III Algebraic Geometry

L1.1
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Plan
1. Spectrum of a ring ; sheaves
2. Define scheme & morphisms
3. Properties of schemes
4. Introduction to sheaf cohomology

Resources · Videos · Course page (Google Dhruv) ; notes
- Texts : Hartshorne ; Vakil ; ...
- web : Math Stack X ; Mathoverflow
- YouTube : AGITTOC, "pseudolectures"

Why suffer through this ?

Weil Conjectures : $f \in \mathbb{Z}[X_0, \dots, X_n]$, homogeneous

Two worlds :

World I over the complex numbers, this gives a projective variety
$$X_{\mathbb{C}} \subseteq \mathbb{P}^n_{\mathbb{C}}$$
Think of $X_{\mathbb{C}}$ as a 'Euclidean' subspace

Assumption $X_{\mathbb{C}} = \mathbb{V}(f)$ is non-singular

$\Rightarrow X_{\mathbb{C}}$ is a manifold

New invariants of $X_{\mathbb{C}}$ from topology

$$\left.\begin{array}{ll} \chi_{top}(X_{\mathbb{C}}) & \text{top. Euler characteristic} \\ b_0, b_1, \dots, b_{2n} & \text{Betti numbers} \end{array}\right\} \text{integers}$$

World II fix a prime number p

Assume $\mathbb{V}(f) \subseteq \mathbb{P}^n_{\mathbb{F}_p}$ is non-singular

Define $N_m = \#$ points in $X(\mathbb{F}_{p^m})$

Weil zeta function $\zeta(X, t) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} t^m\right)$

Weil's Conjecture / Grothendieck's Theorem

1 $\zeta(X,t)$ is a ratio of polynomials in t (!)
$$= \frac{P_0(t)\, P_2(t) \cdots P_{2n}(t)}{P_1(t)\, P_3(t) \cdots P_{2n-1}(t)}$$

2. Degree of $P_i(t)$ is the Betti number b_i of $X_{\mathbb{C}}$

§1. Beyond Algebraic Varieties

1.1. Summary $k = \bar{k}$ algebraically closed field

Affine varieties $\left\{ \begin{array}{l} \text{subsets of } \mathbb{A}^n_k \text{ given as} \\ \mathbb{V}(S),\ S \subseteq k[X_1, \dots, X_n] \end{array} \right\} /$ isomorphism

$\xleftrightarrow{\text{Correspondence}} \left\{ \begin{array}{l} \text{finitely generated } k\text{-algebras} \\ \text{w/o } \underline{\underline{\text{nilpotent}}} \text{ elements} \end{array} \right\}$

1.1.1 Basic Structures

Given an affine variety V ;

coordinate ring = functions on V = $\dfrac{k[X_1, \dots, X_n]}{\mathcal{I}} = \mathcal{O}_V = k[V]$

$V \hookrightarrow \mathbb{A}^n_k$ with ideal $\mathcal{I}$

ideal associated to a variety (with chosen embedding)

$\mathcal{I}(V) = \{ f \in k[X_1, \dots, X_n] \mid f(p) = 0 \ \forall p \in V \}$

1.1.2 Topology

Given $V \subset \mathbb{A}^n_k$, equip it with a Zariski topology

affine variety

Declare that $\mathbb{V}(S)$ are closed for

$S \subseteq k[V] = \mathcal{O}_V$

(Check if $V \subseteq \mathbb{A}^n_k$; the restriction of Z-topology from $\mathbb{A}^n_k$ coincides with the intrinsic one)

1.1.3 Nullstellensatz

Given $p \in V$, we have $\left. \begin{array}{l} ev_p : k[V] \to k \\ \quad f \mapsto f(p) \end{array} \right\}$ surjective ring hom

$\therefore \ker(ev_p) = m_p$

Nullstellensatz: the points of V are in natural bijection with the maximal ideals of $k[V]$

1.1.4 Function theory / morphisms

Given $f \in k[V]$ we get $f: V \to \mathbb{A}^1_k = k$;

similarly $(f_1, \dots, f_m): V \longrightarrow \mathbb{A}^m_k$

Morphisms Given V, W affine varieties ; $W \subseteq \mathbb{A}^m_k$

a morphism $V \to W$ is a morphism $\varphi: V \to \mathbb{A}^m_k$

s.t. $\mathrm{im}(\varphi) \subseteq W$

Correspondence $\{\varphi : V \to W\} \longleftrightarrow \{\varphi^*: k[W] \longrightarrow k[V]\}$
maps of k-algebras

§1.2 Limitations

Ex 1.2.1 (non-algebraically-closed fields)

$I = (X^2 + Y^2 + 1) \subseteq \mathbb{R}[X, Y]$

$\mathbb{V}(I) = \emptyset$ in $\mathbb{R}^2$ but I is prime

Nullstellensatz fails

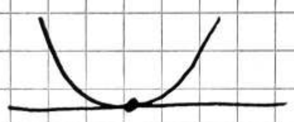
Question 1.2.2 On what topological space is $\mathbb{R}[X,Y]/I$ (or even $\mathbb{R}[X,Y]$)

NATURALLY the space of functions

Q1.2.3 Similar question, for the ring $\mathbb{Z}$, $\mathbb{Z}[X]$?

Example 1.2.3 (Why restrict to radical ideals?) $k = \mathbb{C}$

Take $C = \mathbb{V}(y - x^2)$, $D = \mathbb{V}(y)$



solve $y = 0$, $y - x^2 = 0$ in 2-variables

$k[x,y]/(y, y-x^2) \cong k[t]/t^2$

§ Spectrum of a ring (1.3)

Let A be a commutative ring with identity.

Goal: Identify / construct a "space" X_A on which A is naturally the ring of functions

Requirement: given $p \in X_A$, we should have an evaluation map

$$ev_p : A \longrightarrow K_p \quad ; \quad ev_p \text{ is a ring hom}$$

⚠ Not asking $K_p = K_{p'}$ for $p, p' \in X_A$; nor for ev_p to be surj

Definition 1.3.1 The Zariski spectrum Spec A of a ring A is the set of prime ideals.

Remark (Why not only maximal ideals?)

Given a ring hom $A \to B$, there is always an induced map

$$f^*: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$
$$\mathfrak{p} \longmapsto f^{-1}(\mathfrak{p})$$

In general, no such map on the set of maximal ideals

Example 1.3.2 $A = \mathbb{Z}$, $\operatorname{Spec}(\mathbb{Z}) = \underbrace{(0)}_{\text{prime}} \cup \underbrace{\{(p) \mid p \text{ prime in } \mathbb{Z}\}}_{\text{maximal ideals}}$

Pick a function $132 \in \mathbb{Z}$

This is a function on Spec $\mathbb{Z}$

Let $\mathfrak{p} \subseteq \mathbb{Z}$ a prime. Then

$$132(\mathfrak{p}) = 132 \bmod \mathfrak{p} \in \mathbb{Z}/\mathfrak{p}$$

The function vanishes at the points $(11), (2), (3), \ldots$

Funny: $\mathfrak{p} = (0)$, $132((0)) = 132 \in \mathbb{Q}$

Example 1.3.3 $A = \mathbb{R}[X]$, what is the spectrum ?

Claim (0) is a prime ideal

$(X - a)$, $a \in \mathbb{R}$ are prime (& maximal)

$(X^2 + aX + b)$ w/ $a^2 - 4b$ negative are prime (& max)

PICTURE $\mathbb{C}/$complex conj $\cup$ (0) 

Exercise 1.3.4 Describe Spec A, $A = \mathbb{Z}[x]$, $k[x]$ for k arbitrary (field)

§1.4 Topology (A is a ring)

The set Spec A has a Zariski topology

Fix $f \in A$, define vanishing locus $\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid f(\mathfrak{p}) = 0 \Leftrightarrow f \in \mathfrak{p}\}$

Similarly if $J \subseteq A$ is an ideal, $\mathbb{V}(I)$ defined similarly

Proposition 1.4.1 The sets $\mathbb{V}(J) \subseteq$ Spec A (ranging over ideals J) form the closed sets of a Topology on Spec A.

Proof 4 axioms to check

Spec A & $\emptyset$ are both of the form $\mathbb{V}(J)$

Since $\mathbb{V}(\sum_\alpha J_\alpha) = \bigcap_\alpha \mathbb{V}(J_\alpha)$, arbitrary intersections of closed sets remain closed.

Requires thought: $\mathbb{V}(J_1 \cap J_2) = \mathbb{V}(J_1) \cup \mathbb{V}(J_2)$

$\supseteq$: clear

Claim $\mathbb{V}(J_1 \cap J_2) \subseteq \mathbb{V}(J_1) \cup \mathbb{V}(J_2)$

Notice $J_1 \cap J_2 \supseteq J_1 J_2$

So if $J_1 J_2 \subseteq \mathfrak{p}$ then either J_1 or J_2 is contained in $\mathfrak{p}$. $\square$

can prove easily, come on Dhruv

Example 1.4.2 (Compare with old school stuff)

Let $k = \bar{k}$. Examine Spec $k[x,y]$.

This is $\underbrace{k^2}_{\text{Type I}} \cup \cancel{\{(0)\}}$ $\underbrace{\text{one point for each irred affine curve}}_{\text{Type II}}$ $\cup \{\underbrace{(0)}_{\text{Type III}}\}$

← that this is all of them is non-trivial, see 3.2.E in Vakil

Verify:

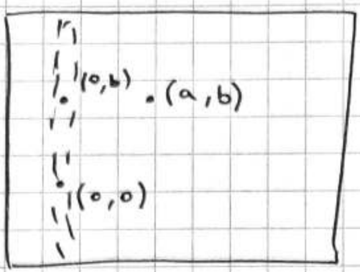
the type II, III points are not closed!

The closure ... Let f be an irreducible 2-variable polynomial e.g. $x^2 - y^3$. This determines a point $\mathfrak{p}_{(x^2-y^3)} \in$ Spec $k[x,y]$.

Let $(a,b) \in k^2$ be a point on which $x^2 - y^3$ vanishes.

Then in fact $\mathfrak{p}_{(a,b)} = (x-a, y-b) \supseteq \mathfrak{p}_{(x^2-y^3)}$.

Bad Picture



k^2

$$\mathfrak{P}(a,b) = (x-a, y-b) \subseteq k[x,y]$$

$$\mathfrak{P}(x) \in \operatorname{Spec} k[x,y]$$

☆ Points are not closed in Zariski topology of Spec A

§1.5 Functions on open sets

Let $f \in A$. Then we define the distinguished open corresponding to f

$$U_f = \operatorname{Spec}(A) \setminus \mathbb{V}(f)$$

Lemma 1.5.1 The distinguished opens form a basis for the Zariski topology on Spec A

Pf Exercise.

Lemma 1.5.2 The Zariski topological subspace U_f is naturally homeomorphic to $\operatorname{Spec} A[1/f]$

Pf Primes in $A[1/f]$ are the primes of A that miss f (Check details). □

Example 1.5.3 Take $A = \mathbb{C}[X,Y]$ and $f = XY$. Then

XY is invertible away from the axes.

Then $A[\frac{1}{f}] = \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$.

Fix a ring A. Let $X = \operatorname{Spec} A$.

$$\begin{aligned} \text{Distinguished Opens} &\longrightarrow \text{Ring} \\ U_f &\longmapsto A[1/f] \end{aligned}$$

If $U_f \subseteq U_{f'}$, then there is a restriction map

$A_{f'} \to A_f$, ring hom

▷ A word about references — Geometry of schemes, Eisenbud–Harris

▷ Last time $X = \operatorname{Spec} A$ | Distinguished opens

$U_f \cong \operatorname{Spec} A_f \curvearrowright$ Rings $A_f = \mathcal{O}_X(U_f)$

General opens

Let $\mathcal{U} \subseteq X$ be an arbitrary open.

By Lemma 1.5.1, we can write $\mathcal{U} = \bigcup_\lambda B_\lambda$

where B_λ are distinguished opens.

What is the natural ring of functions on $\mathcal{U}$?

Define $\mathcal{O}_X(\mathcal{U}) = \{$ families $(f_V)_{V \subseteq \mathcal{U}}$; V a distinguished open s.t. if $V \subseteq W \subseteq \mathcal{U}$ with V, W distinguished then f_W restricts to f_V $\}$

Exercise: $\mathcal{O}_X(\mathcal{U})$ is a ring

Moreover if $\mathcal{U}' \subseteq \mathcal{U}$ is an inclusion of opens then the restriction maps on distinguished opens determines a restriction map $\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_X(\mathcal{U}')$

§2 SHEAVES

2.1 Presheaves

Keep the following example in mind:

If X is a top space,

$$\begin{aligned} \text{Open sets in } X &\longrightarrow \text{Ab Groups} \\ \mathcal{U} &\longmapsto \{ f : \mathcal{U} \to \mathbb{R} \mid \text{continuous} \} \end{aligned}$$

Def^n 2.1.1 (Presheaf) A presheaf of abelian groups on a topological space X is an association

$$\begin{aligned} \mathcal{F} : \text{Opens in } X &\longrightarrow \text{Ab Groups} \\ \mathcal{U} &\longmapsto \mathcal{F}(\mathcal{U}) \end{aligned}$$

such that we have restriction maps

$\operatorname{res}^{\mathcal{U}}_V$; if $\mathcal{U} \supseteq V$ opens, there is a hom of Ab Groups

$$\operatorname{res}^{\mathcal{U}}_V : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V)$$

subject to basic compatibilities:

(i) $res^U_U = id_{\mathcal{F}(U)}$

(ii) $res^V_W \circ res^U_V = res^U_W$ for $U \supseteq V \supseteq W$

<u>Def^n 2.1.3</u> A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves on X is the data of: for each open $U \subseteq X$ a hom of Ab Groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$

commuting with restriction maps:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "res^U_V"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d, "res^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

⚠ res^U_V different

<u>§2.2 Sheaf - Definition & Examples</u>

<u>Def^n 2.2.1</u> (Sheaf axioms) A sheaf $\mathcal{F}$ is a presheaf satisfying:

(S1) If $U \subseteq X$ is open, $\{U_\alpha\}$ an open cover of U, then for $s \in \mathcal{F}(U)$, if $\forall \alpha$, $res^U_{U_\alpha} s = 0$ then $s = 0$

(S2) If U, $\{U_\alpha\}$ as above, and we have $s_\alpha \in \mathcal{F}(U_\alpha)$ s.t. $s_\alpha = s_\beta$ on $U_\alpha \cap U_\beta$ (! abuse !) then there exists $s \in \mathcal{F}(U)$ s.t. $\forall \alpha$, $res^U_{U_\alpha} s = s_\alpha$

<u>Notation</u> $res^U_V(s) = s|_V$; refer to elements of $\mathcal{F}(U)$ as the "sections of $\mathcal{F}$ over U"

<u>Example 2.2.2</u> / Exercise

X top space, $\mathcal{F}(U) = \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$ with obvious res. maps ; $\mathcal{F}$ is a sheaf

<u>Example 2.2.3</u> $X = \mathbb{C}$, standard topology

$\mathcal{F}(U) = \{ f : U \to \mathbb{C} \mid \text{bounded \& analytic} \}$

is <u>not</u> a sheaf

<u>Example 2.2.5</u> (constant sheaf)

Fix a group G; set $\mathcal{F}(U) = \left\{ f : U \to G \mid f \text{ locally constant} \right\}$

Check this is a sheaf.

Note that if U is connected then $\mathcal{F}(U) \cong G$

However the association $U \mapsto G$ for all U is NOT a sheaf.

<u>Example 2.2.6</u> (structure sheaf of an irreducible variety)

Let X be a [affine / projective / quasi-projective] variety and assume X is irreducible. Set

$$\mathcal{O}_X(U) = \{ f \in \underbrace{k(X)}_{\text{FF(coord ring)}} \mid f \text{ regular at all points of } U \}$$

f is regular at p if f can be written as $f = \frac{r}{s}$ with $r, s \in k[X]$ and s is non-zero at p.

⌜Check $U \mapsto \mathcal{O}_X(U)$ is a sheaf⌟

<u>§2.3 Basic Constructions</u> If $\mathcal{F}$ is a sheaf on X, and p is a point of X, there is an abelian group $\mathcal{F}_p$ — the <u>stalk</u> of $\mathcal{F}$ at p — defined as follows

$$\mathcal{F}_p = \left\{ (s, U) \;\middle|\; \begin{matrix} s \in \mathcal{F}(U) \\ U \ni p \text{ open} \end{matrix} \right\} \Big/ \sim$$

where $(s, U) \sim (s', U')$ if there exists $W \subseteq U \cap U'$ with $W \ni p$ s.t. $s|_W = s'|_W$.

<u>Exercise 2.3.2</u> Calculate $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}, 0}$

⌜rational functions $f \in \mathbb{C}(x)$ s.t. $f(0) \neq \infty$⌟

Proposition 2.3.3 If $f: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves that induces an isomorphism $f_p: \mathcal{F}_p \to \mathcal{G}_p$ for every point $p \in X$, then f is an isomorphism.

Remark 2.3.4 (i) The morphism $\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$ is injective

(ii) Given $\mathcal{F} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathcal{G}$ with $\varphi_p = \psi_p$ for all $p \in X$, then $\varphi = \psi$.

Proof of 2.3.3 If $f: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves inducing iso's on stalks $f_p: \mathcal{F}_p \to \mathcal{G}_p$

we will show that $f_U: \mathcal{F}(U) \to \mathcal{G}(U)$

are isomorphisms.

Given this, we define $f^{-1}: \mathcal{G} \to \mathcal{F}$

via f^{-1} on U given by $(f_U)^{-1}$.

[Check this gives an inverse of f] ✓

Injectivity Let $s \in \mathcal{F}(U)$ be s.t. $f_U(s) \in \mathcal{G}(U)$ is zero.

Then the $\underbrace{\text{germ}}_{\text{image of } s \text{ in } \mathcal{F}_p}$ of s at p is zero for all $p \in U$.

If all the stalks of s are zero then is zero. ⌜Rk 2.3.4 (i)⌟

Surjectivity Let $t \in \mathcal{G}(U)$, and try to build $s \in \mathcal{F}(U)$.

Write t_p for the germ of t at p.

So now let $s_p \in \mathcal{F}_p$ be the preimage of t_p.

Choose a pair (V_p, s_{V_p}) representing s_p.

By shrinking V_p if necessary, we can assume

$$t|_{V_p} = f(s_{V_p})$$

Need to show that the s_{V_p} glue across the opens V_p.

Given V_p, V_q, need to show $s_{V_p}|_{V_p \cap V_q} = s_{V_q}|_{V_p \cap V_q}$.

By injectivity, done:

$$f_{V_p \cap V_q}\left(s_{V_p}|_{V_p \cap V_q} - s_{V_q}|_{V_p \cap V_q} \right) = 0$$

Then the s_{V_p}'s, glue & we get $s \in \mathcal{F}(U)$.

By the sheaf axioms, $s \mapsto t$ under f on $\mathcal{U}$. $\square$

<u>Definition 2.3.5</u> (Sheafification) If $\mathcal{F}$ is a presheaf, then a morphism $sh : \mathcal{F} \to \mathcal{F}^{sh}$
is called a sheafification if $\mathcal{F}^{sh}$ is a sheaf & for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf there is a unique map

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d, dotted, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

making diagram commute.

<u>Remark 2.3.6</u> (1) $\mathcal{F}^{sh}$ is unique if it exists
(2) Presheaf morphisms induce morphisms of sheafifications

<u>Proposition / Construction 2.3.7</u> (Sheafifications exist)
Given a presheaf $\mathcal{F}$ on X, define

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (f_p)_{p \in \mathcal{U}} \;\middle|\; \begin{array}{l} f_p \in \mathcal{F}_p \text{; for every } p \text{ there is open } V_p \ni p \\ \text{and a section } s \in \mathcal{F}(V_p) \text{ s.t. } s_q = f_q \text{ in } \mathcal{F}_q \\ \text{for } q \text{ in } V_p \end{array} \right\}$$

This is a sheaf. ⌜restrictions are obvious⌟

<u>Proof that this works</u>
(i) The map $\mathcal{F} \to \mathcal{F}^{sh}$ is clear ✓
(ii) The sheaf axioms are tautologically satisfied ✓
(iii) <u>Exercise / observation</u>: this satisfies the universal property

<u>Corollary 2.3.8</u> The stalks of $\mathcal{F}$ and $\mathcal{F}^{sh}$ coincide

<u>Exercise 2.3.9</u> Find a non-zero presheaf on a topological space X such that its sheafification is zero.

<u>§2.4 Kernels, Cokernels, Etc.</u>

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves.
The presheaf <u>kernel / image / cokernel</u> assigns

$$\mathcal{U} \mapsto \ker / \operatorname{img} / \operatorname{coker} \big(\varphi(\mathcal{U}) : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U}) \big)$$

<u>Exercise 2.4.1</u> If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the presheaf kernel is a sheaf.

⚠ Beware of the cokernel

<u>Example 2.4.2</u> $X = \mathbb{C}$ with Euclidean topology

$\mathcal{O}_X = $ (sheaf of holomorphic functions, $+$)

$\mathcal{O}_X^* = $ (sheaf of nowhere zero holomorphic functions, $\times$)

Now define $\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*$, sheaf map.

by sending

$$\begin{array}{ccc} f & \longmapsto & e^f \\ \cap & & \cap \\ \mathcal{O}_X(U) & & \mathcal{O}_X^*(U) \end{array}$$

Then $\ker(\exp)$ is the constant sheaf $2\pi i \mathbb{Z}$.

But $\operatorname{coker}(\exp)$ is not a sheaf. (Why?)

Take $U = \mathbb{C} \setminus \{0\}$, $U_1 = \mathbb{C} \setminus [0, -\infty]$

$U_2 = \mathbb{C} \setminus [0, +\infty]$

Now take $f = z$ in $\mathcal{O}_X^*(U)$.

On restriction to U_1, U_2, logarithm of f exists (so zero in coker)

but on U, the f is non-zero in the cokernel.

▷ Ex Sheet 1 (up!)

<u>Def^n 2.4.3</u> For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves on X, the sheaf image and sheaf cokernel are the <u>sheafification</u> of the presheaf

$$U \longmapsto \text{image}\left(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\right)$$

resp. $U \longmapsto \text{coker}\left(\varphi(U)\right)$

<u>Remark 2.4.4</u> If $X =$ a complex manifold (or $X = \mathbb{C}$), there is an exact sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$ <u>Exponential Sequence</u>

<u>Remark 2.4.5</u> Not!

<u>Def^n 2.4.3 (cont.)</u> A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective / surjective if $\ker\varphi$ is the constant $\underline{0}$ / $\text{im}(\mathcal{F}) = \mathcal{G}$

<u>Remark 2.4.5</u> (What properties should the kernel satisfy?)

The kernel of a morphism $\varphi : A \to B$ is a pair $(\ker\varphi, \ker\varphi \to A)$ with the following universal property:

for any diagram

$$\begin{tikzcd} & K \arrow[dl, dashed, "\exists!"'] \arrow[d] \arrow[dr, "0"] & \\ \ker\varphi \arrow[r] & A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

can fill in dotted arrow.

<u>Exercise</u> Cokernel / image

Note kernels etc. of morphisms of sheaves satisfy this universal property.

<u>Proximate Notions 2.4.6</u>

(1) <u>Subsheaf</u> $\mathcal{F} \subseteq \mathcal{G}$ is a subsheaf if for every open U there are inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with restriction

(2) <u>Quotient sheaf</u> Given $\mathcal{F} \subseteq \mathcal{G}$ a subsheaf, the quotient sheaf $\mathcal{G}/\mathcal{F}$ is

$$\left(U \mapsto \mathcal{G}(U)/\mathcal{F}(U)\right)^{sh}$$

Warning 2.4(7-8) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a surjective map of sheaves then for any particular U,

$\mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective.

Facts 2.4.7 (1) The stalks of the kernel / image of a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ are the kernels/images of the maps on stalks.

(2) Injectivity and surjectivity are stalk local properties, i.e. they can be checked on stalks. (but this does not mean maps on sections are surjective if stalk maps are surjective)

§2.5 Moving between spaces

Given $f : X \to Y$ a continuous map and sheaves $\mathcal{F}$ on X, $\mathcal{G}$ on Y.

Definition 2.5.1 (pushforward) Define the presheaf on Y $f_* \mathcal{F}$ by

$$\underset{\substack{\text{in} \\ Y}}{U} \mapsto \mathcal{F}(f^{-1}(U))$$

Proposition 2.5.2 $f_* \mathcal{F}$ is a sheaf on Y

Definition 2.5.3 (inverse image sheaf) First define the presheaf inverse image

$(f^{-1}\mathcal{G})^{pre}$ by the assignment

$$\underbrace{V}_{\text{open on } X} \mapsto \{ (s_U, U) \mid U \text{ is open containing } f(V),\ s_U \in \mathcal{G}(U) \} / \sim$$

where $(s_U, U) \sim (s_W, W)$

if they agree on some open $W' \subseteq U \cap W$ containing $f(V)$.

Now this defines $((f^{-1}\mathcal{G})^{pre})^{sh}$.

Remark 2.5.4 (Why is sheafification necessary)

Take $f : X \to Y$ ~~xxxxxxxx~~ where $X = Y \amalg Y$.

Take $\mathcal{G}$ to be any sheaf on Y.

Take $\mathcal{F} = (f^{-1}\mathcal{G})^{pre}$.

Then for $V \subseteq Y$ open, and $U = f^{-1}(V)$ so

$$\mathcal{F}(U) = \mathcal{G}(V)$$

But $U = V \amalg V$, $\mathcal{F}^{sh}(U) = \mathcal{G}(V) \oplus \mathcal{G}(V)$

§3. SCHEMES

A - ring; we will globalise (Spec A, sheaf of rings) to define a scheme.

§3.1 Affine Schemes

Localisation of rings. Let $S \subseteq A$ is a multiplicatively closed subset — $1 \in S$ and if $f, g \in S$ then $fg \in S$.

Example 3.1.1 (i) Take $f \in A$ and $S = \{1, f, f^2, \dots\}$

(ii) Take $\mathfrak{p} \subset A$ prime and $S = A \setminus \mathfrak{p}$.

For $S \subseteq A$ as above, define

$$S^{-1}A = \left\{ (a, s) \;\middle|\; \begin{matrix} a \in A \\ s \in S \end{matrix} \right\} / \sim$$

with $(a, s) \sim (a', s')$ if

$(as' - a's)t = 0$ for some $t \in S$

Notation: we write A_f for $S^{-1}A$ with $S = \{1, f, f^2, \dots\}$

$A_{\mathfrak{p}}$ for $S^{-1}A$ with $S = A \setminus \mathfrak{p}$

⚠ Conflict: $\mathbb{Z}_{(2)}$ vs $\mathbb{Z}_2$

Recall a baes/basis for a topology on X is a collection $\mathcal{B} = \{B_i\}$ s.t any open $U \subseteq X$ is a union of opens in $\mathcal{B}$.

<u>Claim</u> A sheaf is determined by its sections over a basis

<u>Conversely</u>: Given only the data of a sheaf on $B_i \in \mathcal{B}$ how can we construct a sheaf on X?

X a topological space

$\mathcal{B} = \{B_\alpha\}$ a base of open sets for X

Consider an assignment $F: \mathcal{B} \longrightarrow$ Ab Groups (or whatever...)

$$B \longmapsto F(B)$$

with restriction maps $\mathrm{res}^B_{B'} : F(B) \to F(B')$ for $B' \subseteq B$

s.t. $\mathrm{res}^B_B = \mathrm{id}$ and $\mathrm{res}^B_{B''} = \mathrm{res}^{B'}_{B''} \circ \mathrm{res}^B_{B'}$

<u>Remark</u> This determines a presheaf on X

Assume further that F satisfies

<u>SB1</u>: if $B = \bigcup_i B_i$ and $f, g \in F(B)$ verify

$f|_{B_i} = g|_{B_i}$ for all i

then $f = g \in F(B)$

<u>SB2</u>: if $B = \bigcup_i B_i$ and $f_i \in F(B_i)$ verify

$$f_i|_{B''} = f_j|_{B''} \quad \text{for all } B'' \subset B_i \cap B_j$$

then $\exists\, f \in F(B)$ s.t.

$$f|_{B_i} = f_i \in F(B_i) \quad \text{for all } i$$

<u>Terminology</u>: Call all this data a <u>sheaf on a base</u>

<u>Proposition</u> 3.12 ("this is enough data")

If F is a sheaf on the base $\mathcal{B}$ then there exists a sheaf $\mathcal{F}$ on X with identifications

$\mathcal{F}(B) = F(B)$ commuting with restriction

Moreover $\mathcal{F}$ is unique up to unique isomorphism.

<u>Proof</u> (1) Define the stalks $\mathcal{F}_p$ via $F(B_i)$ for B_i basis opens containing p i.e. take

$$\mathcal{F}_p = \{(s, B) \mid s \in F(B),\ B \ni p\} / \sim$$

where $(s, B) = (s', B')$

if $\exists\, B'' \subset B \cap B'$ s.t. $s|_{B''} = s'|_{B''}$, $p \in B''$

(colimit operation)

(2) Use the stalks $\mathcal{F}_p$ to define $\mathcal{F}(U)$ via the "sheafification trick":

$$\mathcal{F}(U) = \left\{ (f_p \in \mathcal{F}_p)_{p \in U} \;\middle|\; \begin{array}{l} \forall p \in U \ \exists \text{ basis open } B \ni p \text{ s.t.} \\ \exists\, s \in F(B) \text{ with } \forall q \in B,\ s_q = f_q \end{array} \right\}$$

(obvious meaning)

[sheaf axioms on U are satisfied]

(3) Natural maps: $F(B) \longrightarrow \mathcal{F}(B)$ are isomorphisms [Exercise]. $\square$

Specialise to rings, Zariski spectra, Z-topology, distinguished opens U_f.

<u>Notation</u>
$$\begin{cases} A \text{ - ring} \\ f \in A \\ A_f = S^{-1}A \text{ for } S = \{1, f, f^2, \dots\} \end{cases}$$
$$U_f = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) \neq 0 \}$$

waffle see 4.1.1 in Vakil

<u>Observation</u> (i) U_f only depends on the smallest multiplicatively closed set containing f, so e.g. $U_f = U_{f^2}$

(ii) $U_f \cap U_g = U_{fg}$

<u>Proposition 3.13</u> The assignment

$$U_f \longmapsto A_f$$

is a sheaf on the base of Spec A given by the distinguished opens, with restriction maps given by localisation.

restriction maps are subtle!

<u>Terminology</u> A topological space X is quasi-compact if every open cover has a finite subcover.

<u>Lemma 3.1.4</u> Spec A is quasi-compact

<u>Proof</u> (for distinguished opens)

$$\operatorname{Spec} A = \bigcup_{\alpha \in I} U_{f_\alpha}$$

Then $\displaystyle\bigcap_\alpha \mathbb{V}(f_\alpha) = \emptyset$

i.e. $(f_\alpha)_{\alpha \in I}$ generate the ideal (1)

But then $1 = \sum_{\text{finite}} r_i f_i \ , \ r_i \in A$

$$\Rightarrow \bigcup_{\text{finite}} U_{f_i} = \operatorname{Spec} A \qquad \square$$

Proof of Prop 3.1.3 We will check SB1 & SB2

SB1: Write $\operatorname{Spec} A = \bigcup_{i=1}^{n} U_{f_i}$

Given $s \in A$ with the property that

$$s|_{U_{f_i}} = 0 \quad \forall i$$

Then $f_i^m s = 0$ for some fixed m, all i (used quasi)

But $U_{f_i^m} = U_{f_i} \Rightarrow (f_1^m, \dots, f_n^m) = (1)$

$$1 = \sum r_i f_i^m \Rightarrow s = \sum r_i f_i^m s = 0$$

SB2: Say $\operatorname{Spec} A = \bigcup_{i \in I} U_{f_i}$

$\rightsquigarrow$ first assume that I finite

Given elements in each A_{f_i} that agree in $A_{f_i f_j}$, does there exist an element of A restricting to these?

On U_{f_i}, we have an element

$$\frac{a_i}{f_i^{L_i}} \in A_{f_i} \ ; \quad \text{write } g_i = f_i^{L_i}$$

Overlap agreement on $U_{g_i g_j}$ is

$$(g_i g_j)^{m_{ij}} (a_i g_j - a_j g_i) = 0 \quad \text{in } A$$

Take $M = \max_{i,j} m_{ij} < +\infty$.

Write

$b_i = a_i g_i^{M}, \quad h_i = g_i^{M+1}$; on each U_{h_i} we have

$\frac{b_i}{h_i}$ overlap condition $h_j b_i = h_i b_j$

But $\mathrm{Spec}\, A = \bigcup U_{h_i}$. So

$$1 = \sum r_i h_i \quad \text{with } r_i \in A$$

Take $r = \sum r_j b_j$.

This restricts correctly by the overlap condition

When I is infinite, pick finitely many $\{f_1, \dots, f_n\}$ s.t. $(f_1, \dots, f_n) = (1)$

Construct r as above. But if we were given $\{f_1, \dots, f_n, f_\beta\}$ we would get a new element r' via the same construction.

But $r - r'$ would have to be 0 by SB1. $\square$

Terminology 3.1.5 A ringed space $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings.
An isomorphism of ringed spaces

$$\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a homeomorphism $\pi : X \to Y$ and an isomorphism of sheaves of rings $\mathcal{O}_Y \cong \pi_* \mathcal{O}_X$
[or equivalently $\mathcal{O}_X \cong \pi^* \mathcal{O}_Y$]

An affine scheme is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.
Definition 3.1.6 A scheme $(X, \mathcal{O}_X)$ is a ringed space such that every point $p \in X$ has an open nghbd U_p such that $(U_p, \mathcal{O}_X|_{U_p})$ is an affine scheme.

§ 3.2 Examples (of schemes)

Example 3.2.1 (some interesting rings)

(i) $k[x_1, \ldots, x_n]$ or $\mathbb{R}[x_1, \ldots, x_n]$
and quotients by ideals

(ii) Among such are monoid rings. A toric monoid P is the positive integer span of a finite set $\{v_1, \ldots, v_k\} \subseteq \mathbb{Z}^n$ for some n

The monoid ring of P over the integers, $\mathbb{Z}[P]$, is

$$\{ \sum a_u \underset{\substack{\uparrow \\ \text{dummy}}}{\chi}^u \mid u \in P, a_u \in \mathbb{Z} \}$$

- If $P = \mathbb{N}^2 = \operatorname{span}_+((1,0), (0,1)) \subseteq \mathbb{Z}^2$, then $\mathbb{Z}[P] = \mathbb{Z}[\mathbb{N}^2] \cong \mathbb{Z}[X, Y]$
- If $P = \mathbb{Z}^2$ then $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[X^{\pm}, Y^{\pm}]$

schemes of the form $\mathbb{Z}[P]$, even $\operatorname{Spec} k[P]$, are called affine toric schemes

(iii) Hypersurface rings $\mathbb{Z}[\underline{X}]/(f)$
(& more generally complete intersections)
(iv) Invariant rings: let G act on $k[X_1, \ldots, X_n] = R$
and consider R^G the subring of invariant polynomials;
these are the models for coordinate rings / structure sheafs of quotients of schemes by groups

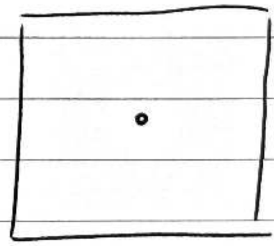
Examples 3.2.2 (Open subschemes) Let X be a scheme.
Let $U \subseteq X$ be open.
Then $(U, \mathcal{O}_X|_U)$ is a scheme [why?]
~~Take $U \equiv$~~
Notation: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, more generally $\mathbb{A}^n_R$ for arbitrary ring R
Take $U \subseteq \mathbb{A}^{n^2}_k$ to be $\mathbb{A}^{n^2}_k \setminus \{\det = 0\}$; $GL_n(k)$ is defined as $(U, \mathcal{O}_{\mathbb{A}^{n^2}_k}|_U)$

This is an example of a group scheme

Example 3.2.4 Take $U = \mathbb{A}^2_k \setminus \{(0,0)\}$ $\hookrightarrow$ ideal (x,y)
Claim: this is not affine
Pf: $\mathbb{A}^2_k \setminus \{(0,0)\}$

$$= U = \operatorname{Spec} k[x, x^{-1}, y] \cup \operatorname{Spec} k[x, y, y^{-1}]$$
$$\operatorname{Spec} k[x,y]$$

⌈If $p \in \mathbb{A}^2_k$ meets both $\{1, x, x^2, \ldots\}$ $\{1, y, y^2, \ldots\}$ then $p = (x,y)$⌋



$\mathbb{A}^2$

Ring inclusions

$$\begin{tikzcd} & k[x,y,x^{-1}] \arrow[dr, hook] & \\ k[x,y] \arrow[ur, hook] \arrow[dr, hook] & & k[x,y,x^{-1},y^{-1}] \subseteq k(x,y) \\ & k[x,y,y^{-1}] \arrow[ur, hook] & \end{tikzcd}$$

Now evaluate $\mathcal{O}_U(U) \cong k[x,y]$
But if U were affine then $U \overset{!}{\cong} \operatorname{Spec} k[x,y]$
But in $U \cong \mathbb{A}^2_k$ but $\mathbb{V}((x,y)) = \emptyset$ in U
$\neq \emptyset$ in $\mathbb{A}^2_k$. □

explained in 4.4.3 Vakil

where $i_0 : \Delta^n \longrightarrow [0,1]\times\Delta^n$, $i_1 : \Delta^n \longrightarrow [0,1]\times\Delta^n$

$x \mapsto (0,x)$ $\qquad$ $x \mapsto (1,x)$

and

$$\delta_j : \Delta^{n-1} \longrightarrow \Delta^n$$
$$(t_0,\dots,t_{n-1}) \mapsto (t_0,\dots,t_{j-1},0,t_j,\dots,t_n) .$$

Then define $H_n : C_n(X) \longrightarrow C_{n+1}(Y)$

$$(\sigma:\Delta^n \to X) \mapsto \big(H\circ([0,1]\times\sigma)\big)_\#(P_n)$$

then calculate

$$\begin{aligned} dH_n(\sigma) &= \big(H\circ([0,1]\times\sigma)\big)_\#(dP_n) \\ &= \big(H\circ([0,1]\times\sigma)\big)_\#\Big(i_1-i_0-\sum_{j=0}^{n}(-1)^j\big([0,1]\times\delta_j\big)_\#(P_{n-1})\Big) \\ &= g_\#(\sigma)-f_\#(\sigma)-\sum_{j=0}^{n}(-1)^j H_\#\circ\big([0,1]\times\sigma\circ\delta_j\big)_\#(P_{n-1}) \\ &= g_\#(\sigma)-f_\#(\sigma)-H_{n-1}(d\sigma) \end{aligned}$$

so H_n's are a chain homotopy from $g_\#$ to $f_\#$.

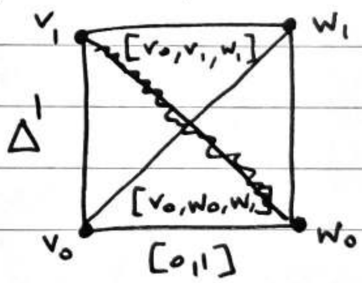
<u>It remains to invent the P_n's</u>

$P_n = \sum \pm (n+1)$-simplices in a triangulation of $[0,1]\times\Delta^n$

Let $v_0,\dots,v_n$ be the vertices of $\{0\}\times\Delta^n$

and $w_0,\dots,w_n$ those of $\{1\}\times\Delta^n$.

Regard $[0,1]\times\Delta^n \subset \underset{\text{convex}}{\mathbb{R}\times\mathbb{R}^{n+1}}$



For any sequence $\{x_0,\dots,x_{n+1}\}$ of v's and w's, let

$$[x_0,\dots,x_{n+1}] : \Delta^{n+1} \longrightarrow [0,1]\times\Delta^n$$
$$(t_0,\dots,t_{n+1}) \mapsto \sum t_i x_i \qquad \text{using convexity}$$

Set $$P_n = \sum_{i=0}^{n}(-1)^i[v_0,\dots,v_i,w_i,\dots,w_n] \in C_{n+1}([0,1]\times\Delta^n)$$

torturous → Check(!) that $dP_n = i_1 - i_0 - \sum_{j=0}^{n}(-1)^j\big([0,1]\times\delta_j\big)_\#(P_{n-1})$

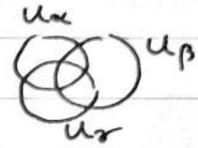
Gluing: · **Sheaves** X top space

$\{U_\alpha\}$ open cover

$\mathcal{F}_\alpha$ sheaf on U_α $\forall \alpha \in \Lambda$

$\varphi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$

transition maps



condition: $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

$\iff \varphi_{\gamma\alpha} \circ \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \mathrm{id}$

Prop $\exists\ \mathcal{F}$ on X s.t. $\mathcal{F}|_{U_\alpha} = \mathcal{F}_\alpha$

Pf Given $V \subseteq X$ define

$V = \bigcup_\alpha (V \cap U_\alpha)$ open cover

$$\mathcal{F}(V) = \left\{ \left(s_\alpha \in \mathcal{F}_\alpha(V \cap U_\alpha)\right)_{\alpha\in\Lambda} : \varphi_{\alpha\beta}(s_\alpha) = s_\beta \ \forall \alpha,\beta \right\}$$

Exercise: finish proof $\square$

Schemes: $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$

$U \subseteq X$ open, $V \subseteq Y$ open

$$(U, \mathcal{O}_X|_U) \overset{\varphi}{\cong} (V, \mathcal{O}_Y|_V)$$

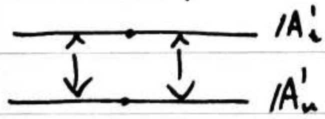
Glued scheme $X \cup_\varphi Y = (X \amalg Y)/\sim$ via $(u \in U) \sim (\varphi(u) \in V)$

$\{X, Y\}$ open cover of $X \cup_\varphi Y$ $\overset{?}{\wedge}$

$\rightsquigarrow$ get scheme $(Z, \mathcal{O}_Z)$

Examples line w/ two origins / bug-eyed line $\{k = \bar{k}\}$

$X = \mathbb{A}^1_t = \operatorname{Spec} k[t]$, $Y = \mathbb{A}^1_u = \operatorname{Spec} k[u]$

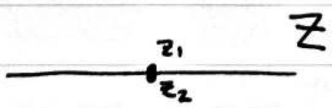


$U \subseteq X$ is $\operatorname{Spec} k[t, t^{-1}] = k^*$

$V \subseteq Y$ is $\operatorname{Spec} k[u, u^{-1}] = k^*$

$\wr$

iso $U \to V$ via $t \mapsto u$



$Z \setminus \{z_1, z_2\} = k^*$

Claim Z is not affine

Pf Assume $(Z, \mathcal{O}_Z) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

We can recover $A = \mathcal{O}_Z(Z)$.

$$\begin{array}{c} \mathcal{O}_Z(X) \searrow \\ \mathcal{O}_Z(Y) \nearrow \end{array} \mathcal{O}_Z(X \cap Y) \quad \text{is} \quad \begin{array}{ccc} k[t] & \longrightarrow & k[t, t^{-1}] \\ & & \wr\wr \\ k[u] & \longrightarrow & k[u, u^{-1}] \end{array} \qquad (\dagger)$$

$\therefore \mathcal{O}_Z(Z) = k[t]$

$\divideontimes$ because $Z \not\cong \mathbb{A}^1_t$

<u>General fact</u> $(X, \mathcal{O}_X)$ scheme $\rightsquigarrow \mathcal{O}_X(X)$ ring

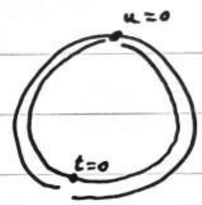
$\exists$ morphism $X \xrightarrow{p} \operatorname{Spec} \mathcal{O}_X(X)$

- X affine $\iff p$ is an isomorphism
- P initial amongst maps to affine schemes

[alternatively (t) vanishes at 2 points]

<u>Example</u>

$$\begin{array}{ccc} \mathbb{A}^1_t & \xrightarrow{\sim} & \mathbb{A}^1_u \\ \cup\!\! \mid & & \cup\!\! \mid \\ k^\times_t & & k^\times_u \\ t & \longmapsto & u^{-1} \\ k[t, t^{-1}] & \longrightarrow & k[u, u^{-1}] \end{array}$$



projective line $\rightsquigarrow \mathbb{P}^1_k$

<u>Claim</u> $\mathbb{P}^1_k$ not affine; repeat $(\dagger)$

$$\begin{array}{ccc} k[t] \longrightarrow & k[t, t^{-1}] & t \\ & \wr\wr & \downarrow \\ & k[u, u^{-1}] & u^{-1} \\ k[u] \nearrow & & \end{array}$$

Need $f(t) \in k[t]$, $g(t) \in k[u]$
s.t. $f(t) \in k[t, t^{-1}]$ maps to $g(u) = f(\frac{1}{t})$

$\therefore \mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k \qquad p : \mathbb{P}^1_k \to \operatorname{Spec} k$

<u>Example</u> $\mathbb{A}^2$ with two origins

$X = \mathbb{A}^2_{x,y}$, $Y = \mathbb{A}^2_{s,t}$

glue $\underbrace{\mathbb{A}^2_{xy} \setminus 0}_{U =} \xrightarrow{\sim} \underbrace{\mathbb{A}^2_{st} \setminus 0}_{= V}$

via $x \mapsto s$
$y \mapsto t$

$\mathcal{U} = \mathbb{A}^2_{xy} \setminus 0$
$V_1 = \mathbb{A}^2_{xy} \setminus V(x) = \operatorname{Spec} k[x, x^{-1}y]$
$V_2 = \mathbb{A}^2_{xy} \setminus V(y) = \operatorname{Spec} k[x, y, y^{-1}]$
$V_1 \cap V_2 = (k^*)^2 = \operatorname{Spec} k[x, y, x^{-1}, y^{-1}]$

$$\mathcal{O}_{\mathcal{U}}(\mathcal{U}) = k[x, y]$$

Now $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \longrightarrow \mathcal{O}_{\mathbb{A}^2}(\mathcal{U})$ is surjective,
an isomorphism of rings.

<u>Proj</u> Most interested in projective varieties

↳ Not affine

↳ construct projective vars/schemes using <u>one ring</u>

<u>Idea</u>: $\mathbb{P}^n_k = (\underset{①}{\mathbb{A}^{n+1}_k} \setminus \underset{③}{\{0\}}) / \underset{②}{k^\times}$

<u>Note</u>: work over k

<u>Defn</u>: A ($\mathbb{Z}$-)<u>graded k-algebra</u> A is a k-algebra w/ a decomposition $A = \bigoplus_{d \in \mathbb{Z}} A_d$ (A_d: k vector spaces)

s.t. $A_d \cdot A_e \subseteq A_{d+e}$.

<u>E.g.</u> $k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$ (S_d: degree d homogeneous polys)

<u>Prop</u>: Given a k-algebra A, there's a natural bijection

$$\left\{ \begin{matrix} \text{choice of} \\ \text{grading} \\ \text{on } A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} k^\times\text{-action} \\ \text{on} \\ \operatorname{Spec} A \end{matrix} \right\}$$

<u>Pf</u>: $(\to)$ $\lambda \in k^\times \rightsquigarrow \lambda : \operatorname{Spec} A \to \operatorname{Spec} A$

$$\Updownarrow$$

$$\boxed{\lambda : A \to A}$$

So let $\lambda(a \in A_d) := \lambda^d \cdot a$.

Well def ring hom from $A_d \cdot A_e \subseteq A_{d+e}$.

$(\leftarrow)$ $k^\times \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$

$(\lambda, p) \longmapsto \lambda(p)$

$\mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec} k[t, t^{-1}]$

⌜need some more condition on the morphism $k^\times \times \operatorname{Spec} A \to \operatorname{Spec} A$⌟

$$\operatorname{Spec} k[t, t^{-1}] \times \operatorname{Spec} A \to \operatorname{Spec} A$$

$$\overset{!}{\Vert}$$

$$\operatorname{Spec} A[t, t^{-1}] \qquad ⌜=⌟ \qquad A[t, t^{-1}] \xleftarrow{(*)} A$$

$$\sum_{d \in \mathbb{Z}} a_d t^d \longleftarrow a$$

□

Given $V \subseteq \mathbb{P}^n_k$ proj. variety

$V = \mathbb{V}(I)$ $\qquad x_0 - x_1^2 = 0$??

$\quad I \trianglelefteq k[x_0, \dots, x_n]$

$\boxed{\text{homogeneous ideals}} \longrightarrow$ Def^n: has a set of homogeneous generators

$\Updownarrow$ exercise

$\forall f$, $f \in I$ iff in $f = \sum f_d$ each $f_d \in I$ too

$\Rightarrow k[x_0, \dots, x_n]/I$ inherits grading ✓

$$\begin{array}{ccc} \pi^{-1}(V) & \longrightarrow & \mathbb{A}^{n+1}_k \setminus \{0\} \\ \downarrow & & \downarrow \pi \\ V = \mathbb{V}_p(I) & \subseteq & \mathbb{P}^n_k \end{array}$$

Fact: $\overline{\pi^{-1}(V)} \subseteq \mathbb{A}^{n+1}_k$

$\qquad \parallel$

$\qquad \mathbb{V}_a(I) \leftarrow$ cone over $\mathbb{V}_p(I)$

$$\boxed{\mathbb{V}_p(I) = \left(\mathbb{V}_a(I) \setminus \{0\}\right) / k^\times}$$

From now on: only consider $\mathbb{Z}_{\geq 0}$-graded k-algebras $A = \bigoplus_{d \geq 0} A_d$

$\Rightarrow A_+ = \bigoplus_{d > 0} A_d \subsetneq A$

Ex: A_+ is an ideal; the irrelevant ideal

$\mathbb{V}(A_+) \subseteq \operatorname{Spec} A$

$\qquad \parallel$

$\operatorname{Spec} A/A_+ = \operatorname{Spec} A_0 \qquad (\text{often } A_0 = k)$

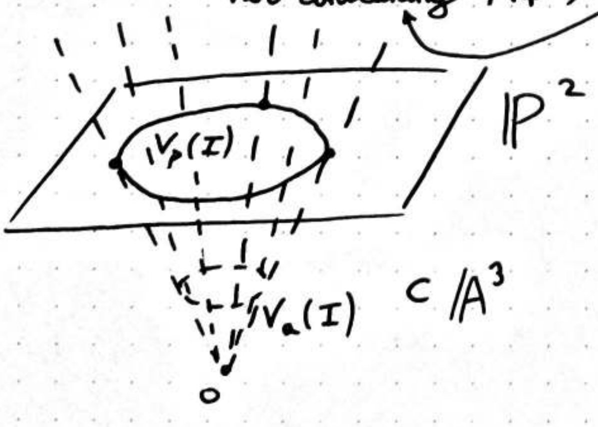
" $\operatorname{Proj} A = (\operatorname{Spec} A \setminus \operatorname{Spec} A_0) / k^\times$ "

As a set,

$$\operatorname{Proj} A = \left\{ \begin{array}{c} \mathfrak{p} \trianglelefteq A \\ \text{homogeneous primes} \\ \text{not containing } A_+ \end{array} \right\}$$

subvarieties preserved by $k^\times$-action

subvarieties not contained in $\mathbb{V}(A_+)$



Topology: closed sets are defined as

$$V(I) = \{ \mathfrak{p} \in \operatorname{Proj} A \mid \mathfrak{p} \supseteq I \} \subseteq \operatorname{Proj} A$$

for I homogeneous ideals

Structure sheaf: define by gluing affine opens

Given homogeneous $f \in A_+ \rightsquigarrow I \underset{\text{hom}}{=} (f)$

$\underbrace{\operatorname{Proj} A \setminus \mathbb{V}(f)}_{\substack{\text{claim:} \\ \text{this is an} \\ \text{affine scheme}}}$ open set in $\operatorname{Proj} A$

Consider $A_f = A\left[\frac{1}{f}\right]$

Has a grading

$$\deg\left(\frac{a}{f^k}\right) = \deg(a) - k\deg(f)$$

Consider $(A_f)_0$

In $k[x_0, \dots, x_n]$
$f = x_0$ get
$k[x_0, x_0^{-1}, x_1, \dots, x_n]$
$= k[x_0, x_0^{-1}, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$
$\cup$
$k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$

Exercise: show $\operatorname{Proj} A \setminus \mathbb{V}(f) = \operatorname{Spec}(A_f)_0$

(hints in notes)

Check structure sheaves on overlaps ☺

A graded ring, $A_0 = k$

Assume A generated by $f_1, \dots, f_k \in A_1$ (as a k-algebra)

$$k[x_1, \dots, x_n] \overset{\varphi}{\twoheadrightarrow} A$$
$$x_i \mapsto f_i$$

$\ker \varphi = I$ is homogeneous

$A \cong k[x_1, \dots, x_n]/I$

⚠ Proj not functorial

$$k[x,y] \to k[s,t]$$
$$x \mapsto s$$
$$y \mapsto s$$

$$\mathbb{P}^1_{xy} \leftarrow \mathbb{P}^1_{st}$$

$$[s:s] \leftarrow\!\!\shortmid [s:t]$$
$$☹ \leftarrow\!\!\shortmid [0:1]$$

Ex Sh II – soon™ ; feedback from I this week™

§4. Morphisms

[Why?] $\mathbb{P}^n_k \supseteq X$ variety theory

Replace k with a ring A (e.g. $\mathbb{Z}$)

Natural to view this as studying schemes X with a morphism to Spec A.

e.g. $\mathbb{P}^n_A \longrightarrow \underbrace{\operatorname{Spec} A}_{\text{scheme}}$

More generally, why only study $\{X \to \operatorname{Spec} A\}$ rather than fixing a scheme & studying morphisms $\{X \to S\}$

So far: If $(X, \mathcal{O}_X)$ is a scheme then it is a ringed space

Notion of morphism of ringed spaces is "not right"

§4.1 Locally ringed spaces & morphisms

Idea varieties & manifolds have tangent spaces

If $(X, \mathcal{O}_X)$ is a projective variety, the stalk $\mathcal{O}_{X,x}$ for any $x \in X$ is a local ring – [non-unit elements form an ideal]

↳ automatically $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$

▷ $\mathfrak{m}_x$ is functions in a ngbd of x vanishing at x

▷ $\mathfrak{m}_x/\mathfrak{m}_x^2$ = Zariski cotangent space

$\varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_x^k = \hat{\mathcal{O}}_{X,x}$ [Taylor series expansion]

DEFINITION 4.1.1 A morphism of ringed spaces is a map

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consisting of two data:

(i) $f_{top} : X \to Y$ continuous

(ii) $f^\# : \mathcal{O}_Y \longrightarrow (f_{top})_* \mathcal{O}_X$ morphism of sheaves on Y

A <u>locally ringed space</u> $(X, \mathcal{O}_X)$ is a ringed space s.t. the stalks $\mathcal{O}_{X,x}$ are all local rings

<u>Definition 4.1.2</u> A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces s.t. at every point $x \in X$, the map

$$f^{\#}_x : \mathcal{O}_{Y, f_{top}(x)} \longrightarrow \mathcal{O}_{X,x}$$

of rings has the property

$$f^{\#}_x(m_{f_{top}(x)}) \subseteq m_x .$$

<u>Exercise</u> Equivalent to saying that the preimage of m_x is the maximal $m_{f_{top}(x)}$. ⌜preimage is prime containing $m_{f_{top}(x)}$⌟

<u>Fact</u> Schemes are automatically LRS

<u>Definition 4.1.3</u> A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of the corresponding locally ringed spaces.

<u>Theorem 4.1.4</u> There is a natural bijection

$$\left\{ \begin{matrix} \text{scheme morphisms} \\ \operatorname{Spec} B \to \operatorname{Spec} A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{ring morphisms} \\ A \longrightarrow B \end{matrix} \right\}$$

<u>Proof</u> ① A ring map $A \to B$ induces a morphism of LRS

② Every scheme map $\operatorname{Spec} B \to \operatorname{Spec} A$ aries (uniquely) this way

Given $\varphi : A \to B$; set $\varphi_{top} : \operatorname{Spec} B \to \operatorname{Spec} A$

$$\mathfrak{p}_B \mapsto \varphi^{-1}(\mathfrak{p}_B)$$ ← prime in A, easy exercise

This is continuous :

$$(\varphi_{top})^{-1}(\mathbb{V}(I)) = \mathbb{V}(\varphi(I))$$ ⌜symbol chasing⌟

Now we build the map on sheaves

$$\varphi_{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow (\varphi_{top})_* \mathcal{O}_{\operatorname{Spec} B}$$

[Think at stalk level]

Let $A_{\varphi^{-1}(\mathfrak{p}_B)} \to B_{\mathfrak{p}_B}$

$$\frac{a}{s} \longmapsto \frac{\varphi(a)}{\varphi(s)}$$

⌈why does this make sense⌋

Notice this is automatically a local homomorphism.
⌈Maximal in $B_{\mathfrak{p}_B}$ is $\mathfrak{p}_B B_{\mathfrak{p}_B}$ ⌋

Think on opens Given $U \subseteq \operatorname{Spec} A$ open; define

$$\varphi^{\#} : \mathcal{O}_{\operatorname{Spec}A}(U) \longrightarrow \mathcal{O}_{\operatorname{Spec}B}(\varphi_{top}^{-1}(U))$$

$$s = [\underset{\underset{U}{\cap}}{\mathfrak{p}} \mapsto s(\mathfrak{p}) \text{ : in } A_{\mathfrak{p}}] \longmapsto [\underset{\underset{(\varphi_{top}^{-1})(U)}{\cap}}{q} \mapsto \varphi_q(s(\varphi_{top}(q)))]$$

This glues If s in $\mathcal{O}_{\operatorname{Spec}A}(U)$ is written locally as $\frac{a}{h}$, then $\varphi_{\#}(s)$ is $\frac{\varphi(a)}{\varphi(h)}$.

Automatically implies that $\varphi_{\#}$ glues

$\boxed{A \to B \text{ gives a scheme map}}$

Conversely, given $(f_{top}, f^{\#}) : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ scheme map

Take $g : \mathcal{O}_{\operatorname{Spec}A}(\operatorname{Spec}A) \to \mathcal{O}_{\operatorname{Spec}B}(\operatorname{Spec}B)$

ring map given by $f^{\#}$.
Need to check it agrees with f_{top}, $f_{\#}$ using the previous construction.

The maps on stalks are compatible with the map on global sections.

$$\begin{array}{ccc} \Gamma(\operatorname{Spec}A, \mathcal{O}_{\operatorname{Spec}A}) & \xrightarrow{f_{\#}} & \Gamma(\operatorname{Spec}B, \mathcal{O}_{\operatorname{Spec}B}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\operatorname{Spec}A, f_{top}(\mathfrak{p}_B)} & \longrightarrow & \mathcal{O}_{\operatorname{Spec}B, \mathfrak{p}_B} \end{array}$$ commutes

$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ notation

By commutativity, and locality, lower horizontal sends $\mathfrak{p}_B B_{\mathfrak{p}_B}$ to $f_{top}(\mathfrak{p}_B) A_{f_{top}(\mathfrak{p}_B)}$

So $g^{-1}(\mathfrak{p}_B) = f_{top}(\mathfrak{p}_B)$ i.e. topological maps agree.

Sheaf maps agree on stalks.

House-keeping : Ex Sheets I & II ; drop-in sessions

Notation

$$\mathcal{O}_X \quad \mathcal{O}_Y \qquad X \xrightarrow{f} Y \qquad \text{\& adjunction} \qquad f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \text{ OR } \mathcal{O}_Y \to f_*\mathcal{O}_X$$

denoted $f^{\#}$

§4.2 A few basic notions

Def^n 4.2.1 $f: X \to Y$ is a morphism of schemes

Then f is an open immersion if f induces an isomorphism between $(X, \mathcal{O}_X)$ and $(U, \mathcal{O}_Y|_U)$ where $U \subseteq Y$ is open

A morphism $g: X \to Y$ is a closed immersion if the topological map $g_{top}: X_{top} \to Y_{top}$ is a homeomorphism onto a closed subset and $g^{\#}: \mathcal{O}_Y \to g_*\mathcal{O}_X$ is surjective

Example 4.2.2 Take $k[t] \longrightarrow k[t]/t^2$ & take Spec

This gives a closed immersion :

$$\operatorname{Spec} k[t]/(t^2) \longrightarrow \mathbb{A}^1_k \qquad \text{homeo onto point } 0 \in \mathbb{A}^1_k$$

(Awkward) Definition 4.2.3 A closed subscheme is an equivalence class of closed immersions $[X \to Y]$, with equivalence given by $X \to Y \sim X' \to Y$ if there is an isomorphism $X \to X'$ commuting w/ maps to Y :

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & X' \arrow[dl] \\ & Y & \end{tikzcd}$$

Notation : People will write "let $X \hookrightarrow Y$ be a closed subscheme"

Scheme-Theoretic Point : Let K be any field. A K-valued scheme-theoretic point of X is a morphism $\operatorname{Spec} K \longrightarrow X$.

The set of K-valued points is typically denoted $X(K)$.

Example 4.2.6 $X = \mathbb{P}^n_{\mathbb{C},sch}$

then $X(\mathbb{C}) = \mathbb{P}^n_{\mathbb{C},sch}(\mathbb{C}) = \mathbb{P}^n_{\mathbb{C},\,\substack{\text{set of the}\\ \text{top space}}}$

For any ring R, the R-valued points of X is the set of maps $\operatorname{Spec} R \longrightarrow X$.

Therefore, every scheme determines

$$\begin{aligned} F_X : \text{Rings} &\longrightarrow \text{Sets} \\ R &\longmapsto X(R) \\ (R \to R') &\longmapsto (\pi^* : X(R) \to X(R')) \end{aligned}$$

«Functor of points viewpoint»

Very concrete 4.2.5 Given $p \in \underset{\text{scheme}}{X}$, choose any affine open $\mathcal{U} \ni p$. Write $\mathcal{U}$ as $\operatorname{Spec}(A)$ and consider p as a prime in A. Then we get

$$A \longrightarrow A/p \hookrightarrow FF(A/p) = k(p)$$

whence applying Spec gives

$$\operatorname{Spec} k(p) \longrightarrow \mathcal{U} \hookrightarrow X$$

§4.3 Fibre products Common generalisation of

(i) Products in scheme theory

(ii) If $X_1 \hookrightarrow Y$, $X_2 \hookrightarrow Y$ are closed subschemes, then so should "$X_1 \cap X_2$" $\hookrightarrow Y$

(iii) Given a morphism $X \to Y$ & $y \in Y$, the fiber $f^{-1}(y)$ should be a scheme

(iv) "Intuitively" $\mathbb{P}^n_{\mathbb{C}}$ is obtained from $\mathbb{P}^n_{\mathbb{Z}}$ & the morphism $\mathbb{Z} \hookrightarrow \mathbb{C}$.

Def^n 4.3.1 Let

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

be a diagram of schemes.

The fibre product $X \times_S Y$ is a scheme with maps

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

, universal as such.

i.e. given Z with maps s.t.
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$
commutes,

then there is a unique map $Z \to X \times_S Y$ s.t.

$$\begin{tikzcd} Z \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$
commutes.

Remark - unique up to unique isomorphism

Theorem 4.3.2 Fiber products exist in schemes [3.3 in Hartshorne]

Remarks on Proof - Affine case. If X, Y, S are affine, say with rings A, B, R then $X \times_S Y$ is $\mathrm{Spec}(A \otimes_R B)$

⚠ Not totally trivial from $\mathrm{Sch}^{op} = \mathrm{Rings}$ since the test objects Z are greater in number. This is ok

Lemma 4.3.3 A morphism $S \to \mathrm{Spec}\, A$ (S a scheme) is equivalent to a ring map $A \to \Gamma(S, \mathcal{O}_S)$.

[Exercise]

Examples 4.3.3 (i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$

§4.4 Separated Morphisms

<u>Definition 4.4.1</u> Given $X \to S$ a scheme map, we have a diagonal $X \to X \times_S X$ induced by universal property

$$\begin{tikzcd}
X \arrow[dr, "\Delta_{X/S}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_S X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

<u>Proposition 4.4.2</u> Let $X \xrightarrow{f} S$ be a scheme map. The diagonal is locally closed immersion i.e. there is a factorisation

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[rr, bend right, "\Delta_{X/S}"'] & U \arrow[r, hook] & X \times_S X
\end{tikzcd}$$

with $U \subseteq X \times_S X$ an open subscheme, and $X \hookrightarrow U$ a closed immersion.

<u>Proof</u> Cover S by affines $\{V_i\}$ and cover each preimage $f^{-1}(V_i)$ by affines $\{U_{ij}\}$.

We have induced maps $U_{ij} \longrightarrow V_{\cancel{j}i}$ (by restriction)

$$\begin{tikzcd}
f^{-1}(V_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] & \\
V_i \arrow[r, hook] & S & U_{ij} \arrow[ul, bend right] \arrow[ll, bend left]
\end{tikzcd}$$

Now take $U_{ij} \times_{V_{\cancel{j}i}} U_{ij}$

This is an affine open subscheme inside $X \times_S X$.

These opens together contain the image of $\Delta_{X/S}$.

Moreover, $(\Delta_{X/S})^{-1}(U_{ij} \times_{V_j} U_{ij}) = U_{ij}$

so $U_{ij} \hookrightarrow U_{ij} \times_{V_i} U_{ij}$ is a $\underbrace{\text{closed immersion}}_{\substack{\text{claim :} \\ \text{check structure} \\ \text{sheaf data}}}$. $\square$

<u>Proposition 4.4.3</u> If $X \to S$ is a map of affine schemes, then $\Delta_{X/S} : X \to X \times_S X$ is closed (immersion).

[this really is easy: $\operatorname{Spec} A \to \operatorname{Spec} B$ closed if $B \to A$ surj.]

<u>Proof</u> $A \otimes_B A \to A$ is always surjective. $\square$

<u>Remark</u> Use this to complete previous proof.

<u>Definition 4.4.4</u> A morphism $X \to S$ is separated if the diagonal $\Delta_{X/S}$ is a closed immersion

- <u>Easy fact</u> If $Y \to Z$ is a locally closed immersion, i.e. factors $Y \xrightarrow[\text{closed immersion}]{} U \underset{\text{open subscheme}}{\hookrightarrow} Z$ <u>and</u> the image of Y is closed as a (topological) subset of Z, then $Y \to Z$ is a closed immersion. ✓

<u>Remark on terminology</u> Typically we work with a fixed "<u>base scheme</u>" e.g. $\operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} \mathbb{Z}$

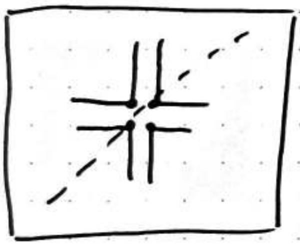
People will say "X is separated" to mean $X \longrightarrow$ base scheme is separated

- <u>Examples 4.4.5</u> (i) For any ring A the morphism $\mathbb{A}^m_A \to \operatorname{Spec} A$ is separated.

(ii) The bug-eyed line isn't separated. Why?

$\mathcal{B} = \mathbb{A}^1_{\mathbb{C}} \cup_{\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}} \mathbb{A}^1_{\mathbb{C}}$ The self-product $\mathcal{B} \times_{\operatorname{Spec} \mathbb{C}} \mathcal{B}$ is the "plane with 4 origins" ~ish

$\mathcal{F}$



The image of the diagonal in $\mathcal{F}$ contains only two origins, but its closure contains all four.

-

(iii) For any ring A, the map $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated $\left(\begin{matrix}\text{think} \\ A = \mathbb{C}\end{matrix}\right)$

<u>Proof of (iii)</u> Ex Sh III, Ravi 10.1.5,

$$\begin{array}{ccc}
\mathbb{P}^n_A & \xrightarrow{\Delta} & \mathbb{P}^n_A \times_{\operatorname{Spec} A} \mathbb{P}^n_A \\
\uparrow & \square & \uparrow \text{open immersion} \\
 & & \text{covered by} \\
U_i \cap U_j & \longrightarrow & U_i \times_{\operatorname{Spec} A} U_j
\end{array}$$

affine

Open cover of $\mathbb{P}^n_A$ ($\operatorname{Proj} A[x_0, \dots, x_n]$) is given by $U_j = \operatorname{Spec} A\left[\frac{x_0}{x_j}, \dots, \widehat{\frac{x_j}{x_j}}, \dots, \frac{x_n}{x_j}\right]$ $(\cong \mathbb{A}^n_A)$

Examining this diagram.

- Explicitly write out the map $U_i \cap U_j \to U_i \times_{\operatorname{Spec} A} U_j$ in terms of rings.

Better to have a criterion for checking separatedness that doesn't require reduction to affines.

Assume all schemes are Noetherian (covered by spectra of noetherian rings)

A <u>discrete valuation ring</u> A is a local PID

<u>Examples 4.4.6</u> $\mathbb{C}[[t]]$, $\mathcal{O}_{\mathbb{A}^1,0}$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$

p-adic integers

Spec A is topologically

DVR

a connected doubleton

$= \{p_0, p_1\}$ where p_0 is open, dense

p_1 is closed

<u>Theorem 4.4.7</u> Let $X \xrightarrow{f} S$ be a scheme map.

Assume X is Noetherian. Then f is separated iff for any DVR A with $FF(A) = k$ given a diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

. there exists at most one lift of the map $\operatorname{Spec} k \to X$ making the diagram commute.

Philosophy Don't cry

Theorem 4.3.2 Fibre products exist in the category of schemes

- Proof (Hartshorne Theorem 3.3 §II)

1. (Last time) If
$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \to & S \end{array}$$
with X, Y, S affine, then the fibre product $X \times_S Y$ exists and is $\mathrm{Spec}(A \otimes_R B)$ where $A = \Gamma(X, \mathcal{O}_X)$, $B = \Gamma(Y, \mathcal{O}_Y)$, $R = \Gamma(S, \mathcal{O}_S)$.

(Globalisation) Slowly turn $X \times_S Y$ into X, Y, S general into affines.

- If $X \times_S Y$ exists and $U \subseteq X$ is open, then $U \times_S Y$ also exists. Namely take
$$U \times_S Y = p_X^{-1}(U) \quad \text{where} \quad p_X : X \times_S Y \longrightarrow X,$$
endowed with the open subscheme structure.

▷ If X is covered by affines $\{X_i\}$ then if $X_i \times_S Y$ exists for all i, then $X \times_S Y$ also exists.

Why? The fibre product as a scheme is constructed by gluing $X_i \times_S Y$.

- By restrictions to opens $X_i \times_S Y$ glues with $X_j \times_S Y$ over $X_{ij} \times_S Y$ by uniqueness of $X_{ij} \times_S Y$.

The morphisms $X \times_S Y \to X, Y$ are also constructed by gluing.
[Easier than you think – no cocycle condition]

▷ X & Y play interchangeable roles, so the existence of fibre products will follow from the existence of $X \times_S Y$ where X, Y are affine.

▷ Cover S by affines $\{S_i\}$. Let X_i be the preimage of S_i under $X \to S$. Similarly for Y_i.

- Key claim $X_i \times_{S_i} Y_i = X_i \times_S Y$ ‖ Exercise (Please)

▷ Now construct $X \times_S Y$ by gluing. □

Remark (later than it should be)

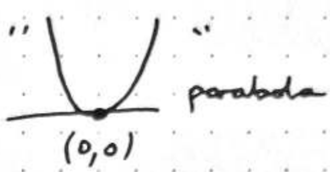
If
$$\begin{tikzcd} & X \arrow[d, "\psi"] \\ Y \arrow[r, "\varphi"] & S \end{tikzcd}$$
is a diagram of sets, $X \times_S Y$ is the subset of $X \times Y$ given by (x,y) s.t. $\psi(x) = \varphi(x)$.

Examples 4.3.3 (i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$

where $\operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}\mathbb{Z}$ is induced by $\mathbb{Z} \hookrightarrow \mathbb{C}$.

(ii) Take $C = \operatorname{Spec} \mathbb{C}[x,y]/(y-x^2)$

$L = \operatorname{Spec} \mathbb{C}[x,y]/(y)$



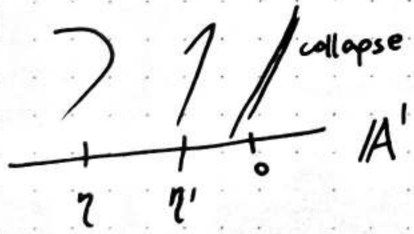
$\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x,y]$, consider

$$\begin{tikzcd} L \times_{\mathbb{A}^2} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ L \arrow[r] & \mathbb{A}^2 \end{tikzcd}$$

$$\begin{tikzcd} L \times_{\mathbb{A}^2} C = \operatorname{Spec}\mathbb{C}[x]/(x^2) \arrow[d] \\ \mathbb{A}^2 \end{tikzcd}$$

⚠ $\mathbb{C}[x]/(x^2)$ is a $\mathbb{C}$-vector space of dimension 2

(iii)
$$\begin{tikzcd} \operatorname{Spec}\mathbb{C}[x,y,t]/(y^2+tx) \arrow[r] & \operatorname{Spec}\mathbb{C}[t] \\ \operatorname{Spec}\mathbb{C}[x,y]/(y^2) \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \operatorname{Spec}\mathbb{C}[t]/(t) \arrow[u] \end{tikzcd}$$
$\mathbb{A}^1_{\mathbb{C},t}$



(iv) Given $X \to S$, given a point $p \in S$, recall that we defined $\kappa(p) = FF(A/p)$ where p is a prime in A, $\operatorname{Spec} A \hookrightarrow S$ is an open ngbd of p.

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{Spec}\kappa(p) \arrow[r] & S \end{tikzcd}$$

Scheme theoretic fibre of $X \to S$ over p.

(v) Take $p \in \mathbb{A}^1_{\mathbb{C}}$ to be the generic point i.e. $(0) \subseteq \mathbb{C}[t]$

X_p (the fiber over p) : $\kappa(p) = \mathbb{C}(t)$

$$\begin{tikzcd}
X_p = \operatorname{Spec} \mathbb{C}(t)[x,y]/(y+tx^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{C}(t) \arrow[d] \\
\operatorname{Spec} \mathbb{C}[x,y,t]/(y+tx^2) \arrow[r] & \operatorname{Spec} \mathbb{C}[t]
\end{tikzcd}$$

§4.4 Separated morphisms

Def^n 4.4.1 Given a scheme map $X \to S$, the diagonal $\Delta_{X/S}$ is the morphism $\Delta_{X/S} : X \to X \times_S X$

defined via the universal property of fiber product:

$$\begin{tikzcd}
X \times_S X \arrow[r] \arrow[d] & X \arrow[d] & \\
X \arrow[r] & S & \\
& & X \arrow[uull, dashed] \arrow[ul, dashed] \arrow[uul, bend right, "id_X"'] \arrow[ull, bend left, "id_X"]
\end{tikzcd}$$

DVR e.g. $\mathbb{C}[[t]]$, $\mathcal{O}_{\mathbb{A}^1,0}$, $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$

Spec(DVR) has two points $\{\bullet, \cdot\}$ – germ of a point on a curve

Thm 4.4.7 (restated for schemes of Spec $\mathbb{C}$)

A scheme $X \xrightarrow{f} \operatorname{Spec}\mathbb{C}$ is ~~proper~~ separated iff ~~tk~~ for any DVR R with fraction field $K = FF(R)$ there is at most 1 lift of

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

e.g. Spec($\mathbb{C}$-algebra)

Spec $\mathbb{C}((t))$

Spec $\mathbb{C}[[t]]$

Separatedness is about the uniqueness of a lift

Example (Bug-eyed line)

$X = \mathbb{A}^1_{\mathbb{C}} \cup_{\mathbb{A}^1_{\mathbb{C}}\setminus 0} \mathbb{A}^1_{\mathbb{C}}$

Claim. There are two maps $\operatorname{Spec} R \rightrightarrows X$, for R a DVR, which agree on the generic point $\operatorname{Spec} K$

Tautology: let $p_1, p_2 \in X$ be the two origins, and $R = \mathcal{O}_{X,p_1} \cong \mathcal{O}_{X,p_2} \cong \mathcal{O}_{\mathbb{A}^1,0}$

$f_1 : \operatorname{Spec} R \to X$ that factors as $\operatorname{Spec} R \to \underbrace{\mathbb{A}^1_{\mathbb{C}}}_{\text{first copy}} \to X$

given by $\mathbb{C}[t] \to R$ the localisation map

Corollaries 4.4.8 (i) Open & closed immersions are separated ✓

(ii) If $X \to Y$ is separated, $Z \to Y$ is arbitrary, then $X \times_Y Z \to Z$ is also separated (stability under base change)

(iii) Separatedness is local on the target;
if $X \to Y$ & $\{U_i\}$ is a cover for Y s.t. $X \times_Y U_i \to U_i$ are all separated, then $X \to Y$ is too

Consequence of (ii) Since $\mathbb{P}^n_{\mathbb{Z}}$ is separated over Spec $\mathbb{Z}$ $\forall n$, get $\mathbb{P}^n_A$ separated over A for any ring A

<u>Proof of Corollary (ii)</u> Suppose $X \to Y$ is separated, $Z \to Y$ arbitrary.

- Consider the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & Z \arrow[r] & Y \end{tikzcd}$$

Composing g_1, g_2 with $X \times_Y Z \to X$

$X \times_Y Z \to Z$

we get two $\operatorname{Spec} R \rightrightarrows X$

which agree as $\operatorname{Spec} K \to X$

so since $X \to Y$ is separated they agree

- By universal property of $X \times_Y Z$ the maps g_1, g_2 agree. $\square$

§4.5 Properness

[Model topological behavior — preimage of compact is compact

$\approx$ image of a closed set is closed]

<u>Defn 4.5.1</u> A scheme map $X \to Y$ is <u>closed</u> if it is closed as a topological map.

It is <u>universally closed</u> if for any base extension i.e. any

- $Z \to Y$, the map $X \times_Y Z \to Z$ is also closed.

A morphism of schemes $X \to Y$ is <u>proper</u> if it is separated, of finite type, and universally closed.

<u>Thm 4.5.1</u> (the valuative criterion of properness) Given $X \xrightarrow{f} Y$ a scheme map. Then f is proper iff for any DVR R with fraction field K, and any diagram

- $$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$ there is a unique lift.

<u>Corollary 4.5.2</u> (i) $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper

(ii) Closed subschemes of $\mathbb{P}^n_A$ are proper [use (i) and that closed embedding is proper]

(iii) ⚠ Open immersions are NOT proper e.g. $\mathbb{A}^n_A \to \operatorname{Spec} A$, $n \geq 1$

<u>Easy Fact 4.5.3</u> Properness is stable under base extension ✓

<u>Remarks on proof of Corollary 4.5.2</u>

(0) [on DVR's] any DVR R has a <u>uniformiser</u> i.e. $\pi \in R$ s.t. $(\pi) = \mathfrak{m}_R$; all ideals in R are of the form (π^k) for some $k \in \mathbb{N}$; every $r \in R$ can be written as $\pi^k r_0$ where $r_0 \in R^\times = R \setminus \mathfrak{m}_R$, $k \in \mathbb{N}$

(1) $\mathbb{P}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is proper via valuation criterion

Pick R a DVR, $FF(R) = K$, uniformiser π

<u>Want</u>: $\mathbb{P}^n_{\mathbb{C}}(K) \xleftrightarrow[\text{bijection}]{} \mathbb{P}^n_{\mathbb{C}}(R)$

Given a K-point of $\mathbb{P}^n_{\mathbb{C}}$, i.e. a tuple $[z_0 : \dots : z_n]$ with $z_i \in K$ in homogeneous coordinates

Multiply by π^k to find an equiv representative $[z_0' ; \dots ; z_n']$ with the property that at least one z_i' is a unit (in R), $k \in \mathbb{Z}$

But not all z_i' lying in $\mathfrak{m}_R$ (?)

$\Rightarrow$ This is now an R-point

<u>Notice</u> We didn't use $\mathbb{C}$ anywhere $\Rightarrow \mathbb{P}^n_{\mathbb{Z}}$ is proper □

<u>Exercise</u> The $\mathbb{C}$-points of $\mathbb{P}^n_{\mathbb{C}}$ are $\mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$.

As explictly as possible describe the $\mathbb{C}[[t]]$ points of $\mathbb{P}^n_{\mathbb{C}}$.

<u>Concluding remarks about properness</u> • "limits exist and are unique" – valuative criterion

• $\mathbb{A}^n_R$ is not proper for any $n \geq 1$

Why is <u>$\mathbb{A}^1_{\mathbb{C}}$ not proper</u>? Using universal closedness:

$$\begin{tikzcd} \mathbb{A}^2_{\mathbb{C}} \arrow[r] \arrow[d, "\text{not closed}"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_{\mathbb{C}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{C}} \arrow[r] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

⌈ $V(xy-1)$ maps to non-closed ⌋

Using valuative criterion:

$R = \mathbb{C}[[t]]$, $k = \mathbb{C}((t))$

Consider $\operatorname{Spec} k \to \mathbb{A}^1_{\mathbb{C}}$ given on rings

$$\mathbb{C}((t)) \longleftarrow \mathbb{C}[x]$$
$$\frac{1}{t} \longmapsfrom x$$

• All closed subschemes of $\mathbb{P}^n_A$ is proper (over $\operatorname{Spec} A$)

<u>Definition 4.5.4</u> A <u>variety</u> is a separated, finite-type, reduced scheme over $k = \bar{k}$.

↳ Religion: irreducibility is optional

A variety is <u>complete</u> if $X \to \operatorname{Spec} k$ is proper.

<u>§4.6 A brief interlude on other types of morphisms</u>

$X \xrightarrow{f} Y$ a scheme map

(i) <u>Finite</u>: f is finite if we can cover Y by open affines $U_i = \operatorname{Spec} B_i$ s.t. $f^{-1}(U_i) = V_i$ is an open affine in X and ($V_i = \operatorname{Spec} A_i$), A_i is a finite B_i-module

<u>Examples</u>: Non-constant maps between smooth curves

(ii) <u>Flat</u>: f is flat if at every $p \in X$, the induced map $f^\#: \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p}$ makes $\mathcal{O}_{X,p}$ a flat $\mathcal{O}_{Y,f(p)}$-module

<u>Utility</u> : Given
(hard)

$$\begin{tikzcd} Z_\eta \arrow[r, hook] & \mathbb{P}^n_{\mathbb{C}((t))} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_{\mathbb{C}[[t]]} \arrow[d] \\ & \operatorname{Spec} \mathbb{C}((t)) \arrow[r] & \operatorname{Spec} \mathbb{C}[[t]] \end{tikzcd}$$

Then there exists a unique scheme map

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{P}^n_{\mathbb{C}[[t]]} \arrow[d, "\mu"] \\ & \operatorname{Spec} \mathbb{C}[[t]] \end{tikzcd}$$

such that π is flat and after base change to $\mathbb{C}((t))$, the subscheme agrees with Z_η.

§5 Modules over $\mathcal{O}_X$

§5.1 Motivation / Examples

<u>Example 5.1.1</u> On $\mathbb{P}^n_{\mathbb{C}}$ the <u>variety</u>, consider the sheaf
(fix $d \in \mathbb{Z}$)

$$\mathcal{O}_{\mathbb{P}^n}(d)(U) = \left\{ \frac{p(\underline{x})}{q(\underline{x})} \;\; \begin{array}{l} \text{homogeneous rational} \\ \text{functions of degree } d \\ \text{s.t. } p/q \text{ has no poles on } U \end{array} \right\}$$

of abelian groups

This is a sheaf, and on any open U, it is a module over $\mathcal{O}_{\mathbb{P}^n}(U)$.

§5.2 Definition of $\mathcal{O}_X$-module

Fix $(X, \mathcal{O}_X)$.

<u>Def 5.2.1</u> A <u>sheaf of $\mathcal{O}_X$-modules</u> is a sheaf $\mathcal{F}$ of groups s.t. for open $U \subseteq X$ there is a multiplication

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U), \text{ making } \mathcal{F}(U) \text{ an } \mathcal{O}_X(U)\text{-module}$$

compatible with restriction maps.

A <u>sheaf of $\mathcal{O}_X$-algebras</u> is defined similarly via a sheaf of rings.

↳ Similar to define a homomorphism of $\mathcal{O}_X$-modules

<u>Standard Notions</u>: kernels, images, cokernels, direct sums, direct products, ... of morphisms of $\mathcal{O}_X$-modules give $\mathcal{O}_X$-modules

<u>Also</u> Tensor product of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ $\Big\}$ sheafification
Hom of $\mathcal{O}_X$-modules ⌜⚠ do hom at sheaf level⌟

<u>Moving between spaces</u>: $X \xrightarrow{f} Y$ ringed space map

Given $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules, the pushforward $f_* \mathcal{F}$ is a sheaf of $f_* \mathcal{O}_X$-modules.

Since we have $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ we also have an $\mathcal{O}_Y$-module.

$\mathcal{O}_X$ - modules

Moving between spaces:

- $$\begin{array}{c} \mathcal{F} \frown \quad \quad \nearrow \mathcal{G} \\ X \xrightarrow[f]{} Y \end{array}$$ a ringed space morphism,

$\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X, \mathcal{O}_Y$ - modules respectively

Pushforward: $f_* \mathcal{F}$ - automatically sheaf of groups on Y

" module over $f_* \mathcal{O}_X$ by functoriality

Since f is a ringed map, we have $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$

Therefore $f_* \mathcal{F}$ has an $\mathcal{O}_Y$ - module structure.

Pullback: $f^{-1} \mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$ - module on X (yes!)

$f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

- Definition of $f^* \mathcal{G}$ $:= f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ "pushout"

Keep in mind: If $\iota: Z \to \mathbb{P}^n$ inclusion of a closed subvariety, then $\mathcal{O}_{\mathbb{P}^n}(d)$ pulls back (using ι^*) to the "natural" notion of homogeneous functions on Z. "rather than ι^{-1}"

Basic Fact 5.2.2 f^* & f_* are adjoint functors on modules over the structure sheaf

§ 5.3 $\mathcal{O}_X$ - modules on schemes

- Fix $(X, \mathcal{O}_X)$ a scheme

Examples of $\mathcal{O}_X$ - modules: ① Take $\mathcal{O}_X^{\oplus n}$

② "Quotients" of free modules over $\mathcal{O}_X$ i.e. take $\mathcal{F}$ on X a sheaf of $\mathcal{O}_X$-modules with a surjection $\mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$

Definition 5.3.1 A sheaf $\mathcal{F}$ of $\mathcal{O}_X$ - modules is quasi-coherent if there exists an open cover $\{U_i\}$ of X such that U_i are affine, and $\mathcal{F}|_{U_i}$ ($:=$ pullback of $\mathcal{F}$ under $U_i \hookrightarrow X$) is the sheaf associated to a module M_i over $\mathcal{O}_X(U_i)$.

- Proposition 5.3.2 An $\mathcal{O}_X$ - module $\mathcal{F}$ is quasi-coherent iff for any affine $U = \operatorname{Spec} A \subseteq X$, the restriction $\mathcal{F}|_U$ is the sheaf associated to a module M over $\mathcal{O}_X(U)$.

<u>(Key) Corollary 5.3.3</u> There is a natural bijection for $X = \operatorname{Spec} A$ between

$$\left\{\begin{matrix}\text{modules} \\ \text{over} \\ \mathcal{O}_X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{modules} \\ \text{over } A\end{matrix}\right\}$$

Natural ≡ compatible w/ morphisms.

<u>Remark</u> A quasi-coherent $\mathcal{F}$ on X is said to be <u>coherent</u> if there is an affine open cover $\{U_i\}$ s.t. $\mathcal{F}|_{U_i}$ is the sheaf associated to a f.g. $\mathcal{O}_X(U_i)$-module. Typically take X noetherian.

<u>"Proj version" of $\mathcal{O}_X$-modules</u>

Let $X = \operatorname{Proj} A_\bullet$ for some $\mathbb{N}$-graded ring $A_\bullet$, a projective scheme.

Given $M_\bullet$ a graded $A_\bullet$-module we can construct a sheaf $M_\bullet^{\text{sheaf}}$ (other people $\tilde{M_\bullet}$) of $\mathcal{O}_X$-modules.

If $f \in A_+$ is a positively graded hom element then

$$M_\bullet^{sh}(\underbrace{\mathbb{V}(f)^c}_{U_f}) = \text{Degree } 0\text{-part of the localization of } M_\bullet \text{ at } \{1, f, f^2, \dots\}$$

Identical arguments give us a sheaf of $\mathcal{O}_X$-modules

<u>Remark</u> Take ~~$S_\bullet$~~ $A_\bullet(1)$ to be the $A_\bullet$-module given as an abelian group by $A_\bullet$ but with a <u>shifted grading</u>:

$$[A_\bullet(1)]_{\text{degree } k} = [A_\bullet]_{\text{degree } k+1}$$

Similarly define $A_\bullet(n)$, $n \in \mathbb{Z}$.

Taking $M_\bullet = A_\bullet(n)$ & applying previous construction, get new sheaves of modules $\mathcal{O}_X(n)$.

<u>Fact</u> $\mathcal{O}_{\mathbb{P}^d}(n)$ for different n are non-isomorphic

<u>Key Lemma 5.3.3</u> $X = \operatorname{Spec} A$, $f \in A$ & $\mathcal{F}$ a quasi-coherent on X. Let $s \in \Gamma(X, \mathcal{F})$ and consider $U_f = \mathbb{V}(f)^c$

① If s restricts to 0 on U_f then $f^n s = 0$ for some n

② If $t \in \Gamma(U_f, \mathcal{F})$ then for some m, ~~f^m~~ $f^m t$ is the restriction of a global section of $\mathcal{F}$ on X.

Proof of prop^n 5.3.2 Given $\mathcal{U} = \operatorname{Spec} A$ on X open affine, we first observe that $\mathcal{F}|_{\mathcal{U}}$ is quasi-coherent.
[quasi-coherence is local]

(Reduces us to $X = \operatorname{Spec} A$. Now we'll take $M = \Gamma(X, \mathcal{F})$

Let M^{sh} be the associated sheaf on X. There is a restriction map $M^{sh} \longrightarrow \mathcal{F}$.

By the lemma, this will locally be an iso $\Rightarrow$ global iso $\square$

Proposition 5.3.2 An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent iff $\mathcal{F}|_U$ for $U = \operatorname{Spec} A$ is the sheaf associated to a module M over A. [If X noetherian then coherent modules restrict to f.g. modules M over A]

Lemma 5.3.3 If $X = \operatorname{Spec} A$, $f \in A$, $\mathcal{F}$ quasi-coherent, $s \in \Gamma(X, \mathcal{F})$, then (i) if $s|_{\mathbb{V}(f)^c} = 0$ then $f^n s = 0$ for some $n > 0$

(ii) if $t \in \Gamma(\mathbb{V}(f)^c, \mathcal{F})$ then $\exists m$ s.t. $f^m t$ is the restriction of a global section $s \in \Gamma(X, \mathcal{F})$

Proof Since sets of the form $\mathbb{V}(g)^c = U_g$ form a basis for the topology, and we have an open cover of $X = \operatorname{Spec} A$ by affines where $\mathcal{F}$ restricts to sheaves of the form M^{sh}.

Then we can find a cover $\{U_{g_i} = \mathbb{V}(g_i)^c\}$ s.t. $\mathcal{F}|_{U_{g_i}}$ is the sheaf associated to a module over A_{g_i}.

Since X is quasi-compact, may take a finite open cover.

The rest follows from standard properties of localizations. (!) $\square$

Proof of 5.3.2 (i) Using the trick from the proof of the lemma, immediately reduce to a quasicoherent sheaf sheaf $\mathcal{F}$ on $X = \operatorname{Spec} A$.

(ii) Take $M = \Gamma(X, \mathcal{F})$, an A-module btw.

Have a restriction map $M^{sh} \xrightarrow{\psi} \mathcal{F}$

By the lemma, the map is injective (i) and surjective (ii) on an open cover, so ψ is an isomorphism. $\square$

Words A sheaf $\mathcal{F}$ on X is an algebraic vector bundle if it is (quasi-coherent) locally on X the sheaf associated to a free $\mathcal{O}_X$-module. If X is connected (noetherian) and $\mathcal{F}$ is coherent then the rank of $\mathcal{F}$ is well-defined, by taking

$\operatorname{rk} \mathcal{F} = \operatorname{rank}_{\mathcal{O}_X(U)} \mathcal{F}(U)$ where U is some open affine s.t. ~~$\mathcal{F}(U)$~~ $\mathcal{F}|_U$ is the sheaf associated to a free module. [Nolbz]

If $\operatorname{rk} \mathcal{F} = 1$ and $\mathcal{F}$ is locally free (i.e. locally $\mathcal{F}$ is isomorphic to $\mathcal{O}_X$) then $\mathcal{F}$ is a line bundle.

▷ An ideal sheaf is a quasi-coherent sheaf of $\mathcal{O}_X$-modules ⌜what⌟

§5.4 Projective Scheme Theory old skewl $\mathbb{P}^n_{\mathbb{C}}$ [sic.]

For each $d \in \mathbb{Z}$,

$$\mathcal{O}_{\mathbb{P}^n}(d)(\mathcal{U}) = \left\{ f/g \;\middle|\; \begin{array}{l}\text{hom. rational functions of} \\ \text{degree } d\text{, no poles on } \mathcal{U}\end{array} \right\}$$

Then $\mathcal{O}_{\mathbb{P}^n}(d)$ is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules on $\mathbb{P}^n$.

Def^n 5.4.1 $A_\bullet$ graded ring, $M_\bullet$ graded $A_\bullet$-module.

Define, for $d \in \mathbb{Z}$, $M_\bullet(d)$ to be the graded module s.t.

$$[M_\bullet(d)]_k = [M_\bullet]_{k+d}$$

Apply the "Proj-style" construction to $A_\bullet(d)$ to get a sheaf on $X = \mathrm{Proj}(A_\bullet)$, denoted $\mathcal{O}_X(d)$.

Proposition 5.4.2 $\mathcal{O}_X(d)$ is a line bundle on X

Proof is trivial. ⌜locally on $\mathbb{V}(f)^c$ free generated by f^d? ⌟

Sources of line bundles Given a morphism $X \xrightarrow{f} \mathbb{P}^n$, we get a line bundle on X, namely $f^*\mathcal{O}_{\mathbb{P}^n}(1) = L$

[verify this is a line bundle]

L that arise in this fashion are called basepoint-free

If $X \hookrightarrow \mathbb{P}^n$ could be a locally closed embedding then L is called very ample

Example 5.4.4 Take $A_{\bullet\bullet} = \mathbb{C}[x, y, z, w]$

Take the bigraded module $M_{\bullet\bullet} = A_{\bullet\bullet}(1,1)$.

Then we get a coherent sheaf (a line bundle) $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$

Sections are rational bihomogeneous functions of degree $(1,1)$.

[Segre embedding] $\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \underset{i}{\hookrightarrow} \mathbb{P}^3$

[FACT] $i^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ ⌜$\mathrm{image}(i) \overset{\shortparallel}{} \mathbb{V}(z_0z_2 - z_1z_3)$⌟

Example 5.4.5 (Veronese)

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^m, \quad m = \binom{n+d}{d}, \quad d \overset{?}{\geq} n$$

given by the tuple of degree d monomials in $n+1$ variables.

In this case $i^*\mathcal{O}_{\mathbb{P}^m}(1) = \mathcal{O}_{\mathbb{P}^n}(d)$.

<u>Vector Bundles</u> Given $\mathcal{E}$ locally free (coherent) of rank r on X, then $\mathcal{E}^\vee = \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}, \mathcal{O}_X)$

$\mathrm{Sym}^\bullet \mathcal{E}^\vee$: sheaf of $\mathcal{O}_X$-algebras

$\underline{\mathrm{Spec}}_X \, \mathrm{Sym}^\bullet \mathcal{E}^\vee \longrightarrow X$ $\left(\begin{array}{c}\text{total space of} \\ \text{a vector space}\end{array}\right)$

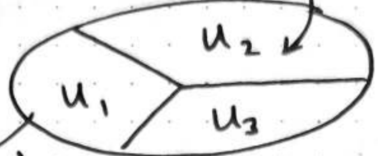
Correction: $\mathcal{O}_X(1)$ on $X = \operatorname{Proj} A$

- hypothesis that A_1 generates $A_\bullet$ as an A_0-algebra

- Global Spec – Ex Sheet 3

$A(U_2)$ = sheaf of rings over $\mathcal{O}_X(U_2)$

$A_\bullet$ - sheaf of [graded] algebras $\mathcal{O}_X$-algebras



X

$A(U_1)$

$\operatorname{Spec} \mathcal{O}_X(U_2) \longleftarrow \operatorname{Spec} A(U_2)$

If U_i are affine

The schemes $A(U_i) \underset{\text{glue}}{\rightsquigarrow} \underline{\operatorname{Spec}} A$

and we get a map down to X

- Example from last time: start with M a locally free sheaf of $\mathcal{O}_X$-modules, pass to $\mathcal{E}^\vee$, pass to $\operatorname{Sym}^\bullet \mathcal{E}^\vee$

sheaf of algebras $\longrightarrow$ $\underline{\operatorname{Spec}} \operatorname{Sym}^\bullet \mathcal{E}^\vee$ = "vector bundle"

§6. Divisors on Schemes – Weil Divisors

hypothesis: noetherian, integral, separated, regular in codimension 1

A scheme X is reg. in codim 1 if every 1-dim local ring $\mathcal{O}_{X,x}$ is a $\underbrace{\text{DVR}}_{\text{i.e. PID}}$. $\mathbb{C}[[t]]$

- ### §6.1 Topological Things

(i) The dimension of X is the length of the longest chain of strict inclusions $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ in X where Z_i are non-empty irred closed subsets.

$\dim \mathbb{A}^n_k = n$

(careful if Z not irred)

(ii) Given $Z \subset X$ closed, $\operatorname{codim}(Z, X)$ is similarly the length of the longest chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ in X

- Remark on intuition If A is a f.g. k-algebra & is integral then the Krull dim of A = height p + Krull dim A/p

p prime

$\Rightarrow$ topological dim behaves well

(iii) Fact / Exercise If X is a Noetherian top space, then every $Z \subseteq X$ has a finite irred decomposition

§6.2 Weil divisors

Def^n 6.2.1 A prime divisor on X is a closed integral subscheme of codimension 1. A Weil divisor is an element of the free abelian group on the prime divisors.

— · —

X is integral. There is a point $\eta \in X$ – the generic point – defined by the inclusion $\underset{\substack{\uparrow \\ \eta}}{\operatorname{Spec} FF(A)} \longrightarrow \underbrace{\operatorname{Spec} A \longrightarrow X}_{\text{affine open}}$

Equivalently, the prime (0) on any affine open in X.

[Why is this independent of A ?] ⌜$\operatorname{Spec} A \cap \operatorname{Spec} B \neq \emptyset$⌟

The stalk $\mathcal{O}_{X,\eta} = FF(A) := k(X)$ "function field" e.g. if $X = \mathbb{P}^n_{\mathbb{C}}$, $A = \mathbb{C}[y_1, \dots, y_n]$

$\operatorname{Spec} \mathbb{C}(y_1, \dots, y_n) \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ is η

Construction 6.2.2 Let $f \in k(X)$.

Take $\operatorname{div} f = \sum_{\substack{Y \subseteq X \\ \text{prime} \\ \text{divisors}}} n_Y(f) [Y]$

where $n_Y(f)$ = valuation of f in the local ring $\mathcal{O}_{X,\eta_Y}$ at the generic point of Y.

Proposition 6.2.3 The element $\operatorname{div}(f)$ for $f \in k(X)$ is a Weil divisor, i.e. $n_Y = 0$ for all but finitely many Y.

Proof Take $U \subseteq X$ affine such that $f \in \Gamma(U, \mathcal{O}_X)$ i.e. f is regular on U.

Then $Z = X \setminus U$ is closed of codimension ≥ 1. ⌜feels weird to put it this way⌟

Thus only finitely many Y_i's in the sum are contained in Z.

Away from Z, any Y_i for which $n_{Y_i}(f) > 0$ must be contained in $\mathbb{V}(f)$ which is also codim 1 (unless $f = 0$)

Again, only finitely many such Y_i's. "topological things"

⚠ We used that any $Z \hookrightarrow X$ closed has a unique reduced scheme structure [Hartshorne Expl 3.2.6]

Def 6.2.4 A divisor of the form $\mathrm{div}(f)$ is principal
The principal divisors form a subgroup of $\underbrace{\mathrm{Div}\,X}_{\text{Weil divisors}}$
Define the Class group

$$\mathrm{Cl}(X) := \mathrm{Div}X / \mathrm{Prin}\,X .$$

▷ Class group $Cl(X) = Div(X) / \underbrace{Prin(X)}$

rational functions produce Weil divisors
"homologically trivial" (analogy)

Basic calculations 6.2.5

(i) If $X = \operatorname{Spec} A$ with A a $\underbrace{\text{unique factorisation domain}}_{\text{height 1 primes are principal ideals}}$, then $Cl(X) = 0$.

Rmk If $\underset{\text{Spec}A}{X}$ is normal then $Cl(X) = 0 \iff A$ is a UFD

(ii) $Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by the class of the divisor

k: field, $\mathbb{Z}$: degree

$H = \mathbb{V}(f)$, for f linear homogeneous
[proof in a moment]

(iii) If $Z \hookrightarrow X$ is a closed subscheme with $U = X \setminus Z$ then the restriction of divisors:

U: open subscheme of X

$$Cl(X) \overset{res}{\twoheadrightarrow} Cl(U)$$

is a surjective homomorphism. (restriction $Div(X) \longrightarrow Div(U)$ respects principal divisors b/c the function fields of U & X are same)

If $\operatorname{codim}(Z, X) \geq 2$ then res is an isomorphism.

Ex $Cl(\mathbb{P}^2_{\mathbb{C}} \setminus \{p_1, \dots, p_r\}) \cong \mathbb{Z}$

Finally, if Z is codimension 1, irreducible then there is a "very" short exact sequence

$$\mathbb{Z} \to Cl(X) \to Cl(U) \to 0$$
$$1 \mapsto [Z]$$

Corollary 6.2.6 If $X = \mathbb{P}^n_{\mathbb{C}}$ and Z is a degree d irred hypersurface, then $Cl(\mathbb{P}^n \setminus Z) \cong \mathbb{Z}/d$.

About $Cl(\mathbb{P}^n_k)$ Given $D = \sum n_Y [Y]$, then define

$$\deg(D) = \sum n_Y \deg(Y)$$

where $\deg(Y)$ is the degree of the homogeneous polynomial defining $Y \subseteq \mathbb{P}^n_k$.

Then the claim is implied by

(i) $D \sim d \cdot H$ where $d = \deg(D)$, H is a hyperplane

(ii) If $f \in K(\mathbb{P}^n)$, then $\deg(\operatorname{div}(f)) = 0$

(iii) Degree map $Cl(\mathbb{P}^n) \longrightarrow \mathbb{Z}$ is an isomorphism

<u>Proof</u> (Hartshorne Prop 6.4) $\square$

§6.3 Cartier Divisors [no assumptions on X] unless I say 'Weil divisor'

<u>Notice</u> If D is a principal Weil divisor on X, then $D = \operatorname{div}(f)$, $f \in K(X)^*$ which is unique up to globally invertible functions, i.e. $\mathcal{O}_X^*(X)$. $\rightsquigarrow$ Defines a section of

$$\Gamma(X, \underbrace{K(X)^*/\mathcal{O}_X^*(X)}_{\text{sheaf of groups}})$$

$\ulcorner \mathcal{D} \lrcorner$

<u>Definition 6.3.1</u> A <u>Cartier divisor</u> $\mathcal{D}$ is a section of the sheaf of <u>abelian</u> <u>groups</u> on X $K_X^*/\mathcal{O}_X^*$ over X. { ☆ Who is K_X^*? }

For a scheme X, consider the presheaf

$U = \operatorname{Spec} A \longmapsto S^{-1}A$ where $S = \{$ all non-(zero divisors) $\}$

Define the sheaf $\mathcal{K}_X^*$ as the sheafification of this.

Similarly define $\mathcal{O}_X^*$ via

$U = \operatorname{Spec} A \longmapsto A^*$ units

and sheafification. $\therefore \mathcal{K}_X^*/\mathcal{O}_X^*$ makes sense

<u>Remark 6.3.2</u> "Practically"

Cover X by $\{U_i\}$ and give ~~rational~~ functions f_i on each U_i such that on $U_i \cap U_j$ the ratio f_i/f_j is invertible.

<u>Construction 6.3.3</u> If X is regular in codim 1 [& integral, separated, Noetherian] then a Cartier divisor $\mathcal{D}$ produces a Weil divisor D via: given $Y \subseteq X$ codim 1 integral, the generic point η_Y is contained in an affine on which $\mathcal{D}$ determines

an element of the fraction field $FF(A_Y) = K(X)$,
well-defined up to invertibles; therefore by the principal divisor construction we get a number $n_Y(\mathcal{D})$ attached to Y.
[check well-defined]
Ranging over Y we get a Weil divisor $D = \sum n_Y(\mathcal{D})[Y]$.

<u>Proposition 6.3.4</u> If X is [noetherian int sep] with all local rings UFDs (locally factorial) then the association "smooth"

$$\{\text{Cartier divisors}\} \longrightarrow \{\text{Weil divisors}\}$$

is an equivalence respecting the principal divisors. ⌜nose⌟

X a scheme, sheaves $\mathcal{K}_X, \mathcal{K}_X^*, \mathcal{O}_X, \mathcal{O}_X^*, \mathcal{K}_X^*/\mathcal{O}_X^*$

- Weil [hypothesis] $\supseteq$ Principal $\hookrightarrow$ Cartier

$\mathcal{D} \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ Practically: giving $\mathcal{D}$ is giving $\{(U_i, f_i)\}$ s.t. f_i is an element of $\mathcal{K}_X^*(U_i)$ & on $U_i \cap U_j$ their ratio is invertible

If X is $\underbrace{\text{R1}}_{\text{reg in codim 1}}$ (integral etc) then Cartier divisors give Weil divisors by applying $\mathrm{div}(f_i)$ locally.

Example Take $\mathbb{P}^n_{\mathbb{C}}$, $D = H$ (hyperplane)

- Can describe as a Cartier divisor – std open cover by affines U_i & f_i to be an equation cutting out $U_i \cap H \subseteq U_i$.

Prop^n 6.3.5 If X is normal, [integral, sep, noetherian] then Cartier divisors are exactly the locally principal Weil divisors.

Proof Omitted. □

Construction 6.3.6 Given $\mathcal{D}$ rep by $\{(U_i, f_i)\}$ a Cartier divisor. Can consider $L(\mathcal{D})$ to be the $\mathcal{O}_X$-submodule of $\mathcal{K}_X$ obtained on U_i as the $\mathcal{O}_X$-submodule generated by $\frac{1}{f_i}$. This is a locally free rank 1 sheaf "line bundle"

- Ex Take $\mathcal{D} = H$ hyperplane in $\mathbb{P}^n$. Then $\mathcal{O}_{\mathbb{P}^n}(H) =: L(H)$
  $\mathbb{V}(x_0)$, or any $\mathbb{V}(L)$ for L hom degree 1. Explicit description,

$$\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$$ [ok?✓]

Remarks A locally free sheaf of rank 1 – "invertible sheaf"
Given a line bundle $\mathcal{L}$, take $\mathcal{L}^\vee = \mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, this is also a line bundle.

Given $\mathcal{L}$ & $\mathcal{L}'$, $\mathcal{L} \otimes \mathcal{L}'$ is also a line bundle.

- Def $\underbrace{\mathrm{Pic}(X)}_{\text{Picard group}} = (\{\text{line bundles up to iso}\}, \otimes)$

Interesting groups $\mathrm{Cl}(X)$, $\mathrm{CartierCl}(X)$, $\mathrm{Pic}(X)$

Very mild assumptions [e.g. integral, or projective over k, ...]
then the natural map

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X) \quad \text{is an iso, i.e.}$$

the map $\mathrm{Cartier}(X) \longrightarrow \mathrm{Pic}(X)$ is surj, and kernel given by principals.

⌜I can at least see how principals are trivial⌟

§7. Sheaf Cohomology (Hartshorne III)

– Riemann-Roch, Serre duality, ...

<u>Basic Q</u>: Given a surjective map of sheaves $\mathcal{F} \to \mathcal{F}'' \to 0$, the global sections map

$$\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \quad \text{need not be surjective.}$$

Given an exact

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \qquad (*)$$

we get a very short

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \longrightarrow ?$$

Cohomology provides a long exact sequence associated to (*),

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to \cdots$$

§7.1 Resolution

<u>Def^n 7.1.1</u> An abelian group I is <u>injective</u> if given any diagram

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dotted] \end{tikzcd}$$

there exists a lifting making the diagram commute.

<u>Remark 7.1.2</u> For abelian groups, I injective $\Leftrightarrow$ I divisible

<u>Ex</u>: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, any quotient of a divisible group

⌜$\mathbb{Q}$ is flat, vector spaces are nice⌟

[Exercise]

<u>Def^n 7.1.3</u> An <u>injective resolution</u> of an abelian group A is an exact sequence

$$0 \to \underset{\text{nasty}}{A} \to \underset{\text{very simple}}{I_\bullet} \;\equiv\; 0 \to A \to \underbrace{I_0 \to I_1 \to \cdots}_{\text{injective objects}}$$

"Half a million things"

Prop 7.1.4 Injective resolutions of abelian groups exist

Proof Every abelian group injects into a divisible group.
Now construct resolution inductively. $\square$

Corollary 7.1.5 Injective resolutions exist in the category of sheaves of abelian groups.

(But: what is an injective abelian sheaf?)

Proof: For each $x \in X$, take $0 \to \mathcal{F}_x \to I_x$ (using Prop 7.1.4)
Now take $\iota_x : \{x\} \hookrightarrow X$ and consider $(\iota_x)_* I_x$
From earlier,

$$\mathcal{F} \hookrightarrow \prod_{x \in X} (\iota_x)_* I_x$$

Now construct resolution by induction. $\square$

§7.2 Cohomology

Given $\mathcal{F}$ a sheaf on X, "replace $\mathcal{F}$ by a resolution $I_\bullet$"
and define $H^i(\mathcal{F}, X) = \frac{\text{kernel}}{\text{image}}$ of $\Gamma(I_\bullet)$ at the i^{th} step

$$\text{i.e. } H^i(\mathcal{F}, X) = \frac{\ker(\Gamma(I_i) \to \Gamma(I_{i+1}))}{\operatorname{im}(\Gamma(I_{i-1}) \to \Gamma(I_i))}$$

<u>Last time</u>: $\Gamma(X,-)$: sheaves on $X \to$ Ab Groups is not (necessarily) right exact.

It <u>is</u> right exact in two circumstances:

(i) If X is affine (Hartshorne II Prop 6.7) and the sheaves are coherent.
Notes §5.3.

(ii) If $\mathcal{F}'$ is injective then $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ stays exact after applying $\Gamma(X,-)$

Based on (ii) defined $H^i(X,\mathcal{F})$ by resolving $\mathcal{F} \to I_\bullet$, applying $\Gamma(X,-)$ to $I_\bullet$, taking cohomology.

<u>Definition</u>: A sheaf $\mathcal{F}$ is <u>flasque</u> if all restriction maps $res^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ are surjective

<u>Prop</u>: Injective $\Rightarrow$ Flasque

<u>Properties 7.2.1</u>

(i) $H^i(X,-)$ is independent of resolution

(ii) Given $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact, there is a LES

$$0 \to H^0(\mathcal{F}') \to H^0(\mathcal{F}) \to H^0(\mathcal{F}'') \xrightarrow{\partial} H^1(\mathcal{F}') \to H^1(\mathcal{F}) \to H^1(\mathcal{F}'') \to \cdots$$

coboundary maps

(iii) Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0 \end{tikzcd}$$

we get a commutative square

$$\begin{tikzcd} H^i(\mathcal{F}'') \arrow[r] \arrow[d] & H^{i+1}(\mathcal{F}') \arrow[d] \\ H^i(\mathcal{G}'') \arrow[r] & H^{i+1}(\mathcal{G}') \end{tikzcd}$$

(iv) If we resolve $\mathcal{F}$ by flasque sheaves rather than injective sheaves $0 \to \mathcal{F} \to \mathcal{G}_\bullet$, then $H^i(X,\mathcal{F})$ can be computed from the complex $\Gamma(\mathcal{G}_\bullet)$.

(v) $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$

Remark 7.2.2 If $(X, \mathcal{O}_X)$ is ringed, $\mathcal{F}$ is q.coh $\mathcal{O}_X$-module, define sheaf cohomology via same definition.

Theorem 7.2.3 (Grothendieck vanishing) X – noetherian top space of dimension n and $\mathcal{F}$ a sheaf of ab groups on X.
Then $H^i(X, \mathcal{F}) = 0$ for $i > n$.

Theorem 7.2.4 (Serre's affineness criterion) X noetherian, then X is affine iff $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all q.coh $\mathcal{O}_X$ modules $\mathcal{F}$.

Typical use * If $\pi : X \to Y$ a morphism & $\mathcal{F}$ a locally free $\mathcal{O}_X$-module, then is $\pi_* \mathcal{F}$ locally free?

* Given X, Y projective over Spec k, there is a scheme $\mathrm{Mor}(X, Y)$. What is dim of $\mathrm{Mor}(X, Y)$?
* Prove that a scheme X is smooth?
* If X is a smooth projective curve over $\mathbb{C}$, then $\underset{\text{genus}}{g(X)} = \underset{\dim_{\mathbb{C}} H^1(\cdot)}{h^1(X, \mathcal{O}_X)}$

§7.3 Čech cohomology

Let X be a top space, $\mathcal{F}$ a sheaf of abelian groups on X.
Fix $\mathcal{U} = \{U_i\}_{i \in I}$, [with a well-ordering on I]

Write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

The group of p-cochains is $C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$

There is a 'differential'
$$C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$
$$\alpha \longmapsto d\alpha$$
where $(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i}_k \cdots i_{p+1}} \big|_{U_{i_0 \cdots i_{p+1}}}$

Exercise $d^2 = 0$

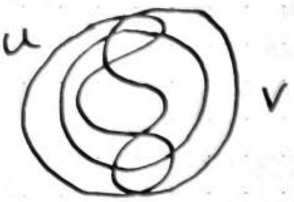
Definition 7.3.1 The Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ are the cohomology of the complex $C^\bullet(\mathcal{U}, \mathcal{F})$.

⚠ If $\mathcal{U}$ sucks then $\check{H}^p(\mathcal{U}, \mathcal{F})$ will also suck

Calculations (Flavours of)

$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ more generally $X \underset{j}{\hookrightarrow} \mathbb{P}^n$, $H^i(X, j^*\mathcal{O}_{\mathbb{P}^n}(d))$

Examples 7.3.2 $X = S^1$ with $\mathcal{F} = \underline{\mathbb{Z}}$ constant sheaf



$U = S^1 \setminus \{p_1\}$,

$V = S^1 \setminus \{p_2\}$, $p_1 \neq p_2$

$U \cap V$ = disjoint union of two intervals

$$C^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \quad , \quad C^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$
$$= \mathcal{F}(U \cap V)$$
$$(a, b) \xmapsto{d} (b-a, b-a)$$

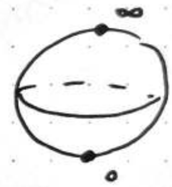
So $\check{H}^0 \cong \check{H}^1 \cong \mathbb{Z}$

kernel cokernel

Example 7.3.3 $X = \mathbb{P}^1_k$ with $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-2)$

$\mathcal{U} = \{U_0, U_1\}$ standard open cover

say $\mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$



$$C^0(\mathcal{U}, \mathcal{F}) \cong k\left[\frac{x_1}{x_0}\right] \times k\left[\frac{x_0}{x_1}\right]$$
$$d \downarrow$$
$$C^1(\mathcal{U}, \mathcal{F}) \cong k\left[\frac{x_0}{x_1}\right]_{\frac{x_1}{x_0}} = k\left[\left(\frac{x_0}{x_1}\right)^{\pm}\right]$$

$$\left[d(f, g) = g - f\frac{x_1^2}{x_0^2}\right]$$

Do calculation, get

$$\check{H}^*(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(-2)) = \begin{cases} 0, & * = 0 \\ k, & * = 1 \end{cases}$$