PART II REPRESENTATION THEORY

How groups act as groups of linear transformations on vector spaces.

Groups are <u>finite</u> or <u>compact</u> topological.

Vector spaces are fd and (usually) over $\mathbb{C}$.

<u>BOOKS</u> · Jones-Liebeck [JL], · Alperin-Bell

· online notes of SM / Teleman

§1 <u>Group action</u>

· F field – usually $F = \mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$: <u>ordinary</u> rep^n theory

– sometimes $F = \mathbb{F}_p$ or $\overline{\mathbb{F}_p}$ (alg cl) : <u>modular</u> rep^n theory

· V vector space over F – always f.d over F

· $GL(V) = \{ \theta : V \to V : \theta \text{ linear, invertible} \}$

– group operation is composition, id op is identity element

<u>Basic Linear Algebra</u>

If $\dim_F V = n < \infty$, choose basis $e_1, \dots, e_n$, identify with F^n.

Then $\theta \in GL(V)$ corresp to a $n \times n$ matrix $A_\theta = (a_{ij})$

$$\theta(e_j) = \sum a_{ij} e_i \quad (1 \le j \le n)$$

and in fact $A_\theta \in GL_n(F)$

(1.1) $GL(V) \cong GL_n(F)$ as groups

$$\theta \mapsto A_\theta$$

Choosing diff basis gives diff isom to $GL_n(F)$

(1.2) A_1, A_2 represent the same elt of $GL(V)$ wrt diff bases iff they are conjugate, i.e. $A_2 = X A_1 X^{-1}$ for $X \in GL_n(F)$

Recall $\mathrm{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$

(1.3) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$

independent of basis

(1.4) Let $\alpha \in GL(V)$ where V is fd / $\mathbb{C}$ with $\alpha^m = \iota$

Then α is diagonalisable cf IB Lin Alg

(1.5) Prop Take V f.d. / $\mathbb{C}$, $\alpha \in End(V)$

Then α is diag^{ble} iff $\exists$ poly f with distinct linear factor $f(\alpha) = 0$

Rmk $\alpha^m - \iota = \prod (\alpha - e^{2\pi j i/m} \iota)$

In fact, (1.4)* a finite family of commuting separately diag^{ble} is simultaneous diag^{ble}

Basic Group Theory

(1.6) Sym group S_n or $Sym(X)$ on set $X = \{1, \dots, n\}$, $n!$

Alt group A_n on X is , $\frac{1}{2} n!$

- set of products of an even number of transpositions

(1.7) Cyclic group order m: $C_m = \langle x : x^m = 1 \rangle$

e.g. $\mathbb{Z}/m\mathbb{Z}$ under $+$, it's also m^{th} roots of unity in $\mathbb{C}$

also rot syms of regular m-gon in $\mathbb{R}^2$

(1.8) Dihedral group D_{2m}, order $2m = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$

Group of rotations & reflections preserving regular m-gon

(1.9) Quaternion group $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$

In $GL_2(\mathbb{C})$, put $\underline{i} = \begin{bmatrix} i & \\ & -i \end{bmatrix}$, $\underline{j} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, $\underline{k} = \begin{bmatrix} & i \\ i & \end{bmatrix}$

(1.10) The conj class (ccls) of $g \in G$ is

$$\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\} = g^G$$

Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$ where $C_G(g) = \{x \in G : xgx^{-1} = g\}$

centraliser of g

(1.11) G group, X set

G acts on X if $\exists$ map $\cdot : G \times X \to X$
for $g \in G, x \in X$ s.t. $(g, x) \mapsto g \cdot x$ (or $gx \in X$)

$1 . x = x \quad \forall x$

$(gh) . x = g . (h . x) \quad \forall g, h \in G, x \in X$

(1.12) Given an action of G on X, obtain hom $\theta : G \to \mathrm{Sym}(X)$
the perm repⁿ of G ⇔ for $g \in G$, the fⁿ $\theta_g : X \to X$, $x \mapsto gx$ is a permⁿ
Indeed, $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$

In this course, we want the action to be linear, namely

(1.13) $g(v_1 + v_2) = gv_1 + gv_2 \quad \forall v_1, v_2 \in V$

$g(\lambda v) = \lambda g v \quad \forall \lambda \in F, v \in V$

2 Basic definitions

G finite group ; F field, usually $\mathbb{C}$

(2.1) Defⁿ Let V be a f.d. vector space $/F$

A ~~(linear)~~ repⁿ of G on V is a gp hom

$$\rho = \rho_V : G \longrightarrow GL(V)$$

[Write ρ_g for $\rho_V(g)$; so for $g \in G$, $\rho_g \in G(V)$, $\rho_1 = id_V$

$\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$; $\rho_{g^{-1}} = (\rho_g)^{-1}$]

The dimension (or degree) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \trianglelefteq G$ and $G/\ker\rho \cong \rho(G) \leq GL(V)$ (iso thm)

We say ρ is faithful if $\ker \rho = \{1\}$

Alternative (equiv) approach is to observe that a repⁿ of G on V is "the same as" a linear action of G

(2.3) Defⁿ G acts linearly on V if $\exists$ linear action $G \times V \to V$

Now, if G acts linearly on V, the map $G \to GL(V)$, $g \mapsto \rho_g$
with $\rho_g : v \mapsto gv$, is a representation of G.

Conversely, given a repⁿ $G \to GL(V)$, have lin action $gv = \rho_g(v)$

Rmk (2.4) We also say that V is a G-space or that V is a G-module. In fact, if you define the group algebra

$FG = \{ \sum_{g \in G} \alpha_g g : \alpha_g \in F \}$ then V is actually a FG-module.

(FG is an example of a F-algebra i.e. ring which is F-module st mult bilin)

(2.5) R is a matrix repⁿ of G of degree n if R is a hom $G \to GL_n(F)$

Γ Given lin rep $\rho : G \to GL(V)$ with $\dim_F(V) = n$, fix basis $\mathcal{B}$;

get a matrix rep $G \to GL_n(F)$, $g \mapsto [\rho(g)]_{\mathcal{B}}$

Conversely, given matrix repn obtain rep $\rho : G \to GL(F^n)$

(2.6) Example Given a group G; take $V = F$ and $\rho : G \to GL(V)$, $g \mapsto id$

is known as the trivial repⁿ of G. So deg $\rho = 1$

(2.7) Ex $G = C_4 = \langle x : x^4 = 1 \rangle$

Let $n = 2$, $F = \mathbb{C}$

$R : x \mapsto X$ will determine $x^j \mapsto X^j$

We need $X^4 = I$: so you can take

- X diagonal, any such with entries $\in \{\pm 1, \pm i\}$ (16 choices); or
- X not diagonal, then conj to a diag, via (1.4)

Equiv representations

(2.8) Defⁿ Fix G, F. Let V, V' be F-spaces, and

$\rho : G \to GL(V)$
$\rho' : G \to GL(V')$ repⁿs of G.

The linear map $\varphi : V \to V'$ is a G-hom / intertwining hom if

$\varphi \rho(g) = \rho'(g) \varphi \quad (\forall g \in G)$

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \varphi \downarrow & & \downarrow \varphi \\ V' & \xrightarrow[\rho'_g]{} & V' \end{array}$$

We say φ intertwines ρ, ρ'

Write $\mathrm{Hom}_G(V, V') = (V, V')^G$

φ is a <u>G-isomorphism</u> if also φ is bijective

If such φ exists, ρ and ρ' are <u>isomorphic / equiv</u>

If φ is a G-isom, can write $\rho' = \varphi \rho \varphi^{-1}$

(2.9) <u>Lemma</u> The relation of "being isomorphic" is an equivalence relation on the set of all lin reps of G (over F)

<u>Remark</u> (2.10) If ρ, ρ' are isom rep^{n}s, they have same dim

Converse maybe false: C_4 has 4 non-iso 1-dim rep

then they are $\rho_j(x^i) = \omega^{ij}$

(2.11) <u>Rmk</u> Given G, V over F of dim n and $\rho : G \to GL(V)$.

Recall def^n of representation $\rho: G \to GL(V)$

Equivalent to linear action, matrix rep^n, G-module

Recall equivalence of rep^{n}s; intertwining maps

Any rep^n isomorphic to a matrix representation

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \varphi \downarrow & & \downarrow \varphi \\ F^n & \xrightarrow[\rho']{} & F^n \end{array}$$

(2.12) • In terms of matrix rep^{n}s,

$R: G \to GL_n(F)$

$R': G \to GL_n(F)$ are <u>G-isom</u> if $\exists$ matrix $X \in GL_n(F)$ with $R'(g) = X R(g) X^{-1}$ $\forall g \in G$

• In terms of G-actions, the actions of G on V, V' are <u>G-isom</u> if $\exists$ isom $\varphi: V \to V'$ s.t. $g\varphi(v) = \varphi(gv)$ $\forall g \in G, v \in V$

<u>Subrep^n</u> <u>Def</u> Let $\rho: G \to GL(V)$ be a rep^n.

Say $W \le V$ is a <u>G-subspace</u> if it's a lin subspace and it is $\rho(G)$ invariant, i.e. $\rho_G(W) \subseteq W$ $\forall g \in G$

Obviously, $V, \{0\}$ are G-subspaces

ρ is <u>irreducible</u>/simple if have no proper G-subspace

(2.14) <u>Ex</u> 1-dim rep^n $\Rightarrow$ irreducible (but not conversely e.g. D_8 on $\mathbb{C}^2$)

(2.15) In Def (2.13), let $g \mapsto \rho(g)|_W$ be $G \to GL(W)$ get a <u>subrep^n</u> of ρ.

(2.16) <u>Lemma</u> Let $\rho: G \to GL(V)$ rep^n. If W is G-subspace, $\mathcal{B} = \{v_1, \dots, v_n\}$ basis containing $\{v_1, \dots, v_m\}$ basis of W, then matrix of $\rho(g)$ is of form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

(2.17) <u>Ex</u> $F = \mathbb{C}$ The irred rep^n of C_4 are all 1-dim

$x \mapsto i$, $x \mapsto -1$, $x \mapsto -i$, $x \mapsto 1$ (only 2 faithful)

In general, $C_m = \langle x \mid x^m = 1 \rangle$ has precisely m irred complex rep^n, all of dim^n 1. Put $\omega = e^{2\pi i/m}$, define $\rho_c: x^j \mapsto \omega^{cj}$ $(0 \le j \le m-1)$

Actually all cx irred rep^n of finite <u>abelian</u> group are 1-dim (see 4.4)

(ii) $G = D_6$, claim irred $\mathbb{C}$-rep^n has dim ≤ 2

Let $\rho : G \to GL(V)$ be irred G-rep^n. Let r = rotation $(\neq 1)$ and s = reflec^n.

Take e-vector v of $\rho(r)$, so $\rho(r)v = \lambda v$, $\lambda \neq 0$.

Let $W = \langle v, \rho(s)v \rangle \leq V$. Since

$$\rho(s)\rho(s)v = v \;,\; \rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v,$$

W is G-invariant, hence $W = V$.

Exercise| Classify all ired rep^n of D_6 up to isom (use fact $\lambda^3 = 1$)

(2.18) Def Say $\rho : G \to GL(V)$ is decomposable if there are G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is a direct sum $\rho_U \oplus \rho_W$.
If no such U, W exist, say ρ is indecomposable.

(2.19) Lemma Spse $\rho : G \to GL(V)$ is decomp with G-inv decomp $V = U \oplus W$.
If $\mathcal{B}$ is a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ formed from bases of U, W, then $\rho(g)$ wrt $\mathcal{B}$ is $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $\forall g \in G$

(2.20) Def^n $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$. The direct sum of ρ, ρ' is

$$\rho \oplus \rho' : G \to GL(V \oplus V') \text{ where } (\rho \oplus \rho')(g)(v + v') = \rho(g)(v) + \rho'(g)(v')$$

is a block diagonal action.

For matrix rep^{n}s $R : G \to GL_n(F)$, $R' : G \to GL_{n'}(F)$ define

$$R \oplus R' : G \to GL_{n+n'}(F) \;,\quad g \mapsto \left[\begin{array}{c|c} R(g) & 0 \\ \hline 0 & R'(g) \end{array}\right] \quad \forall g \in G$$

3. Complete irreducibility & Maschke's Theorem

(3.1) <u>Def^n</u> Rep^n $\rho: G \to GL(V)$ is <u>completely reducible</u> if it is a direct sum of irred rep^{n}s.

- Evidently, irred $\Rightarrow$ comp red, however not all rep^{n}s are comp red (ExSh1)

So <u>from now on</u> take G <u>finite</u> and <u>char $F = 0$</u>

(3.3) <u>Thm</u> Say fd rep^n V of a finite gp over a field of char 0 is comp red, i.e. $V \cong V_1 \oplus \cdots \oplus V_r$ is a direct sum of irred rep^n.

STP

(3.4) <u>Theorem</u> (Maschke's Thm 1899.)

G finite, $\rho: G \to G(V)$ with V an F-space, char $F = 0$.

If W is a G-subspace, $\exists$ G-subspace U of V s.t. $V = W \oplus U$.

(a direct sum of G-subspaces)

<u>Proof 1</u> Let W' be any <u>vector subsp</u> complement of W in V, i.e. $V = W \oplus W'$ ($W \cap W' = \{0\}$, $W + W' = V$)

Let $q: V \to W$ be projection onto W along W' i.e. $v = w + w' \mapsto w$

Define $\bar{q}: v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)\, q(\rho(g^{-1}) v)$ average over G.

need char $F = 0$ to invert
only really need char $F \nmid |G|$

<u>(Drop the ρs)</u>

<u>Claim (i)</u> $\bar{q}: V \to W$

(for $v \in V$, $q(g^{-1}v) \in W$ and $gW \subseteq W$)

<u>Claim (ii)</u> $\bar{q}(w) = w$ for $w \in W$

$$\left(\bar{q}(w) = \frac{1}{|G|} \sum_g g \underbrace{q(g^{-1}w)}_{\in W} = \frac{1}{|G|} \sum_g g\, g^{-1}(w) = \frac{1}{|G|} \sum_g w = w \right)$$

So (i), (ii) $\Rightarrow$ $\bar{q}$ projects V onto W

<u>Claim (iii)</u> If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ $\forall v \in V$

$$\Big(h\bar{q}(v) = h \frac{1}{|G|} \sum_g g\, q(g^{-1}(v)) = \frac{1}{|G|} \sum_g hg\, q(g^{-1}v)$$

$$= \frac{1}{|G|} \sum_g (hg)\, q((hg)^{-1}hv) = \frac{1}{|G|} \sum_g g\, q(g^{-1}hv) = \bar{q}(hv) \Big)$$

Claim (iv) $\ker \bar{q}$ is G-invariant

(if $v \in \ker \bar{q}$, $h \in G$, then $h\bar{q}(v) = \bar{q}(hv) = 0 \therefore hv \in \ker \bar{q}$)

Conclude: $V = \operatorname{im} \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is G-subsp decomp $\square$

Rmk Complements not necessarily unique

Alternative proof uses inner products: take $F = \mathbb{C}$; this can be generalised to certain compact groups (see §15)

Recall, $\langle , \rangle$ is a Hermitian inner product if

(a) $\langle w, v \rangle = \overline{\langle v, w \rangle} \quad \forall v, w$

(b) linear on RHS

(c) $\langle v, v \rangle > 0$ if $v \neq 0$

(d) Additionally, $\langle , \rangle$ is G-invariant if $\langle gv, gw \rangle = \langle v, w \rangle$

Rmk If W is G-invariant subsp, then $W^\perp$ also invariant, $G = W \oplus W^\perp$

Need to show $\forall v \in W^\perp, g \in G, gv \in W^\perp$

But $v \in W^\perp \iff \langle v, w \rangle = 0 \; \forall w \in W$.

By (d), $\langle gv, gw \rangle = 0 \; \forall g \in G$

Hence $\langle gv, w' \rangle = 0 \; \forall w' \in W$, choosing $w = g^{-1} w' \in W$ using G-invariance of W. Then $gv \in W^\perp$ as required.

Hence done if $\exists$ G-inner product

(3.4)* Thm (Weyl's unitary trick) Let ρ be a cx rep^n of finite gp G on the $\mathbb{C}$-space V. Then $\exists$ G-invariant IP on V

Exercise Every finite subgroup of $GL_n(\mathbb{C})$ conj to subgp of $U(n)$

Proof 2 $\exists$ an IP on V, take basis $e_1, \dots, e_n$ define $(e_i, e_j) = \delta_{ij}$

Now define
$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

Claim $\langle , \rangle$ is Herm sesq, pos def, G-invariant,

e.g. if $h \in G$, $\langle hv, hw \rangle = \frac{1}{|G|} \sum_g (ghv, ghw)$

$$= \frac{1}{|G|} \sum_{g'} (g'v, g'w)$$

$$= \langle v, w \rangle \qquad \square$$

(3.5) Def^n (the left regular rep^n of G)

Recall group algebra is the F-space $FG = sp\{e_g : g \in G\}$

There is a linear G-action:

$$h \in G: \quad h \sum_g a_g e_g = \sum_g a_g e_{hg} = \sum_{g'} a_{h^{-1}g'} e_{g'}$$

(~~3.6~~) ρ_{reg} is the corresp rep^n – the regular rep^n of G

This is faithful of dimension $|G|$ $\}$ FG is reg module

It turns out that every irred repn of G is a subrep of ρ_{reg}

(3.6) Prop Let ρ be an irred rep^n of the finite gp G over a field of char 0. Then ρ isom to a subrep of ρ_{reg}

Pf Take $\rho: G \to GL(V)$ irred and let $0 \neq v \in V$

Let $\theta: FG \longrightarrow V$

$$\sum_g a_g e_g \mapsto \sum a_g gv \qquad \text{(a } G\text{-homom)}$$

Now V irred and $\operatorname{im}\theta = V$ (since $\operatorname{im}\theta$ is a G-subspace)

Then $\ker\theta$ is subspace of FG. Let W be a G complement of $\ker\theta$ in FG,

so $FG = \ker\theta \oplus W$

Thus $W \cong FG/\ker\theta \underset{G\text{-isom}}{\cong} \operatorname{im}\theta = V \qquad \square$

More generally (3.7) Def^n Let G act on a set X. Let $FX = \langle e_x : x \in X \rangle$ with G-action $g(\sum a_x e_x) = \sum a_x e_{gx}$

so we have a G-space on FX. The rep^n $G \to GL(V)$ with $V = FX$ is the corresponding permutation representation.

L3.1

§4 Schur's Lemma

Thm (Schur's Lemma) (4.1)

(a) Assume V, W are irred G-spaces (over field F). Then any G-hom $\theta : V \to W$ is either 0 or is an isomorphism.

(b) Assume F is alg closed and let V be an irred G-space. Then any G-endom $V \to V$ is a scalar multiple of 1_V (a "homothety")

Proof (a) Let $\theta : V \to W$ be a hom. Then $\ker\theta$ is a G-subspace.
Since V is irred, either $\ker\theta = 0$ or $\ker\theta = V$.
And $\operatorname{im}\theta$ is G-subspace of W, so similarly $\operatorname{im}\theta = 0$ or $\operatorname{im}\theta = W$.
So either $\theta = 0$ or θ is inj, surj so an isom.

(b) Since F is alg closed, θ has eval λ. Then $\theta - \lambda 1_V$ is singular G-endom. On V, so must be 0 i.e. $\theta = \lambda 1_V$. □

Recall from (2.8) the F-space $\operatorname{Hom}_G(V, W)$ of all G-hom $V \to W$.
Write $\operatorname{End}_G(V)$ for the end algebra $\operatorname{Hom}_G(V, V)$.

Cor If V, W are irred complex G-spaces then (4.2)

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \begin{cases} 1, & \text{if } V, W \text{ isom} \\ 0, & \text{if not} \end{cases}$$

Pf If V, W not isom then only G-hom is 0 by (4.1).
Assume $V \underset{G}{\cong} W$ and $\theta_1, \theta_2 \in \operatorname{Hom}_G(V, W)$, both $\neq 0$.
Then θ_2 is invertible by (4.1) and $\theta_2^{-1}\theta_1 \in \operatorname{End}_G(V)$ is $\neq 0$.
So $\theta_2^{-1}\theta_1 = \lambda 1_V$ for some $\lambda \in \mathbb{C}$, i.e. $\lambda\theta_2 = \theta_1$. □

(4.3) Cor If finite gp G has a faithful complex irred rep^n then $Z(G)$ cyc.

Remark converse is false

Pf Let $\rho : G \to GL(V)$ faithful irred cx rep^n.
Let ~~g~~ $\gamma \in Z(G)$, so $\gamma g = g\gamma$ $\forall g \in G$.
Then the map $\varphi_\gamma : v \mapsto \gamma v$ is a G-endom on V, hence is mult by some scalar μ_γ say (Schur).

Then the map $Z(G) \longrightarrow \mathbb{C}^\times$ is a faithful rep^n of Z.
$\qquad\qquad\qquad\qquad \gamma \longmapsto \mu_\gamma$

Then $Z(G)$ isom to a finite subgroup of $\mathbb{C}^\times$, hence cyclic. $\square$

<u>Rep^n of finite abelian gps</u>

(4.4) <u>Cor</u> The irred $\mathbb{C}$-repns of a finite abelian group are all 1-dim

<u>Pf</u> <u>Either</u> $(1.4)^*$ to invoke simult diag: if v is an eigenval for each $g \in G$ and if V is irred, then $V = \langle v \rangle$

<u>Or</u> Let V be irred $\mathbb{C}$-rep. For $g \in G$, the map ~~α~~ $\theta_g : V \to V$, $v \mapsto gv$
is a G-endom of V, so $\theta_g = \lambda_g \iota$ (Schur)

Then, as V is irred, $V = \langle v \rangle$. $\square$

<u>Remark</u> fails over $\mathbb{R}$, e.g. C_3 has 2 irred $\mathbb{R}$-rep^n, one of dim 1, one of dim 2 (see ExSheet)

Recall every finite is isom to product of cyclic groups (GRM)

(4.5) <u>Prop^n</u> The finite abelian gp $G = C_{n_1} \times \cdots \times C_{n_r}$ has precisely $|G|$ irred $\mathbb{C}$-repns, as described below.

<u>Pf</u> Write $G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ where $o(x_j) = n_j$.

Spose ρ irred, so 1-dim $\rho : G \to \mathbb{C}^\times$

Let $\rho(1, \ldots, x_j, \ldots, 1) = \lambda_j$. Then $\lambda_j^{n_j} = 1$, so is an n_j^{th} root of 1.

Now the values $(\lambda_1, \ldots, \lambda_r)$ determine ρ: any $x \in G$ has form

$$\rho(g) = \rho(x_1^{j_1}, \ldots, x_r^{j_r}) = \lambda_1^{j_1} \cdots \lambda_r^{j_r}$$

Thus $\rho \longleftrightarrow (\lambda_1, \ldots, \lambda_r)$ where $\lambda_j^{n_j} = 1 \ \forall j$

Have $n_1 n_2 \cdots n_r$ such r-tuples, each giving 1-dim rep^n, no two equiv $\square$

<u>Examples</u> (a) $G = C_4 = \langle x \rangle$

	1	x	x^2	x^4
ρ_1	1	1	1	1
ρ_2	1	i	-1	$-i$
ρ_3	$+1$	-1	$+1$	-1
ρ_4	1	$-i$	-1	i

(b) $G = C_2 \times C_2 = \langle x_1, x_2 \rangle$

	1	x_1	x_2	$x_1 x_2$
ρ_1	1	1	1	1
ρ_2	1	1	-1	-1
ρ_3	1	-1	1	-1
ρ_4	1	-1	-1	1

Remark There is no 'natural' one-one corresp between the elts of G and the rep^{n}s of G (if G finite abelian). If you choose an isom $G \cong C_{a_1} \times \cdots \times C_{a_r}$ then we can identify, but depends on choice.

Isotypical decomposition (sketch)

Recall any diag endom $\alpha : V \to V$ gives eigenspace decomp of V,

$$V \cong \bigoplus_\lambda V(\lambda) \quad \text{where} \quad V(\lambda) = \{\underline{v} : \alpha \underline{v} = \lambda \underline{v}\}$$

Canonical in that it depends on α alone

(But no canonical eigenbasis)

Know, in char 0; every rep decomposes $V = \bigoplus_i V_i$, V_i irred.

How unique is this?

(a) (uniqueness) for each V there is only one way to decompose V with V_i irred (eg orbit decomp for gp acting on set)

(b) (isotypes) for each V $\exists!$ subreps $U_1, \ldots, U_k$ s.t. $V = \bigoplus U_i$ and if $V_i \leq U_i$ and $V_j' \leq U_j$ are irred subreps then $V_i \cong V_j' \iff i = j$ (cf eigenspaces of a diag linear map)

(c) (factors) If $\bigoplus_{i=1}^{k} V_i \cong \bigoplus_{i=1}^{k'} V_i'$ with V_i, V_i' irred then $k = k'$ and $\exists \pi \in S_k$ s.t. $V'_{\pi(i)} \cong V_i$ (cf dims of eigenspaces of diag maps)

Evidently (a) too strong ($G = 1$ acting on any V, dim V; every line gives irred subrep). (b), (c) work for $\mathbb{Z}/2\mathbb{Z}$; the U_i are eigenspaces of $\rho(1)$. See [T, §5] ← Teleman

(4.7) Lemma V, V_1, V_2 are reps of G over F. Then

(i) $\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$

(ii) $\mathrm{Hom}_G(V_1 \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$

Pf (i) $\pi_i : V_1 \oplus V_2 \to V_i$ is G-linear projection onto V_i, kernel V_{1-i}

$$\varphi \mapsto (\pi_1 \varphi, \pi_2 \varphi) \quad \text{has inverse} \quad (\psi_1, \psi_2) \mapsto \psi_1 + \psi_2$$

(ii) The map $\varphi \mapsto (\varphi|_{V_1}, \varphi|_{V_2})$ has inverse

$$(\psi_1, \psi_2) \mapsto (\psi_1 \pi_1 + \psi_2 \pi_2) \qquad \underline{\text{Check this!}}$$

□

(4.8) <u>Cor</u> Assume F is alg closed. Let $V = \bigoplus_{i=1}^{n} V_i$ be ~~a~~ decomp of the rep^n G into irreducible summands. Then for each irred rep^n S of G,

$$\# \{ j : V_j \cong S \} = \dim \mathrm{Hom}_G (S, V)$$

(called the <u>multiplicity</u> of S in V)

<u>Pf</u> Use induction on n. If $n=0,1$, OK. If $n>1$, write

$$V = \bigoplus_{i=1}^{n-1} V_i \oplus V_n$$

nice by Schur

By (4.7) $\dim \mathrm{Hom}_G(S,V) = \dim \mathrm{Hom}_G(S, \bigoplus_{i=1}^{n-1} V_i) + \dim \mathrm{Hom}_G(S, V_n)$ □

(4.9) <u>Def^n</u> A decomp of V as $\bigoplus W_j$ where each $W_j \cong n_j$ copies of the irred repn S_j (each non-iso for each j) is the <u>canonical</u> or <u>isotypical</u> decomposition. The W_j are <u>isotypical components</u>. If F is closed, $n_j = \dim \mathrm{Hom}_G (S_j, V)$.

<u>Exercise</u> Show, for G abelian, that every cx rep of G has unique ↺ decomp.

L4.1

5. Character Theory

Want to attach invariants to a rep^n ρ of a finite group G on V. Matrix coeffts of $\rho(g)$ are basis dependant, so not a true invariant.

Take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \to GL(V)$ rep^n of G.

(5.1) Def The character $\chi_\rho = \chi_V = \chi$ is defined as

$$\chi(g) = \mathrm{tr}\,\rho(g)$$

$$[= \mathrm{tr}\, R(g) \text{ where } R(g) \text{ is any matrix for } \rho(g)]$$

The degree of χ is $\dim V$.

Thus χ is a function $G \to \mathbb{C}$. χ is linear if $\dim V = 1$, in which case χ is a hom $G \to \mathbb{C}^\times$.

χ is irred if ρ is. χ is faithful if ρ is.

χ is trivial or principal if ρ is the trivial rep^n. We write $\chi = 1_G$.

χ is a complete invariant in the sense that it determines ρ up to isom (!)

(5.2) Thm (first properties of χ)

(i) $\chi_V(1) = \dim V$ (clear: $\mathrm{tr}\, I_n = n$)

(ii) χ_V is a class function, viz it is a conjugation invariant $\chi(hgh^{-1}) = \chi(g)$

Thus χ_V is constant on the ccls of G

(iii) $\chi_V(g^{-1}) = \overline{\chi(g)}$

(iv) For two rep^{n}s V, W, $\chi_{V \oplus W} = \chi_V + \chi_W$

Proof (ii) $\mathrm{tr}(R_h R_g R_{h^{-1}}) = \mathrm{tr}(R_g) = \chi(g)$

(iii) $g \in G$ has finite order, so can assume $\rho(g)$ is rep by a diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where each λ_i is a root of unity. Then $\chi(g) = \sum \lambda_i$.

Now g^{-1} is rep by $\begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}$ and $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\chi(g)}$

(iv) Spose $V = V_1 \oplus V_2$, $\rho_i : V \to GL(V_i)$, $\rho : V \to GL(V)$

Take basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ of V. Wrt $\mathcal{B}$, $\rho(g)$ has matrix

$\uparrow$ of V_1 $\quad$ $\uparrow$ of V_2

$$\begin{pmatrix} [\rho_1(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_2(g)]_{\mathcal{B}_2} \end{pmatrix}$$ and so $\chi(g) = \mathrm{tr} = \mathrm{tr}\,\rho_1(g) + \mathrm{tr}\,\rho_2(g)$

$= \chi_1(g) + \chi_2(g)$

$\square$

L4.2

<u>Remark</u> We see later that

χ_1, χ_2 chars of $G \Rightarrow \chi_1\chi_2$ also char of G

(uses tensor products – see 9.6)

(5.3) <u>Lemma</u> Let $\rho: G \to GL(V)$ be a (complex) rep^n affording the character χ. Then for $g \in G$, $|\chi(g)| \leq \chi(1)$, with equality iff $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, root of unity.

Moreover $\chi(g) = \chi(1) \iff g \in \ker\rho$, i.e. $\overset{\rho}{\cancel{\chi}}(g) = \overset{\rho}{\cancel{\chi}}(1)$ hahaha T.T

<u>Proof</u> Fix g. Wrt a basis of V of eigenvecs of $\rho(g)$, the matrix of $\rho(g)$ is $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Then $|\chi(g)| \leq \sum |\lambda_i| = \sum 1 = \dim_{\mathbb{C}} V = \chi(1)$, with equality iff all the λ_i are equal, to λ say.

And if $\chi(g) = \chi(1)$ then $\rho(g) = \lambda 1_V \Rightarrow \chi(g) = \lambda\chi(1) \Rightarrow \lambda = 1$ $\square$

(5.4) <u>Lemma</u> a) If χ is a complex (irred) char of G, so is $\overline{\chi}$

b) " , so is $\varepsilon\chi$

for any <u>linear</u> char ε of G.

<u>Pf</u> If $R: G \to GL_n(\mathbb{C})$ is a complex matrix rep, then so are

$$\overline{R}: G \to GL_n(\mathbb{C}), \quad g \mapsto \overline{R(g)} \qquad \text{and} \qquad \varepsilon R: G \to GL_n(\mathbb{C}), \quad g \mapsto \varepsilon(g)R(g)$$

Check the details, verify irreducibility. $\square$

(5.5) <u>Def</u> $\mathcal{C}\ell(G) = \{ f: G \to \mathbb{C} : f(hgh^{-1}) = f(g) \ \forall h, g \in G \}$

the $\mathbb{C}$-space of <u>class functions</u> (naturally a vector space)

Let $k = k(G)$ be the number of ccls of G. ~~Let~~ List ccls as $\mathcal{C}_1, \dots, \mathcal{C}_k$. Choose $g_1 = 1, g_2, \dots, g_k$ reps of the ccls.

($\mathcal{C}_1 = \{1\}$) Note that $\dim_{\mathbb{C}} \mathcal{C}\ell(G) = k$ (the char functions δ_j of the ccls form a basis, where $\delta_j(g) = \begin{cases} 1, & g \in \mathcal{C}_j \\ 0, & \text{o/w} \end{cases}$

Define Hermitian inner product on $\mathcal{C}\ell(G)$

$$\langle f, f' \rangle = \frac{1}{|G|} \sum_g \overline{f(g)} f'(g)$$

$$= \frac{1}{|G|} \sum_j |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) = \sum_j \frac{1}{|C_G(g_j)|} \overline{f(g_j)} f'(g_j)$$

centraliser

For <u>characters</u>, from (5.2)(iii)

$$\langle \chi, \chi' \rangle = \sum_{j=1}^{k} \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

← symmetric cause inversion respects ccls (ch)

a <u>real symmetric</u> form. (We show later $\langle \chi, \chi' \rangle \in \mathbb{Z}$)

Main result follows (proof later)

(5.6) <u>Theorem</u> (Completeness of characters) The $\mathbb{C}$-irred chars of G form an orthonormal basis of $\mathcal{C}_G(G)$. Moreover,

(a) If $\rho: G \to GL(V)$, $\rho': G \to GL(V')$ are irred repns of G, affording characters χ, χ', then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho, \rho' \text{ isom} \\ 0 & \text{o/w} \end{cases}$$

(b) Each class f^n of G can be expressed as a lin combi of the irred characters of G.

(5.7) <u>Cor</u> Complex repns of finite gps are determined by their chars

<u>Pf</u> $\rho: G \to GL(V)$ affording χ

CRT (3.3) says $\rho = m_1\rho_1 \oplus \cdots \oplus m_k\rho_k$ where $\rho_1, \ldots, \rho_k$ are irred, $m_j \geq 0$

Then $m_j = \langle \chi, \chi_j \rangle$, where χ_j is afforded by ρ_j, for

$\chi = m_1\chi_1 + \cdots + m_k\chi_k$ and $\langle \chi, \chi_j \rangle = m_j$ by (5.6)(a) □

(5.8) <u>Cor</u> (Irred criterion)

If ρ is a $\mathbb{C}$-repn of G, affording χ, then ρ irred $\iff \langle \chi, \chi \rangle = 1$

<u>Pf</u> $\Rightarrow$ by orthog (orthon)

$\Leftarrow$ Assume $\langle \chi, \chi \rangle = 1$. CRT says $\chi = \sum m_j \chi_j$, $m_j \in \mathbb{Z}_{\geq 0}$.

Then $\sum m_j^2 = 1$ so $\chi = \chi_j$ for some j. □

(5.9) <u>Theorem</u> If the irred $\mathbb{C}$-reps of G, $\rho_1, \ldots, \rho_k$ have dim $n_1, \ldots, n_k$ then $|G| = \sum n_i^2$

<u>Pf</u> Recall from (3.5) $\rho_{reg}: G \to GL(\mathbb{C}G)$ the regular repn, of dim $|G|$. Let π_{reg} be its character, the <u>regular character</u> of G.

L4.4

<u>Claim 1</u> $\pi_{reg}(1) = |G|$ (5.2)(i)

$\pi_{reg}(h) = 0$ if $h \neq 1$ (only zero on diagonal)

<u>Claim 2</u> $\pi_{reg} = \sum n_j \chi_j$ with $n_j = \chi_j(1)$,

$n_j = \langle \pi_{reg}, \chi_j \rangle = \frac{1}{|G|} \sum_g \overline{\pi_{reg}(g)}\, \chi_j(g) = \frac{1}{|G|} |G| \chi_j(1) = \chi_j(1)$

Evaluate π_{reg} at 1, done. □

(5.10) <u>Cor</u> # irred chars of G (up to equiv) $= k$

(5.11) <u>Cor</u> Elts $g_1, g_2 \in G$ are conj iff $\chi(g_1) = \chi(g_2)$ for all chars χ

<u>Pf</u> $\Rightarrow$ chars are class fns

$\Leftarrow$ Let δ be char function of g_1. Then δ is a class function, so is a linear combi of irred chars of G. Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G$ □

Recall from (5.5) the inner product on $\mathcal{C}_\mathbb{C}(G)$ and the real symmetric form $\langle , \rangle$ for chars.

(5.12) <u>Def</u> G finite, over $\mathbb{C}$

The <u>character table</u> of G is the $k \times k$ matrix $X = [\chi_i(g_j)]$

where $1 = \chi_1, \chi_2, \ldots, \chi_k$ are the irreds of G and

$\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \ldots, \mathcal{C}_k$ ccls with $g_j \in \mathcal{C}_j$

The (i,j) entry is $\chi_i(g_j)$

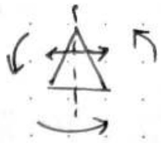
<u>Examples</u> (a) $C_3 = \langle x \mid x^3 = 1 \rangle$

Note orthog rows

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

L4.5

(b) From (4.7), $G = D_6 = \langle r, s \mid r^3 = s^2 = 1,\ rs^{-1} = sr^{-1} \rangle$



Conj classes $\mathcal{C}_1 = \{1\}$, $\{r, r^{-1}\}$, $\{s, sr, sr^2\} = \mathcal{C}_2$

3-cycles transpositions

3 irred repns: $\mathbb{1}$ (trivial)

S (sign) $x \mapsto 1$ (even), $x \mapsto -1$ (odd)

W: 2dim

sr^j acts by matrix, evals ± 1, $\chi_W(sr^j) = 0\ \forall j$

r^k acts via $\begin{pmatrix} \cos\frac{2\pi}{3} & -\sin \\ \sin & \cos \end{pmatrix}$, $\chi_W(r^k) = 2\cos\frac{2\pi}{3} = -1\ \forall k$

	1	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathbb{1}$	1	1	1
χ_S	1	-1	1
χ_W	2	0	-1

$2^2 + 1^2 + 1^2 = 6$ ☺

$\langle \chi_W, \chi_W \rangle = \frac{2^2}{6} + \frac{0^2}{2} + \frac{(-1)^2}{3} = 1$

centraliser sizes

§6 Proofs of orthogonality

Want to prove (5.6)

<u>Proof</u> (of 5.6a) Fix bases of V, V'. Write $\underset{n\times n}{R(g)}$, $\underset{n'\times n'}{R'(g)}$ for matrices of $\rho(g)$, $\rho'(g)$ wrt these.

$$\langle \chi', \chi \rangle = \frac{1}{|G|} \sum_g \chi'(g^{-1}) \chi(g) = \frac{1}{|G|} \sum_{g,i,j} R'(g^{-1})_{ii} R(g)_{jj}$$

Let $\varphi : V \to V'$ be linear and define

$$\varphi_{avg} = \tilde{\varphi} : V \longrightarrow V'$$
$$v \mapsto \sum_g \frac{1}{|G|} \underbrace{\rho'(g^{-1}) \varphi(\rho(g)) v}_{g^{-1}\varphi g . v}$$

Then $\tilde{\varphi}$ is a G-homom [if $h \in G$,

$$\rho'(h^{-1}) \tilde{\varphi} \rho(h) v = \frac{1}{|G|} \sum_g \rho'((gh)^{-1}) \varphi \underset{\rho}{(gh)} v = \frac{1}{|G|} \sum_{g'} \rho'(g'^{-1}) \varphi \rho(g') v$$
$$= \tilde{\varphi}(v) \;]$$

<u>Case 1</u> ρ, ρ' <u>not</u> isom. Schur's lemma says $\tilde{\varphi} = 0$ for any lin $\varphi : V \to V'$.

Take $\varphi = \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ (zero with one in (α,β) pos^n)

Then $\tilde{\varepsilon}_{\alpha\beta} = 0$ so $\frac{1}{|G|} \sum_g \left(R'(g^{-1}) E_{\alpha\beta} R(g) \right)_{ij} = 0$

So $\frac{1}{|G|} \sum_g R'(g^{-1})_{i\alpha} R(g)_{\beta j} = 0 \;\; \forall i,j$.

With $\alpha = i$, $\beta = j$ we get $\langle \chi', \chi \rangle = 0$ as desired.

<u>Case 2</u> ρ, ρ' isom, $\therefore \chi = \chi'$; take $V = V'$, $\rho = \rho'$ (recall WTS $\langle \chi, \chi' \rangle = 1$)

If $\varphi : V \to V$ is linear, then $\tilde{\varphi} \in \mathrm{Hom}_G(V, V)$.

Now $\mathrm{tr}\, \varphi = \mathrm{tr}\, \tilde{\varphi}$: $\left(\mathrm{tr}\, \tilde{\varphi} = \frac{1}{|G|} \sum_g \mathrm{tr}\left(\rho(g^{-1}) \varphi \rho(g) \right) = \frac{1}{|G|} \sum_g \mathrm{tr}\, \varphi = \mathrm{tr}\, \varphi \right)$

By Schur, $\tilde{\varphi} = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$ (dep on φ).

Then $\lambda = \frac{1}{n} \mathrm{tr}\, \varphi$. Let $\varphi = \varepsilon_{\alpha\beta}$, so $\mathrm{tr}\, \varphi = \delta_{\alpha\beta}$. Then

$$\tilde{\varepsilon}_{\alpha\beta} = \frac{1}{n} \delta_{\alpha\beta} \iota = \frac{1}{|G|} \sum_g \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$$

In terms of matrices, take (i,j) entry

$$\frac{1}{|G|} \sum_g R(g^{-1})_{i\alpha} R(g)_{\beta j} = \frac{1}{n} \delta_{\alpha\beta} \delta_{ij}$$

and put $\alpha = i$, $\beta = j$ to get $\frac{1}{|G|} \sum_g R(g^{-1})_{ii} R(g)_{jj} = \frac{1}{n} \delta_{ij}$

Sum over i, j; $\langle \chi, \chi' \rangle = 1$. $\square$

(6.1) Theorem (Column orthogonality relation) assuming (5.10)

$$\sum_{i=1}^{k} \overline{\chi_i(g_j)}\, \chi_i(g_\ell) = \delta_{j\ell}\, |C_G(g_j)|$$

Cor $|G| = \sum_{i=1}^{k} \chi_i^2(1)$

Proof $\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum_{l} \overline{\chi_i(g_l)}\, \chi_j(g_l) \,/\, |C_G(g_l)|$

Consider the char table $X = (\chi_i(g_j))$. Then

$$\bar{X} D^{-1} X^t = I_{k\times k} \quad \text{where} \quad D = \begin{pmatrix} |C_G(g_1)| & & \\ & \ddots & \\ & & |C_G(g_k)| \end{pmatrix}$$

Since X is square, it follows that $D^{-1}\bar{X}^t$ is inverse of X, so $\bar{X}^t X = D$. □

Proof of (b) "span"

List all irred characters $\chi_1, \dots, \chi_l$ of G.

It's enough to show that the orthog complement to $sp\langle \chi_1, \dots, \chi_l \rangle$ in $\mathcal{C}\ell(G)$ is $\{0\}$. To see this, assume $f \in \mathcal{C}\ell(G)$ with $\langle f, \chi_j \rangle = 0 \ \forall j$.

Let $\rho: G \to GL(V)$ irred rep affording $\chi \in \{\chi_1, \dots, \chi_l\}$. Then $\langle f, \chi \rangle = 0$.

Consider $\frac{1}{|G|} \sum_g \overline{f(g)}\, \rho(g) : V \to V$.

This is a G-hom$^{\text{exercise}}$, so as ρ is irred, must be $\lambda\iota_V$ for some $\lambda \in \mathbb{C}$

Now, $n\lambda = \operatorname{tr}\left(\frac{1}{|G|} \sum_g \overline{f(g)}\, \rho(g) \right)$

$$= \frac{1}{|G|} \sum_g \overline{f(g)}\, \chi(g) = 0 \quad (= \langle f, \chi \rangle)$$

So $\lambda = 0$. Hence in fact $\sum_g \overline{f(g)}\, \rho(g) = 0$ for all repns ρ.

Take $\rho = \rho_{reg}$ where $\rho_{reg}(g) : e_1 \mapsto e_g \quad (g \in G)$

So $\sum_g \overline{f(g)}\, \rho_{reg}(g) : e_1 \longmapsto \sum_g \overline{f(g)}\, e_g$

and it follows that $\sum \overline{f(g)}\, e_g = 0 \Rightarrow \overline{f(g)} = 0 \ \forall g \in G$, so $f = 0$. □

Example $G = D_6$ column orthog $\sum_i \overline{\chi_i(g_a)}\, \chi_i(g_b)$

	6	3	2
	1	r	s
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

$a=1, b=2$, $1\cdot1 + 1\cdot1 + 2\cdot(-1) = 0$

$a=2, b=2$, $1\cdot1 + 1\cdot1 + (-1)\cdot(-1) = 3 = |C_G(g_a)|$

$a=1, b=3$, $1\cdot1 - 1\cdot1 = 0$

L5.3

§7 Permutation representations

Preview given in (3.7). Recall

- G finite acting on a finite set $X = \{x_1, \dots, x_n\}$ (called G-set)
- $\mathbb{C}X = \mathbb{C}$-space, basis $\{e_{x_1}, \dots, e_{x_n}\}$ dimension $|X|$

$$= \left\{ \sum_j a_j e_{x_j} : a_j \in \mathbb{C} \right\}$$

$\rho(g) : e_{x_j} \longmapsto e_{g x_j}$; extend linearly, so

$\rho_X(g) : \sum_{x \in X} a_x e_x \longmapsto \sum_{x \in X} a_x e_{gx}$

- ρ_X is the <u>perm repn</u> corresp to the action of G on X
- matrices rep $\rho_X(g)$ wrt basis $\{e_x\}_{\in X}$ are <u>perm matrices</u>,

zero except for one 1 in each row and colum, $(\rho(g))_{ij} = 1$ iff $g x_j = x_i$

(7.1) <u>perm character</u> π_X is

$$\pi_X(g) = |\underbrace{\mathrm{fix}_X(g)}_{\text{fixator}}| = |\{x \in X : gx = x\}|$$

(7.2) π_X always contains 1_G:

$sp\langle e_{x_1} + \cdots + e_{x_n} \rangle$ is a trivial G-subspace of $\mathbb{C}X$, with G-invariant complement $sp\langle \sum_x a_x e_x : \sum a_x = 0 \rangle$

(7.3) <u>Lemma</u> (Burnside, due to Cauchy, Frobenius)

$$\langle \pi_X, 1 \rangle = \# \text{ orbits of } G \text{ on } X$$

<u>Proof</u> If $X = X_1 \cup \cdots \cup X_l$ disjoint union of orbits, then

$$\pi_X = \pi_{X_1} + \cdots + \pi_{X_l} \quad \text{with } \pi_{X_j} \text{ perm char of } G \text{ on } X_j$$

So to prove claim, it's enough to show that if G is transitive then $\langle \pi_X, 1 \rangle = 1$.

$$\langle \pi_X, 1 \rangle = \frac{1}{|G|} \sum_g \pi_X(g) = \frac{1}{|G|} \# \{(g,x) \in G \times X : gx = x\}$$

$$= \frac{1}{|G|} \sum_{x \in X} \underbrace{|G_x|}_{\text{stabiliser}} \underbrace{=}_{\text{orbit-stab}} \frac{1}{|G|} |X| |G_x| = \frac{1}{|G|} |G| = 1 \qquad \square$$

(7.4) <u>Lemma</u> Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1 \times X_2} = \pi_{X_1} \pi_{X_2}$, and so $\langle \pi_{X_1}, \pi_{X_2} \rangle = \# \mathrm{orb}(G, X_1 \times X_2)$.

Pf If $g \in G$ then $\pi_{X_1 \times X_2}(g) = \pi_{X_1}(g)\,\pi_{X_2}(g)$ (fix points)

And, $\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1}\pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle \underset{(7.3)}{=} \# \text{orb}(G, X_1 \times X_2)$ □

Def (7.5) Let G act on X, $|X| > 2$. Then G is 2-trans on X if G has just two orbits on $X \times X$: $\{(x,x) : x \in X\}$ & $\{(x_1, x_2) : x_1 \neq x_2\}$

(7.6) Lemma Let G act on X, with $|X| > 2$. Then $\pi_X = 1 + \chi$, with χ irred $\iff$ G is 2-trans on X.

Pf $\pi_X = m_1 1 + m_2 \chi_2 + \cdots + m_l \chi_l$ with $1, \chi_2, \ldots, \chi_l$ distinct irred, $m_i \in \mathbb{N}_{\geq 0}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum m_i^2$$

hence G is 2 trans on X iff $l = 2$, $m_1 = m_2 = 1$. □

Examples

(7.7) S_n acting on $X = \{1, \ldots, n\}$, 2-trans. Hence $\pi_X = 1 + \chi$, with χ irred of deg $n-1$. Similarly for A_n, if $n > 3$.

(7.8) $G = S_4$

ccl sizes	1	3	8	6	6	
reps →	1	(12)(34)	(123)	(1234)	(12)	
$id = \chi_1$	1	1	1	1	1	(trivial)
χ_2	1	1	1	-1	-1	(sign)
χ_3	3	-1	0	-1	1	$(\pi_X - 1)$
χ_4	3	-1	0	+1	-1	$(\chi_3 \cdot \chi_2)$
χ_5	2	2	-1	0	0	

know $24 = 1+1+9+9+d^2$
$\Rightarrow \deg \chi_5 = 2$

column orthog (6.1)
$$\begin{cases} 1+1-3-3+2x = 0 \\ 1+1+2y = 0 \\ 1-1-3+3+2z = 0 \\ 1-1+3-3+2w = 0 \end{cases}$$

or use (5.9) $\chi_{reg} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$

$$\Rightarrow \chi_5 = \tfrac{1}{2}(\chi_{reg} - \chi_1 - \chi_2 - 3\chi_3 - 3\chi_4)$$

or can use the lifting procedure (see §8)

observing that $S_4 / V_4 \cong S_3$

Alternating group

Suppose $g \in A_n$. Then $\mathcal{C}\ell_{A_n}(g) \subseteq \mathcal{C}\ell_{S_n}(g)$ and might not be equal,

e.g. $g = (123) \in A_3$ where $\mathcal{C}\ell_{A_3}(g) = \{g\}$.

Lemma Let $g \in A_n$, $n > 1$

(a) If g commutes with some odd perm in S_n then $\mathcal{C}\ell_{S_n}(g) = \mathcal{C}\ell_{A_n}(g)$

(b) If not, $\mathcal{C}\ell_{S_n}(g)$ splits into two classes of equal size, $(g \,;\, (12)^{-1} g (12))$

(7,10) EX $G = A_4$, $|G| = 12$ has 4 classes

	1	3	4	4	
	1	(12)(34)	(123)	$(123)^{-1}$	
$\chi_1 = 1_G$	1	1	1	1	
χ_2	3	-1	0	0	$= \pi_X - 1$
χ_3	1	1	ω	ω^2	
χ_4	1	1	ω^2	ω	

Last two chars come from lifting from $G/G' = G/V_4 \cong C_3$: see §8

§8 Normal subgroups & lifting characters

(8.1) <u>Lemma</u> Let $N \trianglelefteq G$, let $\tilde{\rho} : G/N \to GL(V)$ be a repn of G/N.

Then $\rho : G \underset{\text{natural hom}}{\longrightarrow} G/N \xrightarrow{\tilde{\rho}} GL(V)$, $g \mapsto \tilde{\rho}(gN)$

is a repn of G. Moreover, ρ is irred iff $\tilde{\rho}$ is. The corresp chars satisfy

$\chi(g) = \tilde{\chi}(gN)$, and $\deg \chi = \deg \tilde{\chi}$.

We say $\tilde{\chi}$ <u>lifts to</u> χ. The lifting $\tilde{\chi} \mapsto \chi$ is a bijection between

$\{$ all irreds of $G/N \} \longleftrightarrow \{$ all irreds of G with $N \subset$ kernel $\}$

<u>Pf</u> Easy. Note $\chi(g) = \operatorname{tr} \rho(g) = \operatorname{tr} \tilde{\rho}(gN) = \tilde{\chi}(gN)$,

and $\chi(1) = \tilde{\chi}(N)$ $\therefore$ same degree

(Bijection) If $\tilde{\chi}$ is char of G/N and χ is its lift to G then

$\tilde{\chi}(N) = \chi(1)$. Also if $k \in N$ then

$$\chi(k) = \tilde{\chi}(kN) = \tilde{\chi}(N) = \chi(1)$$

$\therefore N \leq \ker \chi$ (by 5.3)

Now let χ be char of G with $N \leq \ker \chi$. Spose $\rho : G \to GL(V)$ affords χ.

Define $\tilde{\rho} : G/N \to GL(V)$

$gN \mapsto \rho(g)$

Well-defined (as $N \leq \ker \rho$) and $\tilde{\rho}$ is a homom, hence a repn of G/N.

If $\tilde{\chi}$ is the char of $\tilde{\rho}$, then $\tilde{\chi}(gN) = \chi(g)$ so $\tilde{\chi}$ lifts to χ.

⌈ For irred: take $W \leq \mathbb{C}^n$. Then $\rho(g)w \in W \ \forall w \in W \iff \tilde{\rho}(gN) w \in W \ \forall W$

$\therefore W$ is G-invariant subspace of $\mathbb{C}^n \iff W$ is G/N-invariant subspace ⌋ □

(8.2) <u>Lemma</u> The <u>derived subgroup</u> $G' = \langle \underset{aba^{-1}b^{-1}}{[a,b]} : a, b \in G \rangle$ of G is the unique minimal normal subgroup of G s.t. G/G' is abelian

(i.e. G/N abelian $\Rightarrow G' \leq N$ and $G^{ab} := G/G'$ abelian)

G has precisely $\ell = |G/G'|$ repns of dim 1, all with kernel containing G' and obtained by lifting from G/G'.

In particular $\ell \mid |G|$.

<u>Proof</u> $G' \trianglelefteq G$ easy exercise. Let $N \trianglelefteq G$.

Let $g, h \in G$. $g^{-1}h^{-1}gh \in N \iff ghN = hgN$

$\iff (gN)(hN) = (hN)(gN)$

$\therefore G' \leq N$ iff G/N abelian. Since $G' \trianglelefteq G$, deduce G/G' abelian.

By (4.5), G/G' has exactly l irred chars $\tilde{\chi}_1, \ldots, \tilde{\chi}_l$ all of deg 1.

The lift of these to G also have deg 1 and by (8.1) there are precisely the irred chars χ of G s.t. $G' \leq \ker \chi$.

But any linear char of G is a homom $\chi : G \to \mathbb{C}^\times$ hence $\forall g, h \in G$,

$\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g)^{-1}\chi(h)^{-1} = 1$, so $G' \leq \ker \chi$ and $\chi_1, \ldots, \chi_l$ are <u>all</u> the <u>linear</u> chars of G. $\square$

<u>Examples</u> (8.3)

(a) Let $G = S_n$. Show $S_n' = A_n$. Thus, since $G/G' \cong C_2$, S_n must have exactly 2 linear chars. [JL, 17.12]

(b) $G = A_4$

	1	3	4	4	
	id	(12)(34)	(123)	(132)	
1_G	1	1	1	1	lifted from C_3
χ_2	1	1	ω	ω^2	
χ_3	1	1	ω^2	ω	
χ_4	3	-1	0	0	$\leftarrow$ (orthog)

Let $V = \{1, (12)(34), (13)(24), (14)(23)\} \leq G$; here $G' = V_4$

$\therefore V \trianglelefteq G$ and $G/V \cong C_3$

Linear chars of A_4, all of them trivial on V

(8.4) <u>Lemma</u> G is <u>not</u> simple iff $\chi(g) = \chi(1)$ for some irred char $\chi \neq 1_G$ and some $g \neq 1 \in G$.

Any normal subgroup of G is the intersection of the kers of some of the irreds of G:
$$N = \bigcap_{i \in I} \ker \chi_i$$

<u>Proof</u> If $\chi(g) = \chi(1)$ for some non-princ irred χ (afforded by ρ, say), then $g \in \ker \rho$ (5.3) $\therefore$ if $g \neq 1$ then $1 \neq \ker \rho \underset{\neq}{\trianglelefteq} G$.

If $1 \neq N \underset{\neq}{\trianglelefteq} G$, take an irred $\bar{\chi}$ of G/N $(\neq 1_{G/N})$

Lift to an irred χ, afforded by ρ of G. Then $N \leq \ker \rho \trianglelefteq G$

$\therefore \chi(g) = \chi(1)$ for $g \in N$

Claim if $1 \neq N \triangleleft G$, then N is the intersection of the kernels of the lifts of all the irreds of G/N.

Now, clearly (8.1) $\leq$; if $g \in G \setminus N$ then $gN \neq N$ $\therefore$ $\tilde{\chi}(gN) \neq \bar{\chi}(N)$ for some irred $\bar{\chi}$ of G/N. Lifting $\bar{\chi}$ to χ, we have $\chi(g) \neq \chi(1)$ □

↳ not so obvious, use char table

§9 Dual space, tensor products of repns

Recall from §5 that

- $\mathcal{C}\ell(G)$ is $\mathbb{C}$-space of class functions on G endowed with inner product $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_g \overline{f_1(g)} f_2(g)$
- $\dim_{\mathbb{C}} \mathcal{C}\ell(G) = k$, with o.n. basis $\chi_1, \dots, \chi_k$ the irred chars of G
- commutative ring with pointwise operations
- $\exists$ involution $f \mapsto f^*$ where $f^*(g) = f(g^{-1})$

Duality (9.1) Lemma Let $\rho: G \to GL(V)$ be a repn over F and let $V^* = \mathrm{Hom}_F(V, F)$ dual space of V. Then V^* is a G-space under

$$(\rho^*(g)\varphi)(v) = \varphi(\rho(g^{-1})v) ,$$

the dual repn to ρ. Its character is $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$.

Proof
$$\begin{aligned}
\rho^*(g_1)(\rho^*(g_2)\varphi)(v) &= (\rho^*(g_2)\varphi)(\rho(g_1^{-1})v) \\
&= \varphi(\rho(g_2^{-1})\rho(g_1^{-1})v) \\
&= \varphi(\rho(g_1g_2)^{-1}v) \\
&= (\rho^*(g_1g_2)\varphi)v
\end{aligned}$$

(char) fix $g \in G$, let $e_1, \dots, e_n$ be basis of V of evecs of $\rho(g)$, say $\rho(g)e_j = \lambda_j e_j$

Let $\varepsilon_1, \dots, \varepsilon_n$ be dual basis of V^*. Then $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$:

$$(\rho^*(g)\varepsilon_j)(e_i) = \varepsilon_j(\rho(g^{-1})e_i) = \varepsilon_j(\lambda_i^{-1}e_i) = \underset{\lambda_j^{-1}\varepsilon_j(e_i)}{\overset{\shortparallel}{\lambda_i^{-1}\varepsilon_j(e_i)}} \quad \forall i$$

$$\therefore \chi_{\rho^*}(g) = \sum_i \lambda_i^{-1} = \chi_\rho(g^{-1})$$ □

(9.2) Defn Let $\rho: G \to GL(V)$. Say ρ is self-dual if $V \cong V^*$ (isom as G-spaces)

Over $F = \mathbb{C}$ this holds $\iff \chi_\rho(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)} \quad \forall g \in G$

$\iff \chi_\rho(g) \in \mathbb{R} \quad \forall g$

Exs • All irred repns of S_n are self-dual (the ccls are det by cycle type)

• Not always true for A_n, ok for A_5, not so for A_7

• Perm repns $\mathbb{C}X$ are always self-dual

Tensor Products

V, W F-spaces ($F = \mathbb{C}$), $\dim V = m$, $\dim W = n$

Fix bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ of V, W rep

The tensor product space $V \otimes W$ is an mn-dimensional F-space with basis

$$\underset{V \otimes_F W}{\overset{\parallel}{V \otimes W}} \qquad \{ v_i \otimes w_j : 1 \le i \le m,\ 1 \le j \le n \}$$

Thus (a) $V \otimes W = \left\{ \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in F \right\}$ with 'obvious' addn and scalar multn

(b) If $v = \sum \alpha_i v_i \in V$, $w = \sum \beta_j w_j \in W$, define

$$v \otimes w \underset{\text{"bilinear"}}{:=} \sum_{i,j} \alpha_i \beta_j (v_i \otimes w_j)$$

Remark Not all elts of $V \otimes W$ are of this form — some are combinations

e.g. $v_1 \otimes w_1 + v_2 \otimes w_2$, which can't be further simp

(9.3) Lemma

(i) For $v \in V, w \in W, \lambda \in F$, $(\lambda v) \otimes w = \lambda (v \otimes w) = v \otimes (\lambda w)$

(ii) If $x, x_1, x_2 \in V$, $y, y_1, y_2 \in W$ then

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

Proof (i) $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$

$$\left.\begin{aligned} (\lambda v) \otimes w &= \sum_{i,j} (\lambda \alpha_i) \beta_j\, v_i \otimes w_j \\ \lambda (v \otimes w) &= \lambda \sum_{i,j} \alpha_i \beta_j\, v_i \otimes w_j \\ v \otimes (\lambda w) &= \sum_{i,j} \alpha_i (\lambda \beta_j)\, v_i \otimes w_j \end{aligned}\right\} \text{all equal}$$

(ii) similar $\square$

Remark $V \times W \to V \otimes W$ is bilinear map

$$(v, w) \mapsto v \otimes w$$

(9.4) <u>Lemma</u> If $\{e_1, \dots, e_m\}$ is any basis for V, $\{f_1, \dots, f_n\}$ any basis for W, then $\{e_i \otimes f_j : 1 \le i \le m, 1 \le j \le m\}$ basis of $V \otimes W$.

<u>Proof</u> Writing $v_k = \sum_i \alpha_{ik} e_i$, $w_l = \sum_j \beta_{jl} f_j$ we have

$$v_k \otimes w_l = \sum_{i,j} \alpha_{ik} \beta_{jl} \, e_i \otimes f_j$$

Hence $\{e_i \otimes f_j\}$ spans $V \otimes W$, and by dimension is a basis □

<u>Remark</u> One can define $V \otimes W$ in a basis indep way in the first place. See Part III Comm Alg course

(9.5) <u>Propn</u>

(9.5) Propn Let $\rho: G \to GL(V)$, $\rho': G \to GL(V')$ be repns of G.

Define $\rho \otimes \rho': G \to GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g): \sum \lambda_{ij} v_i \otimes w_j \longmapsto \sum \lambda_{ij}\, \rho(g) v_i \otimes \rho'(g) w_j$$

Then $\rho \otimes \rho'$ is a repn of G with character

$$\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g)\, \chi_{\rho'}(g) \quad \forall g \in G$$

Hence the product of two chars of G is a char of G.

(9.6) Remarks We know that ρ irred, ρ' of deg 1 $\Rightarrow$ $\rho \otimes \rho'$ irred; if ρ' not of deg 1 this is usually false (since $\rho \otimes \rho'$ can be reducible)

$r(v \otimes w) \neq rv \otimes rw$ for most elts $r \in \mathbb{C}G$ (e.g. $r = \lambda g$)

Proof Clear that $(\rho \otimes \rho')(g) \in GL(V \otimes V')$ $\forall g \in G$ and so (check homom via $(\rho\otimes\rho')(g)(v \otimes w) = (\rho(g) v \otimes \rho'(g) w)$)

$\rho \otimes \rho'$ is a homom $G \longrightarrow GL(V \otimes V')$

Let $g \in G$. Let $v_1, \dots, v_m$ be basis of V of evecs of $\rho(g)$; let $w_1, \dots, w_n$ be basis of V' of evecs of $\rho'(g)$. Say

$$\rho(g) v_j = \lambda_j v_j, \quad \rho'(g) w_j = \mu_j w_j$$

Then $(\rho \otimes \rho')(g)(v_i \otimes w_j) = \rho(g)(v_i) \otimes \rho'(g) w_j = \lambda_i v_i \otimes \mu_j w_j$

$$= \lambda_i \mu_j (v_i \otimes w_j)$$

So $\chi_{\rho \otimes \rho'}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum_{i=1}^m \lambda_i \right)\left(\sum_{j=1}^n \mu_j \right) = \chi_\rho(g)\, \chi_{\rho'}(g)$ $\square$

Powers of characters

Work over $\mathbb{C}$. Take $V = V'$ and define $V^{\otimes 2} = V \otimes V$

Let $\tau: \sum \lambda_{ij} v_i \otimes v_j \longmapsto \sum \lambda_{ij} v_j \otimes v_i$ – a linear G-endom of $V^{\otimes 2}$ s.t. $\tau^2 = 1$ ($\therefore$ evals ± 1) $[\tau(v \otimes w) = w \otimes v]$

(9.7) Defn $S^2 V = \{ x \in V^{\otimes 2} : \tau(x) = x \}$ symmetric square of V

$\Lambda^2 V = \{ x \in V^{\otimes 2} : \tau(x) = -x \}$ exterior square of V

antisym / wedge

(9.8) Lemma $S^2 V$ and $\Lambda^2 V$ are G-subspaces of $V^{\otimes 2}$, and

$$V^{\otimes 2} = S^2 V \oplus \Lambda^2 V$$

$S^2 V$ has basis $\{ v_i v_j := v_i \otimes v_j + v_j \otimes v_i \text{ for } 1 \le i \le j \le n \}$

$\Lambda^2 V$ has basis $\{ v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i \text{ for } 1 \le i < j \le n \}$

NB

and hence $\dim S^2V = \frac{n(n+1)}{2}$, $\dim \Lambda^2 V = \frac{n(n-1)}{2}$

Proof. Elementary linear algebra.

To show $V^{\otimes 2}$ reducible write $x \in V^{\otimes 2}$ as

$$x = \underbrace{\tfrac{1}{2}(x + \tau(x))}_{\in S^2V} + \underbrace{\tfrac{1}{2}(x - \tau(x))}_{\in \Lambda^2 V}$$

In fact $V^{\otimes 2}, V^{\otimes 3}, \ldots$ are never irred if $\dim V > 1$ $\square$

(9.9) Lemma. If $\rho: G \to GL(V)$ is a repn affording character χ, then χ^2 is $\chi_S + \chi_\Lambda$ where $(S^2\chi :=)\ \chi_S$ is the char of G in the subrepn on S^2V and $(\Lambda^2\chi =)\ \chi_\Lambda$ same on $\Lambda^2 V$. Moreover, for $g \in G$

$$\chi_S(g) = \tfrac{1}{2}(\chi^2(g) + \chi(g^2))$$

$$\chi_\Lambda(g) = \tfrac{1}{2}(\chi^2(g) - \chi(g^2))$$

Pf. Compute the chars χ_S, χ_Λ. Fix $g \in G$. Let $v_1, \ldots, v_n$ be a basis of V of evecs of $\rho(g)$; say $\rho(g) v_i = \lambda_i e_i$. Then

$$g \cdot v_i v_j = \lambda_i \lambda_j v_i v_j$$

(drop ρ)

$$g \cdot v_i \wedge v_j = \lambda_i \lambda_j v_i \wedge v_j$$

Hence $\chi_S(g) = \sum_{1 \le i \le j \le n} \lambda_i \lambda_j$ and $\chi_\Lambda(g) = \sum_{1 \le i < j \le n} \lambda_i \lambda_j$

Now $(\chi(g))^2 = (\sum \lambda_i)^2$

$$= \sum \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j$$

$$= \chi(g^2) + 2\chi_\Lambda(g)$$

So $\chi_\Lambda(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$ and $\chi^2 = \chi_S + \chi_\Lambda$, whence $\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$. $\square$

(9.10) Example $G = S_4$

	1	$(12)(34)$	(123)	(12)	(1234)	
1_G	1	1	1	1	1	
sgn	1	1	1	-1	-1	
$\chi_3 = \pi_X - 1$	3	-1	0	1	-1	
$\Lambda^2\chi_3 = \overline{\chi}_3$	3	-1	0	-1	1	
χ_5	2	2	-1	0	0	$\leftarrow S^2\chi_3 - 1 - \chi_3$

χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

$S^2\chi_3 = 1 + \chi_3 + \chi_5$, $\quad \Lambda^2\chi_3 = \overline{\chi_3}$ (irred $\langle\chi,\chi\rangle = 1$)

ip = 3 & contains χ_3

Actually; given 1_G, sgn and χ_3 you can construct remaining 2 irreds using $S^2\chi_3$ and $\Lambda^2\chi_3$

Characters of $G\times H$ - cf (4.5) for abelian groups

(9.11) Propn If G, H are finite gps, with their irred chars $\chi_1, \dots, \chi_k$ and $\psi_1, \dots, \psi_r$ resp, then the irred chars of the direct product $G\times H$ are precisely $\{\chi_i\psi_j : 1\le i\le k,\ 1\le j\le r\}$ where

$$\chi_i\psi_j(g,h) = \chi_i(g)\psi_j(h)$$

Proof If $\rho: G\to GL(V)$ affording χ

$\rho': H \to GL(W)$ " ψ, then

$$\begin{tikzcd} G\times H \arrow[r] \arrow[d] & G \arrow[d] \\ H \arrow[d] & GL(V) \\ GL(W) & \end{tikzcd}$$

$$\rho\otimes\rho' : G\times H \longrightarrow GL(V\otimes W)$$
$$(g,h) \longmapsto \rho(g)\otimes\rho'(h)$$
$$v_i\otimes w_j \longmapsto \rho(g)v_i \otimes \rho'(h)w_j$$

is a repn of $G\times H$ on $V\otimes W$ by (9.5). And $\chi_{\rho\otimes\rho'} = \chi\psi$ as in (9.5)

Claim $\chi_i\psi_j$ are distinct and irred

$$\begin{aligned} \langle \chi_i\psi_j, \chi_r\psi_s\rangle_{G\times H} &= \frac{1}{|G\times H|}\sum_{(g,h)} \overline{\chi_i\psi_j(g,h)}\ \chi_r\psi_s(g,h) \\ &= \frac{1}{|G|}\sum_g \overline{\chi_i(g)}\chi_r(g) \cdot \frac{1}{|H|}\sum_h \overline{\psi_j(h)}\psi_s(h) \\ &= \delta_{ir}\delta_{js} \end{aligned}$$

Complete set $\sum_{i,j}\chi_i\psi_j(1)^2 = \sum_i \chi_i(1)^2 \sum_j \psi_j(1)^2 = |G||H| = |G\times H|$ $\square$

EX $D_6\times D_6$ has 9 chars (JL Ex 19.6)

L7.4

Symmetric & exterior powers

V repn of G, $\rho: G \to GL(V)$ degree d ($=\dim V$)

$\{v_1, \dots, v_d\}$ basis of V

Let $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}$; basis $\{v_{i_1} \otimes \cdots \otimes v_{i_n}\}$ where indices $i_1, \dots, i_n$ ranges over $\{1, \dots, d\}$. So $\dim V^{\otimes n} = d^n$.

Recall any n-tuple of $u_1, \dots, u_n \in V$ defines elt $u_1 \otimes \cdots \otimes u_n \in V^{\otimes n}$, general vector in $V^{\otimes n}$ is a linear combi of such

(9.12) Prop S_n acts on $V^{\otimes n}$ by permuting the factors

$$\sigma: u_1 \otimes \cdots \otimes u_n \longmapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

Pf This defines a perm action on standard basis vectors which extends linearly to entire space. Expressing general $u_1 \otimes \cdots \otimes u_n$ in terms of basis, can check σ maps it to $u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$ as claimed. □

(9.13) Prop Defining $\rho^{\otimes n}(g)(v_1 \otimes \cdots \otimes v_n) = \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_n$ and extending linearly defines an action of G on $V^{\otimes n}$, which commutes with the action of S_n

Pf Check this is an action. Action with S_n commutes as applying $\rho^{\otimes n}(g)$ before or after perm of factors has the same effect. □

(9.14) Defn Consider the trivial 1 and sign repns $\varepsilon: S_n \to \{\pm 1\}$

The sets $S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \ \ \forall \sigma \in S_n\}$

$$\Lambda^n V = \{x \in V^{\otimes n} : \sigma(x) = \varepsilon(\sigma) x \ \ \forall \sigma \in S_n\}$$

are G-subspaces, the symmetric and exterior powers of $V^{\otimes n}$, resp

Note $\dim S^n V = \binom{d+n-1}{n}$ and $\Lambda^n V = 0$ if $n > d$.

Bases given in [T, Prop 7.7]

Tensor algebra

char$(F) = 0$

(9.15) Def Let $T^n V = V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$. The tensor algebra of V is

$$TV = \bigoplus_{n=0}^{\infty} T^n V \quad \text{with} \quad T^0 V = F \ \ (1 \text{ dim space})$$

L7.5

Multiplication

$$T^n \times T^m V \longrightarrow T^{n+m} V$$

$$x \cdot y \longmapsto x \otimes y$$

extended linearly. TV is naturally a graded algebra. Also

$SV = TV / \langle$ ideal gen by all $x \otimes y - y \otimes v \rangle$, symm alg

$\Lambda V = TV / \langle$ ideal gen by all $x \otimes x \rangle$, Grassmanian or exterior algebra

↑ contains $\langle x \otimes y + y \otimes v \rangle$ and coincides iff char $F \neq 2$

Note $SV \cong \bigoplus_{n \geq 0} S^n V$ is a comm algebra over F

of idk

$\Lambda V \cong \bigoplus_{n \geq 0} \Lambda^n V$ is graded comm ($x \in \Lambda^r V, y \in \Lambda^s V \Rightarrow x \wedge y = (-1)^{rs} y \wedge x$)

Character rings

$\mathcal{C}(G)$ is a (comm) ring: sum & product of cls fns are cls fns

In fact we verified that chars are in fact the chars afforded by the direct sum and tensor product of their coresp repns.

(9.16) Def Denote by $\mathcal{R}(G)$ the $\mathbb{Z}$-submodule of $\mathcal{C}(G)$ spanned by irred chars of G. $\mathcal{R}(G)$ is the Grothendieck or character ring, and its elts are generalized or virtual characters, $\psi = \sum_{\chi \text{ irred}} n_\chi \chi$, $n_\chi \in \mathbb{Z}$

- $\mathcal{R}(G)$ is a comm ring and any gen char is a difference of two chars $\psi = \alpha - \beta$, α, β chars $\alpha = \sum_{n_\chi \geq 0} n_\chi \chi$; $\beta = \sum_{n_\chi < 0} -n_\chi \chi$

The irred chars $\{\chi_i\}$ form a $\mathbb{Z}$-basis for $\mathcal{R}(G)$ as a $\mathbb{Z}$-module.

- If α virtual and $\langle \alpha, \alpha \rangle = 1$ and $\alpha(1) > 0$ then α is ~~virtually~~ actually a character of an irred repn of G (Write $\alpha = \sum n_i \chi_i$. Orthonormality says $\langle \alpha, \alpha \rangle = \sum n_i^2 \therefore n_i = \pm 1$ for exactly one i. Also $\alpha(1) > 0 \Rightarrow n_i = +1$)

Henceforth we don't distinguish between a character and its negative, we study generalised chars of norm 1 rather than irred chars.

10 Restriction and induction

As usual $H \leq G$ and $F = \mathbb{C}$.

(10.1) Def Spose $\rho: G \to GL(V)$ is rep^n affording χ. We can think of V as H-space by restriction only to $h \in H$. Get

$$\rho_H = \rho\downarrow_H = \mathrm{Res}^G_H \rho : H \to GL(V),$$

the restriction of ρ to H. It affords the character $\mathrm{Res}^G_H \chi = \chi\downarrow_H = \chi_H$.

(10.2) Lemma Let ψ be a non-zero char of H. Then $\exists$ irred char χ of G s.t. $\langle \chi\downarrow_H, \psi\rangle_H \neq 0$

Pf List irred chars $\chi_1, \dots, \chi_k$ of G. Recall from (5.9)

$$\chi_{reg}(g) = \begin{cases} |G|, & g = 1 \\ 0, & g \neq 1 \end{cases}$$

and $\chi_{reg}(g) = \sum \chi_i(1)\chi_i$. Now

$$0 \neq \frac{|G|}{|H|}\psi(1) = \langle \chi_{reg}\downarrow_H, \psi\rangle_H = \sum \chi_i(1)\langle \chi_i\downarrow_H, \psi\rangle_H$$

Conclude $\langle \chi_i\downarrow_H, \psi\rangle \neq 0$ for some i. $\square$

(10.3) Lemma χ irred char of G, $\chi\downarrow_H = \sum c_i \chi_i$, χ_i irred chars of H, $c_i \in \mathbb{Z}_{\geq 0}$. Then $\sum c_i^2 \leq |G:H|$

with equality iff $\chi(g) = 0$ $\forall g \in G \setminus H$.

Pf $\sum c_i^2 = \langle \chi\downarrow_H, \chi\downarrow_H\rangle = \frac{1}{|H|}\sum_{h\in H} |\chi(h)|^2$

But
$$1 = \langle \chi, \chi\rangle_G = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2$$
$$= \frac{1}{|G|}\Big(\sum_{h\in H}|(\chi(h))|^2 + \sum_{g\in G\setminus H}|\chi(g)|^2\Big)$$
$$= \frac{|H|}{|G|}\sum c_i^2 + \underbrace{\frac{1}{|G|}\sum_{g\in G\setminus H}|\chi(g)|^2}_{\geq 0, \text{ and } = 0 \iff \chi(g) = 0\ \forall g \in G\setminus H}$$

$\therefore \sum c_i^2 \leq |G:H|$

and equality iff $\chi(g) = 0$ $\forall g \in G\setminus H$. $\square$

Example $G = S_5$, $H = A_5$

$\deg \chi_i$:	1	1	4	4	5	5 ($\neq 0$ somewhere outside H)	6 ($\chi(g) = 0\ \forall g \in S_5 \setminus A_5$)
Res^G_H							
$\deg \psi$:	1		4		5		3, 3

Induced characters

(10.4) <u>Def^n</u> If $\psi \in \mathcal{C}_\ell(H)$ is a class f^n of H, define

$$\mathrm{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(x^{-1} g x)$$

where
$$\mathring{\psi}(g) = \begin{cases} \psi(g), & y \in H, \\ 0, & y \notin H. \end{cases} \qquad \psi\uparrow^G = \psi^G$$

(10.5) <u>Lem</u> If $\psi \in \mathcal{C}_\ell(H)$ is a class f^n of H, then $\mathrm{Ind}_H^G \psi \in \mathcal{C}_\ell(G)$ and $\mathrm{Ind}_H^G \psi(1) = |G:H|\, \psi(1)$.

<u>Pf</u> Clear (noting $\mathrm{Ind}_H^G \psi(1) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(1) = |G:H|\,\psi(1)$) □

Let $n = |G:H|$. Let $1 = t_1, \dots, t_n$ be a <u>left transversal</u> of H in G (complete set of coset reps), so that $\underset{H}{\overset{\shortparallel}{t_1H}}, \dots, t_nH$ are precisely the n left cosets of H in G.

(10.6) <u>Lemma</u> Given left transversal as above,

$$\mathrm{Ind}_H^G \psi(g) = \sum_{i=1}^{n} \mathring{\psi}(t_i^{-1} g t_i)$$

<u>Pf</u> For $h \in H$, $\mathring{\psi}((t_i h)^{-1} g (t_i h)) = \mathring{\psi}(t_i^{-1} g t_i)$ as ψ is class f^n. □

(10.7) <u>Theorem</u> (Frobenius Reciprocity) $H \leq G$

ψ class f^n of H, φ class f^n of G. Then

$$\langle \mathrm{Res}_H^G \varphi, \psi \rangle_H = \langle \varphi, \mathrm{Ind}_H^G \psi \rangle_G$$

[so $\langle \varphi_H, \psi \rangle_H = \langle \varphi, \psi^G \rangle_G$]

<u>Proof</u>
$$\begin{aligned}
\langle \varphi, \psi^G \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\, \psi^G(g) \\
&= \frac{1}{|G||H|} \sum_{g, x \in G} \overline{\varphi(g)}\, \mathring{\psi}(x^{-1} g x) \\
&= \frac{1}{|G||H|} \sum_{\underset{\uparrow \text{ class } f^n}{x, y \in G}} \overline{\varphi(y)}\, \mathring{\psi}(y) \underset{\text{indep of } x}{=} \frac{1}{|H|} \sum_{y \in H} \overline{\varphi(y)}\, \mathring{\psi}(y) \\
&= \langle \varphi_H, \psi \rangle_H . \quad \square
\end{aligned}$$

<u>Cor</u> If ψ is a char of H then $\mathrm{Ind}_H^G \psi$ is a char of G.

<u>Pf</u> Let χ be an irred char of G.

$$\langle \underset{\uparrow \psi}{\mathrm{Ind}_H^G}, \chi \rangle_G \underset{(10.7)}{=} \langle \psi, \mathrm{Res}_H^G \chi \rangle_H \in \mathbb{Z}_{\geq 0}$$

since ψ, $\mathrm{Res}_H^G \chi$ are chars. Hence $\mathrm{Ind}_H^G \psi$ is a lin combi of irred chars... □

L8.3

(10.9) Prop^n Let ψ be a char of $H \leq G$, and let $g \in G$.

Let $s = \#$ conj classes of H whose members are conj in G to g:

if $s = 0$ then $\psi\uparrow^G(g) = 0$;

otherwise if we let $x_1, \dots, x_s$ be reps of these s conj classes of H, then

$$\psi\uparrow^G(g) = \sum_{i=1}^{s} \frac{|C_G(g)|}{|C_H(x_i)|}\psi(x_i)$$

Pf If $s=0$ then $\{x \in G : x^{-1}gx \in H\} = \emptyset$, and follows that $\chi\uparrow^G(g) = 0$ by def^n (10.4). Assume $s > 0$.

Let $X_i = \{x \in G : x^{-1}gx \in H$ and is conjugate in H to $x_i\}$ $(1 \leq i \leq s)$

The X_i are pairwise disjoint, union is $\{x \in G : x^{-1}gx \in H\}$

By def^n,

$$\begin{aligned} \mathrm{Ind}_H^G\psi(g) &= \frac{1}{|H|}\sum_{x \in G}\mathring{\psi}(x^{-1}gx) \\ &= \frac{1}{|H|}\sum_{\substack{x \in G \\ x^{-1}gx \in H}}\psi(x^{-1}gx) \\ &= \frac{1}{|H|}\sum_{i=1}^{s}\sum_{x \in X_i}\psi(x^{-1}gx) \\ &= \sum_{i=1}^{s}\frac{|X_i|}{|H|}\psi(x_i) \end{aligned}$$

Fix some $1 \leq i \leq s$ and choose $g_i \in G$ s.t. $g_i^{-1}gg_i = x_i$

So $\forall c \in C_G(g)$ and $h \in H$,

$$\begin{aligned} (cg_ih)^{-1}g(cg_ih) &= h^{-1}g_i^{-1}c^{-1}gcg_ih \\ &= h^{-1}g_i^{-1}gg_ih \\ &= h^{-1}x_ih \quad (\in H) \end{aligned}$$

i.e. $cg_ih \in X_i$; hence $C_G(g)g_iH \subseteq X_i$

Conversely, if $x \in X_i$, then $x^{-1}gx = h^{-1}x_ih = h^{-1}g_i^{-1}gg_ih$ for some $h \in H$; thus $xh^{-1}g_i^{-1} \in C_G(g)$, and $\therefore$ $x \in C_G(g)g_ih \subseteq C_G(g)g_iH$

Conclude $X_i = C_G(g)g_iH$

Thus $|X_i| = |C_G(g)g_iH| = \dfrac{|C_G(g)||H|}{|H \cap g_i^{-1}C_G(g)g_i|}$

Finally, $g_i^{-1}C_G(g)g_i \underset{\text{FACT}}{=} C_G(g_i^{-1}gg_i) = C_G(x_i)$, thus

$$\begin{aligned} |X_i| &= |H : H \cap C_G(x_i)||C_G(g)| \\ &= |H : C_H(x_i)||C_G(g)| \end{aligned}$$

Thus $\dfrac{|X_i|}{|H|} = \dfrac{|H : C_H(x_i)|\,|C_G(g)|}{|H|}$

$= \dfrac{|C_G(g)|}{|C_H(x_i)|}$ for $i = 1, \dots, s$ as desired (!) $\square$

<u>Lemma</u> (10.10) If $\psi = 1_H$ (the principal char of H), then $\mathrm{Ind}_H^G 1_H = \pi_X$ the perm^n char of G on the set X ($= G/H$) of left cosets of H in G.

<u>Pf</u> $\mathrm{Ind}_H^G 1_H(g) \overset{(10.6)}{=} \sum_{i=1}^{n} \mathring{1}_H(t_i^{-1} g t_i)$ transversal of H in G

$= |\{i : t_i^{-1} g t_i \in H\}|$

$= |\{i : g \in t_i H t_i^{-1}|$

stabiliser in G of the point $t_i H \in X$

$= |\mathrm{fix}_X(g)|$

$= \pi_X$. $\square$

<u>Remark</u> $\langle \pi_X, 1_G \rangle_G \overset{(10.10)}{=} \langle \mathrm{Ind}_H^G 1_H, 1_G \rangle_G$

$\underset{(10.7)}{=} \langle 1_H, 1_H \rangle = 1$ (as predicted in §7)

<u>Examples</u> (a) $G = S_5$ acting on $X = \mathrm{Syl}_5(G)$; same as $\pi_X = \mathrm{Ind}_H^G 1_H$ where $H = \langle (12345), (2354) \rangle$ ($|H| = 20$, $|G:H| = 6$) aaah!

(b) $H = C_4 = \langle (1234) \rangle \leq G = S_4$, index 6

Char of induced rep^n $\mathrm{Ind}_{C_4}^{S_4}(\alpha)$ where α is faithful 1-dim rep of C_4

If $\alpha((1234)) = i$ then char of α is

	1	(1234)	(13)(24)	(1432)
χ_α	1	i	-1	$-i$

Induced rep of S_4:

$\lvert\mathcal{C}_G\rvert$	1	6	8	3	6
cls	1	(12)	(123)	(12)(34)	(1234)
$\mathrm{Ind}_{C_4}^{S_4}\alpha$	6 (10.5)	0	0	-2	0

For (12)(34) only one of the 3 elts in S_4 it's conj to lies in H.

So $\mathrm{Ind}_H^G((12)(34)) = 8 \cdot \left(\frac{-1}{4}\right) = -2$

For (1234): it's conj to 6 elts of S_4, of which two lie in C_4: viz $\mathcal{C}_G(1234)$ and $\mathcal{C}_G(1432)$

So $\mathrm{Ind}_{C_4}^{S_4} \alpha((1234)) = 4\left(\frac{i}{4} - \frac{i}{4}\right) = 0$

Induced representations

$H \leq G$ index n. $1 = t_1, \dots, t_n$ transversal. W H-space

(10.11) Def^n Let $V := W \oplus t_2 \otimes W \oplus \cdots \oplus t_n \otimes W = \bigoplus_{t_i} t_i \otimes W$

where $t_i \otimes W = \{ t_i \otimes w : w \in W \}$.

So $\dim V = n \dim W$ and write $V = \mathrm{Ind}_H^G W = \{ \sum t_i \otimes w_i : w_i \in W \}$

G-action $g \in G$ $\forall i$ $\exists!$ j with $t_j^{-1} g t_i \in H$ (namely $t_j H$ coset containing $g t_i$)

Define $g(t_i \otimes w) = t_j \otimes ((\underbrace{t_j^{-1} g t_i}_{\in H}) w)$ ⌜drop $\otimes$⌟

Check it is a G-action:

$$\begin{aligned} g_1(g_2 t_i w) &= g_1 (t_j (t_j^{-1} g_2 t_i) w) \\ {\scriptstyle g_2 t_i H = t_j H} \quad &= t_l ((t_l^{-1} g_1 t_j)(t_j^{-1} g_2 t_i) w) \\ {\scriptstyle g_1 t_j H = t_l H} \quad &= t_l (t_l^{-1} g_1 g_2 t_i) w \\ &= (g_1 g_2)(t_i w) \end{aligned}$$

⌜indeed, $(g_1 g_2) t_i H = t_l H$⌟

It has the "right" char:

$$g : t_i w \mapsto t_j (t_j^{-1} g t_i) w$$

so the contribution to the char is 0 unless $j = i$, i.e.

if $t_{\cancel{j}i}^{-1} g t_i \in H$, then it contributes $\psi(t_i^{-1} g t_i)$. So

$$\mathrm{Ind}_H^G \psi(g) = \sum \psi^\circ (t_i^{-1} g t_i)$$

as per (10.6).

§11 Frobenius groups

Thm (Frobenius 1891) Let G be a transitive perm$^{\underline{n}}$ gp on a finite set X, say $|X| = n$. Suppose each non-1 elt of G fixes at most one elt of X. Then $K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \ \forall \alpha \in X\}$ is a normal subgroup of G of order n.

Note: G must be finite, being isom to a subgp of $Sym(X)$

Pf (due to Isaacs)

To show $K \trianglelefteq G$. Let $H = G_\alpha$, some $\alpha \in X$. Conjugation of H are stabilisers of single elts of X. By hyp, no two conj can share a non-1 elt.

H has n distinct conjs and G has $n(|H|-1)$ elts that fix ecatly one elt of X. But $|G| = |X||H| = n|H|$ (X and G/H are isom G-sets since action trans)

Thus $|K| = |G| - n(|H|-1) = n$

Let $1 \neq h \in H$. Spose $h = gh'g^{-1}$, some $g \in G$, $h' \in H$. Then h lies in both $G_\alpha = H$ and $gHg^{-1} = G_{g\alpha}$; hyp $\Rightarrow g\alpha = \alpha$, hence $g \in H$.

So the conj cls in G of ~~##~~ h in precisely the conj cls in H of h. (well...)

Similarly if $g \in C_G(h)$ then $h = ghg^{-1} \in G_{g\alpha}$, hence $g \in H \Rightarrow C_G(h) = C_H(h)$.

Every elt of G either $\in K$ or lies in one of the n stabs, each of which is conj to H. So every elt of $G \setminus K$ is conj with non-1 elt of H. So

$$\{\underbrace{\overset{"h_1"}{1}, h_2, \dots, h_t}_{\text{reps of } H\text{-ccls}}, \underbrace{y_1, \dots, y_u}_{\text{reps of ccls which comprise } K \setminus \{1\}}\}$$

is set of ccl reps of G.

Problem to show $K \leq G$

Take $\theta_1 = 1_G$, $\{1_H = \psi_1, \dots, \psi_t\}$ irred chars of H.

Fix some $1 \leq i \leq t$. Then, if $g \in G$:

$$\mathrm{Ind}_H^G \psi_i(g) = \begin{cases} |G:H|\psi_i(1) = n\psi_i(1), & g = 1 \\ \psi_i(h_j), & g = h_j \ (2 \leq j \leq t) \\ 0, & g = y_k \ (1 \leq k \leq u) \end{cases}$$

$C_G(h_j) = C_H(h_j)$ (10.9)

L9.2

Fix some $2 \le i \le t$ and put $\theta_i = \psi_i^G - \psi_i(1)\psi_1^G + \psi_i(1)\,\theta_1 \in \mathcal{R}(G)$ (9.15)

Values: for $2 \le j \le t$, $1 \le k \le u$

	1	h_j	y_k
ψ_i	$n\psi_i(1)$	$\psi_i(h_j)$	0
$\psi_i(1)\psi_i^G$	$n\psi_i(1)$	$\psi_i(1)$	0
$\psi_i(1)\theta_1$	$\psi_i(1)$	$\psi_i(1)$	$\psi_i(1)$
θ_i	$\psi_i(1)$	$\psi_i(h_j)$	$\psi_i(1)$

$$\therefore \langle \theta_i, \theta_i \rangle = \frac{1}{|G|}\sum_{g\in G} |\theta_i(g)|^2 = \frac{1}{|G|}\Big(\sum_{g\in K} |\theta_i(g)|^2 + \sum_{\alpha\in X}\ \sum_{1\ne g\in G_\alpha} |\theta_i(g)|^2 \Big)$$

$$= \frac{1}{|G|}\Big(n\,\psi_i(1)^2 + n \sum_{1\ne h\in H} |\theta_i(h)|^2 \Big)$$

$$= \frac{1}{|H|}\sum_{h\in H} |\psi_i(h)|^2 = \langle \psi_i, \psi_i \rangle = 1 \qquad \text{(row orthog)}$$

Thus either θ_i or $-\theta_i$ is an irred char of G; since $\theta_i(1) > 0$ it's θ_i.

Let $\theta := \sum_{i=1}^{t} \theta_i(1)\,\theta_i$. Col orthog $\Rightarrow \theta(h) = \sum_{1}^{t} \overset{\psi}{\phi_i}(1)\,\overset{\psi}{\phi_i}(h) = 0 \quad (1 \ne h \in H)$

and for any $y \in K$, $\theta(y) = \sum_{i=1}^{t} \psi_i(1)^2 = |H|$. Hence

$$\theta(g) = \begin{cases} |H| & (g \in K) \\ 0 & (g \notin K) \end{cases} \qquad \therefore K = \{ g \in G : \theta(g) = \theta(1) \} \trianglelefteq G$$

(15.3) $\square$

NB, now see θ is just lift of χ_{reg} in G/K

(11.2) <u>Def</u> G <u>Frobenius gp</u> is a gp G having sgp H s.t. $H \cap gHg^{-1} = 1$ $\forall g \notin H$: H is a <u>Frob complement</u> of G.

(11.3) <u>Prop^n</u> Any finite Frob gp satisfies the hyp of (11.1). The normal sgp K is a <u>Frobenius kernel</u> of G.

<u>Pf</u> G Frob, compl H then action of G on G/H is transitive & faithful. Further, if $1 \ne g \in G$ fixes both xH and yH, then we get

$g \in xHx^{-1} \cap yHy^{-1}$. This implies $H \cap (y^{-1}x)H(x^{-1}y) \ne 1$, which means $xH = yH$

$\square$

L9.3

§12 Mackey theory

(Work over $\mathbb{C}$) This describes restriction to a sgp $K \leq G$ of an induced repⁿ $\mathrm{Ind}_H^G W$. K, H are unrelated but usually we take $K = H$, in which case we can characterise when $\mathrm{Ind}_H^G W$ is irred.

Special case $W = 1_H$ (trivial H-space of dim 1)

Thus $\mathrm{Ind}_H^G 1_H \overset{(10.10)}{=}$ perm repn of G on G/H ($\equiv$ action on the set of left cosets of H in G), $\mathbb{C}(G/H)$.

Recall if G is trans on a set X and $H = G_\alpha$ (some $\alpha \in X$) then the action of G on X is isom to the action on G/H, viz

(12.1) $$\underbrace{g \cdot \alpha}_{\in X} \longleftrightarrow \underbrace{gH}_{\in G/H}$$

well-def bij & commutes with G-actions ($x(g\alpha) = (xg)\alpha \longleftrightarrow x(gH) = (xg)H$)

Consider (left action) of G on G/H and let $K \leq G$. Then G/H splits into K-orbits; these corresp to double cosets KgH: namely the K-orbit containing gH contains precisely all kgH, $k \in K$. (bunches some gH cosets together)

(12.2) Notation $K \backslash G/H$ = set of (K,H)-double cosets, partition G

note $\# K\backslash G/H = \langle \pi_{G/K}, \pi_{G/H} \rangle_G$ (7.4) + Frob reciprocity

Clearly $G_{gH} = gHg^{-1}$ $\therefore K_{gH} = gHg^{-1} \cap K =: H_g$

So by (12.1) the action of K on the orbit containing gH is isom to the action of K on $K/gHg^{-1} \cap K = K/H_g$.

From this discussion, using $\mathrm{Ind}_H^G 1_H \underset{(10.10)}{=} \mathbb{C}(G/H)$, and if $X = \bigcup X_i$ into orbits then $\mathbb{C}X \cong \bigoplus \mathbb{C}X_i$,

(12.3) Propⁿ (over $\mathbb{C}$) G finite gp, $H, K \leq G$. Then

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G 1 \cong \bigoplus_{\substack{\text{reps} \\ g \in K\backslash G/H}} \mathrm{Ind}_{gHg^{-1}\cap K}^K 1$$

Let $\mathcal{G} = \{1, \ldots, g_r\}$ be s.t. $G = \bigcup K g_i H$ (set of reps of $K\backslash G/H$)

Write $H_g := gHg^{-1} \cap K \leq K$

(ρ, W) repn of H

- for $g \in G$, define (ρ_g, W_g) to be repn of H_g with <u>same</u> underlying vec space W; but now the H_g-action is

$$\rho_g(x) = \rho(h) \quad \Big(= \rho(\underbrace{g^{-1}xg}_{\in H}) \Big)$$

$x \in gHg^{-1}$ (well def because $g^{-1}xg \in H$ for $x \in gHg^{-1}$)

Since $H_g \leq K$ we obtain an induced repn $\mathrm{Ind}_{H_g}^{K} W_g$ from this

(12.4) <u>Thm</u> (Mackey's Restriction Formula)

G finite; $H, K \leq G$ and W is H-space. Then

$$\mathrm{Res}_K^G \, \mathrm{Ind}_H^G \, W \cong \bigoplus_{g \in S} \mathrm{Ind}_{H_g}^{K} W_g \qquad \text{(as } K\text{-modules)} \qquad \text{(proof omitted)}$$

(12.5) <u>Cor</u> (Mackey's Irred Criterion)

If ψ is a char of a repn of H then

$$\mathrm{Res}_K^G \, \mathrm{Ind}_H^G \, \psi = \sum_{g \in S} \mathrm{Ind}_{H_g}^{K} \psi_g$$

where ψ_g is the cls f^n on H_g given as $\psi_g(x) = \psi(g^{-1}xg)$

(12.6) <u>Cor</u> (Mackey version char)

$H \leq G$, W H-space. Then $V = \mathrm{Ind}_H^G W$ is irred iff W is irred and for each $g \in S \setminus H$, the two H_g-spaces W_g and $\mathrm{Res}_{H_g}^{H} W$ has no irred constituents in common

<u>Remark</u> The set S of reps was arbitrary, so we could demand $g \in G \setminus H$, though it's enough to check for $g \in S \setminus H$.

<u>Pf of Cor</u> Use chars. Recall W irred iff $\langle \psi, \psi \rangle = 1$, ψ afforded.

Take $K = H$ in Mackey's. So $H_g = gHg^{-1} \cap H$.

$$\begin{aligned}
\langle \mathrm{Ind}_H^G \psi, \mathrm{Ind}_H^G \psi \rangle_G &= \langle \psi, \mathrm{Res}_H^G \, \mathrm{Ind}_H^G \psi \rangle_H && \text{(Frob)} \\
&= \sum_{g \in S} \langle \psi, \mathrm{Ind}_{H_g}^{H} \psi_g \rangle_H && (12.5) \\
&= \sum_{g \in S} \langle \mathrm{Res}_{H_g}^{H} \psi, \psi_g \rangle_{H_g} && \text{(Frob)} \\
&= \underbrace{\langle \psi, \psi \rangle}_{1} + \sum_{\substack{g \in S \\ g \notin H}} d_g && \text{where } d_g = \langle \mathrm{Res}_{H_g}^{H} \psi, \psi_g \rangle_{H_g} \\
& && (g \neq 1)
\end{aligned}$$

For $g = 1$ we have $H_g = H$, hence this is a sum of non-neg integers, which is ≥ 1. So $\mathrm{Ind}_H^G \psi$ is irred iff $\langle \psi, \psi \rangle = 1$ and all the other terms in the sum are 0. In other words W is an irred repn of H $\forall g \notin H$, W, W_g are disjoint repns of $H \cap gHg^{-1}$. $\square$

(12.7) <u>Cor</u> If $H \triangleleft G$, assume ψ irred char of H. Then $\mathrm{Ind}_H^G \psi$ irred

$\iff$ ψ distinct from all its conjugates ψ_g $(g \notin H)$ [recall $\psi_g(h) = \psi(g^{-1}hg)$]

<u>Pf</u> Take $K = H$, double cosets = left or right cosets.

Now $H_g = gHg^{-1} \cap H = H$ $\forall g$ (as $H \triangleleft G$)

Also W_g is irred since W is irred.

So by (12.6), $\mathrm{Ind}_H^G W$ irred precisely if $W \not\cong W_g$ $\forall g \in G \setminus H$.

Equivalent to $\psi \neq \psi_g$. $\square$

<u>Proof of (12.4)</u>

Write $V = \mathrm{Ind}_H^G W$ and recall from (10.11) that V is direct sum of images $x \otimes W = \{x \otimes w : w \in W\}$ with x running through a set J of reps of left cosets of H in G: $V = \bigoplus_{x \in J} x \otimes W$

Consider a particular double coset KgH.

The terms $V(g) = \bigoplus_{\substack{x \in J \\ x \in KgH}} x \otimes W$ form a subspace which is invariant under the action of K (it's a direct sum of an orbit of subspaces permuted by K).

Now, viewing V as as K-space (forgetting G-structure)

$$\mathrm{Res}_K^G V = \bigoplus_{g \in S} V(g)$$

so need to show $V(g) \cong \mathrm{Ind}_{H_g}^K W_g$ as K-spaces, for each $g \in S$.

Now, $\mathrm{Stab}_K(g \otimes W) = \{k \in K : kg \otimes W = g \otimes W\} = \{k \in K : g^{-1}kg \in H\}$

$$= K \cap gHg^{-1} = H_g$$

Also, if $x = kgh$, $x' = k'gh'$ then $x \otimes W = x' \otimes W$ iff k, k' lie in the same coset in K/H_g, hence $V(g) = \bigoplus_{\substack{\text{reps} \\ k \in K/H_g}} k \otimes (g \otimes W)$

So as a repⁿ for K this subspace is

$$V(g) \cong \mathrm{Ind}_{H_g}^K (g \otimes W)$$

Now $g \otimes W$ is repⁿ of H_g $(ghg^{-1}(g \otimes W) = ghg^{-1}g \otimes w = gh \otimes w = g \otimes hw)$ and $W_g \cong g \otimes W$ as H_g-repⁿs (the isom is $w \mapsto g \otimes w$). $\square$

13. Integrality in the group algebra

Arithmetic results

(13.1) Def^n $a \in \mathbb{C}$ is an algebraic integer if a is the root of a monic poly in $\mathbb{Z}[X]$. Equivalently, the subring of $\mathbb{C}$

$$\mathbb{Z}[a] = \{ f(a) : f(X) \in \mathbb{Z}[X] \}$$

is a f.g. $\mathbb{Z}$-module

(($\Rightarrow$) a alg integer $\Rightarrow a^m + a_{m-1}a^{m-1} + \cdots + a_1 a + a_0 = 0$
$\Rightarrow$ $\mathbb{Z}$-submod of $\mathbb{C}$ gen by $1, a, \dots, a^{m-1}$
is stable under mult^n by a $\therefore = \mathbb{Z}[a]$

($\Leftarrow$) $\mathbb{Z}[a]$ f.g. $\mathbb{Z}$-module. Let R_n be $\mathbb{Z}$ submod gen by $1, a, \dots, a^{n-1}$
Then R_n increasing chain and since $\mathbb{Z}[a]$ is f.g. have $R_n = \mathbb{Z}[a]$ for some large enough n. Hence $a^n \in R_n$)

FACT 1 The alg integers form a subring of $\mathbb{C}$ (Proof: J-L 22.3)

FACT 2 If $a \in \mathbb{C}$ is both alg int and rational then $a \in \mathbb{Z}$

FACT 3 Any subring S of $\mathbb{C}$ which is a f.g. $\mathbb{Z}$-module consists of alg ints
(if $s_1, \dots, s_n$ generators of S, let $a \in S$. Then $a s_i = \sum_j a_{ij} s_j$
put $A = (a_{ij})$; $\therefore A\underline{v} = a\underline{v}$ where $\underline{v} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$, so a is a root of the char poly)

(13.2) Prop^n If χ is a char of G and $g \in G$ then $\chi(g)$ is an alg int

Cor there are no entries in char tables of any finite group which are rational but not integers

Pf $\chi(g)$ is the sum of n^{th} roots of 1 $(n = |g|)$
Each root of 1 is an alg int and so is any sum by Fact 1. $\square$

The Centre of $\mathbb{C}G$

Recall from (2.4) the group algebra $\mathbb{C}G = \{ \alpha_g g : \alpha_g \in \mathbb{C} \}$ of a finite group G; the $\mathbb{C}$-spaces with basis G. Also a ring, so $\mathbb{C}$-algebra.

List $\{1\} = \mathcal{C}_1, \dots, \mathcal{C}_k$ the G-cls. Define the class sums

$$C_j = \sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G$$

$C_j \in Z(\mathbb{C}G)$, the centre of $\mathbb{C}G$ (NOT the same as $\mathbb{C}(Z(G))$)

(13.3) <u>Prop^n</u> $C_1, \dots, C_k$ is a basis of $Z(\mathbb{C}G)$. There exist non-neg integers a_{ijl} with

$$C_i C_j = \sum_l a_{ijl} C_l$$

these are $\left\{ \frac{\text{class algebra}}{\text{structure}} \right\}$ constants for $Z(\mathbb{C}G)$.

<u>Pf</u> Check $g C_j g^{-1} = C_j \ \forall g \in G$, $\therefore C_j \in Z(\mathbb{C}G)$

Clearly the C_j are lin indep (because $\mathcal{C}_{e_j}$ disjoint)

<u>Span</u>: Now spose $z \in Z(\mathbb{C}G)$, $z = \sum_{g \in G} \alpha_g g$.

Then $\forall h \in G$ we have $\alpha_{h^{-1}gh} = \alpha_g$ so the f^n $g \mapsto \alpha_g$ is constant on G-cls. Writing $\alpha_g = \alpha_j$ $(g \in \mathcal{C}_{e_j})$ then $z = \sum_{j=1}^{k} \alpha_j C_j$.

Finally, $Z(\mathbb{C}G)$ is a $\mathbb{C}$-algebra, so $C_i C_j = \sum_{l=1}^{k} a_{ijl} C_l$ as the C_l span. We claim $a_{ijl} \in \mathbb{Z}_{\geq 0}$.

Fix $g_l \in C_l$. Then $a_{ijl} = \# \{ (x, y) \in \mathcal{C}_i \times \mathcal{C}_{e_j} : xy = g_l \} \in \mathbb{Z}_{\geq 0}$ $\square$

<u>Integrality props of chars</u>

(13.4) <u>Def^n</u> Let $\rho: G \to GL(V)$ be an irred repn over $\mathbb{C}$ affording char χ. Extend to $\rho: A = \mathbb{C}G \longrightarrow \operatorname{End} V$, an alg homom.

This is a homom of algs into End V, a <u>representation</u> of A.

A <u>central homom</u> of A is a ring homom $Z(A) \to \mathbb{C}$

Let $z \in Z(\mathbb{C}G)$. Then $\rho(z)$ commutes with all $\rho(g)$ $(g \in G)$

So by Schur's lemma, $\rho(z) = \lambda_z I$ for some $\lambda_z \in \mathbb{C}$.

Consider the central homom $\omega_\chi := \omega : Z(\mathbb{C}G) \to \mathbb{C}$

$$z \mapsto \lambda_z$$

Now $\rho(C_i) = \omega(C_i) I$, so taking traces

$$\chi(1) \, \omega_\chi(C_i) = \sum_{g \in \mathcal{C}_i} \chi(g) = |\mathcal{C}_i| \, \chi(g_i)$$

$$\therefore \omega_\chi(C_i) = \frac{\chi(g_i)}{\chi(1)} |\mathcal{C}_i|$$

(13.5) <u>Lemm</u> o

(13.5) Lem The values of $\omega_\chi(C_i) = \frac{\chi(g)}{\chi(1)} |\mathcal{C}_i|$ are alg integers

Pf Since ω_χ is an alg hom and (13.3),

$$\omega_\chi(C_i)\,\omega_\chi(C_j) = \sum_{l=1}^{k} a_{ijl}\,\omega_\chi(C_l), \quad a_{ijl} \in \mathbb{Z}_{\geq 0}$$

Thus the span$\{\omega(C_j) : 1 \leq j \leq k\}$ is a subring of $\mathbb{C}$ and is f.g. abelian group; so by Fact 3 consists of alg integers. □

Try: Exercise Show $a_{ijl} = \#\{(x,y) \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_l\}$ can be obtained from the char table. In fact $\forall i,j,l$

$$a_{ijl} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{s=1}^{k} \frac{\chi_s(g_i)\,\chi_s(g_j)\,\chi_s(g_l^{-1})}{\chi_s(1)}$$

See [JL, 30.4]

(13.6) Thm Over $\mathbb{C}$, the deg of any irred char of G divides $|G|$

Pf Given irred χ,

$$\frac{|G|}{\chi(1)} = \frac{1}{\chi(1)} \sum_g \chi(g)\,\chi(g^{-1}) = \frac{1}{\chi(1)} \sum_{i=1}^{k} |\mathcal{C}_i|\,\chi(g_i)\,\chi(g_i^{-1})$$

$$= \sum \underbrace{\frac{|\mathcal{C}_i|\,\chi(g_i)}{\chi(1)}}_{\text{alg int (13.5)}} \underbrace{\chi(g_i^{-1})}_{\text{sum of roots of unity}}$$

is an alg integer. It is rational, so an integer by Fact 2. □

(13.7) Ex (a) G p-group. Then $\chi(1)$ is p-power (χ irred) In ptc, if $|G| = p^2$ then $\chi(1) = 1$, hence G abelian

(b) $G = S_n$: every prime p dividing the degree of an irred char also divides $n!$; hence $p \leq n$

(13.8) Thm If χ irred then $\chi(1) \mid |G : Z|$ (Burnside 1904)

Pf (cf $|\mathcal{C}_i| = \frac{|G|}{|C_G(g_i)|} \mid |G:Z|$)

Let $\rho : G \to GL(V)$ afford χ.

Recall that if $z \in Z(G)$, $\rho(z)$ commutes with all $\rho(g)$, so by Schur's Lemma $\rho(z) = \lambda_z \,\mathrm{id}$ ($\lambda_z \in \mathbb{C}$). Then $\omega_\chi : Z(G) \to \mathbb{C}^\times$, $z \mapsto \lambda_z$ central homom.

Given any integer $m \geq 2$, consider

$\rho_m := \rho^{\otimes m} : G^m \longrightarrow GL(V^{\otimes m})$, an irred rep^n of G^m by (9.11)

Note $\rho_m(z_1,\dots,z_m) = \omega_\chi(z_1)\cdots\omega_\chi(z_m)\,\mathrm{id} = \prod_{i=1}^m \lambda_{z_i}\,\mathrm{id}$ $(z_1,\dots,z_m)\in Z^m$

So if $\prod_{i=1}^m z_i = 1$ show $z \in \ker \rho_m$

Now $\ker \rho_m$ contains a subgroup

$$H_m = \{(z_1,\dots,z_m)\in Z^m : z_1\cdots z_m = 1\}, \quad |H_m| = |Z|^{m-1}$$

By (8.1) ρ_m is the lift of an irred rep^n of G^m/H_m

By (13.6) $\dim \rho_m = (\dim\rho)^m \mid |G^m/H_m| = \dfrac{|G|^m}{|Z|^{m-1}}$

i.e. $\left(\dfrac{|G:Z|}{\chi(1)}\right)^m \in \dfrac{1}{|Z|}\mathbb{Z}$ $\forall m$

Since this is true for any m, $\dfrac{|G:Z|}{\chi(1)} \in \mathbb{Z}$:

indeed if $a = \dfrac{|G:Z|}{\chi(1)}$ we have $\mathbb{Z}[a]$ is a ring contained in a f.g. $\mathbb{Z}$-mod

Thus a an alg int, so since $a\in\mathbb{Q}$, we have $a\in\mathbb{Z}$. □

§14 Burnside's $p^a q^b$ theorem

(14.1) Thm (Burnside, 1904)

Let p, q be primes and G a gp of order $p^a q^b$ with a, b non-neg integers s.t. $a+b \geq 2$. Then G is not simple.

Remarks
- In fact such gps are soluble ($\exists$ chain of subgps
  $G = G_0 \geq G_1 \geq \cdots \geq G_r = 1$ with $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} abelian $\forall i$)
  [JL, (31.5)]
- Best possible in the sense that $|A_5| = 2^2\cdot 3\cdot 5$ has exactly three prime factors (and A_5 simple)
- If either $a, b = 0$ then G is p-gp, so $\exists$ elt of order p which generates a proper normal subgp
- First purely gp theoretic proof appeared only in 1972
- In 1963 Feit & Thompson, in a 255 page paper proved every gp of odd order is soluble (!)

L11.3.

(14.2) <u>Lem</u> Spse $0 \neq \alpha = \frac{\lambda_1 + \cdots + \lambda_m}{m}$ with λ_j roots of unity $\forall j$ is an alg integer. Then $|\alpha| = 1$.

<u>Pf</u> (non-ex) The min poly of α over $\mathbb{Q}$: $p_\alpha(X) = \prod_\beta (X - \beta)$ where β ranges over all distinct transforms of α under the action of Galois gp $\mathcal{G}$ of the alg closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ (actually the extension generated by the roots of unity appearing in α would do.)

This is because p_α, having rational coeffs, is $\mathcal{G}$-invariant; thus has all β's as roots. Since the given expression is $\mathcal{G}$-invariant (by construction), it has rational coeffs and has α as a root, so must agree with min poly.

Now Gauss lemma says any monic poly with $\mathbb{Z}$-coeffs is irred over $\mathbb{Q}$ if it is over $\mathbb{Z}$. So if α alg int, $P_\alpha(X)$ must have $\mathbb{Z}$-coeffs, among which the one of least degree must be irred over $\mathbb{Q}$, and hence must agree with the min poly p_α.

If $\alpha \neq 0$, no alg conjugate can be zero. So product of all β, which is integral, has modulus ≥ 1. But alg conj of a root of unity is also a root of unity, so every β is an average of roots of unity, and has modulus ≤ 1. Contradiction to alg integrality, unless $\alpha = 0$ or $|\alpha| = 1$. $\square$

(14.3) <u>Lem</u> Spse χ irred char of G, and $\mathcal{C}$ is ccl in G s.t. $\chi(1)$ and $|\mathcal{C}|$ coprime. For $g \in \mathcal{C}$, $|\chi(g)| = \chi(1)$ or zero.

<u>Pf</u> By Bezout, $\exists a, b \in \mathbb{Z}$ with $a\chi(1) + b|\mathcal{C}| = 1$.

Define $\alpha := \frac{\chi(g)}{\chi(1)} = a\chi(g) + b\frac{\chi(g)}{\chi(1)}|\mathcal{C}|$, alg integer (13.5)

Then α satisfies conditions of previous lemma. $\square$

Assuming

(14.4) <u>Prop</u> If in a finite gp G, the number of elts in a ccl $\mathcal{C}_i \neq 1$ is of prime power order, then G is not (non-abelian) simple.

Granted this we can prove (14.1): if $a, b > 0$, let $Q \in Syl_q(G)$.

So $Q \neq 1$ else G p-group. Then $1 \neq Z(Q)$

$\therefore \exists 1 \neq g \in Z(Q)$. Then as $C_G(g) \geqslant Q$, we have

$$|\mathcal{C}_G(g)| = |G : C_G(g)| = p^r \quad (\text{some } 0 \leq r \leq a)$$

<u>Proof of (14.4)</u> Spose G non-ab simple, and $\exists 1 \neq g \in G$ living in ccl $\mathcal{C}$ of order p^r. If $\chi \neq 1$ non-trivial irred char of G, then

$|\chi(g)| < \chi(1)$ (o/w $\rho(g)$ scalar matrix hence in $Z(\rho(G)) \cong Z(G)$)

By (14.3), for every non-trivial irred char, either $p \mid \chi(1)$ or $|\chi(g)| = 0$.

By col orthog applied to $\mathcal{C}$ and $\{1\}$

$$0 = 1 + \sum_{\substack{1 \neq \chi \text{ irred} \\ p \mid \chi(1)}} \chi(1)\chi(g)$$

Thus $-\frac{1}{p} = \sum_{\chi \neq 1} \frac{\chi(1)}{p} \chi(g)$ is an alg int in $\mathbb{Q}$.

Thus $\frac{1}{p} \in \mathbb{Z}$; a contradiction ✳

§15 Representations of compact groups

(15.1) A **topological group** is a gp G which is also a top space and for which mult^n $G \times G \to G$, $(h, g) \mapsto hg$ and inversion $G \to G$, $g \mapsto g^{-1}$ are cts.

It is compact if it is so as a top space.

(15.2) Examples

- $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ are top gps (as open subgps of $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$ resp)

  Non-compact (not closed); $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$ non-compact (not bdd $n>1$)

- $O(n) = \{n \times n$ real orthog matrices$\}$ rotations & reflections

  $\vee$I

  $SO(n) = O(n) \cap \{$matrices $\det = 1\}$ rotations

  Both compact (orthog means $\|$col vector$\| = 1$ $\therefore |A_{ij}| \leq 1$ $\forall\, 1 \leq i, j \leq n$)

- $U(n)$; $SU(n)$, $Sp(n)$ compact

  symplectic

- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under mult^n, circle gp

  $\cong SO(2) \cong U(1)$ (rot^n of θ about $0 \longleftrightarrow e^{i\theta}$) cpct

- G finite (discrete topology)

- $SU(2) = \{(z_1, z_2) \in \mathbb{C}^2 : z_1\bar{z}_1 + z_2\bar{z}_2 = 1\} \leq \mathbb{R}^4 \cong \mathbb{C}^2$

  $\overset{h}{\cong} S^3$ (the glome) (homeomorphic)

  Actually the only spheres with cts group laws are S^1, S^3

(15.3) Def^n A rep^n of a top gp G on a vector space V is a cts gp homom
$$G \to GL(V)$$

Remarks
- If X is a top space then $X \xrightarrow{\alpha} GL_n(\mathbb{C})$ is cts iff the maps
$$x \mapsto \alpha_{ij}(x) = \alpha(x)_{ij} \text{ are cts } \forall i, j$$
- If G finite with discrete top, "cts f^n" $G \to X$ just means f^n $G \to X$

The compact gp U(1)

Considering $\mathbb{R}$ as $\mathbb{Q}$-space, as abelian gps $S^1 \cong \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{x \in X} \mathbb{Q}$ for some uncountable set X (Hamel's basis thm). Using this [T(19.8)] constructs uncountably many 1-dim (irred) repns of S^1 (and indeed there are even more). However S^1 is not just a gp, it comes with a topology as a subset of $\mathbb{C}$ and S^1 acts naturally on $\mathbb{C}$-vector spaces in a cts way.

(15.4) <u>Thm</u> Every 1-dim (cts) repn of S^1 is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$. Clearly these maps are repns — need to show they are the only ones: Need two lemmas from real analysis.

(15.5) <u>Lem</u> If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a cts gp endom, then $\exists c \in \mathbb{R}$ s.t. $\psi(x) = cx \quad \forall x \in \mathbb{R}$ (i.e. ψ just mult^n by scalar)

<u>Pf</u> Given $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ cts. Put $c = \psi(1)$.

Claim $\psi(x) = cx \;\; \forall x$.

Since ψ homom, $\psi(nx) = \psi(x + \dots + x) = n\psi(x) \quad (x \in \mathbb{R},\ n \in \mathbb{Z}_{\geq 0})$

In ptc, $\psi(n) = cn$.

Also $\psi(-n) = -\psi(n) = -cn$

Thus $\psi(n) = nc \;\; \forall n \in \mathbb{Z}$

Put $x = \frac{m}{n} \in \mathbb{Q}$. $n\psi(x) = \psi(nx) = \psi(m) = cm$

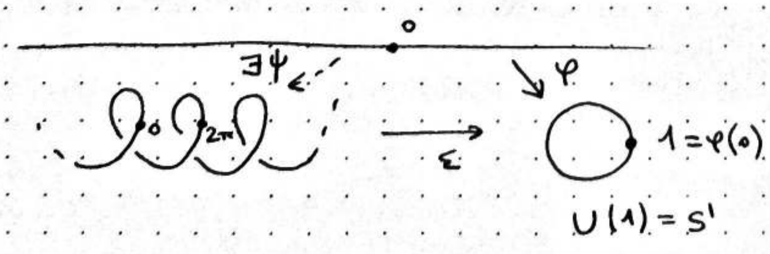
Dividing by n: $\psi(x) = cx \;\; \forall x \in \mathbb{Q}$

But $\overline{\mathbb{Q}} = \mathbb{R}$ and $\underline{\psi \text{ cts}}$ $\therefore \psi(x) = cx \;\; (\forall x \in \mathbb{R})$ $\square$

(15.6) <u>Lem</u> Cts homom $\varphi : (\mathbb{R}, +) \to S^1$ are of the form

$$x \mapsto e^{icx} \quad \text{for some } c \in \mathbb{R}$$

<u>Pf</u> Let $\varepsilon : (\mathbb{R}, +) \to S^1$, $x \mapsto e^{ix}$. This homom wraps the real line around S^1 with period 2π:



<u>Claim</u> Given any cts $\varphi : (\mathbb{R}, +) \to S^1$ s.t. $\varphi(0) = 1$, $\exists!$ cts (lifting) homom

$\psi : \mathbb{R} \to \mathbb{R}$ s.t. $\psi(0) = 0$ (making diagram commute).

In other words $\exists !$ cts f^n $\psi : \mathbb{R} \to \mathbb{R}$ s.t. $\psi(0) = 0$ & $\varphi(x) = \varepsilon(\psi(x))$ $\forall x$

(Alg top)

<u>Claim</u> If φ is a homom then so is its lifting ψ. If true then we can conclude $\psi(x) = cx$ for some $c \in \mathbb{R}$ by (15.5) whence $\varphi(x) = e^{icx}$.

<u>Pf</u> ψ homom: $\psi(x+y) \overset{?}{=} \psi(x) + \psi(y)$

Now $\varphi(x+y) = \varphi(x)\varphi(y) \Rightarrow \varepsilon(\psi(x+y) - \psi(x) - \psi(y)) = 1$

$$\Rightarrow \psi(x+y) - \psi(x) - \psi(y) = 2k\pi$$

for some integer k depending (ctsly) on x, y. Varying continuously, k must be constant. Set $x = y = 0$ to get $k = 0$. $\square$

<u>Proof of (15.4)</u> Let $\rho : S^1 \to \mathbb{C}^\times$ be cts repn. Then $\rho : S^1 \to S^1$: since S^1 cpct, $\rho(S^1)$ is closed and bdd. Since $\rho(z^n) = \rho(z)^n$ $\forall n \in \mathbb{Z}$, we must have $\rho(S^1) \subseteq S^1$.

We get cts homom $\mathbb{R} \to S^1$

$$x \mapsto \rho(e^{ix})$$

So by (15.6) $\exists c \in \mathbb{R}$ s.t. $\rho(e^{ix}) = e^{icx}$

Now $1 = \rho(e^{2\pi i}) = e^{2\pi i c}$ hence $c \in \mathbb{Z}$

Finally put $n = c$: $\rho(z) = z^n$. $\square$

In studying repns of finite gps we "averaged" over the gp via the operation $\frac{1}{|G|}\sum_{g \in G}$

An analogous opn exists for top gps if we replace sum by $\int_G dg$.

(15.7) <u>Def</u> G top gp, Let $\mathcal{C}_\ell(G) = \{ f : G \to \mathbb{C} \mid f \text{ cts}, f(gxg^{-1}) = f(x) \ \forall x, g \}$

Then a non-triv linear functional $\int_G : \mathcal{C}_\ell(G) \to \mathbb{C}$ (written $\int_G f = \int_G f(g)\,dg$) is called a <u>Haar measure</u> if

(i) $\int_G 1 \, dg = 1$ ($\int_G$ is normalised so total volume $= 1$)

(ii) $\int_G f(xg)\,dg = \int_G f(g)\,dg = \int_G f(gx)\,dg$ $\forall x \in G$ (left/right translation inv)

Exs (a) G finite $\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g)$

(b) $G = S^1$, $\int_G f = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta$

(c) $G = SU(2)$, see later

Thm G cpct Hausdorff $\Rightarrow$ $\exists$ unique Haar measure on G

Pf Omitted. $\square$

We compute Haar measure for $SU(2)$ below. Henceforth "compact" means "compact Hausdorff".

(15.9) Cor (Weyl's unitary trick) G cpct. Every repn (ρ, V) has a G-invariant Hermitian inner product.

Pf As for finite gps, take any inner product $(\cdot,\cdot)$ on V. Then

$$\langle v, w \rangle = \int_G (\rho(g)v, \rho(g)w)\, dg$$

is G-inv inner prod. $\square$ ($\Rightarrow$ every repn of cpct gp equiv to unitary repn)

(15.10) Thm (Maschke) If G cpct then every repn of G is completely red

Pf Given repn (ρ, V), choose G-inv inner prod. If W subrepn of V then $W^\perp$ is G-inv complement. $\square$

Use Haar measure to endow $\mathcal{C}_G(G)$ with inner prod

$$\langle f, f' \rangle = \int_G \overline{f(g)}\, f'(g)\, dg$$

If $\rho: G \to GL(V)$ repn then $\underset{\chi_V}{\underset{\shortparallel}{\chi_\rho}} := \mathrm{tr}(\rho)$ is cts cls f^n since each $\rho(g)$ is cts.

(15.11) Thm (Orthog of chars) G cpct, V, W irreps of G.

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1, & V \cong W \\ 0, & V \not\cong W \end{cases}$$

Pf $\langle \chi_V, \chi_W \rangle = \int_G \overline{\chi_V(g)}\, \chi_W(g)\, dg = \dim \mathrm{Hom}_G(\mathbb{1}, \mathrm{Hom}(V, W))$

$= \dim \mathrm{Hom}_G(V, W)$

Then use Schur.

[NB: need $\chi_V(g^{-1}) = \overline{\chi_V(g)}$; clear since assume $\rho_V(G) \subset U(V)$, evals of $\rho_V(g)$ are contained in S^1 $\forall g$

L12.5

- need proj^n $\pi : U \to U^G$ for $U = \mathrm{Hom}_k(V, W)$. Just choose basis $u_1, \dots, u_n$ of U and def^n π as linear map represented by matrix $\pi_{ij} = \int_G \rho(g)_{ij}\, dg$ ⌋ □

Characters basis of $\mathcal{C}_e(G)$? Needs some Hilbert space theory (Peter-Weyl thm)

<u>Ex (15.12)</u> $G = S^1$

1-dim repns are $\rho_n : z \mapsto z^n$. As S^1 abn these are all the irreps.

Given any repn ρ, can find simult evec for each $\rho(g)$

"Char table" has rows χ_n indexed by $\mathbb{Z}$ with $\chi_n(e^{i\theta}) = e^{in\theta}$

V given rep^n of S^1. (15.10) breaks V as a direct sum of 1-dim subrepns $\therefore$ char χ_V of form

$$\chi_V(g) = \sum_{n \in \mathbb{Z}} a_n z^n \quad (a_n \in \mathbb{Z}_{\geq 0} \text{ fin many } \neq 0)$$

Actually $a_n = \#$ copies of ρ_n in decomp of V. Hence

$$a_n = \langle \chi_n, \chi_V \rangle = \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\theta}) e^{-in\theta}\, d\theta$$

$$\therefore \chi_V(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\theta'}) e^{-in\theta'} d\theta' \right) e^{in\theta}$$

<u>Fourier decomp^n</u> gives decomp of χ_V into irred chars and <u>Fourier mode</u> is the multiplicity of the irred char

⌈ <u>FACT</u> Any cts f^n on S^1 can be unif approx by a finite $\mathbb{C}$-combn of the χ_n. The χ_n form complete o.n set in Hilbert space $L^2(S^1)$ i.e. every f^n f on S^1 s.t. $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$ exists and has a unique series exp^n $f(e^{i\theta}) = \sum_{\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta'}) e^{-in\theta'} d\theta' \right) e^{in\theta}$

converging in $\|\cdot\|_2$ norm. ⌋

L13.1

Repns of $G = SU(2)$

$$G = \{A \in GL_2(\mathbb{C}) : \bar{A}^T A = I, \det A = I\}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ then since $\det A = 1$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. So $d = \bar{a}$; $c = -\bar{b}$.

Moreover $a\bar{a} + b\bar{b} = 1$. Hence

$$G = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

Topologically; $G \overset{h}{\cong} S^3 \;(\subseteq \mathbb{C}^2 \cong \mathbb{R}^4)$ More precisely, let

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : w, z \in \mathbb{C} \right\} \leq M_2(\mathbb{C}),$$

<u>Hamilton's quaternion algebra</u> $\mathbb{H}$ is 4-dim Euclid space and $\|A\|^2 := \det A$ defines norm on $\mathbb{H} \cong \mathbb{R}^4$ with G the <u>unit sphere</u> in $\mathbb{H}$.

If $A \in G$ and $x \in \mathbb{H}$ then $\|Ax\| = \|x\|$ since $\|A\| = 1$, so after normalisation (by $\frac{1}{2\pi^2}$) usual integration of f^n's on S^3 defines <u>Haar measure on G</u>. (Left / right translations give isometries of sphere)

Let $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : a \in \mathbb{C}, |a| = 1 \right\} \cong S^1$, <u>maximal torus</u> in G and let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$.

(15.13) <u>Lem</u> (G-ccls)

(i) $t \in T \Rightarrow sts^{-1} = t^{-1}$

(ii) $s^2 = -I \in Z(G)$

(iii) $N_G(T) = T \cup sT = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} & a \\ -a^{-1} & \end{pmatrix} : a \in \mathbb{C}, |a| = 1 \right\}$ ✓

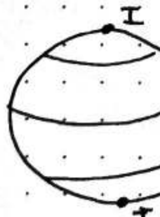
(iv) Every ccls $\mathcal{C}_G$ in G contains an element of T i.e. $\mathcal{C}_G \cap T \neq \emptyset$.

In fact (v) $\mathcal{C}_G \cap T = \{t, t^{-1}\}$ for some $t \in T$. Moreover

$$t = t^{-1} \iff t = \pm I \text{ when } \mathcal{C}_G = \{t\}$$

(vi) $\exists$ bijn $\{\text{ccls in } G\} \longleftrightarrow [-1, 1]$ given by $\underset{\mathcal{C}_G^?}{A} \mapsto \frac{1}{2}\operatorname{tr} A$ ← "normalised trace"

Picture of ccls:



$\mathbb{R} \hookrightarrow \mathbb{H}$

(proj^n onto real axis)

[see Artin p 275]

2 spheres of constant latitude on unit sphere (plus 2 poles)

3-sphere in $\mathbb{R}^4$

$q = a + ib + jc + kd \mapsto a$

$x_1 = c$

$x_2^2 + x_3^2 + x_4^2 = (1 - c^2)$

given $c \in (-1, 1)$, ccl contains all matrices $g \in SU(2)$ w/ $\operatorname{tr} g = 2c$

Pf (iv) Every unitary matrix X has an o.n. basis of evecs hence is conjugate in $U(2)$ to one in T: say $QX\bar{Q}^t \in T$.

[seek Q with $\det Q = 1$ ($\therefore Q \in SU(2)$)] Put $\delta = \det Q$.

Since $Q\bar{Q}^t = I$, get $|\delta| = 1$. If ε is a square root δ, then

$$Q_1 = \bar{\varepsilon} Q \in SU(2) \quad \text{and} \quad Q_1 X \bar{Q}_1^T \in T$$

(v) Let $g \in G$ and spose $g \in \mathcal{C}_e$.

If $g = \pm I$ then $\mathcal{C}_e \cap T = \{g\}$. Otherwise g has distinct evals λ, λ^{-1} and

$$\mathcal{C}_e = \left\{ h \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} h^{-1} : h \in G \right\}$$

Thus $\mathcal{C}_e \cap T = \left\{ \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix} \right\}$ by noting $s \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} s^{-1} = \begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix}$; further if $\begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix} \in \mathcal{C}_e$ then $\{\mu, \mu^{-1}\} = \{\lambda, \lambda^{-1}\}$ (evals are preserved under conjugation).

(vi) Consider $\frac{1}{2}\mathrm{tr} : \{\text{ccls}\} \longrightarrow [-1, 1]$

By (v) matrices conj in G iff evals agree up to order. Now

$$\tfrac{1}{2}\mathrm{tr}\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} = \tfrac{1}{2}(\lambda + \lambda^{-1}) = \mathrm{Re}(\lambda) = \cos\theta$$

hence the map is <u>onto</u> $[-1, 1]$.

Finally, it is injective: if g, g' have $\frac{1}{2}\mathrm{tr}\, g = \frac{1}{2}\mathrm{tr}\, g'$ then g, g' have the same char poly hence same evals so are conjugate. $\square$

Write $\mathcal{C}_{e_t} = \{ g \in SU(2) : \frac{1}{2}\mathrm{tr}\, g = t \}$ for $t \in [-1, 1]$. We've shown the $\mathcal{C}_{e_t}$ are the $SU(2)$ conj-classes. Note $\mathcal{C}_{e_1} = \{I\}$, $\mathcal{C}_{e_{-1}} = \{-I\}$.

<u>Exe</u> If $t \in (-1, 1)$, show $\mathcal{C}_{e_t}$ is homeo to S^2

(Hint: recall $SU(2)$ acts on $S^2 \cong \mathbb{C} \cup \{\infty\}$ by Möbius transf.)

Repns of $SU(2)$

Let V_n = $\mathbb{C}$-space of all homogeneous polys of deg n in two variables x, y i.e. $V_n = \{ \sum_i r_i x^{n-i} y^i : r_i \in \mathbb{C} \}$. So $\dim V_n = n+1$, standard basis $x^n, x^{n-1}y, \ldots, y^n$.

Then $GL(\mathbb{C}^2)$ $(\cong GL_2(\mathbb{C}))$ acts on V_n via

$$\rho_n : GL(\mathbb{C}^2) \longrightarrow GL(V_n)$$

given by the rule $\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x,y) = f(ax+cy, bx+dy)$

(15.14) <u>Exs</u>

- $n=0$: ρ_0 = trivial $(V_0 = \mathbb{C})$
- $n=1$: natural 2-dim repn $(\rho_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ wrt standard basis x, y of $V_1 \cong \mathbb{C}^2$
- $n=2$: $\rho_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has matrix

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

wrt standard basis x^2, xy, y^2 of $V_2 \cong \mathbb{C}^3$

Now G is a sgp of $GL_2(\mathbb{C})$ $\therefore$ can view V_n as a repn of G by <u>restriction</u>. We will show the V_n are all irred repns of G and every irred repn of G is isom to one of these

<u>Characters of G</u>

(15.15) <u>Lem</u> A (cts) class f^n $f : G \to \mathbb{C}$ is determined by its restriction to T, and $f|_T$ is <u>even</u>.

(i.e. $f\left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\right) = f\left(\begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix}\right)$)

<u>Pf</u> Each ccls in G meets T so a cts f^n determined by rest to T. Evenness follows from $T \cap \mathcal{C}_t = \{t, t^{-1}\}$ for some $t \in T$. $\square$

So char of repn (ρ, V) of G is an <u>even</u> f^n $\chi_\rho : S^1 \to \mathbb{C}$

(1.5.16) <u>Lem</u> If χ is a char of a rep^n of G then $\chi|_T$ is a Laurent poly ($\equiv$ finite $\mathbb{N}_0$-linear combi of f's

$$\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \mapsto \lambda^n \quad \text{for } n \in \mathbb{Z})$$

<u>Pf</u> If V is repn of G then $\mathrm{Res}^G_T V$ is repn of T and its char $\chi_{\mathrm{Res}_T V}$ is the rest^n of χ_V to T. But every repn of T has char of the given form. $\square$

Put $\mathbb{N}_0[z, z^{-1}] = \{ \sum_{\mathbb{Z}} a_n z^n : a_n \in \mathbb{N}_0, \text{ only fin many } a_n \neq 0 \}$

$$\mathbb{N}_0[z, z^{-1}]_{ev} = \{ f \in \mathbb{N}_0[z, z^{-1}] : f(z) = f(z^{-1}) \}$$

By Lemmas, for every cts repn V of G, the char $\chi_V \in \mathbb{N}[z, z^{-1}]_{ev}$

We calculate char χ_n of (ρ_n, V_n). Now

$$\chi_{V_n}(g) = \mathrm{tr}\, \rho_n(g) \quad \text{with } g \sim \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \in T$$

and $\rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} (x^i y^j) = (zx)^i (z^{-1} y)^j = z^{i-j} (x^i y^j)$

so $x^i y^j$ is an eigenvec for each elt of T and T acts on V_n as

$$\rho_n\left(\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right) = \begin{pmatrix} z^n & & & \\ & z^{n-2} & & \\ & & \ddots & \\ & & & z^{-n} \end{pmatrix}$$

Hence

$$(\dagger) \quad \chi_n \left(\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$$

$$\in \mathbb{N}[z, z^{-1}]_{ev}$$

<u>Thm</u> (15.17) V_n is irred as a rep^n of $G = SU(2)$

<u>Pf</u> We use elementary combinatorics

● Assume $0 \neq W \leq V_n$, G invariant; WTS $W = V_n$

<u>Claim</u> If $0 \neq w = \sum_j r_j x^{n-j} y^j \in W$ with (some) $r_i \neq 0$ then $x^{n-i} y^i \in W$

We argue by induction on the # of non-zero r_j

If unique $r_i \neq 0$ then it is clear (mult by inverse). So assume more than one, and choose one i s.t. $r_i \neq 0$

Pick $z \in S^1$ with $z^n, z^{n-2}, \dots, z^{2-n}, z^{-n}$ <u>distinct</u> in $\mathbb{C}$.

Now $\rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \cdot w = \sum_j r_j z^{n-2j} x^{n-j} y^j \in W$ (G-space)

Then $\rho_n \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \cdot w - z^{n-2i} w = \sum_j r_j (z^{n-2j} - z^{n-2i}) x^{n-j} y^j \in W$

● Now $r_j (z^{n-2j} - z^{n-2i}) \neq 0$ precisely when ($r_j \neq 0$ and $j \neq i$)

By induction, $x^{n-j} y^j \in W$ $\forall j \neq i$ with $r_j \neq 0$.

<u>Hence</u> also $x^{n-i} y^i = \frac{1}{r_i} \left(w - \sum_{j \neq i} r_j x^{n-j} y^j \right) \in W$, as required. Proved claim.

We know $x^{n-i} y^i \in W$ for <u>some</u> i. We find matrices in G, the action of which give all the $x^{n-i} y^i \in W$.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : x^{n-i} y^i \longmapsto \frac{1}{\sqrt{2}} \left((x+y)^{n-i} (-x+y)^i \right) \in W$$

& can use claim to deduce $x^n \in W$. Repeating this calculation for $i = 0$,

$(x+y)^n \in W$ and so, by claim $x^{n-i} y^i \in W$ $\forall i$. Thus $W = V_n$. $\square$

● [T, 21.1] alternative approach

(15.18) <u>Thm</u> Every (f.d. cts) irred repn of G is one of the $\rho_n : G \to GL(V_n)$

<u>Pf</u> Assume $\rho_V : G \to GL(V)$ is irred repn affording char $\chi_V \in \mathbb{N}[z, z^{-1}]_{ev}$

We show $\chi_V = \chi_n$ for some n.

Now $\chi_0 = 1$, $\chi_1 = z + z^{-1}$, $\chi_2 = z^2 + 1 + z^{-2}$, …, form a basis of (non f.d.) vector space $\mathbb{Q}[z, z^{-1}]_{ev}$, hence $\chi_V = \sum_n a_n \chi_n$, a (finite) sum with $a_n \in \mathbb{Q}$.

Clearing denominators & moving all summands with neg. coeffs to LHS yields relation $m \chi_V + \sum_{i \in I} m_i \chi_i = \sum_{j \in J} m_j \chi_j$

with I, J finite disjoint subsets of $\mathbb{N}$ and $m, m_i, m_j \in \mathbb{N}$.

The left & right hand sides are chars of repns of G, (15.10) (15.11) gives $mV \oplus \bigoplus_I m_i V_i \cong \bigoplus_J m_j V_j$

Since V is irred we must have $V \cong V_n$ for some $n \in J$. $\square$

[T; 21.1.] [Thomas 6.4] Another method

Tensor products of repns of G

Know (15.15), for V, W repns of G s.t.

$$\mathrm{Res}^G_T V \cong \mathrm{Res}^G_T W \Rightarrow V \cong W$$

Want to understand $\otimes$ for repns of G

(15.19) Propn If $G \cong SU(2)$ (or S^1), V, W repns of G

$$\chi_{V \otimes W} = \chi_V \chi_W \quad \text{(bruh ;;)}$$

Pf ET consider $G \cong S^1$

V, W e-bases $e_1, \dots, e_n$, $f_1, \dots, f_m$ s.t. $z e_i = z^{n_i} e_i$, $z f_j = z^{m_j} f_j$ then

$$z(e_i \otimes f_j) = z^{n_i + m_j}(e_i \otimes f_j)$$

$$\therefore \chi_{V \otimes W}(z) = \sum_{i,j} z^{n_i + m_j} = \Big(\sum_i z^{n_i}\Big)\Big(\sum_j z^{m_j}\Big) = \chi_V(z)\,\chi_W(z). \quad \square$$

Exs $\chi_{V_1 \otimes V_1}(z) = (z + z^{-1})^2 = z^2 + 2 + z^{-2} = \chi_{V_2} + \chi_{V_0}$

$\chi_{V_1 \otimes V_2}(z) = (z^2 + 1 + z^{-2})(z + z^{-1}) = z^3 + 2z + 2z^{-1} + z^{-3} = \chi_{V_3} + \chi_{V_1}$

(15.20) Propn (Clebsch-Gordan) For $n, m \in \mathbb{N}$,

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \dots \oplus V_{|n-m|+2} \oplus V_{|n-m|}$$

Pf Check chars. WLOG $n \geq m$

$$(\chi_n \cdot \chi_m)(z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}(z^m + z^{m-2} + \dots + z^{-m})$$

$$= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{2j-n-m-1}}{z - z^{-1}}$$

$$= \sum_{j=0}^{m} \chi_{n+m-2j}(z)$$

(note $n \geq m$ ensures no cancellation in the sum). $\square$

L14.3

Repns of SO(3)

Amongst other things, [T, 22.1] shows

(15.21) Propⁿ There is an isom of top gps

$$SU(2)/\{\pm id\} \cong SO(3)$$

and indeed the gp isom is in fact a homeom,

We'll sketch this below using fact that any cts bijection from a cpct space to a Hausdorff space is a homeo.

First an easy corollary of (15.21)

(15.22) Cor Every irred repn of SO(3) is of the form $\rho_{2m} : SO(3) \to GL(V_{2m})$ for some $m \geqslant 0$

Pf Irred repns of SO(3) correspond to repns of SU(2) s.t. $-I$ acts trivially. But $-I$ acts on V_n as -1 when n odd and as $+1$ when n is even, i.e.

$$\rho_{V_n}(-I) = \begin{pmatrix} (-1)^n & & \\ & (-1)^{n-2} & \\ & & \ddots \\ & & & (-1)^{-n} \end{pmatrix} = (-1)^n I. \qquad \square$$

Sketch proof that $G/\{\pm I\} \cong SO(3)$

Recall the Left-right multiplication action of G, viewed as a subspace of unit norm quaternions on $\mathbb{H} \cong \mathbb{R}^4$. Restricting the Left-right action to the diagonal copy of G gives a conjugation action of G on the space

$$\begin{aligned} \mathbb{H}^0 &= \{ A \in \mathbb{H} : \operatorname{tr} A = 0\} \\ &= \mathbb{R}\langle \underbrace{\begin{pmatrix} i & \\ & -i\end{pmatrix}}_{\underline{i}}, \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}}_{\underline{j}}, \underbrace{\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}}_{\underline{k}} \rangle \end{aligned}$$

of pure quaternions, spanned by $\underline{i}, \underline{j}, \underline{k}$. This is 3-dim Euclidean space with unit norm $\|A\|^2 = \det A$. G acts as isometries on $\mathbb{H}^0$:

$$X \cdot A = XAX^{-1}$$

giving gp homom $\phi : G \to O(3)$ with kernel $Z(G) = \{\pm I\}$.

Now G cpct, $O(3)$ Hausdorff, so cts homeo $G/Z(G) \longrightarrow \operatorname{im}\phi$ is homeo.

To show $\operatorname{im}\phi = SO(3)$.

Now G is (simply) connected which $\Rightarrow$ $\operatorname{im}\phi \leq SO(3)$

since only one of the two possible values ± 1 can be taken by the cts function

$\det \phi(p)$; since $\phi(I_2) = I_3$ has $\det = 1$, value must be $+1$.

For other inclusion:

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \begin{pmatrix} ai & b \\ -\bar{b} & -ai \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ai & be^{2i\theta} \\ -\bar{b}e^{-2i\theta} & -ai \end{pmatrix}$$

and so $\operatorname{diag}(e^{i\theta}, e^{-i\theta})$ acts on $\mathbb{R}\langle \underline{i}, \underline{j}, \underline{k} \rangle$ by rotation in the $(\underline{j}, \underline{k})$ plane

through angle 2θ.

Check similarly that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by rot^n by 2θ in $(\underline{i}, \underline{k})$ plane

$\begin{pmatrix} \cos\theta & i\sin\theta \\ +i\sin\theta & \cos\theta \end{pmatrix}$ acts by rot^n by 2θ in $(\underline{i}, \underline{j})$ plane

We deduce $\operatorname{im}\phi = SO(3)$. $\square$