^\nu e_\nu) \\
&= V^\mu \nabla_{e_\mu}(W^\nu e_\nu) \\
&= V^\mu \partial_\mu(W^\nu) e_\nu + V^\mu W^\nu \nabla_{e_\mu} e_\nu \\
&= V^\mu W^\nu{}_{,\mu} e_\nu + V^\mu W^\nu \Gamma_{\nu\mu}^{\ \ \rho} e_\rho \\
&= V^\mu (W^\rho{}_{,\mu} + \cancel{V^\mu} W^\nu \Gamma_{\nu\mu}^{\ \ \rho}) e_\rho
\end{aligned}$$

$\therefore (\nabla_V W)^\rho = V^\mu (W^\rho{}_{,\mu} + W^\nu \Gamma_{\nu\mu}^{\ \ \rho})$

$\therefore W^\rho{}_{;\nu} =: (\nabla_{e_\nu} W)^\rho =: \nabla_\nu W^\rho = \partial_\nu W^\rho + \Gamma_{\mu\nu}^{\ \ \rho} W^\mu$

Consider coordinate transformation,

$x^\mu \rightsquigarrow \tilde{x}^\alpha$, get

$$\tilde{\Gamma}_{\mu\nu}^{\ \ \sigma} = \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \Gamma_{\alpha\beta}^{\ \ \rho} + \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}$$

i.e. Γ not a tensor, neither is $W^\rho{}_{,\nu}$, but together $W^\rho{}_{;\nu}$ is

Further, the difference of 2 connections is a tensor

e.g. torsion tensor $T_{\mu\nu}^{\ \ \sigma} = \Gamma_{\mu\nu}^{\ \ \sigma} - \Gamma_{\nu\mu}^{\ \ \sigma}$

If $T_{\mu\nu}^{\ \ \lambda} = 0$ we say the connection is <u>torsion free</u>

More generally, a rank $\binom{r}{s}$ tensor T has ∇T a $\binom{r}{s+1}$ tensor

Can work out from Leibniz product rule.

e.g. a 1 form $\nabla_V(\eta(W)) = (\nabla_V \eta)(W) + \eta(\nabla_V W)$

$\quad \therefore (\nabla_V \eta) W = \nabla_V(\eta(W)) - \eta(\nabla_V W)$ determines $(\nabla_V \eta)$

$\nabla \eta$ is a $\binom{0}{2}$ tensor

$$\begin{aligned}
\nabla\eta(V,W) =: (\nabla_V \eta)(W) &= \nabla_V(\eta_\mu W^\mu) - \eta_\mu \nabla_V W^\mu \\
&= V^\rho \partial_\rho(\eta_\mu W^\mu) - \eta_\mu (V^\rho \partial_\rho W^\mu + V^\rho \Gamma_{\nu\rho}^{\ \ \mu} W^\nu) \\
&= V^\rho W^\mu \partial_\rho \eta_\mu - V^\rho W^\nu \Gamma_{\nu\rho}^{\ \ \mu} \eta_\mu \\
&= (\partial_\rho \eta_\mu - \Gamma_{\mu\rho}^{\ \ \nu} \eta_\nu) V^\rho W^\mu
\end{aligned}$$

So in components $(\nabla \eta)_{\rho\mu} =: \eta_{\rho;\mu} = \partial_\rho \eta_\mu - \Gamma_{\mu\rho}^{\ \ \nu} \eta_\nu$

More generally, the covariant derivative of $\binom{r}{s}$ tensor

$$\nabla_\rho T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \partial_\rho T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \Gamma^{\mu_1}_{\sigma\rho} T^{\sigma\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots + \Gamma^{\mu_r}_{\sigma\rho} T^{\mu_1\cdots\sigma}{}_{\nu_1\cdots\nu_s} - \Gamma^{\sigma}_{\nu_1\rho} T^{\mu_1\cdots\mu_r}{}_{\sigma\cdots\nu_s} - \cdots - \Gamma^{\sigma}_{\nu_s\rho} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\sigma}$$

Rembember : index we take derivative with is $\Gamma^{\cdot}_{\cdot\sigma}$ second downstairs.

Levi-Civita connection

don't need a metric to define ∇, given a connection

but a metric g singles out a special connection

Theorem On a manifold M with metric g, $\exists$ unique connection that is (i) torsion free, i.e. $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$

(ii) metric compatible, i.e. $\nabla g = 0$

This connection is the Levi-Civita connection and its components are given by $\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}$.

Proof "$\Rightarrow$" Let $\Gamma^\alpha_{\mu\rho}$ be torsion free, metric compatible.

Then $\nabla_\alpha g_{\mu\nu} = 0$ so

$$\partial_\alpha g_{\beta\gamma} - \Gamma^\rho_{\beta\alpha} g_{\rho\gamma} - \Gamma^\rho_{\gamma\alpha} g_{\beta\rho} = 0$$

Christoffel symbols $\left\{ {\mu \atop \beta\gamma} \right\} = \frac{1}{2} g^{\mu\alpha} (\partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma})$

$$\therefore \left\{ {\mu \atop \beta\gamma} \right\} = \frac{1}{2} g^{\mu\nu} \left(\Gamma^\rho_{\gamma\beta} g_{\rho\nu} + \Gamma^\rho_{\nu\beta} g_{\gamma\rho} + \Gamma^\rho_{\nu\gamma} g_{\rho\beta} + \Gamma^\rho_{\beta\gamma} g_{\nu\rho} - \Gamma^\rho_{\beta\nu} g_{\rho\gamma} - \Gamma^\rho_{\gamma\nu} g_{\beta\rho} \right)$$

$$= \Gamma^\mu_{\beta\gamma} \quad \text{as desired}$$

"$\Leftarrow$" For $\Gamma^\mu_{\beta\gamma} = \left\{ {\mu \atop \beta\gamma} \right\}$ have torsion free (easy) and $\nabla g = 0$. $\square$

In GR, we use the L-C connection.

Parallel Transport along a curve

"a tensor that does not change along a curve"

Def | Let V be a vector field & C an integral curve of V. Then a tensor T is parallel transported along C if $\nabla_V T = 0$ along the curve C. (only depends on V along C)

Tangent of an affinely parametrised geodesic is parallel transported along itself.

$$U^\mu U^\nu{}_{;\mu} = 0 \quad \text{indeed LHS} = U^\mu U^\nu{}_{,\mu} + U^\mu \Gamma^\nu_{\rho\mu} U^\rho$$

$$= \dot{x}^\mu \partial_\mu(\dot{x}^\nu) + \dot{x}^\mu \Gamma^\nu_{\rho\mu} \dot{x}^\rho = 0 \quad \text{by geodesic eq}^n$$

along C. ... transported along C $\Leftrightarrow \nabla_V T = 0$ ... curve of V.

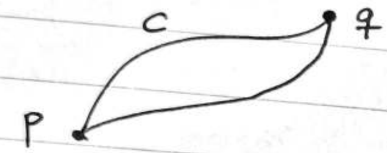
The equation $\nabla_V T = 0$ determines T uniquely given T at some $p \in C$

In coords x^μ the curve is $X^\mu(\lambda)$,

$$V^\sigma \nabla_\sigma T^\mu{}_\nu = V^\sigma \partial_\sigma T^\mu{}_\nu + V^\sigma \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu - V^\sigma \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho$$

$$= \frac{d}{d\lambda} T^\mu{}_\nu + \text{terms in } T = 0$$

ODE $\Rightarrow$ unique sol^n for $T^\mu{}_\nu$



isomorphism between T_pM, T_qM

" " tensors at p, at q

Unlike in Special Relativity, this isomorphism is path dependent.

Preserves length of vectors:

$$\frac{d}{d\lambda}(W_\alpha W^\alpha) = V^\mu \nabla_\mu (W_\alpha W^\alpha)$$

$$= V^\mu \nabla_\mu (g_{\alpha\beta} W^\alpha W^\beta)$$

$$= 2 W^\alpha V^\mu g_{\alpha\beta} \underbrace{\nabla_\mu W^\beta}_{\substack{=0 \\ \text{def of P.T.}}} = 0$$

$\therefore$ geodesics don't change whether they are timelike / spacelike / null

<u>Def</u> acceleration along a curve

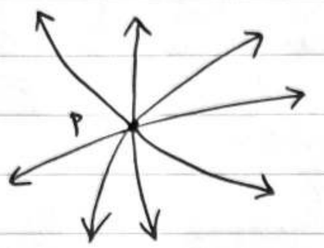
$$a^\mu := U^\rho \nabla_\rho U^\mu$$

the curve is geodesic if $a^\mu = 0$ (U normalised)

$a^\mu = f U^\mu$ (U not ")

<u>Normal coords</u> Let M be a manifold, Γ a connection, $p \in M$

The <u>exponential map</u> $e: T_pM \to M$ sends X_p to the point q a unit affine parameter distance along the geodesic through p with tangent X_p



1) e is a local diffeomorphism (not always globally)

2) the vector X_p fixes the param of the geodesic ; λX_p for $0 \leq \lambda \leq 1$ gets mapped to the point a distance λ along the X_p geodesic

Let $\{e_\mu\}$ be a basis of T_pM.

Normal coordinates in a ngbd of p assign to $q = e(X)$ the coords X^μ

Lemma In normal coordinates, $\Gamma^\mu_{(\nu\rho)} = 0$ at p.

So if Γ is torsion free then $\Gamma^\mu_{\nu\rho} = 0$.

Proof $\Rightarrow$ affinely parametrised geodesic is given by $X^\mu(\lambda) = \lambda X^\mu$

geod eq^n gives $\Gamma^\mu_{\rho\nu} \dot{x}^\nu \dot{x}^\rho = \Gamma^\mu_{\rho\nu} X^\nu X^\rho = 0$ (in normal coords)

for all $X \in T_pM$

deduce $\Gamma^\mu_{(\rho\nu)} = 0$ at p. $\square$

L-C connection $\Rightarrow$ in normal coords at p, $\partial_\rho g_{\mu\nu} = 0$

$\Gamma^\rho_{\mu\nu} = 0$ and $\nabla g = 0$

$2 g_{\sigma\rho} \Gamma^\rho_{\mu\nu} = \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}$ (sym σ, μ)

$\therefore 0 = \partial_\nu g_{\sigma\mu}$

Lemma Let (M, g) be spacetime w/ L-C connection.

Then $\exists$ coordinates at p s.t. $\partial_\mu g_{\nu\rho} = 0$,

$g_{\mu\nu} = \eta_{\mu\nu}$

cf Equivalence principle

Pf Choose o.n. basis $\{e_\mu\}$ for $T_p(M)$ and use that to introduce normal coordinates. $\square$

Def Local inertial frame

$n=4$ spacetime	# components	coord freedom	
$g_{\mu\nu}$	10	16	$\xi^{\mu}{}_{,\nu}$ ch $\frac{\partial \tilde{x}^\mu}{\partial x^\nu}$
$g_{\mu\nu,\sigma}$	40	40	ch $\frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\sigma}$
$g_{\mu\nu,\sigma\tau}$	100	80	ch $\frac{\partial^3 \tilde{x}^\mu}{\partial x^\nu \partial x^\sigma \partial x^\tau}$

Commutator of 2 vector fields

$$[V,W]^\alpha = V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha$$

Transforms as a vector under coord transformations

$$\tilde{V}^\nu \tilde{\partial}_\nu \tilde{W}^\mu - \tilde{W}^\nu \tilde{\partial}_\nu \tilde{V}^\mu = \cdots = \frac{\partial \tilde{x}^\mu}{\partial x^\beta}\left(V^\gamma \partial_\gamma W^\beta - W^\gamma \partial_\gamma V^\beta\right)$$

Some properties:

- $[V,W] = -[W,V]$
- bilinear
- $[V, fW] = f[V,W] + V(f)W$
- $[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0$ "Jacobi"

$$\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0 \qquad (\text{coord basis} \Rightarrow \text{commute})$$

Conversely if $V_0, \dots, V_{m-1}$, $m \le \dim(M)$ are vectors which are linearly indep $\forall p \in M$ & all $[V_i, V_j] = 0$
then in a ngbd of p $\exists$ coords x^μ s.t. $V_i = \frac{\partial}{\partial x^i}$, $i = 0, \dots, m-1$

Scalar field f: $\nabla_\nu \nabla_\mu f = \nabla_\mu \nabla_\nu f - 2\Gamma^\rho_{[\mu\nu]} \partial_\rho f$

$\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$ $\qquad \overset{\shortparallel}{\nabla_\mu \nabla_\nu f}$ (for torsion free Γ)

Vector V: $\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma$

$$= \partial_\alpha \Gamma^\gamma_{\rho\beta} V^\rho + \Gamma^\gamma_{\rho\alpha} \Gamma^\rho_{\sigma\beta} V^\sigma - (\alpha \leftrightarrow \beta)$$

Def Riemann tensor

$$R^\gamma{}_{\rho\alpha\beta} = \partial_\alpha \Gamma^\gamma_{\rho\beta} - \partial_\beta \Gamma^\gamma_{\rho\alpha} + \Gamma^\mu_{\rho\beta} \Gamma^\gamma_{\mu\alpha} - \Gamma^\mu_{\rho\alpha} \Gamma^\gamma_{\mu\beta}$$

Ricci identity:

$$\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R^\gamma{}_{\rho\alpha\beta} V^\rho$$

Def For 3 vector fields U, V, W:

$$R(U,V)(W) = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

Given a function f,

$$R(fU,V)W = fR(U,V)W$$
$$R(U,V)fW = fR(U,V)W$$

Comp $[e_\alpha, e_\beta] = 0$, $\nabla_\alpha e_\beta = \Gamma^\mu_{\beta\alpha} e_\mu$

$$\therefore R(e_\alpha, e_\beta) e_\rho = \nabla_\alpha \nabla_\beta e_\rho - \nabla_\beta \nabla_\alpha e_\rho$$

$$= \nabla_\alpha (\Gamma^\mu_{\rho\beta} e_\mu) - \nabla_\beta (\Gamma^\mu_{\rho\alpha} e_\mu)$$

$$= (\partial_\alpha \Gamma^{\cancel{\mu}\gamma}_{\rho\beta} - \partial_\beta \Gamma^\gamma_{\rho\alpha} + \Gamma^\mu_{\rho\beta} \Gamma^\gamma_{\mu\alpha} - \Gamma^\mu_{\rho\alpha} \Gamma^\gamma_{\mu\beta}) e_\gamma$$

Symmetries of Riemann

(1) $R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \iff R^\alpha{}_{\beta(\gamma\delta)} = 0$

torsion $= 0$, let $p \in M$ (x^μ) normal coords

(2) $\hookrightarrow \Gamma^\mu_{\nu\rho} = 0$ at p

$\hookrightarrow R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$ | antisymmetrize on $\rho\nu\sigma$

$\hookrightarrow R^\mu{}_{[\nu\rho\sigma]} = 0$ tensorial eq^n

$$\begin{cases} R = \partial\Gamma + \Gamma\Gamma \\ \partial R = \partial\partial\Gamma + \Gamma\partial\Gamma \end{cases}$$

$$(3)\quad \nabla_\tau R^\mu{}_{\nu\rho\sigma} = \partial_\tau R^\mu{}_{\nu\rho\sigma}$$
$$= \partial_\tau \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\tau \partial_\sigma \Gamma^\mu_{\nu\rho}$$

| antisymmetrise on $\rho\sigma\tau$

$$\therefore R^\mu{}_{\nu[\rho\sigma;\tau]} = 0 \quad \text{"Bianchi id."}$$
(tensoral)

L-C connection of normal coords at p:

$$(4)\quad 0 = \partial_\mu \delta^\nu{}_\rho = \partial_\mu (g^{\nu\sigma} g_{\sigma\rho})$$
$$= (\partial_\mu g^{\nu\sigma}) g_{\sigma\rho} + g^{\nu\sigma} \underbrace{\cancel{(\partial_\mu g_{\sigma\rho})}}_{\text{zero}} \qquad \Big|\cdot g^{\rho\tau}$$
$$\therefore \nabla_\mu g^{\sigma\tau} = 0 \quad \text{(tensoral)}$$

$$\partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2} g^{\tau\mu} \left(\partial_\rho \partial_\sigma g_{\mu\nu} + \partial_\rho \partial_\nu g_{\sigma\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} \right)$$

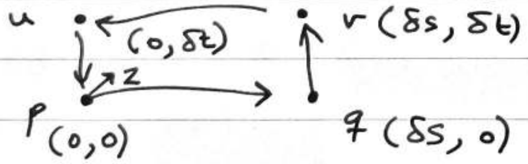
$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\Big(\partial_\rho \partial_\nu g_{\sigma\mu} + \partial_\sigma \partial_\mu g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\mu\rho} \Big)$$

$$\therefore R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$$

$n=4$ dimensions, Riemann has 20 components lin indep

Let X, Y be VF: lin indep and $[X,Y]=0$

Choose coordinates $(s, t, \dots)$ s.t. $X = \partial_s$, $Y = \partial_t$



Let $p, q, r, u \in M$ lie along integral curves of X, Y with coords $(0,0)$ to $(\delta s, 0, \dots)$ to $(\delta s, \delta t, \dots)$ to $(0, \delta t, \dots)$ to $(0, 0, \dots)$
...

Let $Z_p \in T_p(M)$; parallel transp along $pqrup$
$Z'_p \in T_p(M)$

$$\Rightarrow \lim_{\delta s, \delta t \to 0} \frac{(Z'-Z)^\alpha}{\delta s\, \delta t} = R^\alpha{}_{\beta\gamma\delta} Z^\beta Y^\gamma X^\delta$$

Geodesic Deviation: goal to quantify the relative acceleration of nearby geod

Def $(\mathcal{M}, \Gamma)$ is a mfd w/ connection.

A "1-parameter family of geodesics" is a map

$$\gamma : I \times I' \to \mathcal{M}, \quad I, I' \subset \mathbb{R} \text{ open s.t.}$$

(i) for fixed s, $t \mapsto \gamma(s,t)$ a geodesic w/ affine t

(ii) locally $(s,t) \mapsto \gamma(s,t)$ is smooth, 1-1, smooth inverse

$\Rightarrow$ the family of geodesics form a 2d-surface locally

Let T tangent vector $\gamma(s = \text{const}, t)$

S tangent vector $\gamma(s, t = \text{const.})$

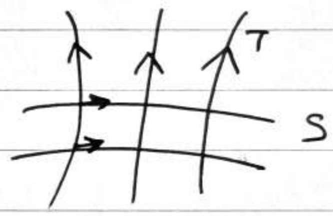
i.e. $$S^\mu = \frac{dx^\mu}{ds}, \quad T^\mu = \frac{dx^\mu}{dt}$$

$$\therefore x^\mu(x + s\delta s, t) = x^\mu(s,t) + \delta s \; S^\mu(s,t) + O(\delta s^2)$$

$\delta s\, S$ points from one geodesic to a nearby one

$\therefore$ define relative velocity $\nabla_T(\delta s\, S) = \delta s\, \nabla_T S$

$\therefore$ rel acceleration $\delta s\, \nabla_T \nabla_T S$



Thm The geodesic deviation is given by

$$\nabla_T \nabla_T S = R(T,S)T$$

Pf In components, $T^\nu \nabla_\nu (T^\mu \nabla_\mu S^\alpha) = R^\alpha{}_{\lambda\mu\nu} T^\lambda T^\mu S^\nu$

We use Lemma $V^\mu \nabla_\mu W^\alpha - W^\mu \nabla_\mu V^\alpha = [V, W]^\alpha$

which holds for a torsion-free connection:

$$LHS = V^\mu \partial_\mu W^\alpha + V^\mu \cancel{\Gamma^\alpha_{\rho\mu} W^\rho} - W^\mu \partial_\mu V^\alpha - \cancel{\Gamma^\alpha_{\rho\mu} V^\rho W^\mu} \checkmark$$

Since $S = \frac{\partial}{\partial s}$, $T = \frac{\partial}{\partial t}$, $[S, T] = 0$

So by Lemma, $\nabla_T S - \nabla_S T = 0$

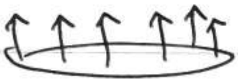
Hence $$\nabla_T \nabla_T S = \nabla_T \nabla_S T \underset{\text{def}^\text{n} \text{ of } R}{=} \nabla_S \nabla_T T + R(T,S)T$$

$$\underset{\nabla_T T = 0 \text{ since geodesic.}}{=} R(T,S)T$$

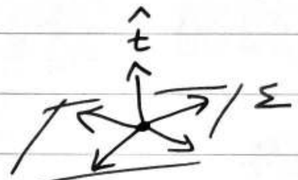
$\square$

<u>Comments</u> $R^{\alpha}{}_{\beta\gamma\delta}$ measures rel acceleration of nearby geodesics
$R^{\alpha}{}_{\beta\gamma\delta} = 0 \iff$ rel acceleration $=0$ for all families of geod
Tidal forces arise from geodesic deviation

<u>Def</u> The "Ricci tensor" is $\boxed{R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}}$
The "Ricci scalar" is $\boxed{R = g^{\mu\nu} R_{\mu\nu}}$
The "Einstein tensor" is $\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}$

R_{tt}; time↑  ← sphere volume V

$$= \lim_{V \to 0} \frac{\ddot{V}}{V}\Big|_{t=0}$$



$$G_{\hat{t}\hat{t}} = \frac{1}{2} R(\Sigma)$$

$\sum_{\hat{x}} R_{\hat{t}\hat{x}\hat{t}\hat{x}} - \frac{1}{2} R_{\hat{t}\hat{x}\hat{t}\hat{x}} - \frac{1}{2} R_{\hat{x}\hat{t}\hat{x}\hat{t}}$
only terms in R with space

Recall Bianchi
$R_{\alpha\beta[\gamma\delta;\mu]} = 0 \quad \Big| \cdot g^{\alpha\gamma} g^{\beta\delta}$

$$\Rightarrow \frac{1}{3} g^{\alpha\gamma} g^{\beta\delta} \left[R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} \right] = 0$$

$$\Rightarrow R_{;\mu} - g^{\alpha\gamma} R_{\alpha\mu;\gamma} - g^{\beta\delta} R_{\beta\mu;\delta} = 0$$

$$\Rightarrow \nabla_{\mu} R - 2 \nabla_{\gamma} R^{\gamma}{}_{\mu} = -2 \nabla^{\gamma} \left(R_{\gamma\mu} - \frac{1}{2} g_{\gamma\mu} R \right) = 0$$

$$\Rightarrow \boxed{\nabla^{\mu} G_{\mu\alpha} = 0}$$ "contracted Bianchi identity"

<u>COVARIANCE PRINCIPLE</u>
- Equivalence Pr: freely falling frame = interial Mink frame
- Normal coords: $\exists$ coords s.t. $g_{\alpha\beta} = \eta_{\alpha\beta}$, $\Gamma^{\alpha}{}_{\beta\gamma} = 0$ at a point P
- Laws of SR inv under <u>Lorentz</u>

"Laws of GR should be tensoral and reduce to laws of SR in normal coords" In particular, replace $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and ∂ by ∇
- Assume no extra R terms in matter equations (minimal coupling)

E-M field in SR: $F_{\mu\nu} = F_{[\mu\nu]}$ w/ $F_{0i} = -E_i$

$F_{ij} = \varepsilon_{ijk} B_k$

"permutation" tensor

vacuum Maxwell eq^{n}s:

$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho} = 0, \quad \partial_{[\alpha} F_{\beta\gamma]} = 0 \quad \text{SR}$$

$$\downarrow \qquad\qquad \downarrow$$

$$g^{\mu\nu}\nabla_\mu F_{\nu\rho} = 0, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0 \quad \text{GR}$$

why not $g^{\mu\nu}\nabla_\mu (F_{\nu\rho} + R_{\nu\rho\alpha\beta}F^{\alpha\beta})$? minimal coupling assumption

Postulate Energy / momentum / stress in GR described by "energy momentum tensor" $T^{\alpha\beta}$, where

$T^{\alpha\beta} :=$ flux of α momentum across a sfc of const. x^β

(1) $T_{\alpha\beta} = T_{\beta\alpha}$ i.e. symmetric

(2) $\nabla^\mu T_{\mu\nu} = 0$ i.e. conserved

Go to locally Mink frame of reference

$$T^{\mu\nu} = \left(\begin{array}{c|c} \text{E dens} & \text{mom dens} \\ \hline \text{E flux} & \text{Stress tensor} \end{array}\right)$$

T^{00} flux of 0-momentum (energy) across $t=0$ surface, i.e. energy density

T^{0i} momentum density

flux of i-momentum in i-direction = pressure in i-direction

negative pressure = tension

Particles: point object w/ 4 momentum $P^\mu = mU^\mu = (E, P^i)$

4-velocity of observer in their own frame $(1,0,0,0)$

$\therefore$ particle energy measured by observer $E = -\eta_{\mu\nu} W^\mu P^\nu$

rest mass $-m^2 = \eta_{\mu\nu} P^\mu P^\nu = -E^2 + p^2$

$\Bigg\}$ SR

In GR, $P^\alpha = mu^\alpha \Rightarrow g_{\alpha\beta}P^\alpha P^\beta = -m^2$

$E = -g_{\mu\nu} W^\mu P^\nu$ ← evaluate at some point p

Pre-relativistic Expressions

$$\rho = \frac{1}{8\pi}(E_i E_i + B_i B_i) \quad , \quad j_i = \frac{1}{4\pi}\varepsilon_{ijk} E_j B_k$$

stress tensor $S_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}(E_k E_k + B_k B_k)\delta_{ij} - E_i E_j - B_i B_j\right]$

Maxwell $\Rightarrow$ $\frac{\partial \rho}{\partial t} + \partial_i j_i = 0$, $\frac{\partial j_i}{\partial t} + \partial_j S_{ij} = 0$

SR: $T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\eta_{\mu\nu}\right) = T_{\nu\mu}$

$$\partial^\mu T_{\mu\nu} = \eta^{\mu\lambda}\partial_\lambda T_{\mu\nu} = 0$$

GR: $\eta \to g$, $\partial \to \nabla$ ALSO tensors with raised indices

$$F_\nu{}^\rho = g^{\rho\sigma}F_{\nu\sigma}$$

Dust continuum limit of non-interacting freely falling particles
w/ rest mass m & number density n
comoving $u^\mu = (1,0,0,0)$ locally $g_{\alpha\beta} = \eta_{\alpha\beta}$
$\Rightarrow \rho = mn$, $T^{i0} = T^{0i} = T^{ij} = 0$
$T^{\alpha\beta} = \rho u^\alpha u^\beta = mnu^\alpha u^\beta$

Perfect fluid can be completely characterized by its rest frame u^α, mass density ρ & an isotropic pressure P.

$\Rightarrow$ $\left.\begin{array}{l}\text{(1) no heat conduction}\\ \text{(2) no viscosity}\end{array}\right\}$ in comoving frame

$T^{\alpha\beta} = \mathrm{diag}(\rho, P, P, P)$, $u^\alpha = (1,0,0,0)$, $g^{\alpha\beta} = \eta^{\alpha\beta}$ in comov

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta}$$

Conservation Law $\nabla_\alpha T^{\alpha\beta} = 0$ implies

$u^\alpha \nabla_\alpha \rho + (\rho + P)\nabla_\alpha u^\alpha = 0$ $\longrightarrow$ looks like mass conservation in Newton limit

$(\rho + P)u^\alpha \nabla_\alpha u^\beta = -(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\alpha P$ $\longrightarrow$ Euler eq^n

To solve eq^n's dynamics, need eq^n of state $P(\rho)$
If $P = 0$ get dust, only left with $u^\alpha \nabla_\alpha u^\beta = 0$ i.e. geodesic motion

Einstein Eq^n $\quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right)T_{\alpha\beta}$

determined by Newtonian limit

Einstein Eq^{n}s $\quad G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \left(\frac{8\pi G}{c^4}\right) T_{\alpha\beta}$

Postulate's of General Relativity

(1) Spacetime is a four-dimensional manifold with metric & LC connection

(2) Free test particles move on timelike or null geodesics

(3) Energy / momentum / stess of matter described by symmetric, conserved $T_{\alpha\beta}$

(4) Curvature is related to matter by Einstein's equations

Vacuum: $T_{\alpha\beta} = 0 \Rightarrow R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0 \quad \Big| \cdot g^{\alpha\beta}$

$$\Rightarrow R - \frac{1}{2} \cdot 4R = 0$$

$$\text{i.e. } R = 0$$

$$\Rightarrow R_{\alpha\beta} = 0 \quad \text{again by Einstein}$$

10 different coupled non-linear PDEs

Lovelock's theorem says

Let $H_{\alpha\beta}$ be symmetric, s.t.

(i) in any coord system, $H_{\mu\nu} = H_{\mu\nu}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu})$ at every point $p \in \mathcal{M}$

(ii) $\nabla^\alpha H_{\alpha\beta} = 0$

(iii) $H_{\mu\nu}$ linear in $\partial_\sigma \partial_\rho g_{\mu\nu}$ [not needed in $n \leq 4$ dim^{n}s]

Then $\exists\, a, b \in \mathbb{R}$ s.t. $H_{\alpha\beta} = a G_{\alpha\beta} + b g_{\alpha\beta}$

$\therefore$ can modify to $\quad G_{\alpha\beta} + \underset{\substack{\uparrow \\ \text{cosmological constant}}}{\Lambda} g_{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right) T_{\alpha\beta}$

Λ term equivalent to a perfect fluid with $\rho = -P = \frac{\Lambda}{8\pi G}$

Key assumption: symmetry

Def A spacetime $(\mathcal{M}, g)$ is "symmetric in a variable s" if $\exists$ coordinates s.t. $x^\alpha = s$ for some α and $g_{\alpha\beta}$ is indep of s.

Def A spacetime is stationary if $\exists$ coordinates s.t. x^0 is a timelike coordinate and $g_{\alpha\beta}$ indep of x^0.

<u>Def</u> A spacetime (M, g) is <u>static</u> if it is stationary and in that coordinate system $g_{0i} = 0$ for $i = 1, 2, 3$

$\Rightarrow$ time reversal symmetry

<u>Def</u> A <u>spherically symmetric</u> spacetime has a coordinate system taking the form

$$ds^2 = -A\,dt^2 + B\,dt\,dr + C\,dr^2 + D\,d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on S^2

- Lorentzian $D > 0$; if $D(r)$ is monotonic, can redefine r coord so that $D = r^2$
- also possible to redefine t so that $B = 0$

so $ds^2 = -j(t,r)\,dt^2 + k(t,r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

Now vacuum Einstein eq^{n}s $R_{\alpha\beta} = 0$ determine j, k as well

Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

<u>Birkhoff's Theorem</u>: any spherically symmetric sol^n to the vacuum Einstein equations can be described by the Schwarzschild metric $\Rightarrow$ static & asymptotically flat

- $\lim_{r\to\infty} g_{\alpha\beta} = \eta_{\alpha\beta}^{(s)}$
- M can be shown to be related to mass-energy of the spacetime
- describes spacetime that is <u>exterior</u> to spherically symmetric stars

$R_{\alpha\beta} \neq 0$ inside, $R_{\alpha\beta} = 0$ outside

<u>Geodesics</u> $\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$

WLOG rotate so $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0 \Rightarrow$ stay in $\theta = \frac{\pi}{2}$

Noether's theorem (i) $\frac{\partial \mathcal{L}}{\partial t} = 0$ so $\frac{\partial \mathcal{L}}{\partial \dot{t}} = -2\left(1 - \frac{2M}{r}\right)\dot{t} =: -2E$

(ii) $\frac{\partial \mathcal{L}}{\partial \phi} = 0$ so $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2r^2\dot{\phi} =: 2L$ $\leftarrow$ use $\theta = \frac{\pi}{2}$

(iii) $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ so $H = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$

$=: Q$ is conserved

Also WLOG $Q \in \begin{cases} +1 & \text{space like} \\ 0 & \text{null} \\ -1 & \text{time like} \end{cases}$

Static metric (symmetric in t, no g_{ti} terms)

i.e. $ds^2 = -g_{tt}\, dt^2 + g_{ij}\, dx^i dx^j \qquad 1 \le i,j \le 3$

indep of t

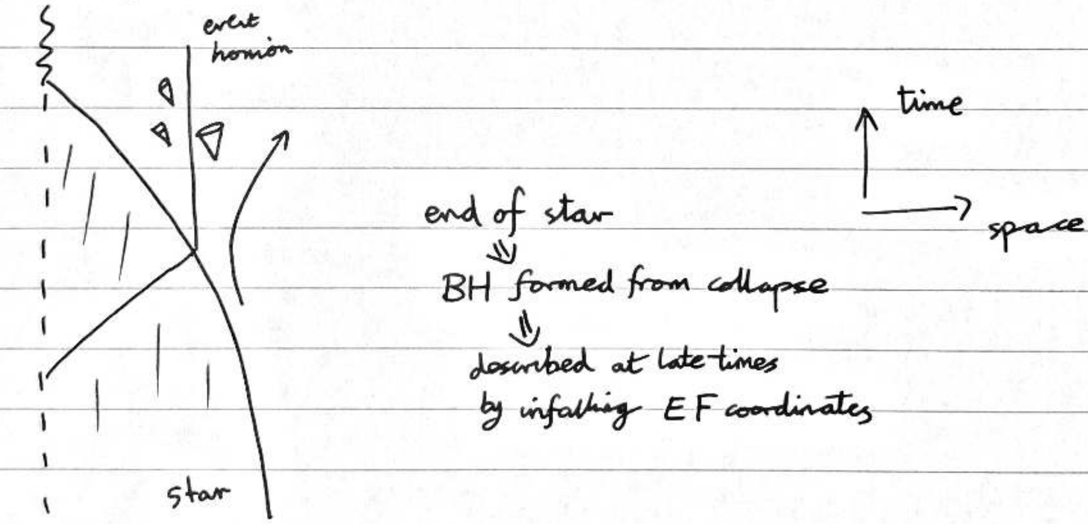
Consider a t = const slice outside event horizon H:

<u>Israel's Theorem</u>: Given an asymptotically flat spacetime solving the vacuum equations,

assume:
1) $g_{tt} = 0$ at H, $g_{tt} < 0$ outside of H
2) g_{tt} has non-zero gradient
3) level sets of g_{tt} are compact
4) geometry is regular outside H & approaching H

Then the spacetime is Schwarzchild.

/ more generally, stationary BM mass M, angular momentum J, charge Q



end of star

$\Downarrow$

BH formed from collapse

$\Downarrow$

described at late times by infalling EF coordinates

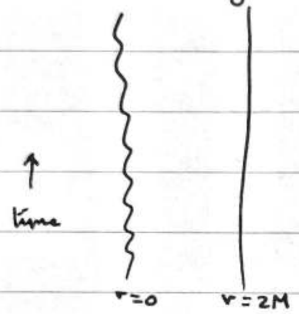
Classically, Hawking proves, for positive energy

$$T_{ab} k^\alpha k^\beta \geq 0$$

k = null vector "null energy condition"

$\Rightarrow$ Area of horizon non-decreasing

QM: Hawking radiation ; semiclassical approx



QFT in spacetime

Hawking temperature

$$T = \frac{\hbar c^3}{8\pi G m k_B}$$

$$-\frac{dM}{dt}\frac{1}{A} \sim \sigma T^4$$

$$\frac{dM}{dt} \sim \frac{\hbar c^6}{15360\pi G^2 M^2} \Longrightarrow t \sim 5120 \frac{\pi G^2}{\hbar c^6} M^3$$

$M_\odot$ gives $t = O(10^{60})$ yrs

Observation: EM observations of Universe out to $\sim 10^{11}$ pc

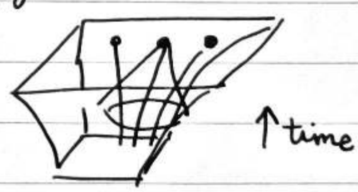
Galaxies $R \sim 10^5$ pc $\rightsquigarrow$ point like

On scales $\sim 10^9$ pc, the universe looks the same

at every location $\equiv$ "homogeneity"

in every direction $\equiv$ "isotropy"

Hubble redshift $\rightarrow$ universe is expanding



<u>Cosmological principle</u>: at a given time, universe is spatially homogeneous & isotropic

<u>Weyl's postulate</u>: the worldlines of matter fluid are orthogonal to hypersurfaces Σ_t of constant time

coords: x^i, $i=1,2,3$ spatial components, comoving w/ fluid

$\dot{x}^i(\text{galaxy}) = 0$, t = proper time along fluid element

$$ds^2 = -dt^2 + \underbrace{g_{0i}\, dt\, dx^i}_{\text{zero by isotropy}} + a(t)^2 h_{ij}(x^k)\, dx^i dx^j$$

spatial part (spherical symm) $dl^2 = C(t,r)\,dr^2 + D(t,r)\,d\Omega^2$

$C \to a(t^2)\,e^{2\beta(r)}$ $\quad D \to a^2(t)\,r^2$

3D Ricci scalar

$$R = R^i{}_i = \frac{2}{r^2}\left(1 - \partial_r\left(r e^{-2\beta}\right)\right) = \tilde{k} = \text{const.}$$

integrates to $e^{2\beta} = \dfrac{1}{1 - \frac{1}{6}\tilde{k}r^2 - \frac{A}{r}}$ where $A = \text{const.}$

demand no singularity at $r=0$

$\lim_{r\to 0} dl^2 \propto (dr^2 + r^2 d\Omega^2)$ so $A = 0$

$\tilde{k} = 6k$ obtain <u>Robertson - Walker metric</u>

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Rescale $r, a \to k = +1, 0, -1$

1) case $k=0$ has $dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = dx^2 + dy^2 + dz^2$

i.e. the <u>flat</u> metric on $\mathbb{R}^3$

2) case $k=1$, set $r = \sin\chi$ so

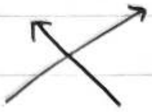
$$dl^2 = d\chi^2 + \sin^2\chi\, d\Omega^2$$ metric on 3 sphere <u>closed</u>

<u>$r=1$: equator</u>

3) case $k=-1$, $r = \sinh\psi$ so

$$dl^2 = d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$ <u>open</u>

$\mathbb{H}^3$



x-directions

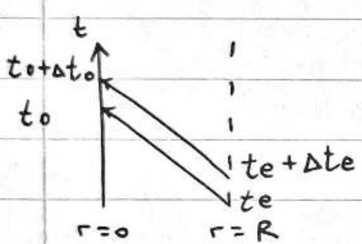
$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

vs

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho/3 & 0 & 0 \\ 0 & 0 & \rho/3 & 0 \\ 0 & 0 & 0 & \rho/3 \end{pmatrix}$$

radial null geodesics : $\quad -dt^2 + \dfrac{a^2}{1-kr^2}\,dr^2 = 0$

$$\therefore \quad \frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1-kr^2}}$$



$$\int_{t_e}^{t_o} \frac{dt}{a} = -\int_R^0 \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a}$$

$$\therefore \int_{t_o}^{t_o+\Delta t_o} \frac{dt}{a} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a}$$

$$\therefore \frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)} \quad \text{to first order}$$

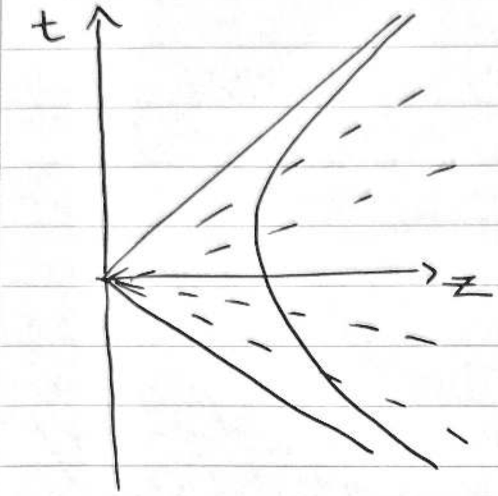
$$\therefore \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} \quad \text{cf } \lambda \propto \Delta t$$

Define z via

$$1+z := \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} \approx 1 + (t_o - t_e)\frac{\dot{a}(t_o)}{a(t_o)} = 1 + (t_o - t_e)H(t_o)$$

redshift factor

for nearby galaxies

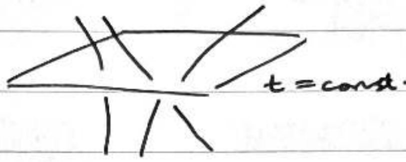


<u>Def</u> A <u>Cauchy slice</u> Σ in a manifold M is a set of points s.t. every <u>complete</u>* timelike or null curve intersects Σ exactly once.

$\Rightarrow$ Σ is a codimension 1 spacelike submanifold

e.g. in Mink

*actually / inextendible c.f. not geodesic



$\Rightarrow$ suitable for specifying initial data for wave equations

<u>Def</u> A <u>globally hyperbolic</u> manifold M is one which has at least one Cauchy slice.

<u>Geroch's Thm</u> A GH manifold is topologically $\Sigma \times \mathbb{R}[t]$ where ∂_t is everywhere timelike

<u>Def</u> Consider a closed 2d surface $\sigma \in \Sigma$ w/ coords (φ, σ)

Let the null congruences N_{in} (N_{out}) be the 3d null surface foliated by light rays γ shot out normal to σ inward (outward)

Each N has coords

- (θ, φ) constant on each γ
- v is an affine parameter along each γ

$\therefore g_{vv} = g_{v\theta} = g_{v\varphi} = 0$

<u>Def</u> On a null surface N, have the

null extrinsic curvature $B_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial v}$

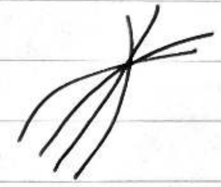
expansion $\theta = B_{ij} g^{ij}$

shear $\sigma_{ij} = B_{ij} - \frac{1}{2} g_{ij} \theta$

<u>Raychaudhuri</u>: $$\frac{d\theta}{dv} = -\frac{\theta^2}{2} - \sigma_{ij}\sigma^{ij} - \underset{\substack{\uparrow \\ (8\pi G) T_{vv}}}{R_{vv}} \quad (\Leftarrow \text{def}^n \text{ of } R_{\mu\nu})$$

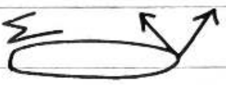
Assume $T_{vv} \geq 0$ (null energy condition)
true for classical KG, Maxwell, p.t. $P+\rho \geq 0$
$\Rightarrow$ RHS ≤ 0
If a γ has $\theta = \theta_0 < 0$, then $\theta \to -\infty$ at $v_* \leq -1/\theta_0$

$\Rightarrow$ either ① γ crosses nearby light rays
② singularity



<u>Def</u> A 2d closed spacelike σ in a Cauchy slice Σ
is called 'trapped' if $\theta < 0$ for both N_{in} and N_{out}

<u>Penrose sing. theorem</u>
Assume i) M has noncompact Cauchy slice Σ



ii) a trapped surface σ in Σ
iii) M satisfies null energy condition $T_{ab}k^a k^b \geq 0$
for all null k, all points
Then M is null geodesically incomplete