II LINEAR ANALYSIS

Source / Material : • lecture notes for past years (Dafermos, Bauerschmidt, Körner, Green / Schein)

• books RUDIN, real & complex analysis

(Clément Mouhot cm612)

[I] General Introduction

Area borne with the study of eq^s for functions that studies vector spaces of functions

Treat fcts as "points" of generalised spaces cf $\mathbb{R}^n$

Main difference : infinite - dimensional spaces

⇒ Analysis matters more !

• In finite dimension, often analysis-based arguments could be replaced by algebraic arguments

• "Measurement tools" (topologies, norms, distances) are no more all comparable

• These tools are the 'microscope' of the analyst and must be studied for themselves

In other mathematical programs, and online, this course often the first half of a "functional analysis" course - [Study of $C(K)$, ℓ^2]

Main application of functional analysis is Lebesgue spaces $L^2(\mathbb{R})$; see the Lent course Analysis of Functions

[II] Normed Vector Spaces

[III] The finite dimensional case

[IV] The Hahn - Banach Theorem, consequences

[V] Completeness and Baire's Theorems

[VI] Detailed study of the topology of $C(K)$

[VII] Hilbert spaces (ℓ^2)

[VIII] Introduction to Spectral Theory

Normed Vector Spaces

Recall # V vector space (VS) is a set of "vectors" endowed with two algebraic structures, $A : V \times V \to V$, $(v,w) \mapsto v+w$

$M : \mathbb{F} \times V \to V$, $(\lambda, v) \mapsto \lambda v$

Revise the axioms.

Here always take $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Span of a set $S \subset V$ is the smallest subspace (= sub vector space) containing S, i.e. the set of linear combinations of elements of S

$\left\{ \sum_{i=1}^{m} \lambda_i s_i \mid m \in \mathbb{N}, \lambda_i \in \mathbb{F}, s_i \in S \right\}$ Note that by def^n is finite

$S \subset V$ is linearly independent if $\forall m \in \mathbb{N}, \forall \alpha_i \in \mathbb{F}, \forall s_i \in S, i = 1, \dots, m$

$\sum_{i=1}^{m} \alpha_i s_i = 0 \Rightarrow \forall i, \alpha_i = 0$

$B \subset V$ is a basis of V if it is lin indep and $\text{span}(B) = V$

FACT (see later) Zorn's lemma $\Rightarrow$ all VS have a basis, and any two bases of a vector space have the same cardinality

DEF^n If VS V has a finite basis B then V is said to be finite dimensional. Otherwise it is infinite dimensional.

We now add a third layer of structure analytical, that is (see later) with previous layers.

DEF^n A Normed Vector Space (NVS) V is a VS endowed with a f^n $\|\cdot\| : V \to \mathbb{R}_+$, $v \mapsto \|v\|$ ("norm") such that

1) $\|v\| \geq 0$, $\forall v \in V$; $\|v\| = 0 \Leftrightarrow v = 0$ (positive definiteness)

2) $\forall \lambda \in \mathbb{F}, v \in V$, $\|\lambda v\| = |\lambda| \|v\|$

3) $\forall v, w \in V$, $\|v+w\| \leq \|v\| + \|w\|$ (triangle inequality "$\Delta \leq$")

BASIC EXAMPLE : $V = \mathbb{R}^n$, $v = (x_1, \dots, x_n)$

$\|v\| = (x_1^2 + \dots + x_n^2)^{1/2}$ (Euclidean norm)

FACT $(V, \|\cdot\|)$ NVS then $d(v,w) \doteq \|v-w\|$ gives a distance

So induces a topology. So have a natural norm-induced topology

Recall Def^n of topology (Jesus Christ mate)

Def^n of topology from a metric

QUESTION Is this topology compatible with the algebraic structure?

Prop $(V, \|\cdot\|)$ NVS. Then A, M are continuous for the norm-induced topology. As a consequence (EXE), translations $T: V \to V$, $v \mapsto v + v_0$ and dilatations $D: V \to V$, $v \mapsto \lambda v$ (λ non-zero) are homeomorphisms.

PROOF Use charac^n of continuity via open sets, i.e. cts $\iff$ preimage (open) = open

- $A^{-1}(\mathcal{U})$, $\mathcal{U}$ open set in V

$(v_1, v_2) \in A^{-1}(\mathcal{U})$, $\quad v_1 + v_2 \in \mathcal{U}$, $B(v_1 + v_2, \varepsilon) \subset \mathcal{U}$

$A(\underbrace{B(v_1, \varepsilon/2), B(v_2, \varepsilon/2)}_{\text{open set in } V \times V}) \subset \mathcal{U}$

- $M^{-1}(\mathcal{U})$, $\mathcal{U}$ open set in V

$(\lambda, v) \in M^{-1}(\mathcal{U})$, $\quad B(\lambda v, \varepsilon) \subset \mathcal{U}$.

$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3\max(1, \|v\|)}\right), B_V\left(v_\xi, \frac{\varepsilon}{3\max(|\lambda|, 1)}\right)\right) \subset B(\lambda v_\xi, \varepsilon)$

$\therefore$ open set in $\mathbb{F} \times V$. $\square$

CHARACTERISATION OF NORMABLE TOPOLOGICAL VECTOR SPACES

Last video (oh bruh)

More topologies possible than those from a norm, when to tell?

Def^n A topological vector space (TVS) V is a VS endowed with a topology τ such that A, M (vector space operations) are continuous; and points are closed.

Rmk Last condition implies V is Hausdorff

Def^n # $C \subset V$ where V is VS, said to be convex if

$\forall x, y \in C, \forall \lambda \in [0,1], \ \lambda x + (1-\lambda) y \in C$

A TVS (V, τ) is locally convex if every ngbd of zero contains a convex neighbourhood of zero.

$B \subset V$ where (V, τ) is a TVS, is <u>bounded</u> if any ngbd $\mathcal{U} \ni 0$ can be dilated so $B \subset s\mathcal{U}$ for some $s \geq t > 0$

(V, τ) TVS is <u>locally bounded</u> if it has $\mathcal{U}$ open containing 0 and bounded

<u>BASIC EXAMPLE</u> $(V, \|\cdot\|)$ NVS

Then $\mathcal{U} = B(0, r)$, $r > 0$, is open, convex (check it) and bounded

Reciprocal statement ?

<u>PROP</u> (V, τ) TVS with $C \subset V$ a bounded, convex ngbd of zero, then V is normable : there is a norm $\|\cdot\|$ inducing topology τ

<u>PROOF</u> <u>Step 1</u> Update C to $\tilde{C}$ a <u>balanced</u> bdd convex ngbd of zero.

means $\lambda\tilde{C} \subset \tilde{C}$ for $|\lambda| \leq 1$

<u>Proof of Step 1</u>

$M : \mathbb{F} \times V \longrightarrow V$ continuous

so $M^{-1}(C)$ is a ngbd of $(0,0)$ in $\mathbb{F} \times V$

so $\exists\ \underset{\varepsilon > 0}{B_{\mathbb{F}}(0, \varepsilon)} \times \underset{\substack{\uparrow \\ \text{open around} \\ 0 \text{ in } V}}{\mathcal{U}}$ s.t. $M(B_{\mathbb{F}}(0, \varepsilon), \mathcal{U}) \subset C$

Define then $\tilde{C} : \underbrace{\text{cvx}}_{\substack{\text{convex} \\ \text{hull}}} (B_{\mathbb{F}}(0, \varepsilon)\mathcal{U})$

Convex hull of E : $\bigcup_{\lambda \in [0,1]} (\lambda E + (1-\lambda)E)$

<u>Check</u> : $\tilde{C}$ is (convex), bounded : $B_{\mathbb{F}}(0, \varepsilon)\mathcal{U} \subset C$ cvx

$\Rightarrow \text{cvx}(B_{\mathbb{F}}(0, \varepsilon)\mathcal{U}) \subset C$ bounded

$\tilde{C}$ is balanced since

$\lambda B_{\mathbb{F}}(0, \varepsilon) \subset B_{\mathbb{F}}(0, \varepsilon)$, $|\lambda| \leq 1$

and so $B_{\mathbb{F}}(0, \varepsilon)\mathcal{U}$ is balanced

$\underset{\text{check}}{\Rightarrow}$ $\text{cvx}(B_{\mathbb{F}}(0, \varepsilon)\mathcal{U})$ is balazed too

<u>Step 2</u> Define Minkowski functional (gauge) of $\tilde{C}$:

$\forall v \in V,\ \mu_{\tilde{C}}(v) \doteq \inf\{t \geq 0 ;\ v \in t\tilde{C}\}$

- Well-defined (not $+\infty$) : Any $v \in V$ satisfies $\frac{v}{t} \to 0$

- Check $\mu_{\tilde{C}}(v) = \|v\|$ satisfies axioms of a norm

1) $\mu_{\tilde{C}}(v) = 0 \overset{?}{\Rightarrow} v = 0$

Otherwise $\exists\ \mathcal{U}$ open around 0 s.t. $v \notin \mathcal{U}$

$\exists\ t > 0$ s.t. $\tilde{C} \subset t\mathcal{U}$ (boldness)

$v \in \frac{1}{t}\tilde{C} \subset \mathcal{U}$ absurd

2) $\mu_{\tilde{C}}(\lambda v) = |\lambda|\, \mu_{\tilde{C}}(v)$ (balanced)

3) $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$ (convex) $\square$

OOH BOY

Examples of NVS

(Last lecture: VS, NVS, TVS, TVS locally cvx, locally bd = NVS)

TERMINOLOGY $(V, \|\cdot\|)$ NVS complete is called Banach Space

EX 1 $\mathbb{R}^n$ or $\mathbb{C}^n$ [Any fd NVS reduces to them]

EX 2 $C(X)$

S set,
$$\mathcal{F}_{\mathbb{F}}(S) = \{ f : S \to \mathbb{F} \}$$
$$\cup$$
$$\mathcal{B}_{\mathbb{F}}(S) = \{ f : S \to \mathbb{F} \text{, bdd} \}$$
$$\cup$$
$$C_{\mathbb{F}}(S) = \{ f : X \to \mathbb{F} \text{, cts} \}$$

here $X = S$ compact Hausdorff

EXE $C_{\mathbb{F}}(X) \subset \mathcal{B}_{\mathbb{F}}(X)$

On $C_{\mathbb{F}}(X)$ and $B_{\mathbb{F}}(X)$ we can define

$$\|f\|_\infty \doteq \sup_{x \in X} |f(x)|$$

EXE It is a norm

Prop^n $(C(X), \|\cdot\|_\infty)$ is a Banach Space

Rmk It implies in particular that $(C(X), \|\cdot\|_\infty)$ is closed in $(\mathcal{B}(X), \|\cdot\|_\infty)$ – recall IB A&T

Proof Consider $(f_k)_{k \geq 0}$ Cauchy in $(C(X), \|\cdot\|_\infty)$

$$\forall \varepsilon > 0, \exists k_0 \text{ s.t. } \forall k_1, k_2 \geq k_0, \|f_{k_1} - f_{k_2}\|_\infty < \varepsilon \quad (*)$$

Then
- $\forall x \in X$, $f_k(x)$ is Cauchy in $\mathbb{F}$, so converges to $f(x)$
- $(*) \Rightarrow \|f_k - f\|_\infty \to 0$
- $\Rightarrow f$ is continuous and $f_k \to f$ in $(C(X), \|\cdot\|_\infty)$ $\square$

EXAMPLE 3 $C^m(\overline{\mathcal{U}})$ where $m \in \mathbb{N}$, $\mathcal{U}$ open bdd set in $\mathbb{R}^n$

$C^m(\overline{\mathcal{U}}) = \{ f : \mathcal{U} \to \mathbb{R}$ s.t. all $\partial^\alpha f$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $(\alpha) = \alpha_1 + \dots + \alpha_n \leq m$ are continuous and uniformly bounded on $\mathcal{U} \}$

EXE 1) $\partial^\alpha f$ can be extended as a cts function on $\overline{\mathcal{U}}$

2) $C^m(\overline{\mathcal{U}})$ is a Banach space

Ex 4 $C_{\mathbb{R}}([0,1])$ can be endowed with the norm

$$\|f\|_p \doteq \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}, \quad 1 \le p \le \infty$$

EXE 1) It is a norm

2) It is not a Banach Space

Hint Construct $(f_k)_{k\geq 0}$ cts in $[0,1]$ such $f_k \xrightarrow{\|\cdot\|_p} \mathbb{1}_{[\frac{1}{2},1]}$

Can you build a completion of $(C([0,1]), \|\cdot\|_p)$?

Yes Can be done abstractly, but not useful

Fact Can be realised as Lebesgue-measurable functions f

s.t. $\int_0^1 |f|^p < +\infty$, with classes of equivalence $f = g$ almost everywhere

Denoted $L^p([0,1])$. See AoF.

Ex 5 Functions over $\mathbb{N}$ instead of $[0,1]$

$\ell^p_{\mathbb{F}} = \{ (x_n)_{n\geq 0} : \sum_n |x_n|^p < \infty \} \qquad p \in [1,\infty)$

$\ell^\infty_{\mathbb{F}} = \{ (x_n)_{n\geq 0} : \sup_n |x_n| < \infty \}$

EXE ℓ^p, $p \in [1,\infty]$ are subspaces of $\mathcal{F}_{\mathbb{F}}(\mathbb{N})$

2) $\|\cdot\|_{\ell^p}$, $p \in [1,\infty]$ are norms

3) $p \in (0,1)$: $\ell^p = \{ (x_n)_{n\geq 0} : \sum |x_n|^p < \infty \}$

is still a subspace of $\mathcal{F}_{\mathbb{F}}(\mathbb{N})$ but $\|\cdot\|_{\ell^p}$ is not a norm

use $(|x_n| + |y_n|)^p \le |x_n|^p + |y_n|^p$ $\diamond$ oh oh

Examples of TVS

Rmk $C^m(\overline{\mathcal{U}})$ for $m \geq 1$

Exe Prove $\partial^\alpha f$ for $f \in C^m(\overline{\mathcal{U}})$ extends to cts f^α on $\overline{\mathcal{U}}$ for $|\alpha| \le m-1$

EX1 $C_{\mathbb{R}}(\mathcal{U})$ and $\mathcal{U}$ open, bdd subset of $\mathbb{R}^n$

TVS for the following topology

$(K_n)_{n\geq 1}$ increasing sequence of compact sets in $\mathcal{U}$ s.t. $\bigcup_n K_n = \mathcal{U}$

EXE Build such a sequence [Hint: $K_n = \{ x \in \mathcal{U} : d(x, \mathcal{U}^c) \geq \frac{1}{n} \}$]

Base of neighbourhoods of 0

$\mathcal{V}_n = \{ f \in C(\mathcal{U}) : \sup_{K_n} |f| < \frac{1}{n} \}$

EXE Check that the topology generated makes $C(\mathcal{U})$ a TVS ☐

2) This topology is locally convex

3) It is not locally bounded

[Hint: If B bdd and nbgd of 0, then $B \supset \mathcal{V}_{n_0}$ for some $n_0 \in \mathbb{N}$

$\mathcal{V}_{n_0}$ includes f's taking values as high as wanted on $K_{n_0+1} \setminus K_{n_0}$

So $B \subset t\mathcal{V}_{n_0+1}$ for $t>0$ is impossible.] □

EXE Check that $d(f,g) \doteq \sum_{n\geq 1} \frac{1}{2^n} \frac{\|f-g\|_{\infty,K_n}}{1+\|f-g\|_{\infty,K_n}}$

is a distance that induces

the same topology as above, making metric space complete.

Rmk It is a Fréchet space (i.e. a locally convex metrizable TVS with a complete translation-invariant metric)

EXE Define, as above, TVS topologies on $C^m(U)$, $C^\infty(U)$

Ex 2 $\underbrace{C_c}_{\text{cts, compact support}}(U)$, U open subset of $\mathbb{R}^n$

Claim TVS for the topology generated by

$$\mathcal{V}_{\substack{(\varepsilon_n)_{n\geq 1}\\ \varepsilon_n \to 0}} = \{ f \in C_c(U) : \|f\|_{\infty,K_n} \leq \varepsilon_n \}$$

EXE $(C_c(U), \tau)$ is TVS, locally convex, not locally bounded (as before), with no countable basis of neighbourhoods hence not metrizable (?)

Rk In same spirit, $C_c^\infty(U)$ is a locally convex TVS for the topology generated

$$\mathcal{V}_{(\varepsilon_n)_{n\geq 1},(m_n)_{n\geq 1}} = \{ f \in C_c^\infty(U) : \sup_{|\alpha|\leq m_n} \|\partial^\alpha f\|_{\infty,K_n} \leq \varepsilon_n \}$$

This is the space $\mathcal{D}(U)$ of "test functions" in the theory of L. Schwartz. letter "d"

Bounded Linear Maps

We have defined objects : NVS, TVS

We want the natural morphisms between them

<u>DEFN</u> (V, τ_V), (W, τ_W) TVS, $T: V \to W$ linear map

T is <u>bdd</u> if it maps bounded sets to bounded sets

$B \subset V$ bdd $\Rightarrow T(B) \subset W$ bdd

<u>PROP</u> (V, τ_V), (W, τ_W) TVS, locally bounded

Then $T: V \to W$ linear is bdd $\Longleftrightarrow$ cts

<u>PROOF</u> <u>Step 1</u> T bdd $\Rightarrow T$ cts at 0

Consider $\mathcal{U}_W$ open set around 0 in W,

$\mathcal{U}_V$ open <u>bdd</u> around 0 in V

$T(\mathcal{U})_V$ bdd hence $\exists t>0 \mid T(\mathcal{U}_V) \subset t\,\mathcal{U}_W$

$\Rightarrow T^{-1}(\mathcal{U})_W \supset \underbrace{t^{-1}\mathcal{U}_V}_{\text{open set around zero}}$ [using that dilatation is a homeomorphism]

<u>Step 2</u> T linear cts at $0 \Rightarrow T$ cts everywhere

$T(v) = w$ and consider $w \in \mathcal{U}_W$ open

Then $\mathcal{U}_W - w$ open around 0 [translation is homeo]

$T^{-1}(\mathcal{U}_W - w)$ open around 0 (T cts at 0)

$\Rightarrow v + T^{-1}(\mathcal{U}_W - w) = T^{-1}(\mathcal{U}_W)$ open around v

<u>Step 3</u> T continuous $\Rightarrow T$ bdd

Consider $\mathcal{B}_V \subset V$ bdd and $\mathcal{U}_W$ open around 0 in W

Then $T^{-1}(\mathcal{U}_W)$ open around 0 in V

Hence $\exists t>0$ s.t. $\mathcal{B}_V \subset t\,T^{-1}(\mathcal{U}_W)$ ← need some more details

$\Rightarrow T(\mathcal{B}_V) \subset t\,\mathcal{U}_W \Rightarrow T(\mathcal{B}_V)$ bdd in W $\square$

We specialise the def^n to NVS

<u>DEF</u> $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ NVS, $T: V \to W$ linear, T bdd if

$T(B(0,1)) \subset B(0;t)$ for some $t>0$,

the infimum of such t is denoted $|||T|||$. ö

EXE 1) Check this def^n agrees with previous when particularised

2) $|||T||| = \sup_{\|v\| \leq 1} \|Tv\|_W = \sup_{\|v\|<1} \|Tv\|_W = \sup_{\|v\|=1} \|Tv\|_W$ $(*)$

Notation $\mathcal{L}(V,W) = \{T: V \to W \text{ linear}\}$

$\mathcal{B}(V,W) = \{T: V \to W \text{ linear bdd}\}$

EXE $\mathcal{B}(V,W)$ is a subspace of $\mathcal{L}(V,W)$

PROP $|||\cdot|||$ is a norm on $\mathcal{B}(V,W)$, making it a TVS

PROOF
- $|||T||| \geq 0$
- $|||0||| = 0$
- $|||T||| = 0 \Rightarrow T(v) = 0$ for $v \in B(0,1)$
  $\Rightarrow T(v) = 0$ for $v \in V$ (linearity)
- $|||\lambda T||| = |\lambda| \cdot |||T|||$ & $|||T_1 + T_2||| \leq |||T_1||| + |||T_2|||$
  immediate from $(*)$

$\square$

L3.1

Duality for NVS

Intro Last time, discussed bdd linear maps between NVS

Particular case when arrival space is $\dim^n$ 1 gives dual space

Prop $(V, \|\cdot\|_V)$ NVS, $(W, \|\cdot\|_W)$ Banach space

Then $\mathcal{B}(V,W)$ is a Banach space

Proof only need completeness

Consider $(T_k)_{k\geq 1}$ Cauchy in $\mathcal{B}(V,W)$.

$(*)$ $\forall \varepsilon > 0, \exists k_0 \geq 1 \mid k_1, k_2 \geq k_0 \Rightarrow |||T_{k_1} - T_{k_2}||| < \varepsilon$

Implies $\forall v \in V$

$$(**) \quad \sup_{k_1,k_2\geq k_0} \|T_{k_1}(v) - T_{k_2}(v)\|_W \leq \|v\| \sup_{k_1,k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \underset{\text{as } k_0\to\infty}{\longrightarrow} 0$$

Therefore $(T_k(v))_{k\geq 1}$ Cauchy in W

So W Banach $\Rightarrow \exists\, T(v) = \lim_{k\to\infty} T_k(v)$

Limit operator T is linear (follows from continuity of VS operations)

$k_2 \to \infty$ in $(**)$

$$\Rightarrow \forall v \in V, \|v\| \leq 1, \quad \|T_{k_1}(v) - T(v)\| \leq \varepsilon(k_1) \underset{k_1\to\infty}{\longrightarrow} 0$$

$$\Rightarrow |||T_{k_1} - T||| \underset{k_1\to\infty}{\longrightarrow} 0$$

□

EXE In fact $|||T||| = \lim_{k\to\infty} |||T_k|||$

DEFN $(V, \|\cdot\|_V)$ NVS then $\mathcal{L}(V, \mathbb{F})$ is called the algebraic dual, and $\mathcal{B}(V, \mathbb{F})$ is called the dual space, denoted $(V^*, \|\cdot\|_{V^*})$.

FACT Previous PROP $\Rightarrow (V^*, \|\cdot\|_{V^*})$ is a Banach space

DEFN $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ NVS, $T \in \mathcal{B}(V,W)$

Adjoint of T is
$$T^* : W^* \to V^*, \quad \psi \mapsto \psi \circ T \qquad \text{i.e. } \underset{v}{T^*(\psi)(v)} = \psi(\underbrace{T(v)}_{\in W})$$

Prop Such T^* is linear and bounded with $|||T^*||| \leq |||T|||$

Proof Linearity follows by composition

$$\begin{aligned} T^*(\lambda_1\psi_1 + \lambda_2\psi_2)(v) &= (\lambda_1\psi_1 + \lambda_2\psi_2)(Tv) \\ &= \lambda_1\psi_1(Tv) + \lambda_2\psi_2(Tv) \\ &= \lambda_1 T^*(\psi_1)(v) + \lambda_2 T^*(\psi_2)(v) \end{aligned}$$

L3.2

Boundedness: $\left|T^*(\psi)(v)\right|_{\mathbb{F}} = \left|\psi(T(v))\right|_{\mathbb{F}}$

$$\leq \|\psi\|_{W^*} \|Tv\|_W$$

$$\leq \|\psi\|_{W^*} \, |||T||| \, \|v\|_V$$

$$\therefore \|T^*(\psi)\|_{V^*} = \sup_{\|v\|\leq 1} |T^*(\psi)(v)| \leq \|\psi\|_{W^*} \, |||T|||$$

$$\therefore |||T^*||| = \sup_{\|\psi\|\leq 1} \|T^*(\psi)\|_{V^*} \leq |||T|||$$ □

Remark In fact $|||T^*||| = |||T|||$, see later with Hahn-Banach theorem

FACT-RK $(V, \|\cdot\|_V)$ NVS, then $(V^*, \|\cdot\|_{V^*})$ is again a NVS, so its dual can be defined

TERMINOLOGY Denote $(V^{**}, \|\cdot\|_{**})$ bi-dual of V, i.e. dual of V^*

PROP The "bi-dual embedding" $\Phi : V \longrightarrow V^{**}$,

$$v \longmapsto \Phi(v)$$

$$\Phi(v)(\underset{V^*}{\underset{\cap}{\varphi}}) \doteq \varphi(v)$$ is linear, bdd with $|||\Phi||| \leq 1$

PROOF Linearity by composition (EXE)

Bddness $|||\Phi||| = \sup_{\substack{v\in V \\ \|v\|\leq 1}} \|\Phi(v)\|_{V^{**}} = \sup_{\|v\|\leq 1} \sup_{\|\varphi\|\leq 1} |\Phi(v)(\varphi)|$

$$= \sup_{\|v\|\leq 1} \sup_{\|\varphi\|\leq 1} |\varphi(v)| \leq \|v\| < 1$$ □

Rk Φ is actually isometric (w/ in particular $|||\Phi||| = 1$), and injective See later with Hahn-Banach. However, it is not always surjective.

Examples Finite dimension: V, W fd NVS with bases $(v_i)_{i=1}^n$, $(w_j)_{j=1}^{n'}$, and $T : V \longrightarrow W$ linear ($\Rightarrow$ bdd linear)

And if $v_i^* \in V^*$ s.t. $v_i^*(v_{i'}) = \delta_{ii'}$,

$w_j^* \in W^*$ s.t. $w_j^*(w_{j'}) = \delta_{jj'}$,

(v_i^*), (w_j^*) are bases of V^*, W^*

T written in $(v_i), (w_j)$ is matrix A

T^* written in $(w_j^*), (v_i^*)$ is transposed ${}^tA$

L3.3

Infinite dimension

$\ell^p,\ p \in [1,\infty]$

EXE $\ell^1 \subset \ell^p \subset \ell^\infty$, $p \in [1,\infty]$

$\|(x_n)\|_{\ell^\infty} \leq \|(x_n)\|_{\ell^p} \leq \|(x_n)\|_{\ell^1}$ ← this is just not true? nvm nontrivial XD

- $Id : \ell^1 \to \ell^p$, $p \geq 1$ is bounded linear
- $T : \ell^2 \to \ell^2$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$

  linear, bdd, injective but not surjective
- $M = \{(x_n) \in \ell^2 : \sum_n n^2 x_n^2 < \infty\}$

  EXE $M \subset \ell^2$ dense not closed

  $T : (M, \|\cdot\|_{\ell^2}) \longrightarrow (\ell^2, \|\cdot\|_{\ell^2})$

  $(x_n) \longmapsto (n x_n)$ unbounded
- $Id : (C^1[0,1], \|\cdot\|_{C^0}) \longrightarrow (C^1[0,1], \|\cdot\|_{C^1})$

  linear unbounded

L5.1 (4.1)

Finite dimensional NVS

Announcement: Bookmarks / notes on videos

Blackboard notes

End of Lecture 2b

$C_c(\mathcal{U})$ $\quad \mathcal{V}((\varepsilon_n)_{n\geq 1}) = \{ f \in C_c(\mathcal{U}),\ \|f\|_{K_n^c,\infty} \leq \varepsilon_n \}$

$C_c^\infty(\mathcal{U})$ $\quad \mathcal{V}((\varepsilon_n)_{n\geq 1},(m_n)_{n\geq 1}) = \{ f \in C_c^\infty(\mathcal{U}),\ \sup_{|\alpha|\leq m_n} \|\partial^\alpha f\|_{K_n^c,\infty} \leq \varepsilon_n \}$

$K_n \subset K_{n+1}^\circ$ [L. Schwartz Theory of Distributions]

c'est en français

Introduction

This course is focused on infinite dimension

Discuss how finite dimension is so much simpler from the viewpoint of analysis.

DEFN V VS with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$

These two norms are said to be equivalent $\|\cdot\|_1 \sim \|\cdot\|_2$ if there are $C, C' > 0$ s.t. $\forall v \in V,\ C\|v\|_1 \leq \|v\|_2 \leq C'\|v\|_1$

EXE 1) This defines an equivalence relation on the set of norms on V

2) $\|\cdot\|_1 \sim \|\cdot\|_2 \Rightarrow$ their induced topologies are the same $\tau_1 = \tau_2$

3) Notions of bounded operators for each norm coincide [can see directly or via bdd $\Leftrightarrow$ cts]

4) Set of converging sequences is the same for $\|\cdot\|_1$, $\|\cdot\|_2$ (limits too)

Set of Cauchy sequences is the same as well

Finite dim identity through a basis, $V \simeq \mathbb{F}^n$, n dim, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\|v\|_\infty \doteq \sup_{i=1}^n |v_i|$ where $v = \sum_{i=1}^n v_i e_i$

Prop (i) Let $\|\cdot\|$ norm on $\mathbb{F}^n$, then $\|\cdot\| \sim \|\cdot\|_\infty$

(ii) All norms are equivalent in finite dimension

PROOF (i) $\Rightarrow$ (ii) Easy

(i) $\|v\| = \|\sum v_i e_i\| \leq \sum |v_i| \|e_i\| \leq \|v\|_\infty \underbrace{\left(\sum_{i=1}^n \|e_i\|\right)}_{C'} \leq C' \|v\|_\infty$

$\|\cdot\| : (V, \|\cdot\|_\infty) \to \mathbb{R}_+$

$S_{\|\cdot\|_\infty}(0,1) \longrightarrow \mathbb{R}_+^*$ is continuous

$\| \|v\| - \|w\| \| \leq \|v-w\| \leq C' \|v-w\|_\infty$ so continuous ✓

Next $S_{\|\cdot\|_\infty}(0,1)$ cpct set in $(V, \|\cdot\|_\infty)$

EXE $(v^k)_{k\geq 1}$ in $S_{\|\cdot\|_\infty}(0,1)$ then each $(v_i^k)_{k\geq 1}$ bdd in $\mathbb{F}$, hence has a convergent subsequence:

$(v_1^{\varphi_1(k)})_{k\geq 1}$ CVG for a $\varphi_1 : \mathbb{N}^* \to \mathbb{N}^*$ increasing

$(v_2^{\varphi_1 \circ \varphi_2})_{k\geq 1}$ CVG for $\varphi_2 : \mathbb{N}^* \to \mathbb{N}^*$ increasing too

$\vdots$

$(v_n^{\varphi_1 \circ \dots \circ \varphi_n})_{k\geq 1}$ CVG for $\varphi_n : \mathbb{N}^* \to \mathbb{N}^*$ increasing too

Then v^ψ is CVG in $(V, \|\cdot\|_\infty)$ $\square$

$\|\cdot\| : S_{\|\cdot\|_\infty}(0,1) \to \mathbb{R}_+^*$ attains infimum

$\exists v_0 \in S_{\|\cdot\|_\infty}(0,1)$ s.t. $\|v_0\| = \min_{\|v\|_\infty = 1} \|v\| = C > 0$

$\forall v \neq 0$, $\|v\| = \|v\|_\infty \left\| \frac{v}{\|v\|_\infty} \right\| \geq C \|v\|_\infty$ $\square$

Proof $(V, \|\cdot\|)$ finite dim NVS

Then closed bounded sets are compact [closed bounded indep of norm]

And it is a Banach space

Finally, any ~~set~~ finite dim subspace of an NVS is closed

Proof left as exercise

PROP $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ NVS with V finite dim

Then any $T : V \to W$ linear is bounded

Proof $(e_i)_{i=1}^n$ basis for V

$$\|T(v)\|_W = \left\|T\left(\sum_{i=1}^n v_i e_i\right)\right\|_W \leq \sum_{i=1}^n |v_i| \, \|T(e_i)\| \leq \|v\|_{\infty,V} \left(\sum_{i=1}^n \|T(e_i)\|_W \right)$$

$$\leq C \|v\|_V \left(\sum_{i=1}^n \|T(e_i)\|_W \right) \underset{\substack{\uparrow \\ \text{with} \\ \text{constant}}}{\lesssim} \|v\|_V$$ $\square$

PROP $(V, \|\cdot\|)$ NVS with $\overline{B}(0,1)$ compact

Then V finite dimensional

<u>Proof</u> $\overline{B}(0,1) \subset \bigcup_{v \in \overline{B}(0,1)} \mathring{B}(v, \frac{1}{2})$ open covering

compact $\Rightarrow \exists v_1, \dots, v_n \mid \overline{B}(0,1) \subset \bigcup_{i=1}^{n} \mathring{B}(v_i, \frac{1}{2})$

$$\overline{B}(0,1) \subset \bigcup_{i=1}^{n} \{ v_i + \mathring{B}(0, \tfrac{1}{2}) \}$$
$$\subset \underset{\substack{\| \\ \mathrm{Span}(x_1, \dots, x_n)}}{W} + \mathring{B}(0, 1/2) \subset W + \overline{B}(0, \tfrac{1}{2})$$

<u>Iterate</u> $\overline{B}(0,1) \subset W + \frac{1}{2}\overline{B}(0,1) \subset \dots \subset W + \frac{1}{2^n}\overline{B}(0,1)$
$$\subset \bigcap_{k \geq 1} (W + \overline{B}(0, 2^{-k})) \subset \overline{W} = W \qquad \square$$

<u>EXE</u> Even more subtle properties of finite dim

Series $\sum v_n$ is converging <u>unconditionally</u> (CVGUC)

if $\forall \sigma : \mathbb{N}^* \to \mathbb{N}^*$ bijective, $\sum v_{\sigma(n)}$ CVG

1) Prove that if $\sum v_n$ CVGUC then all limits are the same

2) <u>Finite dim</u> CVGUC $\Rightarrow$ abs. CVG $(*)$
$\mathbb{F} = \mathbb{R}$

3) $(*)$ false in infinite dimension

<u>ZORN'S LEMMA</u> (preliminary to Hahn-Banach)

<u>Intro</u> V NVS $\longrightarrow V^*$ dual

Finite dimension : easy to construct V^*

Infinite dimension : not clear that $V^* \neq \{0\}$

Hahn-Banach Theorem answers that by constructing "little by little" a linear map $V \to \mathbb{F}$ that respect some "boundedness"

HB Thm will show that V^* is "large enough" to separate well V

Two main ingredients in Hahn-Banach

(1) Extending a continuous linear map $V \to \mathbb{F}$ to $W \supset V$ with codimension 1

(2) Transfinite induction

<u>Rk</u> If $V = \bigcup_{n \geq 1} V_n$, countably many vs, $V_n \subset V_{n+1}$, V_n dimension n, could replace transfinite induction by regular induction.

L4.4

<u>Def</u> S set is <u>partially ordered</u> if endowed with a binary relation $\leq$ s.t.

$\forall x, y \in S$, $x \leq y$ or $x \not\leq y$ (wat)

$\forall x \in S$, $x \leq x$

$\forall x, y, z \in S$, $x \leq y$, $y \leq z \Rightarrow x \leq z$

$\forall x, y \in S$, $x \leq y$, $y \leq x \Rightarrow x = y$

$(S, \leq)$ a partially ordered set

\# Subset $S' \subset S$ is <u>totally ordered</u> if

$\forall x, y \in S'$, $x \leq y$ or $y \leq x$

\# $S' \subset S$, $b \in S$ is an <u>upper bound</u> of S' if

$\forall x \in S'$, $x \leq b$

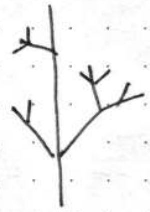
\# $S' \subset S$, $l \in S$ is a <u>least upper bound</u> of S' if

l is an upper bound of S', and for all b' upper bounds of S', $l \leq b'$

\# $(S, \leq)$ is said to have the <u>least upper bound property</u> (LUBP) if

every $S' \subset S$ non-empty, totally ordered has a least upper bound

\# $m \in S$ is <u>maximal</u> if $\forall x \in S$, $x \geq m \Rightarrow x = m$



<u>Lemma</u> (Zorn) $(S, \leq)$ partially ordered, non-empty with the least upper bound property, then there is a maximal element $m \in S$

<u>Application to construction of bases</u>

<u>Prop</u> $V \neq \{0\}$ is a vector space, then V has basis,

Moreover, $S \subset V$ lin indep $\Rightarrow \exists$ basis β containing S

<u>Proof</u> (ii) $\Rightarrow$ (i), $S = \{v_0\}$, $v_0 \neq 0$

<u>Proof of (ii)</u> $\mathcal{S} = \{$ linearly independent S' of V containing $S \}$

$\mathcal{S} \neq \emptyset$, $\mathcal{S} \ni S$

$(\mathcal{S}, \subset)$ is a partially ordered set (EXE)

$\mathcal{T} \subset \mathcal{S}$ non-empty totally ordered, then it has the LUB

$$\bar{S} = \bigcup_{S' \in \mathcal{T}} S'$$

$\bar{S} \supset S$, $\bar{S}$ still linearly indep, for if $\sum_{i=1}^{n} \alpha_i v_i = 0$ with $v_i \in \bar{S}$, can find a single $S' \in \mathcal{T}$ to which the v_i all belong

- S therefore satisfies assumption of Zorn's lemma, and has a maximal element $\mathcal{B}$
- $\mathcal{B}$ is lin indep, and it spans V

Else $v_0 \in V \setminus \mathrm{span}(\mathcal{B})$ and $\overline{\mathcal{B}} = \mathcal{B} \cup \{v_0\}$ would still be lin indep, contradicting maximality of $\mathcal{B}$. $\square$

EXE (harder) Show that any two bases have the same cardinality

L5.1

The Hahn-Banach Theorem

Intro HB theorem is about building linear functionals and showing that the dual space is large. NVS based on $\mathbb{R}$ here.

Thm (HB) V NVS with $p: V \to \mathbb{R}$ s.t.

$$p(v_1+v_2) \leq p(v_1)+p(v_2), \quad \forall\, v_1, v_2 \in V$$

$$p(\lambda v) = \lambda p(v), \quad \forall\, v \in V,\ \forall\, \lambda > 0$$

W subspace of V, f linear on W with $f(w) \leq p(w)$ $\forall\, w \in W$

Then $\exists\, \tilde{f}$ extending f, $(\tilde{f}|_W = f)$ linear, with $\tilde{f}(v) \leq p(v)$ $\forall\, v \in V$

Proof Step 1 The codimension 1 case

$V = \operatorname{span}(v_0, W)$, $v_0 \in V \setminus W$

Any $v \in V$ writes $v = w + av_0$, $a \in \mathbb{R}$, $w \in W$ (uniquely)

To extend f from W to V, ST specify $\tilde{f}(v_0)$, since then

$$\tilde{f}(v) = \tilde{f}(w+av_0) = \tilde{f}(w) + a\tilde{f}(v_0)$$

We want $\tilde{f}(v_0)$ s.t.

$$\underbrace{\tilde{f}(w+av_0)}_{f(w)+a\tilde{f}(v_0)} \leq p(w+av_0) \qquad \forall\, w \in W,\ a \in \mathbb{R}$$

$a>0$ $\tilde{f}(v_0) \leq \frac{1}{a}\left[p(w+av_0) - f(w)\right]$

$$= p\left(\tfrac{w}{a}+v_0\right) - f\left(\tfrac{w}{a}\right)$$

i.e. $\tilde{f}(v_0) \leq p(w'+v_0) - f(w')$ $\quad \forall\, w' \in W$ $\qquad$ (1)

$a<0$ $\tilde{f}(v_0) \geq \frac{1}{a}\left[p(w+av_0) - f(w)\right]$

$$= -p\left(-\tfrac{w}{a}-v_0\right) + f\left(-\tfrac{w}{a}\right)$$

i.e. $\tilde{f}(v_0) \geq f(w'') - p(w''-v_0)$ $\quad \forall\, w'' \in W$ $\qquad$ (2)

(1),(2) $\Leftrightarrow$ $\sup_{w'' \in W}\left[f(w'') - p(w''-v_0)\right] \leq \inf_{w' \in W}\left[p(w'+v_0) - f(w')\right]$

'RHS' holds since $f(w'') - p(w''-v_0) \leq p(w'+v_0) - f(w')$ $\quad \forall\, w', w'' \in W$,

for it's just $f(w'') + f(w') \leq p(w''-v_0) + p(w'+v_0)$

$$\underset{\displaystyle f(w''+w')}{\parallel} \qquad \leq \qquad \underset{\displaystyle p(w''+w')}{\vee\!/}$$

Step 2 The general case (transfinite induction)

$\mathcal{J} = \{$ all (lin) extensions $(\tilde{f}, \tilde{V})$ of (f, V) s.t.

$(\tilde{f}|_V = f,\ V \subset \tilde{V}) \quad \tilde{f} \leq p \}$

$\mathcal{J}$ non-empty : $(f, V) \in \mathcal{J}$, partially ordered by extension

$\mathcal{J}$ has LUBP, if $\mathcal{C} \subset \mathcal{J}$ totally ordered non-empty then

$\tilde{V}_0 = \bigcup_{(\tilde{f}, \tilde{V}) \in \mathcal{C}} \tilde{V}$ and $\tilde{f}_0$ defined by $\forall v \in \tilde{V}_0$ s.t. $v \in \tilde{V}$ for $(\tilde{f}, \tilde{V}) \in \mathcal{C}$,

$\tilde{f}_0(v) \doteq \tilde{f}(v)$

Well-defined because of total order
Upper bound, least one
} exercises (also prove it's in $\mathcal{J}$)

Therefore Zorn's Lemma implies $\exists\ (\tilde{f}, \tilde{V})$ maximal element in $\mathcal{J}$

Let us prove $\tilde{V} = V$.

Otherwise $\exists\ v_0 \in V \setminus \tilde{V}$ and Step 1 with $W = \tilde{V}$ applies and allows to construct $\tilde{\tilde{f}}$ extending $\tilde{f}$ to $\mathrm{span}(\tilde{V}, v_0)$ with $\tilde{\tilde{f}} \leq p$

$(\tilde{\tilde{f}}, V) \in \mathcal{J}$ and is strictly $>$ for the partial order than $(\tilde{f}, \tilde{V})$ ⋇

Therefore $\tilde{V} = V$ and $\tilde{f}$ is as desired. $\square$

Rk Conditions on the bound p are reminiscent but weaker than those of a semi-norm : p valued in $\mathbb{R}$, not necessarily $\mathbb{R}_+$

$p(\lambda v) = \lambda p(v)$ only for $\lambda > 0$

Consequences of Hahn-Banach Theorem

Intro - Dual Space - Separation by hyperplanes

Prop (i) V NVS, $W \leq V$, $f \in W^*$

Then $\exists\ \tilde{f} \in V^*$ with $\tilde{f}|_W = f$ and $\|\tilde{f}\|_{V^*} = \|f\|_{W^*}$

(ii) $V \neq \{0\}$ then $V^* \neq \{0\}$

(iii) If $v, w \in V$, $v \neq w$ then $\exists\ f \in V^*$ s.t. $f(v) \neq f(w)$

Proof (i) Apply HB with $p(v) \doteq \|f\|_{W^*} \cdot \|v\|_V$

(ii) Consider $v_0 \in V \setminus \{0\}$ and define f on $\mathbb{R}v_0 = W$ linear with $f(v_0) = \|v_0\|$, Then $\|f\|_{W^*} = 1$ and (i) constructs an $\tilde{f}$ such that

$|\tilde{f}(v)| \leq \|v\| \ \forall v \in V$, $|\tilde{f}(v_0)| = \|v_0\| \implies \|\tilde{f}\|_{V^*} = \|f\|_{W^*} = 1$

Rk This particular $\tilde{f}$ is known as a support functional at v_0, denote f_{v_0}

(iii) Apply (ii) to $v - w \neq 0$ $\square$

Prop $\Phi : V \to V^{**}$, $v \mapsto \Phi(v)$ s.t. $\Phi(v)(f) = f(v)$ for $f \in V^*$

Φ is an isometry ($|||\Phi||| = 1$ in particular) ⌈Recall $|||\Phi||| \leq 1$⌋

Proof Given $v \in V$, $\|v\|_V = 1$,

$$\underset{\substack{\uparrow \\ \text{supp functional;} \\ \|f_v\|_{V^*} = 1 \\ f_v(v) = \|v\|}}{\Phi(v)(f_v)} = f_v(v) = \|v\| = 1$$

So $\sup_{\|f\|_{V^*} \leq 1} \Phi(v)(f) \geq 1$ i.e. $\|\Phi(v)\|_{V^{**}} = 1$ ⌈Recall $\|\Phi(v)\| \leq \|v\|$⌋ $\square$

Prop V, W NVS, $T : V \to W$ bded linear

Then $T^* : W^* \to V^*$ adjoint satisfies $|||T^*||| = |||T|||$

Proof We proved earlier that $|||T^*||| \leq |||T|||$

Consider $v \in V$, $\|v\| = 1$, $w = Tv \in W$ $(w \neq 0)$ ⌈If $T = 0$ then $T^* = 0$⌋

$$T^*(f_w)(v) = f_w(T(v)) = f_w(w) = \|w\|$$

supp funct at w $\begin{cases} \|f_w\|_{W^*} = 1 \\ f_w(w) = \|w\| \end{cases}$

Hence $\|T^*(f_w)\|_{V^*} \geq \|w\|_W$; and $\|f_w\|_{W^*} = 1$

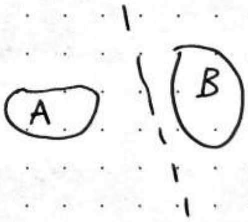
$$\Rightarrow |||T^*||| = \sup_{\|f\|_{W^*} \leq 1} \|T^*(f)\| \geq \|w\|_W = \|Tv\|_W$$

$$\Rightarrow |||T^*||| \geq \sup_{\|v\| = 1} \|Tv\|_W = |||T|||$$ $\square$

Geometric form of HB

Thm (HB bis) V NVS

(i) $\left.\begin{array}{l} A \subset V \text{ open cvx not empty} \\ B \subset V \quad \text{cvx not empty} \end{array}\right\} A \cap B = \emptyset$



There is a closed hyperplane weakly separating A, B.

i.e. $\exists \underset{f \neq 0}{f \in V^*}$, $\alpha \in \mathbb{R}$ s.t. $\sup_A f \leq \alpha \leq \inf_B f$

(ii) $\left.\begin{array}{l} A \subset V \text{ closed cvx not empty} \\ B \subset V \ \underline{\text{cpct}} \text{ cvx not empty} \end{array}\right\} A \cap B = \emptyset$

L5.4

There is a closed hyperplane strictly separating A and B,

i.e. $\exists\, f \in V^*, \alpha \in \mathbb{R}, \varepsilon > 0$ s.t.

$$\sup_A f \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \inf_B f$$

<u>EXE</u> 1) A hyperplane being defined as $\{\overset{H}{\overset{\shortparallel}{f = \alpha}}\}$ for some f linear and $\alpha \in \mathbb{R}$, then H is closed iff f is continuous

2) C open cvx (not empty) containing 0 then

$p \doteq \mu_C, \quad \mu_C(v) = \inf\{t > 0, \frac{v}{t} \in C\}$

is well-defined $p : V \to \mathbb{R}_+$ and such that $p(v_1+v_2) \leq p(v_1) + p(v_2)$, $p(\lambda v) = \lambda p(v)$ for $\lambda > 0$,

and $\exists\, M > 0$ s.t. $0 \leq p(v) \leq M\|v\| \quad \forall v \in V$

<u>Proof</u> (i) $C_0 = A \setminus B \overset{\text{is}}{}$ not empty <u>open</u> (EXE) (I think he means $A - B$)

and $0 \notin C_0 \quad (A \cap B = \emptyset)$

Let $v_0 \in C_0$ and $C = C_0 - v_0$

Now C as in exercise above, and $p \doteq \mu_C$ satisfies assumptions of HB & $0 \leq p(v) \leq M\|v\|$ for some M

The define f linear on $\mathbb{R}v_0 = W$ by $f(v_0) = -1$

HB $\Rightarrow \exists\, \tilde{f}$ on V extending f with $\tilde{f} \leq p$

$\tilde{f}|_C < 1 \quad (p(v) < 1 \ \forall v \in C)$

$\tilde{f}|_{C_0} < 0 :\ c \in C_0$ writes as $c_0 - v_0,\ c_0 \in C_0$

This means $\tilde{f}(a) - \tilde{f}(b) < 0 \quad \forall a \in A, \forall b \in B$

$\sup_A \tilde{f} \leq \inf_B \tilde{f}$ concludes the proof

WOAH

(ii) $C = B - A$, cvx non-empty and <u>closed</u>

<u>Closed</u> since if $(b_n - a_n)_{n \geq 1}$ sqce in $B - A$ converges,

have subseq $b_{\varphi(n)}$ converging in B, then by difference $a_{\varphi(n)}$ converges

now A closed $\Rightarrow a_{\varphi(n)} \to a \in A,\ b_{\varphi(n)} \to b \in B \Rightarrow b_{\varphi(n)} - a_{\varphi(n)} \to b - a \in B - A$

<u>$0 \notin C$</u> $(A \cap B = \emptyset)$

$\exists\, \mathring{B}(0, r) \subset C^c$ for some $r > 0$

Apply (i) to separate (weakly)

$\mathring{B}(0,r)$ open cvx non-empty

C cvx non-empty

$\exists\ f \in V^*,\ \alpha_0 \in \mathbb{R}$ s.t. $\cancel{\sup_f \mathring{B}}$ $\underbrace{\sup_{\mathring{B}(0,r)} f}_{r\|f\|_{V^*}} \leq \alpha_0 \leq \inf_C f$

$(f \neq 0)$

$[\alpha_0 \geqslant r\|f\|_{V^*} > 0]$

$\inf_C f \geqslant \alpha_0 > 0 \Rightarrow \sup_A f + \alpha_0 \leq \inf_B f$

Then one can find $\alpha \in \mathbb{R}, \varepsilon > 0$ as in the statement □

Rk Zorn's lemma not really considered "practical"

HB theorem nice and general but for concrete spaces we usually know the dual already or we can build large subspaces of it

EXPLES 1) $(\ell^p)^* = \ell^q$, $p \in q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$

2) $(C([0,1]), \|\cdot\|_\infty)^*$ part of it represented by

g fct continuous on $[0,1]$

through $f \mapsto \int_0^1 fg$

Then norm is $\int |g|$

However $(C[0,1], L^1)$ not complete so cannot be the dual

[It does not represent $f \in V \mapsto f(\frac{1}{2})$]

In fact the dual is {finite signed Borel measures} normed with total variation

$\cancel{\tilde{f}(c) = \tilde{f}(c_0) - \tilde{f}(x_0) \leq 1 - 1 = 0}$

L6.1

The Baire Category Theorem

Intro Last lecture(s) exploited the structure of convexity of norm of a NVS (in other words the convexity of unit ball), see formulation of HB thm with p function, to prove that dual V^* is "big", or equivalently (geometric HB); space V has plenty of closed hyperplanes to separate sets.

* Today we shall exploit the structure of completeness to prove complete metric spaces (and other subsets of NVS...) are necessarily "big" by virtue of "having no holes".

Baire useful for proving existence of objects or local-to-global bounds, see next lectures

Defn (X, τ) topological space

1) $B \subset X$ is rare or nowhere dense in X if
$\overline{B}$ has empty interior, i.e.
(EXE) $\forall\ \mathcal{U} \subset X$ open, $B \cap \mathcal{U}$ not dense in $\mathcal{U}$

2) $B \subset X$ is meagre in X (or of first category in X) if it is a countable union of rare sets.
Otherwise it is non-meagre (or co-meagre), or of second category

3) (X, τ) is said to be meagre (or of first cat) if it is meagre in itself, otherwise it is non-meagre / comeagre / of second category

EXE (X, τ) of second category iff either of the following statements is true: (1) $\forall\ (\mathcal{U}_n)_{n \geq 1}$ countable collection of open dense subsets of X,
$\bigcap_{n \geq 1} \mathcal{U}_n$ is non-empty
(2) $\forall\ (C_n)_{n \geq 1}$ countable collection of closed sets s.t. $\bigcup_{n \geq 1} C_n = X$,
$\exists n_0 \geq 1$ s.t. C_{n_0} has non-empty interior

Prop (X, d) complete metric space is of second category

In fact, it is a "Baire space": a countable intersection of open dense subsets is dense in X

L6.2

Proof Consider $(U_n)_{n\geq 1}$ countable collection of dense open subsets, and O an arbitrary open subset of X, and let us prove

$$\left(\bigcap_{n\geq 1} U_n\right) \cap O \neq \emptyset$$

Construct by induction $x_1 \in U_1 \cap O$ (U_1 dense)

- $\overline{B(x_1, r_1)} \subset U_1 \cap O$ ($U_1 \cap O$ open)
- $x_2 \in U_2 \cap B(x_1, r_1)$ (U_2 dense)
- $\overline{B(x_2, r_2)} \subset U_2 \cap B(x_1, r_1)$ ($U_2 \cap B(x_1, r_1)$ "open"), $r_2 \leq \frac{r_1}{2}$

$\vdots$

- $\overline{B(x_{k+1}, r_{k+1})} \subset U_{k+1} \cap B(x_k, r_k)$, $r_{k+1} \leq \frac{r_k}{2}$

$r_k \leq \frac{r_1}{2^{k-1}}$ so $(x_k)_{k\geq 1}$ is Cauchy:

$$\forall k_1, k_2 \geq k_0, \quad d(x_{k_1}, x_{k_2}) \leq \frac{r_1}{2^{k_0 - 1}} \longrightarrow 0 \text{ as } k_0 \to \infty$$

Completeness $\Rightarrow$ $x_k \xrightarrow[k\to\infty]{} l \in X$ and

$$l \in \bigcap_{n\geq 1} U_n, \quad l \in B(x_1, r_1) \subset O$$ «bruh» □

EXE Prove cpct Hausdorff topological spaces are also Baire spaces pretty cool ✓

EXAMPLES 1) Existence of irrationals in $\mathbb{R}$

$(\mathbb{R}, |\cdot|)$ is a complete metric space and each point $\{x\}$ is closed with empty interior$^{(\text{in } \mathbb{Q})}$. Therefore $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meagre, proper subset

EXE Prove that every complete metric space with no isolated point is unctble

[Isolated point $x_0 \iff \{x_0\}$ is open]

2) Existence of continuous f^ns nowhere differentiable

Prop There is $f \in C([0,1])$ that is nowhere differentiable

Pf (EXE) We can prove more with Baire theory:

$$D \doteq \{ f \in C([0,1]) \text{ s.t. } \exists x \in [0,1] \text{ s.t. } f \text{ diff at } x \}$$

is meagre (first category) in the Baire space $(C([0,1]), \|\cdot\|_\infty)$

Steps (i) Prove $D \subset \bigcup_{m,n \geq 1} A_{mn}$ where

$$A_{mn} = \{ f \in C([0,1]),\ \exists x \in [0,1] \text{ with } \forall y \in [0,1] \cap [x - \tfrac{1}{m}, x + \tfrac{1}{m}],\ |f(x) - f(y)| \leq n|x-y| \}$$

L6.3

(ii) $A_{m,n}$ is closed in $C([0,1])$ can circumvent (hard methinks) assuming ← mm hmm $\exists t$'s $C\{\epsilon\}$ (D?)

(iii) $A_{m,n}$ has empty interior in $C([0,1])$ Mouhot: "actually quite easy"

(use the subset of piecewise linear functions)

(iv) Conclude with BCT, ⌜from Wikipedia, the free encyclopaedia, at en.wikipedia.org, ... ⌟

Remark Give another constructive proof of Prop inspired by (iii)

Relation between Baire theory and Measure theory in terms of measuring (quantifying) "smallness"

Small for Baire = meagre

Small for Lebesgue measure = null set, i.e. a set that can be covered by ctble union of intervals with arbitrarily small total length ⌜outer measure⌟

Ex $\{0\}$ is meagre and null set in $\mathbb{R}$

$\mathbb{Q}$ is meagre and null set in $\mathbb{R}$

However these two notions do not coincide, there are meagre sets which have positive measure and null sets of second category

Illustration $\mathbb{Q} = \{q_k\}_{k\geq 1}$

$n \geq 1$, $\quad D_n \doteq \bigcup_{k\geq 1} I_{k,n}$, $\quad I_{k,n} = (q_k - \frac{1}{2^{n+k}}, q_k + \frac{1}{2^{n+k}})$

Total length $|D_n| \leq \sum_{k\geq 1} |I_{k,n}| \leq 2 \sum_{k\geq 1} \frac{1}{2^{n+k}} = \frac{1}{2^{n-1}}$

D_n is open, dense in $\mathbb{R}$

$\bigcap_{n\geq 1} D_n$ is a null set but is 2nd category (call it D)

D^c full measure but is meagre ← this is the wonky one

Exe 1) Prove that if $f: [0,1] \to \mathbb{R}$ is pointwise limit of $f_n: [0,1] \to \mathbb{R}$ with each f_n cts, then set of points where f is discontinuous is meagre

2) Reciprocal is true (pretty hard) If f s.t. $\{$discontinuity of $f\}$ is meagre then f is pointwise limit of continuous functions

L7.1 **Uniform Bddness Ppl & Open Mapping Theorem**

__Intro__ # Last lecture Baire Cat Thm at abstract level + exples

Today combine Baire & linearity: linear maps in NVS

__Theorem__ (Unif Bddness Ppl; Banach 1922, Hahn 1922, Banach-Steinhaus 1927)

V, W Banach spaces

(i) Given $(T_i)_{i \in I}$ collection (not necessarily countable) of bounded linear maps $\in \mathcal{B}(V,W)$ such that is locally bdd

$$\forall v \in V, \quad \sup_{i \in I} \|T_i v\|_W < \infty$$

Then $\sup_{i \in I} |||T_i||| < \infty \quad \left(\sup_{i \in I} \sup_{\|v\|=1} \|T_i v\|_W < \infty \right)$

(ii) Given $(T_n)_{n \geq 1}$ sequence in $\mathcal{B}(V,W)$ converging pointwise to $L \in \mathcal{L}(V,W)$

i.e. $\forall v \in V, \quad \|T_n v - Tv\| \xrightarrow[n\to\infty]{} 0$

Then T is bounded, with $|||T||| \leq \overset{\text{lim inf}}{\underline{\lim}} |||T_n|||$, $T \in \mathcal{B}(V,W)$

(iii) $B \subset V$ is bounded iff $\forall f \in V^*$, $f(B)$ bdd (in $\mathbb{R}$)

(iv) $B' \subset V^*$ is bounded iff $\forall v \in V$, $\Phi(v)(B')$ bdd in $\mathbb{R}$

__Proof__ "(i) $\Rightarrow$ (ii)"

Aply (i) to $(T_n)_{n \geq 1}$; use pointwise convergence implies local bddness (DETAILS ARE A THING)

"(i) $\Rightarrow$ (iii)"

$\Rightarrow$ is trivial, $\Leftarrow$ apply (i) with Banach spaces V^* and $\mathbb{R}$,

and collection of linear maps $(\Phi(v))_{v \in B}$

$$\forall f \in V^*, \ |\Phi(v)(f)| = |f(v)| \leq \sup_{v \in B} |f(v)| < \infty$$

local bddness $\underset{(i)}{\Longrightarrow} \sup_{v \in B} \|\Phi(v)\|_{V^{**}} < \infty \underset{\Phi \text{ isom}}{\Longrightarrow} \sup_{v \in B} \|v\| < \infty$

"(i) $\Rightarrow$ (iv)"

$\Rightarrow$ is trivial, $\Leftarrow$ apply (i) to Banach spaces V and $\mathbb{R}$,

and collection of linear maps $(f)_{f \in B'}$

$\therefore \sup_{f \in B'} |\Phi(v)(f)| = \sup_{f \in B'} |f(v)| < \infty$ for each v

deduce from (i), $\sup_{f \in B'} \|f\|_{V^*} < \infty \Rightarrow B'$ bounded

Finally, let us prove (i)

Define $C_n = \{ v \in V : \forall i \in I, \ \|T_i v\|_W \leq n \}$

Then C_n is closed using continuity of each T_i (intersecc)

Local boundedness $\Rightarrow$ $V = \bigcup_n C_n$

BCT $\Rightarrow$ $\exists n_0 \geqslant 1$ s.t. C_{n_0} has non-empty interior

$\exists v_0 \in V, \ \varepsilon > 0$ s.t. $\forall i \in I, \ \forall v \in B(v_0, \varepsilon), \ \|T_i v\|_W \leq n_0$

$\therefore \ \forall v' \in B(0, \varepsilon), \ \forall i \in I, \ \|T_i v'\|_W \leq n_0 + \|T_i v_0\|$

$\therefore \ |||T_i||| \leq \varepsilon^{-1} \big[n_0 + \sup_{i \in I} \|T_i v_0\|_W \big] < \infty$

indep of i.

$\square$

Rmks 1) Points (ii) - (iii) - (iv) not always included, but give nice illustrations

2) Point (iii) means "strongly bdd" <=> "weakly bounded";

it replaces the idea of checking that each coordinate is bounded

EXE Prove, using (iii) above, that if $(v_n)_{n \geqslant 1}$ in V Banach space is "weakly converging" to v : $\forall f \in V^*$, $f(v_n - v) \longrightarrow 0$,

then $(v_n)_{n \geqslant 1}$ is bdd in V

Thm (Open Mapping Thm, consequences, Banach - Schauder)

V, W Banach spaces

(i) If $T \in \mathcal{B}(V, W)$ is surjective then T is an open map,

i.e. T maps open sets to open sets

(ii) (Inverse Mapping Theorem) If $T \in \mathcal{B}(V, W)$ is bijective, then

T^{-1} is bdd linear

(iii) (Closed Graph Theorem) $T \in L(V, W)$ is bounded iff

$\mathrm{Graph}(T) = \{ (v, T(v)) \in V \times W : v \in V \}$ is closed

(note $V \times W$ Banach space for $\|(v, w)\| = \|v\|_V + \|w\|_W$)

Proof (i) $\Rightarrow$ (ii) Just need T^{-1} bdd <=> T^{-1} cts <=> $(T^{-1})^{-1}$ open = open

(ii) $\Rightarrow$ (iii) Part bdd => closed clear from continuity of T

Part closed $\Rightarrow$ bdd via $\mathcal{P} : \mathrm{Graph}(T) \longrightarrow V$, $(v, Tv) \mapsto v$

Observe $\mathcal{P}$ is linear, bijective, bdd

If $\mathrm{Graph}(T)$ is closed then it is ~~Banach~~ Banach $\Rightarrow$ $\mathcal{P}^{-1}$ bdd $\Rightarrow$ T bdd

Proof of (i):

EXE $T \overset{\text{lin map}}{}$ open iff $\exists r>0$ s.t. $T(B(0,1)) \supset B(0,r)$

Note that $W = \bigcup_{n\geq 1} T(B(0,n))$ (surjective)

$$= \bigcup_{n\geq 1} n\, T(B(0,1))$$

$$= \bigcup_{n\geq 1} n\, \overline{T(B(0,1))}$$

BCT $\Rightarrow$ $\exists n_0 \geq 1$ s.t. $\overline{n_0\, T(B,01)}$ has non-empty interior

$\Rightarrow \overline{T(B(0,1))}$ has non-empty interior,

$\exists w_0 \in W, r>0$ s.t.

$$\overline{T(B(0,1))} \supset w_0 + B(0;2r)$$

Using $\overline{T(B(0,1))} = -\overline{T(B(0,1))}$ and convexity

$$\overline{T(B(0,1))} \supset \frac{1}{2}(w_0 + B(0,2r)) + \frac{1}{2}(-w_0 + B(0,2r))$$

$$\supset B(0;2r)$$

Finally let us prove that $T(B(0,1)) \supset B(0,r)$

$w \in B(0,r) \Rightarrow w \in \overline{T(B(0,\tfrac{1}{2}))}$,

hence $\exists v_1 \in B(0,\tfrac{1}{2})$ s.t. $\|\underbrace{w_1 - Tv_1}_{w_2}\|_W < \frac{r}{2}$ ($w_1 = w$)

$w_2 \in B(0;\frac{r}{2}) \Rightarrow \exists v_2 \in B(0,\frac{1}{4})$ s.t. $\|\underbrace{w_2 - Tv_2}_{w_3}\|_W < \frac{r}{4}$

$\vdots$

$w_k \in B(0,\frac{r}{2^{k-1}}) \Rightarrow \exists v_k \in B(0,2^{-k})$ s.t. $\|w_k - Tv_k\|_W < \frac{r}{2^k}$

Define $\bar{v} = \sum_{n\geq 1} v_n$, $\|\bar{v}\|_V < 1$, $T(\bar{v}) = w$

because $\|w - T(\sum_{n=1}^{N} v_n)\|_W < \frac{r}{2^N} \to 0$ $\square$

Rmks 1) Completeness enters the proof both for V and W but differently,
V complete so $\sum_{n\geq 1} v_n$ converges, W complete then BCT

2) Point (iii) is an improvement for proving continuity of linear maps between Banach spaces; only need

$$\left.\begin{matrix} v_n \to v \\ Tv_n \to w \end{matrix}\right\} \Rightarrow w = Tv$$

L 8.1

Topology of $C(K)$ – Extension Theorems

Intro # After several abstract thms, turn to study more concrete spaces

$C(K)$ fundamental in functional analysis, and we will illustrate that with applications, examples

First of a series of lectures on $C(K)$:

- extension thms: $C(K)$ is rich enough to separate points and construct many fncts
- Arzela-Ascoli thm: characterizing compact sets in $C(K)$
  apply to Cauchy-Peano thm in ODEs
- Stone-Weierstrass thm: abstract method for finding large and useful dense subsets
  Application to Fourier series ☺

DEFN (Axioms of separation)

(X, τ) a topological space is

- T_0 if any two distinct points are topologically distinguishable (i.e. do not have the exact same neighbourhoods)
- T_1 if any two distinct points are separated, as in $x_1 \notin \overline{\{x_2\}}$, $x_2 \notin \overline{\{x_1\}}$
- T_2 (Hausdorff) if any two distinct points x_1, x_2 are separated by neighbourhoods: $\exists$ open U_1, U_2 s.t. $x_i \in U_i$, $U_1 \cap U_2 = \phi$
- Normal if any two disjoint, non-empty, closed subsets C_1, C_2 can be separated by ngbds: $\exists$ open U_1, U_2 s.t. $C_i \subseteq U_i$, $U_1 \cap U_2 = \phi$

EXE Prove (X, τ) normal iff $\forall\, C_1 \subset U_1$, C_1 closed non-empty, U_1 open there exist U_2 open, C_2 closed with $C_1 \subset U_2 \subset C_2 \subset U_1$

PROP (K, τ) compact Hausdorff top space

Then it is normal

Proof Consider C_1, C_2 closed non-empty disjoint in K

K cpct $\Rightarrow C_1, C_2$ are cpct

$\forall x \in C_1, y \in C_2$ apply Hausdorff to get

U^1_{xy}, U^2_{xy} open s.t. $x \in U^1_{xy}$, $y \in U^2_{xy}$, $U^2_{xy} \cap U^1_{xy} = \phi$

Fix $y \in C_2$. Then $C_1 \subset \bigcup_{x \in C_1} \mathcal{U}^1_{xy}$ open cover

$\underset{\text{cpct}}{\Rightarrow}$ $\exists x_1, \dots, x_m \in C_1$

s.t. $C_1 \subset \underbrace{\bigcup_{i=1}^{m} \mathcal{U}^1_{x_i y}}_{\text{open}}$ $\leftarrow$ finite — call this V^1_y

$y \in V^2_y := \bigcap_{i=1}^{m} \mathcal{U}^2_{x_i y}$ finite intersection $\Rightarrow$ open

$C_2 \subset \bigcup_{y \in C_2} V^2_y \underset{\text{cpct}}{\Rightarrow} C_2 \subset \underbrace{\bigcup_{j=1}^{n} V^2_{y_j}}_{W^2 \text{ open}}$

$W^1 =: \bigcap_{j=1}^{n} V^1_{y_j}$ finite intersection $\Rightarrow$ open

$C_1 \subset V^1_{y_j} \Rightarrow C_1 \subset W^1$, and W^1 as desired $\square$

<u>Rk</u> Note we only used cptness of C_1, C_2 not X itself

(could separate by nghbds any two cpct subsets of X Hausdorff)

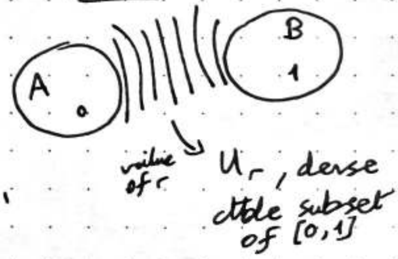
<u>Lemma</u> (Urysohn) $(X, \mathcal{T})$ normal topological space,

A, B disjoint non-empty closed subsets

$\exists f \in C(X)$ with range $[0,1]$ s.t.

$f = 0$ on A, $f = 1$ on B

<u>Idea</u>



<u>Proof</u> <u>Step 1</u> Interpolating between two open sets $\leftarrow$ need $\overline{\mathcal{U}_0} \subset \mathcal{U}_1$

$\mathcal{U}_0 \subset \mathcal{U}_1$ open sets, not ϕ, ~~not X~~, closure not X,

Then $\exists\, \mathcal{U}_{1/2}$ open s.t. $\mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}_{\frac12} \subset \overline{\mathcal{U}_{\frac12}} \subset \mathcal{U}_1$

<u>Indeed</u>, Normality with $C_1 = \overline{\mathcal{U}_0}$, $C_2 = {}^c\mathcal{U}_1$

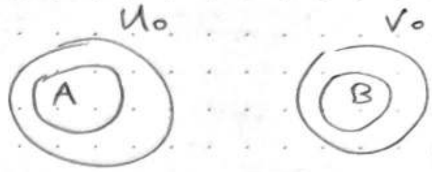
closed, non-empty, disjoint, $\exists\, \mathcal{U}_{\frac12}, V_{\frac12}$ open such that $\mathcal{U}_{\frac12} \cap V_{\frac12} = \phi$

$\overline{\mathcal{U}_0} \subset \mathcal{U}_{\frac12}$, ${}^c\mathcal{U}_1 \subset V_{\frac12}$

$\overline{\mathcal{U}_{\frac12}} \subset {}^cV_{\frac12} \subset \mathcal{U}_1$

Step 2 | Inductive construction

Set $\mathcal{U}_0, V_0$ open sets given by normality with



$C_1 \doteq A,\ C_2 = B$

Define $\mathcal{U}_1 = {}^cB$ open

Then $\mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}_1$ (since $\overline{\mathcal{U}_0} \subset {}^cV_0 \subset \mathcal{U}_1$)

Apply Step 1 inductively to construct $\mathcal{U}_r$ for all $r \in D = \{\frac{k}{2^n};\ n \geq 1,\ k = 0, \dots, 2^n\}$

with $\overline{\mathcal{U}_p} \subset \mathcal{U}_q$ for $p < q$ in D

$A \subset \mathcal{U}_p\ \forall p \in [0,1]$

$\mathcal{U}_p \cap B = \emptyset\ \ \forall p \in [0,1]$

Step 3 | Construction of the f^n

$$f(x) \doteq \begin{cases} \inf \{q \in D,\ x \in \mathcal{U}_q\}, & x \in X \setminus B = \mathcal{U}_1 \\ 1, & x \in B \end{cases}$$

Well-defined $x \in X \setminus B$, then $x \in \mathcal{U}_1$ at least $\qquad (\overline{D} = [0,1])$

$f = 0$ on A since $A \subset \mathcal{U}_0$

Step 4 | f is continuous

Equivalent to $\forall a \in [0,1),\ f^{-1}((a,1])$ open $\quad (1)$

and $\forall b \in (0,1],\ f^{-1}([0,b))$ open $\quad (2)$

(1) $x \in f^{-1}((a,1]) \Rightarrow f(x) > q' > q > a$ for some dyadic $q, q' \in D$

$\Rightarrow x \notin \mathcal{U}_{q'} \Rightarrow x \notin \overline{\mathcal{U}_q}$

$\Rightarrow x \in {}^c(\overline{\mathcal{U}_q}) \subset f^{-1}((a,1])$

(2) $x \in f^{-1}([0,b)) \Rightarrow f(x) < q < b$ for some $q \in D$

$\Rightarrow x \in \mathcal{U}_q \subset f^{-1}([0,b))$ $\square$

Corollary (K, τ) cpct Hausdorff, then $C(K)$ separates points:

if $x \neq y$ in K then $\exists f \in C(K)$ s.t. $f(x) \neq f(y)$.

Proof Apply Urysohn's Lemma to $A = \{x\}$, $B = \{y\}$. $\square$

L8.4

Thm (Tietze Extension)

(X, τ) normal top space, C closed subset of X not empty,

$f: C \longrightarrow \mathbb{C}$ bdd and continuous

Then $\exists\, \tilde{f}: X \longrightarrow \mathbb{C}$ extension continuous with $\|\tilde{f}\|_{\infty, X} = \|f\|_{\infty, C}$

Reduction to f valued in $[0,1]$ scaling

Proof Construct inductively $f_1 = f$ on C

$A = f^{-1}([0, 1/3])$, $B = f^{-1}([\frac{2}{3}, 1])$ closed in C by continuity $\Rightarrow$ closed in X

By Urysohn, $\exists\, g_1 : X \longrightarrow [0, \frac{1}{3}]$ with $g_1|_A = 0$, $g_1|_B = \frac{1}{3}$

$f_2 \doteq f_1 - g_1|_C$ has range $[0, \frac{2}{3}]$

Continue : $f_k : C \longrightarrow [0, (\frac{2}{3})^{k-1}]$

UL $A = f_k^{-1}([0, \frac{1}{3}(\frac{2}{3})^{k-1}])$, $B = f_k^{-1}([\frac{2}{3}(\frac{2}{3})^{k-1}, (\frac{2}{3})^{k-1}])$

closed and $\exists\, g_k : X \longrightarrow [0, \frac{1}{3}(\frac{2}{3})^{k-1}]$ continuous

with $g_k|_A = 0$, $g_k|_B = \frac{1}{3}(\frac{2}{3})^{k-1}$

$f_{k+1} \doteq f_k - g_k|_C$

Conclusion $\sum_{k \geq 1} g_k|_C \xrightarrow{\text{unif}} f$ on C

$\tilde{f} = \sum_{k \geq 1} g_k : X \to [0,1]$ continuous since $\sum_{k \geq 1} \|g_k\|_\infty \leq 1$ □

Exe Deduce that (X, d) metric space is compact iff all its functions $X \to \mathbb{R}$ are bounded

L9.1

Topology of $C(K)$: The Arzelà - Ascoli Theorem

Intro Compact subsets of $C(K)$

Application to differential equations

Start with a useful reformulation of (relative) cptness in complete metric spaces

DEFN (X,d) metric space. $Y \subset X$ is totally bdd if $\forall \varepsilon > 0$,

$\exists$ finite $N = \{x_1, \dots, x_n\} \subset X$ s.t. $Y \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

Rk Such set N (finite or not) is called an ε-net

Exe 1) X NVS, $Y \subset X$ tot bdd then Y bdd

2) (X,d) metric space, $Z \subset Y \subset X$ with Y tot bdd, then Z tot bdd

3) Prove that in def^n of tot bdd may take $N \subset Y$

Prop (X,d) complete metric space, $Y \subset X$

Y relatively cpct (i.e. $\overline{Y}$ cpct) iff Y tot bdd

Proof Step 1 $\overline{Y}$ cpct $\Leftrightarrow$ $\forall (y_n)$ seq in Y, (y_n) has a Cauchy subseq (EXE)

Step 2 $\underset{\Leftrightarrow Y \text{ tot bdd}}{\overline{Y}}$ tot bdd $\Leftrightarrow$ $\forall (y_n)$ seq in Y, (y_n) has a Cauchy subseq

"$\Rightarrow$" Induction, $\varepsilon_k = \frac{1}{k}$

At each step we shall make use of a finite ε_k-net

Y tot bdd $\Rightarrow$ $\exists\, y_1^1, \dots, y_{n_1}^1 \in Y$ s.t. $Y \subset \bigcup_{i=1}^{n_1} B(y_i^1, 1)$

$\Rightarrow$ infinitely may indices $n \geq 1$ s.t. $y_n \in Y \cap B(y_{i_1}^1, 1)$ for some i_1

Choose $\varphi(1)$ smallest such index

$Y \cap B(y_{i_1}^1, 1)$ tot bdd $\Rightarrow$ $\exists\, y_1^2, \dots, y_{n_2}^2 \in Y \cap B(y_{i_1}^1, 1)$

s.t. $Y \cap B(y_{i_1}^1, 1) \subset \bigcup_{i=1}^{n_2} B(y_i^2, \frac{1}{2})$

$\exists\, i_2 \in \{1, \dots, n_2\}$ s.t. infinitely many indices $n \geq \varphi(1) + 1$

s.t. $y_n \in Y \cap B(y_{i_1}^1, 1) \cap B(y_{i_2}^2, \frac{1}{2})$

Pick $\varphi(2)$ smallest such n

Continue to build $(y_{\varphi(k)})$, a subsequence such that $\forall k_0 \geq 1$,

$\{y_{\varphi(k)} : k \geq k_0\} \subset B(y_{i_{k_0}}^{k_0}, \frac{1}{k_0})$ hence is Cauchy

Step 3 "<=" Y not tot bdd, then for some $\varepsilon > 0$, there is no finite collection $y_1, \dots, y_n$ s.t. $Y \subset \bigcup_{i=1}^{n} B(y_i, \varepsilon)$

Start with any $y_1 \in Y$ and construct inductively

$(y_n)_{n \geq 1}$ in Y s.t. $y_{n+1} \notin \bigcup_{i=1}^{n} B(y_i, \varepsilon)$

Then $d(y_n, y_{n'}) \geq \varepsilon$ when $n \neq n'$, so no Cauchy subsequence $\square$

Rk Relatively cpct $\Rightarrow$ tot bdd can also be proved directly from def^n of compactness via finite open covers (try it)

Defn (K, τ) compact Hausdorff, $\mathcal{F} \subset C(K)$ subset of functions, is said to be (i) equi-bdd at $x \in K$ if $\sup_{f \in \mathcal{F}} |f(x)| < \infty$

(ii) pointwise equi-bdd on K if equi-bdd at all $x \in K$

(iii) uniformly equi-bdd on K if $\sup_{f \in \mathcal{F}} \|f\|_\infty < \infty$

(iv) equi-continuous at $x \in K$ if $\forall \varepsilon > 0$, $\exists\, \mathcal{U}$ open ngbd of x s.t.

$$\sup_{y \in \mathcal{U}} \sup_{f \in \mathcal{F}} |f(x) - f(y)| < \varepsilon$$

(v) equi-continuous on K if equi-continuous at each $x \in K$

Exe Prove that any finite $\mathcal{F} \in C(K)$ is uniformly equi-bdd and equicontinuous on K

Theorem (Arzelà - Ascoli)

K cpct Hausdorff, then $\mathcal{F} \subset C(K)$ is relatively compact iff

$\mathcal{F}$ is pointwise equi-bdd and equi-continuous

Proof We'll prove the equivalent statement

$\mathcal{F}$ tot bdd $\iff$ $\mathcal{F}$ pw equi-bdd and equi-cts

"$\Rightarrow$" $\mathcal{F}$ bdd for norm of $C(K)$ $\Rightarrow$ pw equi-bdd

Given $x \in K$, $\varepsilon > 0$, total bddness implies

$\exists\, f_1, \dots, f_n \in \mathcal{F}$ s.t. $\mathcal{F} \subset \bigcup_{i=1}^{n} B(f_i, \varepsilon)$ (balls of $(C(K), \|\cdot\|_\infty)$)

$\forall i = 1, \dots, n$, f_i cts hence $\exists\, \mathcal{U}_i$ open ngbds of x s.t.

$\sup_{y \in \mathcal{U}_i} |f_i(x) - f_i(y)| < \varepsilon$

$\forall f \in \mathcal{F}$, $\forall y \in \mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_i$ open ngd of x, have f_{i_0} s.t. $f \in B(f_{i_0}, \varepsilon)$

$$|f(y) - f(x)| \leq |f(y) - f_{i_0}(y)| + |f_{i_0}(y) - f_{i_0}(x)| + |f_{i_0}(x) - f(x)| < 2\varepsilon + \varepsilon$$

L9.3

"<=" Given $\varepsilon>0$, for $x\in K$, have open ngbd U_x of x s.t.

$$f(U_x) \subset B(f(x),\varepsilon) \quad \forall f\in\mathcal{F} \text{, by equi-cts}$$

$$K = \bigcup_{x\in K} U_x \underset{\text{cpct}}{\Rightarrow} \exists x_1,\dots,x_n \text{ s.t. } K \subset \bigcup_{i=1}^{n} U_{x_i}$$

The ptwise equi-bdd means that for $i=1,\dots,n$,

$(f(x_i))_{f\in\mathcal{F}}$ bdd in $\mathbb{R}$ hence $A=\{(f(x_1),\dots,f(x_n)),\ f\in\mathcal{F}\}$

is bounded in $\mathbb{R}^n$, meaning that

$$\exists f_1,\dots,f_m\in\mathcal{F} \text{ s.t. } A\subset \bigcup_{j=1}^{m} B_{\ell^\infty(\mathbb{R}^n)}((f_j(x_1),\dots,f_j(x_n)),\varepsilon)$$

Finally, we prove that $N=\{f_1,\dots,f_m\}$ is a finite 3ε-net for $\mathcal{F}$

For $x\in K$, take $i_0\in\{1,\dots,n\}$ s.t. $x\in U_{x_{i_0}}$,
$f\in\mathcal{F}$, $j_0\in\{1,\dots,m\}$ s.t. $((f(x_1),\dots,f(x_n))\in B(f_{j_0}(x_1),\dots,f_{j_0}(x_n)),\varepsilon)$

$$|f(x)-f_{j_0}(x)| \le |f(x)-f(x_{i_0})| + |f(x_{i_0})-f_{j_0}(x_{i_0})| + |f_{j_0}(x_{i_0})-f_{j_0}(x)|$$

$$< \varepsilon + \underset{\substack{\uparrow \\ \text{caputre } x_{i_0}}}{\varepsilon} + \varepsilon$$

$\square$

L 10.1 The Peano Existence Theorem

Intro # Application of Arzelà-Ascoli theorem which is fundamental in the theory of diff^{l} equations

Compactness used to prove existence of objects by approximation, in particular when contraction mapping theorem, Picard iteration cannot be applied

This is the case for $f'(t) = \varphi(t, f(t))$ when φ is merely continuous

Recall (IB Analysis & Topology)

Thm (Picard–Lindelöf, Cauchy 1841, Lipschitz 1868,
Modern form: Picard 1893, Lindelöf 1894)

Given $\varphi : [a,b] \times \overline{B}(y_0, R) \longrightarrow \mathbb{R}^n$,

continuous, $[a,b] \subset \mathbb{R}$, $a<b$; $y_0 \in \mathbb{R}^n$, $R>0$, $n \geq 1$, and $K>0$

such that $\forall t \in [a,b]$, $y, y' \in \overline{B}(y_0, R)$,

$$\| \varphi(t,y) - \varphi(t,y') \| \leq M \underbrace{\| y - y' \|}_{\text{norm on } \mathbb{R}^n},$$

then $\exists \varepsilon > 0$ s.t. $\forall t_0 \in [a,b]$, the initial-value problem

$$(*) \begin{cases} f'(t) = \varphi(t, f(t)), \\ f(t_0) = y_0, \end{cases}$$ has a unique solution on $[t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$:

there is a unique $f : [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b] \longrightarrow \mathbb{R}^n$ C^1

such that $(*)$ satisfied at all $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$

Rks # Modern proof done in IB based on Banach fixed point theorem and Picard iteration. $\mathbb{R}^n$ could be generalized to a Banach space

Iteration is explicit and give explicit error and approximation of the solution

↑ (not true for Peano)

$\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap (a,b)$, unique solution $f(t)$ depends continuously (Lipschitz even) on initial data y_0

If φ defined on whole $\mathbb{R} \times \mathbb{R}^n$ is locally Lipschitz in second variable and continuous on each compact (constant can vary w/ cpct) then $\exists!$ maximal solution on maximal interval of existence (T_-, T_+)

\# If $T_+ < \infty$ (resp $T_- > -\infty$) then solution leaves any compact set as $t \nearrow T_+$ (resp as $t \searrow T_-$)

EXAMPLES \# $\varphi(t,y) = y$ on $\mathbb{R} \times \mathbb{R}$

then $f(t) = y_0 e^t$

\# $\underset{y_0>0}{\varphi(t,y)} = y^2$ then $f(t) = \dfrac{1}{\frac{1}{y_0} - t}$ on $(-\infty, \frac{1}{y_0})$ i.e. $T_+ < \infty$

\# $\varphi(t,y) = \sqrt{|y|}$ and $y_0 = 0$, φ not Lipschitz at $y=0$

Picard Lindelöf does not apply. Are there solutions ?

Point of the following theorem, proved here in dim $n=1$ for simplicity, valid in $\mathbb{R}^n$

Theorem (Peano 1886, 1890)

Given $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous ; $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$,

there is $\varepsilon > 0$ and $f : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$ C^1 s.t.

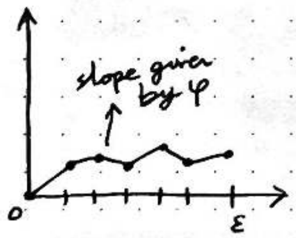
$$f'(t) = \varphi(t, f(t)) \quad \text{on } (t_0 - \varepsilon, t_0 + \varepsilon)$$

$$f(t_0) = y_0$$

Rk Contrast with Picard-Lindelöf theorem

- no uniqueness $f \equiv 0$, $f(t) = \begin{cases} t^2/2, & t \geq 0 \\ -t^2/2, & t \leq 0 \end{cases}$ solve $f'(t) = \sqrt{|f(t)|}$ (almost)
- no control of ε

Proof Strategy Euler approximation / polygons (cf Euler explicit scheme)



$f_n(t)$, $n \to \infty$?

Prove that $f_{\psi(n)} \longrightarrow \tilde{f}$ for subsequence,

and $\tilde{f}$ desired solution of IVP

Step 1 Preliminaries Wlog assume $t_0 = y_0 = 0$

Enough to construct solution C^1 on $[0, \varepsilon_+)$

Then symmetrical argument $(t \to -t)$ constructs a solution on $(-\varepsilon; 0]$, and the merging gives a solution on $(-\varepsilon, \varepsilon)$, $\varepsilon = \min(\varepsilon_+, \varepsilon_-)$

Choose $\delta > 0$, then since φ bounded on $[0,1] \times \overline{B(0,\delta)}$ (continuous), say by M, choose $\varepsilon \in (0,1)$ such that $\varepsilon M < \delta$.

Step 2 Approx solution

Given $n \geq 1$, define $f_n(t)$ on $[0, \varepsilon]$ as

$$\begin{cases} f_n(0) = 0 \\ \forall k = 1, \dots, n-1, \quad f_n(t) = f_n\left(\frac{k_t \varepsilon}{n}\right) + t\,\varphi\left(\frac{k_t \varepsilon}{n}, f_n\left(\frac{k_t \varepsilon}{n}\right)\right) \end{cases}$$ inductively

where k_t such that $\frac{k_t \varepsilon}{n} \leq t \leq \frac{(k_t+1)\varepsilon}{n}$,

well-defined by finite induction

Step 3 $(f_n)_{n\geq 1}$ uniformly equi-bounded on $[0, \varepsilon]$

Finite induction on $k = 0, \dots, n$ with $|f_n(\frac{k\varepsilon}{n})| \leq \frac{k\delta}{n}$

(f interpolates, suffices to check there)

$\underline{k=0}$ $f_n(0) = 0$

$\underline{k \Rightarrow k+1}$ $f_n\left(\frac{(k+1)\varepsilon}{n}\right) = f_n\left(\frac{k\varepsilon}{n}\right) + \frac{\varepsilon}{n}\varphi\left(\frac{k\varepsilon}{n}, \underset{\in\, \overline{B}(0,\delta)}{f_n\left(\frac{k\varepsilon}{n}\right)}\right)$

$$\left|f_n\left(\frac{(k+1)\varepsilon}{n}\right)\right| \leq \frac{k\delta}{n} + \frac{\varepsilon}{n}M \leq \frac{(k+1)\delta}{n}$$ so done

Finally, $\sup_{n\geq 1} \sup_{t\in[0,\varepsilon]} f_n(t) \leq \delta$ as desired

Step 4 $(f_n)_{n\geq 1}$ is equicontinuous

f_n is continuous piecewise affine, and

$\forall t \in [0,\varepsilon] \setminus \{\frac{k\varepsilon}{n}, k = 0, \dots, n\}$, f_n diff'ble with $f_n'(t) = \varphi\left(\frac{k_t\varepsilon}{n}, f_n\left(\frac{k_t\varepsilon}{n}\right)\right)$

$\Rightarrow |f_n'(t)| \leq M$ (use above bound on f_n)

$\Rightarrow \forall t_1, t_2 \in [0, \varepsilon]$, $\forall n \geq 1$,

$$|f_n(t_1) - f_n(t_2)| \leq M|t_1 - t_2|$$ implying equicontinuity

Step 5 Convergence

Arzelà-Ascoli thm $\Rightarrow \exists$ subseq $\psi(n)$ converging uniformly on $[0, \varepsilon]$

to some $\tilde{f} \in C([0, \varepsilon])$

Define $R_n(t) \doteq f_n(t) - \int_0^t \varphi(s, f_n(s))\, ds$

Claim $\sup_{t\in[0,\varepsilon]} |R_n(t)| \leq \varepsilon\, \varpi(n)$ where $\varpi = \sup_{\substack{t,t' \in [0,\varepsilon] \\ y,y' \in \overline{B}(0,\delta) \\ |t-t'| \leq \varepsilon/n \\ |y-y'| \leq \delta/n}} \dfrac{|\varphi(t,y) - \varphi(t',y')|}{\text{what}}$

Observe that $\varpi(n) \xrightarrow[n\to\infty]{} 0$ because φ unif continuous on $[0,\varepsilon] \times \overline{B}(0;\delta)$

Proof of claim $R_n(t)$ continuous piecewise ~~affine~~ diff'ble with

$\forall t \in [0,\varepsilon] \setminus \{\frac{k\varepsilon}{n}, k=0,\dots,n\}$, $R_n'(t)$ exists and

$$R_n'(t) = \varphi\left(\tfrac{k_t\varepsilon}{n}, f_n\left(\tfrac{k_t\varepsilon}{n}\right)\right) - \varphi(t, f_n(t))$$

$$\begin{cases} \left|t - \frac{k_t\varepsilon}{n}\right| \leq \frac{\varepsilon}{n} \\ \left|f_n\left(\frac{k_t\varepsilon}{n}\right) - f_n(t)\right| \leq M\left|\frac{k_t\varepsilon}{n} - t\right| \leq \frac{M\varepsilon}{n} \leq \delta/n \end{cases}$$

Finally, $|R_n(t)| \leq \underbrace{|R_n(0)|}_{\text{zero}} + \varepsilon \sup_{\substack{t\in[0,\varepsilon] \\ \setminus \text{bad}}} |R_n'(t)|$

$$\leq \varepsilon\, \cancel{A}(n) \quad \text{—}$$

Finally $f_{\psi(n)} \xrightarrow[n\to\infty]{} \tilde{f}$ unif on $[0,\varepsilon]$

$R_{\psi(n)} \xrightarrow[n\to\infty]{} 0$ "

$\varphi(t, f_{\psi(n)}(t)) \longrightarrow \varphi(t, \tilde{f}(t))$ "

$$\Rightarrow \tilde{f}(t) = \int_0^t \varphi(s, \tilde{f}(s))\, ds \ , \quad t \in [0,\varepsilon]$$

Hence $\tilde{f}$ is C^1 and satisfies the diff equation. $\square$

L11.1

Topology of $C(K)$ – The Stone-Weierstrass Theorem

Intro # Follow-up study on $C(K)$

Finding good dense subsets of relevant function spaces is crucial / useful in Analysis: it allows to reduce many proofs to checking a desired property on simpler objects and then argue by density

Everything done with $\mathbb{F} = \mathbb{R}$ except for the last statement

DEFN V is an algebra over field $\mathbb{F}$ it if is a vector space over $\mathbb{F}$ with an additional algebraic structure, a product $V \times V \longrightarrow V$ s.t.

- distributive on left $(v_1 + v_2) \times w = v_1 \times w + v_2 \times w$
  " " right $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$
- compatibility with scalar multiplication $(\lambda_1 v_1) \times (\lambda_2 v_2) = (\lambda_1 \lambda_2)(v_1 \times v_2)$

If V also a NVS and

$\forall v_1, v_2 \in V, \quad \|v_1 \times v_2\| \leq \|v_1\| \|v_2\|$

then V is called a normed algebra

If V normed algebra is complete, it is a Banach algebra

If $\times$ commutative, V called commutative algebra

If there is an identity element for $\times$, it is called a unitary algebra

EX ① $C(K)$, K cpct Hausdorff,
is a unitary, commutative Banach algebra for $\times$ the pointwise product

② Given V NVS, $\mathcal{L}(V,V)$ unitary algebra for $\times$ composition,
and $\mathcal{B}(V,V)$ is a unitary normed algebra;
when V is Banach then so is $\mathcal{B}(V,V)$.

Theorem (Stone-Weierstrass, Weierstrass 1885, Stone 1937, 1948)

$\mathcal{A} \subset C_{\mathbb{R}}(K)$ subalgebra that separates points:

$\forall x, y \in K,\ x \neq y,\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$

Then $\overline{\mathcal{A}} = C_{\mathbb{R}}(K)$ or $\exists x_0 \in K$ s.t.

$\overline{\mathcal{A}} = \{ f \in C_{\mathbb{R}}(K) : f(x_0) = 0 \}$

<u>Cor</u> The set of polynomials on $[0,1]$ is dense in $(C([0,1]), \|\cdot\|_\infty)$

<u>Proof</u> $\mathcal{A} \doteq \{$ polynomial f's on $[0,1]\} \subseteq C([0,1])$

Then $\mathcal{A}$ is a subalgebra that separates points (consider affine linears).

Moreover, $\mathcal{A}$ not included in any $\{f \in C([0,1]) : f(x_0) = 0\}$ for $x_0 \in K$.

Therefore, $\overline{\mathcal{A}} = C_{\mathbb{R}}([0,1])$. □

<u>Rks</u> 1) NOT the only proof: Fourier analysis on $C_{\mathbb{R}}([0,1])$ gives another proof for instance, but general method is useful

2) Cor and its proof are in fact valid more generally for $C(\overline{\mathcal{U}})$ where $\mathcal{U}$ open bounded set of $\mathbb{R}$

<u>Proof of Stone-Weierstrass</u>

<u>Step 1</u> Given $\mathcal{L} \subset C_{\mathbb{R}}(K)$ a subset "closed under finite min & max":

$$f, g \in \mathcal{L} \Rightarrow \min(f,g) \in \mathcal{L},\ \max(f,g) \in \mathcal{L}$$

Then any $g \in C_{\mathbb{R}}(K)$ s.t. $\forall x, y \in K$, $\forall \varepsilon > 0$, $\exists f \in \mathcal{L}$ s.t.

$$|f(x) - g(x)| < \varepsilon, \quad |f(y) - g(y)| < \varepsilon$$

belongs to $\overline{\mathcal{L}}$ (closure for $\|\cdot\|_\infty$)

<u>Proof of Step 1</u> Let $g \in C_{\mathbb{R}}(K)$ as above, $\varepsilon > 0$.

$\forall x, y \in K$, $\exists f_{x,y} \in \mathcal{L}$ s.t.

$$|f_{x,y}(x) - g(x)| < \varepsilon, \quad |f_{x,y}(y) - g(y)| < \varepsilon$$

$f_{x,y}$, g are continuous: $\exists\ \mathcal{U}_{x,y}, V_{x,y}$ open ngbds of x, y resp s.t.

$$|f_{x,y} - g| < \varepsilon \text{ on } \mathcal{U}_{x,y} \cup V_{x,y} \qquad (*)$$

Given x fixed, $K \subset \bigcup_{y \in K} V_{x,y}$

Cpct $\Rightarrow \exists y_1, \dots, y_m \in K$ s.t. $K \subset \bigcup_{i=1}^{m} V_{x,y_i}$

Dedine $\tilde{\mathcal{U}}_x \doteq \bigcap_{i=1}^{m} \mathcal{U}_{x,y_i}$ open ngbd of x,

$$f_x = \min(f_{x,y_1}, \dots, f_{x,y_m}) \in \mathcal{L}$$

Then $(*) \Rightarrow \begin{cases} f_x(y) < g(y) + \varepsilon & \forall y \in K \\ f_x(y) > g(y) - \varepsilon & \forall y \in \tilde{\mathcal{U}}_x \end{cases}$

L11.3

$K \subset \bigcup_{x \in K} \tilde{\mathcal{U}}_x$, so cpct $\Rightarrow$ $\exists x_1, \dots, x_n \in K$ s.t.

$K \subset \bigcup_{j=1}^{n} \tilde{\mathcal{U}}_{x_j}$ define $f = \max(f_{x_1}, \dots, f_{x_n})$

which satisfies $|f-g| < \varepsilon$ on K

This concludes step 1.

Step 2 Given $\mathcal{A} \subset C_{\mathbb{R}}(K)$ subalgebra, if closed for the topology ~~and~~ then it is closed under the lattice operations

Proof of Step 2 $\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$

$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$

$\mathcal{A}$ has vectorial structure, so STP that $f \in \mathcal{A} \Rightarrow |f| \in \mathcal{A}$

WLOG $\|f\|_\infty \leq 1$, let $\varepsilon > 0$

Define $\varphi_\varepsilon(z) = \sqrt{\varepsilon^2 + z}$ on $z \in [0,1]$ for some duh

Note $|\varphi_\varepsilon(z^2) - |z|| \leq \varepsilon$ on $[-1,1]$

φ_ε analytic in $[0,1]$ and expanding around $\frac{1}{2}$

$$\varphi_\varepsilon(z) = \sum_{k=0}^{N} a_k^\varepsilon (z - \tfrac{1}{2})^k + R_N^\varepsilon(z)$$

$R_N^\varepsilon(z) \xrightarrow[N\to\infty]{} 0$ uniformly (depending on ε)

Let $G_N^\varepsilon(z) = \sum_{k=0}^{N} a_k^\varepsilon (z - \tfrac{1}{2})^k$

$G_N^\varepsilon(0) \xrightarrow[N\to\infty]{} \varepsilon$ and $G_N^\varepsilon(z) - G_N^\varepsilon(0)$ is poly without constant term

$$|f| = \underbrace{[G_N^\varepsilon(f^2) - G_N^\varepsilon(0)]}_{\substack{\in \mathcal{A} \text{ from} \\ \text{algebra structure}}} + \underbrace{G_N^\varepsilon(0)}_{\substack{\to \varepsilon \\ \text{as } N\to\infty}} + \underbrace{R_N^\varepsilon(f^2)}_{\substack{\to 0 \\ N\to\infty}} + \underbrace{|\varphi_\varepsilon(f^2) - |f||}_{\leq \varepsilon}$$

So can find $||f| - \text{element of } \mathcal{A}| \leq 3\varepsilon$

$|f| \in \overline{\mathcal{A}} = \mathcal{A}$

Conclude (with) step 2

Step 3 Synthesis & conclusion

Given $\mathcal{A}$ as in theorem (algebra that separates points), note that $\overline{\mathcal{A}}$ is still an algebra (continuity of vector space and product operations) and still separates points

need K cpct? eh, needs f^n bdd

Step 2 $\Rightarrow$ $\overline{\mathcal{A}}$ closed under lattice operations

L11.4

Distinguish two cases

① $\forall x \in K,\ \exists f \in \overline{\mathcal{A}}$ s.t. $f(x) \neq 0$

Then $\forall x, y \in K,\ x \neq y,\ \exists f_x, f_y, f_{xy} \in \overline{\mathcal{A}}$

s.t. $f_x(x) \neq 0,\quad f_y(y) \neq 0,\quad f_{xy}(x) \neq f_{xy}(y)$

Note $\exists \alpha, \beta \in \mathbb{R}$ s.t.

$\tilde{f} \doteq f_x + \alpha f_y + \beta f_{xy}$ satisfies

$\tilde{f}(x) \neq 0,\ \tilde{f}(y) \neq 0,\ \tilde{f}(x) \neq \tilde{f}(y)$

$\tilde{f}, \tilde{f}^2 \in \overline{\mathcal{A}}$ and $\operatorname{span}\left((\tilde{f}(x), \tilde{f}(y)),\ (\tilde{f}^2(x), \tilde{f}^2(y)) \right) = \mathbb{R}^2$

Hence assumption of Step 1 always satisfied and $\overline{\mathcal{A}} = C_{\mathbb{R}}(K)$

② Assume $\exists x_0 \in K$ s.t.

$$\overline{\mathcal{A}} \subset \{ f \in C_{\mathbb{R}}(K) : f(x_0) = 0 \}$$

Define $\mathcal{A}' = \{ \mathcal{A} + \lambda \mathbb{1},\ \lambda \in \mathbb{R} \}$ where $\mathbb{1}$ is constant f^n 1

Then $\mathcal{A}'$ is an algebra, separates points and satisfies ~~Step~~ case ①

So $\overline{\mathcal{A}'} = C_{\mathbb{R}}(K)$

This implies that $\overline{\mathcal{A}} = \{ f \in C_{\mathbb{R}}(K) : f(x_0) = 0 \}$

since any $g \in$ RHS is in $\overline{\mathcal{A}'}$, i.e. $\forall \varepsilon > 0$,

$\exists f \in \mathcal{A},\ \lambda \in \mathbb{R}$ s.t. $|g - f - \lambda \mathbb{1}| < \varepsilon$ on K

Evaluate at $x_0 \in K$ to get $|\lambda| < \varepsilon$

$\Rightarrow |g - f| < 2\varepsilon$ and so $g \in \overline{\mathcal{A}}$ $\square$

<u>Statement in the complex case</u>

<u>Thm</u> $\mathcal{A} \subset C_{\mathbb{C}}(K)$, K cpct Hausdorff, subalgebra that separates points and closed under complex conjugation.

Then same conclusion as with $\mathbb{F} = \mathbb{R}$, i.e.

$\overline{\mathcal{A}} = C_{\mathbb{C}}(K)$ or $\overline{\mathcal{A}} = \{ f \in C_{\mathbb{C}}(K) : f(x_0) = 0 \}$ for some $x_0 \in K$

Topology of $C(K)$: Approximation & Applications

Intro # Building upon the Stone-Weierstrass theorem, we explore various approximations & applications

Includes in particular initialising study of vigorous theory of Fourier series of

$$f \in C_p([0,2\pi]) = C(\mathbb{T})$$

Some applications of Stone-Weierstrass

① Density of polynomials in $C_{\mathbb{R}}([0,1])$

SW thm with $\mathcal{A} = \operatorname{span}\{x^k, k \in \mathbb{N}\}$ subalgebra ; $\mathbb{1} \in \mathcal{A}$

$\Rightarrow \mathcal{A} \not\subseteq \{f \in C([0,1]), f(x_0) = 0\}$ for no x_0

it separates points in x

② Density of linear combination of exponential f^n's in $C([0,1])$

SW thm with $\mathcal{A} = \operatorname{span}\{e^{kx}, k \in \mathbb{N}\}$ subalgebra, $\mathbb{1} \in \mathcal{A}$,

separates points with e^x

③ Density of linear combination of e^{ikx}, $k \in \mathbb{Z}$ i.e. trigonometric polynomials in $C_{\mathbb{C}}(\mathbb{T})$

$\mathbb{C}$-SW thm with $\mathcal{A} = \operatorname{span}\{e^{ikx}, k \in \mathbb{Z}\}$ subalgebra, $\mathbb{1} \in \mathcal{A}$,

separates points e.g. combining $e^{\pm ix}$, $\mathbb{1}$, $\operatorname{conj}(\mathcal{A}) = \mathcal{A}$

hence $\overline{\mathcal{A}} = C_{\mathbb{C}}(\mathbb{T})$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ i.e. the circle as in $[0,2\pi]$ with endpoints identified

Note that $\mathbb{T}$ is a compact Hausdorff (in fact metric) topological space, and $C(\mathbb{T})$ identifies with complete (Banach) subspace $\{f \in C([0,2\pi]), f(0) = f(2\pi)\}$

④ Density of linear combination of continuous product functions in $C([0,1]^2)$

($F = \mathbb{R}$ or $\mathbb{C}$; no difference here)

$\mathcal{A} \doteq \operatorname{span}\{(x,y) \mapsto f_1(x) f_2(x),\ f_1; f_2 \in C([0;1])\}$

subalgebra of $C([0,1]^2)$, includes $\mathbb{1}$ and it separates points :

$(x_1, y_1) \neq (x_2, y_2)$ then wlog $x_1 \neq x_2$ and find $f_1(x_1) \neq f_1(x_2)$, $f_2 = \mathbb{1}$

⌈ example generalises ⌋ Hence $\overline{\mathcal{A}} = C([0,1]^2)$

As an application, we deduce the so-called <u>Fubini theorem</u> in the case of continuous functions. Using only Riemann integration theory in one dimension,

if $f \in C([0,1]^2)$, $\cdot\ \forall x \in [0,1]$; $x \mapsto \int_0^1 f(x,y)\,dy$ exists, is cts

$\cdot\ \forall y \in [0,1]$, $y \mapsto \int_0^1 f(x,y)\,dx$ exists, is cts

Thus $A \doteq \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx$ exists,

$B \doteq \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$ exists

Do they agree? Yes: given any $\varepsilon > 0$, there is

$$h(x,y) = \sum_{i=1}^{n} u_i(x) v_i(y), \quad u_i, v_i \in C([0,1])$$

such that $\| f - h \|_\infty < \varepsilon$.

Then
$$|A - B| \leq \int_0^1 \left(\int_0^1 |f-h|\,dy \right) dx + \int_0^1 \left(\int_0^1 |f-h|\,dx \right) dy + \underbrace{\left| \int_0^1 \left(\int_0^1 h\,dy \right) dx - \int_0^1 \left(\int_0^1 h\,dx \right) dy \right|}_{=0}$$
$$\leq 2\varepsilon, \quad \text{thus } (\varepsilon \to 0);\ A = B$$

These examples show the abstract / generality power of the Stone-Weierstrass, but it is, to some extent, at the cost of "concreteness". Let us explore more concrete approximations.

<u>Theorem</u> (Weierstrass Approximation)

$f \in C_{\mathbb{R}}([0,1])$ and $\varepsilon > 0$, there is a polynomial p such that

$\| p - f \|_\infty < \varepsilon$

<u>Proof</u> (!) due to Bernstein (1912), makes use of the important concept of interpolating polynomials, here we use Bernstein polynomials

Given $f \in C([0,1])$, define $p_n(x) \doteq \sum_{i=0}^{n} b_{i,n}(x) f\left(\frac{i}{n}\right)$

where $b_{i,n}(x) \doteq \binom{n}{i} x^i (1-x)^{n-i}$ on $x \in [0,1]$

Let us prove that $\| f - p_n \|_\infty \to 0$

Note that $b_{i,n}(x) = \text{Prob}(X_1 + \cdots + X_n = i)$ for X_j iid Bernoulli(x)

$b_{i,n}(x) \geq 0$ and $\sum_{i=0}^{n} b_{i,n}(x) = 1 \quad \forall x \in [0,1]$

Given $M > 0$ (to be fixed later), consider

$$| f(x) - p_n(x) | \leq \sum_{i=0}^{n} b_{i,n}(x) \left| f(x) - f(i/n) \right|$$

L12.3

$$|f(x)-p(x)| \leq \underbrace{\sum_{i:|i-nx|\leq M}}_{S_1^N(x)} + \underbrace{\sum_{i:|i-nx|>M}}_{S_2^N(x)}$$

$$|S_1^N(x)| \leq \sup_{i:|\frac{i}{n}-x|<\frac{M}{n}} |f(x)-f(\tfrac{i}{n})| \quad \text{which} \xrightarrow[n\to\infty]{} 0 \quad \text{provided } \frac{M}{n}\to 0$$

$$|S_2^N(x)| \leq 2\|f\|_\infty \sum_{i:|i-nx|>M} b_{i,n}(x)$$

$$\leq 2\|f\|_\infty \,\text{Prob}(|X_1+X_2+\dots+X_n-nx|>M)$$

Note that $X_1+\dots+X_n-nx$ has mean zero, variance $\leq \frac{n}{4}$ (each has $x(1-x)$) (ch)

Chebyshev's ineq $\Rightarrow$ $\text{Prob}(|X_1+\dots+X_n-nx|>M) \leq \frac{(n/4)}{M^2}$

Choose $M=n^{2/3}$ to get $|S_2^N(x)| \leq \frac{\|f\|_\infty}{4} n^{-1/3}$

and $\|S_1^N(x)\|_\infty \to 0$ since $\frac{M}{n}=n^{-1/3}\to 0$ $\square$

Rks ① Nice argument w/ probability interpretation, could be reframed without tho

② If $f\in C^1$, then $\|S_1^N\|_\infty \leq (\sup|f'|)\frac{M}{n} \leq \|f'\|_\infty n^{-1/3}$

③ Striking power of Bernstein theory is that it extends to higher derivatives; if $f\in C^k([0,1])$, $p_n^{(k)}(x) = \sum_{i=0}^n b_{i,n}^{(k)}(x) f(\frac{i}{n}) \xrightarrow{\text{unif}} f^{(k)}(x)$

④ There are other interpolating polynomials, such as Lagrange polynomials; but latter are not helpful for proving Weierstrass thm

Rigorous theory of Fourier Series in $C(\mathbb{T})$

Particular type of approximation by trigonometric polynomials, seen in IB, introduced by Fourier.

$$f\in C(\mathbb{T}), \quad \hat{f}(k) \doteq \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx, \quad k\in\mathbb{Z}$$

$$S_N(f)(x) \doteq \sum_{k=-N}^{N} \hat{f}(k)e^{ikx}$$ Does this converge to f?

From density of trigonometric polys, we know that for $\varepsilon>0$, $\exists\, g$ with

$$g=\sum_{k=-N}^{N} a_k e^{ikx} \quad \text{s.t.} \quad \|f-g\|_\infty<\varepsilon$$

However coeffs a_k not necessarily Fourier coeffs

In fact Fourier coefficients are constrained, e.g.

<u>Lemma</u> (Riemann-Lebesgue)

Given $f \in C(\mathbb{T})$, $\int_0^{2\pi} f(x)\, e^{ikx}\,dx \longrightarrow 0$ as $|k| \to \infty$

<u>Proof</u> Weierstrass approx thm $\Rightarrow$ $\forall \varepsilon > 0$, $\exists g \in C(\mathbb{T})$ that is C^1 s.t.

$\|f-g\|_\infty < \varepsilon$, whence

$$\left|\int_0^{2\pi} f(x)\, e^{ikx}\,dx\right| \leq 2\pi\varepsilon + \left|\int_0^{2\pi} g(x)\, e^{ikx}\,dx\right|$$

by parts

$$\leq 2\pi\varepsilon + \|g'\|_\infty \frac{2\pi}{|k|} \quad (k \neq 0)$$

$|k|$ large enough $\Rightarrow$ $\left|\int_0^{2\pi} f(x)\, e^{ikx}\,dx\right| \leq 4\pi\varepsilon$ $\square$

<u>Theorem</u> (Kolmogorov) There is $f \in C(\mathbb{T})$ s.t.

$S_N(f)(0)$ diverges as $N \to \infty$

<u>Proof</u> (Sketch)

$$S_N(f)(0) = \sum_{k=-N}^{N} \hat{f}(k)$$

Define $\Psi_N : \underbrace{C(\mathbb{T})}_{\text{Banach}} \longrightarrow \underbrace{\mathbb{C}}_{\text{Banach}}$, $f \mapsto S_N(f)(0)$.

linear

Ψ_N linear bdd for each $N \in \mathbb{N}$ ~~$f \in C(\mathbb{T})$~~

Then UBP $\Rightarrow$ $\left[\text{ if } \sup_{N\geq 1} |\Psi_N(f)| < +\infty \Rightarrow \sup_{N \geq 1} \|\Psi_N\| < \infty \right]$

Let us prove that $\sup_{N\geq 1} \|\Psi_N\| = \infty$, which will conclude the proof.

$$\Psi_N(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x) \underbrace{D_N(x)}_{\substack{\text{Dirichlet} \\ \text{kernel}}}\,dx$$

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$$

and $\|\Psi_N\| = \frac{1}{2\pi}\int_0^{2\pi} |D_N(x)|\,dx$

by testing on $f = e^{-i\arg D_N(x)}$ ← waffle, D_N real valued does sign changes

Finally prove that $\int_0^{2\pi} |D_N(x)|\,dx \xrightarrow[N\to\infty]{} +\infty$ by showing integral $\geq \frac{\alpha}{\ell}$ around each peak $x = \frac{\pi(\ell+\frac{1}{2})}{N+\frac{1}{2}}$, $\ell = 1, \dots, N$

$$\int_0^{2\pi} |D_N(x)|\,dx \geq \alpha\left(1 + \frac{1}{2} + \dots + \frac{1}{N}\right)$$

$\square$

L13.1 The inner product

Intro # First lecture of a series on "Hilbertian analysis", i.e. in Hilbert spaces (complete NVS whose norm has an "inner product structure")

"Hilbert space" is terminology coined by Von Neumann when formulating quantum mechanics

Aims at generalizing (part of) the finite-dim^l old Euclidean geometry

Today focus on inner product structure. Then one lecture on orthonormal families and Bessel inequality; one lecture on completion and the Riesz-Fisher theorem; and one lecture on projections in Hilbert spaces and the Riesz representation theorem

Def Let V be a vector space (on $\mathbb{F} = \mathbb{R}, \mathbb{C}$).

An inner product is a map $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$ such that

(i) $\forall v, w \in V, \quad \langle v ; w \rangle = \overline{\langle w, v \rangle}$

(ii) $\forall v_1, v_2, w \in V, \lambda_1, \lambda_2 \in \mathbb{F}, \quad \langle \lambda_1 v_1 + \lambda_2 v_2 ; w \rangle = \lambda_1 \langle v_1 , w \rangle + \lambda_2 \langle v_2 , w \rangle$

(iii) $\forall v \in V, \quad \langle v, v \rangle \geqslant 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

A vector space V with such a structure is called an inner product space

Rmk # (i)-(ii) are just saying, when $\mathbb{F} = \mathbb{R}$, that $\langle \cdot , \cdot \rangle$ is a symmetric bilinear form; when $\mathbb{F} = \mathbb{C}$ it is called sesquilinearity and define Hermitian forms

$\mathbb{F} = \mathbb{C}$, $\langle \cdot , \cdot \rangle$ sometimes called "antilinear" in second variable:

$$\langle v ; \lambda_1 w_1 + \lambda_2 w_2 \rangle = \overline{\lambda}_1 \langle v, w_1 \rangle + \overline{\lambda}_2 \langle v, w_2 \rangle$$

(sometimes opposite convention between first and second variable)

Prop Let $(V ; \langle \cdot , \cdot \rangle)$ an inner product space.

Then (i) Cauchy-Schwarz inequality holds:

$$\forall v, w \in V, \quad |\langle v, w \rangle| \leqslant \sqrt{\langle v, v \rangle} \sqrt{\langle w ; w \rangle}$$

(ii) $\|v\| \doteq \sqrt{\langle v, v \rangle}$ defines a norm on V

Proof (i) $\exists \alpha \in \mathbb{C}$, $|\alpha|=1$ s.t. if $\tilde{v} = \alpha v$ then

$$\langle \tilde{v}, w \rangle = \alpha \langle v, w \rangle \in \mathbb{R}$$

Then (i) $\iff |\langle \tilde{v}, w \rangle| \leq \sqrt{\langle \tilde{v}, \tilde{v} \rangle} \sqrt{\langle w, w \rangle}$

Consider $P(t) = \langle \tilde{v} + tw ; \tilde{v} + tw \rangle \geq 0$, real poly.

P is a 2^{nd} order polynomial, real-valued and ≥ 0,

$$P(t) = t^2 \langle w, w \rangle + 2t \langle \tilde{v}, w \rangle + \langle \tilde{v}, \tilde{v} \rangle \qquad \ulcorner \text{use } \langle \tilde{v}; w \rangle = \langle w; \tilde{v} \rangle \lrcorner$$

$\Rightarrow$ Discriminant $\Delta = 4 \langle \tilde{v}, w \rangle^2 - 4 \langle \tilde{v}, \tilde{v} \rangle \langle w, w \rangle \leq 0$ ~ baby

which proves (i)

Case of equality in (i) $\iff P(t)$ has one real root t_0

$$\iff v, w \text{ colinear}$$

(ii) Axioms of norm all implied by those of the inner product, sauf the triangle ineq: Prove the latter:

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \langle w; v \rangle \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\ &= (\|v\| + \|w\|)^2 \end{aligned}$$

$\square$

Def A NVS $(V, \|\cdot\|)$ where $\|\cdot\| = \sqrt{\langle \cdot ; \cdot \rangle}$ with $\langle \cdot, \cdot \rangle$ inner product on V is called a Euclidean space

Prop If $(V; \|\cdot\|)$ is a Euclidean space, there is only one inner product inducing the norm $\|\cdot\|$; it is given by the polarisation identities

$(\mathbb{F} = \mathbb{R})$ $\langle v; w \rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 \right)$

$(\mathbb{F} = \mathbb{C})$ $\langle v; w \rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 + i \|v + iw\|^2 - i \|v - iw\|^2 \right)$

This unique $\langle \cdot, \cdot \rangle$ is a continuous map $V \times V \to \mathbb{F}$

Proof Check that $(\mathbb{F}=\mathbb{R})$ any inner product giving $\|\cdot\|$ has

$$\|v+w\|^2 - \|v-w\|^2 = \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle$$
$$- \|v\|^2 - \|w\|^2 + 2\langle v,w\rangle$$
$$= 4\langle v,w\rangle$$

$(\mathbb{F}=\mathbb{C})$ $\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2$

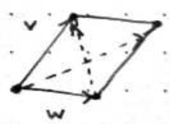
$$= \|v\|^2 + \|w\|^2 - \|v\|^2 - \|w\|^2 + i\left(\|v\|^2 + \|w\|^2 - \|v\|^2 - \|w\|^2\right)$$
$$+ 2\left[\langle v,w\rangle + \langle w,v\rangle\right] + 2i\left[\langle v,w\rangle - \langle w,v\rangle\right]$$
$$= 4\langle v,w\rangle$$

Continuity clear from Cauchy-Schwarz and bilinearity / sesquilinearity. □

Theorem (Jordan - Von Neumann)

Let $(V,\|\cdot\|)$ a NVS. Then it is a Euclidean space iff the norm satisfies the parallelogram law:

$$\forall v,w \in V, \quad \|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$



Proof Easy part "$\Rightarrow$". If V Euclidean, then

$$\|v+w\|^2 + \|v-w\|^2 = \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \langle w,v\rangle$$
$$+ \|v\|^2 + \|w\|^2 - \langle v,w\rangle - \langle w,v\rangle$$
$$= 2\|v\|^2 + 2\|w\|^2$$

Hard part "$\Leftarrow$". Define $\langle\cdot,\cdot\rangle$ via polarisation identities. Observe that this tentative inner product is continuous on $V\times V$ because the norm is.

Then axioms of inner product

(i) $\langle v,w\rangle = \overline{\langle w,v\rangle}$

(iii) $\langle v,v\rangle \geq 0$ and zero only when $v=0$

are clear from the polarization identities (check) ✓

What remains to be proved is axiom (ii)

$$(*)\begin{cases} \langle v_1+v_2, w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle \\ \langle \lambda v, w\rangle = \lambda\langle v,w\rangle \end{cases} \qquad \begin{matrix} v_1,v_2,v,w\in V \\ \lambda\in\mathbb{F}\end{matrix}$$

Reduce complex case to real case:

Since $\langle iv, w\rangle = i\langle v, w\rangle$ and $\langle v, w\rangle = \overline{\langle w, v\rangle}$ from polarization identities, it is enough to prove $(*)$ separately on $\mathrm{Re}\langle\cdot,\cdot\rangle$ and $\mathrm{Im}\langle\cdot,\cdot\rangle$, which reduces proof to the real case. [takes some care: of note that $\mathrm{Re}\langle\cdot,\cdot\rangle$ is sym inner product giving same norm]

Proof of $(*)$ for real case:

To prove $\langle v_1+v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, compute with <u>parallelogram law</u>

$$(\text{Par Law})\quad \|v_1+v_2+w\|^2 = 2\|v_1+w\|^2 + 2\|v_2\|^2 - \|v_1-v_2+w\|^2$$

$$(v_1 \leftrightarrow v_2)\quad \|v_1+v_2+w\|^2 = 2\|v_2+w\|^2 + 2\|v_1\|^2 - \|v_2-v_1+w\|^2$$

$$(\text{half sum})\quad \|v_1+v_2+w\|^2 = \|v_1\|^2 + \|v_2\|^2 + \|v_1+w\|^2 + \|v_2+w\|^2 - \tfrac{1}{2}\left(\|v_1-v_2+w\|^2 + \|v_2-v_1+w\|^2\right)$$

$$(w \leftrightarrow -w)\quad \|v_1+v_2-w\|^2 = \|v_1\|^2 + \|v_2\|^2 + \|v_1-w\|^2 + \|v_2-w\|^2 - \tfrac{1}{2}\left(\|v_1-v_2-w\|^2 + \|v_2-v_1-w\|^2\right)$$

note these unchanging

Then

$$\|v_1+v_2+w\|^2 - \|v_1+v_2-w\|^2 = \|v_1+w\|^2 + \|v_2+w\|^2 \,\cancel{+}\!- \|v_1-w\|^2 \,\cancel{+}\!- \|v_2-w\|^2$$

i.e. $4\langle v_1+v_2, w\rangle = 4\langle v_1, w\rangle + 4\langle v_2, w\rangle$ as desired.

By induction on $v_1 = v_2 = \cdots = v_n = v$,

$$\langle nv, w\rangle = n\langle v, w\rangle \quad \text{for } n \in \mathbb{N}$$

Hence $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$ for $\lambda \in \mathbb{Z}$ since $\langle -v, w\rangle = -\langle v, w\rangle$, def^{n}

Consider $\lambda = \frac{p}{q}$, $p, q \in \mathbb{Z}$; $q \neq 0$, so

$$q\langle \lambda v, w\rangle = q\langle p(\tfrac{v}{q}), w\rangle = pq\langle (\tfrac{v}{q}), w\rangle = p\langle v, w\rangle$$

Thus $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$ for $\lambda \in \mathbb{Q}$

By continuity, we finally deduce that

$$\langle \lambda v, w\rangle = \lambda\langle v, w\rangle \quad \text{for } \lambda \in \mathbb{R}. \qquad \square$$

Orthogonality and decomposition

Intro # 2^{nd} of 5 lectures on Hilbertian analysis: today study concepts of orthogonality / orthonormality, prove Bessel inequality

Applications to study of Fourier series $C(\mathbb{T})$ endowed with $(f,g) \mapsto \frac{1}{2\pi}\int f\,\bar{g}$

Discuss also $(C([0,1]), \langle\cdot,\cdot\rangle)$ and ℓ^2 endowed with $\langle(x),(y)\rangle \mapsto \sum x_n \bar{y}_n$

$(\ell^2, \langle\cdot,\cdot\rangle)$ is complete (so is a Hilbert space) but $(C(\mathbb{T}), \langle\cdot,\cdot\rangle)$ and $(C([0,1]), \langle\cdot,\cdot\rangle)$ are not complete. See completion and Hilbert spaces in next lecture

Def^n Given $(V, \langle\cdot,\cdot\rangle)$ an inner product space,

(i) $S \subset V$ is orthogonal if

$$\forall v, w \in S,\ v \neq w,\ \langle v, w\rangle = 0$$

(ii) $S \subset V$ is orthonormal if

$$\forall v, w \in S;\ \langle v, w\rangle = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

(iii) Given $S \subset V$, the orthogonal space $S^\perp$ is

$$S^\perp \doteq \{ v \in V,\ \forall w \in S;\ \langle v; w\rangle = 0 \}$$

Rk $\langle v, w\rangle = 0$ also written $v \perp w$ (v orthogonal to w)

Prop (i) Given $(V, \langle\cdot,\cdot\rangle)$ inner product space, $S \subset V$,

$S^\perp$ is a subspace

(ii) Given V Euclidean space, $S \subset V$,

$$S^\perp = (\overline{\text{Span}(S)})^\perp$$

Proof (i) $v_1, v_2 \in S^\perp$, $\lambda_1, \lambda_2 \in \mathbb{F}$ then

$$\forall w \in S,\ \langle \lambda_1 v_1 + \lambda_2 v_2; w\rangle = \lambda_1 \underbrace{\langle v_1; w\rangle}_{\text{zero}} + \lambda_2 \underbrace{\langle v_2, w\rangle}_{\text{zero}} = 0$$

$$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in S^\perp$$

(ii) Just observe that $S \subset \overline{\text{Span}(S)}$; then $\overline{\text{Span}(S)}^\perp \subset S^\perp$

We need to prove $S^\perp \subset \overline{\text{Span}(S)}^\perp$

Let $v \in S^\perp$ and $w \in \overline{\text{Span}(S)}$.

Then $\exists (w_n)_{n\geq 1}$ seq in $\text{Span}(S)$ s.t. $w_n \to w$, and $w_n = \sum_{k=1}^{N_n} a_{k,n} u_{k,n}$

where $a_{k,n} \in \mathbb{F}$, $u_{k,n} \in S$

Then $\langle v, w_n \rangle = \sum_{k=1}^{N_n} a_{k,n} \underbrace{\langle v, u_{k,n} \rangle}_{\text{zero}} = 0$

Inner product continuous $\Rightarrow$ $\langle v, w \rangle = \lim_n \langle v, w_n \rangle = 0$

hence $v \in \overline{\mathrm{Span}(S)}^{\perp}$. $\square$

<u>Def</u> Given E Euclidean space, an <u>orthonormal system</u> $S \subset E$ (orthonormal collection of vectors) is said to be <u>maximal</u> if it cannot be extended to a strictly larger orthonormal system.

<u>Rk</u> In next lecture, we'll see that when E is complete (aka Hilbert space), maximality is equivalent to $\overline{\mathrm{span}(S)} = E$, and S is called then a Hilbert (space) basis.

Be careful that it is <u>not</u> in infinite dimension an "algebraic" basis.

<u>Prop</u> Proof is conceptually identical to that of existence of bases.

Given $E \neq \{0\}$ Euclidean space, then E has a maximal o.n. family

<u>Proof</u> Zorn's lemma. Sketch:

Given $S \subset V$ orthonormal, build $S \subset S^m \subset V$ maximal orthonormal

- $\mathcal{T} \doteq \{$orthonormal systems $S' \supset S\}$ poset for inclusion
- $\mathcal{T}$ has the LUBP, $\mathcal{C} \subset \mathcal{T}$ non-empty totally ordered, then
  $S^{\#} = \bigcup_{S' \in \mathcal{C}} S'$ least upper bound
- $\Rightarrow$ existence of maximal element $\square$

Using Zorn's lemma gives powerful and general results but at the cost of concreteness. In the "countable" case we prefer instead:

<u>Prop</u> (Gram-Schmidt orthogonalisation)

Given E Euclidean space and $(v_n)_{n=1}^{N}$ an independent (linearly) family of vectors with $N \in \mathbb{N}^* \cup \{\infty\}$ (finite or countably infinite)

Then there is $(e_n)_{n=1}^{N}$ orthonormal with

$$\forall k = 1, \dots, N, \quad \mathrm{Span}((e_n)_{n=1}^{k}) = \mathrm{Span}((v_n)_{n=1}^{k})$$

L14.3

Proof Induction $\cdot$ $e_1 = \frac{v_1}{\|v_1\|}$ ($v_1 \neq 0$ by lin indep)

- if $e_1, \dots, e_n$ built define

$$e_{n+1} \doteq \frac{v_{n+1} - \sum_{k=1}^{n} \langle v_{n+1}, e_i \rangle e_i}{\|v_{n+1} - \sum_{k=1}^{n} \langle v_{n+1}, e_i \rangle e_i\|} \longrightarrow \text{not zero since } v_{n+1} \notin \operatorname{span}((e_i)_{i=1}^{n}) = \operatorname{span}((v_i)_{i=1}^{n})$$

Check that $(e_n)_{n=1}^{N}$ built this way answers requirements. $\square$

Before discussing examples, let us estimate that are abstract generalizations of Pythagoras's theorem.

Theorem E Euclidean space

(i) Pythagoras's theorem: $\forall v_1, v_2 \in E$, $v_1 \perp v_2$

$$\Rightarrow \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$$

(ii) Extension of Pythagoras, finite: Given $(e_i)_{i=1}^{n}$, $n \in \mathbb{N}^*$ orthonormal system and $v \in E$, then $\|v\|^2 = \|v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i\|^2 + \sum_{i=1}^{n} |\langle v_i, e_i \rangle|^2$

(iii) Extension of Pythagoras, countable (Bessel inequality):

Given $(e_n)_{n \geq 1}$ countably infinite orthonormal system and $v \in E$, then $\sum_{i=1}^{\infty} |\langle v, e_i \rangle|^2 \leq \|v\|^2$ with equality iff $\sum_{i=1}^{\infty} \langle v, e_i \rangle e_i = v$.

Proof (i) $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 + \underbrace{\langle v_1, v_2 \rangle}_{\text{zero}} + \underbrace{\langle v_2, v_1 \rangle}_{\text{zero}}$ ✓

(ii)
$$\begin{aligned}
\|v\|^2 &= \|(v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i) + \sum_{i=1}^{n} \langle v, e_i \rangle e_i\|^2 \\
&= \|v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i\|^2 + \|\sum_{i=1}^{n} \langle v, e_i \rangle e_i\|^2 \\
&\quad \left.\begin{aligned} &+ \langle v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i, \sum_{i=1}^{n} \langle v, e_i \rangle e_i \rangle \\ &+ \langle \sum_{i=1}^{n} \langle v, e_i \rangle e_i, v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \rangle \end{aligned}\right\} \text{zero} \\
&= \|v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i\|^2 + \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 \leftarrow \text{Pythagoras}
\end{aligned}$$

(iii) Taking limit as $n \to \infty$ in (ii) $\square$

Examples

(I) $\ell^2 = \{ (x_n)_{n\geq 1} : \sum_{n\geq 1} |x_n|^2 < +\infty \}$

is Euclidean space for $\langle x, y \rangle \doteq \sum_{n\geq 1} x_n \overline{y_n}$

- It is complete so a Hilbert space
- Hölder $\leq$ (proved in revision lecture from Young's ineq) here reproved in the guise of Cauchy-Schwarz
- Family $S = \{ (e_k) : k \geq 1 \}$, $e_k = (\delta_{kn})_{n\geq 1}$ is orthonormal maximal,

since $\overline{\text{Span}(S)} = \ell^2$; $f \perp S \Rightarrow f \perp \overline{\text{Span}(S)} = \ell^2 \Rightarrow f \perp f \Rightarrow f = 0$

(II) $C([0,1])$ is inner product space for

$$\langle f, g \rangle \doteq \int_0^1 f(x) \overline{g(x)} \, dx$$

Rk Proof of axioms of inner product easy exercise

In particular, $\langle f, f \rangle = 0 \Rightarrow f = 0$ since f cts

$(C([0,1]), \|\cdot\|_2)$ with $\|f\|_2 = \left(\int_0^1 |f|^2 \right)^{1/2}$ is not complete

Consider $f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ n(x - (\frac{1}{2} - \frac{1}{n})), & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$

f_n continuous ; (f_n) Cauchy in $\|\cdot\|_2$ and any limit $f \in C([0,1])$ should satisfy $\int_0^{1/2} |f|^2 = 0$, $\int_{1/2}^1 |f-1|^2 = 0$ absurd

(III) $C(\mathbb{T})$ is inner product space with $(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx$

Check axioms as in (II)

Not complete for similar reasons

As an application of the Stone-Weierstrass theorem and Bessel's inequality, let us prove the following:

Prop Given $f \in C(\mathbb{T})$, then

(i) $\| S_N(f) - f \|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} | S_N(f)(x) - f(x) |^2 dx \right)^{1/2} \xrightarrow[N\to\infty]{} 0$

(ii) $\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx$

Rk (i) does not contradict Kolmogorov thm because it does not imply pointwise convergence

L14.5

Proof (i) $\mathbb{C}$-SW thm $\Rightarrow$ for any $\varepsilon > 0$, $\exists\, p$ trigonometric poly $\in C(\mathbb{T})$

such that $\|f - p\|_\infty = \sup_{x \in \mathbb{T}} |f(x) - p(x)| < \varepsilon$

Observe that $S_N(p) \overset{p}{=}$ for N large enough ($N \geqslant \deg p$)

$$\|S_N(p) - S_N(f)\|_2 = \|S_N(f-p)\|_2 \leq \|f - p\|_2 \quad \text{(Bessel ineq)}$$

Hence $\|f - S_N(f)\|_2 \leq \|f - p\|_2 + \|p - S_N(p)\|_2$

$$+ \|S_N(p) - S_N(f)\|_2$$

$$\leq 2\|f - p\|_2$$

$$\leq 2\sqrt{2\pi}\,\varepsilon$$

(ii) Case of equality in Bessel $\leq$:

$\|\lim_{N\to\infty} S_N(f)\|_2^2 = \|f\|_2^2$ means exactly

$$\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx. \quad \square$$

Completion and (abstract) Riesz-Fischer Theorem

Intro # Third of 4 lectures on Hilbertian analysis

Today we add completeness to the inner product structure

Since we have seen natural inner product spaces that are not complete, we study first the completion

We end with introducing Hilbert spaces and (Riesz-Fischer) characterize separable Hilbert spaces

Def $(V, \|\cdot\|)$ NVS. A completion of V is a triple $(\bar{V}, \|\cdot\|_*; \varphi)$ such that $(\bar{V}, \|\cdot\|_*)$ is a complete NVS and $\varphi: V \to \bar{V}$ linear isometric and $\overline{\varphi(V)} = \bar{V}$.

Rk There is a more general theory of completion in metric spaces

Theorem Given V NVS, its completion exists and is uniquely determined up to an isometric isomorphism

Proof (uniqueness) Consider two completions of V:

$(\bar{V}_1, \|\cdot\|_1, \varphi_1)$, $(\bar{V}_2, \|\cdot\|_2, \varphi_2)$

Note that $\varphi_1: V \to \bar{V}_1$, bij onto $\varphi_1(V)$

$\varphi_2: V \to \bar{V}_2$, bij onto $\varphi_2(V)$

Hence $\varphi_0 = \varphi_2 \circ \varphi_1^{-1}: \underset{\bar{V}_1}{\underset{\cap}{\varphi_1(V)}} \to \underset{\bar{V}_2}{\underset{\cap}{\varphi_2(V)}}$

φ_0 is linear, isometric and domain, range dense

φ_0 isometric $\|\varphi_0(v) - \varphi_0(w)\|_2 = \|v-w\|_1$,

hence φ_0 is uniformly continuous and has a unique extension $\bar{\varphi}_0: \bar{V}_1 \to \bar{V}_2$ (using here that $\bar{V}_1; \bar{V}_2$ are complete)

Note that by continuity of vector space operations and norm, $\bar{\varphi}_0$ linear, isometric. Last thing to prove is $\bar{\varphi}_0(\bar{V}_1) = \bar{V}_2$. (*)

Claim $\bar{\varphi}_0(\bar{V}_1)$ is complete subspace hence closed in $\bar{V}_2$

(which proves (*) since $\underbrace{\varphi_2(V)}_{\text{dense}} \subset \bar{\varphi}_0(\bar{V}_1) \subset \bar{V}_2$)

Given $(w_n)_{n\geq 1}$ Cauchy seq in $\overline{\varphi_0}(\overline{V_1})$, then $\exists$ $(v_n)_{n\geq 1}$ seq in $\overline{V_1}$ s.t. $\overline{\varphi_0}(v_n) = w_n$. Isometry $\Rightarrow$ (v_n) Cauchy as well.

Therefore $v_n \xrightarrow[n\to\infty]{} \bar{v}$ in $\overline{V_1}$ since $\overline{V_1}$ complete.

Since ($\overline{\varphi_0}$ continuous) $w_n \xrightarrow[n\to\infty]{} \overline{\varphi_0}(\bar{v}) \in \overline{\varphi_0}(\overline{V_1})$.

(existence) Define $\mathcal{C}_V \doteq \{(v_n)_{n\geq 1}$ Cauchy sqce in $V\}$

$\mathcal{C}_V$ is a vector space for $\lambda(v_n) + \mu(w_n) = (\lambda v_n + \mu w_n)$, $(v_n), (w_n) \in \mathcal{C}_V$, $\lambda, \mu \in \mathbb{F}$ ⌈civilised⌋

\# Define $\mathcal{N}_V \doteq \{(v_n)$ sqce of V s.t. $v_n \to 0\}$

Note $\mathcal{N}_V$ subspace of $\mathcal{C}_V$ for the same operations

Define finally $\overline{V} \doteq \mathcal{C}_V / \mathcal{N}_V$ which is a vector space for the operations above.

Denote element of $\overline{V}$ by $[(v_n)]$ the class of equivalence of (v_n).

\# Endow $\overline{V}$ with norm

$$\| [(v_n)] \|_* \doteq \underbrace{\lim_{n\to\infty} \|v_n\|}_{\substack{\text{exists since} \\ (\|v_n\|) \text{ Cauchy in } \mathbb{R}}}$$

Well-defined since $\|[(v_n)]\|_*$ does not depend on representative of class $[(v_n)]$.

$$[(v_n)] = [(w_n)] \Rightarrow v_n - w_n \xrightarrow[n\to\infty]{} 0$$
$$\Rightarrow \lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|w_n\|$$

Axioms of norm easy to check; exercise

\# Define $\varphi: V \to \overline{V}$, $v \mapsto [(v)]$ where (v) means stationary sqce v

$\|[(v)]\|_* = \lim_{n\to\infty} \|v\| = \|v\|$ so φ is linear, isometric

\# <u>Claim</u> $\overline{\varphi(V)} = \overline{V}$

Given $[(v_n)] \in \overline{V}$ and $\varepsilon > 0$; $\exists\, n_0 \geq 1$ s.t. $\|v_m - v_n\| < \varepsilon$ for $m, n \geq n_0$

Then $\|\varphi(v_{n_0}) - [(v_n)]\|_* = \lim_{n\to\infty} \|v_n - v_{n_0}\| \leq \varepsilon$

\# <u>Claim</u> $(\overline{V}; \|\cdot\|_*)$ is complete

Let $([(v_n^k)])_{k\geq 1}$ Cauchy sequence in $\overline{V}$

For each $k \geq 1$, $\exists\, w_k \in V$ s.t. $\|\varphi(w_k) - [(v_n^k)]\|_* \leq \frac{1}{2^k}$ (previous claim)

Consider sequence (w_k), is Cauchy in V:

$$\|w_{k_1} - w_{k_2}\|_V = \|\varphi(w_{k_1}) - \varphi(w_{k_2})\|_*$$
$$\leq \|\varphi(w_{k_1}) - [(v_n^{k_1})]\|_* + \|[(v_n^{k_1})] - [(v_n^{k_2})]\|_*$$
$$+ \|[(v_n^{k_2})] - \varphi(w_{k_2})\|_*$$
$$\leq \frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \underbrace{\|[(v_n^{k_1})] - [(v_n^{k_2})]\|_*}_{\substack{\to 0 \text{ as} \\ k_1, k_2 \to \infty}}$$

Hence $[(w_k)] \in \overline{V}$ and

\# Claim $\|[(w_n)] - [(v_n^k)]\|_* \to 0$ as $k \to \infty$

Given $\varepsilon > 0$, and calculate

$$\|w_n - v_n^k\| \leq \underbrace{\|w_n - w_k\|}_{\substack{\leq \varepsilon/2 \\ \text{for } k,n \text{ large}}} + \underbrace{\|w_k - v_n^k\|}_{\leq 2 \cdot \frac{1}{2^k} \text{ for } n \text{ large}}$$

$\leq \varepsilon/2$ for k, n large enough

Hence for k large enough,

$$\lim_{n \to \infty} \|w_n - v_n^k\| \leq \varepsilon. \quad \square$$

Def^n A complete Euclidean space is called a Hilbert space

Rk If V is Euclidean not complete, its completion $\overline{V}$ is a Hilbert space: the inner product extends by continuity to $\overline{V}$

Theorem (Abstract Riesz-Fischer)

Given H infinite dim separable ($\exists$ ctble dense subset) Hilbert space,

(i) There is a countable Hilbert basis $(e_n)_{n \geq 1}$; i.e. a countable orthonormal system $S = \{e_n : n \geq 1\}$ such that $\overline{\text{Span}(S)} = H$

(ii) Parseval identity

$$\forall v, w \in H, \ \langle v, w \rangle = \sum_{n=1}^{\infty} v_n \overline{w_n} \quad \text{where } v_n = \langle v, e_n \rangle, \ w_n = \langle w, e_n \rangle$$

(iii) $\phi: H \to \ell^2$, $v \mapsto (v_n)_{n \geq 1}$ with $v_n = \langle v, e_n \rangle$ is an isometric isomorphism (surjective in particular)

Proof (i) Consider $(y_n)_{n\geq 1}$ countable dense and go down the list removing inductively each y_n in span of the previous terms.

It yields $(z_n)_{n\geq 1}$ countable linearly independent with $\overline{\mathrm{Span}((z_n)_n)} = H$

Apply Gram-Schmidt orthogonalisation process to conclude.

(ii) Define $S_n \doteq \sum_{i=1}^{n} v_i e_i$

(S_n) is Cauchy in H:

$\| S_m - S_n \|^2 \leq \| \sum_{i=m+1}^{n} v_i e_i \|^2 \leq \sum_{i=m+1}^{n} |v_i|^2 \longrightarrow 0$ from Bessel $\leq$

$\left(\sum_{i\geq 1} |v_i|^2 < +\infty \right)$

$\langle S_n - v, e_i \rangle = 0$ for $n \geq i$

$\Rightarrow S = \lim S_n$ satisfies $S - v \perp \overline{\mathrm{Span}(e_n)} = H \Rightarrow S = v$

Similarly $S_n' = \sum_{i=1}^{n} w_i e_i \longrightarrow w$

Finally, $\langle S_n, S_n' \rangle \underset{Pyth}{=} \underbrace{\sum_{i=1}^{n} v_i \overline{w_i}}_{\text{abs converging by Hölder}} \Rightarrow \sum_{i=1}^{\infty} v_i \overline{w_i}$

$\langle S_n, S_n' \rangle \xrightarrow{n\to\infty} \langle v, w \rangle$.

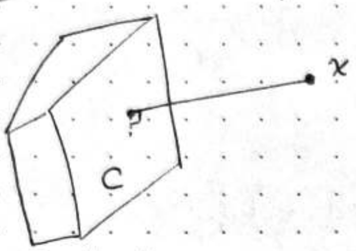
(iii) ϕ linear and isometric by case of equality in Bessel $\leq$, surjective by $v \doteq \sum_{n\geq 1}^{\infty} x_n e_n$ $\square$

L16.1

<u>Projections and Riesz-Fréchet Representation Theorem</u>

<u>Intro</u> Last of 4 lectures on Hilbertian analysis

- Fundamental property related to inner product structure = orthogonal projection



C non-empty convex complete subset of E Euclidean

Particularise to $C = F$ closed subset of H Hilbert space, deduce orthogonal decomposition criteria & Riesz-Fréchet theorem

<u>Thm</u> E Euclidean space, $C \subset E$ non-empty convex complete (as a metric subspace of E). Then

- (i) $\forall x \in E$, $\exists ! \; y = p_C(x) \in C$ s.t.

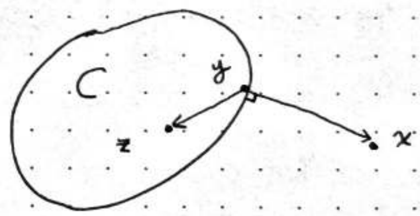
$$d(x, C) \doteq \inf_{z \in C} \|x - z\| = \|x - y\|$$

(ii) This $y = p_C(x)$ (the "<u>orthogonal projection</u>" of x on C) is characterised by $\forall z \in C$, $\operatorname{Re}\langle z - y, x - y\rangle \leq 0$.

(iii) $p_C : E \to C$ is 1-Lipschitz:

$\forall x, x' \in E$, $\|p_C(x) - p_C(x')\| \leq \|x - x'\|$

in particular p_C continuous



- <u>Proof</u> <u>(i) - Existence</u>

If $x \in C$, take $y = p_C(x) = x$.

If $x \notin C$, take a minimising sequence (by def^n of infimum) $(y_n)_{n \geq 1}$ s.t.

$$\|y_n - x\|^2 \leq d(x, C)^2 + \frac{1}{n}, \quad y_n \in C$$

Let us prove that (y_n) is Cauchy:

<u>Parallelogram law</u>: $\|y_m - y_n\|^2 + \|y_m + y_n - 2x\|^2 = 2\|y_m - x\|^2 + 2\|y_n - x\|^2$

$$\Rightarrow \frac{\|y_m - y_n\|^2}{2} = \|y_m - x\|^2 + \|y_n - x\|^2 - 2\Big\| \underbrace{\frac{y_m + y_n}{2}}_{\in C} - x \Big\|^2$$

$$\leq d(x,C)^2 + \frac{1}{m}$$

$$+ d(x,C)^2 + \frac{1}{n} - 2\, d(x,C)^2$$

$$= \frac{1}{m} + \frac{1}{n}$$ So (y_n) is Cauchy.

So since C is complete, $y_n \to y$ for some $y \in C$,

and $\|y - x\| = \lim_n \|y_n - x\| = d(x; C)$

(i) – Uniqueness

Consider $y_1, y_2 \in C$ s.t.

$$\|y_1 - x\| = \|y_2 - x\| = d(x, C)$$

Parallelogram Law: $\dfrac{\|y_1 - y_2\|^2}{2} + 2\left\|\dfrac{y_1+y_2}{2} - x\right\|^2 = \|y_1 - x\|^2 + \|y_2 - x\|^2$

$\therefore \dfrac{\|y_1 - y_2\|^2}{2} \leq 2d(x,C)^2 - 2d(x,C)^2 = 0$ $\quad \ddot{o} \quad \therefore \|y_1 - y_2\| = 0$

(ii) Let us prove that $y = p_C(x)$ satisfies

$\forall z \in C, \ \mathrm{Re}\langle y - z, y - x\rangle \leq 0$

[other direction straightforward]

Consider $z \in C$ and $\lambda \in (0,1]$ and

$$\|\underbrace{\lambda z + (1-\lambda) y}_{\in C} - x\|^2 \geq d(x,C)^2 = \|y - x\|^2$$

$$\|y - x + \lambda(z - y)\|^2 \geq \|y - x\|^2$$

$$\cancel{\|y - x\|^2} + 2\lambda \,\mathrm{Re}\langle y - x; z - y\rangle + \lambda^2 \|z - y\|^2 \geq \cancel{\|y - x\|^2}$$

$\therefore 2\,\mathrm{Re}\langle y - x, z - y\rangle + \lambda \|z - y\|^2 \geq 0$ $\quad$ ($\lambda > 0$ divided out)

Take $\lambda \searrow 0^+$ ($\lambda = 1/k$ say) to deduce

$$\mathrm{Re}\langle y - x, z - y\rangle \geq 0$$

Hence $\forall z \in C$, $\mathrm{Re}\langle y - x, z - y\rangle \geq 0$ (heh)

(iii) Consider $x, x' \in E$, y, y' the projections on C, and use what we just proved in (ii):

$$\begin{cases} \forall z \in C, & \mathrm{Re}\langle y - x, y - z\rangle \leq 0 \quad ① \\ \forall z' \in C, & \mathrm{Re}\langle y' - x', y' - z'\rangle \leq 0 \quad ② \end{cases}$$

① with $z = y'$ + ② with $z' = y$ gives

$$\mathrm{Re}\langle (y - x) - (y' - x'), y - y'\rangle \leq 0$$

$$\mathrm{Re}\left[\|y - y'\|^2 + \langle x' - x, y - y'\rangle\right] \leq 0$$

$$\Rightarrow \|y - y'\|^2 \leq |\langle x' - x, y - y'\rangle| \leq \|x' - x\| \|y - y'\|$$

$\Rightarrow \|y-y'\| \leq \|x-x'\|$ as desired $\square$

Examples 1. E Euclidean ; C finite dim subspace

2. $E = H$ Hilbert space ; C closed subspace

3. $E = H$ Hilbert space, $C = \overline{B}(0,1)$

then $p_C(x) = \begin{cases} x, & \|x\| \leq 1, \\ \frac{x}{\|x\|}, & \|x\| > 1. \end{cases}$ ✓

Application to orthogonal decomposition (particularising to $C = F$ complete subspace)

Thm E Euclidean, $F \leq E$ (closed); complete subspace

Then $F \oplus F^\perp = E$, and the unique decomposition in this sum is

$$\forall x \in E, \quad x = \underbrace{p_F(x)}_{\in F} + \underbrace{(x - p_F(x))}_{\in F^\perp}$$

(Rk) Moreover, p_F is linear $E \to F$; $p_F|_F = id_F$, $p_F|_{F^\perp} = 0$;

$p_F^2 = p_F$

Rk 1) F closed subspace of a Hilbert space or F finite dimensional of a Euclidean space satisfy the assumption

2) $\|p_F\| \leq 1$ (equality if F non-zero)

$\|1 - p_F\| \leq 1$ (equality if $F \neq E$)

Pf p_F exists (and unique) from previous theorem-(i)

Moreover, (ii) $\Rightarrow$

$\forall z \in F, \quad \mathrm{Re}\langle z - p_F(x), x - p_F(x)\rangle \leq 0 \quad (*)$

F subspace $\Rightarrow F - p_F(x) = F$

$(*) \iff \forall z \in F, \quad \mathrm{Re}\langle z, x - p_F(x)\rangle \leq 0$

F subspace $\Rightarrow F = -F$

$(*) \iff \forall z \in F, \quad \mathrm{Re}\langle z, x - p_F(x)\rangle = 0$

$\Rightarrow x - p_F(x) \perp F$ (i.e. get rid of Re)

Hence $x = \underset{F}{\underset{\Uparrow}{p_F(x)}} + \underset{F^\perp}{\underset{\Uparrow}{(x - p_F(x))}}$ and so $E = F \oplus F^\perp$

(sum always direct)

Moreover, uniqueness and linearity of decomposition

$\Rightarrow$ P_F is linear

Other properties are standard for projections $\square$

<u>Cor</u> H Hilbert space, $S \subset H$ subset, then $\overline{\mathrm{Span}(S)} = H$ iff $S^\perp = \{0\}$

<u>Rk</u> Powerful criterion for checking if an orthonormal system $S = \{(e_n)_{n\geq 1}\}$ is maximal (and thus a Hilbert basis)

<u>Pf</u> $F = \overline{\mathrm{Span}(S)}$ is complete subspace

$F^\perp = S^\perp$ from results in last lectures $\square$

<u>Ex</u> ① $F = \mathrm{Span}(e_1, \dots, e_n)$ in E Euclidean with (e_i) orthonormal, then $P_F(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

② $F = \overline{\mathrm{Span}\{(e_n)_{n\geq 1}\}}$ in H Hilbert and $(e_n)_{n\geq 1}$ orthonormal,

$P_F(x) = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$

Application to dual space of a Hilbert space

<u>Theorem</u> (Riesz-Fréchet representation)

H Hilbert, define $\phi : H \to H^*$ (dual of H)

$v \mapsto \phi_v$ where $\phi_v(w) = \langle w, \cancel{x}\, v \rangle$ $\forall w \in H$

Then ϕ is an isometric (anti-) isomorphism

<u>Rks</u> 1) ϕ well-defined even in Euclidean spaces (but not surjective in general)

2) "Anti"-isomorphism means when H complex

$\phi(\lambda v) = \bar{\lambda}\, \phi(v)$, $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$

3) Thm $\Rightarrow$ H^* also a Hilbert space, and realized in a very explicit way.

<u>Proof</u> Isometry and (sesqui)-linearity clear: Only need to show surjectivity.

Let $f \in H^* \setminus \{0\}$, $F = \mathrm{Ker} f$ closed since f cts

Then $H = \mathrm{Ker} f \oplus V$ for V 1-dim subspace ($V = F^\perp$)

Hence $F^\perp = \mathrm{Span}(x_0)$ for some $x_0 \neq 0$. Define $x = a x_0$, $a \in \mathbb{C}$ (or $\mathbb{R}$), such that $\|x\|^2 = f(x)$. Then $\varphi_x = f$ on F, on $F^\perp$ hence on H. $\square$

The Spectrum in Infinite Dimension

Intro # Given $T: B \to B$ linear bdd on B a Banach space, can we define eigenvalues? Can we "reduce" T like we diagonalise some matrices?

Spectrum turns out to be more subtle in infinite dimension than for matrices
[Even more general theory for unbounded operators with closed graph defined on a domain, e.g. Δ on L^2...]

Define / classify spectrum Lecture 1

Extend notion of symmetry / hermitian to infinite dimension, introduce compact operators Lecture 2

The Spectral theorem for self-adjoint cpct operators Lecture 3

Applications of Spectral (Sturm-Liouville & Schrödinger eq^{n}s) Lecture 4

Note From now on, Banach & Hilbert spaces are complex

Def^n B Banach space, $T \in \mathcal{B}(B) = \mathcal{B}(B, B)$ a linear bounded operator

Resolvent set of T is

$$\rho(T) \doteq \{ \lambda \in \mathbb{C} : T - \lambda = T - \lambda \mathrm{Id} \text{ is invertible, i.e. bijective with bdd inverse} \}$$

Spectrum of T is

$$\sigma(T) \doteq \mathbb{C} \setminus \rho(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not invertible} \}$$

Resolvent map of T is

$$R_T : \rho(T) \to \mathcal{B}(B) \; ; \; \lambda \mapsto R_T(\lambda) = (T - \lambda)^{-1}$$

Rks
1) Given $T: B \to B$ linear bijective, T^{-1} well-defined linear map
2) Given $T: B \to B$ linear bounded bijective, T^{-1} bdd by the inverse mapping thm, hence $\lambda \in \rho(T) \iff (T - \lambda)$ bijective
[not the case for unbounded operators ...]

L17.2

<u>Prop</u> Given $\mathcal{B}$ Banach space, $T \in \mathcal{B}(B)$

(i) $\rho(T)$ open, equiv $\sigma(T)$ closed, with more precise bound:

if $\lambda_0 \in \rho(T)$ then $D(\lambda_0; \|R_T(\lambda_0)\|^{-1}) \subset \rho(T)$

↑ open disk

(ii) $R_T : \rho(T) \to \mathcal{B}(B)$ is analytic; i.e. can be represented by an absolutely convergent power series in any small enough disc, and has a $\mathbb{C}$-derivative

(iii) $\sigma(T) \neq \emptyset$ and $\sigma(T) \subseteq \{\lambda \in \mathbb{C},\ |\lambda| \leq \|T\|\}$

<u>Proof</u> Proof of (i) and (ii) relies on a deep & general argument due to Carl Neumann (1877):

<u>Claim</u> (1) Given $T \in \mathcal{B}(B)$ with $\|T\| < 1$, then $1 - T$ is invertible

with $(1-T)^{-1} = \sum_{n \geq 0} T^n \quad (T^0 = 1)$ XD

$$\|(1-T)^{-1}\| \leq \frac{1}{1 - \|T\|}$$

(2) As a csqce, subset $\mathcal{I}(B) \subseteq \mathcal{B}(B)$ of bdd invertible operators is open in $\mathcal{B}(B)$.

<u>Proof of (1) ⇒ (2)</u> $T \in \mathcal{I}(B)$ and $U \in \mathcal{B}(B)$ with $\|U - T\| < \frac{1}{\|T^{-1}\|}$

then $U = T + (U - T)$

$$= T\left(1 + T^{-1}(U-T)\right)$$

$$\|T^{-1}(U-T)\| \leq \|T^{-1}\| \, \|U - T\| < 1$$

Point (1) ⇒ $1 + T^{-1}(U-T)$ invertible, hence U invertible, as a composition of two invertible operators.

<u>Proof of (1)</u> Note that $\|T^n\| \leq \|T\|^n$

Hence $A_N = \sum_{n=0}^{N} T^n$ is Cauchy in $\mathcal{B}(B)$ *yawn*

$N_1 < N_2$, $\|A_{N_2} - A_{N_1}\| = \left\| \sum_{n=N_2+1}^{N_1} T^n \right\| \leq \sum_{n=N_1+1}^{N_2} \|T\|^n \leq \|T\|^{N_1+1} \frac{1}{1 - \|T\|}$

So A_N converges in $\mathcal{B}(B)$, denote

$$A = \sum_{n=0}^{\infty} T^n$$

Now $(1-T)A_N = A_N(1-T) = \sum_{n=0}^{N} T^n - \sum_{n=1}^{N+1} T^n = 1 - T^{n+1} \to 1$ in $\mathcal{B}(B)$

So A inverse of $(1-T)$.

<u>Application of the claim</u>

$\lambda_0 \in \rho(T)$, then if $\lambda \in D(\lambda_0, \|R_T(\lambda_0)\|^{-1})$,

$$(T-\lambda) = (T-\lambda_0) - (\lambda-\lambda_0)$$
$$= (T-\lambda_0)\left[1 - \underbrace{R_T(\lambda_0)(\lambda-\lambda_0)}_{\|\cdot\|<1}\right]$$

Hence by claim $[-]$ invertible hence $T-\lambda$ as well; and

$$\|(T-\lambda)^{-1}\| \leq \frac{\|R_T(\lambda_0)\|}{1-\|R_T(\lambda_0)\|^{+1}|\lambda-\lambda_0|}$$ This proves (i)

Moreover, we have the Neumann series

$$(1-R_T(\lambda_0)(\lambda-\lambda_0))^{-1} = \sum_{n\geq 0} (\lambda-\lambda_0)^n R_T(\lambda_0)^n \in \mathcal{B}(B)$$

Hence $(T-\lambda)^{-1} = \sum_{n\geq 0} (\lambda-\lambda_0)^n R_T(\lambda_0)^{n+1}$ for $\lambda \in D(\lambda_0, \|R_T(\lambda_0)\|^{-1})$

which proves (ii).

<u>What is the $\mathbb{C}$-derivative?</u>

$\lambda_0 \in \rho(T)$; $\lambda \in D(\lambda_0, \|R_T(\lambda_0)\|^{-1})$

$$(T-\lambda_0)^{-1} - (T-\lambda)^{-1} = (T-\lambda_0)^{-1}\left[\underbrace{(T-\lambda)-(T-\lambda_0)}_{(\lambda_0-\lambda)}\right](T-\lambda)^{-1}$$

$$\Rightarrow \frac{(T-\lambda_0)^{-1}-(T-\lambda)^{-1}}{\lambda_0-\lambda} = (T-\lambda_0)^{-1}(T-\lambda)^{-1}$$

check!

$(T-\lambda)^{-1} \xrightarrow[\lambda\to\lambda_0]{} (T-\lambda_0)^{-1}$ in $\mathcal{B}(B)$, then

$$\exists \underbrace{\lim_{\lambda\to\lambda_0} \frac{(T-\lambda_0)^{-1}-(T-\lambda)^{-1}}{\lambda_0-\lambda}}_{\frac{d}{d\lambda}(T-\lambda)^{-1}\big|_{\lambda=\lambda_0}} = (T-\lambda_0)^{-2}$$

<u>Proof of (iii)</u> Given $\lambda \in \mathbb{C}$, $|\lambda| > \|T\|$, then

(*) $T-\lambda = -\lambda[1-\lambda^{-1}T]$, $\|\lambda^{-1}T\| < 1$

hence (claim) $(T-\lambda)$ invertible with

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\lambda\left(1-\frac{\|T\|}{\lambda}\right)} = \frac{1}{\lambda-\|T\|}$$

hence $\sigma(T) \subset \{\lambda \in \mathbb{C},\ |\lambda| \leq \|T\|\}$

L17.4

To prove that $\sigma(T) \neq \emptyset$, argue by contradiction and assume $\sigma(T) = \emptyset$, hence $\rho(T) = \mathbb{C}$.

<u>Claim</u> Then $\forall x \in B$, $\varphi \in B^*$, the map

$$F_{x,\varphi} : \mathbb{C} \to \mathbb{C}, \ \lambda \mapsto \varphi(R_T(\lambda)x) \quad \text{is } \underline{\text{constant}}$$

<u>Pf of claim</u> • $F_{x,\varphi}(\lambda)$ is entire on $\mathbb{C}$ because $R_T(\lambda)$ is and taking on x / against φ is linear

• $|F_{x,\varphi}(\lambda)| \leq \|\varphi\| \, \|R_T(\lambda)\| \, \|x\|$

$$\leq \frac{\|\varphi\| \, \|x\|}{\lambda - \|T\|} \quad \text{for } |\lambda| > \|T\|$$

$$\longrightarrow 0 \quad \text{as } |\lambda| \to \infty$$

<u>Liouville thm</u> $\Rightarrow$ $F_{x,\varphi}$ is constant

This implies that $\mathbb{C} \to \mathcal{B}(B)$, $\lambda \mapsto R_T(\lambda)$ is constant:

if not, $\exists \lambda_1 \neq \lambda_2 \in \mathbb{C}$ s.t. $R_T(\lambda_1) \neq R_T(\lambda_2)$

$\therefore \exists x \in B$ s.t. $R_T(\lambda_1)x \neq R_T(\lambda_2)x$

$\therefore$ by Hahn-Banach $\Rightarrow \exists \varphi \in B^*$ s.t. $\varphi R_T(\lambda_1)x \neq \varphi R_T(\lambda_2)x$,

in contradiction to our claim above

Finally observe that $\lambda_1 \neq \lambda_2$ always implies $R_T(\lambda_1) \neq R_T(\lambda_2)$ since $T - \lambda_1$ is not $T - \lambda_2$, lmao. $\square$

<u>Rk</u> $\sigma(T) \neq \emptyset$ is the equivalent in infinite dimension of the fundamental theorem of algebra when formulated on char poly of a matrix. Just like the latter, its proof requires some analysis.

Classification of the Spectrum & Examples

<u>Def^n</u> B Banach space; $T \in \mathcal{B}(B)$, then spectrum of T

$\sigma(T)$ splits as:

\# <u>point spectrum</u> (set of eigenvalues)

$$\sigma_p(T) \doteq \{ \lambda \in \sigma(T) : (T-\lambda) \text{ not injective} \}$$

\# <u>continuous spectrum</u>

$$\sigma_c(T) \doteq \{ \lambda \in \sigma(T) : (T-\lambda) \text{ injective}, \ \mathrm{Im}(T-\lambda) \text{ dense in } B \}$$

\# <u>residual spectrum</u>

$$\sigma_r(T) \doteq \{ \lambda \in \sigma(T) : (T-\lambda) \ \overset{?}{\cancel{\text{not}}} \text{ injective}, \ \mathrm{Im}(T-\lambda) \text{ not dense in } B \}$$

<u>Prop</u> (i) $T \in \mathcal{B}(B)$ invertible iff the two following propeties are satisfied:

(1) T bdd below, $\exists \delta > 0$ s.t.

$$\forall x \in B, \ \|Tx\| \geq \delta \|x\|$$

(2) $\mathrm{Im}(T)$ dense in B

(ii) $\sigma_c(T)$ is included in approximate point spectrum:

$$\sigma_{ap}(T) = \{ \lambda \in \sigma(T) : \exists (x_n) \text{ in } B, \|x_n\| = 1 \text{ s.t. } (T-\lambda)x_n \to 0 \}$$

<u>Proof</u> (i) $\Rightarrow$ (ii) since $\lambda \in \sigma_c(T)$ then $\mathrm{Im}(T-\lambda)$ is dense and $(T-\lambda)$ is injective, therefore since $(T-\lambda)$ non-invertible, $(T-\lambda)$ is not bdd below, which implies $\lambda \in \sigma_{ap}(T)$.

<u>Pf of (i)</u> "$\Rightarrow$" clear

"$\Leftarrow$" $(T-\lambda)$ injective (bdd below), there is map $(T-\lambda)^{-1}$ defined on $\mathrm{Im}(T-\lambda)$, and if $\mathrm{Im}(T-\lambda) = B$, is bdd since $(T-\lambda)^{*}$ bdd below.

What remains to prove is $\mathrm{Im}(T-\lambda) = B$.

STP $\mathrm{Im}(T-\lambda)$ is <u>closed</u>, given it is dense in B

Consider (y_n) converging sqce in $\mathrm{Im}(T-\lambda)$, $y_n \to y$ ($y \overset{\text{a priori}}{\notin} \mathrm{Im}(T-\lambda)$)

Then $\exists (x_n)$ s.t. $(T-\lambda)x_n = y_n$

(y_n) Cauchy hence ($(T-\lambda)$ bdd below),

$$\|x_m - x_n\| \leq C \|y_m - y_n\| \text{ and thus } (x_n) \text{ Cauchy}$$

B complete hence $x_n \to x$; $(T-\lambda)x = y$ by continuity. $\square$

Examples

(I) Finite-dimension $B = \mathbb{R}^d$. Then $(T-\lambda)$ injective $\Leftrightarrow$ bijective
$\Leftrightarrow$ invertible

Hence $\sigma(T) = \sigma_p(T)$ and $\sigma(T) = \{\lambda \in \mathbb{C} : \det(T-\lambda) = 0\}$ contains at most d points (hence $\rho(T)$ is dense in $\mathbb{C}$)

(II) Infinite-dimension, $B = \ell^2_{\mathbb{C}}$

Left shift operator $T_\ell((x_1, x_2, \dots)) = (x_2, x_3, \dots)$

Then $\|T_\ell\| = 1$ hence $\sigma(T_\ell) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

Every $\lambda \in \mathbb{C}$, $|\lambda| < 1$ is an eigenvalue with eigenvector
$(1, \lambda, \lambda^2, \dots) \in \ell^2$

On $|\lambda| = 1$, $(T_\ell - \lambda)$ injective but $\mathrm{Im}(T_\ell - \lambda)$ dense in ℓ^2
(for instance it reaches all sequences with finite support)

Therefore $\sigma_p(T_\ell) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$

$\sigma_c(T_\ell) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

$\sigma_r(T_\ell) = \emptyset$

Right shift operator $T_r((x_1, x_2, \dots)) = (0, x_1, x_2, \dots)$

Again $\|T_r\| = 1$ and so $\sigma(T_r) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

$\sigma_p(T_r) = \emptyset$ since $(T_r - \lambda)((x_n)) = 0 \Rightarrow (x_n) = 0$
(check separately $\lambda = 0, \lambda \neq 0$)

for $|\lambda| \leq 1$?
$\mathrm{Im}(T_r - \lambda)$ not dense in ℓ^2 since, e.g. with $\lambda = 0$, $(1, 0, 0, \dots)$ is not reached and isolated in $\ell^2 \setminus \mathrm{Im}(T)$

$\Rightarrow \sigma_c(T_r) = \emptyset$

$\sigma_r(T_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

can check $\sigma_{ap}(T_r) = \text{😊}$

Note that T_r is "adjoint" of T_ℓ

(III) Given any $K \subseteq \mathbb{C}$ cpct, there is $T \in \mathcal{B}(B)$; $B = \ell^2 = H$ separable Hilbert ; so that $\sigma(T) = K$.

[Argue as for diagonal matrices]

Consider $(e_n)_{n \geq 1}$ Hilbert basis of H, and set

$$T(e_n) = \lambda_n e_n \quad , n \geq 1$$

where $(\lambda_n)_{n \geq 1}$ countable set in K, dense in K

T extends to $\mathcal{B}(H)$ by linearity

$\sigma(T) \supset \{\lambda_n : n \geq 1\}$

So $\sigma(T) \supset K = \overline{\{\lambda_n\}}$

If $\lambda \notin K$, $\exists\ \alpha > 0$ s.t. $d(\lambda, K) \doteq \alpha > 0$

and $(T - \lambda)$ is invertible with

$(T-\lambda)^{-1}$ defined by $(T-\lambda)^{-1}(e_n) = \dfrac{1}{\lambda_n - \lambda} e_n$

$\|(T-\lambda)^{-1}\| \leq \dfrac{1}{\alpha}$

Normal and Self-adjoint operators

Intro # "Add" inner product structure and Riesz representation theorem
Study effect on spectrum when the operator has specific relations to adjoint

Def V, W NVS and $T \in \mathcal{B}(V,W)$. Then ~~dual~~ adjoint of T,
$T^* \in \mathcal{B}(W^*, V^*)$ is defined by $(T^*f)(x) = f(Tx)$
We proved that $\|T^*\| = \|T\|$.

Let us particularise this def^n to Hilbert spaces.

Def Let H Hilbert space and $T \in \mathcal{B}(H)$, and denote $\theta: H \longrightarrow H^*$ the isometric (anti-) isomorphism from Riesz representation theorem.

Then we define $\tilde{T}^* x \doteq \theta^{-1} T^*(\theta x)$
(Well-defined linear bdd operator on H)

Prop $\tilde{T}^* \in \mathcal{B}(H)$ and $\forall x, y \in H$, $\langle Tx, y \rangle = \langle x, \tilde{T}^* y \rangle$

Proof $\langle x, \tilde{T}^* y \rangle = \langle x, \theta^{-1} T^* \theta y \rangle$
$= T^*(\theta y)(x)$ def^n of θ^{-1}
$= (\theta y)(Tx)$
$= \langle Tx, y \rangle$ def^n of θ
$\tilde{T}^* \in \mathcal{B}(H)$ by composition. $\square$

We always, from now on, identify $\tilde{T}^*$ with T^* in Hilbert spaces.

BASIC EXAMPLES

$Id^* = 1^* = 1$
$(\lambda Id)^* = \bar{\lambda}\, Id$
$\underset{\substack{\lambda, \mu \in \mathbb{C},\\ S, T \in \mathcal{B}(H)}}{(\lambda S + \mu T)^*} = \bar{\lambda} S^* + \bar{\mu} T^*$
$(ST)^* = T^* S^*$
$(T^*)^* = T$
In finite dimension, this is the transposed conjugate of a matrix

Def H Hilbert space; $T \in \mathcal{B}(H)$ is

- normal if $T^*T = TT^*$ (commutes w/ adjoint)
- self-adjoint (aka Hermitian) if $T^* = T$
- unitary if $T^* = T^{-1}$ (T invertible)

Prop (i) self-adjoint and $\sim$ $\Rightarrow$ normal
unitary $\Rightarrow$ normal

(ii) if T normal, then

(a) $\|Tx\| = \|T^*x\| \quad \forall x \in H$

(b) $\operatorname{Ker} T = \operatorname{Ker} T^* = (\operatorname{Im} T)^\perp = (\operatorname{Im} T^*)^\perp$

(c) $\overline{\operatorname{Im} T} = \overline{\operatorname{Im} T^*}$ (top closure)

(iii) if T normal, then $\sigma_r(T) = \phi$

(iv) if T normal, $(T-\lambda)x = 0 \iff (T^* - \bar{\lambda})x = 0$

and $\sigma_p(T^*) = \overline{\sigma_p(T)}$ (cx conj)

(v) if T normal, ~~$\forall x$~~ $(T-\lambda)x_n \xrightarrow[n\to\infty]{} 0$ iff $(T^* - \bar{\lambda})x_n \xrightarrow[n\to\infty]{} 0$

and $\sigma_{ap}(T^*) = \overline{\sigma_{ap}(T)}$

(vi) if T normal and (x, λ), $(y; \mu)$ two eigenpairs: $x, y \neq 0$, $\lambda \neq \mu$

then $x \perp y$

(vii) T self-adjoint iff $\langle Tx, x \rangle \in \mathbb{R} \ \forall x \in H$

(viii) if T self-adjoint, then $\sigma(T) \subset \mathbb{R}$

Proof (i) clear

(ii) - (a)
$$\langle Tx, Tx \rangle = \langle T^*Tx ; x \rangle = \langle TT^*x, x \rangle = \langle T^*x, T^*x \rangle$$

(ii) - (b) $\operatorname{Ker} T = \operatorname{Ker} T^*$ from $\|Tx\| = \|T^*x\|$

then $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$, $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ (check)

which proves (b)

(ii)-(c) note if V subspace of H, then $(V^{\perp})^{\perp} = \overline{V}$ (top closure)

[Indeed: $\overline{V} \oplus \underset{\substack{\uparrow \\ V^{\perp}}}{\overline{V}^{\perp}} = H$ by orthog decomposition]

Thus $\overline{\operatorname{Im} T} = ((\operatorname{Im} T)^{\perp})^{\perp} = (\operatorname{Ker} T^*)^{\perp} = (\operatorname{Ker} T)^{\perp} = ((\operatorname{Im} T^*)^{\perp})^{\perp}$

$= \overline{\operatorname{Im} T^*}$

(iii) Follows from (ii)-(b) since

$\operatorname{Ker} T = \{0\} \Rightarrow \overline{\operatorname{Im} T} = H$

(iv) & (v) T normal $\Rightarrow$ $(T-\lambda)$ normal

$$
\begin{aligned}
(T-\lambda)^*(T-\lambda) &= (T^*-\bar{\lambda})(T-\lambda) \\
&= T^*T + |\lambda|^2 - \bar{\lambda}T - \lambda T^* \\
&= TT^* + |\lambda|^2 - \bar{\lambda}T - \lambda T^* \\
&= (T-\lambda)(T^*-\bar{\lambda})
\end{aligned}
$$

Hence $\|(T-\lambda)x\| = \|(T^*-\bar{\lambda})x\|$

(vi) T normal, $x \neq 0$, $y \neq 0$, $\lambda \neq \mu$ in $\mathbb{C}$

$Tx = \lambda x$, $Ty = \mu y$

Then (iv) $\Rightarrow$ $T^*y = \bar{\mu} y$, and

$$
\begin{aligned}
\lambda\langle x, y\rangle &= \langle Tx, y\rangle = \langle x, T^*y\rangle \\
&= \langle x, \bar{\mu} y\rangle = \mu\langle x, y\rangle \quad \text{done ✓}
\end{aligned}
$$

(vii) T s.a. iff $\langle Tx, x\rangle \in \mathbb{R}$ $\forall x \in H$ (?)

$A \doteq T - T^*$, then $\forall x \in H$,

$$
\begin{aligned}
(*) \quad \langle Ax, x\rangle &= \langle Tx, x\rangle - \langle T^*x, x\rangle \\
&= \langle Tx, x\rangle - \langle x, Tx\rangle \\
&= \langle Tx, x\rangle - \overline{\langle Tx, x\rangle} \\
&= 0 \quad \text{if } \langle Tx, x\rangle \in \mathbb{R}
\end{aligned}
$$

Then $(*) \Rightarrow A = 0$ by expanding $\langle A(x-y), (x-y)\rangle$

$\langle A(x-iy), (x-iy)\rangle$

(viii) T s.a. $\Rightarrow \sigma(T) \subset \mathbb{R}$

Given $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then

$$U \doteq (T-\lambda)^*(T-\lambda) = (T^* - \bar{\lambda})(T-\lambda)$$
$$= (T-\bar{\lambda})(T-\lambda) = T^2 + |\lambda|^2 - 2(\mathrm{Re}\,\lambda)\,T$$
$$= |\mathrm{Im}\,\lambda|^2 + (T - \mathrm{Re}\,\lambda)^*(T - \mathrm{Re}\,\lambda)$$

Therefore, $\|(T-\lambda)x\|^2 = \langle Ux, x\rangle$
$$= |\mathrm{Im}\,\lambda|^2 \|x\|^2 + \|(T - \mathrm{Re}\,\lambda)x\|^2$$
$$\geq |\mathrm{Im}\,\lambda|^2 \|x\|^2$$

Hence $(T-\lambda)$ bdd below, hence $\mathrm{Ker}(T-\lambda) = \{0\}$ and (ii) implies $\overline{\mathrm{Im}(T-\lambda)} = H$, so by char^n of invtble, $T-\lambda$ invertible with $\|(T-\lambda)^{-1}\| \leq \frac{1}{|\mathrm{Im}\,\lambda|}$ ☒

Rk Unitary operators have their spectrum in $\{\lambda \in \mathbb{C}, |\lambda| = 1\}$ ExSh 4

Examples 1) If $A \in \mathcal{B}(H)$ self-adjoint, then $S = iA$ satisfies
$$S^* = -S \text{ (skew-adjoint or skew-Hermitian)}$$

2) If $A \in \mathcal{B}(H)$ self-adjoint then
$$e^{iA} \doteq \sum_{n \geq 0} \frac{1}{n!}(iA)^n \in \mathcal{B}(H) \text{ and is unitary}$$

e^{iAt} defined similarly for $t \in \mathbb{R}$, and $e^{iAt}\psi_0 \doteq \psi_t$ solves $-i\partial_t \psi_t = A\psi_t$ (c.f. Schrödinger eq^n)

3) Diagonal matrices with real coeffts and operators "diagonal" on a Hilbert basis $Te_n = \lambda_n e_n$; $\lambda_n \in \mathbb{R}$ are self-adjoint

4) When you will have studied $L^2([0,1])$, key example of operator is
$$Kf(x) = \int_{[0,1]} k(x,y)\, f(y)\, dy$$

and K self adjoint if $k(x,y) = \overline{k(y,x)}$

<u>Prop</u> H Hilbert space:

(i) If $T \in \mathcal{B}(H)$ self-adjoint then

$$\|T\| = \sup_{\|x\| \le 1} \sup_{\|y\| \le 1} |\langle Tx, y\rangle| = \sup_{\|x\| \le 1} |\langle Tx, x\rangle|$$

(ii) Given $T \in \mathcal{B}(H)$, $\|T^*T\| = \|T\|^2$

(note that $\|T^2\| \le \|T\|^2$ but <u>not</u> equal in general)

(iii) Given $T \in \mathcal{B}(H)$, self-adjoint, then one of $\|T\|$ and $-\|T\|$ must be an approximate eigenvalue.

<u>Proof</u> (i) $\Rightarrow$ (ii) since T^*T is self-adjoint

thus $\|T^*T\| = \sup_{\|x\| \le 1} \langle T^*Tx, x\rangle = \sup_{\|x\| \le 1} \langle Tx, Tx\rangle = \|T\|^2$

(i) $\Rightarrow$ (iii) Replacing T by $-T$ if needed;

$\exists (x_n)$ on H, $\|x_n\| = 1$ s.t. $\langle Tx_n, x_n\rangle \longrightarrow \|T\| \doteq \lambda$

no abs value

then $\|(T-\lambda)x_n\|^2 = \|Tx_n\|^2 + \lambda^2\|x_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle$

$$\le \lambda^2 + \lambda^2 - \underbrace{2\lambda\langle Tx_n, x_n\rangle}_{\to -2\lambda^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

(i) We prove first $\|T\| = \sup_{\|x\| \le 1} \sup_{\|y\| \le 1} |\langle Tx, y\rangle|$

Take (y_n), $\|y_n\| = 1$ s.t. $\|Ty_n\| \longrightarrow \|T\|$ (assume $T \ne 0$, $\|T\| \ne 0$, so $\|Ty_n\| \ne 0$ $\forall n$)

Define $x_n \doteq \frac{Ty_n}{\|Ty_n\|}$ and

$$\langle Tx_n, y_n\rangle = \frac{1}{\|Ty_n\|}\langle T(Ty_n), y_n\rangle$$

$$= \frac{1}{\|Ty_n\|} \|Ty_n\|^2 = \|Ty_n\| \longrightarrow \|T\|$$

so done with the first bit.

Finally we prove $\sup_{\|x\| \le 1} \sup_{\|y\| \le 1} |\langle Tx, y\rangle| = \sup_{\|x\| \le 1} |\langle Tx, x\rangle|$

($\ge$) clear.

($\le$) $|\langle Tx, y\rangle| = \frac{1}{4}\left|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right|$ wlog $\langle Tx, y\rangle \in \mathbb{R}$

$$\le \frac{1}{4}\left[\sup_{\|z\| \le 1} |\langle Tz, z\rangle|\right]\left(\|x+y\|^2 + \|x-y\|^2\right)$$

L19.6.

$$\leq \sup_{\|z\|\leq 1} |\langle Tz, z\rangle|. \quad \square$$

Compact operators and a Theorem of Schauder

<u>Intro</u>: # Last 2 lectures, we have seen many notions that those in finite dimension

Spectrum is more subtle but many intutions are similar

Go back to initial question: are there classes of operators that we can understand / reduce (spectrum)?

Today we look at <u>compact</u> operators which are a natural extension of matrices in finite dimension (in fact in Hilbert spaces, compact operators <u>are</u> limit of finite rank operators, but false in general for Banach spaces)

Motivated by theory of integral equations, Fredholm, Schauder, Hilbert, ...

<u>Def</u> Given V, W Banach spaces and $T \in \mathcal{B}(V, W)$, T is said to be <u>compact</u> if $T(\bar{B}_V(0,1))$ is relatively compact in W.

<u>Rmks</u> ① Relatively compact means that the topological closure is compact. Since W is a complete metric space, this is equivalent to $T(\bar{B}_V(0,1))$ totally bounded

② When V is reflexive (notions seen in AoF next term: $V \sim V^{**}$ via the canonical embedding Φ) then $T(\bar{B}_V(0,1))$ is closed and "relatively" can be dropped.

In general, $T(\bar{B}_V(0,1))$ is not closed as shown in Ex 4 below.

④ If W has infinite dimension, saying that $T(\bar{B}_V(0,1))$ is relatively compact means that it is <u>very</u> small (strong assumption)

⑤ Thanks to the metric structure, we can (re)formulate the equivalent criterion $T \in \mathcal{B}(V,W)$, T cpct $\Longleftrightarrow$ $\forall (x_n)$ bdd in V, (Tx_n) has a convergent subsequence

⑥ Note $T(\bar{B}_V(0,1))$ rel cpct $\Longleftrightarrow$ $T(\bar{B}_V(0,1))$ tot bdd $\Rightarrow$ $T(\bar{B}_V(0,1))$ bdd

hence assumption T "bdd" could be dropped...

Examples ① If W finite-dimensional and $T \in \mathcal{B}(V; W)$ then T cpct ;
$(T(\bar{B}_V(0,1))$ has closed bdd closure in W hence cpct by B-W.

② If V, W f. dim all linear operators $V \to W$ are bdd compact

② Given V, W Banach spaces and $T \in \mathcal{B}(V; W)$ with $\mathrm{Im}(T)$ f.d. (T has finite rank) then T is compact

③ [Integral operator] Interval operator

$V = W = C_{\mathbb{C}}([0,1])$ and

$$T : V \to V, \; f \longmapsto \left(Tf(x) = \int_0^1 K(x,y) f(y) \, dy \right)$$

where $K : [0,1]^2 \to \mathbb{C}$ is continuous.

Then T is compact on V.

Indeed, $T(\bar{B}_V(0,1)) = \{ Tf \, , \, \|f\|_\infty \leq 1 \}$

is relatively compact iff (Arzelà-Ascoli) it is equi-bdd & equi-cts

- equi-bdd : $\|Tf\|_\infty \leq \|f\|_\infty \|K\|_\infty$
- equi-cts : $x_1, x_2 \in [0,1]$, $f \in \bar{B}_V(0,1)$

$$|Tf(x_1) - Tf(x_2)| \leq \|f\|_\infty \int_0^1 |K(x_1,y) - K(x_2,y)| \, dy$$

and since K is uniformly cts on the compact $[0,1]^2$,

$$\sup_{\substack{x_1, x_2 \in [0,1] \\ |x_1 - x_2| < \delta \\ y \in [0,1]}} |K(x_1,y) - K(x_2,y)| \longrightarrow 0 \quad \text{as } \delta \to 0$$

hence
$$\sup_{\substack{f \in \bar{B}_V(0,1) \\ |x_1 - x_2| < \delta}} |Tf(x_1) - Tf(x_2)| \longrightarrow 0 \quad \text{as } \delta \to 0.$$

④ [Integral operator] Anti-derivative operator

$V = C([0,1])$ and

$$T : V \to V, \; f \mapsto \left(Tf(x) = \int_0^x f(t) \, dt \right)$$

is compact : $\{ Tf \, , \, \|f\|_\infty \leq 1 \}$ is relatively compact

since (A-A thm) since it is

- equi-bdd $\|Tf\|_\infty \leq \|f\|_\infty$
- equi-cts $|Tf(x_1) - Tf(x_2)| \leq \int_{x_1}^{x_2} |f(t)| \, dt \leq \|f\|_\infty |x_1 - x_2| \leq |x_1 - x_2|$

Observe that $T(\bar{B}_V(0,1)) = \{ f \in C^1([0,1]) \ ; \ f(0)=0$ and $\|f'\|_\infty \leq 1 \}$ ← sth sth right derivative

is not closed in V

Facts ("lego" structure results)

Prop V, W, Z Banach spaces

(i) Space of compact operators $V \to W$

$\mathcal{B}_0(V,W)$ is a closed subspace of $\mathcal{B}(V,W)$

hence a Banach space itself

(ii) If $T \in \mathcal{B}(V,W)$, $S \in \mathcal{B}(W,Z)$ and one of S, T is compact, then $ST \in \mathcal{B}(V,Z)$ is compact

Proof (i) # S, T bdd $\in \mathcal{B}_0(V,W)$, $\lambda, \mu \in \mathbb{F}$, let us prove that $\lambda S + \mu T$ is cpct

$\lambda S + \mu T \in \mathcal{B}(V,W)$ already known

Use sequential characterisation: given (x_n) bdd sequence in V,

$(S(x_n))$ has a convergence subseq $(S x_{\varphi(n)})$ and

$(T x_{\varphi(n)})$ has then a further convergent subsequence

$(T x_{\varphi \circ \psi(n)})$ convergent ($\varphi, \psi : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing)

Then $(\lambda S + \mu T)(x_{\varphi \circ \psi(n)})$ converges and hence $\lambda S + \mu T$ compact

Topological closedness: $T_n \in \mathcal{B}_0(V,W)$, $T_n \xrightarrow[n\to\infty]{} T$ i.e.

$\|T_n - T\| \xrightarrow[n\to\infty]{} 0$

Let us prove that T is compact.

We prove that $T(\bar{B}_V(0,1))$ is totally bounded: given $\varepsilon > 0$,
$\exists n_0 \geq 1$ s.t. $\|T_{n_0} - T\| \leq \frac{\varepsilon}{2}$ and T_{n_0} cpct $\Rightarrow \exists y_1, \dots, y_N \in W$
s.t. $T_{n_0}(\bar{B}_V(0,1)) \subseteq \bigcup_{i=1}^{N} \bar{B}(y_i, \varepsilon/2)$

Δ-ineq $\Rightarrow T(\bar{B}_V(0,1)) \subseteq \bigcup_{i=1}^{N} \bar{B}(y_i, \varepsilon)$

(ii) If $T \in \mathcal{B}(V,W)$, $S \in \mathcal{B}(W,Z)$ and T cpct :

$\forall (x_n)$ bdd in ~~W?~~ V, $\exists$ subseq $(Tx_{\varphi(n)})$ converging in W

hence $(STx_{\varphi(n)})$ converging in Z (S cts)

If S is cpct, use boundedness of T

$\forall (x_n)$ bdd in V, (Tx_n) bdd in W

$\therefore$ $(STx_{\varphi(n)})$ convergent in Z for some φ. $\square$

<u>Theorem</u> (Schauder)

V, W Banach spaces

$T \in \mathcal{B}(V,W)$ is compact iff $T^* \in \mathcal{B}(W^*, V^*)$ is cpct

<u>Proof</u> $\Rightarrow$ Assume T cpct. Consider (f_n) in $\overline{B}_{W^*}(0,1)$ and let us prove that $(T^* f_n)$ has a convergent subsequence in V^*.

Define $K \doteq \overline{T(\overline{B}_V(0,1))}$ cpct

Define $\tilde{f}_n \doteq f_n|_K$

$(\tilde{f}_n)$ sqce in $C(K)$ that is

- <u>equi-bdd</u> : $\forall y \in K$,

$$\|y\|_W \leq \|T\| \, \|x\| \leq \|T\|$$

and $|\tilde{f}_n(y)| \leq \underbrace{\|f_n\|_{W^*}}_{\leq 1} \|y\|_W \leq \|T\|$

- <u>equi-cts</u> : $\forall n \geq 1$, $\forall y_1, y_2 \in K$,

$$|\tilde{f}_n(y_1) - \tilde{f}_n(y_2)| = |f_n(y_1) - f_n(y_2)|$$
$$\leq \underbrace{\|f_n\|_{W^*}}_{\leq 1} \|y_1 - y_2\| \leq \|y_1 - y_2\|$$

Hence Arzelà-Ascoli thm $\Rightarrow$ $\exists$ subsequence $\tilde{f}_{\varphi(n)}$ that converges uniformly on K, and therefore Cauchy, and for $m, n \geq 1$

$$\|T^* f_{\varphi(m)} - T^* f_{\varphi(n)}\| = \sup_{\substack{x \in V \\ \|x\| \leq 1}} |T^* f_{\varphi(m)}(x) - T^* f_{\varphi(n)}(x)|$$
$$= \sup_{\substack{x \in V \\ \|x\| \leq 1}} |\tilde{f}_{\varphi(m)}(Tx) - \tilde{f}_{\varphi(n)}(Tx)|$$
$$\leq \sup_{y \in K} |\tilde{f}_{\varphi(m)}(y) - \tilde{f}_{\varphi(n)}(y)| \leq \|\tilde{f}_{\varphi(m)} - \tilde{f}_{\varphi(n)}\|_\infty$$
$$\to 0 \text{ as } m, n \to \infty$$

hence $T^* f_{\varphi(n)}$ converges.

($\Leftarrow$) As $T^* : W^* \to V^*$ cpct, then by above

$$T^{**} : V^{**} \to W^{**} \text{ cpct}$$

Recall $\Phi_V : V \to V^{**}$, $\Phi_W : W \to W^{**}$

isometric embeddings.

And $T^{**} \circ \Phi_V : V \to W^{**}$

$\Phi_W \circ T : V \to W^{**}$

are the same map (commute):

$\forall x \in V, g \in W^*$,

$$\begin{aligned}(T^{**} \circ \Phi_V)(x)(g) &= \Phi_V(x)(T^*(g)) \\ &= (T^*(g))(x) \\ &= g(T(x)) \\ &= \Phi_W(T(x))(g) \checkmark\end{aligned}$$

Therefore $\Phi_W(T(\overline{B}_V(0,1))) = \underbrace{T^{**}(\Phi_V(\overline{B_V}(0,1)))}_{\substack{\text{rel cpct since} \\ T^{**} \text{ is cpct}}}$

Hence $\forall (x_n) \in \overline{B}_V(0,1)$,

$\Phi_W(Tx_n)$ has a convergent subseq, so Cauchy

hence since Φ_W isometric,

(Tx_n) has a Cauchy subseq so convergent $\square$

The Riesz-Fischer Spectral Theorem

<u>Intro</u> For compact operators, we can understand the non-zero spectrum with the sole finite dimensional concept of <u>eigenvalues</u>

<u>Theorem</u> (Riesz 1918, Schauder 1930)

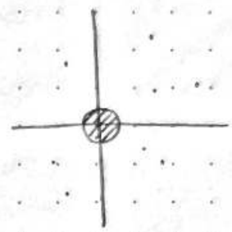
V infinite dimensional Banach Space

$T \in \mathcal{B}_0(V)$ compact operator $V \to V$

Then: (i) $0 \in \sigma(T)$

(ii) $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$

(non-zero $\lambda \in \sigma(T)$ are eigenvalues)



(iii) $\forall \varepsilon > 0$, $\sigma(T) \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geq \varepsilon\}$ is <u>finite</u>

(only accumulation point of $\sigma(T)$ is 0)

(iv) $\forall \lambda \in \sigma_p(T)$, $\lambda \neq 0$,

$\dim(\mathrm{Ker}(T-\lambda)) < \infty$; $\mathrm{codim}(\mathrm{Im}(T-\lambda)) < \infty$

<u>Proof</u> (i) Otherwise, $0 \notin \sigma(T)$ then T is invertible ; therefore $\exists R > 0$ s.t. $\overline{B}(0,1) \subset \underbrace{T(\overline{B}(0,R))}_{\text{rel. cpct.}}$ hence $\overline{B}(0,1)$ is cpct and $\dim V < \infty$ ⋕

(ii) It is enough to prove that if $\lambda \neq 0$ s.t. $\mathrm{Ker}(T-\lambda) = \{0\}$ then $T-\lambda$ is invertible.

<u>claim</u> (a) $(T-\lambda)$ bded below

(b) $\mathrm{Im}(T-\lambda)$ closed

<u>pf</u> (a) If not, $\exists (x_n)$ in V with $\|x_n\| = 1$ s.t. $(T-\lambda)x_n \longrightarrow 0$

Tx_n has a convergent subseq since T cpct, say $(Tx_{\varphi(n)})$

then $x_{\varphi(n)} = \frac{1}{\lambda}[Tx_{\varphi(n)} - y_{\varphi(n)}]$; $y_{\varphi(n)} = (T-\lambda)x_{\varphi(n)} \to 0$

hence $(x_{\varphi(n)})$ converges to some x, hence $\|x\| = 1$ ($\|\cdot\|$ cts)

and since T cts, $(T-\lambda)x = 0$ ⋕

(b) Consider $y_n \in \mathrm{Im}(T-\lambda)$, $y_n = (T-\lambda)x_n$ with $y_n \longrightarrow y$

Then (y_n) is Cauchy hence since $(T-\lambda)$ bded below so is (x_n)

Hence $x_n \to x$ so $y = (T-\lambda)x \in \mathrm{Im}(T-\lambda)$

We want to prove that $T-\lambda$ is invertible

Since $\ker(T-\lambda) = \{0\}$, $(T-\lambda)$ bdd below, $\mathrm{Im}(T-\lambda)$ closed, it is enough to prove that $\mathrm{Im}(T-\lambda) = V$

If not: $V_n = \mathrm{Im}\left((T-\lambda)^n\right)$, $n \geq 1$

$V_0 = V$

Then $(V_n)_{n\geq 0}$ is a strictly decreasing sequence of closed subspaces:

- $V_{n+1} \subseteq V_n$ is clear
- V_n closed by previous claim; induct on n; same as previous claim
- Assume $V_{n+1} = V_n$ for some $n \geq 1$. Then $\forall x \in V_0$, $(T-\lambda)^n x \in V_n = V_{n+1}$

  Hence $\exists x' \in V_0$ s.t. $(T-\lambda)^n x = (T-\lambda)^{n+1} x'$

  And since $(T-\lambda)$ injective deduce $x = (T-\lambda)x' \in V_1$ ⋇

  So $V_{n+1} \subsetneq V_n$ $\forall n \geq 0$

[Riesz lemma] # <u>claim</u> $\exists x_n \in V_n \setminus V_{n+1}$ s.t. $\|x_n\| = 1$,

$$d(x_n; V_{n+1}) = \inf_{y \in V_{n+1}} \|x_n - y\| \geq \frac{1}{2}$$

Consider $\tilde{x}_n \in V_n \setminus V_{n+1}$; exists since $V_{n+1} \subsetneq V_n$

Then $(V_{n+1}$ closed$) \Rightarrow d(\tilde{x}_n, V_{n+1}) > 0$

~~Define~~ Find $y_n \in V_{n+1}$ $\cancel{x_n}$ $\cancel{\tilde{x}_n}$ s.t. $\|\tilde{x}_n - y_n\| \leq 2\, d(\tilde{x}_n, V_{n+1})$

Define $x_n = \dfrac{\tilde{x}_n - y_n}{\|\tilde{x}_n - y_n\|}$, well-def, $\|x_n\| = 1$, and $\forall y \in V_{n+1}$

$$\|x_n - y\| = \frac{1}{\|\tilde{x}_n - y_n\|} \| \tilde{x}_n - \underbrace{y_n - y\|\tilde{x}_n - y_n\|}_{\in V_{n+1}} \|$$

$$\geq \frac{d(\tilde{x}_n, V_{n+1})}{\|\tilde{x}_n - y_n\|} \geq \frac{1}{2}.$$

Finally, we have (x_n) in V s.t. $\|x_n\| = 1$ and $\forall m < n$,

$$\|Tx_m - Tx_n\| = |\lambda| \, \| x_m + \underbrace{\frac{1}{\lambda}(T-\lambda)x_m - x_n - \frac{1}{\lambda}(T-\lambda)x_n}_{\in V_{m+1}} \|$$

$$\geq \frac{|\lambda|}{2}$$

Therefore (Tx_n) has no convergent subsequence, contradicting compactness.

L21.3

(iii) Otherwise $\exists\, \varepsilon > 0$ and $(\lambda_n)_{n\geq 1}$ s.t. $|\lambda_n| \geq \varepsilon$, $\lambda_n \neq \lambda_m \ \forall\, m \neq n$ with $\lambda_n \in \sigma_p(T)$. Define $E_n = \mathrm{Span}\{x_1, \dots, x_n\} \subseteq V$

with $\|x_n\| = 1$ s.t. $(T - \lambda_n)\, x_n = 0$

Note that $(x_1, \dots, x_n)$ linearly independent $\forall\, n \geq 1$, E_n finite dim subspace hence closed; $E_n \subsetneq E_{n+1}$

Arguing as above (Riesz lemma), $\exists\, (y_n)$ a sqce s.t. $y_n \in E_n \setminus E_{n-1}$, $n \geq 2$

$\|y_n\| = 1$, $d(y_n, E_{n-1}) \geq \frac{1}{2}$

Observe that $y_n \in \sum_{k=1}^{n} a_{n,k}\, x_k$ for some coeffs $(a_{n,k})_{k=1}^{n}$ in $\mathbb{F}$; and

$$(T - \lambda_n)\, y_n = \sum_{k=1}^{n-1} a_{n,k} (\lambda_k - \lambda_n)\, x_k \in E_{n-1}$$

Hence $\forall\, m < n$,

$$\|Ty_n - Ty_m\| = |\lambda_n| \, \Big\| y_n + \underbrace{\frac{1}{\lambda_n}(T - \lambda_n) y_n - \frac{\lambda_m}{\lambda_n} y_m - \frac{1}{\lambda_n}(T - \lambda_m) y_m}_{\in E_{n-1}} \Big\|$$

$$\geq \frac{|\lambda_n|}{2} \geq \frac{\varepsilon}{2}$$

Hence (Ty_n) has no convergent subsqce contradicting T cpct $\#$

(iv) Want to prove that for $\lambda \in \sigma_p(T) \setminus \{0\}$, $\begin{cases} \dim \mathrm{Ker}(T - \lambda) < \infty & \text{(a)} \\ \mathrm{codim}\ \mathrm{Im}(T - \lambda) < \infty & \text{(b)} \end{cases}$

<u>Pf of (a)</u> $\mathrm{Ker}(T - \lambda)$ is closed since $(T - \lambda)$ continuous, hence a Banach space. If (x_n) sqce in $\mathrm{Ker}(T - \lambda)$ with $\|x_n\| = 1$ then $\exists$ subsqce $(T x_{\varphi(n)})$ convergent (T cpct) and $x_{\varphi(n)} = \frac{1}{\lambda} T x_{\varphi(n)}$ converges too. Hence $\overline{B}_{\mathrm{Ker}(T-\lambda)}(0, 1)$ cpct $\Rightarrow \dim \mathrm{Ker}(T - \lambda) < \infty$

Let us prove that $\mathrm{codim}(\mathrm{Im}(T - \lambda)) < \infty$

$\#$ <u>claim</u>: Any finite dim subspace $E = \mathrm{span}\{x_1, \dots, x_n\}$ ($x_1, \dots, x_n$ lin indep) in V ~~has~~ is complemented by a closed subspace F s.t. $V = E \oplus F$

<u>pf</u> $x \in E \iff x = \sum_{i=1}^{n} \alpha_i(x)\, x_i$ with unique decomposition and α_i linear form on E: Hahn-Banach Theorem $\Rightarrow \exists\, \tilde{\alpha}_i \in V^*$ extending α_i; take $F = \bigcap_{i=1}^{n} \mathrm{Ker}\, \tilde{\alpha}_i$

Consider F closed s.t. $\mathrm{Ker}(T-\lambda) \oplus F = V$

and $\widetilde{T-\lambda} : F \to \mathrm{Im}(T-\lambda)$ restriction to F.

The same reasoning as in the first claim shows that

$\widetilde{T-\lambda}$ bded below and $\mathrm{Im}(T-\lambda)$ closed

Define $V_n = \mathrm{Im}(T-\lambda)^n$

$V_0 = V$

$V_1 \subsetneq V_0$ with (assume by contradiction) <u>infinite</u> codimension

<u>claim</u> Then $V_{n+1} \subsetneq V_n \ \forall n \geq 1$

Indeed if $V_{n+1} = V_n$ for some $n \geq 1$, observe that then $V = V_0 = V_1 + \underbrace{\mathrm{Ker}(T-\lambda)^n}_{\text{finite dim}}$

which contradicts infinite codimension

Finally, arguing as before, $\exists (x_n)$ sqce in V s.t.

$\|x_n\| = 1$, $x_n \in V_n \setminus V_{n+1}$, $d(x_n; V_{n+1}) \geq \frac{1}{2}$ and then for $m<n$,

$\|Tx_m - Tx_n\| \geq |\lambda|/2$ and we contradict T cpct. $\square$

<u>Rks</u> ① Other proof will be discussed briefly next lecture, when V is a Hilbert space and T can be approximated by finite rank operators

② "Fredholm" alternative is (ii) : either eigenvalue or invertible

In fact, it is a more powerful / general theory: because of (iv) one can define $\mathrm{ind}(T-\lambda) = \dim \mathrm{Ker}(T-\lambda) - \mathrm{codim}\, \mathrm{Im}(T-\lambda) \in \mathbb{Z}$

and for $\lambda \neq 0$, it is continuous hence const. i.e. $\dim \mathrm{Ker}(T-\lambda) = \mathrm{codim}\, \mathrm{Im}(T-\lambda)$

(try to prove it !)

③ When V is Hilbert and T self-adjoint cpct then have stronger version of this theorem ; can be proved by variational method; see next lecture.

Compact operators in Hilbert Spaces

Intro We now revisit the concept of compact operator when adding the inner product structure

Def B Banach Space; $T \in \mathcal{B}(B)$ is said to have finite rank if
$$\dim \operatorname{Im}(T) < \infty$$

Prop Characterizations of compactness in Hilbert spaces:
H Hilbert space, $T \in \mathcal{B}(H)$ then
(i) T cpct iff T is a limit of finite rank operators
(ii) T cpct iff for any orthonormal system $(e_n)_{n \geq 1}$ (countable),
$$\|Te_n\| \xrightarrow[n\to\infty]{} 0$$

Rk (i) is FALSE in general Banach spaces, however this is hard to exhibit explicitely; done first by Per Enflo in 1973

Proof (i) "$\Leftarrow$" If $\exists (T_n)_{n\geq 1}$ finite rank operators s.t. $\|T_n - T\| \longrightarrow 0$, then each T_n is cpct and the closedness of compactness implies T is too

"$\Rightarrow$" If T cpct, then given any $\varepsilon > 0$; since $T(\overline{B}(0,1))$ totally bounded, $\exists y_1, \dots, y_n \in H$ s.t. $T(\overline{B}(0,1)) \subseteq \bigcup_{i=1}^{m} \overline{B}(y_i, \frac{\varepsilon}{2})$

Define $Y_n \doteq \operatorname{span}(y_1, \dots, y_n)$ and

Π_n orthogonal projection on Y_n, and $T_n = \Pi_n T$

Then T_n has finite rank: $\operatorname{Im}(T_n) \subseteq Y_n$

And $\forall x \in \overline{B}(0,1)$,
$$\|Tx - T_n x\| \underset{\substack{\uparrow \\ \text{some } y_{i_0}}}{\leq} \|Tx - y_{i_0}\| + \|T_n x - \underset{\substack{\uparrow \\ = \Pi_n y_{i_0}}}{y_{i_0}}\|$$
$$\leq \frac{\varepsilon}{2} + \|\Pi_n(Tx - y_{i_0})\|$$
$$\leq \frac{\varepsilon}{2} + \|Tx - y_{i_0}\| \leq \varepsilon$$

use Hilbertian structure to get norm 1 projections

Hence $\|T_n - T\| \leq \varepsilon$ and $T_n \to T$. $\ulcorner$

(ii) "$\Rightarrow$" If T cpct and $(e_n)_{n\geq 1}$ s.t (argue by contradiction) $T(e_n) \not\to 0$ then renaming the sequence if necessary (subseq) we can assume
$$\exists \varepsilon > 0, \quad \|T(e_n)\| \geq \varepsilon \quad \forall n \quad , \quad T(e_n) \longrightarrow y$$

Then $\|y\| \geq \varepsilon$, so $y \neq 0$.

But $\forall x \in H$, $\langle x, Te_n \rangle \xrightarrow[n\to\infty]{} \langle x, y \rangle$

and $\langle x, Te_n \rangle = \langle T^* x, e_n \rangle$

Bessel ineq : $\sum_{n \geq 1} |\langle T^* x, e_n \rangle|^2 \leq \|T^* x\|^2$

Hence $|\langle T^* x, e_n \rangle| \to 0$, hence $\langle x, Te_n \rangle \to 0$

We deduce that $\langle x, y \rangle = 0 \ \forall x$, so $y = 0$ #

"$\Leftarrow$" Assume $Te_n \to 0$ for any (e_n) orthonormal system.

Assume T not compact.

Then (i) $\Rightarrow$ T not limit of finite rank operators, hence $\exists \varepsilon > 0$ s.t.

$\forall R \in \mathcal{B}(H)$ of finite rank, $\|T - R\| > \varepsilon$

We construct (e_n) orthonormal system such that $\|Te_n\| \geq \varepsilon \ \forall n$ as follows:

- $R = 0$, $\|T\| > \varepsilon$ hence $\exists e_1 \in H$, $\|e_1\| = 1$ s.t. $\|T(e_1)\| \geq \varepsilon$
- If $(e_1, \dots, e_n)$ built, then

$E_n \doteq \mathrm{span}(e_1, \dots, e_n)$ and Π_n orthogonal proj^n on E_n

and note that $T\Pi_n$ has finite rank ; $\mathrm{Im}(T\Pi_n) \subset T(E_n)$ finite dim

So $R = T\Pi_n$ gives $\|T - T\Pi_n\| > \varepsilon$, hence

$\exists y_n \in H$ s.t. $\|y_n\| = 1$, $\|Ty_n - T\Pi_n y_n\| > \varepsilon$

$y_n \notin E_n$ otherwise $Ty_n = T\Pi_n y_n$

We define $e_{n+1} = (y_n - \Pi_n y_n) / \|y_n - \Pi_n y_n\|$

Then $e_{n+1} \perp E_n$, $\|e_{n+1}\| = 1$, and

$$\|T(e_{n+1})\| \geq \frac{1}{\|y_n - \Pi_n y_n\|} \|Ty_n - T\Pi_n y_n\|$$

$$\geq \frac{\varepsilon}{\|y_n - \Pi_n y_n\|} \geq \frac{\varepsilon}{\|y_n\|} = \varepsilon$$

which concludes the proof. $\square$

Examples

① Hilbert - Schmidt operators

H separable Hilbert space with Hilbert basis (e_n)

Then $T \in \mathcal{B}(H)$ is called a Hilbert - Schmidt operator if

$$\|T\|_{HS} \doteq \Big(\sum_{n\geq 1} \|T(e_n)\|^2\Big)^{1/2} < \infty$$

Facts · $\|\cdot\|_{HS}$ well-defined independently of $(e_n)_{n\geq 1}$: if (e'_n) is another Hilbert basis; then (Parseval)

$$\sum_{n\geq 1} \|T(e_n)\|^2 = \sum_{m,n\geq 1} |\langle e_m', Te_n\rangle|^2$$

$$= \sum_{m,n\geq 1} |\langle T^* e_m', e_n\rangle|^2$$

$$= \sum_{m\geq 1} \|T^*(e_m')\|^2$$

· $\|\cdot\|_{HS}$ is a norm on the subspace where it is finite $(HS, \|\cdot\|_{HS})$ complete

· T Hilbert - Schmidt $\Rightarrow$ T cpct

See question 16 on Ex Sheet 4

Idea $Tx = T(\sum x_n e_n) = \sum x_n T(e_n)$

Use diagonal argument and strong control on tail for large n

Picture $\{$finite rank$\} \subseteq \{$Hilbert - Schmidt$\} \subseteq \{$cpct op$\} \subseteq \{$bdd op$\}$

· HS not closed for $\|\cdot\|$ ⚠

(unlike compactness) if dimension infinite

② Diagonal operators

H separable Hilbert space and $(e_n)_{n\geq 1}$ a Hilbert basis

$T \in \mathcal{B}(H)$ defined by $Te_n = a_n e_n \ \forall n \geq 1$

Then · T bdd iff (a_n) bdd

· T cpct iff $a_n \longrightarrow 0$

[See question 4 on Ex Sheet 4]

Examples of alternative arguments in Hilbert spaces for Schauder & Riesz-Schauder theorems:

<u>Theorem</u> H Hilbert space, $T \in \mathcal{B}(H)$ cpct

(i) T^* cpct

(ii) If $\lambda \in \sigma(T) \setminus \{0\}$ then $\lambda \in \sigma_p(T)$ i.e. eigenvalue,

with $\dim \operatorname{Ker}(T-\lambda) < \infty$, $\operatorname{codim} \operatorname{Im}(T-\lambda) < \infty$

<u>Proof</u> (i) <u>Argument 1</u>

T cpct $\Rightarrow$ $\exists (T_n)$ finite rank operators s.t. $\|T_n - T\| \to 0$

Then $T_n^* \longrightarrow T^*$ since $\|T_n^* - T^*\| = \|T_n - T\|$, and the adjoint of a finite rank operator $\mathcal{U}$ is finite rank:

$\mathcal{U}|_{\operatorname{Ker}\mathcal{U}^\perp} : (\operatorname{Ker}\mathcal{U})^\perp \longrightarrow \operatorname{Im}(\mathcal{U})$ invertible, (inv mapp thm)

and $\mathcal{U}^* = \begin{cases} (\mathcal{U}|_{(\operatorname{Ker}\mathcal{U})^\perp})^* & \text{on } \operatorname{Im}(\mathcal{U}) \\ 0 & \text{on } \operatorname{Im}(\mathcal{U})^\perp \end{cases}$

is finite rank.

Hence T^* limit of finite rank operators hence compact.

<u>Argument 2</u> Consider $(x_n)_{n \geq 1}$ in H with $\|x_n\| \leq 1$, then since T compact, we can consider subseq s.t. $(TT^* x_{\varphi(n)})$ converges, so is Cauchy

TT^* is cpct

Then $\|T^* x_m - T^* x_n\|^2$

$$= \langle T^*(x_m - x_n), T^*(x_m - x_n) \rangle$$
$$= \langle TT^*(x_m - x_n), x_m - x_n \rangle$$
$$\leq 2 \|TT^*(x_m) - TT^*(x_n)\|$$

Hence $(T^* x_{\varphi(n)})$ converges, so T^* is compact.

(ii) If $\lambda \in \sigma_p(T)$, $\lambda \neq 0$, so $\operatorname{Ker}(T-\lambda)$ has finite dimension and unit ball cpct; and $\operatorname{Ker}(T-\lambda)^*$ finite dimension since T^* cpct; and

$$\operatorname{Im}(T-\lambda)^\perp = \operatorname{Ker}(T-\lambda)^*$$

hence $\operatorname{codim} \operatorname{Im}(T-\lambda) < \infty$

Remains to prove: if $\lambda \neq 0$ s.t. $\mathrm{Ker}(T-\lambda) = \{0\}$ then $(T-\lambda)$ inv^+ble

We proved that $(T-\lambda)$ bdd below: $\exists\, c>0$ s.t.

$$\forall x \in H, \quad \|(T-\lambda)x\| \geq c\|x\|$$

$\exists\, (T_n)_{n\geq 1}$ finite rank s.t. $T_n \to T$

Then for n large, $\|(T_n-\lambda)x\| \geq \frac{c}{2}\|x\|$ ($n \geq N$ say)

Hence $\mathrm{Ker}(T_n-\lambda) = \{0\}$

Claim $T_n - \lambda$ is invertible $n \geq N$

The claim concludes the proof.

Indeed given $y \in H$, $\forall n \geq N$, $\exists\, x_n$ s.t.

$(T_n-\lambda)(x_n) = y$ (surjective)

$(T_n-\lambda)$ bdd below $\Rightarrow \|x_n\| \leq \frac{2}{c}\|y\|$

Bdd seq $\Rightarrow \exists$ subseq $Tx_{\varphi(n)} \longrightarrow$ convergent

Then $x_{\varphi(n)} = \frac{1}{\lambda}\left[Tx_{\varphi(n)} - y - \underbrace{(T-T_n)x_{\varphi(n)}}_{\to 0}\right]$

Hence $x_{\varphi(n)}$ converges to some $x \in H$

And $(T-\lambda)x = y$; whence $T-\lambda$ surjective so invertible.

Pf of claim Complement $\mathrm{Ker}(T_n)$ with closed subspace F

Rank theorem $\Rightarrow \dim F = \dim \mathrm{Im}(T_n) = n_0 < \infty$

Define $\tilde{T}_n = T_n|_F : F \to \mathrm{Im}(T_n)$ invertible

$\tilde{T}_n (T_n-\lambda)|_F : F \to \mathrm{Im}(T_n)$ is injective

Hence (rank thm) invertible, and

$\dim(\mathrm{Im}(T_n-\lambda) \cap F) = n_0 = \dim F$

hence $\mathrm{Im}(T_n-\lambda) \cap F = F$ and $T_n - \lambda$ is surjective $\square$

Rk This approximation method could also be used to prove that

$0 = \mathrm{ind}(T_n-\lambda) \longrightarrow \mathrm{ind}(T-\lambda)$

$(\mathrm{ind}(T-\lambda) = \dim \mathrm{Ker}(T-\lambda) - \mathrm{codim}\, \mathrm{Im}(T-\lambda))$

hence $\dim \mathrm{Ker}(T-\lambda) = \mathrm{codim}\, \mathrm{Im}(T-\lambda)$

The Spectral Theorem for Compact Self-Adjoint operators

Intro We had seen

- results on spectrum for cpct operators
- orthogonality of eigenspaces for self-adjoint (in fact normal) operators

We now combine these structure; give a new specifically 'Hilbertian' proof and apply to so-called Sturm-Liouville theory.

Theorem (Spectral Theorem for cpct s.a. operators)

Consider H Hilbert space and $T \in \mathcal{B}(H)$ compact and self-adjoint.

Then
- $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$
- $\sigma(T) \subseteq \mathbb{R}$
- $\forall \lambda, \lambda' \in \sigma_p(T)$, $\lambda \neq \lambda'$ non-zero then $E_\lambda \perp E_{\lambda'}$ (eigenspaces)
- If $(\lambda_n)_{n\geq 1}$ sequence of non-zero eigenvalues, then $\lambda_n \to 0$ if the sequence is infinite, and

$$H = (\operatorname{Ker} T) \overset{\perp}{\oplus} \Big[\overset{\perp}{\bigoplus_{n\geq 1}} E_{\lambda_n} \Big]$$

$$T = \sum_{n\geq 1} \lambda_n \Pi_{\lambda_n} \quad \text{with } \Pi_{\lambda_n} \text{ orthogonal proj}^n \text{ on } E_{\lambda_n},$$

with $\dim E_{\lambda_n} < \infty$

Proofs Proof 1 Use the Riesz-Schauder Theorem plus results on self-adjoint operators... Exercise

Proof 2 Self-contained (?)

- $\sigma(T) \subseteq \mathbb{R}$ proved before
- $\forall \lambda \neq 0$, $\dim E_\lambda < \infty$ proved as before

Construct inductively a sequence (λ_n) of eigenvalues s.t.

$\lambda_n \neq 0 \ \forall n \geq 1$, $(|\lambda_n|)_{n\geq 1}$ non-increasing, $|\lambda_{n+1}| < |\lambda_n|$

- One of $\pm \|T\|$, at least, has to be an eigenvalue, call it λ_1 (*)
- If $\lambda_1, \dots, \lambda_n$ constructed, define

$F_n = \overset{\perp}{\bigoplus_{i=1,\dots,n}} E_{\lambda_i}$ and P_n the $\perp$-proj^n on F_n

Then $T(F_n) \subseteq F_n$ hence $T(F_n^\perp) \subset F_n^\perp$

$\forall x \in F_n^\perp$, $\forall y \in F_n$, $\langle Tx, y\rangle = \langle x, \underset{\in F_n}{Ty}\rangle = 0 \Rightarrow Tx \in F_n^\perp$

Define $T_{n+1} = T(1-P_n)$ then apply step (*) to T_{n+1} to build λ_{n+1}, i.e. $\pm\|T_{n+1}\|$ is an eigenvalue that we call λ_{n+1}

\# If (λ_n) infinite not going to zero, then $\exists\ (x_{\varphi(n)})$, $\|x_{\varphi(n)}\| = 1$, $\|\lambda_{\varphi(n)}\| \geq \delta > 0$, $Tx_{\varphi(n)} = \lambda_{\varphi(n)} x_{\varphi(n)}$ and for $m < n$

$$\begin{aligned} \|Tx_{\varphi(n)} - Tx_{\varphi(m)}\| &= \|\lambda_{\varphi(m)} x_{\varphi(m)} - \lambda_{\varphi(n)} x_{\varphi(n)}\| \\ &= \left(|\lambda_{\varphi(m)}|^2 + |\lambda_{\varphi(n)}|^2\right)^{1/2} \\ &\geq \sqrt{2}\,\delta > 0 \end{aligned}$$

contradicts T cpct

\# Define $F = \overset{\perp}{\bigoplus_{n\geq 1}} E_{\lambda_n}$, then $T|_{F^\perp} : F^\perp \to F^\perp$ and no non-zero eigenvalue by construction

But since $\pm\|T|_{F^\perp}\|$ is an eigenvalue, we deduce that $T|_{F^\perp} = 0$ and finally, $H = \operatorname{Ker} T \overset{\perp}{\oplus} [F]$ as desired.

\# If set of non-zero eigenvalues is finite, then representation

$$T = \sum_{n=1}^{N} \lambda_n \Pi_{\lambda_n} \text{ is clear}$$

If (λ_n) infinite and $\lambda_n \to 0$, then

$$\begin{aligned} \left\|T - \sum_{n=1}^{N} \lambda_n \Pi_{\lambda_n}\right\| &= \left\|T\left(1 - \sum_{n=1}^{N} \Pi_{\lambda_n}\right)\right\| \\ &= \|T(1-P_n)\| \\ &\leq |\lambda_n| \quad \text{by construction of the sqce,} \end{aligned}$$

hence it goes to zero, and $T = \sum_{n\geq 1} \lambda_n \Pi_{\lambda_n}$

\# Last thing to prove is that if $\lambda \neq 0$ not an eigenvalue ($\operatorname{Ker}(T-\lambda) = \{0\}$) then $(T-\lambda)$ invertible ($\lambda \in \sigma(T)$)

Proof now simple: if $\nu \in \mathbb{C} \setminus \{0, \lambda_n : n \geq 1\}$ then

$$T - \nu = \sum_{n\geq 1} (\lambda_n - \nu)\Pi_{\lambda_n} - \nu\,\Pi_{\operatorname{Ker} T}$$

which is invertible clearly $\square$

Rk In particular if H separable Hilbert space, ∃ Hilbert basis made up of eigenvalues (∴ spoiler wlf)

● Application The Sturm-Liouville theory

Consider the Sturm-Liouville operators (that could be called 1-dimensional Schrödinger operator today)

$$L : C^2([a,b]) \longrightarrow C^0([a,b])$$
$$f \longmapsto Lf(x) = -f''(x) + q(x)f(x)$$

where $a<b \in \mathbb{R}$; $q \in C^0([a,b])$ a potential function, $q \geqslant 0$

We want to understand the spectrum of the operator L with boundary conditions $f(a) = f(b) = 0$

● In quantum mechanics, this means searching for possible energy states of a quantum system

The Sturm-Liouville Theorem says that these states are quantified

Rk Spectral theorem for cpct s.a operators implies that ∃ Hilbert basis made up of eigenvectors of L, provided separability

② If $[a,b] = [0,\pi]$ and $q=1$, complete set of eigenvectors given by $(\sin(nx))_{n\in\mathbb{Z}}$

Theorem (Sturm-Liouville, 1836)

● There is (f_n) in $C^0_0([a,b]) = \{ f : [a,b] \to \mathbb{C} \text{ cts}, f(a) = f(b) = 0\}$ and $(\lambda_n)_{n\geqslant 1}$, $\lambda_n > 0$ increasing to infinity such that

$$\begin{cases} Lf_n = \lambda_n f_n \\ \int_a^b f_m \overline{f_n} = \delta_{mn} \end{cases} \quad \left(\text{orthonormal for } \|f\|_2 = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}\right)$$

Rk The Sturm-Liouville theory covers more general equations / operators

$$Lf(x) = A(x)f''(x) + B(x)f'(x) + C(x)f(x)$$

with suitable boundary conditions.

Proof L not bdd on $C^2([a,b]) \to C^2([a,b])$

or $C^0([a,b]) \to C^0([a,b])$

hence we cannot apply the spectral theorem to L.

The idea is to construct a right-inverse T and prove that T is cpct s.a. on $H = \overline{C_0^0([a,b])}^{\|\cdot\|_2}$ (completion for $\|\cdot\|_2$)

Then the spectral theorem for T in H will give the result

(Linear) Picard-Lindelöf theorem shows that

- $\begin{cases} \phi_-''(x) = q(x)\,\phi_-(x) \\ \phi_-(a) = 0, \ \phi_-'(a) = 1 \end{cases}$

has a unique C^2 solution $\phi_-(x)$ on $[a,b]$

- $\begin{cases} \phi_+''(x) = q(x)\,\phi_+(x) \\ \phi_+(b) = 0, \ \phi_+'(b) = 1 \end{cases}$

has a unique C^2 solution $\phi_+(x)$ on $[a,b]$

Observe that $\phi_-(b) \neq 0$ and $(\phi_+(a) \neq 0)$

since otherwise

$$0 = \int_a^b \phi_-(x)\left[\, q(x)\,\phi_-(x) - \phi_-''(x)\,\right] dx$$

$$\underset{\substack{\text{IBP} \\ \phi_-(a)=\phi_-(b) \\ =0}}{=} \int_a^b \left(q(x)\,\phi_-(x)^2 + \phi_-'(x)^2 \right) dx$$

This implies $\phi_- \equiv 0$ absurd since $\phi_-'(a) = 1$

[Proof that $\phi_+(a) \neq 0$ analogous]

Wronskian $W(x) = \phi_+(x)\,\phi_-'(x) - \phi_-(x)\,\phi_+'(x)$ then $W \in C^1([a,b])$,

$W'(x) = 0$, so $W(x) = \text{const} = W(b) = \underbrace{-\phi_-(b)}_{=W} \neq 0$

Define
$$K(x,y) = \begin{cases} \frac{1}{W}\,\phi_+(x)\,\phi_-(y), & x \geq y \\ \frac{1}{W}\,\phi_-(x)\,\phi_+(y), & x \leq y \end{cases}$$

Then $K \in C^0([a,b]^2)$, $K(x,y) = K(y,x)$ and define

$$Tf(x) \doteq \int_a^b K(x,y)\, f(y)\, dy$$

\# Then: • $T : C_0^0([a,b]) \longrightarrow C_0^0([a,b])$

i.e. $Tf(a) = Tf(b) = 0$

• T bdd cpct: $(C_0^0([a,b]), \|\cdot\|_2)$ to itself (cf example of integral operators in L20!)

• $\forall f, g \in C_0^0([a,b])$,

$$\langle Tf, g\rangle = \int_a^b \int_a^b K(x,y) f(y) \overline{g(x)}\, dx\, dy$$

$$= \langle f, Tg\rangle$$

• $$Tf(x) = \frac{1}{W}\int_a^x \phi_+(x)\phi_-(y) f(y)\, dy + \frac{1}{W}\int_x^b \phi_-(x)\phi_+(y) f(y)\, dy$$

is C^1 by observation,

$$(Tf)'(x) = 0 + \frac{1}{W}\int_a^x \phi_+'(x)\phi_-(y) f(y)\, dy + \frac{1}{W}\int_x^b \phi_-'(x)\phi_+(y) f(y)\, dy$$

$(Tf)'$ still C^1, and

$$(Tf)''(x) = -f(x) + q(x)\, Tf(x)$$

Hence $LTf = f \quad \forall f \in C_0^0([a,b])$

• $\langle Tf, f\rangle = \langle Tf, LTf\rangle$

$$= \int_a^b (Tf)^2 q + (Tf')^2 \geq 0$$

To conclude, T extends to $\tilde{T}$ cpct self-adjoint on H = completion of $C_0^0([a,b])$ for $\|\cdot\|_2$ (which is $L^2([a,b])$)

Therefore we can apply Spectral Theorem:

$\exists (f_n)_{n\geq 1}$ Hilbert basis, $(\mu_n)_{n\geq 1} \in \mathbb{R}_+$ with all $\mu_n > 0$ since

$Tf = 0 \Rightarrow LTf = f = 0$

So finally f_n and $\lambda_n = \frac{1}{\mu_n}$ answers the statement in theorem $\square$

<u>Rk</u> Inverting $\begin{cases} Lf = 0 \text{ on } [a,b] \\ f(a) = 0,\ f'(a) = 1 \end{cases}$

would <u>not</u> give a self-adjoint operators

* The Tychonoff and Banach–Alaoglu Theorems

<u>Intro</u> 1) Transition to AoF course next term

2) We have seen that in NVS the unit ball is compact for the topology of the norm iff dimension is finite.

However we can salvage a notion of local compactness by weakening the topology.

Some terminology / definitions from topology

Given (X, τ) a topological space, we say $\mathcal{B}$, a collection of open sets is a <u>base</u> of τ if any $\mathcal{U} \in \tau$ is a union of elements of $\mathcal{B}$.

We say that $\mathcal{S} \subset \tau$ is a <u>subbase</u> if the collection of all finite intersections of elements of $\mathcal{S}$ are a base of τ

Let A a set (with arbitrary cardinal) and (X_α, τ_α) topological spaces for each $\alpha \in A$. Then we define the <u>product topology</u> on $X = \prod_{\alpha\in A} X_\alpha$ as the smallest topology that includes all the $p_\alpha^{-1}(\mathcal{U}_\alpha)$ for $\alpha \in A$, $\mathcal{U}_\alpha \in \tau_\alpha$ and $p_\alpha : X \longrightarrow X_\alpha$ the coordinate projection: $f \in X$, $f : A \to \bigsqcup_{\alpha\in A} X_\alpha \mapsto p_\beta(f) = f(\beta)$

In other words, the product topology τ has the subbase

$$\mathcal{S} = \{ p_\alpha^{-1}(\mathcal{U}_\alpha), \mathcal{U}_\alpha \in \tau_\alpha, \alpha \in A \}$$

The product topology is the smallest topology that makes all the coordinate maps continuous.

<u>Theorem</u> (Tychonoff, 1930)

Given a set A and (X_α, τ_α) compact topological spaces for each $\alpha \in A$, then (X, τ) the product topological space is compact.

<u>Rmks</u> ① If $A = \mathbb{N}$ and $(X_n, \tau_n) = (X_n, d_n)$ for $n \geq 1$ are compact metric spaces, then the theorem can be proved by showing that the product topology is metrizable,

and using that in a metric space compactness is equivalent to sequential compactness. Indeed, can then use a diagonal argument.

② Compact not always equivalent to sequentially compact:

Take $A = [0,1]$, $(X_\alpha, \tau_\alpha) = (\{0,1\}, \text{discrete})$ for each α.

Then $f_n \in X = \prod_{\alpha \in A} X_\alpha = \{0;1\}^{[0,1]}$

defined by $f_n(\alpha) = n\text{th digit in base 2 of } \alpha \in [0,1]$

For any subsequence $(f_{\varphi(n)})_{n \geq 1}$ with $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing, $\exists\, \alpha \in [0,1]$ such that $(f_{\varphi(n)}(\alpha))$ alternates between 0 and 1 and is not convergent.

Hence no subsequence is convergent.

Still (Tychonoff), X is compact for the product topology.

<u>Proof</u> Two lemma / claims

<u>Claim 1</u> Any cover $\mathcal{R}$ of X consisting solely of elements of

$\mathcal{S} = \{ p_\alpha^{-1}(U_\alpha),\ U_\alpha \in \tau_\alpha,\ \alpha \in A \}$

(subbase of the product topology)

has a finite subcover.

<u>Pf of Claim 1</u> Define $\mathcal{U}_\alpha = \{ U_\alpha \in \tau_\alpha : p_\alpha^{-1}(U_\alpha) \in \mathcal{R} \}$

Then $\exists\, \alpha_0 \in A$ s.t. $\mathcal{U}_{\alpha_0}$ covers X_{α_0},

otherwise $\forall \alpha \in A$, $\exists\, x_\alpha \in X_\alpha$ not covered by $\mathcal{U}_\alpha$, and $f \in X$ defined by $f(\alpha) = x_\alpha$ is not covered by $\mathcal{R}$.

Then (X_{α_0} cpct), $\mathcal{U}_{\alpha_0}$ has a finite subcover:

$\exists\, U_{\alpha_0}^1, \dots, U_{\alpha_0}^n \in \mathcal{U}_{\alpha_0}$ s.t.

$X_{\alpha_0} \subseteq \bigcup_{i=1}^n U_{\alpha_0}^i$, and

$\mathcal{R}_0 = \{ p_{\alpha_0}^{-1}(U_{\alpha_0}^1), \dots, p_{\alpha_0}^{-1}(U_{\alpha_0}^n) \}$

is a finite subcover of $\mathcal{R}$ that covers X. $\square$

Claim 2 (Alexander's subbase theorem)

Given (X, τ) a topological space with $\mathcal{S}$ subbase of τ and such that every cover of X with elements in $\mathcal{S}$ has a finite subcover, then X is compact.

Proof of the theorem with claims 1 & 2:

Consider $\mathcal{S} = \{ p_\alpha^{-1}(U_\alpha),\ U_\alpha \in \tau_\alpha;\ \alpha \in A \}$ which is a subbase of product topology τ on $X = \prod_{\alpha \in A} X_\alpha$.

Then claim 1 shows that any cover "from" $\mathcal{S}$ has a finite subcover when claim 2 gives compactness of X. $\square$

Proof of claim 2 (by contradiction)

Assume that every cover of X from $\mathcal{S}$ has a finite subcover but X is not compact.

Define Λ = collection of open covers of X that do not have a finite subcover. Then Λ is not empty since X assumed non-compact.

partial order by inclusion

It satisfies the least upper bound property:

Consider $\Lambda_0 \subseteq \Lambda$ totally ordered non-empty.

Define $\mathcal{C}_0$ the cover made up of all open sets in covers in Λ_0.

Then $\mathcal{C}_0$ is the smallest cover containing all covers in Λ_0.

It remains to prove that $\mathcal{C}_0 \in \Lambda$.

Consider $U_1, \dots, U_n \in \mathcal{C}_0$. Then by def^n of $\mathcal{C}_0$, $\forall i = 1, \dots, n$, $U_i \in \mathcal{C}_{\alpha_i} \in \Lambda_0$. Take the largets such $\mathcal{C}_{\alpha_i}$ (total order), say $\mathcal{C}_{\alpha_{i_0}}$. Then $U_1, \dots, U_n \in \mathcal{C}_{\alpha_{i_0}}$ hence are not an open cover.

Hence $\mathcal{C}_0 \in \Lambda$ and is a least upper bound for Λ_0.

Apply Zorn's Lemma: $\exists\, \mathcal{C}_m \in \Lambda$ a maximal element

Define $\mathcal{M} = \mathcal{C}_m \cap \mathcal{S} = \{ U \text{ open},\ U \in \mathcal{C}_m,\ U \in \mathcal{S} \}$.

$\mathcal{M}$ is a cover of X: Otherwise $\exists\, x \in X$ not in any set of $\mathcal{M}$.

Since $\mathcal{C}_m$ is a cover, $\exists\, U_0 \in \mathcal{C}_m$ s.t. $x \in U_0$.

Since $\mathcal{S}$ subbase, $\exists\, U_1, \dots, U_m \in \mathcal{S}$ s.t. $x \in \bigcap_{i=1}^{m} U_i \subset U_0$.

$\mathcal{C}_m$ maximal in $\Lambda \Rightarrow \mathcal{C}_m \cup \{U_i\}$ has a finite subcover of X.

(Note that none of the $U_1, \dots, U_m$ are in $\mathcal{C}_m$, otherwise x covered) by $\mathcal{M}$

$$X = U_i \cup \bigcup_{j=1}^{n_i} V_{ij} \quad , \quad V_{ij} \in \mathcal{C}_m$$

for each $i = 1, \dots, m$

But then $X = U_0 \cup \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} V_{ij}$ is a finite subcover of $\mathcal{C}_m \in \Lambda$.

$\divideontimes$

Hence $\mathcal{M}$ covers X.

Assumption $\Rightarrow$ it has a finite subcover ($\mathcal{M} \subset \mathcal{S}$), which is a finite subcover of $\mathcal{C}_m$ $\divideontimes$

Hence X is compact $\square$

Application to Linear Analysis

Def $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, X Banach space over $\mathbb{F}$

The weak-$*$ topology on X^* is the smallest topology that makes all coordinate maps

$$\varphi_x : X^* \longrightarrow \mathbb{F} \quad \text{continuous.}$$
$$f \longmapsto f(x)$$

Remark τ_{w*} weak-$*$ topology is the restriction of the product topology on $\mathbb{F}^X$ to $X^* \subset \mathbb{F}^X$ (adding conditions of linearity and boundedness)

Theorem (Banach-Alaoglu, 1932 and 1938 resp)

If X Banach space, then the closed unit ball of X^*, denoted B_{X^*}, is compact for the weak-$*$ topology.

Proof $B_{X^*} \subset Y = \prod_{x \in X} \underbrace{\{\lambda \in \mathbb{F} : |\lambda| \leq \|x\|\}}_{\text{cpct}}$

(Y, τ) with product topology is compact by Tychonoff's theorem

B_{X^*} is closed in Y for product topology, and the restriction of the product topology to B_{X^*} is the weak-$*$ topology.

Let us prove that B_{X^*} is weak-$*$ closed.

Be careful: cannot be proven with sequences in general / product topology not always first countable, i.e. doesn't always have a ctble nghd basis around each point.)

So instead let us prove that

$Y \setminus B_{X^*}$ is open: $f \in Y \setminus B_{X^*}$,

f has to violet linearity, so either

(a) $\exists \lambda \in \mathbb{F}, x \in X$ s.t. $f(\lambda x) \neq \lambda f(x)$

(b) $\exists x, y \in X$ s.t. $f(x+y) \neq f(x) + f(y)$

Case (a): define $r_1 \doteq \frac{1}{2}(f(\lambda x) - \lambda f(x))$

and $f \in \underbrace{(\varphi_{\lambda x} - \lambda \varphi_x)^{-1}(\{|\lambda| > r_1\})}_{\text{open for product topology}} \subset Y \setminus B_{X^*}$

Case (b): define $r_2 \doteq \frac{1}{2}(f(x+y) - f(x) - f(y))$

and $f \in \underbrace{(\varphi_{x+y} - \varphi_x - \varphi_y)^{-1}(\{|\lambda| > r_2\})}_{\text{open for product topology}} \subset Y \setminus B_{X^*}$

Hence closed B_{X^*} in Y compact so B_{X^*} compact. $\square$

<u>Rk</u> When X separable, there is a proof without Tychonoff's theorem (Banach) arguing as follows:

1) Prove that B_{X^*} metrizable for the weak* topology

2) Use diagonal argument to prove sequential compactness: extract subseq of $(f_n)_{n \geq 1}$ in B_{X^*} converging on a countable dense subset of X, extend to whole of X.