II Riemann Surfaces

Analytic & meromorphic functions

'What do we really mean by multi-valued functions, such as the complex logarithm or m^{th} roots?'

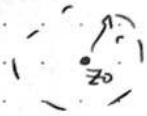
~ Branch cuts define log or $\sqrt[n]{\cdot}$ as a collection of functions

But we talk about a single, multi-valued function!

1.1 Analytic functions & their zeroes

Domain : open, connected $D \subseteq \mathbb{C}$

eg. discs, punctured disc



$\mathbb{D}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$

$D_*(z_0, r) = \{ z \in \mathbb{C} : 0 < |z - z_0| < r \}$

Also ngbds, punctured ngbhds

Def^n Let $D \subseteq \mathbb{C}$ be a domain. $f: D \to \mathbb{C}$ is holomorphic or analytic if either of the two equiv def^{n}s are satisfied:

(i) f is $\mathbb{C}$-diff at every $z_0 \in D$

(ii) for any $z_0 \in D$, have $r > 0$ s.t. f has power series exp^n on $D(z_0, r)$

Prop 1.2 (Principle of isolated zeroes)

Let $f: D \to \mathbb{C}$ be analytic. If $f(z_0) = 0$, then either $f \equiv 0$ in a ngbd of z_0, or f is non-zero on a punctured ngbd of z_0

Proof Consider Taylor series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Take smallest $m \geq 1$ s.t. $a_m \neq 0$ ← if f not id zero near z_0

Then $f(z) = (z - z_0)^m \underbrace{\sum_{n \geq 0} a_{n+m} (z - z_0)^n}_{g(z)} = (z - z_0)^m g(z)$

Now $g(z_0) \neq 0$, g analytic. $\square$

Def^n $C \subseteq D$ is discrete if subspace topology is discrete topology

Cor 1.3 (Identity principle) Let $f, g : D \to \mathbb{C}$ be analytic. Unless $\{ z \in D \mid f(z) = g(z) \}$ is discrete, $f \equiv g$ on D.

Proof $A = \{z_0 \in D \mid f \equiv g \text{ in ngbd of } z_0\}$

$B = \{z_0 \in D \mid f \neq g \text{ on a pnct ngbd of } z_0\}$

Both A, B are open, disjoint. Use connectedness, isolated zeros. $\square$

1.3 Meromorphic f^n's & singularities

$f : D_*(z_0, r) \longrightarrow \mathbb{C}$ has a singularity at z_0

Prop 1.5 (Laurent series) If an analytic f^n has an isolated singularity at z_0, then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ in pctd ngbd of z_0

Def^n 1.6 (class^n of singularities)

(i) $a_n = 0$ for $n < 0 \Rightarrow$ removable singularity

(ii) $a_n = 0$ for $n < -m < 0$, $a_{-m} \neq 0 \Rightarrow$ pole of order m

(iii) $a_n \neq 0$ for ∞-many $n < 0 \Rightarrow$ essential singularity

Thm 1.7 (removable singularity)

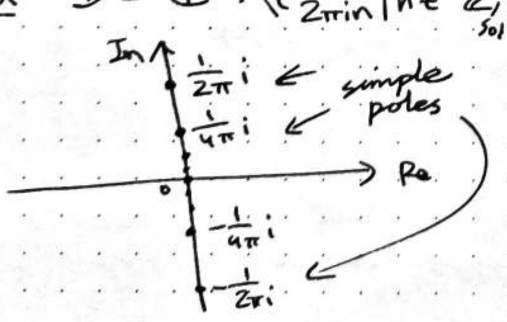
Analytic f^n has removable at z_0 iff f is bdd on a $D_*(z_0, r)$

Thm 1.8 (Casorati-Weierstrass)

Analytic f^n has essential at z_0 iff $f(D_*(z_0, r))$ is dense in $\mathbb{C}$ for any $r > 0$ such that $D_*(z_0, r) \subseteq D$.

Def^n 1.9 Let D domain. If A discrete $\subseteq D$, and f holo on $D \setminus A$ with poles at pts of A, then say f is a meromorphic function on D

Ex $D = \mathbb{C} \setminus \left(\left\{ \frac{1}{2\pi i n} \,\middle|\, n \in \mathbb{Z}_{\neq 0} \right\} \cup \{0\} \right)$, $f(z) = \frac{1}{e^{1/z} - 1}$



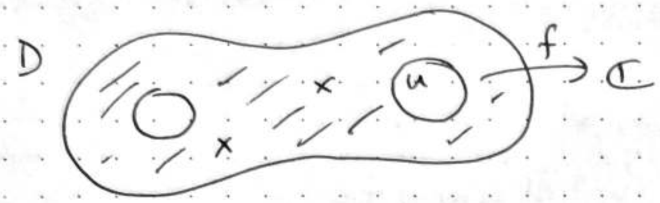
However, singularity at zero is essential.

1.3 Analytic continuation

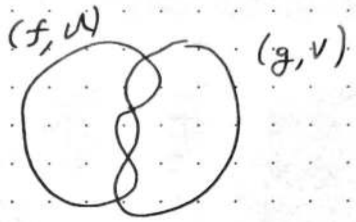
Def^n A function element $F = (f, \mathcal{U})$ is:

(i) a subdomain $\mathcal{U} \subseteq D$;

(ii) an analytic function $f : \mathcal{U} \to \mathbb{C}$



Direct analytic continuation



$(f, \mathcal{U}) \sim (g, V)$

- $\mathcal{U} \cap V \neq \emptyset$
- $f|_{\mathcal{U}\cap V} = g|_{\mathcal{U} \cap V}$

Id principle $\Rightarrow$ f determines g!

Analytic continuation

$(f, \mathcal{U}) \approx (g, V)$ means $(f, \mathcal{U}) \sim (f_1, \mathcal{U}_1) \sim \dots \sim (f_n; \mathcal{U}_n) \sim (g, V)$

Remark $\approx$ is an equivalence relation

Def^n 1.12 A $\approx$-equiv class $\mathcal{F}$ of function elements on a domain D is called a complete analytic function on D.

1.4 The complex logarithm

Let $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$.

We want to INVERT $\exp : \mathbb{C} \to \mathbb{C}_*$

BUT $\exp(0) = 1 = \exp(2\pi i)$

$\rightsquigarrow$ log is just a collection of function elements

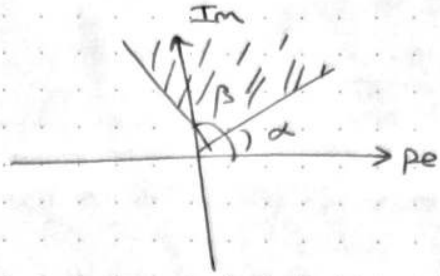
i.e. a complete analytic function

For $(\alpha,\beta)\subseteq\mathbb{R}$ with $|\alpha-\beta|<2\pi$,

$U_{(\alpha,\beta)} = \{ re^{i\theta} \mid r>0,\ \alpha<\theta<\beta \}$

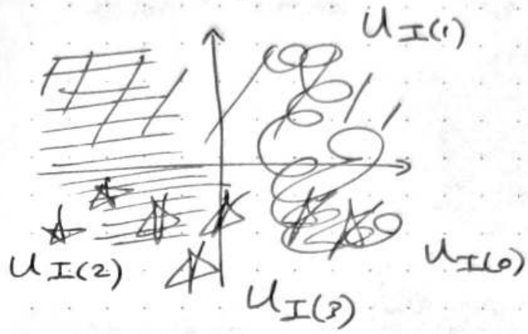
$f_{(\alpha,\beta)}(z) = \log r + i\theta$

$z=re^{i\theta}$ where $\boxed{\alpha<\theta<\beta}$



$\rightsquigarrow$ f^n elts $F_{(\alpha,\beta)} = (f_{(\alpha,\beta)}, U_{(\alpha,\beta)})$

$I(n) = \left((n-1)\frac{\pi}{2}, (n+1)\frac{\pi}{2}\right)$



Case analysis:

$F_{I(m)} \sim F_{I(n)}$ iff $|m-n|\le 1$

Details: 4 cases

- $m-n\equiv 0 \mod 4$

  $\hookrightarrow U_{I(n)} = U_{I(m)}$

  $I(m)\cap I(n)=\phi$ unless $m=n$

  $\therefore$ agree only if $m=n$

- $m-n\equiv 2 \mod 4$

  $U_{I(n)} \cap U_{I(m)} = \phi$

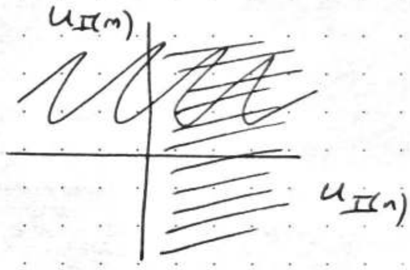
  $\therefore$ no direct continuation

- $m-n \equiv \pm 1 \mod 4$

  $\hookrightarrow U_{I(n)}\cap U_{I(m)}$ is a quadrant

  $I(m)\cap I(n)=\phi$ unless $m=n\pm 1$

  $\therefore F_{I(m)}\sim F_{I(n)}$ $\Updownarrow$ $|m-n|=1$



Conclusion

$F_{I(m)} \approx F_{I(n)}$ $\forall m,n\in\mathbb{Z}$

$\therefore \{F_{I(n)} \mid n\in\mathbb{Z}\}$

determine a complete analytic function

This is the complex logarithm

Remark $f_{I(0)}(1) = 0$

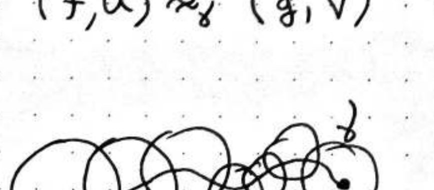
$f_{I(4)}(1) = 2\pi i$

$\therefore$ analytic continuation on $\mathbb{C}^*$ is not unique

Def^n $\gamma:[0,1]\longrightarrow D$ path

$(f,U)\approx_\gamma (g,V)$ means $\exists\ 0=t_0<t_1<\dots<t_n=1$,

subdomains $U_\phi,\dots,U_n$ s.t. $\gamma([t_i,t_{i+1}])\subseteq U_{i+1}$



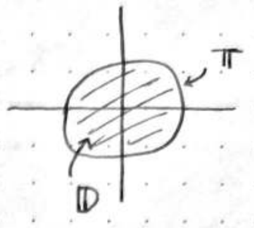
Later we will see analytic cont along a path is unique

CLASSICAL MONODROMY THEORY

L2.1 Natural boundary, gluing constructions and roots

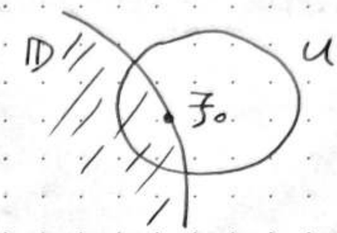
2.1 Natural boundary

Sometimes it's impossible to analytically continue.



NOTATION : $\mathbb{D} = D(0,1)$; $\mathbb{T} = \partial\mathbb{D} = S^1$

$f(z) = \sum_{n\geq 0} a_n z^n$ with r.o.c. $= 1$



$z_0 \in \mathbb{T}$ is REGULAR if $\exists\,(g, U)$ s.t.

$$f|_{U\cap\mathbb{D}} \equiv g|_{U\cap\mathbb{D}}$$

Otherwise, $z_0 \in \mathbb{T}$ is SINGULAR

Remark $\{\text{regular points}\} \subseteq \mathbb{T}$ is OPEN, by def^n.

Therefore, $\{\text{singular points}\}$ is CLOSED

BEWARE! Ex 2.3, $f(z) = \frac{1}{1-z} = \sum_{n\geq 0} z^n$

$\{\text{regular points}\} = \mathbb{T}\setminus\{1\}$

BUT $\sum(-1)^n$ does not converge

ALSO Ex 2.4 $g(z) = \sum_{n\geq 2} \frac{z^n}{n(n-1)}$

$\sum_{n\geq 2} \frac{1}{n(n-1)}$ is convergent

BUT 1 is a singular point of g, because it is singular for $g'' = f$

MORAL z_0 regular $\not\Rightarrow$ $\sum a_n z_0^n$ converges

$\sum a_n z_0^n$ converges $\not\Rightarrow$ z_0 regular

Prop 2.5 If a power series has radius of convergence 1, then some point of $\mathbb{T}$ is singular for f.

Proof Compactness of $\mathbb{T}$ $\Rightarrow$ covered by finitely many balls $D(z_1, \delta_1), \dots, D(z_n, \delta_n)$ such that f extends analytically over each $D(z_i, \delta_i)$

$\Rightarrow$ R.O.C. > 1 ※ $\square$

Def^n If $\{\text{singular points}\} = \mathbb{T}$, then $\mathbb{T}$ is the NATURAL BOUNDARY

Rmk This makes sense for other curves in $\mathbb{C}$, e.g. $\mathbb{R}$

L 2.2

<u>Example 2.7</u> $f(z) = \sum_{n \geq 0} z^{n!}$, ROC $= 1$

<u>Claim</u> $e^{2\pi i p/q} = \omega$ is singular

Since $\{ e^{2\pi i p/q} \mid p, q \in \mathbb{Z} \}$ is dense in $\mathbb{T}$ $\Rightarrow$ f has natural boundary $\mathbb{T}$

Whenever $0 < r < 1$, we have

$$f(r\omega) = \sum_{n=0}^{q-1} r^{n!} \omega^{n!} + \sum_{n \geq q} r^{n!}$$

For any M,

$$\sum_{n=q}^{M+q} r^{n!} \longrightarrow M+1 \quad \Rightarrow \quad \sum_{n=q}^{M+q} r^{n!} > M$$

as $r \to 1$ for r large enough

Therefore, $f(r\omega) = \text{const.} + \sum_{n \geq q} r^{n!} \longrightarrow \infty$ as $r \longrightarrow 1$.

So $\nexists$ analytic g extending f at ω. $\square$

2.2 A gluing construction

Last time, we saw that log is a COMPLETE ANALYTIC FUNCTION

But this is still just a collection of functions

Now we'll construct a space to realise log as a genuine function, but on a bigger domain R.

IDEA: Glue the function elements in log together to give R

$$R = \Big(\coprod_{n \in \mathbb{Z}} \mathcal{U}_{I(n)} \Big) / \sim$$

$\sim$ is defined by $\underset{\mathcal{U}_{I(m)}}{\zeta_1} \sim \underset{\mathcal{U}_{I(n)}}{\zeta_2}$ $\iff$ $\zeta_1 = \zeta_2$ in $\mathbb{C}$ AND $f_{I(m)}(\zeta_1) = f_{I(n)}(\zeta_2)$

R is given the QUOTIENT TOPOLOGY

<u>Remark</u> Since $F_{I(m)} \approx F_{I(n)}$ for all $m, n \in \mathbb{Z}$, it follows that R is path-connected.

" INFINITE MULTI-STOREY CAR PARK "

GLOBAL FUNCTIONS

The function elements $f_{I(n)}$ on the $\mathcal{U}_{I(n)}$ descend to a global function $f : R \to \mathbb{C}$, $f([\zeta]) = f_{I(n)}(\zeta)$ for $\zeta \in \mathcal{U}_{I(n)}$

<u>Note</u> This is well-defined by the def^n of the gluing relation.

Similarly, the natural inclusions

$$\mathcal{U}_{I(n)} \hookrightarrow \mathbb{C}$$

can be used to define a global function

$$\pi : R \longrightarrow \mathbb{C}, \quad \pi([\zeta]) = \zeta \quad \text{for } \zeta \in \mathcal{U}_{I(n)}$$

<u>Note</u> $\exp \circ f([\zeta]) = \pi([\zeta])$ for all $[\zeta] \in R$

The global functions can be used together

$$\Phi : R \longrightarrow \mathbb{C}^2$$
$$[\zeta] \longmapsto (\pi([\zeta]), f([\zeta]))$$

By the def^n of $\sim$, Φ is injective, continuous

$\therefore$ R is HAUSDORFF because $\mathbb{C}^2$ is too

<u>Remark</u> Φ identifies R with the graph $\{(w, \zeta) \in \mathbb{C}^2 \mid w = \exp \zeta\}$

This gives an alternative point of view on R, since the graph should be obtained by flipping the graph of exp.

<u>2.3 Complex roots</u>

Consider the k^{th} power map $p_k : z \mapsto z^k$

The 'inverse' of this is $\sqrt[k]{z} = \exp\left(\frac{1}{k} \log z\right)$

This multi-valued function can be analysed just like log.

Take $I(n) = ((n-1)\pi/2, (n+1)\pi/2)$ and $U_{I(n)}$ as before, equipped with $f_{I(n)} : \mathcal{U}_{I(n)} \longrightarrow \mathbb{C}$ branch of log.

Set $g_{I(n)}(\zeta) = \exp\left(\frac{1}{k} f_{I(n)}(\zeta)\right)$ to define function elts $G_{I(n)} = (g_{I(n)}, \mathcal{U}_{I(n)})$

This time, $G_{I(n)}$ only depends on n modulo k ($\otimes$ doubt), so wlog $n \in \mathbb{Z}/k\mathbb{Z}$.

Everything else is as before : $G_{I(n)} \sim G_{I(m)}$ iff $m - n \in \{0, \pm 1\}$ modulo k

$\therefore$ the $\{G_{I(n)} \mid n \in \mathbb{Z}/k\mathbb{Z}\}$ define a complete analytic function, the k^{th} root.

Furthermore, a similar gluing construction defines a path-connected, Hausdorff space R_k and maps

$$\begin{array}{l} R_k \xrightarrow{g} \mathbb{C}_* \\ R_k \xrightarrow{\pi} \mathbb{C}_* \end{array} \quad \text{s.t } g([\zeta])^k = \pi([\zeta])$$

L3.1 Riemann Surfaces & Analytic Maps

3.1 Covering Maps

In § 2.2 we realized the complex logarithm by constructing functions

$$f, \pi : R \longrightarrow \mathbb{C}$$

satisfying $\exp \circ f = \pi$.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathbb{C} \arrow[d, "\exp"] \\ & \mathbb{C}^* \end{tikzcd}$$

Although π isn't a homeomorphism, it's the next best thing.

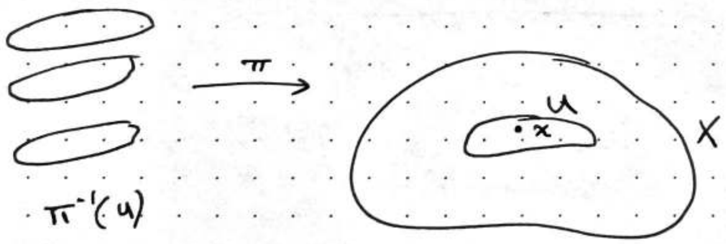
<u>Def</u> (Covering map) Let $X, \tilde{X}$ be path-connected, Hausdorff topological spaces. A covering map is $\pi : \tilde{X} \to X$, a local homeomorphism, that is, each $\tilde{x} \in \tilde{X}$ has an open neighbourhood $\tilde{U}$ such that $\pi|_{\tilde{U}}$ is a homeomorphism onto its image.

<u>Ex</u> Any inclusion of an (open) set $\pi : \tilde{X} \hookrightarrow X$ is a covering map, $(\tilde{U} = \tilde{X})$

A covering map $\pi : \tilde{X} \to X$ is <u>regular</u> if, for each $x \in X$, there is an open nbgd U of x and a discrete set Δ_x such that $\pi^{-1}(U)$ is homeo to the direct product $U \times \Delta_x$ and the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "\pi"'] & U \times \Delta_x \arrow[d] \\ & U \end{tikzcd}$$

commutes, where $U \times \Delta_x \to U$ is projection onto the first factor.



$U \times \Delta_x \cong \coprod_i U_i$

Beware! In AlgTop, covering maps are what we call regular covering maps.

<u>Example 3.3</u> The map $\pi : R \longrightarrow \mathbb{C}_*$ is a regular covering map. Each $z \in \mathbb{C}_*$ lives in at least one $U_{I(n)}$ — and in each case the preimage is of the form $\pi^{-1}(U_{I(n)}) = \coprod_{m \equiv n \bmod 4} U_{I(m)}$

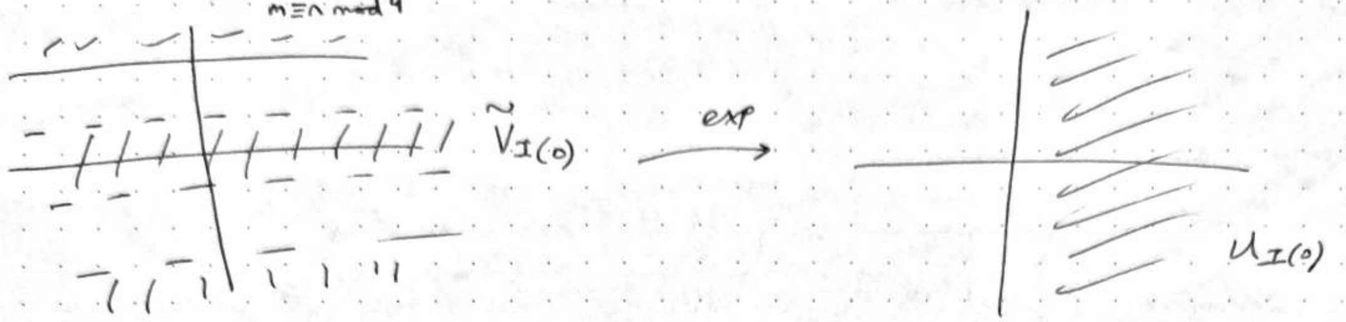
<u>Note</u> If $m \equiv n \bmod 4$ then $U_{I(n)}, U_{I(m)}$ disjoint unless $m = n$, in R

<u>Ex 3.4</u> For each open interval $I \subseteq \mathbb{R}$, let $\tilde{V}_I = \mathbb{R} + iI$.

As long as I is of diameter at most 2π, the exponential map restricts to a homeomorphism $\tilde{V}_I \longrightarrow U_I$, with inverse provided by the map f_I.

As in the previous example, every $z \in \mathbb{C}_*$ is contained in some $U_{I(n)}$, and

$$\exp^{-1}(U_{I(n)}) = \coprod_{m \equiv n \bmod 4} \tilde{V}_{I(m)} \cong U_{I(n)} \times \mathbb{Z}$$



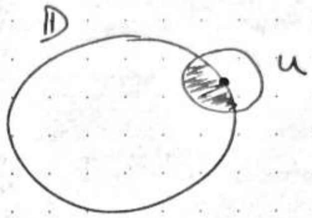
However, not all covering maps are regular

<u>Ex</u> $\pi : \mathbb{D} \hookrightarrow \mathbb{C}$

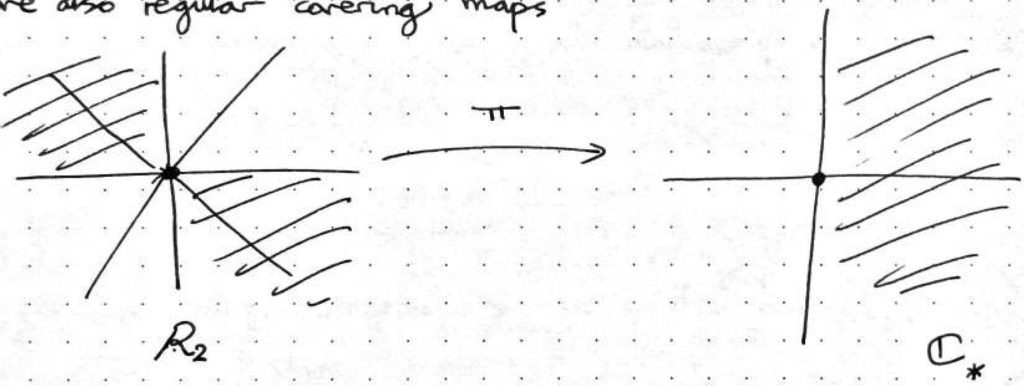
If $\zeta \in \mathbb{T}$, and U a nbgd (in $\mathbb{C}$);

$\pi^{-1}(U) = U \cap \mathbb{D} \xrightarrow{\pi} U$

is never surjective.



<u>Example</u> The maps $\pi : R_k \to \mathbb{C}_*$ we constructed for k^{th} roots are also regular covering maps



<u>3.2 Abstract Riemann surfaces</u>

<u>Def 3.6</u> A <u>chart</u> on R is a pair (ϕ, U), where U is an open subset of R and $\phi : U \to D$ is a homeo to an open subset of $\mathbb{C}$. A set of charts $\mathcal{A}$ is called an <u>atlas on R</u> if the following hold:

(i) $\bigcup_{(\phi,U) \in \mathcal{A}} U = R$

(ii) if $(\phi_1, U_1), (\phi_2, U_2) \in \mathcal{A}$ and $U_1 \cap U_2 \neq \emptyset$, then

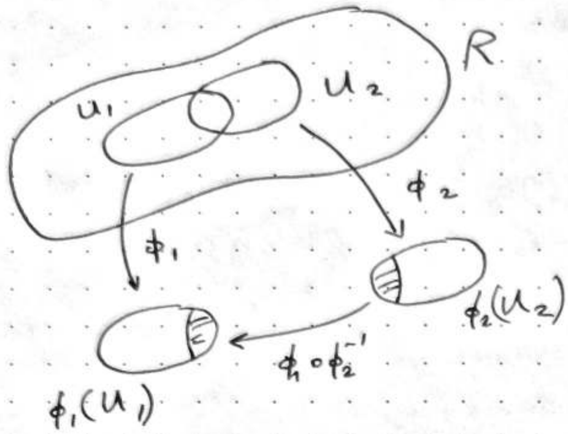
$$\phi_1 \circ \phi_2^{-1} \equiv (\phi_1|_{U_1 \cap U_2}) \circ (\phi_2|_{U_1 \cap U_2})^{-1}$$

is analytic on $\phi_2(U_1 \cap U_2)$.

The composition $\phi_1 \circ \phi_2^{-1}$ is called a <u>transition function</u>

L3.3

Remark $(\phi_1 \circ \phi_2^{-1})^{-1} = \phi_2 \circ \phi_1^{-1}$ so transition functions are INVERTIBLE



R is Hausdorff, path-connected

Example $R = \mathbb{C}$

(i) $\mathcal{A} = \{ id : \mathbb{C} \to \mathbb{C} \}$

(ii) $\mathcal{A}' = \{ z \mapsto z+1 \}$

(iii) $\mathcal{A} \cup \mathcal{A}'$

} atlases of $\mathbb{C}$

But $\mathcal{A} \cup \mathcal{A}'$ contains 'more' information than $\mathcal{A}$ or $\mathcal{A}'$.

Therefore, we should only consider atlases that are "as big as possible"

Def (3.9) A conformal structure on R is an atlas $\mathcal{A}$ on R which is maximal in the following sense: if (ψ, V) is a chart on R such that, for any $(\phi, U) \in \mathcal{A}$, the transition f^n $\phi \circ \psi^{-1}$ is analytic, then $(\psi, V) \in \mathcal{A}$

Def (3.10) A Riemann surface is a pair $(R, \mathcal{A})$ where $\mathcal{A}$ is a conformal structure on R. Abuse notation, denote by R.

Lemma (3.11) Every atlas $\mathcal{A}$ is contained in a unique conformal structure $\hat{\mathcal{A}}$

Proof Let $\hat{\mathcal{A}}$ be the set of charts (ψ, V) s.t. $\psi \circ \phi^{-1}$ is analytic $\forall\ (\phi, U) \in \mathcal{A}$

This is necessarily maximal, so to prove existence, it remains to show that it is an atlas. Take $(\psi_1, V_1), (\psi_2, V_2) \in \hat{\mathcal{A}}$, and $p \in V_1 \cap V_2$.

Since $\mathcal{A}$ an atlas, have $(\phi, U)^{\in \mathcal{A}}$ with $p \in U$. Then

$$\psi_1 \circ \psi_2^{-1} = (\psi_1 \circ \phi^{-1}) \circ (\phi \circ \psi_2^{-1}) \text{ is analytic at } \psi_2(p).$$

Uniqueness is clear, if an atlas $\mathcal{A}'$ contains $\mathcal{A}$ then $\mathcal{A}' \subseteq \hat{\mathcal{A}}$.

So $\hat{\mathcal{A}}$ contains any other conformal structure containing $\mathcal{A}$. □

no proper subset

Moral ATLAS $\rightsquigarrow$ CONFORMAL STRUCTURE

L3.4

Example 3.12 The atlas $\mathcal{A} = \{id : \mathbb{C} \to \mathbb{C}\}$ from Ex 3.9 is contained in a unique conformal structure on $\mathbb{C}$.

Beware! Not the only Riemann surface structure on $\mathbb{C}$.

Ex 3.13 The atlas $\mathcal{A}' = \{z \mapsto \bar{z}\}$ is not contained in $\mathcal{A}$, so defines a different conformal structure. Call this Riem sfc $\overline{\mathbb{C}}$.

Def $\{id : \mathbb{C} \to \mathbb{C}\}$ defines the CANONICAL conformal structure on $\mathbb{C}$

Ex If R is a Riem sfc, $S \subseteq R$ is open, (+ path connected) then

eh!

$$\{ (\phi|_{S \cap U}, U \cap S) : (\phi, U) \in \mathcal{A} \}$$

is an atlas on S.

In particular, any domain $D \subseteq \mathbb{C}$ is a Riemann surface

FAVORITE $\mathbb{D}$ the disc, $\mathbb{C}_*$ the punctured plane

Example (Riemann sphere)

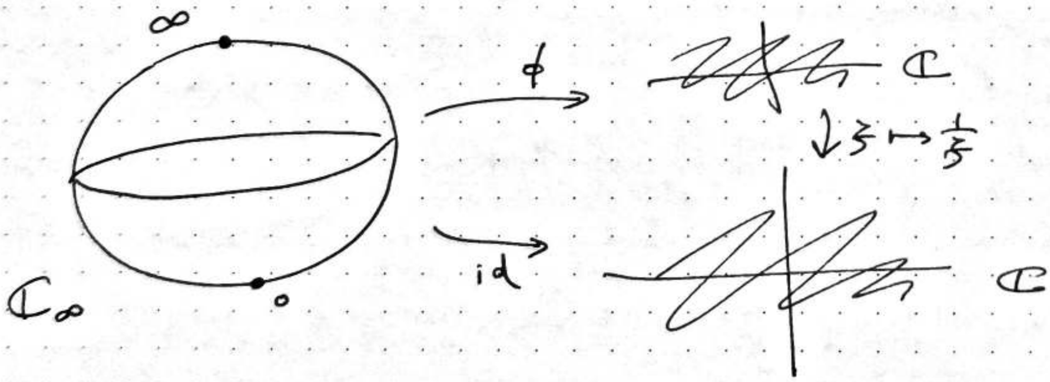
By stereographic projection, let $\mathbb{C} \cup \{\infty\} \cong S^2$. ← statement in topology

Let $\mathcal{A} = \{ (id, \mathbb{C}), (\phi, U) \}$ where

$$U = (\mathbb{C} \setminus \{0\}) \cup \{\infty\} \quad \text{and}$$

$$\phi(z) = 1/z \text{ for } z \in \mathbb{C}_*, \quad \phi(\infty) = 0$$

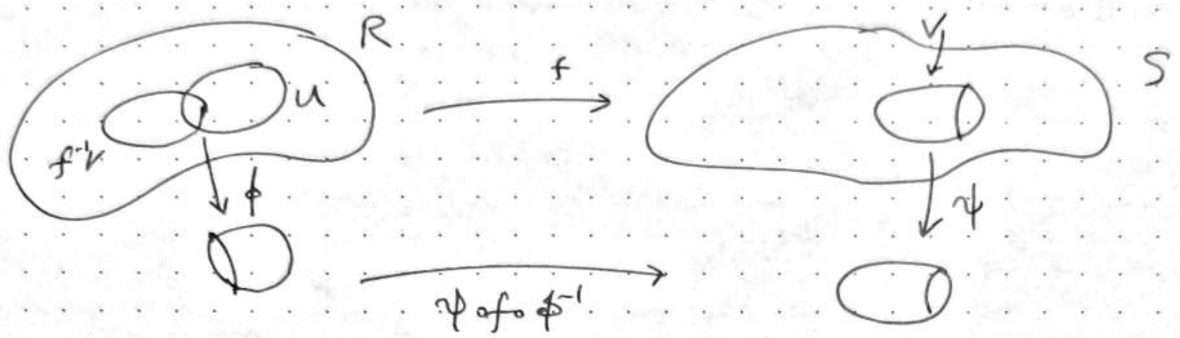
Transition functions are $z \mapsto \frac{1}{z}$, so $\mathcal{A}$ is an atlas



L3.5

3.3 Analytic maps

Def Let R, S be Riem sfcs. A cts map $f: R \to S$ is analytic or holomorphic if, for all charts (ϕ, U) on R and (ψ, V) on S, the map $\psi \circ f \circ \phi^{-1}$ is analytic on $\phi(U \cap f^{-1}V)$



Lemma 3.18 A cts map of Riem sfcs $f: R \to S$ is analytic iff the following holds: for each $p \in R$, there is a chart (ϕ_p, U_p) on R with $p \in U_p$ and a chart (ψ_p, V_p) on S with $f(p) \in V_p$ such that the map of open subsets of $\mathbb{C}$ $\psi_p \circ f \circ \phi_p^{-1} : \phi_p(U_p \cap f^{-1}(V_p)) \to \psi_p(V_p)$ is analytic at $\phi_p(p)$.

[That, need only check at ONE pair of charts for each $p \in R$]

Proof "only if" direct

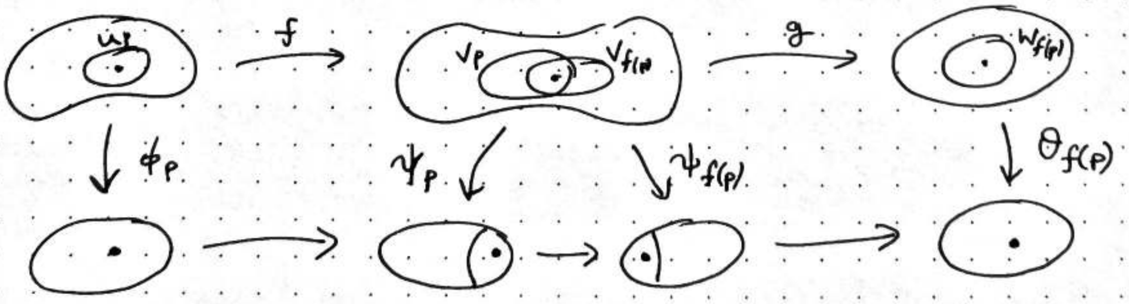
"if", given charts (ϕ, U) on R, (ψ, V) on S, STP $\psi \circ f \circ \phi^{-1}$ analytic at $\phi(p)$ on the domain $\phi(U \cap f^{-1}V)$, do this via charts at ~~$\phi(p)$~~, $\psi_p \circ f \circ \phi_p^{-1}$ analytic,

$$\psi \circ f \circ \phi^{-1} = (\psi \circ \psi_p^{-1}) \circ (\psi_p \circ f \circ \phi_p^{-1}) \circ (\phi_p \circ \phi^{-1}) \qquad \square$$

Key point $\psi \circ f \circ \phi^{-1}$ can be written locally as a composition of $\psi_p \circ f \circ \phi_p^{-1}$ and transition functions.

Lemma 3.19 If $f: R \to S$, $g: S \to T$ are analytic then $g \circ f$ is analytic.

Proof



$\square$

L3.6

Def 3.20 A conformal equiv or biholomorphism is an analytic bijection of Riem sfc, $f: R \to S$ with analytic inverse $f^{-1}: S \longrightarrow R$

Lemma 3.19 $\Rightarrow$ EQUIV RELN

Ex Let $\overline{\mathbb{C}} = (\mathbb{C}, \bar{\mathcal{A}})$

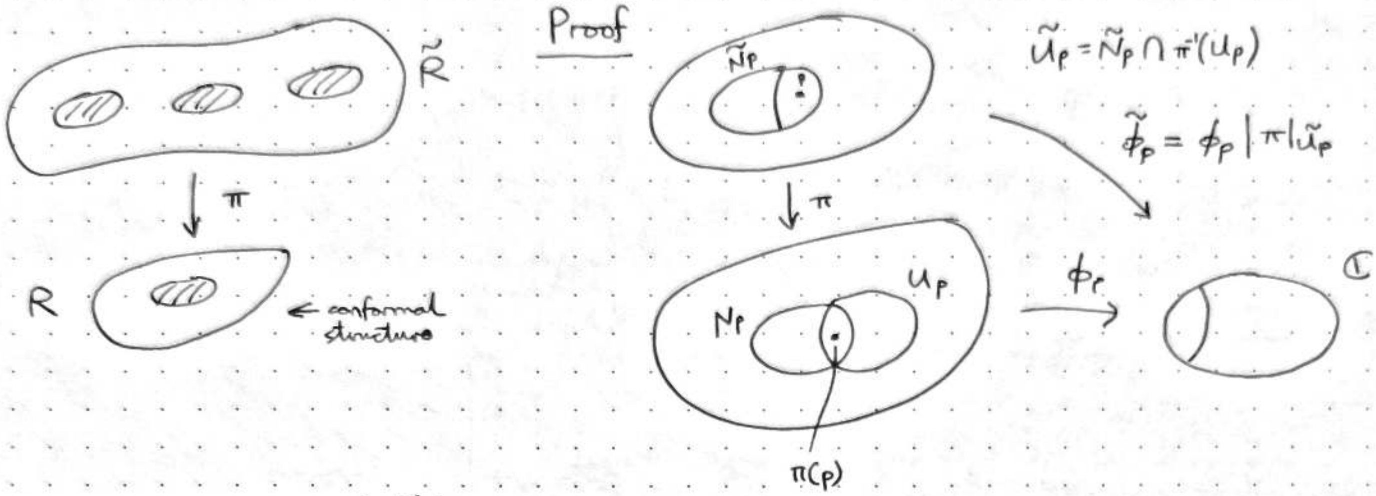
Then $\mathbb{C} \to \overline{\mathbb{C}}$, $\xi \mapsto \overline{\xi}$ is a conformal equivalence.

L4.1

COVERING MAPS AND ANALYTICITY

Lemma 4.1 If $\pi : \tilde{R} \to R$ is a covering map and R is a Riemann surface then there is a unique conformal structure on $\tilde{R}$ such that π is analytic.

Proof



(Transition functions of $\tilde{R}$)

These are of the form

makes sense locally

$$\tilde{\phi}_p \circ \tilde{\phi}_q^{-1} = \phi_p \circ \pi \circ \pi^{-1} \circ \phi_q^{-1} = \phi_p \circ \phi_q^{-1}$$

which are transition functions of R, hence analytic.

$\therefore \tilde{\mathcal{A}} = \{ (\tilde{\phi}_p, \tilde{U}_p) \mid p \in \tilde{R} \}$ is an atlas

(Analyticity of π)

Let $p \in \tilde{R}$. Choosing charts (ϕ_p, U_p) about $\pi(p)$ and $(\tilde{\phi}_p, \tilde{U}_p)$ about p, π has local form

$$\phi_p \circ \pi \circ \tilde{\phi}_p^{-1} = \phi_p \circ \pi \circ \pi^{-1} \circ \phi_p^{-1} = \mathrm{id}_{\mathbb{C}}$$

so π is indeed analytic

(Uniqueness)

Let $\tilde{\mathcal{B}}$ be a conformal structure on $\tilde{R}$ s.t. $\pi : \tilde{R} \to R$ is analytic.

Let $p \in \tilde{R}$ and $(\psi, V) \in \tilde{\mathcal{B}}$ a chart about p.

Then π has analytic local form

$$\phi_p \circ \pi \circ \psi^{-1} = \tilde{\phi}_p \circ \psi^{-1}$$

$\therefore \tilde{\phi}_p$ has analytic transition functions with every $(\psi, V) \in \tilde{\mathcal{B}} \Rightarrow \tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$ □

<u>Ex 4.2</u> Consider f, π, R constructed as in §2.2

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathbb{C} \arrow[d, "\exp"] \\ & \mathbb{C}^* \end{tikzcd}$$

π covering map

$\Rightarrow$ R conformal structure

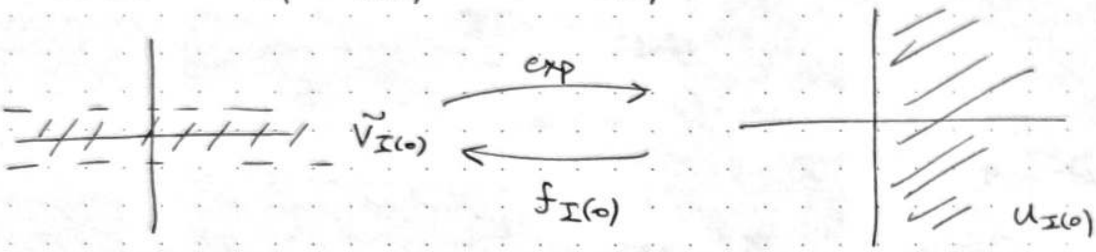
Further, locally,

$f|_{U_{I(n)}} = f_{I(n)} \circ \pi$ so f and π are both analytic

Furthermore, recall that in §3.1 we saw that

$f_{I(n)} : U_{I(n)} \longrightarrow \tilde{V}_{I(n)}$ is a homeomorphism

with inverse $\exp : \tilde{V}_{I(n)} \longrightarrow U_{I(n)}$



These inverses $f_{I(n)}^{-1}$ agree on intersections $\tilde{V}_{I(m)} \cap \tilde{V}_{I(n)}$,

and so piece together to define a GLOBAL analytic inverse

$$\begin{tikzcd} R \arrow[r, shift left, "f"] & \mathbb{C} \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

In particular f is a CONFORMAL EQUIVALENCE

Again, k^{th} roots are similar.

<u>Ex 4.3</u> π covering map

$$\begin{tikzcd} R_k \arrow[r, "g"] \arrow[dr, "\pi"'] & \mathbb{C}_* \arrow[d, "p_k"] \\ & \mathbb{C}_* \end{tikzcd}$$

$\Rightarrow$ conformal structure on R_k

Locally $g|_{U_{I(n)}} = g_{I(n)}$ so analytic

Furthermore, g is a conformal equivalence.

k^{th} roots have an additional nice feature

f, π, p_k extend to analytic maps of COMPACT Riemann surfaces

$$\begin{tikzcd} \hat{R}_k \arrow[r, "\hat{g}"] \arrow[dr, "\hat{\pi}"'] & \mathbb{C}_\infty \arrow[d, "p_k"] \\ & \mathbb{C}_\infty \end{tikzcd}$$

WARNING: $\hat{\pi}$ is NOT a covering map

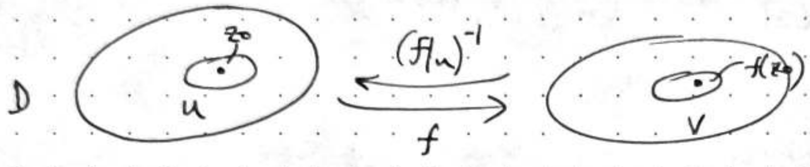
L4.3

4.2 Analytic function

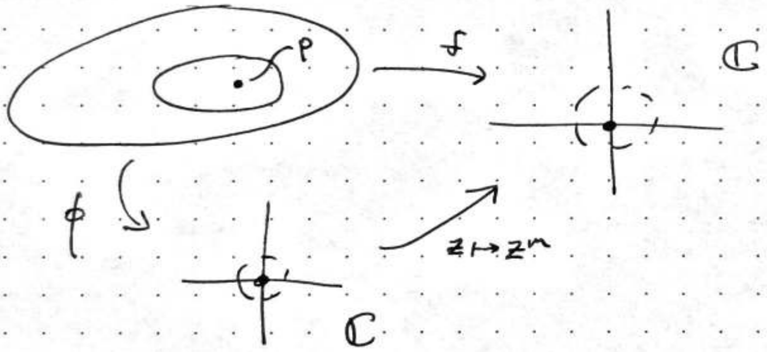
Def 4.4 An analytic function on a Riemann surface R is an analytic map $R \to \mathbb{C}$

The Riemann surfaces setting means we can put analytic functions into a NICE FORM.

Thm 4.5 (Inverse function theorem) Let f be analytic on domain $D \subseteq \mathbb{C}$. If $f'(z_0) \neq 0$ for $z_0 \in D$, then there are open ngbds U of z_0 and V of $f(z_0)$ such that f restricts to a biholomorphism $U \to V$.



Prop 4.6 Let f be a non-constant analytic function on a Riem sfc R and let $p \in R$ be a zero of f. Then there is a chart (ϕ, U) about p with $\phi(p) = 0$ such that $f \circ \phi^{-1}(z) = z^m$ for some $m > 0$.



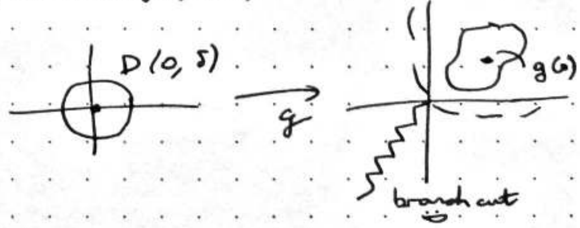
Proof Let $\psi : V \to \mathbb{C}$ be a chart about p with $\psi(p) = 0$. (WLOG)
There is analytic g s.t.

$$f \circ \psi^{-1}(z) = z^m g(z)$$

on a nbgd of 0. IDENTITY PRINCIPLE FOR RIEMANN SURFACES

$\Rightarrow$ f not locally constant $\Rightarrow$ $m > 0$ (more like $g(0) \neq 0$)

Since $g(0) \neq 0$, there is $\delta > 0$ s.t.



$$g(D(0,\delta)) \subseteq D(g(0), |g(0)|)$$

$\therefore$ there is an analytic m^{th} root on $g(D(0,\delta))$

Therefore $h(z) = z \sqrt[m]{g(z)}$ makes sense on $D(0, \delta)$,

so $f \circ \psi^{-1}(z) = (h(z))^m$

L4.4

Now differentiate h:

$$h'(z) = \sqrt[m]{g(z)} + z\frac{d}{dz}\left(\sqrt[m]{g(z)}\right)$$

so $h'(0) = \sqrt[m]{g(0)} \neq 0$

$\therefore$ by INVERSE FUNCTION THEOREM

h has an analytic inverse h^{-1} on some $D(0,\varepsilon)$ ✓

Let $\phi = h \circ \psi$ and $U = \phi^{-1}(D(0,\varepsilon))$

to get the required chart:

$$f \circ \phi^{-1}(z) = f \circ \psi^{-1} \circ h^{-1}(z)$$
$$= [h(h^{-1}(z))]^m = z^m$$

because $f \circ \psi^{-1} = h(z)^m$

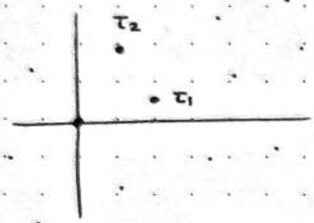
L 5.1

Complex Tori & the Open Mapping Theorem

5.1 Complex tori

So far, $\mathbb{C}_\infty$ is our only example of a COMPACT Riemann surface.

Ex 5.1 Let $\tau_1, \tau_2 \in \mathbb{C}_*$ s.t. $\frac{\tau_2}{\tau_1} \notin \mathbb{R}$

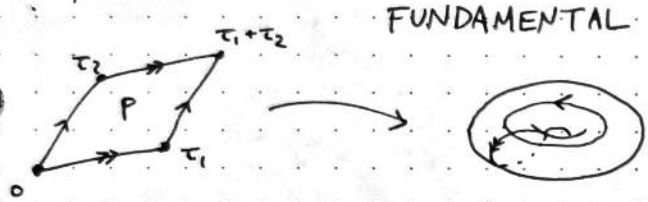


Let $\Lambda = \langle \tau_1, \tau_2 \rangle \leq \mathbb{C}$

and $T = \mathbb{C}/\Lambda$.

We can study the topology of T via the

FUNDAMENTAL PARALLELOGRAM P



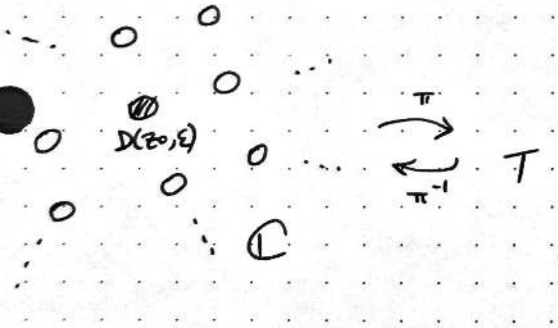
$$\therefore T \cong S^1 \times S^1$$

The quotient map $\pi : \mathbb{C} \to T = \mathbb{C}/\Lambda$

is a regular covering map.

If $\varepsilon < \frac{1}{2} \min \{ |\lambda| \mid \lambda \in \Lambda \setminus \{0\} \}$, then

$$\pi^{-1}(\pi(D(z,\varepsilon))) = \bigcup_{\lambda\in\Lambda} (D(z_0;\varepsilon) + \lambda) = (\bigsqcup_{\lambda \in \Lambda} D(z_0;\varepsilon) + \lambda) \cong D(z_0,\varepsilon) \times \Delta_\Lambda$$



We can now use π to construct an ATLAS on T.

For $p = \pi(z_0) = z_0 + \Lambda \in T$, let

$$\mathcal{U} = \pi(D(z_0,\varepsilon)) \quad \text{for } \varepsilon > 0 \text{ as before}$$

and let $\phi = (\pi|_{D(z_0,\varepsilon)})^{-1}$ to define a chart

$(\phi, \mathcal{U})$.

Given another chart $(\psi, V) = ((\pi|_{D(z_1,\varepsilon)})^{-1}, \pi(D(z_1,\varepsilon)))$,

the domains $\mathcal{U}$ and V overlap only if there is some (unique) $\lambda \in \Lambda$ s.t.

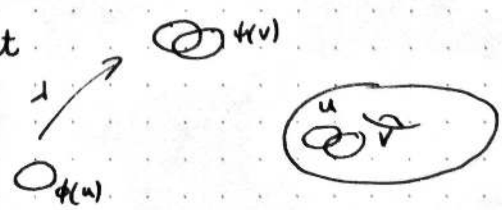
$|z_0 - (z_1 + \lambda)| < 2\varepsilon$

the bounds need tight

In this case, the transition function is just

$$z \mapsto z + \lambda$$

which is analytic!



<u>Conclusion</u> T admits a CONFORMAL STRUCTURE

All these tori are <u>homeomorphic</u>

However, it will turn out that this construction gives

∞-many conformal equivalence classes ☺ (see ExSh2Q5)

5.2 The open mapping theorem

We want to study complex functions $f: R \to \mathbb{C}$

The case when R is COMPACT is especially interesting.

These are governed by the OPEN MAPPING THEOREM.

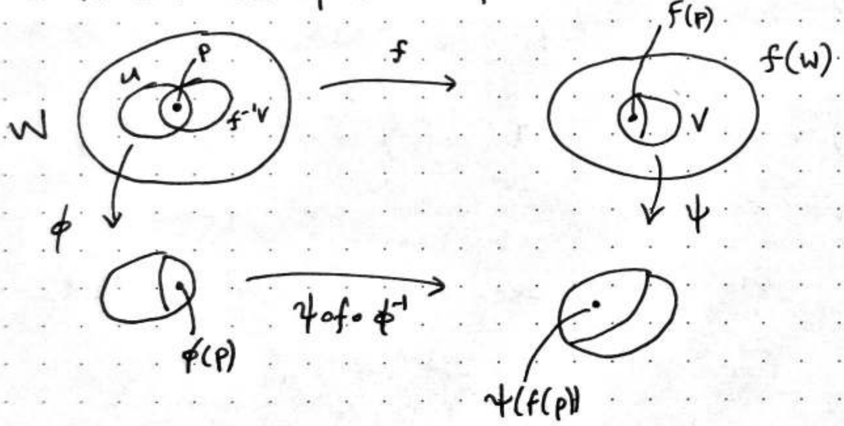
<u>Theorem 5.3</u> (Open mapping theorem for Riemann surfaces) Any ~~of~~ non-constant, analytic map of Riemann surfaces $f: R \to S$ is an open map.

<u>Proof</u> Again, id principle for Riemann surfaces

$\Rightarrow$ f is NOT locally constant

Now the proof is standard: work in charts and appeal to OMT from IB

Let $W \subseteq R$ be open and $p \in W$



Let (ϕ, U) and (ψ, V) be charts about p and $f(p)$ respectively

By OMT from Complex Analysis,

$\psi \circ f(U \cap W \cap f^{-1}V)$ is an open ngbd of $\psi \circ f(p)$

in $\psi(f(W) \cap V)$ so $f(U \cap W \cap f^{-1}V)$ is an open ngbd

of $f(p)$ in $f(W)$. $\square$

L5.3

The OMT is a big constraint on analytic maps from COMPACT Riemann surfaces

Corollary 5.4 Let $f: R \to S$ be a non-constant, analytic map of Riemann surfaces. If R is compact, then f is surjective, and in particular, S is also compact.

Proof OMT $\Rightarrow f(R)$ open

R cpct $\Rightarrow f(R)$ compact $\Rightarrow f(R)$ closed (Hausdorff)

S connected $\Rightarrow f(R) = S$ $\square$

In particular, since $\mathbb{C}$ is not compact, analytic functions are constrained

Corollary 5.5 Every analytic function on a cpct Riemann sfc is constant

5.3 Harmonic functions

This section is a digression

By OMT, a f^n $u: D \to \mathbb{R}$ can't be analytic. But it can be

Def 5.6 Let $D \subseteq \mathbb{C}$ be a domain. A smooth f^n $u: D \to \mathbb{R}$ is called harmonic if $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on D.

(Recall Δu is the Laplacian of u)

Note $z = x + iy$; $x, y \in \mathbb{R}$

Lemma 5.7 Consider a disc $D \subseteq \mathbb{C}$. A f^n $u: D \to \mathbb{R}$ is harmonic iff

$u = Re(f)$ for some analytic f^n f

Proof Let $f = u + iv$

($\Leftarrow$) The Cauchy-Riemann equations tell us

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$ as required

($\Rightarrow$) See Ex Sheet 1, q11 $\square$

With this, we can make sense of harmonic functions on an arbitrary Riemann surface.

<u>Def 5.8</u> Let R be a Riemann sfc. A function $u : R \to \mathbb{R}$ is harmonic if, for any chart (ϕ, U) on R the composition

$$u \circ \phi^{-1} : U \to \mathbb{R}$$

is harmonic.

<u>Lemma 5.7</u> implies that we can check "harmonicity" on any atlas $\mathcal{A}$ on R. Indeed, let $(\phi, U) \in \mathcal{A}$ and $u : \underset{R}{\cancel{R}} \to \mathbb{R}$ s.t. $u \circ \phi^{-1}$ is harmonic.

Then $u \circ \phi^{-1} = \mathrm{Re}\, f$ for analytic f,

so for any $(\psi, V) \in \hat{\mathcal{A}}$,

$$u \circ \psi^{-1} = (u \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = \mathrm{Re}\left(\underbrace{f \circ (\phi \circ \psi^{-1})}_{\text{analytic}} \right)$$

so $u \circ \psi^{-1}$ is also harmonic. $\square$

Facts about analytic f^{n}s often carry over to harmonic functions

For instance

<u>Proposition 5.10</u> (Identity principle for harmonic f^{n}s) Let u, v be harmonic on a Riemann sfc R. Either $u \equiv v$ or the set where they coincide $\{p \in R \mid u(p) = v(p)\}$ is discrete

<u>Proposition 5.11</u> (Open mapping theorem for harmonic f^{n}s) Any non-constant harmonic function u on a Riemann surface R is an open map

<u>Proof</u> The proof is very similar to the analytic case.

Let $W \subseteq R$ be open and $p \in W$. For small $p \in U \subseteq W$ there is a chart $\phi : U \to \mathbb{C}$ and analytic f s.t. $u \circ \phi^{-1} = \mathrm{Re}\, f$. As before, the id principle $\Rightarrow$ u locally non-constant

$\Rightarrow$ f non-constant

By the (IB) OMT, $f \circ \phi(U)$ is open.

Writing $f \circ \phi(p) = a + ib$, we have

$$(a-\delta, a+\delta) \times (b-\varepsilon, b+\varepsilon) \subseteq f \circ \phi(U)$$

for some $\delta, \varepsilon > 0$.

L5.5

In particular, $(a-\delta, a+\delta)$ is an open nbhd of $a = u(p)$ in $u(W)$, as required. $\square$

As in the analytic case, since $\mathbb{R}$ is non-compact, we deduce

<u>Cor 5.12</u> If R is a compact Riemann surface, all harmonic functions on R are constant

L 6.1 Meromorphic Functions

Last time, we saw that there are vetew analytic functions on compact Riemann surfaces. So it's more friutful to consider compact ranges.

Def 6.1 A meromorphic function on a Riemann surface is an analytic map $f : R \to \mathbb{C}_\infty$, where $\mathbb{C}_\infty$ is the Riemann sphere, which is not identically ∞.

The next result justifies the terminology.

Prop 6.2 Let $D \subseteq \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}_\infty$ is meromorphic iff there is a discrete subset $A \subseteq D$ such that $f : D \setminus A \longrightarrow \mathbb{C}$ is analytic, and f has a pole at each $a \in A$.

Proof "$\Rightarrow$" Let $A = f^{-1}(\infty)$.

A is discrete by the identity principle.

It remains to show each $a \in A$ is a pole.

Working in the standard chart on $\mathbb{C}_\infty$ about ∞, we see that

$$1/f(z) = (z-a)^m g(z)$$

on a ngbd of a, where $m \geq 1$ and $g(a) \neq 0$.

So $f(z) = (z-a)^{-m} h(z)$ on a (possibly smaller) ngbd, where $h = \frac{1}{g}$

$\therefore$ a is a pole!

"$\Leftarrow$" If a is a pole of f of order m, we have

$$f(z) = (z-a)^{-m} h(z)$$

on a ngbd of a, with $h(a) \neq 0$.

$\therefore$ on a possibly smaller ngbd we have

$$1/f(z) = (z-a)^m g(z)$$

meaning that f extends at a to an analytic map to $\mathbb{C}_\infty$. $\square$

L6.2 <u>A worked example</u>

We have already constructed Riemann surfaces associated to the multivalued functions $w = \log z$ and $w = \sqrt[k]{z}$.

In this section we treat another one $w = \sqrt{z^3 - z}$

There are several possible approaches. One is to put a conformal structure on the <u>graph</u> $\{(w,z) \in \mathbb{C}^2 \mid w^2 = z^3 - z\}$

(See Ex Sh 1, q 14) Today we will construct a Riem sfc by gluing.

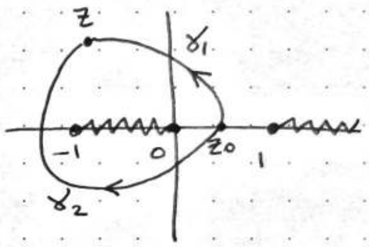
As with log, we need to write down some <u>function elts</u>.

Start with $f(z) = z^3 - z = z(z+1)(z-1)$, so $w = \sqrt{f(z)}$

We can find branches of $\sqrt{\cdot}$ locally unless $f(z) = 0 \iff z = 0, \pm 1$

We also run into trouble at ∞.

Therefore, we make some natural <u>branch cuts</u>.



We need to construct f^n elts g on D s.t.

$$g(z)^2 = f(z)$$

$D = \mathbb{C} \setminus ([-1,0] \cup [1,\infty))$

Use a <u>path integral</u>. Fix any $z_0 \in D$ and let $g(z_0) = w_0$ for some $w_0^2 = f(z_0)$.

Now set
$$g(z) = g(z_0) \exp\left(\frac{1}{2}\int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right)$$
for γ any choice of path from z_0 to z.

First, let's check this is <u>well-defined</u>. Consider γ a <u>loop</u> $\gamma_1 \cdot \overline{\gamma_2}$.

By the <u>ARGUMENT PRINCIPLE</u>,

winding no.s

$$\int_\gamma \frac{f'}{f}\, d\zeta = 2\pi i \left(\sum_{z_i \text{ zeros of } f} n(\gamma, z_i) - \sum_{p_j \text{ poles of } f} n(\gamma, p_j) \right)$$

$$= 2\pi i \left(n(\gamma(0) + n(\gamma, -1) + n(\gamma, 1) \right)$$

OMG WHAT A LAD

On ExSh1, q1 showed $n(\gamma, 1) = 0$, $n(\gamma, -1) = n(\gamma, 0)$

Therefore $\int_\gamma \frac{f'}{f}\, d\zeta = 4\pi i\, n(\gamma, 0) \in 4\pi i \mathbb{Z}$

times $\frac{1}{2}$, take exp $\Rightarrow$ $\exp\left(\frac{1}{2}\int_{\gamma_1} \frac{f'}{f}\, d\zeta\right) = \exp\left(\frac{1}{2}\int_{\gamma_2} \frac{f'}{f}\, d\zeta\right)$

$\implies g(z)$ is well-defined

L6.3

<u>continuity</u> is standard, since f'/f cts away from $0, \pm 1$

We may choose small disc $D(z,\delta)$ s.t.

$$\left| \frac{f'}{f}(\tau) - \frac{f'}{f}(z) \right| < 1 \quad \text{for all } \tau \in D(z,\delta)$$

$$\therefore \left| \int_z^\tau \frac{f'}{f} d\zeta \right| \leq \int_z^\tau \left| \frac{f'}{f} \right| d\zeta \leq |\tau - z| \xrightarrow{} 0 \quad \text{as } \tau \to z$$

const?

Hence $\int_{z_0}^z \frac{f'}{f}$ is cts and so $g(z) = \exp\left(1/2 \int_{z_0}^z \frac{f'}{f} d\zeta \right)$ is too.

<u>analyticity</u> is now clear; we constructed g so that

waffle?

$g(z)^2 = f(z)$ so locally $g(z) = \sqrt{f(z)}$ which is analytic

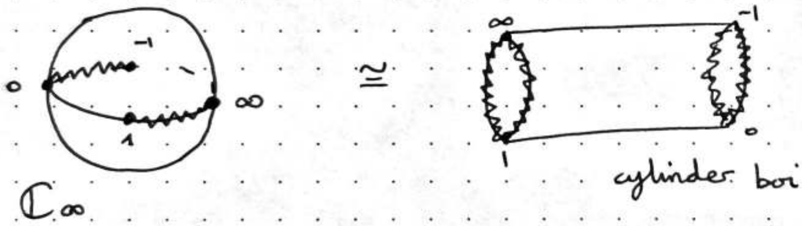
We made one choice: $g(z_0) = w_0$ where $w_0^2 = f(z_0)$

$\therefore$ we have two branches of g

Call them g_+, g_-; so we get two F^n <u>elements</u>

(g_+, D_+) and (g_-, D_-) ü

Let's discuss the topology of D: it's a cylinder $\cong S^1 \times \mathbb{R}$

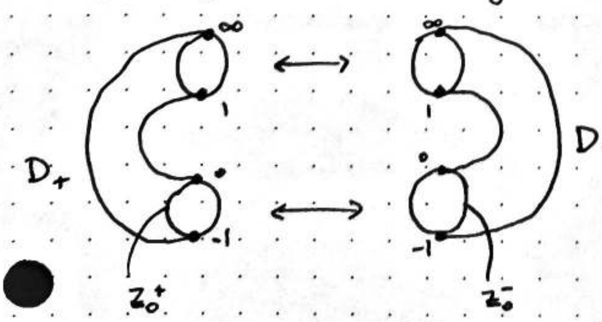


Now we construct a Riemann surface by gluing D_+ and D_- along the branch cuts. For $z_0 \in (-1,0) \cup (1,\infty)$, note that

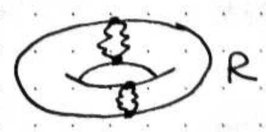
$$\lim_{z \to z_0^-} g_+(z) = \lim_{z \to z_0^+} g_-(z) \quad , \quad \lim_{z \to z_0^+} g_+(z) = \lim_{z \to z_0^-} g_-(z_0)$$

where $z \to z_0^+$ denotes approaching z_0 from the <u>upper</u> half plane, and $z \to z_0^-$ denotes approaching from the <u>lower</u> half plane.

This tells us how to glue:



The result is a <u>torus</u> with 4 points removed



Later will learn how to identify surfaces we can't visualise

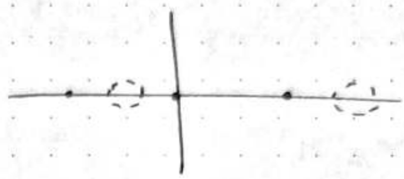
L6.4

CAREFUL

Strictly we also need to exhibit charts about points in the branch cuts.

For instance, we can use small discs

We'll usually suppress this detail.



As in previous examples, we get analytic functions with nice properties

- g_+ and g_- together define $g : R \to \mathbb{C}$
- inclusions $D_\pm \hookrightarrow \mathbb{C}$ define a covering map $\pi : R \to \mathbb{C}$

Locally, hence globally, $g(p)^2 = f \circ \pi(p) = \pi(p)^3 - \pi(p)$

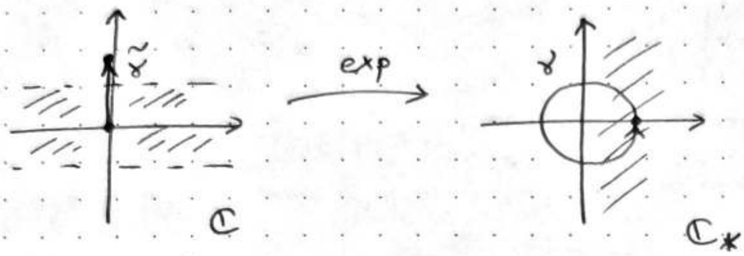
Covering - Space Theory

Def

Def^n Suppose $\pi : \tilde{X} \to X$ is a covering map and $\gamma : [0,1] \to X$ a path.

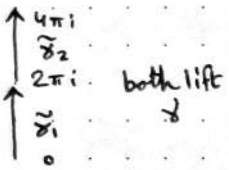
A lift of γ along π is a path $\tilde{\gamma} : [0,1] \to \tilde{X}$ s.t. $\pi \circ \tilde{\gamma} = \gamma$

e.g



[NOT TO SCALE]

Lifts are not usually unique, e.g.



However, lifts are determined by their start points

Prop (Uniqueness of lifts) Suppose $\tilde{\gamma}_1, \tilde{\gamma}_2$ both lift γ along π.

If $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ then $\tilde{\gamma}_1 = \tilde{\gamma}_2$

Proof Follows from

Claim $I = \{t \in [0,1] : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$

is both open and closed

Since $0 \in I$ by hypothesis and $[0,1]$ is connected $\Rightarrow [0,1] = I$.

I closed True for very general reasons!

Consider $\tilde{\gamma}_1 \times \tilde{\gamma}_2 : I \to \tilde{X} \times \tilde{X}$; $t \mapsto (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$

NOTE $I = (\tilde{\gamma}_1 \times \tilde{\gamma}_2)^{-1}(\tilde{X})$ where

$\tilde{X} \hookrightarrow \tilde{X} \times \tilde{X}$ is the diagonal

But $\tilde{X}$ Hausdorff $\iff$ the diagonal $\tilde{X} \subseteq \tilde{X} \times \tilde{X}$ is closed

$\therefore I$ is closed

I open Let $t \in I$. Since π is a covering map, $\tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)$ has a ngbd $\tilde{N}$ s.t. $\pi|_{\tilde{N}} : \tilde{N} \to N$ is a homeomorphism.

Since γ is continuous, there is $\delta > 0$ s.t.

$$\gamma((t-\delta, t+\delta)) \subseteq N.$$

want $\tilde{\gamma}_i((t-\delta, t+\delta)) \subseteq \tilde{N}$ for $i = 1,2$

L7.2

For $t-\delta < s < t+\delta$, then

hmm... $\tilde{\gamma}_1, \tilde{\gamma}_2$ in $\tilde{N}$

$$\pi \circ \tilde{\gamma}_1(s) = \gamma(s) = \pi \circ \tilde{\gamma}_2(s).$$

Since π is invertible on $\tilde{N}$,

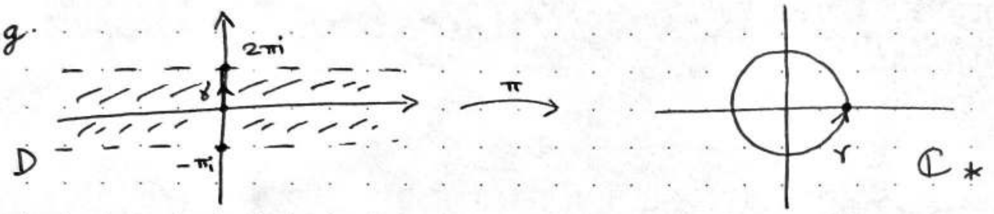
$$\tilde{\gamma}_1(s) = (\pi|_{\tilde{N}})^{-1} \circ \gamma(s) = \tilde{\gamma}_2(s)$$

$\therefore (t-\delta, t+\delta) \subseteq I$, so I is open as required. □

What about EXISTENCE?

Even if π is surjective, lifts may not exist

e.g.



$D = \{-\pi < \operatorname{im} z < 2\pi\}$, $\pi = \exp|_D$

For a positive result, need to work with <u>regular</u> coverings

<u>Prop</u> (Path-lifting lemma) Let $\pi : \tilde{X} \to X$ be a regular covering map.

Let $\gamma : [0,1] \to X$ be a path, and suppose that $\pi(\tilde{x}) = \gamma(0)$.

Then there is a (unique) lift of γ s.t. $\tilde{\gamma}(0) = \tilde{x}$.

<u>Proof</u> Let $I = \{ t \in I[0,1] : \exists \text{ lift } \tilde{\gamma} \text{ of } \gamma|_{[0,t]} \text{ with } \tilde{\gamma}(0) = \tilde{x} \}$

Again, since $0 \in I$ by hypothesis, the result follows if we can prove that I is both open and closed.

<u>I closed</u> Consider $t_n \to \tau$, $t_n \in I$

$*$ { WLOG t_n in intervals s.t. $\gamma(J) \subset U$ }

π regular $\Rightarrow \gamma(\tau) \in U$ s.t.

$$\pi^{-1}(U) \cong \coprod_{\delta \in D} U_\delta$$

Wlog, $\gamma(t_n) \in U$ for all n*. In particular, $\tilde{\gamma}(t_n)$ are all in the same path component $U_\delta \subseteq \pi^{-1}(U)$.

$\therefore$ setting $\tilde{\gamma}(\tau) = (\pi|_{U_\delta})^{-1} \circ \gamma(\tau)$

extends $\tilde{\gamma}$ continuously over τ, so $\tau \in I$

<u>I open</u> Let $\tau \in I$, and let

$$\gamma(\tau) \in U \text{ s.t. } \pi^{-1}(U) \cong \coprod_{\delta \in \Delta} U_\delta$$

Then there is a unique δ s.t. $\tilde{\gamma}(\tau) \in U_\delta$

Since γ is continuous, there is $\varepsilon > 0$ s.t.

$$|t-\tau| < \varepsilon \Rightarrow \gamma(t) \in U$$

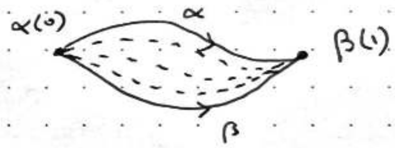
By uniqueness of lifts, $\tilde{\gamma}(t) = (\pi|_{U_\delta})^{-1} \circ \gamma(t)$

whenever $|t-\tau| < \varepsilon$ and $t \in I$.

In particular, $(\pi|_{U_\delta})^{-1} \circ \gamma$ defines a cts extension of $\tilde{\gamma}$ over $(t-\varepsilon, t+\varepsilon)$.

$\therefore$ I is open as required □

<u>Def^n</u> Let X be a top space and $\alpha, \beta : [0,1] \to X$ paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. This pair of paths is <u>homotopic</u> (conveniently $\alpha \simeq \beta$) if there exists a family of paths $(\alpha_s)_{s \in [0,1]}$ s.t.



(i) $\alpha_0 = \alpha$, $\alpha_1 = \beta$

(ii) $\alpha_s(0) = \alpha(0)$, $\alpha_s(1) = \alpha(1)$ $\forall s$

(iii) the map $(t,s) \mapsto (\alpha_s(t))$ is continuous $[0,1]^2 \to X$

<u>Def^n</u> A topological space X is <u>simply connected</u> if

(i) X is path-connected

(ii) every pair of points $\alpha, \beta : [0,1] \to X$ with the same endpoints,
i.e. $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$ is homotopic

<u>Rk</u> Let $D \subseteq \mathbb{C}$ be a convex domain. The formula

$$\alpha_s(t) = (1-s)\alpha(t) + s\beta(t)$$

defines a homotopy between any two paths α, β in D with equal endpoints, so D is simply connected.

<u>Ex</u> $\mathbb{C}$, $\mathbb{D}$, half spaces $\subseteq \mathbb{C}$ are all simply connected

Theorem 7.9 (Monodromy theorem)

Let $\pi : \tilde{X} \to X$ be a covering map; α, β paths in X. Suppose

(i) $\alpha \simeq \beta$ in X

(ii) $\exists$ lifts $\tilde{\alpha}$ of α, $\tilde{\beta}$ of β with $\tilde{\alpha}(0) = \tilde{\beta}(0)$

(iii) every path γ in X with $\gamma(0) = \alpha(0) = \beta(0)$ has a lift to $\tilde{X}$ with $\tilde{\gamma}(0) = \tilde{\alpha}(0) = \tilde{\beta}(0)$

Then $\tilde{\alpha} \simeq \tilde{\beta}$, in particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$

Proof Omitted, but cf the homotopy lifting lemma in Alg Top $\square$

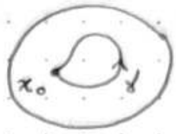
Note (ii), (iii) are automatically satisfied if π is regular

L 8.1

The Monodromy Group

Let $\pi : \tilde{X} \to X$ be a regular covering map.

Pick a basepoint $x_0 \in X$.



Any choice of loop

$\gamma : [0,1] \to X$

s.t. $\gamma(0) = \gamma(1) = x_0$

defines a permutation $\sigma_\gamma : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$

<u>Def</u> Let $\tilde{x} \in \pi^{-1}(x_0)$.

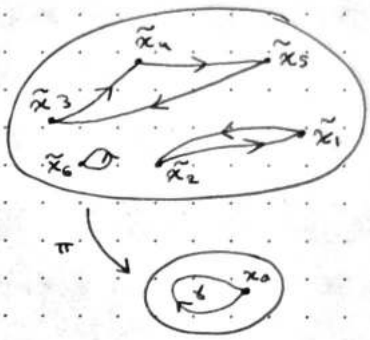
Let $\tilde{\gamma}_{\tilde{x}}$ be the unique lift of γ starting at $\tilde{x}$.

Then

$$\pi(\tilde{\gamma}_{\tilde{x}}(1)) = \gamma(1) = x_0$$

so $\tilde{\gamma}_{\tilde{x}}(1) \in \pi^{-1}(x_0)$.

So define $\sigma_\gamma(\tilde{x}) = \tilde{\gamma}_{\tilde{x}}(1)$



<u>Remarks</u>

(i) The constant loop $t \mapsto x_0$

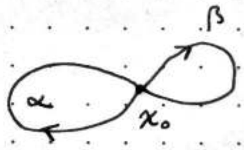
defines the identity permutation.

(ii) Let $\bar{\gamma}(t) = \gamma(1-t)$. If $\tilde{\gamma}_{\tilde{x}_1}$ ends at $\tilde{x}_2$ then $\tilde{\bar{\gamma}}_{\tilde{x}_2}$ ends at $\tilde{x}_1$,

so $\sigma_{\bar{\gamma}} = \sigma_\gamma^{-1}$.

(iii) The <u>concatenation</u> of loops α, β is

$$\alpha.\beta(t) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & : \frac{1}{2} \leq t \leq 1. \end{cases}$$

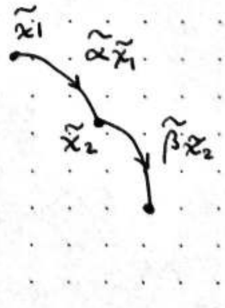


If $\tilde{\alpha}_{\tilde{x}_1}$ is a lift of α with $\tilde{\alpha}_{\tilde{x}_1}(1) = \tilde{x}_2$, then

$(\widetilde{\alpha.\beta})_{\tilde{x}_1} = \tilde{\alpha}_{\tilde{x}_1} . \tilde{\beta}_{\tilde{x}_2}$, by uniqueness of lifts

In particular,

$$\sigma_{\alpha.\beta}(\tilde{x}_1) = \sigma_\beta(\tilde{x}_2) = \sigma_\beta \circ \sigma_\alpha(\tilde{x}_1)$$



L 8.2

Together, (i)–(iii) imply that

$\{ \sigma_\gamma : \gamma \text{ loop based at } x_0 \}$

form a subgroup of $\mathrm{Sym}(\pi^{-1}(x_0))$. This is called the

<u>monodromy group</u> of π. ⌜well def?⌟

Furthermore:

(iv) By the monodromy theorem,

$$\alpha \simeq \beta \Rightarrow \sigma_\alpha = \sigma_\beta$$

(v) The monodromy group is independent of the choice of basepoint (Exe)

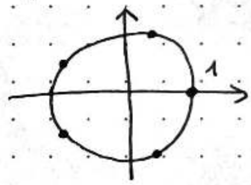
<u>Example 8.1</u>

Recall that $p_k : \mathbb{C}_* \longrightarrow \mathbb{C}_*$
$\qquad z \longmapsto z^k$

is a regular covering map. Take base point 1,

so $\pi^{-1}(1) = \{ k^{th} \text{ roots of unity} \}$

$= \{ \zeta_k^n : n \in \mathbb{Z}/k\mathbb{Z} \}$



Let $\gamma(t) = e^{2\pi i t}$.

For each n, $\tilde{\gamma}_{\zeta_k^n}(1) = \zeta_k^{n+1}$

$\therefore \sigma_\gamma(\zeta_k^n) = \zeta_k^{n+1}$

<u>FACT</u> Every loop in $\mathbb{C}_*$ based at 1 is homotopic to γ^n for some $n \in \mathbb{Z}$.

(No need to justify).

<u>Conclusion</u> The monodromy group of p_k is the cyclic group C_k

See Ex Sheet 2 q9 for a similar computation

L8.3

The space of germs

$D \subseteq \mathbb{C}$ a domain

<u>Def 8.2</u> Let (f, U), (g, V) be function elements on D. For any $z \in \underset{U \cap V}{\cancel{D \cap E}}$, write $(f, U) \equiv_z (g, V)$ if f and g agree on a ngbd of z.

<u>Note</u> $\equiv_z$ is an equivalence relation, so

<u>Def 8.3</u> (Germ) Let (f, U) be a function element and $z \in U$. The equivalence class of (f, U) under $\equiv_z$ is called the germ of f at z, and is denoted $[f]_z$. In summary, two germs $[f]_z = [g]_w$ if and only if $z = w$ and $f = g$ on a ngbd of $z = w$.

To understand analytic continuation, we want to study all possible germs on the domain D.

<u>Def 8.4</u> (space of germs) The space of germs over D is

$$\mathcal{G} =: \{ [f]_z \mid z \in D,\ (f, U) \text{ function elt with } z \in U \}$$

as a set.

Our next task is to endow $\mathcal{G}$ with a <u>topology</u>, and ultimately, a conformal structure.

<u>Topology</u> For any f^n elt (f, U) on D, let

$$[f]_U =: \{ [f]_z \mid z \in U \}$$

The open sets of $\mathcal{G}$ are defined to be all unions of sets of the form $[f]_U$ across any set of f^n elements on D.

<u>Lemma 8.5</u> $[f]_U$ define a topology on $\mathcal{G}$

<u>Proof</u> Suffices to check finite intersections.

Want $[f]_U \cap [g]_V$ to be open.

Consider $[h]_z \in [f]_U \cap [g]_V$

want W domain

This means h agrees with f and g on a ngbd W of z.

$\therefore [h]_z \in [h]_W \subseteq [f]_U \cap [g]_V$ & the result follows. $\square$

L8.4

Next we should check that the topology on $\mathcal{G}$ is nice

Lemma 8.6 The space of germs $\mathcal{G}$ is Hausdorff

Proof Consider $[f]_z \neq [g]_w \in \mathcal{G}$

There are two cases.

(i) $z \neq w$ Then there are $(f, U) \in [f]_z$; $(g, V) \in [g]_w$

s.t. $U \cap V = \emptyset$ and hence $[f]_U$ and $[g]_V$ are disjoint

(ii) $z = w$ In this case, we choose a (connected) ngbd U s.t.

$(f, U) \in [f]_z$ and $(g, U) \in [g]_z$

Unless $[f]_U \cap [g]_{\cancel{z}U} = \emptyset$; there is a germ

$[h]_{\cancel{z}}^{w} \in [f]_U \cap [g]_U$

Therefore, by the identity principle;

$f|_U = h|_U = g|_U$

so $[f]_z = [h]_z = [g]_z$ ⋇ □

Conformal Structure

As in previous examples, we use a covering map.

Def 8.7 Let $\mathcal{G}$ be the space of germs over a domain D. The forgetful map $\pi : \mathcal{G} \to D$ is defined by $[f]_z \mapsto z$

Lemma 8.8 For each component $G \subseteq \mathcal{G}$, the restriction of the forgetful map $\pi : G \to D$ is a covering map

Proof continuity Let $U \subseteq D$ be open. Then

$$\pi^{-1}(U) = \bigcup_{\substack{(f,V) \\ \text{on } U}} [f]_V$$ is open as required

local homeo^n The open sets $\{[f]_U\}$ cover $\mathcal{G}$ by def^n

For each such open set $\pi|_{[f]_U}$ has inverse

$$(\pi|_{[f]_U})^{-1}(z) = [f]_z$$

This inverse is continuous since the preimage of an open set $[g]_V$ is just $U \cap V$, which is open.

So π is indeed a local homeomorphism. □

L8.5

By Lemma 4.1, the forgetful map π induces a unique conformal structure on $\mathcal{G}$ s.t. π is analytic

Explicitly, the charts are of the form

$$(\pi|_{[f]_U}, [f]_U)$$

across all function elements (f, U) on D.

<u>Def 8.9</u> Let $\mathcal{G}$ be the space of germs on a domain D. The <u>evaluation map</u> $\mathcal{E}: \mathcal{G} \to \mathbb{C}$ is defined by $\mathcal{E}([f]_z) = f(z)$.

In a standard chart $(\pi|_{[f]_U}, [f]_U)$
this takes the form

$$\mathcal{E} \circ (\pi|_{[f]_U})^{-1}(z) = \mathcal{E}([f]_z) = f(z)$$

Since f is analytic, it follows that $\mathcal{E}$ is <u>analytic</u>

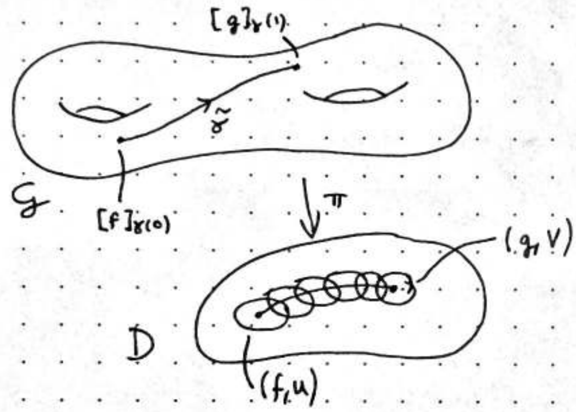
<u>Technical Remark</u> Riemann surfaces are assumed to be connected

$\therefore$ $\mathcal{G}$ not a Riemann surface, but its components $G \subseteq \mathcal{G}$ are

Analytic continuation revisited

The main theorem of this section shows that the space of germs controls analytic continuation.

Theorem 9.1 Let (f, U) and (g, V) be function elements on a domain $D \subseteq \mathbb{C}$, and let $\gamma : [0,1] \to D$ be a path starting in U and ending in V. Then $(f,U) \approx_\gamma (g,V)$ iff the lift $\tilde{\gamma}$ to (a component of $\mathcal{G}$) starting at $[f]_{\gamma(0)}$ exists, and ends at $[g]_{\gamma(1)}$



Proof "$\Rightarrow$" If $(f,U) \approx_\gamma (g,V)$, there are

$$(f,U) = (f_1, U_1) \sim \dots \sim (f_n, U_n) = (g, V)$$

and a dissection $0 = t_0 < t_1 < \dots < t_n = 1$

s.t. $\gamma[(t_{i-1}; t_i)] \subseteq U_i$ for all $1 \le i \le n$.

Define a lift of $\mathcal{G}$ via

$$\tilde{\gamma}(t) = [f_i]_{\gamma(t)}$$

for $t_{i-1} \le t \le t_i$.

First, $\tilde{\gamma}$ is well defined, since

$$(f_i, U_i) \sim (f_{i+1}, U_{i+1})$$

so $f_i|_{U_i \cap U_{i+1}} = f_{i+1}|_{U_i \cap U_{i+1}}$

$\Rightarrow [f_i]_{\gamma(t_i)} = [f_{i+1}]_{\gamma(t_i)}$ since $\gamma(t_i) \in U_i \cap U_{i+1}$

Continuous since

$$\tilde{\gamma}(t) = (\pi|_{[f_i]U_i})^{-1} \circ \gamma(t) \quad \text{on} \quad [t_{i-1}, t_i]$$

Lift since

$$\pi \circ \tilde{\gamma}(t) = \pi([f_i]_{\gamma(t)}) = \gamma(t)$$

for each $t \in [t_{i-1}, t_i]$.

By construction

$$\tilde{\gamma}(0) = [f_1]_{\gamma(0)} = [f]_{\gamma(0)}$$

AND

$$\tilde{\gamma}(1) = [f_n]_{\gamma(1)} = [g]_{\gamma(1)} \quad \text{as required!}$$

"$\Leftarrow$" Suppose such a lift $\tilde{\gamma}$ exists.

By the def^n of the topology on $\mathcal{G}$, every $\tilde{\gamma}(t)$ has a ngbd

$[f_t]_{U_t}$ where (f_t, U_t) a f^n element on D.

Furthermore, we may assume each U_t is a disc

Compactness of $[0,1]$

$\Rightarrow$ we may choose finitely many

$(f_1, U_1), \dots, (f_n; U_n)$

and a dissection $0 = t_0 < t_1 < \dots < t_n = 1$

s.t. $\tilde{\gamma}([t_{i-1}, t_i]) \subseteq [f_i]_{U_i}$ for each $1 \leq i \leq n$.

Since $\tilde{\gamma}$ is a lift,

$$\gamma([t_{i-1}; t_i]) = \pi \circ \tilde{\gamma}([t_{i-1}, t_i])$$
$$\subseteq \pi([f_i]_{U_i}) = U_i \quad \text{as required}$$

Finally, it remains to prove that

$(f_{i-1}, U_{i-1}) \sim (f_i, U_i)$ for each $1 < i \leq n$

Indeed,

$$[f_{i-1}]_{\gamma(t_{i-1})} = \tilde{\gamma}(t_{i-1}) = [f_i]_{\gamma(t_{i-1})}$$

so f_{i-1} and f_i agree on a ngbd of $\gamma(t_{i-1}) \in U_{i-1} \cap U_i$

But $U_{i-1} \cap U_i$ is connected (since U_i discs), so $f_{i-1}|_{U_{i-1} \cap U_n} = f_i|_{U_{i-1} \cap U_i}$

by the identity principle, as required. ☺ $\square$

L9.2

<u>Cor 9.2</u> Let $\mathcal{F}$ be a complete analytic function on a domain $D \subseteq \mathbb{C}$.

Then $\mathcal{G}_{\mathcal{F}} := \bigcup_{(f,U) \in \mathcal{F}} [f]_U$ is a path component of $\mathcal{G}$.

<u>Moral:</u> Complete analytic functions on a domain $D \subseteq \mathbb{C}$ are equivalent to Riemann surfaces R equipped with covering maps $R \to D$.

<u>Def 9.3</u> The component $\mathcal{G}_{\mathcal{F}}$ is the <u>Riemann surface associated to $\mathcal{F}$</u>

<u>§9.2 The classical monodromy theorem</u>

We are finally ready to address uniqueness of analytic continuation

<u>Theorem 9.4</u> (Classical monodromy theorem) Let $D \subseteq \mathbb{C}$ be a domain. Suppose that (f, U) is a function element in D and can be continued along any path in D starting in U. If $(f, U) \approx_\alpha (g_1, V)$ and $(f, U) \approx_\beta (g_2, V)$ and $\alpha \simeq \beta$ then $g_1 \equiv g_2$ on V.

<u>Moral</u> Analytically continuing along a path only depends on the <u>homotopy class</u>

<u>Proof</u> Let $\tilde{\alpha}, \tilde{\beta}$ be the lifts of α and β to $\mathcal{G}$ s.t.

$$\tilde{\alpha}(0) = [f]_{\alpha(0)} = \tilde{\beta}(0)$$

Since $\alpha \simeq \beta$, we also have

$\tilde{\alpha} \simeq \tilde{\beta}$ by the monodromy theorem (regular map?).

In particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$, i.e. $[g_1]_{\alpha(1)} = [g_2]_{\beta(1)}$

so $g_1 \equiv g_2$ on V by the identity principle. □

It immediately follows that, on simply connected domain, analytic continuation is unique.

<u>Cor 9.5</u> Let D be a simply connected domain and (f, U) a function element on D. If (f, U) can be analytically continued along every path in D then (f, U) extends to an analytic function $f : D \to \mathbb{C}$.

L9.4 § 9.3 Gluing Riemann Surfaces

When studying k^{th} roots, we constructed

$$\begin{tikzcd} R_k \arrow[r, "\theta"] \arrow[dr, "\pi"'] & \mathbb{C}_* \arrow[d, "p_k"] \\ & \mathbb{C}_* \end{tikzcd}$$

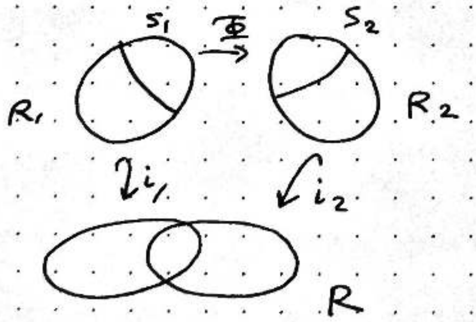
We also said that this can be compactified

$$\begin{tikzcd} \hat{R}_k \arrow[r, "\hat{g}"] \arrow[dr, "\hat{\pi}"'] & \mathbb{C}_\infty \arrow[d, "p_k"] \\ & \mathbb{C}_\infty \end{tikzcd}$$

How would we do this in practice?

Answer: <u>Def 9.6</u> Let X, Y be topological spaces. Suppose we have subspaces $X' \subseteq X$, $Y' \subseteq Y$ and a homeomorphism $\Phi : X' \to Y'$. The quotient space $Z := (X \sqcup Y)/\sim$
where $\sim$ is the smallest equiv relation such that $x \sim \Phi(x)$ for all $x \in X'$, is called the result of <u>gluing X and Y along Φ</u>. It may also sometimes be denoted by $X \cup_\Phi Y$, or even by $X \cup_{X'} Y$ if Φ is implicit.

In this course, we need to understand when the result of gluing gives a Riemann surface: <u>Prop 9.7</u> Let R_1, R_2 be Riemann surfaces. Suppose $S_j \subseteq R_j$ are non-empty, connected, open subsets and $\Phi : S_1 \to S_2$ is a conformal equivalence of Riemann surfaces. There is a unique conformal structure on $R = R_1 \cup_\Phi R_2$ such that the inclusion maps $i_j : R_j \hookrightarrow R$ are analytic. In particular, if the resulting gluing R is Hausdorff then it is a Riemann surface.



<u>Proof</u> First, exhibit an atlas:
(ϕ_j, U_j) a chart on R_j gives $(\phi_j \circ i_j^{-1}, i_j(U_j))$ a chart on R.
Transition f^ns are <u>either</u> transition functions of R_j <u>or</u>

$$\phi_2 \circ i_2^{-1} \circ i_1 \circ \phi_1^{-1} = \phi_2 \circ \Phi \circ \phi_1^{-1}$$

which is analytic because Φ is, $\therefore$ this is indeed an atlas.

Uniqueness If (ϕ_j, U_j) is a chart on R_j and (ψ, V) is a chart on R then i_j analytic means that

- $\psi \circ i_j \circ \phi_j^{-1}$ is analytic (& analytic inverse)

$\therefore$ $\phi_j \circ i_j^{-1}$ has analytic transition functions with all charts on R, so by minimality, is a chart on R

Riemann Surface

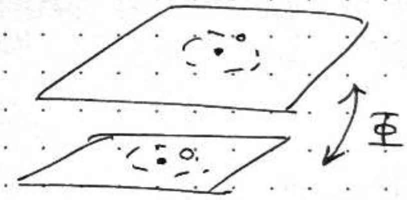
Finally, note that

$$\left.\begin{array}{c} R_1, R_2 \text{ connected} \\ + \\ S_1 \cong S_2 \neq \phi \end{array}\right\} \Rightarrow R \text{ connected}$$

Since R is assumed Hausdorff, it is a Riemann surface $\square$

- It's easy to accidentally construct non-Hausdorff spaces by gluing.

Ex 9.8 Let $R_1 = R_2 = \mathbb{C}$, let $S_1 = S_2 = \mathbb{C}_*$ and let $\Phi : S_1 \to S_2$ be the identity map. The gluing $R = \mathbb{C} \cup_{\mathbb{C}_*} \mathbb{C}$ is not Hausdorff, since the two copies of 0 do not have disjoint ngbds.



$\mathbb{C} \cup_{\mathbb{C}_*} \mathbb{C}$

The easiest example of compactification by gluing is the Riemann sphere

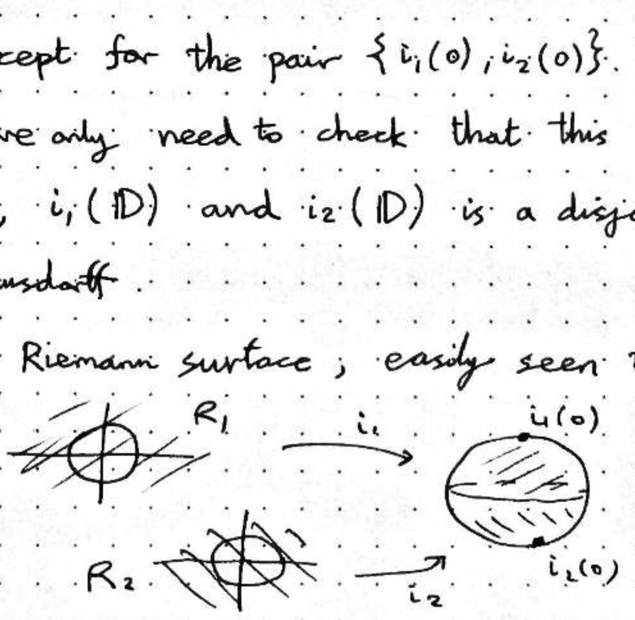
- Ex 9.9 Let $R_1 = R_2 = \mathbb{C}$, let $S_1 = S_2 = \mathbb{C}_*$ and let $\Phi : S_1 \to S_2$ be the inversion map $z \mapsto 1/z$. Every pair of points in the gluing

$$R = \mathbb{C} \cup_{\Phi} \mathbb{C}$$

is contained in either R_1 or R_2, except for the pair $\{i_1(0), i_2(0)\}$. Therefore, to check that R is Hausdorff, we only need to check that this pair have disjoint open ngbds. Indeed, $i_1(\mathbb{D})$ and $i_2(\mathbb{D})$ is a disjoint pair of open ngbds; so R is Hausdorff.

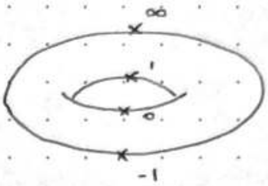
By Prop 9.7, R is therefore a Riemann surface, easily seen to be

- the Riemann sphere $\mathbb{C}_\infty$.



More glue $\leadsto$ compactifying $w^2 = z^3 - z$

In lecture 6, we constructed a Riemann surface R associated to $w^2 = z^3 - z$ and saw it is homeomorphic to $T^2 \setminus \{4 \text{ pts}\}$ as shown.



Goal Compactify R by adding 4 points

In Example Sheet 1, q. 14, you saw that the graph

$$R_1 = \{(z,w) \in \mathbb{C}^2 \mid w^2 = z^3 - z\}$$

is CONFORMALLY EQUIVALENT to $R \cup \{-1, 0, 1\}$.

So it remains to compactify R_1 by adding in ∞.

Ex 10.1 As in $\mathbb{C}_\infty$, perform a change of coordinates to send ∞ to a finite point: $u = \frac{1}{z}$ and $v = \frac{z}{w}$

$$\Longleftrightarrow \quad z = \frac{1}{u}, \quad w = \frac{z}{v} = \frac{1}{uv}$$

Eq^n becomes $\left(\frac{1}{uv}\right)^2 = \frac{1}{u^3} - \frac{1}{u} \iff \boxed{u = v^2(1-u^2)}$ #magic

$\therefore$ set $R_2 := \{(u,v) \in \mathbb{C}^2 : u = v^2(1-u^2)\}$

and let $\pi : R_2 \to \mathbb{C}$, $(u,v) \mapsto u$ and $\tau : R_2 \to \mathbb{C}$, $(u,v) \mapsto v$

be the coordinate projections.

Fact* Restrictions of π and τ define an atlas on R_2.

The maps π and τ are analytic functions on R_2.

We now want to compactify by gluing R_1 to R_2.

The gluing map is $\Phi(z,w) := (u,v) = \left(\frac{1}{z}, \frac{z}{w}\right)$

with

$$\Phi^{-1}(u,v) = (z,w) = \left(\frac{1}{u}, \frac{1}{uv}\right)$$

These are conformal equivalences, since the coordinates are analytic.

In particular, $S_1 = \operatorname{dom}(\Phi) = R_1 \setminus \{(0,0), (\pm 1, 0)\}$,

and $S_2 = \operatorname{dom}(\Phi^{-1}) = R_2 \setminus \{(0,0)\}$.

Therefore,

$$\hat{R} := R_1 \cup_{\Phi} R_2$$

is connected, and has a conformal structure by Prop$^{\text{n}}$ 9.7. Before we check Hausdorff, first note that

$$\hat{\pi}_1 : R_1 \longrightarrow \mathbb{C}_\infty,\ (z,w) \mapsto z$$
and
$$\hat{\pi}_2 : R_2 \longrightarrow \mathbb{C}_\infty,\ (u,v) \mapsto \frac{1}{u}$$

satisfy $\hat{\pi}_1 = \hat{\pi}_2 \circ \Phi$, $\therefore$ they define a <u>meromorphic function</u> $\hat{\pi} : \hat{R} \to \mathbb{C}_\infty$

To prove that $\hat{R}$ is Hausdorff, it now suffices to separate

$$\hat{R} \setminus i_2(R_2) = \{ i_1(0,0),\ i_1(\pm 1, 0))$$

from $\hat{R} \setminus i_1(R_1) = \{ i_2(0,0) \}$

The open sets

$$\hat{\pi}^{-1}(\{|z| < 2\}) \quad \text{and} \quad \hat{\pi}^{-1}(\{|z| > 2\} \cup \{\infty\})$$

do the job!

(Sequential) <u>Compactness</u>

Consider a sequence $(p_n)_{n \geq 0}$ in $\hat{R}$.

After passing to a subsequence, there are two cases.

(i) If $|\hat{\pi}(p_n)| \leq M$ then, after passing to a further subsequence, $\hat{\pi}(p_n) \to z_0$. But there are ≤ 2 values $w_0 \in \mathbb{C}$ s.t. $(z_0, w_0) \in \hat{R}$, $\therefore$ some subsequence p_n converges to one of them

(ii) Alternatively, $|\hat{\pi}(p_n)| \to \infty$ as $n \to \infty$.

But $p_\infty = i_2(0,0)$ is the only point of $\hat{R}$ with $\hat{\pi}(p_\infty) = \infty$.

Therefore(!) $p_n \longrightarrow p_\infty$ as $n \to \infty$.

In either case, $\exists$ convergent subseq so is compact.

In summary, we have embedded $R \hookrightarrow \hat{R}$ a <u>compact torus</u>; and π extends to a meromorphic function $\hat{\pi}$.

Similarly, g (which is just z) extends to a meromorphic $\hat{g} : \hat{R} \to \mathbb{C}_\infty$.

10.3 Branching

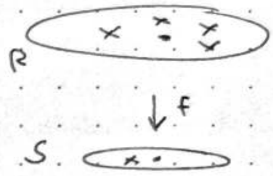
So far, covering maps have played an important role. But, when we compactify, analytic maps often lose the covering property.

Ex $p_k : \mathbb{C}_* \longrightarrow \mathbb{C}_*$ is a covering map
$$z \longmapsto z^k$$

But $\hat{p}_k : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is not $(k \geq 2)$

Def 10.2 Let $f : R \to S$ be analytic. For $p \in R$, recall that we can find charts that put f into a standard local form:

$$\psi \circ f \circ \phi^{-1}(z) = z^m \text{ for some } m \in \mathbb{Z}$$

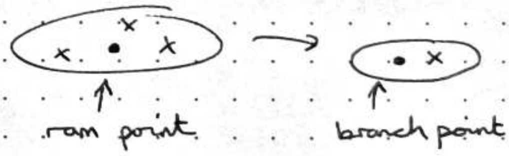


The integer $m =: m_f(p)$ is the multiplicity of f at p

Most points have $m_f(p) = 1$.

The remaining points are especially interesting.

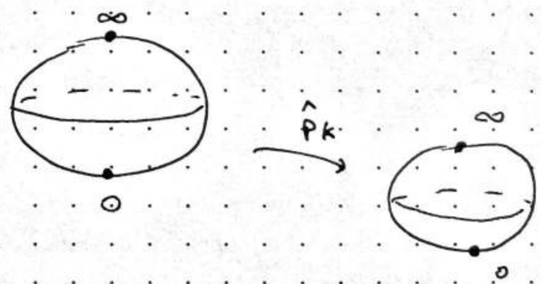
Def 10.3 If $m_f(p) > 1$ then p is called a ramification point, and $f(p)$ is called a branch point. In this case, the multiplicity $m_f(p)$ is also called the ramification index of p.



Ex 10.4 $\hat{p}_k : \mathbb{C}_\infty \to \mathbb{C}_\infty$
$$z \longmapsto z^k = w \quad (k \geq 2)$$

Ramification points $z = 0, \infty$

$\Rightarrow$ branch points $w = 0, \infty$



Example 10.5 $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$

$$f(z) = a_d z^d + \cdots + a_0$$

with $a_d \neq 0$. Near ∞, if $w = \frac{1}{z}$,

$$\frac{1}{f(z)} = \frac{1}{a_d z^d + \cdots + a_0} = \frac{1}{a_d w^{-d} + \cdots + a_0} = \frac{w^d}{a_d + \cdots + a_0 w^d} = w^d g(w)$$

for g analytic and $g(0) \neq 0$; hence $m_f(\infty) = d$.

Rk 10.6 $f: R \to \mathbb{C}$ analytic f^n, $p \in R$, (ϕ, U) a chart at p

Then $F(z) := f \circ \phi^{-1}(z) = (z - \underset{\phi(p)}{\overset{\shortparallel}{z_0}})^m g(z) \quad (m = m_f(\phi p))$

$\Rightarrow$

$$F'(z) = m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)$$
$$= [mg(z) + (z-z_0)g'(z)](z-z_0)^{m-1}$$

$\therefore m=1 \Rightarrow F'(z_0) = g(z_0) \neq 0$

but $m>1 \Rightarrow F'(z_0) = mg(z_0)(z_0-z_0)^{m-1} = 0$

CONCLUSION

$\{$ramification pts$\} = \{$zeros of $F'\}$

Lemma 10.7 If $f: R \to S$ and $g: S \to T$ are analytic maps of Riemann surfaces then $m_{g \circ f}(p) = m_g(f(p))\, m_f(p)$ for any $p \in R$

Pf Find local coords s.t.

$$z \xmapsto{f} z^{m_f(p)} \quad \text{and} \quad w \xmapsto{g} w^{m_g(f(p))}$$

Then $g(f(z)) = (z^{m_f(p)})^{m_g(f(p))}$ as required. $\square$

The Valency Theorem

This is the first step towards relating

branching data $\longleftrightarrow$ topology of R

Theorem 11.1 (valency theorem) Suppose $f: R \to S$ is a non-constant map between compact Riemann surfaces. The function $n: S \to \mathbb{N}$ defined by $n(q) := \sum_{p \in f^{-1}(q)} m_f(p)$ is constant on S.

Proof Id principle $\Rightarrow$ sum is finite

$\therefore$ n is well-defined

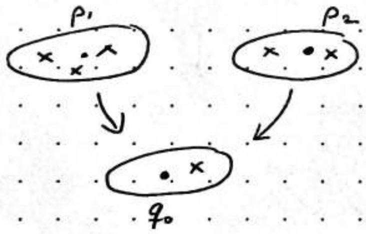
Since ~~R~~ S is connected, it suffices to prove:

n is locally constant,

i.e. every $q_0 \in S$ has ngbd V s.t. $n|_V$ is constant

Let $q_0 \in S$ and $f^{-1}(q_0) = \{p_1, \dots, p_k\}$

Goal: Find simultaneous local coordinates about the p_i and q_0 s.t. f is a power map



Let (ψ, V) be a chart about q_0 s.t. $\psi(q_0) = 0$.

Prop^n 4.6 $\Rightarrow$ $\exists$ disjoint $(\phi_1, \underset{p_1}{\underset{\cup}{U_1}}), \dots, (\phi_k, \underset{p_k}{\underset{\cup}{U_k}})$

s.t. $\psi \circ f \circ \phi_i^{-1}(z) = z^{m_f(p_i)}$ for each i.

Problem: Other points in V might have MORE PREIMAGES

Fix this using COMPACTNESS

Let $U = \bigcup_i U_i$

U open $\Rightarrow$ $U^c \subseteq R$ is closed

$\Rightarrow$ U^c compact

$\Rightarrow$ $K = f(U^c)$ compact

$\Rightarrow$ K closed

$\therefore$ let $V' = V \setminus K$ open

Claim $f^{-1}(V') \subseteq \mathcal{U}$

Pf By def^n,

$V' \subseteq K^c$, so $f^{-1}(V') \subseteq f^{-1}(K^c) = f^{-1}(K)^c$

But $K = f(\mathcal{U}^c) \Rightarrow \mathcal{U}^c \subseteq f^{-1}(K) \Rightarrow \mathcal{U} \supseteq f^{-1}(K)^c$. □

$\therefore$ set $U_i' = U_i \cap f^{-1}(V')$

By the claim, $f^{-1}(V') = \bigcup_i U_i'$

and so, in the charts $(\phi_i; U_i')$ and (ψ, V'), f takes the form of power maps.

Hence $n(q) = n(q_0)$ $\forall$ $q \in V'$. □

The number n produced by the valency theorem is called the valency or degree of f, and denoted by $\deg(f)$

Ex 10.5 $\Rightarrow$ if f is a polynomial, this is the usual (!) notion of degree

The valency theorem now immediately implies

Corollary 11.3 (FTA) Any polynomial f of degree d has exactly d zeros, counted with multiplicity

11.2 Euler Characteristic

Def 11.4 Let S be a cpct Riem sfc. A topological triangulation (sic) is a continuous embedding $\Delta \hookrightarrow S$, where Δ is a closed triangle in the plane $\mathbb{R}^2$. Take finite collection of topological triangles $\{\Delta_i\}$ satisfying the following conditions:

(i) $\bigcup_i \Delta_i = S$

(ii) $i \neq j \Rightarrow \Delta_i \cap \Delta_j$ is $\emptyset$, a vertex, or an edge

(iii) Each edge $\subseteq$ 2 triangles

The Euler characteristic of a triangulation is $\chi := V - E + F$, where

- V = # vertices
- E = # edges
- F = # triangles

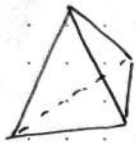
FACTS (i) Every cpct Riem sfc admits a triangulation

(ii) χ does NOT depend on the triangulation

$\therefore$ we may write $\chi(S)$, and call it the Euler characteristic of S

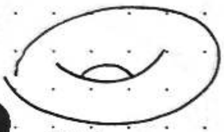
Ex 11.6

$\mathbb{C}_\infty$ $\underset{\text{homeo}}{\cong}$

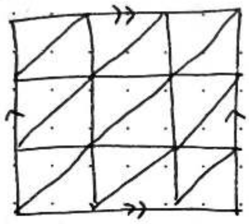


$\therefore \chi(\mathbb{C}_\infty) = 4 - 6 + 4 = 2$ ◡̈

Ex 11.7



$\mathbb{C}/\Lambda$ $\underset{\text{homeo}}{\cong}$



$\therefore \chi(\mathbb{C}/\Lambda) = 9 - 27 + 18 = 0$

More generally, every cpct Riemann surface S is homeo^c to some surface of genus g:

$\chi(S) = 2 - 2g$

Note $\chi(S)$ determines S up to homeo^m

That is $\chi(R) = \chi(S) \iff R \underset{\text{homeo}}{\cong} S$

11.3 The Riemann-Hurwitz Theorem

As promised, connection between branching and topology.

Thm 11.9 (Riemann-Hurwitz) Let $f: R \to S$ be any non-constant analytic map of cpct Riemann surfaces. Then

$$\chi(R) = \deg(f)\,\chi(S) - \sum_{p \in R} (m_f(p) - 1)$$

Note $m_f(p) = 1$ unless p is a ramification point

$\therefore$ the sum is finite

L11.4

Sketch proof

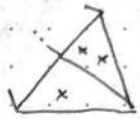
As in the proof of the valency thm, each $q \in S$ has a "power ngbd" $\mathcal{U}$ s.t. f restricts to a union of power maps on $f^{-1}(\mathcal{U})$.

cptness $\Rightarrow$ open cover $\{U_1, \dots, U_k\}$ where U_i is a power ngbd of $q_i \in S$

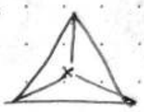
note $\Rightarrow$ # branch points $< \infty$

Triangulate S

Subdivide each triangle until contains ≤ 1 branch pt

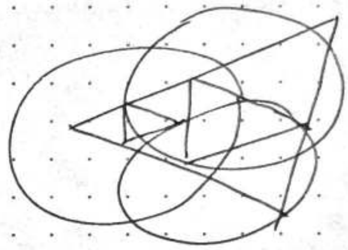


Then, subdivide further until each branch point is a vertex



Finally, subdivide until every triangle Δ is contained in some U_i

Now, the preimages of the triangles in S form a triangulation of R.

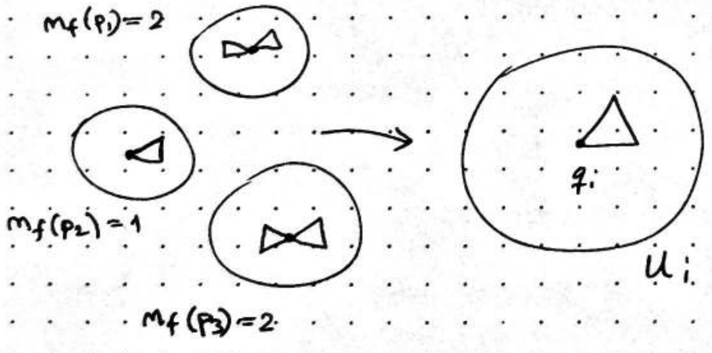


Introduce NOTATION:

$n := \deg(f)$

$V_R :=$ # vertices of R $\quad V_S$
$E_R :=$ # edges of R $\quad E_S$
$F_R :=$ # faces of R $\quad F_S$
$\}$ similarly for S

Above each vertex $q \in S$, the local picture looks like:



Therefore:

- $F_R = n F_S$
- $E_R = n E_S$
- branch vertices have slightly "too few" preimages:

$$\# f^{-1}(q) = n - \underbrace{\sum_{p \in f^{-1}(q)} (m_f(p) - 1)}_{\text{" } 0 \text{ unless } q \text{ is a branch point}}$$

From this,

$$V_R = nV_S - \sum_{q\in S} \sum_{p\in f^{-1}(q)} (m_f(p) - 1)$$

$$V_R = nV_S - \sum_{p\in R} (m_f(p) - 1)$$

Now just add up to finish the proof:

$$\begin{aligned}\chi(R) &= F_R - E_R + V_R \\ &= nF_S - nE_S + nV_S - \sum_{p\in R} (m_f(p) - 1) \\ &= \chi(S)\deg(f) - \sum_{p\in R} (m_f(p) - 1). \qquad \square\end{aligned}$$

Applications of Riemann-Hurwitz

12.1 Immediate consequences

- It's convenient to rearrange Riemann-Hurwitz to

$$2g_R - 2 = n(2g_S - 2) + \sum_{p \in R} (m_f(p) - 1)$$

where g_R = genus of R,
g_S = genus of S,
$n = \deg(f)$

We can use R-H to calculate genera of Riem surfaces!

Ex 12.1 Ex10.1 $w = \sqrt{z^3 - z} \rightsquigarrow \hat{R} \xrightarrow{\hat{\pi}} \mathbb{C}_\infty$

a meromorphic function on a compact Riemann Surface

- FACTS about $\hat{\pi}: \hat{R} \to \mathbb{C}_\infty$
  - $\deg \hat{\pi} = 2$
  - There are 4 branch pts: $0, \pm 1, \infty$
  - Each has 1 preimage (i.e. "TOTALLY RAMIFIED")
  - $\therefore$ there are 4 ramification pts, each with multiplicity $= 2$

Plugging these numbers into Riemann-Hurwitz gives

$$2g_{\hat{R}} - 2 = 2 \times (0 - 2) + \sum_{i=1}^{4} (2 - 1) = 0$$

$$\Rightarrow g_{\hat{R}} = 1$$

- Consistent with $\hat{R} \underset{\text{homeo}}{\cong} \text{torus}$

Remark The correction term $\Sigma_{p \in R} (m_f(p) - 1)$ is even

Let's look one more time at

$$\hat{R} = R_1 \cup_{\Phi} R_2$$

Imagine we know nothing about $\hat{\pi}^{-1}(\infty) \subseteq R_2$.

The correction term in R-H is:

$$3 \times (2 - 1) + \underbrace{\sum_{p \in \pi^{-1}(\infty)} (m_{\hat{\pi}}(p) - 1)}_{= C} = 3 + C$$

-

There are 2 cases:

(i) $\# \hat{\pi}^{-1}(\infty) = 2$

$\Rightarrow$ NO RAMIFICATION

$\Rightarrow C = 0$

(ii) $\# \hat{\pi}^{-1}(\infty) = 1$

$\Rightarrow \infty$ is totally ramified

$\Rightarrow C = 1$

But $3 + C$ is <u>even</u>, so it must be case (ii). (cool)

<u>Conclusion</u> $C = 1$ and we deduce $g_{\hat{R}} = 1$ as before

<u>Moral</u> When $\deg f = 2$, we don't need to understand the branching at ∞, neat.

<u>Remark 12.4</u> If f is a covering map (also called unramified) then the correction term coming from the branching data is zero, so

$$g_R - 1 = n(g_S - 1)$$

(i) $g_S = 0 \Rightarrow g_R = 0, \quad n = 1$

$\Rightarrow f$ is a conformal equivalence

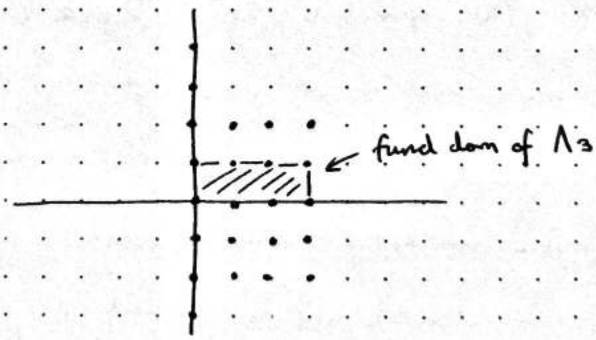
$\Rightarrow f$ Möbius transformation

(ii) $g_S = 1 \Rightarrow g_R = 1$, n unrestricted

(iii) $g_S > 1 \Rightarrow \begin{cases} n = 1 \Rightarrow f \text{ conformal equiv} \\ \text{OR} \\ n > 1 \Rightarrow g_R > g_S \end{cases}$

<u>Example 12.5</u>

Let $\Lambda_n = \langle n, i \rangle \leq \mathbb{C}$, $n \in \mathbb{Z}_{>0}$



Then $\Lambda_n \leq \Lambda_1 \; \forall n$ induces

$$\mathbb{C}/\Lambda_n \to \mathbb{C}/\Lambda_1 \; ;$$

a covering map of degree n.

12.2 Higher genus Riemann surfaces

Let's construct Riemann surfaces with genus > 1

Example 12.8 Consider the Fermat curve of degree d

$$F_d' := \{ x^d + y^d = 1 \} \subseteq \mathbb{C}^2$$

Aside Rational pts $\to$ sol^{n}s to FLT (!)

ATLAS The coord proj

$$\pi_x : (x,y) \mapsto x, \quad \ulcorner \pi_y : (x,y) \mapsto y \lrcorner$$

has local inverse

$$\pi_x^{-1}(x) = (x, \sqrt[d]{1-x^d})$$

which exists in a ngbd of any x_0 unless

$$1 - x_0^d = 0 \iff x_0 = \zeta_d^i, \quad d^{th} \text{ root of unity}$$
$$\iff y_0 = 0$$

$\therefore$ π_x defines charts on $F_d' \setminus \{(\zeta_d^i, 0)\}$

Symmetrically, π_y provides charts on $F_d' \setminus \{(0, \zeta_d^i)\}$

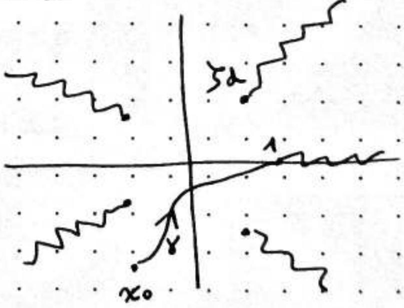
Since the 'bd' (i.e. ramification) points of π_x and π_y are distinct, we have charts that cover F_d'.

TRANSITION FUNCTIONS: The non-trivial ones are

$$\pi_y \circ \pi_x^{-1}(x) = \sqrt[d]{1-x^d} \quad \text{and} \quad \pi_x \circ \pi_y^{-1}(y) = \sqrt[d]{1-y^d}$$

These are analytic $\Rightarrow$ CONFORMAL STRUCTURE

Hausdorff: clear since $F_d' \subseteq \mathbb{C}^2$



Connected: Let

$$D = \mathbb{C} \setminus \bigcup_{i=1}^{d} \{ t \zeta_d^i \mid t \geq 1 \}$$

There are well-defined branches of

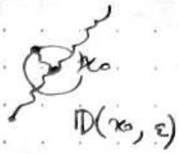
$$y(x) = \sqrt[d]{1-x^d}$$

Let $(x_0, y_0) \in F_d'$, $x_0 \in D$. Choose the branch of $y(x)$ s.t. $y(x_0) = y_0$.

Let γ be any path in D from $x_0 \to 1$.

Then $\tilde{\gamma}(t) = (\gamma(t), y(\gamma(t)))$ joins $(x_0, y_0) \to (1, 0)$

If $x_0 \notin D$ then we use the same argument on a small disc to join $(x_0, y_0) \rightsquigarrow (x_1, y_1)$ with $x_1 \in D$



Hence F_d' is path-connected.

COMPACTIFICATION

See q13, Ex Sheet 3.

- compact $F_d := F_d' \cup_\Phi F_d''$
- meromorphic $\hat{\pi}_x : F_d \to \mathbb{C}_\infty$, $\deg \hat{\pi}_x = d$
- $\# \hat{\pi}_x^{-1}(\infty) = d$

In particular, ∞ is <u>not</u> a brach point.

$\therefore$ The ram pts of $\hat{\pi}_x$ are $\{(\zeta_d^i, 0) : i = 0, \ldots, d-1\}$

There are d of them, of multiplicity d.

$\therefore$ R-H gives

$$2 g_{F_d} - 2 = d \times (-2) + d \times (d-1)$$

$$\Rightarrow g_{F_d} = \frac{(d-1)(d-2)}{2}$$

<u>CONCLUSION</u> There are Riem sfcs of arbitrarily large genus

Rational and periodic functions

In this section §13.1 we study meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$

We notice that they are all of the familiar form

Prop 13.1 Every meromorphic function on the Riemann sphere $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function. That is, it is of the form

$$f(z) = c\,\frac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_n)}$$

for some integers $m, n \geq 0$ and constants $a_i, b_j, c \in \mathbb{C}$.

Proof Clearly we may assume f non-constant

Replacing $f \rightsquigarrow \frac{1}{f}$ if necesseray, wlog $f(\infty) \in \mathbb{C}$

$\therefore$ f has a finite set of poles $b_1, \dots, b_{n'} \in \mathbb{C}$

The Laurent series about b_j is of the form

$$f(z) = \sum_{l=-k_j}^{\infty} c_{jl}\,(z-b_j)^l$$

so we may consider the principle part

$$Q_j(z) = \sum_{l=-k_j}^{-1} c_{jl}\,(z-b_j)^l$$

Now $g(z) := f(z) - \sum_{j=1}^{n'} Q_j(z)$ only has removable singularities on $\mathbb{C}_\infty$,

so $g(z) = c$ constant

$$\therefore\ f(z) = c + \sum_{j=1}^{n'} Q_j(z) \quad \text{is rational.} \quad \square$$

Suppose $f(z) = c\,\dfrac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_n)}$

with $a_i \neq b_j$ for all i, j.

Remark 13.2 In the proof, $f(\infty) \in \mathbb{C}$ means $m \leq n$, and in this case, the proof gives

$$\sum_{j=1}^{n'} k_j = n = \deg(f)$$

$\therefore$ in general $\deg(f) = \max(m, n)$

Simply periodic functions

NEXT: Classify meromorphic f^n's on other Riemann surfaces

Many Riem sfcs are quotients $R = D/\sim$ for some domain $D \subseteq \mathbb{C}$

This is useful, because then

$$\{ f^{ns} \text{ on } R \} \longleftrightarrow \{ \text{periodic } f^n\text{s on } D \}$$

Def 13.3 Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be meromorphic. A period of f is a complex number $\omega \in \mathbb{C}$ s.t. $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$

omega ω, cf w [sic].

Note The periods of f form a subgroup $\Omega \leqslant \mathbb{C}$

Lemma 13.4 Let Ω be the set of periods of a meromorphic function f on $\mathbb{C}$. One of the following holds:

(i) $\Omega = \{0\}$, (ii) $\Omega = \langle \omega \rangle \cong \mathbb{Z}$, $\omega \neq 0$

(iii) $\Omega = \langle \omega_1, \omega_2 \rangle \cong \mathbb{Z}^2$, $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$; (iv) $\Omega = \mathbb{C}$

Proof See Ex Sheet 3, q1. $\square$

Def^n A meromorphic function f on $\mathbb{C}$ for which the group of periods is isomorphic to $\mathbb{Z}$ is called simply periodic

Example $f(z) = \exp(z)$ has $\Omega = \langle 2\pi i \rangle \cong \mathbb{Z}$

Example 3.4 $\exp : \mathbb{C} \to \mathbb{C}_*$ is a covering map

Prop 13.7 If f is a meromorphic function on $\mathbb{C}$ and the periods of f contain an infinite cyclic subgroup $\langle \omega \rangle$, then $\exists!$ meromorphic function $\bar{F}$ on $\mathbb{C}_*$ s.t. $f(z) = \bar{F}(\exp((2\pi i/\omega) z))$ $\forall z \in \mathbb{C}$

Proof Near any $\underset{''}{z} \in \mathbb{C}_*$ choose a branch of log.

Define $\bar{F}(w) = f\left(\frac{\omega}{2\pi i} \log(w)\right)$

As claimed, $\bar{f}\left(\exp\left(\left(\frac{2\pi i}{\omega}\right) z\right)\right) = f\left(\frac{\omega}{2\pi i} \cancel{\log} \cancel{\exp} \log\left(\exp\left(\left(\frac{2\pi i}{\omega}\right) z\right)\right)\right)$

$= f(z)$

It remains to prove that $\bar{F}$ is well defined.

Suppose we chose a different branch of log.

Then we defined

$$\hat{f}(w) = f\left(\left(\tfrac{\omega}{2\pi i}\right)(\log w + 2\pi i n)\right) \quad , n \in \mathbb{Z}$$
$$= f\left(\left(\tfrac{\omega}{2\pi i}\right)\log w + n\omega\right)$$
$$= \bar{f}(w) \qquad \square$$

<u>Moral</u> : $\{\text{simply periodic } f^n\text{'s}\} \leftrightarrow \{\text{functions on } \mathbb{C}_*\}$
$$f \longmapsto \bar{f}$$

The point is that
$$\mathbb{C}_* \cong \mathbb{C} / \langle 2\pi i \rangle$$
and exp is the quotient map. YES!

§13.3 Doubly periodic functions

<u>Def^n 13.8</u> A meromorphic function f on $\mathbb{C}$ with periods $\Omega = \langle \omega_1 \rangle \oplus \langle \omega_2 \rangle \cong \mathbb{Z}^2$ is said to be <u>doubly periodic</u> or <u>elliptic</u>.

In this case, Ω is a <u>lattice</u> Λ, so f descends to a meromorphic f^n on $\mathbb{C}/\Lambda$

<u>Prop 13.9</u> If f is a meromorphic function on $\mathbb{C}$ and the periods of f contain a lattice Λ, then $\exists !$ meromorphic function $\bar{f}$ on the complex torus such that $f(z) = \bar{f} \circ \pi(z) \;\; \forall z \in \mathbb{C}$. ($\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is quotient map)

<u>Proof</u> "Same" as Prop 13.7. $\square$

Again, the Moral is a correspondence
$$\{\text{elliptic functions on } \mathbb{C} \text{ , } \Omega \supseteq \Lambda \subseteq \mathbb{C}\} \longleftrightarrow \{f^n\text{s on } \mathbb{C}/\Lambda\}$$
$$f \longmapsto \bar{f}$$

So we can apply theorems/def^n's from <u>compact</u> Riemann surfaces to elliptic functions.

For instance, because analytic functions on $\mathbb{C}/\Lambda$ are constant, we get

<u>Cor 13.10</u> Any analytic function f on $\mathbb{C}$ that is doubly periodic is const.

Another application:

We write $\deg(f) := \deg(\bar{f})$

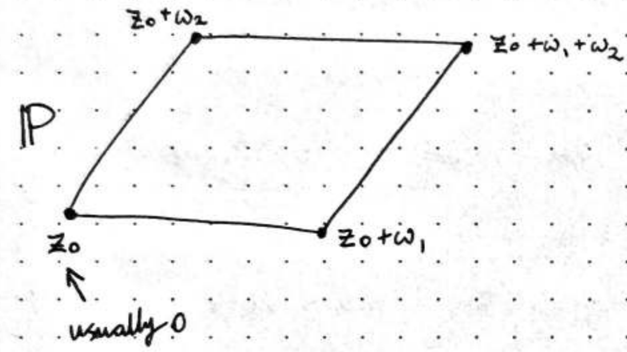
We can apply Riemann–Hurwitz. For instance

<u>Cor 13.11</u> If f is doubly periodic, non-constant, meromorphic, then $\deg(f) \geq 2$

<u>Proof</u> If $\deg(f) = 1$ then f is <u>unramified</u> so R–H gives

$$0 = 1 \times (-2) + 0 \quad \text{\#} \quad \square$$

The <u>period parallelogram</u> is a more concrete way to study a function f with periods $\Lambda = \langle \omega_1, \omega_2 \rangle$.



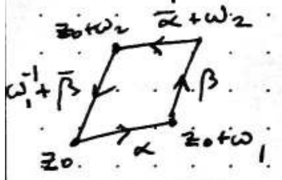
<u>Note</u>: f determined by $f|_{\mathbb{P}}$

<u>Application</u> Alternative proof of Cor 13.11 avoiding R–H

Choose z_0 so that no zeros or poles of f lie on ∂P. The residue theorem gives

$$\sum_{\substack{z \in P \\ z \text{ pole}}} \operatorname{res}_z(f) = \frac{1}{2\pi i} \oint_{\partial P} f(z)\, dz$$

Use periodicity to evaluate the contour integral.



By periodicity, $\int_\alpha f(z)\, dz + \int_{\bar{\alpha}+\omega_2} f(z)\, dz$

$$= \left(\int_\alpha + \int_{\bar{\alpha}}\right) f(z)\, dz = 0$$

Likewise, $\int_\beta f(z)\, dz + \int_{\bar{\beta}-\omega_1} f(z)\, dz = 0$

Hence,
$$\sum_{\substack{z \in P \\ z \text{ pole}}} \operatorname{res}_z(f) = 0$$

Therefore, either f has no poles $\rightarrow$ f constant

or f has ≥ 2 poles (with multiplicity) so $\deg(f) \geq 2$. $\square$

The Weierstrass $\wp$-function

The Definition

We know that a non-constant elliptic function needs deg ≥ 2

Def^n Let Λ be a lattice in $\mathbb{C}$. The associated Weierstrass $\wp$-function is defined by
$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

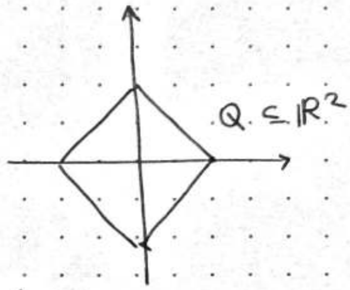
To check this really does converge, we use

Lemma Let $\Lambda = \langle \omega_1, \omega_2 \rangle$ be a lattice in $\mathbb{C}$ and $t \in \mathbb{R}$.

The sum $\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^t}$ converges iff $t > 2$.

Proof Consider the set

$$Q = \{ (t_1, t_2) \in \mathbb{R}^2 \} \{ |t_1| + |t_2| = 1 \}$$



Compactness $\Rightarrow$ the function $Q \longrightarrow \mathbb{R}$
$$(t_1, t_2) \longmapsto \| t_1\omega_1 + t_2\omega_2 \|$$
attains its max M and its min m.

Also, $0 = m = |t_1\omega_1 + t_2\omega_2| \Rightarrow \omega_1, \omega_2$ lin dep over $\mathbb{R}$ ※

$\therefore \; 0 < m \leq |t_1\omega_1 + t_2\omega_2| \leq M < \infty$

for all $(t_1, t_2) \in Q$.

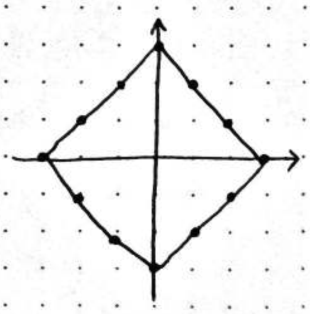
Consider $(k, \ell) \in \mathbb{Z}^2$ and take $t_1 = \frac{k}{|k|+|\ell|}$, $t_2 = \frac{\ell}{|k|+|\ell|}$

By the previous slide, we get
$$m \leq \frac{|k\omega_1 + \ell\omega_2|}{|k| + |\ell|} \leq M \iff m(|k|+|\ell|) \leq |k\omega_1 + \ell\omega_2| \leq M(|k|+|\ell|)$$

The sum we're interested in is
$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^t} = \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|k\omega_1 + \ell\omega_2|^t}$$

By the previous slide, this is bounded on both sides by multiples of
$$\sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(|k|+|l|)^t} \quad (*)$$

Set $n = |k| + |\ell|$ and notice that there are exactly $4n$ such pairs (k, ℓ) for each $n > 0$.



Thus we can rewrite (*):

$$\sum_{(k,l)\neq 0} \frac{1}{(|k|+|l|)^t} = \sum_{n>0} \frac{4n}{n^t} = 4 \sum_{n>0} \frac{1}{n^{t-1}} \cancel{}$$

which converges iff $t>2$ as required. □

Now we can prove that $\wp$ is well-defined.

Theorem 14.3 The function $\wp_\Lambda$ is a well-defined elliptic function with periods Λ. Moreover, $\wp_\Lambda$ is even and of degree 2.

Proof Start with convergence: need to estimate the summands

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{\omega^2 - (z-\omega)^2}{\omega^2 (z-\omega)^2} \right|$$

$$= \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right|$$

$$= \left| \frac{z}{\omega^2} \right| \left| \frac{2\omega - z}{(z-\omega)^2} \right|$$

$$\leq \left| \frac{2(\omega - z)}{(z-\omega)^2} \right| + \left| \frac{z}{(z-\omega)^2} \right|$$

$$= \frac{2}{|z-\omega|} + \frac{|z|}{|z-\omega|^2}$$

Fix $R \geq |z|$. For all but finitely many ω, we have $|\omega| \geq 2R \Rightarrow |\omega - z| \geq \frac{1}{2}|\omega| \geq R$

So we can bound the summand

$$\left|\frac{z}{\omega^2}\right| \left(\frac{2}{|z-\omega|} + \frac{|z|}{|z-\omega|^2} \right) \leq \frac{R}{|\omega|^2} \left(\frac{2}{\frac{1}{2}|\omega|} + \frac{R}{\frac{1}{2}|\omega| R} \right) \quad \text{cheeky!}$$

$$= \frac{R}{|\omega|^2} \frac{6}{|\omega|} = \frac{6R}{|\omega|^3} \quad \leftarrow \text{convergence by the lemma}$$

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

EVEN: Clear from the def^n

ELLIPTIC: Let $\omega_0 \in \Lambda$. Need to show ω_0 is a period of $\wp$.

Clearly it's a period of the derivative:

$$\wp'(z) = \sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}$$

Now look at the function

$$f(z) = \wp(z+\omega_0) - \wp(z)$$

Then $f'(z) = \wp'(z+\omega_0) - \wp'(z) = 0$

$\Rightarrow f(z) = \text{const.}$

$\therefore \wp(z+\omega_0) = \wp(z) + c$ for all $z \in \mathbb{C}$

Now set $z = -\frac{\omega_0}{2}$ and use even-ness

$$\wp\left(\frac{\omega_0}{2}\right) = \wp\left(-\frac{\omega_0}{2} + \omega_0\right)$$
$$= \wp\left(-\frac{\omega_0}{2}\right) + c$$
$$= \wp\left(\frac{\omega_0}{2}\right) + c$$

$\therefore c = 0$ and ω_0 is a period of $\wp$.

$\therefore \Lambda \subseteq \{\text{periods of } \wp\}$

Fro the reverse inclusion, note that

$\{\text{poles of } \wp'\} = \Lambda$

$\therefore \{\text{periods of } \wp\} \subseteq \{\text{periods of } \wp'\} \subseteq \Lambda$

In particular, $\wp$ has a unique pole of order 2 on $\mathbb{C}/\Lambda$

$\Rightarrow \deg(\wp) = 2.$ $\square$

In fact, the proof characterises $\wp$:

Rmk 14.4 (i) $\wp_\Lambda$ is meromorphic with periods Λ

(ii) $\wp_\Lambda$ has poles only at Λ

(iii) $\wp_\Lambda(z) - \frac{1}{z^2} \rightarrow 0$ as $z \rightarrow 0$

These properties uniquely characterize $\wp$. [look at $f - \wp$]

14.2 Branching properties of $\wp_\Lambda$

We know $\wp$ has a unique pole in $\mathbb{C}/\Lambda$ of order 2.

The other ramification points occur where $\wp' = 0$.

$\therefore$ need to investigate $\wp'$

Rmk 14.5 Recall

$$\wp'_\Lambda(z) = \sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^3}$$

So we see:

(i) $\{\text{poles of } \wp'\} = \Lambda$

(ii) $\deg \wp' = 3$

(iii) $\wp'$ is odd

Let's figure out the zeros of $\wp'$!

Let $\omega \in \Lambda$. Then

$$\wp'\left(\tfrac{\omega}{2}\right) = \wp'\left(\tfrac{\omega}{2} - \omega\right) \quad \text{(periodicity)}$$

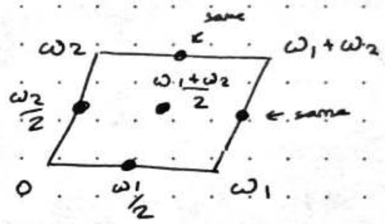
$$= \wp'\left(-\tfrac{\omega}{2}\right)$$

$$= -\wp'\left(\tfrac{\omega}{2}\right) \quad \text{(odd)}$$

$$\Rightarrow \wp'\left(\tfrac{\omega}{2}\right) = 0$$

This gives us three zeros in the period parallelogram P

But $\deg \wp' = 3$ so these are all the zeros and they are all simple



Rmk 14.6 $\wp$ has 4 ram. pts in P, each with multiplicity 2:

$$0, \ \frac{\omega_1}{2}, \ \frac{\omega_2}{2}, \ \frac{\omega_1+\omega_2}{2}$$

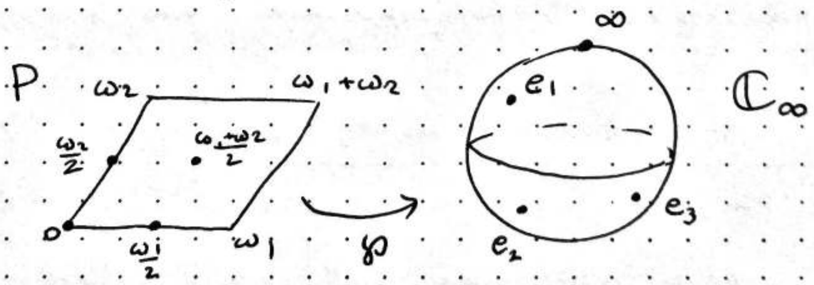
The branch points are

$$\wp(0) = \infty, \quad \wp\left(\tfrac{\omega_1}{2}\right) = e_1, \quad \wp\left(\tfrac{\omega_2}{2}\right) = e_2, \quad \wp\left(\tfrac{\omega_1+\omega_2}{2}\right) = e_3$$

and the e_i are distinct by the valency theorem.

Rmk This is consistent with Riemann–Hurwitz:

$2\times(-2) + 4\times(2-1) = 0$ ✓



L14. (3+2)

14.3 An algebraic relation

Although $\wp$ is just an example of an elliptic function, it will be key to classifying them. First relate $\wp'$ to $\wp$.

Prop^n 14.7 There are constants $g_1, g_2 \in \mathbb{C}$ depending only on Λ s.t.

$$(\wp'_\Lambda)^2 \equiv 4\wp_\Lambda^3 - g_2 \wp_\Lambda - g_3$$

Proof The idea is to cancel the negative terms of the Laurent series.

Near 0, we have $\wp(z) = \frac{1}{z^2} + az^2 + o(z^4)$

since $\wp$ even and $\wp \approx \frac{1}{z^2}$. Cubing this gives

$$(\wp(z))^3 = \frac{1}{z^6} + f(z) + \frac{3a}{z^2} \qquad (1)$$

for an analytic f.

We can also differentiate to get a Laurent series for $\wp'$:

$$\wp'(z) = -\frac{2}{z^3} + 2az + o(z^3)$$

Squaring gives

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{8a}{z^2} + g(z) \qquad (2)$$

for analytic g.

Computing $(1) - 4\times(2)$ gives

$$(\wp'(z))^2 - 4(\wp(z))^3 = -\overset{(-20?)}{\frac{8a}{z^2}} - h(z)$$

where, again, h is analytic.

Finally setting $g_2 = 8a$, we get that

$$(\wp')^2 - 4\wp^3 + g_2\wp$$

is analytic, with no poles, and doubly periodic

$\Rightarrow$ constant, $-g_3$. $\square$

Finally, we notice that the constants g_2, g_3 relate to the branch points.

Rmk When $z \in \frac{1}{2}\Lambda \setminus \Lambda$, we have

- $\wp'(z) = 0$, plugging into the eq^n gives
- $\wp(z) = e_i$, $0 = 4e_i^3 - g_2 e_i - g_3$

So e_1, e_2, e_3 are the 3 zeros of $4X^3 - g_2 X - g_3$,
so it factors as $4(X-e_1)(X-e_2)(X-e_3)$

In particular the relation can be written as

$$\wp' \equiv 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

This also tells us that

$$4(e_1+e_2+e_3) = \text{coeff of } \wp^2 = 0$$

More $\wp$ - function and quotients

15.1 An elliptic curve

We have seen two constructions of Riemann surfaces with genus 1

(i) $\mathbb{C}/\Lambda$

(ii) compactifying $\{w^2 = z^3 - z\}$

Qn: Are these constructions related?

The answer is yes!

Every torus of type (i) is also of type (ii)

Corollary 15.1 Let $\mathbb{C}/\Lambda$ be a complex torus. There are constants g_2, g_3 such that $\mathbb{C}/\Lambda$ is biholomorphic to a one-point compactification of the graph

$$X' := \{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}$$

Sketch Pf Take g_1, g_2 as in Prop^n 14.7

It turns out that X' can be compactified: $X = X' \cup \{\infty\}$ as in previous examples, with charts provided by the coordinate projections. (cf Example 10.1 or Ex Sheet 1 q14)

Define $F : \mathbb{C} \longrightarrow X$

$$z \longmapsto (\wp(z), \wp'(z))$$

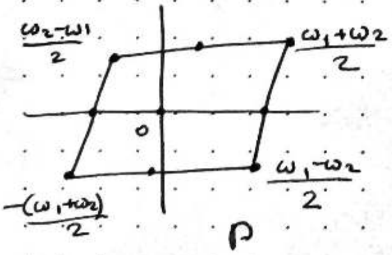
We have $\operatorname{im} F \subseteq X$ by Prop 14.7, and F is analytic because the charts are coord proj^{n}s $\therefore$ F descends to

$$\Phi : \mathbb{C}/\Lambda \rightarrow X$$

It remains to show that Φ is a conformal equivalence.

Surjectivity: Φ non-constant, $\mathbb{C}/\Lambda$ cpct.

Injectivity: centre a period parallelogram on 0



Let $z, z' \in P$ and suppose $f(z) = f(z')$.

$\Rightarrow$ ① $\wp(z) = \wp(z') \Rightarrow z = \pm z'$

because $\wp$ is even of degree 2

AND ② $\wp'(z) = \wp'(z') = \wp'(\pm z) = \pm \wp'(z)$

because $\wp'$ odd $\Rightarrow z = z'$

$\therefore \Phi$ is injective

15.2 Classification of elliptic functions

We already classified the meromorphic functions on $\mathbb{C}_\infty$. Remarkably we can do something similar for the torus $\mathbb{C}/\Lambda$.

Theorem 15.2 Let f be an elliptic function with periods Λ. There exist rational functions Q_1, Q_2 such that

$$f(z) = Q_1(\wp(z)) + Q_2(\wp(z))\,\wp'(z)$$

Furthermore, if f is even then we can take $Q_2 = 0$.

Proof First, assume f is even.

Begin by simplifying. Since f and $\wp$ only have finitely many branch points, we may choose $c, d \in \mathbb{C}$ not branch points of for $\wp$. Now

$$z \longmapsto \frac{f(z)-c}{f(z)-d}$$

is even, with simple zeros & poles, not ramif^n points of $\wp$ ↖ a bit more subtle, want $f^{-1}(c), f^{-1}(d)$ NOT ramif^n $\wp$

So we assume f has this property.

Since f is even, we have

$$\{\text{zeros of } f\} = \{\pm a_1, \dots, \pm a_m\}$$

Likewise

$$\{\text{poles of } f\} = \{\pm b_1, \dots, \pm b_n\}$$

$\therefore$ we can write down an elliptic function with the same zeros and poles

$$g(z) := \frac{(\wp(z)-\wp(a_1))\cdots(\wp(z)-\wp(a_m))}{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))}$$

$\therefore$ $f(z)/g(z)$ is elliptic without zeros or poles $\Rightarrow$ const. non-zero

So $f(z) = c\,g(z)$, a rational f^n of $\wp(z)$ as claimed.

If f is odd, then $f(z)/\wp'(z)$ is elliptic and even

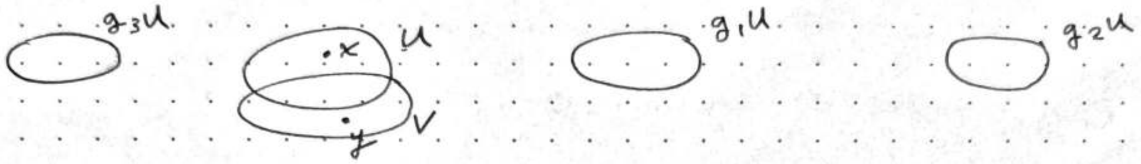
$\Rightarrow$ a rational function of $\wp$ by the above

For arbitrary f: $f(z) = \frac{1}{2}\Big(\underbrace{\frac{f(z)+f(-z)}{2}}_{\text{even}}\Big) + \frac{1}{2}\Big(\underbrace{\frac{f(z)-f(-z)}{2}}_{\text{odd}}\Big)$ $\square$

15.3 Quotients of Riemann Surfaces

Can we think of more complicated Riem surfaces as quotients?

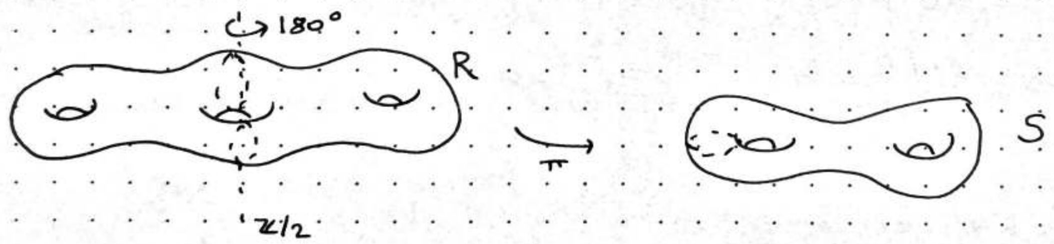
Def^n 15.3 Let a group G act by homeo^{m} on a space X.
The action is said to be properly discontinuous if every $x, y \in X$ have open nghbds U, V respectively s.t. $g(U) \cap V = \phi$ for every $g \in G \setminus 1$



Ex 15.4 If Λ is a lattice in $\mathbb{C}$ then the action of Λ on $\mathbb{C}$ by translation is properly discontinuous.

Prop 15.5 Let R be a Riemann surface, and let G be a group acting properly discontinuously by conformal equivalences on R. Then the quotient $S = G \setminus R$ is a Riemann surface, and the quotient map

$\pi : R \to S$ is analytic and a regular covering map



Proof Connected $\pi : R \twoheadrightarrow S$ surj $\Rightarrow$ S connected

Hausdorff Suppose $\underset{\pi(x)}{\overset{\shortparallel}{p}} \neq \underset{\pi(y)}{\overset{\shortparallel}{q}} \in S$

Prop disc $\Rightarrow \exists\, U \ni x,\ V \ni y$ s.t. $G.U \cap V = U \cap V$

But R Hausdorff $\Rightarrow$ WLOG $U \cap V = \phi$

$\therefore\ GU \cap V = \phi \Rightarrow \pi(U), \pi(V)$ are disjoint open sets

Finally, covering map & conformal structure exactly as for $\mathbb{C}/\Lambda$ (example 5.1). $\square$

Next lecture, we will use this to state a classification of Riemann surfaces.

But first, Hurwitz's Theorem

Thm 15.6 Let R be a cpct Riemann surface of genus $g_R \geq 2$ and suppose that G acts faithfully and properly discontinuously on R by conformal equivalences. Then G is finite, with $|G| \leq g_R - 1$.

Pf Prop 15.5 $\Rightarrow$ $S = G \backslash R$ Riemann surface and $\pi : R \to S$ analytic & unramified

Note that $\deg(\pi) = |G|$. Therefore, R–H gives

$$2g_R - 2 = |G|(2g_S - 2) + 0$$

$$\Rightarrow g_R - 1 = |G|(g_S - 1)$$

Now $g_R \geq 2 \Leftrightarrow \text{LHS} > 0 \Rightarrow \text{RHS} > 0 \Rightarrow g_S - 1 \geq 1$

Hence $g_R - 1 \geq |G|$. $\square$

Finally, note the result fails if $g_R = 1$:

Ex 15.7 The torus $\mathbb{C}/\Lambda$ is a group acting on itself

$$\mathbb{C}/\Lambda \circlearrowright \mathbb{C}/\Lambda$$

$$(z+\Lambda)\cdot(z_0+\Lambda) \mapsto (z+z_0+\Lambda)$$

Any finite subgroup acts properly discontinuously

They can be arbitrarily large, e.g. if $\Lambda = \langle 1, i \rangle$ consider

$\langle \frac{1}{n} + \Lambda \rangle \leq \mathbb{C}/\Lambda$

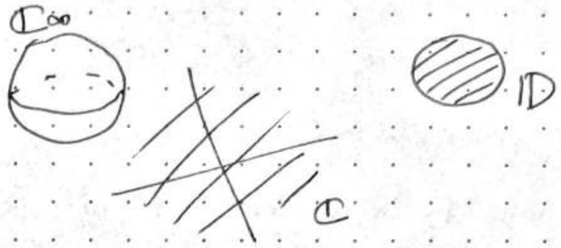
Uniformisation and its consequences

16.1 The uniformisation theorem

We state a classification of simply connected Riemann surfaces.

Theorem 16.1 (Uniformisation) Every simply connected Riemann surface is conformally equivalent to one of:

(i) $\mathbb{C}_\infty$ (ii) $\mathbb{C}$ (iii) $\mathbb{D}$



Remark 16.2 These three are all conformally distinct:

$\mathbb{C}_\infty$ is compact $\Rightarrow$ $\mathbb{C}_\infty$ not homeo to $\mathbb{C}$, $\mathbb{D}$

Any analytic map $f: \mathbb{C} \to \mathbb{D}$ is bounded $\Rightarrow$ constant (Liouville)

so not a conformal equivalence.

Uniformisation makes it possible to classify Riemann surfaces.

Let's start with genus $= 0$

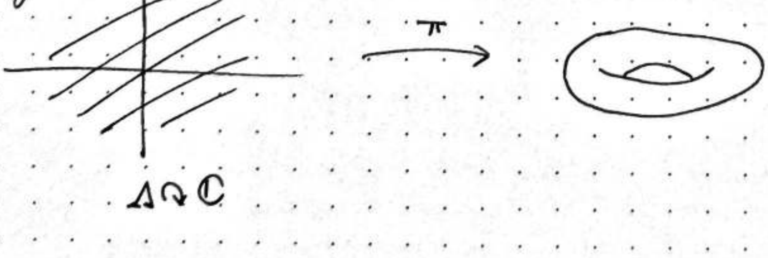
Corollary 16.3 The conformal structure on S^2 is conformally equivalent to $\mathbb{C}_\infty$ Pf S^2 simply connected, compact

$\Rightarrow$ has to be $\mathbb{C}_\infty$. $\square$

For genus > 0, we need tools from Algebraic Topology.

Theorem 16.4 Every Riemann surface R has a regular covering map $\pi: \tilde{R} \to R$ such that $\tilde{R}$ is simply connected. Furthermore, there is a group G acting freely and properly discontinuously by conformal equivalences on $\tilde{R}$, and the covering map π descends to a conformal equivalence $G \backslash \tilde{R} \cong R$

e.g.



Combined with uniformisation, we get a starting point for classifying Riemann surfaces

<u>Corollary 16.5</u> Every Rieman surface R is conformally equivalent to a quotient $R \cong G \backslash \widetilde{R}$

where $\widetilde{R}$ is one of $\mathbb{C}_\infty$, $\mathbb{C}$, or $\mathbb{D}$ and G is a free & properly discontinuous group of conformal equivalences of $\widetilde{R}$.

We say R is <u>uniformised</u> by $\widetilde{R}$

<u>Remark 16.6</u> $G = \{ \phi : \widetilde{R} \xrightarrow{c.e} \widetilde{R} \mid \pi \circ \phi = \pi \}$

<u>16.2 Classification of Riemann surfaces</u>

Classification of R into 3 cases, depending on $\widetilde{R}$

(i) $\widetilde{R} = \mathbb{C}_\infty$

<u>Prop 16.7</u> If a Riemann surface R is uniformised by $\mathbb{C}_\infty$ then R is conformally equivalent to $\mathbb{C}_\infty$

<u>Proof</u> $R = G \backslash \mathbb{C}_\infty$. But

$$\mathrm{Aut}(\mathbb{C}_\infty) = PSL_2 \mathbb{C} \quad \text{(ex sheet 1 q7)}$$

and every $g \in G \setminus 1$ has a fixed point

Action free $\Rightarrow G = 1$ and we are done. $\square$

(ii) $\widetilde{R} = \mathbb{C}$

Again, we can use our understanding of the conf equivs of $\mathbb{C}$

<u>Prop 16.8</u> If a Riemann sfc R is uniformised by $\mathbb{C}$ then one of the following holds:

(i) $G \cong 1$ and $R \cong \mathbb{C}$,

(ii) $G \cong \mathbb{Z}$ and $R \cong \mathbb{C}_*$,

(iii) $G \cong \mathbb{Z}^2$ and $R \cong \mathbb{C}/\Lambda$

L16.3

<u>Pf</u> Q7 of ex sheet 1:

$$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \mid a \in \mathbb{C}_*, b \in \mathbb{C} \} =: \mathbb{C} \rtimes \mathbb{C}_*$$

<u>Note</u> $z = az+b$ has a unique sol^n unless $a=1$

$\therefore$ $G \leq \mathrm{Aut}(\mathbb{C})$ only acts freely if it consists of <u>translations</u>

Q1 of ex sheet 3:

Then there are 3 cases:

(i) $G = 1 \Rightarrow R \cong \mathbb{C}$

(ii) $G = \langle \omega \rangle \cong \mathbb{Z} \Rightarrow R \cong \mathbb{C}_*$ (Prop^n 13.7)

(iii) $G = \Lambda \cong \mathbb{Z}^2 \Rightarrow R \cong \mathbb{C}/\Lambda$ $\square$

(iii) $\tilde{R} = \mathbb{D}$:

EVERYTHING ELSE!

To see this, we need the

<u>Lemma 16.9</u> Let $f: R \to S$ be an analytic map of Riemann surfaces. Suppose that R is simply connected and let $\pi: \tilde{S} \to S$ be the uniformising map of S. Then there is an analytic map $F: R \to \tilde{S}$ s.t. $f = \pi \circ F$

<u>Proof</u> cf Q3 of Ex Sheet 2. $\square$

$$\begin{tikzcd} & \tilde{S} \arrow[d, "\pi"] \\ R \arrow[ur, dashed, "\exists F"] \arrow[r, "f"'] & S \end{tikzcd}$$

Using the lifting lemma, we can prove

<u>Prop 16.10</u> A Riemann surface R is uniformised by at most one of $\mathbb{C}_\infty$, $\mathbb{C}$ and $\mathbb{D}$

<u>Proof</u> Prop^n 16.7 & 16.8

$\Rightarrow$ no R is uniformised by $\mathbb{C}_\infty$ & $\mathbb{C}$

$\therefore$ suppose R is uniformised by $\mathbb{D}$ and $\tilde{R} = \mathbb{C}_\infty$ or $\mathbb{C}$

That is, we have uniformsing maps

$$\begin{tikzcd} \tilde{R} \arrow[r, "F"] \arrow[dr, "f"'] & \mathbb{D} \arrow[d, "\pi"] \\ & R \end{tikzcd}$$

The lifting lemma $\Rightarrow$ $F: \tilde{R} \to \mathbb{D}$ s.t. $f = \pi \circ F$

Now Liouville $\Rightarrow$ F const $\Rightarrow$ f const ※ $\square$

Any attempt to classify surfaces uniformised by $\mathbb{D}$ must rely on an understanding of

$$\mathrm{Aut}(\mathbb{D}) = \{\text{conf equiv of } \mathbb{D}\}$$

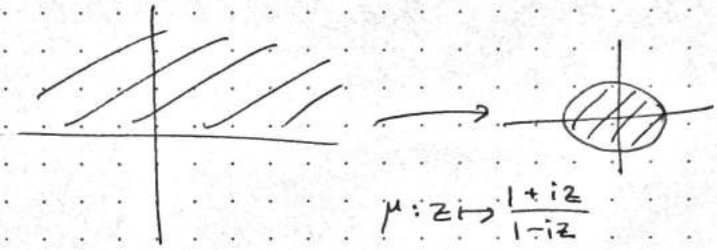
<u>Prop 16.11</u> The group of conformal equiv of $\mathbb{D}$ is the group of Möb transf^n

$$\left\{ z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z} \;\middle|\; a \in \mathbb{C},\ \theta \in \mathbb{R} \right\}$$

acting on $\mathbb{D}$ in the natural way.

The group is easier to understand if we conjugate to

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}\, z > 0 \}$$



From this point of view,

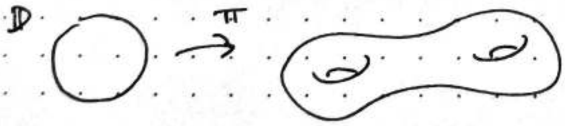
$$\mathrm{Aut}(\mathbb{H}) = PSL_2(\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; \begin{array}{l} ad-bc=1 \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

<u>Def 16.13</u> A subgroup of $PSL_2(\mathbb{R})$ that acts freely & properly discontinuously on $\mathbb{H}$ is called a <u>Fuchsian group</u>

<u>16.3 Consequences of uniformisation</u>

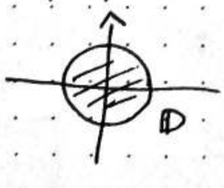
Surfaces uniformised by $\mathbb{C}_\infty$ or $\mathbb{C}$ all had genus ≤ 1

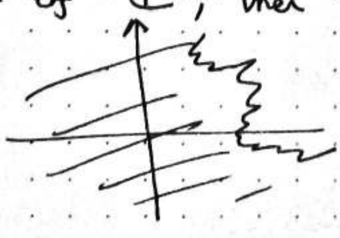
<u>Cor 16.14</u> If R is a cpct Riemann surface and the genus of R is at least 2 then R is uniformised by $\mathbb{D}$



Even the case of $R \subseteq \mathbb{C}$ is a deep theorem!

<u>Cor 16.15</u> (Riemann mapping theorem) If $D \subsetneq \mathbb{C}$ is a simply connected, proper subdomain of $\mathbb{C}$, then D is conformally equivalent to $\mathbb{D}$





Proof By uniformisation, STP $\mathbb{D} \not\cong \mathbb{C}_\infty$ or $\mathbb{C}$

(i) $\mathbb{C}_\infty$ cpt, $\mathbb{D}$ isn't
not the disc, $D \subsetneq \mathbb{C}$

(ii) Suppose $f : \mathbb{C} \xrightarrow[\text{c.e.}]{\sim} D$. Consider the singularity at ∞.

If it's essential:

Casorati-Weierstrass $f(\{|z|>1\})$ dense in $\mathbb{C}$

OMT $\Rightarrow f(\{|z|<1\})$ open (non-empty)

$\therefore$ f not injective $\divideontimes$

So ∞ is removable or a pole and f extends

$$\bar{f} : \mathbb{C}_\infty \longrightarrow D \cup \{\bar{f}(\infty)\} \subseteq \mathbb{C}_\infty$$

But now

$\mathbb{C}_\infty$ compact $\Rightarrow \bar{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ surjective

$\Rightarrow \bar{f}(\infty) = \infty$

$\Rightarrow D = \mathbb{C}_\infty \setminus \{\bar{f}(\infty)\} = \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$

$\therefore$ D is not a proper subdomain of $\mathbb{C}$ $\divideontimes$ $\square$

We finish with some complex analysis

Cor 16.16 (Picard's theorem) Any analytic function

$$f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\} \quad \text{is constant}$$

Proof Q9, Ex Sheet 3 $\Rightarrow$ $\mathbb{C} \setminus \{0,1\}$ unitarised by $\mathbb{D}$

LIFTING LEMMA $\Rightarrow \exists F : \mathbb{C} \to \mathbb{D}$ s.t. $f = \pi \circ F$

Now F const by Liouville $\Rightarrow f$ const. $\square$