III Algebraic Topology

L1.1
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Def A homotopy between two maps $f_0, f_1 : X \to Y$ is a map $F: [0,1] \times X \longrightarrow Y$ s.t. $F(0,x) = f_0(x)$, $F(1,x) = f_1(x)$

| map = cts function

If such F exists, we say f_0, f_1 are homotopic, $f_0 \simeq f_1$

This is an equivalence relation on the set of maps from X to Y.

Def A map $f: X \to Y$ is a homotopy equivalence if there is a map $g: Y \to X$ s.t.

$$f \circ g \simeq id_Y, \quad g \circ f \simeq id_X$$

If this exists we write $X \simeq Y$

Silly example If $f: X \to Y$ is a homeomorphism, let $g = f^{-1}: Y \to X$, and take the constant homotopies b/w $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$

Example: consider $i: \{0\} \longrightarrow \mathbb{R}^n$, let $r: \mathbb{R}^n \longrightarrow \{0\}$ be the only map,

$r \circ i = id_{\{0\}}$ so take constant homotopy

$$i \circ r : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad x \mapsto 0$$

Let $F: [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(t,x) \mapsto tx$.

Then F is continuous, and is a homotopy b/w $i \circ r$, $id_{\mathbb{R}^n}$

$\Rightarrow \mathbb{R}^n \simeq \{0\}$ "$\mathbb{R}^n$ is contractible" (contractible = homotopy equivalent to $*$)

Example: consider $i: S^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$

Let $r: \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n$

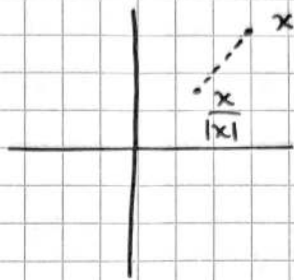
$$x \longmapsto \frac{x}{\|x\|},$$

$r \circ i = id_{S^n}$

$i \circ r \simeq id_{\mathbb{R}^{n+1}\setminus\{0\}}$ via $F: [0,1] \times (\mathbb{R}^{n+1}\setminus\{0\}) \to \mathbb{R}^{n+1}\setminus\{0\}$

$(t, x) \mapsto tx + (1-t)\frac{x}{\|x\|}$

$\Rightarrow \mathbb{R}^{n+1}\setminus\{0\} \simeq S^n$



Many problems in topology
can be reduced to problems in <u>alg topology</u>
"indirectly" ; $\mathbb{R}^n \cong \mathbb{R}^m \Rightarrow \mathbb{R}^n\setminus\{0\} \cong \mathbb{R}^m\setminus\{0\}$

$\Rightarrow \mathbb{R}^n\setminus\{0\} \simeq \mathbb{R}^m\setminus\{0\}$

$\Leftrightarrow S^{n-1} \simeq S^{m-1}$

this can be disproved for $n \neq m$ using "homology"

<u>Chain complexes</u>

<u>Def</u> A <u>chain complex</u> is a sequence of abelian groups and homomorphisms

$$\cdots \to C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

s.t. $d_i d_{i+1} = 0$ for all i

Write $(C_\bullet, d)$

A <u>co-chain complex</u> is

$$\cdots \leftarrow C^3 \leftarrow C^2 \xleftarrow{d^2} C^1 \xleftarrow{d^1} C^0 \leftarrow 0$$

s.t. $d^{i+1} d^i = 0$ for all i

The <u>homology</u> of $(C_\bullet, d)$ is

$$H_i(C_\bullet, d) = \ker(d_i : C_i \to C_{i-1}) / \operatorname{im}(d_{i+1} : C_{i+1} \to C_i)$$

using that $d_i d_{i+1} = 0$.

Similarly the <u>cohomology</u> of $(C^\bullet, d)$ is

$$H^i(C^\bullet, d) = \ker(d^{i+1} : C^i \to C^{i+1}) / \operatorname{im}(d^i : C_{i-1} \to C_i)$$

Singular (co)chains

The standard n-simplex $\Delta^n = \left\{ \begin{matrix} (t_0,\dots,t_n) \in \mathbb{R}^{n+1} : \\ t_i \geq 0,\ \sum t_i = 1 \end{matrix} \right\}$

The i^{th} face of Δ^n is

$$\Delta^n_i = \Delta^n = \{ (t_i)_{i=0}^{n+1} : t_i = 0 \},$$

the map

$$\delta_i : \Delta^{n-1} \to \Delta^n_i$$
$$(t_i)_{i=0}^{n-1} \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

is a homeo, i^{th} face map.

If X is a space, a singular n-simplex in X is a map $\sigma : \Delta^n \to X$.

$$C_n(X) = \left\{ \begin{matrix} \text{free abelian group on} \\ \text{the singular n-sx in } X \end{matrix} \right\}$$

$$= \left\{ \sum h_\sigma \cdot \sigma \;\middle|\; \begin{matrix} \sigma \text{ simplex},\ h_\sigma \in \mathbb{Z}, \\ \text{only finitely many } h_\sigma \neq 0 \end{matrix} \right\}$$

Furthermore, define

$$d_n : C_n(X) \to C_{n-1}(X)$$
$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta_i)$$

$$\ulcorner \Delta^{n-1} \xrightarrow{\delta_i} \Delta^n \xrightarrow{\sigma} X \lrcorner$$

Lemma For $i < j$, the maps $\delta_j \circ \delta_i$, $\delta_i \circ \delta_{j-1}$ from Δ^{n-2} to Δ^n are equal.

Pf Both insert zeros in i^{th}, j^{th} positions, and rearrange remaining terms monotonically. □

Corollary $d_{n-1} \circ d_n : C_n(X) \to C_{n-2}(X)$ is zero

Pf STP each generator goes to zero

$$d_{n-1} \circ d_n(\sigma) = \sum_{i=0}^{n-1} (-1)^i \left(\sum_{j=0}^{n} (-1)^j \sigma \circ \delta_j \right) \circ \delta_i$$

$$= \sum_{i<j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i$$

$$+ \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i$$

$$= \sum_{i<j} (-1)^{i+j} \sigma \circ \delta_i \circ \delta_{j-1} + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i$$

$$= \sum_{i \leq k} (-1)^{i+k+1} \sigma \circ \delta_i \circ \delta_k + \quad " \qquad k = j-1$$

$$= 0 . \qquad \square$$

$C_n(X) = \{$ free abelian group on the set of maps $\sigma : \Delta^n \to X \}$

$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \delta_i$ Checked $d_{n-1} \circ d_n = 0$

$$\Rightarrow H_n(X) = H_n(C_\bullet(X), d) = \frac{\mathrm{Ker}(d_n : C_n(X) \to C_{n-1}(X))}{\mathrm{Im}(d_{n+1} : C_{n+1}(X) \to C_n(X))}$$

(cycles: the numerator; boundaries: the denominator)

the (singular) homology of X

Similarly, let $C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z})$

Let d^{n-1} be the adjoint of d_n, i.e.

$(d^{n-1}\varphi)(\sigma) = \varphi(d_n \sigma)$ $\qquad d^{n-1} : C^{n-1}(X) \to C^n(X)$

$(d^n \circ d^{n-1})(\varphi)(\sigma) = \varphi(d_n \circ d_{n+1}(\sigma))$

$= \varphi(0) = 0 \quad \forall \sigma$

$\Rightarrow (d^n \circ d^{n-1})(\varphi) = 0 \quad \forall \varphi$

$\Rightarrow d^n \circ d^{n-1} = 0$

So $(C^\bullet(X), d)$ is a cochain complex.

The cohomology of X is

$$H^n(X) = H^n(C^\bullet(X), d) = \frac{\mathrm{Ker}(d^n : C^n(X) \to C^{n+1}(X))}{\mathrm{Im}(d^{n-1} : C^{n-1}(X) \to C^n(X))}$$

(cocycles: the numerator; coboundaries: the denominator)

<u>Def</u> If $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ are chain complexes, then a collection of maps $f_n : C_n \to D_n$, $n = 0, 1, \dots$ is called a <u>chain map</u> if $f_n \circ d^C_{n+1} = d^D_{n+1} \circ f_{n+1}$

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1} \\ d^C_{n+1} \downarrow & & \downarrow d^D_{n+1} \\ C_n & \xrightarrow{f_n} & D_n \end{array}$$

"this square commutes"

<u>Lemma</u> If $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, then

$$f_* : H_n(C_\bullet, d^C) \longrightarrow H_n(D_\bullet, d^D)$$
$$[x] \longmapsto [f_n(x)]$$

is a well-defined homomorphism.

<u>Proof</u> If $[x] \in H_n(C_\bullet, d^C)$, then x is a cycle i.e. $d^C_n(x) = 0$.

Thus $d^D_n(f_n(x)) = f_{n-1}(d^C_n(x)) = 0$.

So $f_n(x)$ is a cycle. So $[f_n(x)]$ exists.

If $[x] = [y]$ then $x - y = d^C_{n+1}(z)$ for some $z \in C_{n+1}$.

$$\begin{array}{ccc} f_n(x-y) & = & f_n(d^C_{n+1} z) \\ \| & & \| \\ f_n(x) - f_n(y) & & d^D_{n+1}(f_{n+1} z) \end{array}$$

So $[f_n(x)] = [f_n(y)]$ and we're done. $\square$

If $f: X \to Y$ is a continuous map, then

$$f_\#: C_\bullet(X) \to C_\bullet(Y)$$
$$(\sigma: \Delta^n \to X) \mapsto (f \circ \sigma : \Delta^n \to Y)$$

is a chain map (exercise).

So induces a map $f_*: H_n(X) \to H_n(Y)$.

If $g: Y \to Z$ then $(g \circ f)_* = g_* \circ f_*$

and $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$

Similarly, applying $\mathrm{Hom}(-, \mathbb{Z})$ gives a map

$$f^\#: C^\bullet(Y) \to C^\bullet(X)$$

adjoint to $f_\#$. Again a cochain map.

So induces a map $f^*: H^*(Y) \to H^*(X)$.

$$(g \circ f)^* = f^* \circ g^*$$

<u>Lemma</u> If $f: X \to Y$ is a homeomorphism, then

(i) $f_*: H_n(X) \to H_n(Y)$ and $f^*: H^n(Y) \to H^n(X)$ are isomorphisms $\forall n$

<u>Pf</u> Write $g = f^{-1}: Y \to X$ for the inverse.

Then $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$.

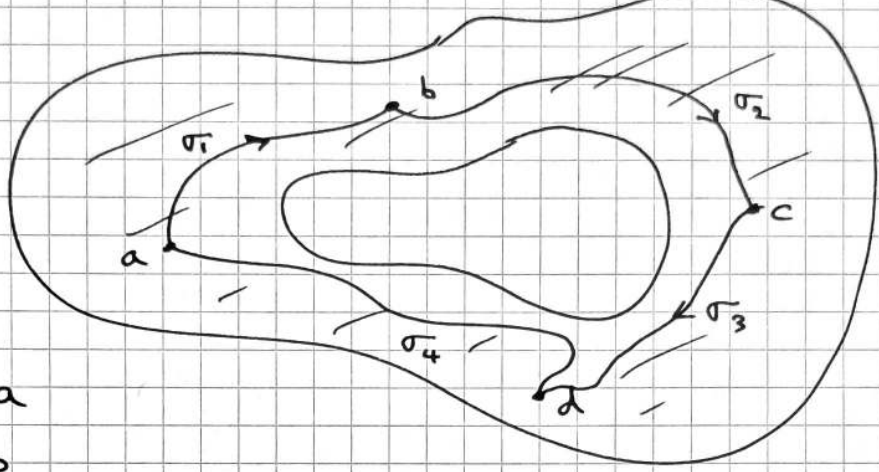
So $f_* \circ g_* = \mathrm{id}_{H_n(Y)}$, $g_* \circ f_* = \mathrm{id}_{H_n(X)}$. $\square$

<u>Remark</u> although cochains $C^\cdot(X) = \mathrm{Hom}(C_\cdot(X), \mathbb{Z})$ are dual, $H^n(X) \neq \mathrm{Hom}(H_n(X), \mathbb{Z})$ usually.

What does singular homology measure?

$$H_n(X) = \frac{\text{cycles}}{\text{boundaries}}$$

A cycle is a collection of singular n-simplices whose $(n-1)$-dim faces cancel.



$$d_1(\sigma_1) = b - a$$
$$d_1(\sigma_2) = c - b$$
$$d_1(\sigma_3) = d - c$$
$$d_1(\sigma_4) = a - d$$

$$\therefore d_1(\underbrace{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4}_{\text{cycle}}) = 0$$

$$dx \stackrel{?}{=} 0 \quad \text{vs} \quad d(-) \stackrel{?}{=} x$$

<u>Example</u> Let $X = \{pt\}$.

$$C_n(X) = \mathbb{Z}\{\underbrace{\sigma_n : \Delta^n \to \{pt\}}_{\text{\underline{the} map}}\}$$

$$\cdots \to \underset{\mathbb{Z}}{\overset{}{C_3(X)}} \xrightarrow[0]{} \underset{\mathbb{Z}}{C_2(X)} \xrightarrow[\cong]{} \underset{\mathbb{Z}}{C_1(X)} \xrightarrow[0]{} \underset{\mathbb{Z}}{C_0(X)} \to 0$$

(each $C_n(X) \cong \mathbb{Z}$)

$$d_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n \circ \delta_i = \left(\sum_{i=0}^{n} (-1)^i\right) \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{n-1} & \text{if } n \text{ even} \end{cases}$$

$$H_0(X) \cong \mathbb{Z}, \quad H_n(X) = 0 \text{ for } n \geq 1$$

$$H_i(*) = \begin{cases} \mathbb{Z} \ ; & i=0 \\ 0 \ ; & o/w \end{cases}$$

$C_\cdot(*)$ was

$$\cdots \to \mathbb{Z} \xrightarrow{\text{zero}} \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{\text{zero}} \mathbb{Z} \to 0$$

$C^\cdot(*)$ is

$$\cdots \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\sim} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

So $H^i(*) = \begin{cases} \mathbb{Z} \ ; & i=0 \\ 0 \ ; & o/w \end{cases}$

<u>Example 2</u> If $X = \coprod_{\alpha \in I} X_\alpha$ is a disjoint union, as Δ^n is connected, any singular n-simplex in X lies in some X_α.

$$\Rightarrow H_i(X) = \bigoplus_{\alpha \in I} H_i(X_\alpha)$$

<u>Example 2.5</u> $H_i(\emptyset) = ?$

<u>Exercise</u> $H^i(X) = ?$

<u>Example 3</u> Let X be path-connected (& non-empty)

$C_0(X) = \{\text{free abelian group on pts of } X\}$
$= \{ \sum n_\sigma \cdot \sigma \mid \sigma : \Delta^0 \to X \text{ only finitely many } n_\sigma \text{ non-zero} \}$

Consider

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$$
$$\sum n_\sigma \cdot \sigma \mapsto \sum n_\sigma$$

This is onto as $X \neq \emptyset$

The composition $\varepsilon \circ d_1 : C_1(X) \longrightarrow \mathbb{Z}$ is zero.

$(\sigma : \Delta^1 \to X) \searrow (\sigma \circ \delta_0 - \sigma \circ \delta_1) \nearrow 1 - 1 = 0$

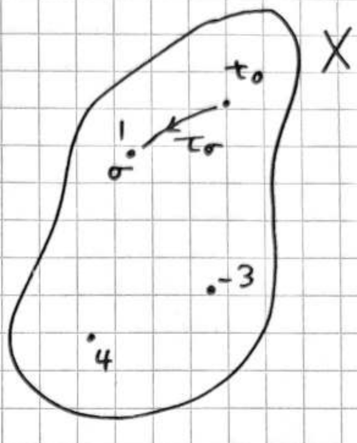
$\therefore \varepsilon : H_0(X) \overset{\text{onto}}{\twoheadrightarrow} \mathbb{Z}$ is well-defined

$$[\sum n_\sigma \sigma] \mapsto \sum n_\sigma$$

Suppose $\varepsilon(\sum n_\sigma \sigma) = 0$.

WTS $\sum n_\sigma \sigma = d_1(\text{qqch})$ so that $\varepsilon : H_0(X) \xrightarrow{\sim} \mathbb{Z}$

For each $\sigma \in X$, <u>choose</u> a path $\tau_\sigma : \Delta^1 \to X$ from x_0 to σ.



Compute

$$d_1\left(\sum n_\sigma \tau_\sigma\right)$$
$$= \sum n_\sigma \, d_1(\tau_\sigma)$$
$$= \sum n_\sigma (\sigma - x_0)$$
$$= \left(\sum n_\sigma \sigma\right) - \underbrace{\left(\sum n_\sigma\right)}_{\text{zero}} x_0 \quad \checkmark$$

$\therefore H_0(X) \cong \mathbb{Z}$ for X path-connected (& non-empty)

Major tools of (co)homology

<u>Homotopy invariance theorem</u>: if $f \simeq g : X \to Y$,

then $f_* : H_i(X) \to H_i(Y)$ equals g_*

$f^* : H^i(Y) \to H^i(X)$ equals g^*

$\therefore$ if f is a homotopy equivalence then f_*, f^* are isos

(same proof as for homeomorphisms)

<u>Def</u> Homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ are <u>exact at B</u> if $\operatorname{im} f = \ker g$.

A longer sequence $\cdots \to A_i \to A_{i-1} \to A_{i-2} \to A_{i-3} \to \cdots$ is <u>exact</u> if it is exact at A_i for all i.

<u>Mayer-Vietoris theorem</u>: let $X = A \cup B$ with A, B open sets

$$\begin{array}{ccccc} & & A & & \\ & \overset{i_A}{\nearrow} & & \overset{j_A}{\searrow} & \\ A \cap B & & & & A \cup B = X \\ & \underset{i_B}{\searrow} & & \underset{j_B}{\nearrow} & \\ & & B & & \end{array}$$

There exist homomorphisms $\partial_{MV} : H_i(X) \to H_{i-1}(A \cap B)$ s.t.

$$\cdots \to H_{i+1}(X) \xrightarrow{\partial_{MV}} H_i(A \cap B) \xrightarrow{(i_A)_* \oplus (i_B)_*} H_i(A) \oplus H_i(B) \xrightarrow{(j_A)_* - (j_B)_*} H_i(X) \xrightarrow{\partial_{MV}} H_{i-1}(A \cap B) \to \cdots$$

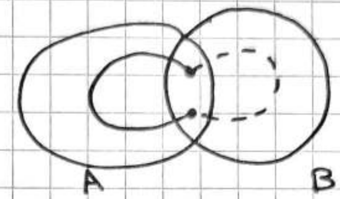
is an exact sequence.

horrid! layout on my part

For ∂_{mv}: suppose $[a+b] \in H_i(X)$
for a a sum of sing. simplices in A
b " in B

then $\partial_{mv}([a+b]) = [d_i(a)]$ in $H_{i-1}(A \cap B)$
$= [-d_i(b)]$

This is natural in $X = A \cup B$:



if $f : A \cup B \to U \cup V$
is s.t. $f(A) \subset U$, $f(B) \subset V$
then f_* gives a map of long exact sequences, e.g.

$$\begin{array}{ccc} H_i(A\cup B) & \xrightarrow{\partial_{mv}} & H_{i-1}(A\cap B) \\ f_* \downarrow & \circlearrowright & \downarrow f_* = (f|_{A\cap B})_* \\ H_i(U\cup V) & \xrightarrow[\partial_{mv}]{} & H_{i-1}(U\cap V) \end{array}$$

Relative homology and excision

If $A \subset X$ is a subspace, $i : A \hookrightarrow X$, observe
$i_\# : C_n(A) \to C_n(X)$ is injective,
so define $C_n(X, A) := C_n(X)/i_\# C_n(A)$
One checks that
$$d_i([x]) = [d_i(x)]$$
is well-defined $C_n(X,A) \to C_{n-1}(X,A)$
and gives a chain complex $(C_\bullet(X,A), d)$.
The quotient map $C_\bullet(X) \xrightarrow[q]{} C_\bullet(X,A)$ is a chain map.

$\Rightarrow H_n(X,A)$ is $H_n(C_\bullet(X,A), d)$, relative homology.

Relative homology exact sequence

There is a $\partial : H_n(X, A) \longrightarrow H_{n-1}(A)$ s.t.

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{q_*} H_{n-1}(X, A) \longrightarrow \cdots$$

is exact.

Relative homology theorem

For $i : A \hookrightarrow X$ an inclusion, $C_\bullet(X, A) = \dfrac{C_\bullet(X)}{i_\# C_\bullet(A)}$, there is a hom $\partial : H_i(X, A) \to H_{i-1}(A)$ s.t.

$$\cdots \to H_i(A) \xrightarrow{i_*} H_i(X) \xrightarrow{q_*} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \xrightarrow{i_*} H_{i-1}(X) \xrightarrow{q_*} H_{i-1}(X, A) \to \cdots$$ is exact.

Here $[[x]] \in H_n(X, A)$,
$\partial([[x]]) = [a]$

$[x] \in C_n(X, A)$ is a cycle
if $d_n[x] = 0$
$\quad \overset{\shortparallel}{[d_n(x)]}$
i.e. $d_n(x) \in i_\# C_{n-1}(A)$
$d_n(x) = i_\#(a)$

Relative homology $H_n(X, A)$ is defined for pairs (X, A) (space, subspace), and if $(X, A) \to (Y, B)$ is a map of pairs i.e. $f : X \to Y$ s.t. $f(A) \subset B$, then there is an induced map $f_* : H_n(X, A) \to H_n(Y, B)$, which participates in a map of LES.

Excision theorem Let (X, A) be a pair and $Z \subset A$ be s.t. $\overline{Z} \subset A^\circ$. Then the map of pairs

$$(X \setminus Z, A \setminus Z) \longrightarrow (X, A)$$

induces an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A).$$

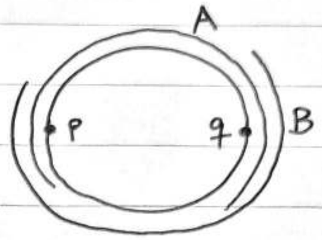
All of these have clear analogues for cohomology:

- homotopy invariant
- M-V, with $\partial_{MV} : H^i(A \cap B) \longrightarrow H^{i+1}(A \cup B)$
- relative cochains $C^\bullet(X, A) = \mathrm{Hom}(C_\bullet(X, A), \mathbb{Z})$

  so relative cohomology + exact seq $\partial : H^i(A) \to H^{i+1}(X, A)$
  + excision holds

Homology of spheres

Example Cover S^1 by open sets A, B as shown, so



$$A \cong (0,1) \simeq *$$
$$B \cong (0,1) \simeq *$$
$$A \cap B \cong (0,1) \amalg (0,1) \simeq * \amalg *$$

Apply Mayer-Vietoris

$$H_{n+1}(S^1) \xrightarrow{\partial} H_n(A\cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(S^1) \to$$
(with $H_n(A\cap B)$, $H_n(A)$, $H_n(B)$ crossed out and marked "zero")

$$\to H_{n-1}(A\cap B) \longrightarrow \cdots \quad \cdots \longrightarrow H_2(S^1) \xrightarrow{\partial}$$
(with $H_{n-1}(A\cap B)$ marked "zero")

$$\to H_1(A\cap B) \to H_1(A) \oplus H_1(B) \longrightarrow H_1(S^1) \xrightarrow{\partial}$$
(with $H_1(A\cap B)$, $H_1(A)$, $H_1(B)$ marked "zero"; $H_1(S^1) \cong \mathbb{Z}$ (!))

$$\to H_0(A\cap B) \xrightarrow{i_A \oplus i_B} H_0(A) \oplus H_0(B) \longrightarrow H_0(S^1) \longrightarrow 0$$
$$H_0(A\cap B) \cong \mathbb{Z}\oplus\mathbb{Z}, \quad H_0(A) \cong \mathbb{Z}, \quad H_0(B) \cong \mathbb{Z}, \quad H_0(S^1) \cong \mathbb{Z}$$

For $k>1$, $H_k(S^1)$ sits between zero abelian groups.
So $H_k(S^1) = 0$ for $k>1$ by exactness.

$$H_0(A\cap B) = \mathbb{Z}[p] \oplus \mathbb{Z}[q]$$
$$H_0(A) = \mathbb{Z}[p] \quad \text{and } [p]=[q]$$
$$H_0(B) = \mathbb{Z}[q] \quad \text{"}$$

So $H_0(A\cap B) \longrightarrow H_0(A) \oplus H_0(B)$
given by $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
Kernel spanned by $[p]-[q]$, iso to $\mathbb{Z}$.

$$\therefore H_1(S^1) \cong \ker(i_{A*} \oplus i_{B*}) = \operatorname{im}(\partial_{mv})$$
$$\cong \mathbb{Z}\{\text{some } x \text{ s.t. } \partial_{mv}(x) = [p]-[q]\}$$

In fact x is represented by (1-sx in A from p to q) − (1-sx in B from p to q)

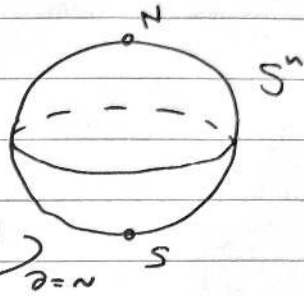
Theorem For $n \geq 1$, $H_i(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{o/w} \end{cases}$

Proof We have the case $n=1$. Proceed by induction.

Let $A = S^n \setminus \{N\} \simeq *$

$B = S^n \setminus \{S\} \simeq *$

$A \cap B = S^n \setminus \{N, S\} \simeq S^{n-1}$ (the equator)



$$H_{i+1}(S^n) \xrightarrow{\partial} H_i(S^{n-1}) \xrightarrow{i_A \oplus i_B} \underbrace{H_i(*)}_{\text{zero}} \oplus \underbrace{H_i(*)}_{\text{zero}} \xrightarrow{j_A - j_B} H_i(S^n) \xrightarrow{\partial = \sim}$$

$$H_{i-1}(S^{n-1}) \xrightarrow{i_A \oplus i_B} \underbrace{H_{i-1}(*)}_{\text{zero}} \oplus \underbrace{H_{i-1}(*)}_{\text{zero}} \xrightarrow{j_A - j_B} H_{i-1}(S^n) \to \cdots$$

$$\cdots \to H_1(S^n) \xrightarrow{\partial = 0}$$

$$H_0(S^{n-1}) \to H_0(*) \oplus H_0(*) \to H_0(S^n) \to 0$$

$$\mathbb{Z} \qquad\quad \mathbb{Z} \qquad \mathbb{Z} \qquad\qquad \mathbb{Z}$$

$$(a, b) \longmapsto a+b$$

$$a \longmapsto (a, -a)$$

$\Rightarrow \partial_{mv} : H_i(S^n) \to H_{i-1}(S^{n-1})$ is an iso for $i > 1$

and $\partial_{mv} : H_1(S^n) \to H_0(S^{n-1})$ is the zero hom for $n \geq 2$

\+ inj $\Rightarrow H_1(S^n) = 0$

It follows by induction that for $i \geq 2$,

$H_i(S^n) \cong H_{i-1}(S^{n-1})$

so that the theorem holds. ▨

Corollary $S^n \not\cong S^m$ if $n \neq m$

Corollary $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$

<u>Def</u>

Def^n If $f: S^n \to S^n$ is a continuous map, then
$$f_* : \underset{\mathbb{Z}}{\underset{\wr\wr}{H_n(S^n)}} \to \underset{\mathbb{Z}}{\underset{\wr\wr}{H_n(S^n)}}$$
is multiplication by some integer; this is called the **degree of f** $\deg(f)$.

Elementary properties

i) $\deg(id : S^n \to S^n) = 1$ as $(id)_* = id_{H_n(S^n)}$

ii) if $f: S^n \to S^n$ is not surjective, say $x \notin f(S^n)$
factorise $f: S^n \xrightarrow{f'} S^n \setminus \{x\} \xrightarrow{inc} S^n$
$$f_* : \underset{\mathbb{Z}}{\underset{\wr\wr}{H_n(S^n)}} \xrightarrow{f'_*} \underset{0}{\underset{\|}{H_n(*)}} \xrightarrow{inc_*} \underset{\mathbb{Z}}{\underset{\wr\wr}{H_n(S^n)}}$$
$\therefore \deg(f) = 0$

iii) if $f \simeq g$ then $\deg(f) = \deg(g)$

Theorem (Brouwer)

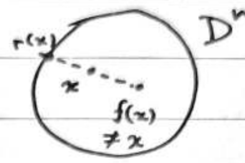
If $f: D^n \to D^n$ is continuous then f has a fixed point.

Brouwer's fixed point theorem: any continuous map $f: D^n \to D^n$ has a fixed point

Proof $n=0$ ✓

$n=1$ IVT (or variant for what follows)

$n \geq 2$ Suppose $f: D^n \to D^n$ does not have a fixed point.



Define $r(x) \in \partial D^n = S^{n-1}$ to be the point on the ray from $f(x)$ through x.

One checks that r is continuous as long as f is.

$$r: D^n \to \partial D^n$$

By construction $r(x) = x$ if $x \in \partial D^n$.

So consider

$$id: S^{n-1} = \partial D^n \xhookrightarrow{\iota} D^n \xrightarrow{r} S^{n-1} = \partial D^n$$

$$\Rightarrow id_*: \underset{\mathbb{Z}}{H_{n-1}(S^{n-1})} \xrightarrow{\iota_*} \underset{=0}{H_{n-1}(D^n)} \xrightarrow{r_*} \underset{\mathbb{Z}}{H_{n-1}(S^{n-1})}$$

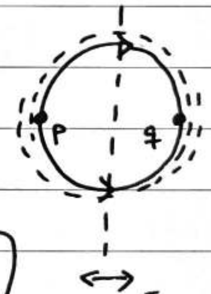
(composite: id) $(n \geq 2)$

As $1 \neq 0 \in \mathbb{Z}$, this cannot happen. ※

So such an f cannot exist. ▨

Proposition A reflection $r: S^n \to S^n$ has degree -1.

Proof $n=1$ Choose A, B, p, q adapted to the reflection r.



By naturality of the M-V sequence

$$0 \to H_1(S^1) \xrightarrow{\partial_{MV}} H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(S^1) \to 0$$

$H_1(S^1) \cong \mathbb{Z}[x]$ where $x \mapsto [p]-[q]$; $H_0(A\cap B) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $[p]$, $[q]$; $H_0(A) \oplus H_0(B) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $[p]$, $[p]$.

$[p] \longmapsto ([p],[p])$

$[q] \longmapsto ([p],[p])$

$[p]-[q] \longmapsto 0$

$$0 \to H_1(S^1) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(S^1) \to 0$$

$$\begin{array}{ccc} x & \longmapsto & [p]-[q] \\ {\scriptstyle !}\downarrow & & \downarrow r_* \\ -x & \longmapsto & [q]-[p] \end{array}$$

$\therefore\ r_* : H_1(S^1) \longrightarrow H_1(S^1)$ is multiplication by -1.

<u>For $n>2$</u>: Cover S^n by $A = S^n \setminus \{N\}$
$B = S^n \setminus \{S\}$

and let r reflect in a plane containing N, S.

$$\begin{array}{ccccccc} 0 \to H_n(S^n) & \xrightarrow[\sim]{\partial_{mv}} & H_{n-1}(A\cap B) & \longrightarrow 0 & & & \\ & & & \overset{\sim}{\longleftarrow} & H_{n-1}(S^{n-1}) \\ \downarrow r_* & & \downarrow r_* & & \downarrow (r|_{S^{n-1}})_* \\ 0 \to H_n(S^n) & \xrightarrow[\partial_{mv}]{\sim} & H_{n-1}(A\cap B) & \longrightarrow 0 & & & \\ & & & \overset{\sim}{\longleftarrow} & H_{n-1}(S^{n-1}) \end{array}$$

$H_{n-1}(A\cap B) \cong H_{n-1}(S^{n-1})$ equator

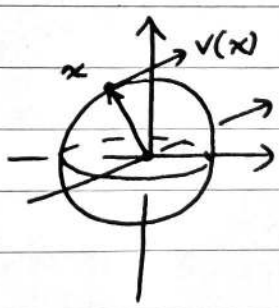
<u>Observe</u>: $r|_{S^{n-1}}$ is again a reflection
$\Rightarrow$ it has degree -1 by induction

two commuting squares $\Rightarrow$ r has degree -1 too. □

<u>Corollary</u>: the antipodal map $a : S^n \longrightarrow S^n$
$(x_0, \dots, x_n) \mapsto (-x_0, \dots, -x_n)$

has degree $(-1)^{n+1}$. □

<u>Corollary</u>: S^n has a nowhere zero vector field
$\Updownarrow$
n is odd



<u>Proof</u>: A vector field is a cts
$v : S^n \to \mathbb{R}^{n+1}$
s.t. $\forall x \in S^n$, $\langle x, v(x) \rangle = 0$
Suppose a nowhere zero v exists,
so let $w = \frac{v}{\|v\|} : S^n \to S^n$

Consider $H : [0,\pi] \times S^n \longrightarrow S^n$
$(t, x) \mapsto (\cos t\, x) + (\sin t)\, w(x)$,

a homotopy from the identity to the antipodal map.

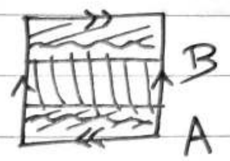
$\Rightarrow 1 = \deg(id_{S^n}) = \deg(a) = (-1)^{n+1}$

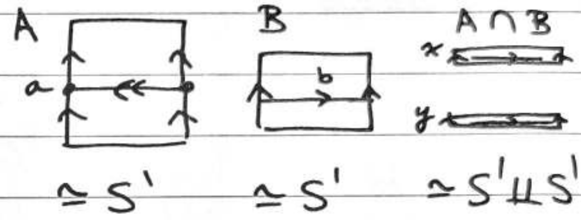
$\Rightarrow$ n is odd

If n is odd, let

$v(x_1, y_1, \dots, x_r, y_r) = (-y_1, x_1, \dots, -y_r, x_r)$. ▨

<u>Example</u>: let K be given as the shown quotient of $[0,1]^2$.





$$0 \to H_2(K) \xrightarrow{\partial_{mv}}$$

$$H_1(A\cap B) \xrightarrow{i_A \oplus i_B} H_1(A) \oplus H_1(B) \xrightarrow{j_A - j_B} H_1(K) \xrightarrow{\partial_{mv}}$$

$H_1(A\cap B) \cong \mathbb{Z}\oplus\mathbb{Z}$, $H_1(A)\oplus H_1(B) \cong \mathbb{Z}_a \oplus \mathbb{Z}_b$

$$H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(K) \to 0$$

$H_0(A\cap B) \cong \mathbb{Z}\oplus\mathbb{Z}$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $H_0(A)\oplus H_0(B) \cong \mathbb{Z}\oplus\mathbb{Z}$, $H_0(K) \cong \mathbb{Z}$

$0 \to \mathbb{Z}([p]-[q]) \to H_0(A\cap B)$

[a, b images of generator of $H_1(S^1)$ under maps $S^1 \to A, B$]

$$H_1(A\cap B) \longrightarrow H_1(A) \oplus H_1(B)$$
$$x \longmapsto -[a] + [b]$$
$$y \longmapsto [a] + [b]$$

$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ is injective, but its cokernel is $\dfrac{\langle a, b\rangle}{(b-a, b+a)} \cong \dfrac{\langle a\rangle}{(2a)} \cong \mathbb{Z}/2\mathbb{Z}$

$\therefore H_2(K) = 0$ and $0 \to \mathbb{Z}/2\mathbb{Z} \to H_1(K) \to \mathbb{Z} \to 0$

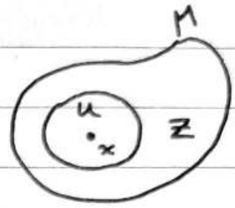
$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Local degree

Lemma Let M be a d-dim manifold (M is Hausdorff + locally homeo to $\mathbb{R}^d$) and let $x \in M$. Then

$$H_i(M, M \setminus x) \cong \begin{cases} \mathbb{Z} & \text{if } i = d, \\ 0 & \text{o/w}. \end{cases}$$

Proof Let $x \in \overset{\text{open}}{U} \subset M$ with $U \cong \mathbb{R}^d$, $x \mapsto 0$.



Then $Z = M \setminus U$ is closed and lies in $M \setminus x$.
So by excision

$$H_i(M \setminus Z, (M \setminus x) \setminus Z) \xrightarrow{\sim} H_i(M, M \setminus x)$$

excision: $\overline{Z} = Z \subset \operatorname{int}(M \setminus x) = M \setminus x$

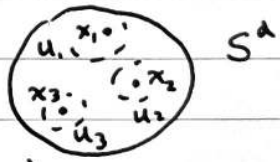
$$H_i(M\setminus Z, (M\setminus x)\setminus Z) = H_i(U, U \setminus x)$$

$\downarrow \cong$ via homeo

$$H_i(\mathbb{R}^d, \mathbb{R}^d \setminus 0)$$

This gives the claimed answer, using
(i) LES for relative H_*
(ii) $\mathbb{R}^d \simeq *$
(iii) $\mathbb{R}^d \setminus 0 \simeq S^{d-1}$ ▨

Suppose $f: S^d \to S^d$ is such that $\exists\, y \in S^d$ with $f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite.
Let $x_i \in U_i$ be disjoint open neighbourhoods s.t. $f(U_i) \subset V$ for some open ngbd $V \ni y$.



Then $(f|_{U_i})_*: H_d(U_i, U_i \setminus x_i) \overset{\sim}{\underset{\text{excise}}{\to}} H_d(S^d, S^d \setminus x_i)$

$$\uparrow \wr \text{ LES}$$

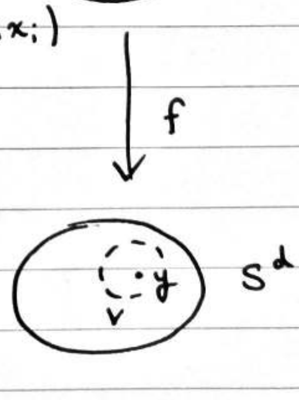
$$H_d(S^d) \text{ ``}=\mathbb{Z}\text{''}$$

$\downarrow$

$H_d(V, V \setminus y) \xrightarrow{\sim} H_d(S^d, S^d \setminus y)$

$\uparrow \wr$

$H_d(S^d) \leftarrow - - -$ mult. by $\deg(f)|_{x_i}$ $- - - H_d(S^d)$



Then define $\deg(f)|_{x_i}$ as above.

Theorem $\deg(f) = \sum_{i=1}^{k} \deg(f)|_{x_i}$

Proof

$$\begin{array}{ccccc}
\mathbb{Z} = H_d(S^d) & \xrightarrow[\cdot \deg f]{f_*} & H_d(S^d) = \mathbb{Z} & & \\
\downarrow & & \searrow \cong & & \text{id} \\
H_d(S^d, S^d \setminus \{x_1, \dots, x_k\}) & \xrightarrow{f_*} & H_d(S^d, S^d \setminus y) \cong \mathbb{Z} & & \\
\uparrow \cong \ \text{excision } Z = S^d \setminus \bigcup_{i=1}^{k} U_i & & \uparrow \cong & & \text{id} \\
\bigoplus_{i=1}^{k} H_d(U_i, U_i \setminus x_i) & \xrightarrow{\oplus (f|_{U_i})_*} & H_d(V, V \setminus y) \cong \mathbb{Z} & &
\end{array}$$

$1 \in \mathbb{Z}$, $1 \mapsto (1, \dots, 1)$

$(1, \dots, 1) \in \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{k \text{ times}} \cong \bigoplus_{i=1}^{k} H_d(U_i, U_i \setminus x_i)$

and $(1, \dots, 1) \mapsto \sum_{i=1}^{k} \deg(f)|_{x_i} \in H_d(V, V\setminus y)$

☒

5-lemma!

Ex: $z \mapsto z^n$ as map $S^1 \to S^1$ has degree n (tricky!)

Relative homology exact sequence

For a space X and subspace $A \subset X$, $i : A \to X$,

$$C_\bullet(X, A) = C_\bullet(X) / i_\# C_\bullet(A)$$

The LES for $H_*(A)$, $H_*(X)$, $H_*(X, A)$ comes from a more general construction.

Theorem Let $0 \to A_\bullet \xrightarrow{i_\#} B_\bullet \xrightarrow{q_\#} C_\bullet \to 0$ be a short exact sequence of chain complexes.

i.e. all maps are chain maps, and

$$0 \to A_n \to B_n \to C_n \to 0 \text{ is exact for all } n.$$

Then there are maps $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ s.t.

$$\to H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{q_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{i_*} H_{n-1}(B_\bullet) \to \cdots$$

is exact.

Proof (outline)

Let $[c] \in H_n(C_\cdot)$, so $c \in C_n$ has $d^C c_n = 0$.

As q_n is onto, choose $b \in B_n$ s.t. $q_n(b) = c$.

$$q_{n-1} d^B b = d^C q_n b = 0$$

$$\begin{array}{ccccccccc}
0 & \to & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{q_n} & C_n & \to & 0 \\
 & & \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\
0 & \to & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & \xrightarrow{q_{n-1}} & C_{n-1} & \to & 0 \\
 & & \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\
0 & \to & A_{n-2} & \xrightarrow{i_{n-2}} & B_{n-2} & \xrightarrow{q_{n-2}} & C_{n-2} & \to & 0
\end{array}$$

So $\exists\, a \in A_{n-1}$ s.t. $i_{n-1} a = d^B b$.

$$i_{n-2} d^A a = d^B i_{n-1} a = 0$$

i_{n-2} injective so $d^A a = 0$.

Let $\partial([c]) = [a]$. (Or try)

We <u>chose</u> b.

Let b' map to c under q_n.

Let a' map to $d^B b'$ under i_{n-1}.

Now $q_n(b - b') = 0 \quad \therefore \quad b - b' = i_n(a'' \in A_n)$

Then
$$\begin{aligned}
i_{n-1}(d^A a'') &= d^B i_n a'' \\
&= d^B(b - b') \\
&= d^B b - d^B b' \\
&= i_{n-1} a - i_{n-1} a'
\end{aligned}$$

$\therefore d^A a'' = a - a'$ since i_{n-1} injective

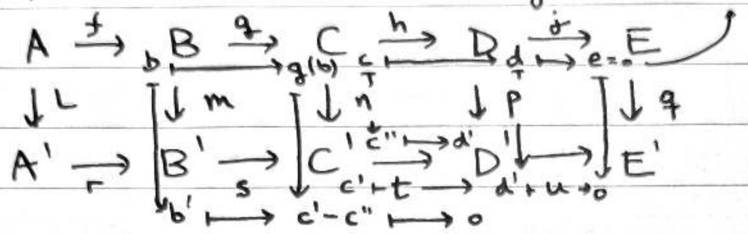
$\therefore [a] = [a']$

Similarly analyse choice of c; take $c' = c + d^C(c'')$.

See Ex sheet for rest of proof. ☒

Theorem (5-lemma)

Consider the commutative diagram



where the rows are exact,

- m, p are isomorphisms
- L is epic (surjective)
- q is monic (injective)

Then n is an isomorphism.

Proof let's show n is surjective.

Trivial diagram chase:

Let $c' \in C'$.

Then $t(c') = p(d)$, as p is epic.

Then $q(j(d)) = u(p(d)) = u(t(c')) = 0$.

Since q is monic, $j(d) = 0$.

By exactness, if $j(d) = 0$ then $d = h(c)$ for some $c \in C$.

Consider $c' - n(c) \in C'$.

Then $t(c' - n(c)) = t(c') - p(h(c))$
$= t(c') - p(d) = 0$.

So by exactness, $c' - n(c) = s(b')$.

Let $b' = m(b)$, using m epic.

Then $n(c + g(b)) = n(c) + s(m(b))$
$= n(c) + s(b')$
$= n(c) + c' - n(c)$
$= c'$. ∎

$n(g(b)) = c' - c''$
$= c' - n(c)$
$\therefore c' = n(c + g(b))$

Corollary If $f: (X, A) \to (Y, B)$ is a map of pairs, and any two of

$f_* : H_*(A) \to H_*(B)$

$f_* : H_*(X) \to H_*(Y)$

$f_* : H_*(X, A) \to H_*(Y, B)$

are iso's, then so is the third.

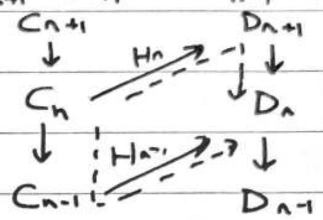
Proof Look at map of LES. 4/5 maps are isos.

The unknown map lies between 4 iso's.
So apply the 5-lemma. ▨

Homotopy invariance

Need to show $f \simeq g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$

Def^n A chain homotopy between chain maps $f_\bullet, g_\bullet$ from $C_\bullet$ to $D_\bullet$ is a collection of homomorphisms $H_n : C_n \to D_{n+1}$ s.t. $g_n - f_n = d^D_{n+1} \circ H_n + H_{n-1} \circ d^C_n \quad \forall n$.



Lemma If $f_\bullet, g_\bullet$ are chain homotopic, then $f_* = g_* : H_n(C_\bullet) \to H_n(D_\bullet)$

Pf Let $[c] \in H_n(C_\bullet)$.
Then

$$g_n(c) - f_n(c) = d^D_{n+1}(H_n(c)) + H_{n-1}(\underbrace{d^C_n(c)}_{\text{zero as } c \text{ cycle}})$$

$$\Rightarrow [g_n(c)] = [f_n(c)]$$

$$\begin{matrix} [g_n(c)] & = & [f_n(c)] \\ \| & & \| \\ g_*([c]) & & f_*([c]) \end{matrix}$$ ▨

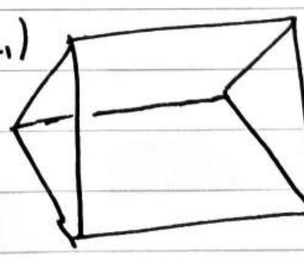
By the Lemma, if $f \simeq g : X \to Y$, enough to show that $f_\#, g_\#$ as maps $C_\bullet(X) \to C_\bullet(Y)$ are chain homotopic.
Let $H : [0,1] \times X \to Y$ be a homotopy from f to g.
If $\sigma : \Delta^n \to X$ is a singular n-simplex, then

$$[0,1] \times \Delta^n \xrightarrow[1\times\sigma]{} [0,1] \times X \xrightarrow[H]{} Y$$

is a homotopy from $f \circ \sigma$ to $g \circ \sigma$

Suppose that there are chains $P_n \in C_{n+1}([0,1] \times \Delta^n)$ such that

$$dP_n = i_1 - i_0 - \sum_{j=0}^{n} (-1)^j \left([0,1] \times \delta_j\right)_\# (P_{n-1})$$
$$\in C_n([0,1] \times \Delta^n)$$



"prism"

where $i_0 : \Delta^n \longrightarrow [0,1]\times\Delta^n$, $i_1 : \Delta^n \to [0,1]\times\Delta^n$
$x \mapsto (0,x)$ $\quad$ $x \mapsto (1,x)$

and

$$\delta_j : \Delta^{n-1} \longrightarrow \Delta^n$$
$$(t_0,\dots,t_{n-1}) \mapsto (t_0,\dots,t_{j-1},0,t_j,\dots,t_n)$$

Then define $H_n : C_n(X) \longrightarrow C_{n+1}(Y)$
$$(\sigma:\Delta^n\to X) \mapsto (H\circ([0,1]\times\sigma))_\#(P_n)$$

then calculate

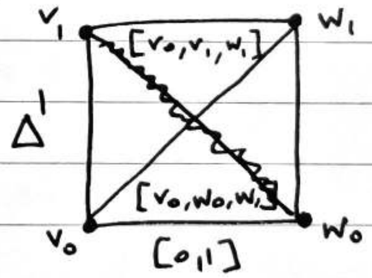
$$dH_n(\sigma) = (H\circ([0,1]\times\sigma))_\#(dP_n)$$
$$= (H\circ([0,1]\times\sigma))_\#\Big(i_1-i_0-\sum_{j=0}^{n}(-1)^j([0,1]\times\delta_j)_\#(P_{n-1})\Big)$$
$$= g_\#(\sigma) - f_\#(\sigma) - \sum_{j=0}^{n}(-1)^j H_\#\circ\big([0,1]\times\sigma\circ\delta_j\big)_\#(P_{n-1})$$
$$= g_\#(\sigma) - f_\#(\sigma) - H_{n-1}(d\sigma)$$

So H_n's are a chain homotopy from $g_\#$ to $f_\#$.

<u>It remains to invent the P_n's</u>

$P_n = \sum \pm (n+1)$-simplices in a triangulation of $[0,1]\times\Delta^n$

Let $v_0,\dots,v_n$ be the vertices of $\{0\}\times\Delta^n$
and $w_0,\dots,w_n$ those of $\{1\}\times\Delta^n$.
Regard $[0,1]\times\Delta^n \subset \mathbb{R}\times\mathbb{R}^{n+1}$ convex



For any sequence $\{x_0,\dots,x_{n+1}\}$ of v's and w's, let
$$[x_0,\dots,x_{n+1}] : \Delta^{n+1} \longrightarrow [0,1]\times\Delta^n$$
$$(t_0,\dots,t_{n+1}) \mapsto \sum t_i x_i \quad \text{using convexity}$$

Set $P_n = \sum_{i=0}^{n}(-1)^i[v_0,\dots,v_i,w_i,\dots,w_n] \in C_{n+1}([0,1]\times\Delta^n)$

torturous → Check(!) that $dP_n = i_1 - i_0 - \sum_{j=0}^{n}(-1)^j([0,1]\times\delta_j)_\#(P_{n-1})$

<u>Want:</u> P_n's s.t. $dP_n = i_1 - i_0 - \sum_{j=0}^{n} (-1)^j ([0,1]\times\delta_j)_\# P_{n-1}$
in $C_n([0,1]\times\Delta^n)$

<u>Propose:</u> $P_n = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n]$

$[0,1]\times\Delta^n$
v_i's are vx of $\{0\}\times\Delta^n$
w_i's are vx of $\{1\}\times\Delta^n$

Calculate

$$dP_n = \sum_{i=0}^{n} (-1)^i \Big[\sum_{j\le i} \overset{\leftarrow (-1)^j}{} [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_i, w_i, \dots, w_n] + \sum_{j\ge i} (-1)^{j+1} [v_0, \dots, v_i, w_i, \dots, w_{j-1}, w_{j+1}, \dots, w_n] \Big]$$

$= [w_0, \dots, w_n]$, $i=j=0$ (other $i=j$ cancel)
$- [v_0, \dots, v_n]$, $i=j=n$
$+ \sum_{j<i} (-1)^{i+j} [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_i, w_i, \dots, w_n]$
$+ \sum_{j>i} (-1)^{i+j+1} [v_0, \dots, v_i, w_i, \dots, w_{j-1}, w_{j+1}, \dots, w_n]$

Now

$$([0,1]\times\delta_j)_\# (P_{n-1}) = \sum_{i=0}^{n} (-1)^i \begin{cases} [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{i+1}, w_{i+1}, \dots, w_n] & j\le i \\ [v_0, \dots, v_i, w_i, \dots, w_{j-1}, w_{j+1}, \dots, w_n] & j>i \end{cases}$$

◪

If $f \simeq g: X \to Y$ then we know $f_\#, g_\#: C_\bullet(X) \to C_\bullet(Y)$ are chain homotopic.
Applying $\mathrm{Hom}(\cdot, \mathbb{Z})$ with H_n gives $H^n = (H_n)^*$ gives a chain homotopy from $f^\#$ to $g^\#$.
$\Rightarrow f^* = g^* : H^*(Y) \to H^*(X)$

<u>Mayer-Vietoris and excision</u>

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a collection of subspaces of X whose interiors cover X.
Let $C_\bullet^{\mathcal{U}}(X) = \langle \sigma: \Delta^n \to X \text{ s.t. } \sigma(\Delta^n) \text{ lies in some } U_\alpha \rangle$
$\cap$
$C_\bullet(X)$

If $\sigma(\Delta^n) \subset U_\alpha$ then $(\sigma\circ\delta_j)(\Delta^{n-1}) \subset \sigma(\Delta^n) \subset U_\alpha$

$\Rightarrow d(\sigma) \in C_{n-1}^{\mathcal{U}}(X)$

This is a sub-chain complex by ↗

Theorem [Small simplices theorem]

$$H_n^{\mathcal{U}}(X) = H_n(C_\bullet^{\mathcal{U}}(X), d) \xrightarrow{\sim} H_n(X) \quad \text{for any } \mathcal{U}$$

Proof of Mayer-Vietoris $X = A \cup B$, A, B open

So let $\mathcal{U} = \{A, B\}$.

Let $C_\bullet(A+B) := C_\bullet^{\mathcal{U}}(X)$.

Consider

$$0 \to C_n(A \cap B) \xrightarrow{i_A \oplus i_B} C_n(A) \oplus C_n(B) \xrightarrow[\substack{\text{surj,}\\ \text{kernel given}\\ \text{by chains in } A\cap B}]{j_A - j_B} C_n(A+B) \to 0$$

get a SES of chain complexes.

$\Rightarrow$ LES on homology

$$\cdots \to H_n(A\cap B) \xrightarrow{i_A \oplus i_B} H_n(A) \oplus H_n(B) \xrightarrow{j_A - j_B} H_n^{\mathcal{U}}(A \cup B) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

with $H_n^{\mathcal{U}}(A\cup B) \xrightarrow{\wr} H_n(X)$, a dashed arrow $H_n(A)\oplus H_n(B) \xrightarrow{j_A - j_B} H_n(X)$, and $H_n(X) \xrightarrow{\partial_{mv}} H_{n-1}(A\cap B)$.

Note $\partial_{mv}([x]) = [d_n a]$ when $[x] = [a+b]$ for $a \in C_n(A)$, $b \in C_n(B)$

The SST guarantees that any $[x]$ is of this form. ▨

Proof of excision: $X \supset A \supset Z$ s.t. $\overline{Z} \subset A^\circ$. Let $B = X \setminus Z$.

Then $B^\circ = X \setminus \overline{Z}$ so $X = A^\circ \cup B^\circ$.

Let $\mathcal{U} = \{A, B\}$. Consider

$$\begin{array}{ccccccccc}
0 & \to & C_\bullet(A) & \to & C_\bullet(A+B) & \to & \dfrac{C_\bullet(A+B)}{C_\bullet(A)} & \to & 0 \\
 & & \| & & \downarrow {\scriptstyle H_n\text{-iso by SST}} & & \downarrow \phi_\bullet & & \\
0 & \to & C_\bullet(A) & \to & C_\bullet(X) & \to & \dfrac{C_\bullet(X)}{C_\bullet(A)} = C_\bullet(X, A) & \to & 0
\end{array}$$

Taking LES's on homology, the 5-lemma implies that $\phi_\bullet$ induces an isomorphism on homology.

$$\frac{C_\bullet(A+B)}{C_\bullet(A)} \xleftarrow[\sim]{\text{onto}} \frac{C_\bullet(B)}{C_\bullet(A\cap B)} = C_\bullet(B, A\cap B)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad C_\bullet(X\setminus Z, A\setminus Z)$$

In total,

$$H_*(X\setminus Z, A\setminus Z) \xrightarrow{\sim} H_*(X, A)$$

the map induced by inclusion of pairs. ∎

<u>Addendum to excision</u>

X space, $x_0 \in X$ basepoint, <u>reduced homology</u> $\tilde{H}_n(X) = H_n(X, x_0)$

Say (X, A) is <u>good</u> if there is an open set $\bar{A} \subset U$ s.t. $A \hookrightarrow U$ is a homotopy equivalence.

Nay, a <u>deformation retraction</u>: $\exists\, r: U \to A$ s.t. $r \circ i = id_A$ and $i \circ r: U \to U$ is homotopic (via H) to id_U, via a homotopy fixed on A

<u>Theorem</u>: if (X, A) is good then the map

$$H_n(X, A) \longrightarrow H_n(X/A, A/A) = \tilde{H}_n(X/A)$$

is an isomorphism.

<u>Proof</u>: As $A \to U$ is a homotopy equivalence, by the 5-lemma the map

$$H_n(X, A) \longrightarrow H_n(X, U)$$

is an isomorphism. As H fixes A inside U, it gives a homotopy from $id_{U/A}$ to the constant map $U/A \to U/A$, $[u] \mapsto [a]$

(classic issue of homotopies on quotients $(A\times I)/\sim$ vs $(A/\sim)\times I$ →)

So $H_*(U/A) \xleftarrow{\sim} H_*(A/A = pt)$

and also $H_*(X/A, U/A) \xleftarrow{\sim} H_*(X/A, A/A) = \tilde{H}_*(X/A)$. (← 5-lemma)

So

$$\begin{array}{ccccc}
H_n(X, A) & \xrightarrow{\sim} & H_n(X, U) & \xleftarrow[\text{excision}]{\sim} & H_n(X\setminus A, U\setminus A) \\
\downarrow \;\therefore \cong & & \downarrow & & \cong \downarrow \text{ homeomorphism of pairs} \\
H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, U/A) & \xleftarrow[\text{excision}]{\sim} & H_n(X/A \setminus A/A,\; U/A \setminus A/A)
\end{array}$$

$X/A \setminus A/A = X\setminus A$ ∎

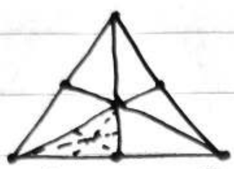
⌜U/A is a subspace of X/A b/c U open, saturated⌟

⌜$q: X \to X/A$ restricts to homeo on $X\setminus A$ if A closed in X⌟

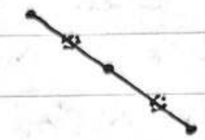
Idea of Small Simplices Theorem

Barycentric subdivision

Δ^2



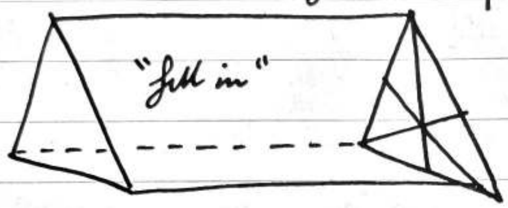
Δ^1



- Construct $\rho : C_\bullet(X) \to C_\bullet(X)$

$$(\sigma : \Delta^n \to X) \mapsto \sum_{\substack{\text{sxs} \\ \text{in bary}}} \pm (\Delta^n \to \Delta^n \xrightarrow{\sigma} X)$$

- Construct chain homotopy from ρ to Id, T say

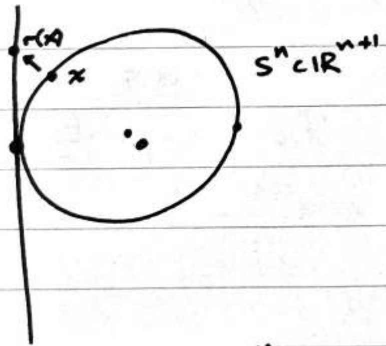


- For a given $[x] \in H_n(X)$, $[\rho^k(x)] \in H_n^{\mathcal{U}}(X)$, $k \gg 0$

So $H_n^{\mathcal{U}} \longrightarrow H_n$ is surj

Similar for inj

Generators $\quad S^n \subset \mathbb{R}^{n+1} \qquad D^n/S^{n-1} = \text{int}(D^n)^+$ (1 point compactifn)



projection gives $r : S^n \to D^n/S^{n-1}$

$\{-1\} \times \mathbb{R}^n \cong \text{int } D^n$

Construct preferred generators $u_n \in \tilde{H}_n(S^n)$ by

i) $u_0 = [+1] - [-1] \in \tilde{H}_0(S^0)$

ii) $\cancel{\tilde{H}_n(D^n)} \to H_n(D^n, S^{n-1})$ (with $\cancel{\tilde{H}_n(D^n)} = 0$)

$H_n(D^n, S^{n-1}) \xrightarrow{\cong \text{ exc}} \tilde{H}_n(D^n/S^{n-1}) = \tilde{H}_n(S^n)$

$H_n(D^n, S^{n-1}) \xrightarrow{\partial} \tilde{H}_{n-1}(S^{n-1}) \to \cancel{\tilde{H}_{n-1}(D^n)} = 0$

$\tilde{H}_n(D^n/S^{n-1}) \longrightarrow \tilde{H}_{n-1}(S^{n-1})$

Choose $u_n \in \tilde{H}_n(S^n)$ mapping to $u_{n-1} \in \tilde{H}_{n-1}(S^{n-1})$ via $\partial \circ (\text{excision})^{-1}$

Ex $n=1$ 

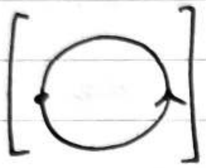
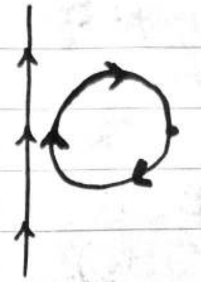
D^1

$\tau : \Delta^1 \to D^1$ is a chain, $d\tau = [+1] - [-1]$

$(t_0, t_1) \mapsto -2t_0+1$ so $[\tau] \in C_1(D^1, S^0)$ is a cycle,

so it represents $[[\tau]] \in H_1(D^1, S^0)$

s.t. $\partial[[\tau]] = [d\tau] = [+1] - [-1] = u_0$

 $\overset{?}{=} u_1$ NOT  is a 1-sx going clockwise around S^1

Cell complexes and cellular homology

A cell complex X is a space constructed by:

- X_0 is a discrete space
- X_n is constructed from X_{n-1} by giving
  $\{\varphi_\alpha : S^{n-1}_\alpha \to X^{n-1}\}_{\alpha \in I_n}$
  and letting
  $$X^n = \Big(X^{n-1} \sqcup \coprod_{\alpha \in I_n} D^n_\alpha\Big)/\sim$$
  where $\sim$ is generated by
  $x \sim \varphi_\alpha(x)$ for $x \in S^{n-1}_\alpha \subset D^n_\alpha$
  Call the image of $D^n_\alpha \setminus S^{n-1}_\alpha$ the open cell e_α
- $X = \bigcup_{n \geq 0} X^n$ with the topology
  ($U \subset X$ open $\iff$ $U \cap X^n \subset X^n$ open $\forall n$)

Call X^n the n skeleton of X

If $X = X^n$ then X is finite dimensional

If also X^0 and I_n are finite, X is called finite

For each open cell e_α, call

$$\Phi_\alpha : D^n_\alpha \to \Big(X^{n-1} \sqcup \coprod_{\alpha'} D^n_{\alpha'}\Big) \twoheadrightarrow X^n \hookrightarrow X$$

the characteristic map of e_α

A <u>subcomplex</u> A of X is a subspace which is a union of open cells e_α of X, s.t. $\overline{e_\alpha}$ is also contained in A.
i.e. $A^0 \subset X^0$ and each $\varphi_\alpha : S^{n-1}_\alpha \to X^{n-1}$
lands in A^{n-1} if $e_\alpha \subset A$.

<u>Lemma</u> If $A \subset X$ is a subcomplex then (X, A) is good.
<u>Pf</u> Exercise. Use (D^n, S^{n-1}) good.

<u>Theorem</u>: let X be a cell complex.

1) $H_*(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } i \neq n \\ \mathbb{Z}[I_n] & i = n, \end{cases}$ free abelian group on set I_n of n-cells

2) $H_i(X^n) = 0$ for $i > n$

3) $H_i(X^n) \longrightarrow H_i(X)$ is an iso for $i < n$

<u>Proof</u> i) As $X^{n-1} \subset X^n$ is a subcomplex, (X^n, X^{n-1}) is good

So $H_i(X^n, X^{n-1}) \xrightarrow{\cong} \tilde{H}_i(X^n/X^{n-1})$

Now $X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n_\alpha = \bigvee_{\alpha \in I_n} (D^n_\alpha / S^{n-1}_\alpha)$

i.e. the quotient of $\coprod_{\alpha \in I_n} S^n_\alpha =: Y$

by the subcx $Z = \{ x_\alpha \mid \alpha \in I_n \}$ for $x_\alpha \in S^n_\alpha$ the basepoint

$$\tilde{H}_i(Y/Z) \cong H_i(Y, Z)$$

$$\cdots H_i(Y, Z) \to H_{i-1}(Z) \to H_{i-1}(Y) \to \cdots$$

$H_{i-1}(Z) = \begin{cases} \mathbb{Z}[I_n] & \text{if } i = 1 \\ 0 & \text{o/w} \end{cases}$ $\quad H_{i-1}(Y) = \begin{cases} \mathbb{Z}[I_n] & \text{if } i = 1, n+1 \\ 0 & \text{o/w} \end{cases}$

at $i = 1$, cancel
end up with $\tilde{H}_i(Y/Z) \cong \begin{cases} \mathbb{Z}[I_n], & i = n \\ 0, & \text{else} \end{cases}$

generator

Lemma X a cell complex

i) $H_i(X^n, X^{n-1}) = \begin{cases} 0 & i \neq n \\ \mathbb{Z}^{n\text{-cells}} & i = n \end{cases}$

ii) $H_i(X^n) = 0, \quad i > n$

iii) $H_i(X^n) \xrightarrow{\sim} H_i(X), \quad i < n$

Proof i) ✓ each cell has a characteristic map Φ_α s.t.

$$\Phi_\alpha : (D^n_\alpha, S^{n-1}_\alpha) \longrightarrow (X^n, X^{n-1})$$

canonical generator in $H_n(D^n, S^{n-1})$ gets sent to $e_\alpha \in H_n(X^n, X^{n-1})$

ii)

$$\cdots \to \underset{\overset{\shortparallel}{0} \text{ by ind}^n}{H_i(X^{n-1})} \longrightarrow H_i(X^n) \longrightarrow \underset{\overset{\shortparallel}{0} \text{ b/c } i \neq n}{H_i(X^n, X^{n-1})} \longrightarrow \cdots$$

Suppose $i > n$.

By induction on n, $H_i(X^{n-1}) = 0$.

$\Rightarrow H_i(X^n) = 0$ too

iii) Under the assumption that X is finite-dim^L, say $X = X^m$ for some m:

Go by downwards induction on n. $n = m$ true

For $i < n < m$, have

$$\underset{\overset{\shortparallel}{0} \text{ as } i+1 \neq n+1}{H_{i+1}(X^{n+1}, X^n)} \xrightarrow{\partial} H_i(X^n) \underset{\therefore \sim}{\longrightarrow} H_i(X^{n+1}) \longrightarrow \underset{\overset{\shortparallel}{0} \text{ as } i \neq n+1}{H_i(X^{n+1}, X^n)}$$

$$H_i(X^n) \overset{\therefore \sim}{\searrow} H_i(X), \qquad H_i(X^{n+1}) \downarrow \wr \text{ by ind}^n \; H_i(X)$$

To generalise to X infinite-dimensional, use that any simplex in X lies in some X^n (see Ex Sheet). ☑

Def^n Let X be a cell complex.

$$C_n^{cell}(X) := H_n(X^n, X^{n-1}) \cong \text{free abelian group on the set of } n\text{-cells of } X$$

and

$$d_n^{cell} : C_n^{cell}(X) = H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{q} H_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}^{cell}(X)$$

Then

$$\underbrace{H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{q}}_{d_n^{cell}} \overbrace{H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\partial}}^{\text{zero}} \underbrace{H_{n-2}(X^{n-2}) \xrightarrow{q} H_{n-2}(X^{n-2}, X^{n-3})}_{d_{n-1}^{cell}}$$

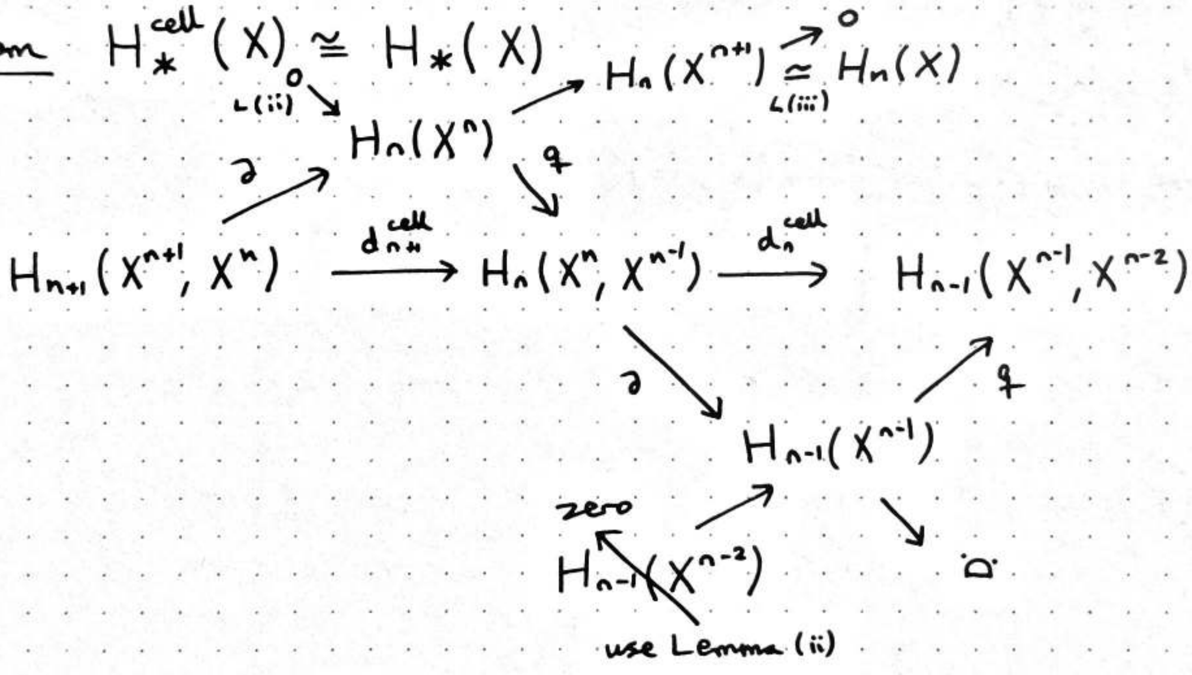
So $(C_\bullet^{cell}(X), d_\bullet^{cell})$ is a chain complex.

— The "cellular chain complex".

Call its homology $H_*^{cell}(X)$.

Theorem $H_*^{cell}(X) \cong H_*(X)$

Proof



Now, $H_n(X) \cong H_n(X^{n+1})$

$\cong H_n(X^n) / \mathrm{Im}(\partial : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n))$

$\cong q(H_n(X^n)) / \mathrm{Im}(d_{n+1}^{cell})$

$\cong \ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})) / \mathrm{Im}(d_{n+1}^{cell})$

$= \ker(d_n^{cell}) / \mathrm{im}(d_{n+1}^{cell})$

$= H_n^{cell}(X)$. ∎

Many immediate applications

i) If X has finitely many n-cells, then $H_n(X)$ is a finitely generated abelian group. (generated by $\leq$ # n-cells elts)

ii) If X has a cell structure with no n cells, $H_n(X) = 0$

iii) If $H_n(X) \neq 0$ then any cell cx $\simeq$ to X must have n-cells

iv) If X only has even-dimensional cells, then

$$H_*(X) \cong C_*^{cell}(X)$$

Can define cellular cohomology via

$$C^n_{cell}(X) = H^n(X^n, X^{n-1})$$

$$d_n^{cell} : H^n(X^n, X^{n-1}) \xrightarrow[q^*]{} H^n(X^n) \xrightarrow[\partial]{} H^{n+1}(X^{n+1}, X^n)$$

This gives $(C^\bullet_{cell}, d_{cell})$ with cohomology $H^*_{cell} \cong H^*$.

Also, $(C^\bullet_{cell}, d_{cell}) \cong \mathrm{Hom}((C_\bullet^{cell}, d), \mathbb{Z})$.

To calculate, need to understand

$$d_n^{cell} : C_n^{cell}(X) = \mathbb{Z}\{e_\alpha \mid n \text{ cells}\}$$
$$\downarrow$$
$$C_{n-1}^{cell}(X) = \mathbb{Z}\{f_\beta \mid (n-1) \text{ cells}\}$$

Let
$$d_n^{cell}(e_\alpha) = \sum_\beta d_{\alpha\beta} f_\beta, \quad d_{\alpha\beta} \in \mathbb{Z}.$$

<u>Lemma</u> $d_{\alpha\beta}$ is the degree of the map

$$S^{n-1}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} = \bigvee_{\beta'} S^{n-1}_{\beta'} \xrightarrow{\text{collapse}} S^{n-1}_\beta$$

also determines d^n_{cell}, as the transpose of the matrix $(d_{\alpha\beta})$

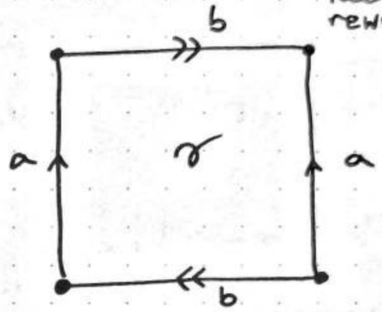
Proof

$$\begin{tikzcd}
 & 1 \in \mathbb{Z} \arrow[d, phantom, "\shortparallel"] \arrow[rr, bend left, mapsto] & & \\
1 \in \mathbb{Z}=H_n(D^n_\alpha,S^{n-1}_\alpha) \arrow[ur, bend left, mapsto] \arrow[r,"\partial"] \arrow[d,"(\Phi_\alpha)_*"] & H_{n-1}(S^{n-1}_\alpha) \arrow[r, dashed] \arrow[d,"(\varphi_\alpha)_*"] & H_{n-1}(S^{n-1}_\beta) & d_{\alpha\beta} \\
e_\alpha \in H_n(X^n,X^{n-1}) \arrow[r,"\partial"] \arrow[dr,"d_n^{cell}"'] \arrow[rrr, bend right=60, mapsto] & H_{n-1}(X^{n-1}) \arrow[r] \arrow[d,"q"] & H_{n-1}(X^{n-1}/X^{n-2})=\mathbb{Z}\{f_\gamma\} \arrow[u,"\text{collapse}"] \arrow[d,"?"] & \sum d_{\alpha\gamma} f_\gamma \ ?\ n-1>0 \arrow[u, mapsto] \\
 & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r,"\sim","\text{exc}"'] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[ur, mapsto] &
\end{tikzcd}$$

$\boxtimes$

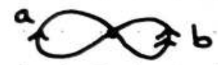
← $n=0$ needs rework

Example K = Klein bottle



As a cell complex,

$K^0 = \{v\}$

$K^1 =$ 

use $\varphi_\alpha, \varphi_\beta : S^0 \to \{v\} = K^0$

$K = K^2$: attach a 2-cell γ along

$\varphi_\gamma : S^1_\gamma \longrightarrow$, the loop $a b a^{-1} b$.

$$C^{cell}_\bullet(K) = \quad \overset{2}{\mathbb{Z}\{\gamma\}} \longrightarrow \overset{1}{\mathbb{Z}\{a,b\}} \longrightarrow \overset{0}{\mathbb{Z}\{v\}}$$

$$a, b \longmapsto v - v = 0$$

$$\gamma \longmapsto 2b$$

$$H^{cell}_\bullet(K) = \quad 0\,\cancel{\mathbb{Z}} \qquad \mathbb{Z}\oplus\mathbb{Z}/2 \qquad \mathbb{Z}$$

<u>Example</u> $\mathbb{RP}^n = S^n/x\sim -x$

$\cong D^n_+ \Big/ {}^{x \in \partial D^n_+}_{\sim -x \in \partial D^n_+}$

$= \mathbb{RP}^{n-1}$ with an n-cell attached along $S^{n-1} \xrightarrow[2:1]{} \mathbb{RP}^{n-1}$
quotient map

So by induction $\mathbb{RP}^n$ is a cell complex with one cell of each dimension $0, 1, \dots, n$.

Let e_i = the i-cell

So $C^{cell}_\bullet(\mathbb{RP}^n) = \overset{n}{\mathbb{Z}e_n} \xrightarrow[\substack{2 \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd}}]{d^{cell}_n} \overset{n-1}{\mathbb{Z}e_{n-1}} \cdots \overset{3}{\mathbb{Z}e_3} \xrightarrow[0]{} \overset{2}{\mathbb{Z}e_2} \xrightarrow[2]{d^{cell}_2} \overset{1}{\mathbb{Z}e_1} \xrightarrow[0]{d^{cell}_1} \overset{0}{\mathbb{Z}e_0}$

To compute $d^{cell}_n(e_n)$,

$$\delta: S^{n-1} \xrightarrow[quo]{} \mathbb{RP}^{n-1} \longrightarrow \frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^{n-2}} = S^{n-1}$$

Need degree.

The open hemispheres $\mathring{D}^{n-1}_+$, $\mathring{D}^{n-1}_-$ are sent homeomorphically to $S^{n-1} \setminus \{\infty\}$.

Observe that the antipodal map on S^{n-1} has degree $(-1)^n$, swaps $\mathring{D}^{n-1}_\pm$, and commutes with δ

Conclusion : local degrees differ by $(-1)^n$

$\Rightarrow d^{cell}_n(e_n) = \pm(1+(-1)^n)\, e_{n-1}$ ⌜thinking cap ⇒ +⌟ ⌜?⌟

So $H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/2, & i \text{ odd}, 0<i<n \\ \mathbb{Z}, & i=n \text{ odd} \\ 0, & \text{else} \end{cases}$

$H^i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/2, & i \text{ even}, 0<i\le n \\ \mathbb{Z}, & i \text{ odd} = n \\ 0, & \text{else} \end{cases}$

<u>Example</u> Let $\varphi : S^{n-1} \to X$ be a map and $Y = X \cup_\varphi D^n$

Now (Y, X) is good, so $H_*(Y, X) \xrightarrow{\sim} \tilde{H}_*(Y/X) = \tilde{H}_*(S^n)$
$$= \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

So get

$$0 \to H_n(X) \xrightarrow{i_*} H_n(Y) \to \mathbb{Z} \to H_{n-1}(X) \xrightarrow{i_*} H_{n-1}(Y) \to 0$$
$$1 \longmapsto \varphi_* u_{n-1}$$

[see that $1 \mapsto \varphi_* u_{n-1}$ via map of pairs $(D^n, S^{n-1}) \to (Y, X)$]

and $H_i(X) \xrightarrow{\sim} H_i(Y)$ for $i \neq n, n-1$.

<u>Cases</u> I) If $\varphi_* u_{n-1} \in H_{n-1}(X)$ has ∞ order,
then $H_n(X) \xrightarrow{\sim} H_n(Y)$, $H_{n-1}(Y) = H_{n-1}(X)/\langle \varphi_* u_{n-1} \rangle$

II) If $\varphi_* u_{n-1}$ has order k, then get
$$0 \to H_n(X) \to H_n(Y) \to k\mathbb{Z} \to 0 \qquad \therefore H_n(Y) \cong H_n(X) \oplus \mathbb{Z}$$
and $H_{n-1}(Y) = H_{n-1}(X)/\langle \varphi_* u_{n-1} \rangle$.

<u>Coefficients</u> If A is an abelian group, let $C_\bullet(X; A) = C_\bullet(X) \otimes_{\mathbb{Z}} A$ [A-linear combis of σ's]

with differential $d \otimes id_A$ and $H_*(X; A) := H_*(C_\bullet(X; A), d \otimes id_A)$

Similarly $H_*^{cell}(X; A) = H_*(C_\bullet^{cell}(X) \otimes_{\mathbb{Z}} A)$ and so on.

[All the tools so far hold for homology $H_*(-; A)$]

Similarly $C^\bullet(X; A) = \mathrm{Hom}(C_\bullet(X), A)$ $\neq \mathrm{Hom}(C_\bullet(X; A), \mathbb{Z})$

+ cellular + ...

Mainly use $A = \mathbb{Z}/n$ or $\mathbb{Q}$

<u>Example</u> In $C_\bullet^{cell}(\mathbb{RP}^n)$ all differentials were 0 or 2

So in $C_\bullet^{cell}(\mathbb{RP}^n; \mathbb{Z}/2)$ all d's are 0.

$$\Rightarrow H_*(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$$

$$\cong$$

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2)$$

On the other hand, $\mathbb{Q} \xrightarrow{2} \mathbb{Q}$ is an isomorphism, so

$$H_*(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i=0 \\ \mathbb{Q} & i=n \text{ odd} \\ 0 & \text{else} \end{cases}$$

<u>Euler characteristic</u> If X is a finite cell complex, let

$$\chi(X) = \sum_i (-1)^i \#\{i\text{-dim}^l \text{ cells of } X\}$$

be the <u>Euler characteristic</u> of X. Let

$$\chi_{\mathbb{Z}}(X) = \sum_i (-1)^i \operatorname{rank} H_i(X)$$

and for $\mathbb{F}$ a field,

$$\chi_{\mathbb{F}}(X) = \sum_i (-1)^i \dim_{\mathbb{F}} H_i(X; \mathbb{F})$$

<u>Theorem</u> $\chi = \chi_{\mathbb{Z}} = \chi_{\mathbb{F}}$

<u>Pf</u> Let $C_i = C_i^{cell}(X)$.

$$Z_i = \operatorname{Ker}(d_i : C_i \to C_{i-1})$$

$$B_i = \operatorname{im}(d_{i+1} : C_{i+1} \to C_i)$$

Have exact sequences

$$0 \to B_n \to Z_n \to H_n(X) \to 0$$

$$0 \to Z_n \to C_n \xrightarrow{d_n} B_{n-1} \to 0$$

If $0 \to A \to B \to C \to 0$ is exact,
then $\text{rank}(B) = \text{rank}\, A + \text{rank}\, C$ (Algebra)

$$\text{rank}\, H_i(X) = \text{rank}(Z_i) - \text{rank}(B_i), \text{ so}$$

$$\chi_{\mathbb{Z}}(X) = \sum_i (-1)^i (\text{rk}\, Z_i - \text{rk}\, B_i)$$

Also $\text{rk}\, B_i = \text{rk}\, C_{i+1} - \text{rk}\, Z_{i+1}$

$$= \sum_i (-1)^i (\text{rk}\, Z_i - \text{rk}\, C_{i+1} + \text{rk}\, Z_{i+1})$$

$$= \sum_i (-1)^{i+1} \underbrace{\text{rk}\, C_{i+1}}_{= \#(i+1)\text{-cells}} = \chi(X)$$

For $\mathbb{F}$ a field it is the same, using

$$\text{rank}\, C_i = \dim_{\mathbb{F}} (C_i \otimes \mathbb{F})$$

□

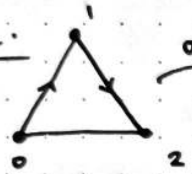
Cup Product

Def^n Let R be a ring, $\phi \in C^k(X;R)$, $\psi \in C^l(X;R)$. Then $\phi \cup \psi \in C^{k+l}(X;\mathbb{R})$ is defined by

$$(\phi \cup \psi)(\sigma : \Delta^{k+l} \longrightarrow X)$$
$$= \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+l}]}) \in R$$

where $v_0,\dots,v_{k+l}$ are the vertices of Δ^{k+l}, and

$$\sigma|_{[x_0,\dots,x_i]} : \Delta^i \longrightarrow \Delta^{k+l} \longrightarrow X$$
$$(t_0,\dots,t_i) \mapsto \sum t_j x_j \mapsto \sigma\left(\sum t_j x_j\right)$$

e.g. 

$\xrightarrow{\sigma} X$ $\quad \phi, \psi \in C^1(X;R)$

$(\phi \cup \psi)(\sigma)$

Lemma $d(\phi \cup \psi) = (d\phi) \cup \psi + (-1)^k \phi \cup (d\psi) \quad (k = \deg \phi)$

Pf.
- $((d\phi) \cup \psi)(\sigma : \Delta^{k+l+1} \longrightarrow X)$

$$= (d\phi)(\sigma|_{[v_0,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$
$$= \phi\left(\sum_{i=0}^{k+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]}\right) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

- $(-1)^k(\phi \cup (d\psi))(\sigma)$

$$= (-1)^k \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot (d\psi)(\sigma|_{[v_k,\dots,v_{k+l+1}]})$$
$$= (-1)^k \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi\left(\sum_{j=0}^{l+1} (-1)^j \sigma|_{[v_k,\dots,\widehat{v_{k+j}},\dots,v_{k+l+1}]}\right)$$

Last term of first calculation and first term of second calculation cancel. So the sum is then

$$(d(\phi \cup \psi))(\sigma) = (\phi \cup \psi)\left(\sum_{i=0}^{k+l+1} (-1)^r \sigma|_{[v_0,\dots,\hat{v}_r,\dots,v_{k+l+1}]}\right)$$

By observation this gives the sum of the previous two calculations. □

Corollary There is a well-defined map

$$- \cup - : H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R)$$
$$([\phi],[\psi]) \longmapsto [\phi \cup \psi]$$

Proof: As ϕ, ψ represent cohomology classes, $d\phi = 0$, $d\psi = 0$

$\Rightarrow d(\phi \cup \psi) = d\phi \cup \psi \pm \phi \cup d\psi = 0$ (both $d\phi$ and $d\psi$ are zero)

So $\phi \cup \psi$ defines a cohomology class.

Suppose $[\phi] = [\phi']$ so $\phi = \phi' + d\tau$.

Then
$$\begin{aligned} \phi \cup \psi &= (\phi' + d\tau) \cup \psi \\ &= \phi' \cup \psi + (d\tau) \cup \psi \\ &= \phi' \cup \psi + d(\tau \cup \psi) \end{aligned}$$

$d(\tau \cup \psi) = (d\tau) \cup \psi \pm \tau \cup (d\psi)$, where $d\psi$ is zero.

So $[\phi \cup \psi] = [\phi' \cup \psi]$. ∎

Notes (i) $1 \in C^0(X;R)$ given by $1(\sigma : \Delta^0 \to X) = 1_R \in R$

$d1 = 0$, so get $[1] \in H^0(X;R)$

From the formula, $1 \cup \psi = \psi \cup 1 = \psi$.

So $[1]$ is a unit for $\cup$ on $H^*(X;R)$.

(ii) $- \cup -$ is associative on cochains, so also on cohomology

(iii) If $f : X \to Y$ then $f^\#(\phi \cup \psi) = f^\#\phi \cup f^\#\psi$

So $f^*([\phi \cup \psi]) = f^*[\phi] \cup f^*[\psi]$.

i.e. $f^* : H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism

(iv) For spaces X, Y, $\quad X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$

the <u>cross product</u> is

$$- \times - : H^k(X;R) \otimes_R H^l(Y;R) \longrightarrow H^{k+l}(X \times Y)$$
$$\alpha \otimes \beta \longmapsto (\pi_X^* \alpha) \cup (\pi_Y^* \beta)$$

(v) There is a <u>relative</u> cup product

$$H^k(X,A;R) \otimes H^l(X;R) \longrightarrow H^{k+l}(X,A;R)$$

given by the same formula.

If $\phi \in C^k(X,A) = \mathrm{Hom}\left(\frac{C_k(X)}{C_k(A)}, R\right)$

$$= \{\phi : C_k(X) \to R \mid \phi|_{C_k(A)} = 0\}$$

Then $(\phi \cup \psi)(\sigma : \Delta^{k+l} \longrightarrow A)$

$$= \phi(\sigma|_{[\dots]}) \cdot \psi(\sigma|_{[\dots]})$$

zero

So $\phi \cup \psi$ also vanishes on $C^{k+l}(A)$ so is a relative cochain.

<u>Proposition</u> If R is commutative, then

$$[\phi] \cup [\psi] = (-1)^{kl} [\psi] \cup [\phi] \in H^{k+l}(X;R)$$

<u>Proof</u> Let $\varepsilon_n = (-1)^{\frac{n\cdot(n+1)}{2}}$, and $\rho_n : C_n(X) \to C_n(X)$

$$\sigma \longmapsto \varepsilon_n \cdot \tilde{\sigma}$$

where $\tilde{\sigma}(t_0, \dots, t_n) = \sigma(t_n, \dots, t_0)$.

<u>Claim</u>: $\rho_\bullet : C_\bullet(X) \to C_\bullet(X)$ is a chain map and is chain homotopic to the identity

Suppose for now that the claim holds.

If $\phi : C_n(X) \to R$ is a cochain, let $\rho^*\phi = \phi \circ \rho_n$.

$$((\rho^*\phi) \cup (\rho^*\psi))(\sigma : \Delta^{k+l} \to X) = \phi(\varepsilon_k \cdot \sigma|_{[v_k, \dots, v_0]})$$
$$\cdot\psi(\varepsilon_l \cdot \sigma|_{[v_{k+l}, \dots, v_k]})$$

$$(\rho^*(\psi \cup \phi))(\sigma) = \varepsilon_{k+l} \psi(\sigma|_{[v_{k+l}, \dots, v_k]}) \cdot \phi(\sigma|_{[v_k, \dots, v_0]})$$

and $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$

So $\phi \cup \psi \sim \rho^*\phi \cup \rho^*\psi = (-1)^{kl} \rho^*(\psi \cup \phi) \sim (-1)^{kl} \psi \sim \phi$

↑ differ by a coboundary

$\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$ $\rho : C_n(X) \to C_n(X)$

$\sigma \mapsto \varepsilon_n \tilde{\sigma}$, $\tilde{\sigma}(t_0, \dots, t_n) = \sigma(t_n, \dots, t_0)$

- <u>Claim</u> $\rho_\bullet : C_\bullet(X) \to C_\bullet(X)$ is a chain map, chain homotopic to the identity

<u>Pf of claim</u>: to see it is a chain map,

$$(d\rho)(\sigma) = \varepsilon_n \sum_{i=0}^{n} (-1)^i \sigma|_{[v_n, \dots, \hat{v}_{n-i}, \dots, v_0]}$$

$$\rho(d\sigma) = \rho\left(\sum_{j=0}^{n} (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_n]}\right)$$

$$= \varepsilon_{n-1}\left(\sum_{j=0}^{n} (-1)^j \sigma|_{[v_n, \dots, \hat{v}_j, \dots, v_0]}\right)$$

$$= \varepsilon_{n-1}\left(\sum_{j=0}^{n} (-1)^{n-j} \sigma|_{[v_n, \dots, \hat{v}_{n-j}, \dots, v_0]}\right) \qquad \varepsilon_n = \varepsilon_{n-1}(-1)^n$$

- are equal.

For the homotopy, let $P_n = \sum_{i=0}^{n} (-1)^i \varepsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, w_i]$

$\in C_{n+1}([0,1] \times \Delta^n)$

similar to proof of homotopy invariance.

Letting $\pi : [0,1] \times \Delta^n \to \Delta^n$ be projection, let

$$F_n : C_n(X) \longrightarrow C_{n+1}(X)$$

$$(\sigma : \Delta^n \to X) \longmapsto (\sigma \circ \pi)_\#(P_n)$$

- So $dF_n(\sigma) = (\sigma \circ \pi)_\#(dP_n)$

$$= (\sigma \circ \pi)_\#\Big(\sum_{i=0}^{n} \sum_{j \le i} (-1)^j (-1)^i \varepsilon_{n-i} [v_0, \dots, \hat{v}_j, \dots, v_i, w_n, \dots, w_i]$$

$$+ \sum_{j \ge i} (-1)^{n+i+1-j} (-1)^i \varepsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, \hat{w}_j, \dots, w_i]$$

The terms with $j = i$ give

$$(\sigma \circ \pi)_\#\Big(\sum_i \varepsilon_{n-i} [v_0, \dots, v_{i-1}, w_n, \dots, w_i] + \sum_i (-1)^{n+i+1} \varepsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, w_{i+1}]$$

$$\underset{\text{cancellation}}{=} (\sigma \circ \pi)_\#(\varepsilon_n [w_n, \dots, w_0] - [v_0, \dots, v_n])$$

- $= \rho(\sigma) - \sigma$

I was thinking about why so many in the radical left partake in speedrunning...

The terms $j \neq i$ are precisely $-F_{n-1}(d\sigma)$ by observation. ← check is pain

$\Rightarrow \quad dF_n + F_{n-1}d = \rho - \mathrm{id}$ as required.

<u>Theorem (Künneth)</u>: Let R be a commutative ring, and Y be a space s.t. each $H^i(Y;R)$ is a free R-module. Then for any finite cell complex X, the cross-product map

$$-\times- : \bigoplus_{i+j=n} H^i(Y;R) \otimes_R H^j(X;R) \to H^n(Y\times X;R)$$
$$y \otimes x \mapsto \pi_Y^*(y) \smile \pi_X^*(x)$$

is an isomorphism.

<u>Notes</u> (1) Convenient to write

$H^*(Y;R) \otimes_R H^*(X;R)$ for the graded R-module which in degree n is the LHS in the statement of the theorem.

Then it says $H^*(Y;R) \otimes_R H^*(X,R) \xrightarrow[\sim]{-\times-} H^*(Y\times X,R)$ is a ring isomorphism. [↖ skew-commutative, associative, graded, R-algebra, $(y\otimes x)\cdot(y'\otimes x') = (-1)^{|x||y'|}(y\smile y')\otimes(x\smile x')$]

(2) If R is a field (usually $\mathbb{Q}$ or $\mathbb{Z}/p$) then every R-module is free.

<u>Proof</u>: Will proceed by (double) induction on ($\dim X$, # cells of X)

$F^n(X) = \bigoplus_{i+j=n} H^i(Y;R) \otimes_R H^j(X;R)$ «functor in X»

$G^n(X) = H^n(Y\times X;R)$ "

We have $-\times- : F^n(-) \to G^n(-)$.

Clear how to define $F^n(X,A)$, $G^n(X,A) = H^n(Y\times X, Y\times A;R)$ relative versions.

and extends to $-\times- : F^n(-,-) \longrightarrow G^n(-,-)$.

Consider $X = X' \cup_f D^d$:

$$\begin{array}{ccccccccccc}
\cdots \to & F^n(X) & \xrightarrow{i^*} & F^n(X') & \xrightarrow{\partial} & F^{n+1}(X,X') & \xrightarrow{q^*} & F^{n+1}(X) & \xrightarrow{i^*} & F^{n+1}(X') & \to \cdots \\
 & \downarrow -\times- & \circlearrowleft & \downarrow -\times- & \circlearrowleft \text{think} & \downarrow -\times- & \circlearrowleft & \downarrow -\times- & \circlearrowleft & \downarrow -\times- & \\
\cdots \to & G^n(X) & \xrightarrow[i^*]{} & G^n(X') & \xrightarrow[\partial]{} & G^{n+1}(X,X') & \xrightarrow[q^*]{} & G^{n+1}(X) & \xrightarrow[i^*]{} & G^{n+1}(X') & \to \cdots
\end{array}$$

Exactness of the top row uses that $H^*(Y;R)$ are free: this sequence is many copies of the LES for $H^i(X,X';R)$

⌈sign problem with top row, put $(-1)^{|y|}$ before ⊕ exacts⌋

The maps $F^*(X') \to G^*(X')$ are iso's by induction.

So if $-\times-: F^*(X,X') \to G^*(X,X')$ are iso's, we'll be done by the 5-lemma.

By excision,

$$\begin{array}{ccc} F^*(X,X') & \xrightarrow{\times - \times} & G^*(X,X') \\ \cong \downarrow \text{res} & & \cong \downarrow \text{res} \\ F^*(D^d,\partial D^d) & \xrightarrow{-\times-} & G^*(D^d,\partial D^d) \end{array}$$

so ETS lower map is an iso.

$$\begin{array}{ccccccccc} F^{n-1}(D^d) & \to & F^{n-1}(\partial D^d) & \to & F^n(D^d,\partial D^d) & \to & F^n(D^d) & \to & F^n(\partial D^d) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ G^{n-1}(D^d) & \to & G^{n-1}(\partial D^d) & \to & G^n(D^d,\partial D^d) & \to & G^n(D^d) & \to & G^n(\partial D^d) \end{array}$$

The maps $F^\cdot(D^d) \to G^\cdot(D^d)$ are iso's by direct calculation.

So are $F^\cdot(\partial D^d) \to G^\cdot(\partial D^d)$ by induction on dimension.

$\Rightarrow$ middle map is iso by 5-lemma, as required. ▨

<u>Ex</u>: If $\mathbb{F}$ is a field, $H^*(Y\times X;\mathbb{F}) \cong H^*(Y;\mathbb{F}) \otimes_{\mathbb{F}} H^*(X;\mathbb{F})$.

<u>Ex</u>: $H^*(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z}1 & i=0 \\ \mathbb{Z}x & i=1 \\ 0 & \text{else} \end{cases}$

$1 \cup x = x = x \cup 1$

$x \cup x = 0$

$= \mathbb{Z}[x]/(x^2)$

$\Rightarrow H^*(T^n;\mathbb{Z})$

$= H^*(\underbrace{S^1\times\cdots\times S^1}_{n};\mathbb{Z}) \cong H^*(S^1;\mathbb{Z})^{\otimes n}$

$\cong \mathbb{Z}\langle x_1,\ldots,x_n\rangle/(x_i^2,\ x_{ij}+x_{ji})$

Ex $H^*(T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}} ; \mathbb{Z})$

$$\cong \mathbb{Z}\langle x_1, \dots, x_n \rangle / (x_i^2, x_i x_j + x_j x_i)$$

⌜ $\pi_i : T^n \to S^1$ proj to i^{th} factor, $x_i = \pi_i^* x$, $x \in H^1(S^1)$ generator ⌟

Follows from Künneth.

Corollary Let $n \geq 2$, $f : S^n \to T^n$.
Then $f^* : H^n(T^n) \to H^n(S^n)$ is zero.

⌜ $x_i^2 = 0$, $x_i x_j = -x_j x_i$ ⌟

Proof $H^n(T^n) = \mathbb{Z}\{x_1 \cup \cdots \cup x_n\}$
So $f^*(x_1 \cup \cdots \cup x_n) = f^*(x_1) \cup \cdots \cup f^*(x_n)$
But $f^*(x_1) \in H^1(S^n) = 0$, so the latter product is 0. □

Theorem (Universal coefficients for homology)
Let R be a PID (e.g. $\mathbb{Z}$, $\mathbb{Z}/p$) and M be an R-module.
Then there is a natural map

$$H_*(X;R) \otimes_R M \longrightarrow H_*(X;M)$$

⌜ obvious R-hom, well-def on H_* ⌟

which if $H_*(X;R)$ is a free R-module for all $*$, is an isomorphism.

Proof Let $C_n = C_n(X;R)$
$Z_n \subseteq C_n$ sub-R-module of cycles
$B_n \subseteq C_n$ sub-R-module of boundaries

There is a SES

$$0 \to Z_n \xrightarrow[\text{inc}]{} C_n \xrightarrow[\text{quot}]{d} B_{n-1} \to 0$$

Consider this as a SES of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

where the outer terms have differential zero.
As R is a PID, $B_{n-1} \subseteq C_{n-1}$ is a free R-module
(C_{n-1} is free R-module on singular simplices
PID $\Rightarrow$ submodule of free is free)

So we can choose a splitting $s : B_{n-1} \to C_n$

- i.e. $q \circ s = id_{B_{n-1}}$
- i.e. $C_n \cong Z_n \oplus B_{n-1}$

It follows that

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0$$

is again a SES. Associated LES

[$C_\bullet = C_\bullet(X;R) = C_\bullet(X) \otimes_{\mathbb{Z}} R$
$C_\bullet \otimes_R M = C_\bullet(X) \otimes_{\mathbb{Z}} M$]

$$\to Z_n \otimes_R M \to \underset{H_n(C_\bullet \otimes_R M)}{H_n(X;M)} \to B_{n-1} \otimes_R M \xrightarrow{\partial = inc \otimes_R M} Z_{n-1} \otimes_R M$$

$$\cdots \to B_n \otimes_R M \xrightarrow{\partial = inc \otimes_R M} Z_n \otimes_R M$$

- Now, also have a SES

$$0 \to B_n \xrightarrow{inc} Z_n \to H_n(X;R) \to 0$$

$\exists$ splitting as $H_n(X;R)$ free R-module

$$\therefore \quad 0 \to B_n \otimes_R M \xrightarrow{inc \otimes_R M} Z_n \otimes_R M \to H_n(X;R) \otimes_R M \to 0$$

is again exact.

This implies the LES above splits into SES's

- $\Rightarrow$ Both $H_n(X;R) \otimes_R M$, $H_n(X;M)$

are identified with the cokernel of the same map, so are (naturally) isomorphic. □

<u>Theorem</u> (Universal Coefficient Theorem for cohomology)

R a PID, ~~$H_n(X;R)$~~ M an R-module. There's a map

$$H^*(X;M) \to Hom_R\big(H_*(X;R), M\big)$$

which is an isomorphism if $H_n(X;R)$ is free R-module for all n.

- <u>Proof obs</u>: $C^n(X;R) = Hom_{\mathbb{Z}}(C_n(X), R)$
$= Hom_R(C_n(X) \otimes_{\mathbb{Z}} R, R)$

then dualise the last argument. ⁙ ⌜$\mathrm{Hom}_R(-,M)$ preserves split SES⌟

Vector bundles

Def^n A real vector bundle (of dimension d) over X is

- a map $\pi: E \to X$
- the structure of a d-dimensional (real) vector space on each fibre $E_x := \pi^{-1}(x)$

s.t. the following local triviality condition holds: each $x \in X$ has an open neighbourhood $U \ni x$ with a homeomorphism

$$\varphi: E|_U = \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^d$$

$$\pi \searrow \; \circlearrowleft \; \swarrow \mathrm{proj}_1$$

$$U$$

over U, which is a linear isomorphism of vector spaces over each fibre (i.e. $E_y \xrightarrow[\varphi|_{E_y}]{} \{y\} \times \mathbb{R}^d$ is linear iso $\forall y \in U$)

Note: complex vector bundles are analogous

Def^n: a map $s: X \to E$ s.t. $\pi \circ s = id_X$ is called a section

The zero section is $s(x) = 0_{E_x} \in E_x$

Some operations (i) Pullback: if $\pi: E \to X$ is a vector bundle and $f: Y \to X$, then we define

$$f^*E = \{(e,y) \in E \times Y \mid \pi(e) = f(y)\} \xrightarrow{\pi_Y := f^*\pi} Y$$

has $(f^*E)_y = E_{f(y)}$, given the structure of a vector space.

And satisfies the local triviality condition.

ii) If $E \xrightarrow{\pi} X$, $F \xrightarrow{p} X$, then

"Whitney sum"

$$E \oplus F = \coprod_{x \in X} E_x \oplus F_x \longrightarrow X$$

$$= \{ (e,f) \in E \times F \mid \pi(e) = p(f) \}$$ topologised as a subspace of $E \times F$

Check, using local trivialisations of E and F, that $E \oplus F \longrightarrow X$ is locally trivial ✓

(Similarly define $E \otimes F$)

iii) If $E \to X$ is a vector bundle and $F \subset E$ is a subspace s.t. for each $x \in X$ there is $U \ni x$ and $\varphi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^d$

s.t. $F|_U$ gets sent homeomorphically to $U \times \mathbb{R}^k \subset U \times \mathbb{R}^d$, then F is called a <u>vector sub-bundle</u> of E

(F is then a vector bundle in its own right)

Can then define

$E/F \to X$ by forming the quotient vector space on each fibre.

<u>Examples</u> 1) $X = Gr_k(\mathbb{R}^n) := \{ k\text{-dim vector subspaces of } \mathbb{R}^n \}$

"Grassmanian"

Topologise this so that

$$GL_n(\mathbb{R}) \xrightarrow{\text{onto}} Gr_k(\mathbb{R}^n)$$

$$A \longmapsto \operatorname{span}\{\text{first } k \text{ columns of } A\} = A \cdot (\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n)$$

is a quotient map.

Let $\gamma^{\mathbb{R}}_{k,n} := E = \{ (V,v) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n \mid v \in V \} \xrightarrow{\pi} Gr_k(\mathbb{R}^n)$

$$(V,v) \longmapsto V$$

The fiber $\pi^{-1}(\{V\})$ is V, which <u>is</u> a vector space.

<u>Proof of local triviality</u> Let $\langle -,- \rangle$ be the standard inner product on $\mathbb{R}^n$.

For $V \in Gr_k(\mathbb{R}^n)$, let $\mathcal{U}_V = \{ W \in Gr_k(\mathbb{R}^n) \mid W \cap V^\perp = \{0\} \}$.

This contains V and is open. ⌈take inner products, get determinant condition⌋

Let $\varphi_V : E|_{\mathcal{U}_V} \longrightarrow \mathcal{U}_V \times V \cong \mathcal{U}_V \times \mathbb{R}^k$

$$(W, w) \longmapsto (W, pr_V(w))$$

$pr_V : \mathbb{R}^d \to V$ is orthogonal projection

On the fibre over $\{W\}$ this map is

$$W \hookrightarrow \mathbb{R}^n \xrightarrow{pr_V} V$$

Which is injective as $W \cap V^\perp = \{0\}$, so a linear iso.

Thus φ_V is a bijection, can also check it is continuous.

Similarly have $Gr_k(\mathbb{C}^n)$ with tautological bundle $\gamma_{n,k}^{\mathbb{C}}$

$$Gr_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n \leftarrow \gamma_{1,n+1}^{\mathbb{R}}$$

$$Gr_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n \leftarrow \gamma_{1,n+1}^{\mathbb{C}}$$

ii) If X is a smooth manifold, then it has a <u>tangent bundle</u> $TX \xrightarrow{\pi} X$ with fibers $\pi^{-1}(x) = T_x X$, the tangent space at x of X.

If $Y^k \overset{i}{\subset} X^n$ is a k-dimensional manifold of an n-manifold, then $TY \subset i^* TX$ is a sub-bundle, the normal bundle of Y in X is $\nu_Y = \dfrac{i^* TX}{TY} \longrightarrow Y$.

<u>Fact from DG</u> (Tubular neighbourhood theorem)

In the situation above there is an open neighbourhood U_Y of Y and a homeomorphism $\nu_Y \xrightarrow{\cong} UY$ which on the zero-section $s_0(Y)$ is the "identity" of Y

<u>Fact from point-set topology</u> If X is compact Hausdorff and $\{U_\alpha\}$ is an open cover, then there are $\lambda_\alpha : X \to \mathbb{R}_{\geq 0}$ s.t.

(i) $supp(\lambda_\alpha) = \overline{\{x \in X \mid \lambda_\alpha(x) > 0\}} \subset U_\alpha$

(ii) each $x \in X$ lies in finitely many $supp(\lambda_\alpha)$ ← ⌈something about $U \ni x$⌋

(iii) $\sum_\alpha \lambda_\alpha(x) = 1$ for all $x \in X$

"Partition of unity"

<u>Lemma</u> Let $\pi : E \to X$ be a vector bundle over X compact Hausdorff. Then

i) E admits an inner product $\langle -,- \rangle : E \oplus E \to \mathbb{R}$ (i.e. restricts to an inner product on each fiber)

ii) There is $F \xrightarrow{p} X$ s.t. $E \oplus F \cong X \times \mathbb{R}^N$

<u>Proof</u> i) Let $\{U_\alpha\}$ be an open cover of X with

$$\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^d \quad \text{trivialisations.}$$

then the usual inner product on $\mathbb{R}^d$ gives an inner product $\langle -,- \rangle_\alpha$ on $E|_{U_\alpha}$.

Define $\langle v, w \rangle = \sum_\alpha \lambda_\alpha(\pi(v)) \langle v, w \rangle_\alpha$

($v, w \in E_x$)

This is an inner product on each E_x, and is continuous on $E \oplus E$

ii) ETS that E is a subbundle of $X \times \mathbb{R}^N$ for some $N \gg 0$

If so, let $F = E^\perp \subset X \times \mathbb{R}^N$ using some inner product coming from (i).

$E^\perp = \{(x, v) \mid \langle v, w \rangle = 0 \ \forall w \in E_x\}$

⌜See Q^n on ES 3 for why $E^\perp$ is a sub-bundle⌟

Continuation of proof

(ii) To show $\exists F \to X$ s.t. $E \oplus F \cong X \times \mathbb{R}^N$, enough to show E is a subbundle of $X \times \mathbb{R}^N$, and then define $F = E^\perp$ where $E^\perp = \{(x,v) \in X \times \mathbb{R}^N \mid \langle v,w \rangle = 0 \ \forall w \in E_x\}$

See Ex Sheet 3 for proof of local triviality.

Let $\{U_\alpha\}_{\alpha \in I}$ be a finite open cover of X over which $E \xrightarrow{\pi} X$ is trivial, i.e. there are $\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^d$

Let $f_\alpha : E|_{U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d$ and define

$$f : E \longrightarrow X \times (\mathbb{R}^d)^n$$
$$v \longmapsto (\pi(v), (\lambda_\alpha(\pi(v)) f_\alpha(v))_{\alpha \in I})$$

where $n = |I|$.

On each fiber E_x this is a linear injection, for if $x \in U_\alpha$ is s.t. $\lambda_\alpha(x) \neq 0$, then $E_x \hookrightarrow E|_{U_\alpha} \xrightarrow[\varphi_\alpha]{\sim} U_\alpha \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d$ is a linear isomorphism and stays being an iso on multiplication by $\lambda_\alpha(x)$. □

[check f is homeo onto image]

Corollary If $\pi : E \to X$ is a d-dimensional vector bundle over X compact Hausdorff, then $\exists N \gg 0$ and $f_E : X \to Gr_d(\mathbb{R}^N)$ s.t. $f_E^* \gamma_{d,N}^{\mathbb{R}} \cong E$.

Proof Choose a linear embedding $\phi : E \longrightarrow X \times \mathbb{R}^N$.

Let
$$f_E : X \longrightarrow Gr_d(\mathbb{R}^N)$$
$$x \longmapsto \phi(E_x)$$

This defines the map, check that $E \cong f_E^* \gamma_{d,N}^{\mathbb{R}}$ via the only possible map. □

Orientations and the Thom isomorphism

- Let R be a commutative ring.

Let $\pi : E \to X$, and $E^{\#} = E \setminus s_0(X)$ where $s_0 : X \to E$ is the zero section.

If the fibers of π are d-dimensional, then

$$H^i(E_x, E_x^{\#}; R) \cong H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) = \begin{cases} R, & i = d \\ 0, & \text{o/w} \end{cases}$$

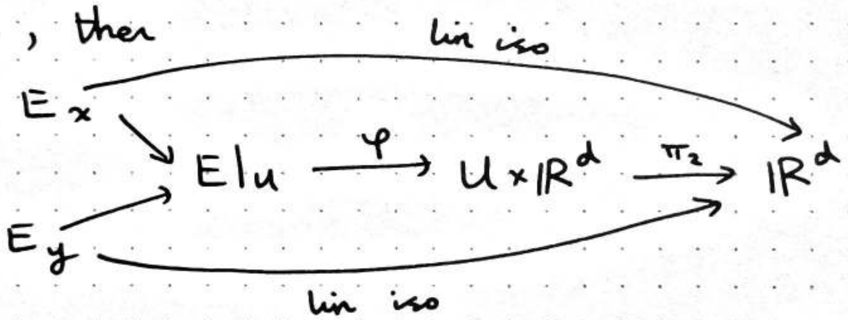
So

$$H^i(E_x, E_x^{\#}; R) = \begin{cases} \text{free } R\text{-module of rank } 1, & i = d \\ 0, & \text{o/w} \end{cases}$$

A <u>local R-orientation</u> of E at $x \in X$ is a choice

- $\varepsilon_x \in H^d(E_x, E_x^{\#}; R)$ of R-module generator.

An <u>R-orientation</u> of E is a choice of local orientation ε_x for each $x \in X$, s.t. if $x, y \in U$ and $\varphi : E|_U \xrightarrow{\cong} U \times \mathbb{R}^d$ is a trivialisation, then



- under the homeomorphisms $E_x, E_y \to \mathbb{R}^d$, the elements ε_x and ε_y correspond to the same element of $H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$.

<u>Lemma</u> i) For $R = \mathbb{F}_2 = \mathbb{Z}/2$ every bundle has a unique orientation.

ii) If $\{U_\alpha\}_{\alpha \in I}$ is a trivialising cover of $\pi : E \to X$ s.t. for all $\alpha, \beta \in I$ the homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^d \xleftarrow[\cong]{\varphi_\alpha|} E|_{U_\alpha \cap U_\beta} \xrightarrow[\cong]{\varphi_\beta|} (U_\alpha \cap U_\beta) \times \mathbb{R}^d$$

- is orientation-preserving on each fiber (in the sense of linear algebra: the matrix has +ve determinant) then E is R-orientable for any R.

Proof i) $\mathbb{Z}/2$ has a unique $\mathbb{Z}/2$-module generator, so all ε_x's are predetermined.

ii) Choose a ~~generator~~ $u \in H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$ and for any $x \in U_\alpha \subset X$, choose ε_x to correspond to u under

$$E_x \hookrightarrow E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d \qquad (*)$$
$$\varepsilon_x \longleftarrow\!\!\!\!\!-\!\!\!-\!\!\!-\!\!\!\mid u$$

The collection $\{\varepsilon_x\}_{x \in X}$ is compatible because if $x \in U_\beta$ too then the homeomorphism analogous to $(*)$ differs from $(*)$ by a post-composing with a linear $\mathbb{R}^d \xrightarrow[\psi]{\sim} \mathbb{R}^d$ of positive determinant. But $\psi^*: H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \circlearrowleft$ is $\psi(u) = u$, as ψ can be connected via homotopy of linear isomorphisms to id. $\square$

Theorem (Thom isomorphism) Let $\pi: E \to X$ be a d-dim^l vector bundle, $\{\varepsilon_x\}_{x \in X}$ an R-orientation. Then

i) $H^i(E, E^\#; R) = 0$ for $i < d$

ii) there is a unique $u_E \in H^d(E, E^\#; R)$ which restricts to $\varepsilon_x \in H^d(E_x, E_x^\#; R)$ $\forall x \in X$

iii) the map $\Phi: H^i(X; R) \xrightarrow[\cong]{\pi^*} H^i(E; R) \xrightarrow{- \cup u_E} H^{i+d}(E, E^\#; R)$

is an isomorphism. ö

Def^n Call u_E the Thom class of E, and its image under $u_E \in H^d(E, E^\#; R) \xrightarrow{q^*} H^d(E; R) \xrightarrow[s_0^*]{\sim} H^d(X; R)$ is called

$e(E)$, the Euler class of E.

If $f: Y \to X$ is a map, then

$$\hat{f}: (f^*E)_y \xrightarrow{\cong} E_{f(y)}$$

gives an orientation of f^*E from one of E.

$$\begin{array}{ccc} f^*E & \xrightarrow{\hat{f}} & E \\ \downarrow f^*\pi & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array} \qquad \{(y,e):\dots\} \mapsto e$$

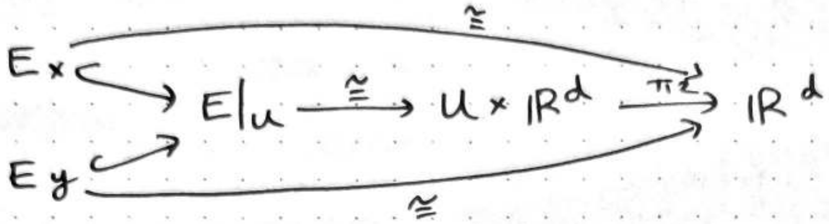
Then $\hat{f}^*(u_E) = u_{f^*E}$ and $f^*e(E) = e(f^*E)$ in $H^d(Y;R)$

<u>Theorem</u> $u_E \cup u_E = \pi^*(e(E)) \cup u_E$

<u>Pf</u> When cupping two relative cohomology classes, only one needs to be relative, so

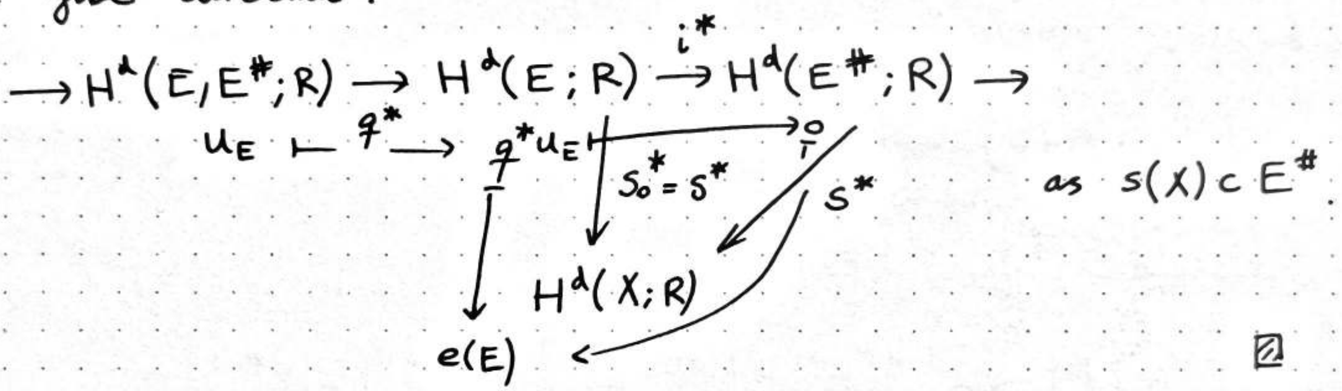
$$u_E \cup u_E = (q^* u_E) \cup u_E = \pi^*(e(E)) \cup u_E. \quad \square$$

<u>Correction</u> An <u>R-orientation</u> of E is a collection $\{\varepsilon_x\}$ of orientations of each E_x s.t. for each $x \in X$ there is a ngbd $U \ni x$ and a trivialisation $\varphi E|_U \xrightarrow{\cong} U \times \mathbb{R}^d$ s.t. for all $y \in U$, ε_y and ε_x are compatible under



<u>Theorem</u> If $\pi : E \to X$ is R-oriented and $s : X \to E$ is a section which is nowhere 0, then $e(E) = 0 \in H^d(X;R)$

<u>Proof</u> s is homotopic to the zero section s_0 by scaling in the fiber direction.



☐

<u>Corollary</u> If $e(E) \neq 0$ then any section $X \to E$ must have a zero. e.g. $E \not\cong X \times \mathbb{R}^d$

<u>Lemma</u> If $\pi : E \to X$ is a d-dimensional vector bundle, d odd, then $2e(E) = 0 \in H^d(X;R)$.

<u>Pf</u> Let $a : E \to E$ be the map over X which is $v \mapsto -v$ on each fibre.

As d is odd, this map has det -1 on each fiber, so it sends u_E to $-u_E$ (by uniqueness of Thom classes)

So pulling back along $s_0 = a \circ s_0$ gives $e(E) = -e(E)$. ☐

Proof of Thom iso theorem

Suppose for simplicity that $\pi : E \to X$ has a finite (s.t. compat) trivialising cover $\{U_\alpha\}_{\alpha \in I}$ (in general resort to Zorn's Lemma)

Base case: Let $\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^d$.

$$- \times - : H^*(U_\alpha ; R) \otimes_R H^*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\} ; R)$$

$$\downarrow \cong \text{ by Kunneth, as } H^*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\} ; R) \text{ is free}$$

$$H^*(U_\alpha \times \mathbb{R}^d, U_\alpha \times (\mathbb{R}^d \setminus \{0\}) ; R)$$

$$\varphi_\alpha^* \downarrow \cong$$

$$H^*(E|_{U_\alpha}, E^\#|_{U_\alpha} ; R)$$

⌜do we need U_α "nice"⌟

⌜uniqueness needs some check on $H^0(U_\alpha ; R)$⌟

⌜uniqueness clear if U_α contractible but ...⌟

(i) LHS vanishes below degree d

(ii) Let $u_{E|_{U_\alpha}}$ correspond to $1 \otimes \varepsilon_x$ using $E_x \xrightarrow[\cong]{\varphi_\alpha} \mathbb{R}^d$

(iii) the composition above *is* the map Φ, and is an isomorphism

Induction step Let $U, V \subset X$ and suppose the Thom iso thm holds for $E|_U$, $E|_V$, $E|_{U \cap V}$

Using (relative) Mayer-Vietoris

$$\cdots \to \underbrace{H^{d-1}(E|_{U\cap V}, E^\#|_{U \cap V})}_{=0} \xrightarrow{\partial^{MV}} H^d(E|_{U \cup V}, E^\#|_{U\cup V}) \ni \therefore \exists!\, u_{E|_{U\cup V}}$$

$$\downarrow i_U^* \oplus i_V^*$$

$$u_{E|_U} \in H^d(E|_U, E^\#|_U) \oplus H^d(E|_V, E^\#|_V) \ni u_{E|_V}$$

$$\downarrow j_U^* - j_V^*$$

$$H^d(E|_{U\cap V}, E^\#|_{U \cap V}) \ni u_{E|_{U\cap V}} \to \cdots$$

⇦ zeros this way

(i) To the left of this position, "almost all" terms are zero by assumption, so $H^i(E|_{U\cup V}, E^\#|_{U\cup V}) = 0$ for $i<d$ too.

(ii) The MV-chase above shows there is a unique $u_{E|_{U\cup V}}$ restricting to $u|_{E_U}$ and $u_{E|_V}$.

So as $u_{E|U}, u_{E|V}$ restrict to ε_x at $x \in X$, so does $u_{E|U \cup V}$.

(iii) Get map of MV sequences of (UuV, U, V) to $(E|_{U\cup V}, E|_U, E|_V)$ by pullback, and from the latter to the sequence above by cupping with the relevant Thom classes. Check this commutes and apply 5-lemma. ⌜△⌟ ☒

<u>Gysin sequence</u> If $\pi : E \to X$ is a vector bundle of dimension d, with $\langle \cdot, \cdot \rangle$ an inner product.

$$\mathbb{S}(E) = \{ v \in E \mid \langle v, v \rangle = 1 \}$$ is the sphere bundle associated to E.

The inclusion $j : \mathbb{S}(E) \hookrightarrow E^{\#}$ has a homotopy inverse, given by $v \mapsto \dfrac{v}{\sqrt{\langle v, v\rangle}}$.

Write $p : \mathbb{S}(E) \longrightarrow X$, $v \longmapsto \pi(v)$ for the projection.

The LES for $(E, E^{\#})$ is

$$\cdots \to H^{i+d}(E, E^{\#}) \to H^{i+d}(E) \to H^{i+d}(E^{\#}) \xrightarrow{\partial} H^{i+d+1}(E, E^{\#}) \to \cdots$$

with vertical maps: $\pi^*\alpha \cup u_E \mapsfrom$ (from below, Φ, $\cong$) into $H^{i+d}(E,E^{\#})$; π^*, $\cong$, s_0^* between $H^{i+d}(E)$ and $H^{i+d}(X)$; j^*, $\cong$ between $H^{i+d}(E^{\#})$ and $H^{i+d}(\mathbb{S}(E))$; Φ, $\cong$ from $H^{i+1}(X)$ to $H^{i+d+1}(E,E^{\#})$.

Suppose E is R-oriented.

$$\cdots \to H^i(X) \xrightarrow{-\cup e(E)} H^{i+d}(X) \xrightarrow{p^*} H^{i+d}(\mathbb{S}(E)) \xrightarrow{p_!} H^{i+1}(X) \to \cdots$$

$\alpha \mapsto \alpha \cup e(E)$

$p_!$ defined to make the square commute

The lower exact sequence is called the <u>Gysin sequence</u>

It is a LES of $H^*(X)$-modules.

Example $L = \gamma^{\mathbb{C}}_{1,n+1} \longrightarrow \mathbb{CP}^n = Gr_1(\mathbb{C}^{n+1})$ is the tautological complex line bundle (1-dim vector bundle)

- Ex Sheet 3: $\mathbb{C}$-vector bundles are oriented ($GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$; $GL_n(\mathbb{C})$ connected, so all have +ve det when considered as real matrices)

Let $x := e(L) \in H^2(\mathbb{CP}^n; \mathbb{Z})$

Note $S(L) = \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ s.t. $v \in l, \|v\| = 1\}$

$$\downarrow \text{forget } l \quad (\cong)$$

$$\{v \in \mathbb{C}^{n+1} \mid \|v\| = 1\} = S^{2n+1}$$

$\left[p: S^{2n+1} \to \mathbb{CP}^n,\ v \mapsto \langle v \rangle \right]$

Consider the Gysin sequence for this bundle

- $$\cdots \to H^{i+1}(S^{2n+1}) \xrightarrow{p_!} H^i(\mathbb{CP}^n) \xrightarrow{-\cup x} H^{i+2}(\mathbb{CP}^n) \xrightarrow{p^*} H^{i+2}(S^{2n+1}) \to$$

$H^{i+1}(S^{2n+1}) = 0$ for $i+1 < 2n+1$; $H^{i+2}(S^{2n+1}) = 0$ for $i+2 < 2n+1$

So $H^0(\mathbb{CP}^n) \xrightarrow[-\cup x]{\cong} H^2(\mathbb{CP}^n)$, with $H^0(\mathbb{CP}^n) = \mathbb{Z}\{1\}$ and $H^2(\mathbb{CP}^n) = \mathbb{Z}\{x\}$

$$\mathbb{Z} \to H^{-1}(\mathbb{CP}^n) \xrightarrow{-\cup x} H^1(\mathbb{CP}^n) \to H^1(S^{2n+1}) \to H^0(\mathbb{CP}^n) \to \cdots$$

$H^{-1}(\mathbb{CP}^n) = 0$, $H^1(S^{2n+1}) = 0$, $\therefore H^1(\mathbb{CP}^n) = 0$

- Discover: $H^i(\mathbb{CP}^n) \xrightarrow{-\cup x} H^{i+2}(\mathbb{CP}^n)$ is an isomorphism for $i < 2n-1$.

$$\Rightarrow \cdots \quad \boxed{H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})} \text{ as a ring.}$$

$$H^{2n+1}(\mathbb{CP}^n) \xrightarrow{p^*} H^{2n+1}(S^{2n+1}) \xrightarrow{p_!} H^{2n}(\mathbb{CP}^n) \to H^{2n+2}(\mathbb{CP}^n)$$

$H^{2n-1}(\mathbb{CP}^n) \xrightarrow{-\cup x} H^{2n+1}(\mathbb{CP}^n)$, with $H^{2n-1}(\mathbb{CP}^n) = 0$ and $H^{2n+1}(\mathbb{CP}^n) = 0$

$H^{2n+1}(S^{2n+1}) \cong \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}\{x^n\} = H^{2n}(\mathbb{CP}^n)$

$H^{2n+2}(\mathbb{CP}^n) = 0$ (by cell)

- $\uparrow$
  $\vdots$

<u>Example 2</u> $L = \gamma^{\mathbb{R}}_{1,n+1} \to \mathbb{RP}^n$ is orientable over $\mathbb{Z}/2$

$\Rightarrow z = e(L) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$

Same analysis shows

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2[z]/(z^{n+1})$$

Compactly Supported Cohomology

If $\varphi \in C^n(X)$ is a cochain, say it has support in a set $S \subset X$ if for all n-simplices $\sigma : \Delta^n \to X \setminus S \subset X$, $\varphi(\sigma) = 0$.

Then $d\varphi$ also has support in $S \subset X$.

<u>Def^n</u>: Let $C^*_c(X)$ be the sub-complex of $C^*(X)$ consisting of those φ's having support in <u>some</u> compact set.

Equivalently, $C^*_c(X) = \bigcup_{K \text{ cpct}} C^*(X, X\setminus K) \subset C^*(X)$.

Let $H^*_c(X) = H^*(C^*_c(X), d)$ be the <u>compactly supported cohomology</u> of X.

<u>Def^n</u> A <u>directed set</u> is a partially ordered set $(I, \leq)$ s.t. $\forall a, b \in I$, $\exists c \in I$ s.t. $a \leq c$ & $b \leq c$.

A <u>directed system of abelian groups</u> indexed by $(I, \leq)$ is

- a collection $\{G_a\}_{a \in I}$ of abelian groups,
- when $a \leq b$ in I, a map $\rho_{ab} : G_a \to G_b$ s.t.

$$\rho_{aa} = id_{G_a}, \quad \text{and} \quad \rho_{ac} = \rho_{bc} \circ \rho_{ab} \text{ for } a \leq b \leq c$$

The <u>direct limit</u> (colimit) is

$$\varinjlim_I G_a = \operatorname*{colim}_I G_a := \Big(\bigoplus_{a \in I} G_a\Big) \Big/ \Big\langle x - \rho_{ab}(x) \;\Big|\; \begin{matrix} x \in G_a, \\ a \leq b \end{matrix} \Big\rangle$$

<u>Observe</u>: if $J \subseteq I$ is a subset s.t. $\forall a \in I$, $\exists b \in J$ s.t. $a \leq b$ then $\operatorname*{colim}_J G_a \xrightarrow{\sim} \operatorname*{colim}_I G_a$.

For any space X, $(\mathcal{K}(X) = \{\text{compact subsets of } X\}, \subseteq)$ is a directed set, as $\forall K, K' \in \mathcal{K}(X)$, $K, K' \subseteq \underbrace{K \cup K'}_{\text{compact too}}$

Then the assignment

$$K \longmapsto C^\bullet(X, X\setminus K)$$

is a directed system of abelian groups,

as is $K \longmapsto H^n(X, X\setminus K)$.

<u>Lemma</u>: $H^n_c(X) \cong \varinjlim_{\mathcal{K}(X)} H^n(X, X\setminus K)$

<u>Proof</u>: $C^\bullet_c(X) = \varinjlim_{\mathcal{K}(X)} C^\bullet(X, X\setminus K)$ by def^n,

and taking homology commutes with direct limits.

See Ex Sheet 4. ☒

<u>Lemma</u>: $H^i_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & i=n \\ 0, & o/w \end{cases}$

<u>Proof</u>: each compact $K \subset \mathbb{R}^n$ lies in some $B_N = \{ v \in \mathbb{R}^n \mid \|v\| \leq N \}$

So $H^i_c(\mathbb{R}^n) \cong \varinjlim_{\mathbb{N}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_N)$

But

$$\begin{array}{ccc} H^i(\mathbb{R}^n, \mathbb{R}^n\setminus B_N) & \longrightarrow & H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_{N+1}) \\ \cong \uparrow \partial & & \cong \uparrow \\ \tilde{H}^{i-1}(\mathbb{R}^n\setminus B_N) & \xrightarrow[\text{(homotopy equiv)}]{\cong} & \tilde{H}^{i-1}(\mathbb{R}^n \setminus B_{N+1}) \\ \| & & \\ \begin{cases} \mathbb{Z}, & i=n \\ 0, & o/w \end{cases} & & \end{array}$$

all the maps in the direct system are isomorphisms, so

$$\begin{cases} \mathbb{Z}, & i=n \\ 0, & \text{else} \end{cases} = H^i(\mathbb{R}^n, \mathbb{R}^n\setminus 0)$$

$$\downarrow$$

$$\varinjlim_{\mathcal{K}(\mathbb{R}^n)} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \quad \text{is an iso.}$$

$$\|$$

$$H^i_c(\mathbb{R}^n)$$

☒

- $H^*_c(-)$ is <u>not</u> homotopy invariant

In fact, a general map $f: X \to Y$ does <u>not</u> induce a map on $H^*_c(-)$

- If $f: X \to Y$ is <u>proper</u> (preimage of compact is compact), then it induces a map $f^*: H^*_c(Y) \to H^*_c(X)$ by the usual formula
- If $\iota: U \hookrightarrow X$ is the inclusion of an open subspace (and X is Hausdorff) then for $K \subset U$ compact, by excision

$$H^n(U, U \setminus K) \xleftarrow[\text{excision}]{\cong} H^n(X, X \setminus K)$$

So on direct limits get a map

$$\begin{array}{c} \iota_*: H^*_c(U) \cong \varinjlim_{\substack{K \subset U \\ \text{cpct}}} H^*(U, U \setminus K) \\ \uparrow \cong \\ \varinjlim_{\substack{K \subset U \\ \text{cpct}}} H^*(X, X \setminus K) \\ \downarrow \\ \varinjlim_{\substack{K \subset X \\ \text{cpct}}} H^*(X, X \setminus K) \cong H^*_c(X) \end{array}$$

called <u>extension by zero</u>.

[$\iota_*[\varphi]$, for φ supported in $K \subset U$, is given by a cochain which agrees with φ on simplices in U, and is 0 on simplices in $X \setminus K$]

<u>Ex</u>: if $U \hookrightarrow \mathbb{R}^d$ is the inclusion of an open ball, then

$$\iota_*: H^*_c(U) \xrightarrow{\cong} H^*_c(\mathbb{R}^d)$$

Let us write $H^*(X|K) := H^*(X, X \setminus K)$, call it <u>local cohomology near K</u>. (Only depends on a ngbd of K in X)

Proposition Let $K, L \subset X$ be compact, X Hausdorff.
Then there is a M-V-like sequence

$$\to H^n(X|K\cap L) \xrightarrow{\text{inc} \oplus -\text{inc}} H^n(X|K) \oplus H^n(X|L) \xrightarrow{\text{inc}+\text{inc}} H^n(X|K\cup L) \xrightarrow{\partial} H^{n+1}(X|K\cap L) \to \cdots$$

Proof Cover $X \setminus (K\cap L)$ by $\mathcal{U} = \{X\setminus K, X \setminus L\}$.
Then form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & \mathrm{Hom}\left(\frac{C_\bullet(X)}{C_\bullet^{\mathcal{U}}(X\setminus K\cap L)}, \mathbb{Z}\right) & \dashrightarrow & C^\bullet(X, X\setminus K) \oplus C^\bullet(X, X\setminus L) & \dashrightarrow & C^\bullet(X, X\setminus K\cup L) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^\bullet(X) & \xrightarrow{\text{id} \oplus -\text{id}} & C^\bullet(X) \oplus C^\bullet(X) & \xrightarrow{\text{id}+\text{id}} & C^\bullet(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X\setminus K\cap L), \mathbb{Z}) & \xrightarrow{\text{inc} \oplus -\text{inc}} & C^\bullet(X\setminus K) \oplus C^\bullet(X\setminus L) & \xrightarrow{\text{inc}+\text{inc}} & C^\bullet(X\setminus K\cup L) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Top row is exact by some non-sense [e.g. LES in cohomology], maps are induced.

Looking at left column, as

$$C^\bullet(X\setminus K\cap L) \longrightarrow \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X\setminus K\cap L), \mathbb{Z})$$

is an iso on cohomology by the small simplices theorem (dualized), by the 5-lemma the map

$$C^\bullet(X, X\setminus K\cap L) \to \mathrm{Hom}\left(\frac{C_\bullet(X)}{C_\bullet^{\mathcal{U}}(X\setminus K\cap L)}, \mathbb{Z}\right)$$

is also an H^*-iso.

Using this, the LES for the top row of our 3×3 diagram is the required LES. ∎

<u>Corollary</u> Let X be a manifold and A, B an open cover of X. Then there is a MV sequence

$$\cdots \to H_c^*(A\cap B) \xrightarrow{(i_A)_* \oplus (i_B)_*} H_c^*(A) \oplus H_c^*(B) \xrightarrow{(j_A)_* - (j_B)_*} H_c^*(A \cup B = X) \xrightarrow{\partial} H_c^{*+1}(A\cap B) \to \cdots$$

<u>Proof</u> For $K \subset A$, $L \subset B$ compact, use

$H^*(X \backslash K) \xrightarrow{\cong} H^*(A|K)$,

$H^*(X|L) \xrightarrow{\cong} H^*(B|L)$,

$H^*(X|K\cap L) \xrightarrow{\cong} H^*(A\cap B|K \cap L)$

The LES from the Prop^n gives

$$\cdots \to H^n(A\cap B|K\cap L) \to H^n(A|K) \oplus H^n(B|L) \to H^n(X|K\cup L) \xrightarrow{\partial} H^{n+1}(A\cap B|K\cap L) \to \cdots$$

Taking direct limits over <u>all</u> compact $K \subset A$, $L \subset B$, get

$$\cdots \to H_c^n(A\cap B) \to H_c^n(A) \oplus H_c^n(B) \to \varinjlim_{\substack{K\subset A \\ L \subset B \\ \text{cpct}}} H^n(X|K\cup L) \to H_c^{n+1}(A\cap B) \to \cdots$$

Any compact $C \subset X$ is contained in some $K \cup L$: we can cover C by interiors of closed balls in X which are contained in A or B, take a finite subcover and $K = \bigcup$ balls in A, $L = \bigcup$ balls in B.

So $\varinjlim_{\substack{K\subset A \\ L \subset B \\ \text{cpct}}} H^n(X|K\cup L) \underset{\text{canonically}}{\cong} H_c^n(X)$. ☒

⌜phew!⌟

(Co)homology of manifolds

For a d-dimensional manifold M, the <u>local homology</u> (with R-coeffs) is $H_*(M|x;R) = H_*(M, M\setminus x; R) \cong \begin{cases} R, & * = d \\ 0, & \text{o/w} \end{cases}$

A <u>local R-orientation of M at x</u> is a choice $\varepsilon_x \in H_d(M|x;R)$ of R-module generator.

An <u>R-orientation</u> of M is a choice ε_x of local orientation for every $x \in M$ s.t.

if $\varphi: \mathbb{R}^d \xrightarrow{\cong} U \subset M$ is a homeo to an open set U and $p, q \in D^d \subset \mathbb{R}^d$, then under

$$\begin{array}{ccccc} H_d(M|\varphi(p);R) & \xleftarrow[\text{exc}]{\cong} & H_d(U|\varphi(p);R) & \xleftarrow[\cong]{\varphi_*} & H_d(\mathbb{R}^d|p;R) \\ H_d(M|\varphi(q);R) & \xleftarrow[\text{exc}]{\cong} & H_d(U|\varphi(q);R) & \xleftarrow[\cong]{\varphi_*} & H_d(\mathbb{R}^d|q;R) \end{array}$$

with both $H_d(\mathbb{R}^d|p;R) \xleftarrow{\cong} H_d(\mathbb{R}^d|D^d;R) \xrightarrow{\cong} H_d(\mathbb{R}^d|q;R)$,

~~ε_x~~ $\varepsilon_{\varphi(p)}$, $\varepsilon_{\varphi(q)}$ correspond to each other.

<u>Lemma</u> (i) For $R = \mathbb{F}_2 = \mathbb{Z}/2$, every manifold is canonically R-oriented.

(ii) If $\{\varphi_\alpha : \mathbb{R}^d \xrightarrow{\cong} U_\alpha \subset M\}_{\alpha \in I}$ is an open cover of M by coordinate charts s.t. for each $x \in U_\alpha \cap U_\beta$: ⌢

<u>Lemma</u> i) If $R = \mathbb{F}_2 = \mathbb{Z}/2$ then any manifold is R-orientable.

ii) If $\{\varphi_\alpha : \mathbb{R}^d \xrightarrow{\cong} U_\alpha \subset M\}_{\alpha \in I}$ is an open cover s.t. for $x \in U_\alpha \cap U_\beta$,

$$\begin{array}{ccccc} H_d(M|x;R) & \xleftarrow[ex]{\sim} & H_d(U_\alpha|x;R) & \xleftarrow{\sim} & H_d(\mathbb{R}^d|\varphi_\alpha^{-1}(x);R) \\ & & & & \wr\| \text{ translation} \\ \big\downarrow id & & & & H_d(\mathbb{R}^d|0;R) \\ & & & & \wr\| \text{ translation} \\ H_d(M|x;R) & \xleftarrow[ex]{\sim} & H_d(U_\beta|x;R) & \xleftarrow{\sim} & H_d(\mathbb{R}^d|\varphi_\beta^{-1}(x);R) \end{array} \quad (*)$$

commutes, then M is R-orientable.

<u>Proof</u> i) $H_d(M|x;\mathbb{F}_2) = \mathbb{F}_2$ has a unique $\mathbb{F}_2$-module generator

ii) For $x \in M$, if $x \in U_\alpha$ then let ε_x correspond to a fixed generator $u \in H_d(\mathbb{R}^d|0;R)$ under the above diagram.
Indep of choice of U_α by assumption.
For compatibility at points near $x \in U_\alpha$, use $(*)$ with each $y \in U_\alpha$. ☒ [Check] ⌜use translations ~ id⌟

<u>Theorem</u> Let A be compact, $A \subset M$. Let M be R-oriented.
Then 1) There is a unique $\varepsilon_A \in H_d(M|A;R) = H_d(M, M\setminus A;R)$ which restricts to ε_x for each $x \in A$.

ii) $H_i(M|A;R) = 0$ for $i > d$.

<u>Proof</u>: Ignore R in the notation.
Call $A \subset M$ compact "good" if the theorem holds for A.
① <u>If $A, B, A\cap B$ are good, then $A \cup B$ is good</u>
By the Mayer-Vietoris sequence for local <u>homology</u> (c.f. last lecture, but for <u>homology</u>).

⌜this is the same as relative Mayer-Vietoris⌟

$H_{d+1}(M|A\cap B) \leftarrow$ zero by (ii), $A\cap B$ good

$$\overset{\shortparallel}{0} \xrightarrow{\partial} H_d(M|A\cup B) \xrightarrow{\text{res,res}} H_d(M|A)\oplus H_d(M|B) \xrightarrow{\text{res}-\text{res}} H_d(M|A\cap B) \to \cdots$$

$$\varepsilon_A,\ \varepsilon_B \longmapsto \varepsilon_{A\cap B},\ \varepsilon_{A\cap B}$$

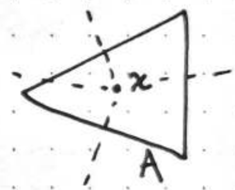
$$\exists!\ \varepsilon_{A\cup B} \longmapsto \varepsilon_A + \varepsilon_B \longmapsto 0$$

The $\varepsilon_{A\cup B}$ defined in this way restricts to ε_x at $x\in A\cup B$ as ε_A does if $x\in A$, and ε_B does if $x\in B$.

To the left, have lots of zeros and $H_i(M|A\cup B)$ for $i>d$ vanishes too.

② <u>Convex subsets of $\mathbb{R}^d$ are good</u> (i.e. inside a coord chart)

If $A\subset\mathbb{R}^d$ is convex, then for $x\in A$, $\mathbb{R}^d\setminus A \hookrightarrow \mathbb{R}^d\setminus x$ is a homotopy equivalence by scaling radially from $x\in\mathbb{R}^d$.



So $H_d(\mathbb{R}^d|A) \xrightarrow{\sim} H_d(\mathbb{R}^d|x)$

$$\varepsilon_A \longmapsto \varepsilon_x$$

choose ε_A to restrict to ε_x.

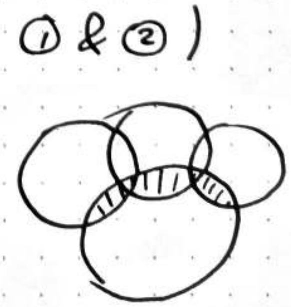
<u>Check</u> that this also restricts to ε_y for any $y\in A$. ⌈translation business⌋

③ <u>All compact subsets of $\mathbb{R}^d$ are good</u> Let $A\subset\mathbb{R}^d$ be compact. For any open $U\supset A$ we can find finitely many balls $B_1,\dots,B_n$ s.t.

$$U\supset B=\textstyle\bigcup_i B_i \supset \operatorname{int}(B)\supset A$$

By induction on # ~~balls~~ convex sets, B is good. (using ① & ②)

⌈$B_n\cap\left(\bigcup_{i=1}^{n-1} B_i\right)$ is a union of $\le n-1$ convex sets⌋



Hence the set $\mathcal{G}:=\{B \mid \operatorname{int}B\supset A,\ B \text{ good}\}$ is non-empty.

Want to show $\mathcal{G}$ is a directed set:

for $B, B' \in \mathcal{G}$, want $B \cap B' \supset \text{int}(B \cap B') \supset B'' \supset \text{int}(B'') \supset A$ for some B'', exists by earlier part.

So $\mathcal{G}$ is directed, and

$$B \longmapsto H_d(M|B)$$

is a directed system of abelian groups.

As $A \subset B$, get

$$\varinjlim_{B \in \mathcal{G}} H_i(M|B) \longrightarrow H_i(M|A) \qquad (*)$$

<u>Claim</u>: this is an iso

<u>Pf</u> Let $[[c]] \in H_i(M|A)$.

Then $dc \in C_{i-1}(M \setminus A)$.

As dc is a finite sum of simplices, it lies in some compact

$$K \subset M \setminus A \quad \text{i.e.} \quad A \subset \underbrace{M \setminus K}_{\text{open}}$$

So there is $M \setminus K \supset B \supset \text{int}\, B \supset A$ with B good, so dc is in $C_{i-1}(M \setminus B)$.

So $[[c]] \in H_i(M|B)$, i.e. $(*)$ is surjective.

Injectivity similar. ▨

Immediately get $H_i(M|A) = 0$ for $i > d$, as LHS of $(*)$ is 0.

The class ε_B for $B \in \mathcal{G}$ give a class ε_A. This restricts to ε_x at $x \in A$ as ε_B does.

The uniqueness follows as $(*)$ is injective. [how does it ?! see Hatcher for more convincing]

④ A compact $A \subset M$ can be written as a finite union of A_i's each $A_i \subset U_\alpha$ for some α.

So each A_i is good, so by ① is good. ▨

<u>Corollary</u> If M is compact R-oriented manifold, then there is a unique $[M] = \varepsilon_M \in H_d(M)$ restricting to ε_x at every $x \in M$. ▨

We call $[M]$ the <u>*fundamental class of M*</u>.

Poincaré Duality

Thm Let M be an R-oriented manifold, of dimension d.

- Then there is a map

$$D_M : H_c^k(M;R) \to H_{d-k}(M;R)$$

which is an isomorphism.

Def The cap product is

$$-\frown- : C_k(X;R) \times C^l(X;R) \longrightarrow C_{k-l}(X;R)$$
$$(\sigma, \varphi) \longmapsto \varphi(\sigma|_{[v_0,\dots,v_l]}) \cdot \sigma|_{[v_l,\dots,v_k]}$$

Lemma $d(\sigma \frown \varphi) = (-1)^l[(d\sigma)\frown\varphi - \sigma \frown d\varphi]$

Proof Similar to $-\smile-$ case. ▨ "yes"

- As with the cup product, this descends to a well-defined map

$$-\frown- : H_k(X;R) \otimes_R H^l(X;R) \longrightarrow H_{k-l}(X;R)$$

As with $-\smile-$, the same formula induces a relative cap product:

$$-\frown- : H_k(X,A;R) \otimes H^l(X;R) \longrightarrow H_{k-l}(X,A;R)$$
$$-\frown- : H_k(X,A;R) \otimes H^l(X,A;R) \longrightarrow H_{k-l}(X;R)$$

Lemma If $f: X \to Y$ is a map, $x \in H_k(X;R)$, $y \in H^l(Y;R)$, then $f_*(x \frown f^*y) = f_*x \frown y$.

- **Proof** Let $x = [c]$, $y = [\varphi]$. Then LHS, if $c = \sum a_\sigma \sigma$, is

$$f_\#((\textstyle\sum a_\sigma\sigma)\frown f^\#\varphi)$$
$$= \sum a_\sigma f_\#(\sigma \frown f^\#\varphi)$$
$$= \sum a_\sigma f_\#(\underbrace{(f^\#\varphi)(\sigma|_{[v_0,\dots,v_l]})}_{\in R}\cdot \sigma|_{[v_l,\dots,v_k]})$$
$$= \sum a_\sigma(\varphi(f_\#\sigma|_{[v_0,\dots,v_l]})\cdot f_\#(\sigma|_{[v_l,\dots,v_k]}))$$
$$= f_\#(\textstyle\sum a_\sigma \sigma)\frown\varphi = \text{a chain representing } f_*x \frown y. \quad ▨$$

- If M is compact, then we have constructed $[M] = \varepsilon_M$ living in $H_d(M;R)$. Then $D_M = [M]\frown - : H^k(M;R) \to H_{d-k}(M;R)$.

In general, if $K \subset L \subset M$ are compact and

$$i : (M, M \setminus L) \longrightarrow (M, M \setminus K) \text{ is inclusion},$$

then $i_*(\varepsilon_L) = \varepsilon_K$.

So,

$$\begin{array}{ccc} H^k(M|K;R) & \xrightarrow{\ i^*\ } & H^k(M|L;R) \\ & \searrow^{\varepsilon_K \cap -} \quad {}^{\varepsilon_L \cap -}\swarrow & \\ & H_{d-k}(M;R) & \end{array}$$

which commutes by the lemma:

$$\begin{aligned} \varepsilon_L \cap i^*(\xi) \\ \| \\ i_*(\varepsilon_L \cap i^*(\xi)) &= i_*(\varepsilon_L) \cap \xi \\ &= \varepsilon_K \cap \xi \end{aligned}$$

So the $(\varepsilon_K \cap -)$'s assemple into a map

$$D_M : H^k_c(M) = \varinjlim_K H^k(M|K;R) \longrightarrow H_{d-k}(M;R).$$

<u>Proof of Poincaré Duality</u>

Say M is <u>good</u> if it satisfies the theorem (D_M is iso)

① <u>If $M = U \cup V$ union of open sets, $U, V, U \cap V$ are good,</u> <u>then M is good:</u> we claim that

$$\begin{array}{ccccccccc} \cdots \to & H^k_c(U\cap V) & \xrightarrow{(i_U)_* + (i_V)_*} & H^k_c(U) \oplus H^k_c(V) & \xrightarrow{(j_U)_* - (j_V)_*} & H^k_c(M) & \xrightarrow{\partial} & H^{k+1}_c(U \cap V) & \to \cdots \\ & \downarrow D_{U\cap V} & ① & \downarrow D_U \oplus D_V & ② & \downarrow D_M & ③ & \downarrow D_{U \cap V} & \\ \cdots \to & H_{d-k}(U \cap V) & \to & H_{d-k}(U) \oplus H_{d-k}(V) & \to & H_{d-k}(M) & \xrightarrow[\partial]{} & H_{d-k-1}(U \cap V) & \to \cdots \end{array}$$

commutes (up to sign).

The top row is by def^n a direct limit of M-V sequences in cohomology local at $K \subset U$, $L \subset V$. So it's enough to show the squares in this larger diagram commute.

For ① and ② this follows from the Lemma. ⌜lol sure, but took me a while to believe⌟

For ③, one finds that

$$D_{U\cap V} \circ \partial = (-1)^{k+1} \partial \circ D_M$$ instead, by spelling out def^{n}s. ⌜quite tricky, see Hatcher⌟

So follows from 5-lemma.

② <u>If $U_1 \subset U_2 \subset U_3 \subset \cdots$ are good, then $M = \bigcup_i U_i$ is good</u>

Have $H^k_c(M) = \varinjlim_i H^k_c(U_i)$ as any compact $K \subset M$ lies in some U_i.

But also $H_{d-k}(M) = \varinjlim_i H_{d-k}(U_i)$.

Then use that a direct limit of isos is an iso.

③ <u>$\mathbb{R}^d$ is good</u>

We know $H^i_c(\mathbb{R}^d; R) \cong H^i(\mathbb{R}^d|0; R) = \begin{cases} R, & i=d \\ 0, & \text{else} \end{cases}$

$$\Big\downarrow \text{UCT} \ \wr\wr$$

$$\mathrm{Hom}_R(H_i(\mathbb{R}^d|0;R), R)$$

Under this isomorphism,

$$D_{\mathbb{R}^n} : H^d_c(\mathbb{R}^d; R) \longrightarrow H_0(\mathbb{R}^d; R) \xrightarrow[\varepsilon]{\cong} R$$

$$\quad \ni \qquad\qquad\qquad\qquad$$

$$f : H_d(\mathbb{R}^d|0; R) \to R \longmapsto f(\varepsilon_0) \qquad \ulcorner \varepsilon_0 \in H_d(\mathbb{R}^d|0) \lrcorner$$

we see that $D_{\mathbb{R}^d}$ is <u>surjective</u> (as $\exists$ a f s.t. $f(\varepsilon_0) = 1$)

so $D_{\mathbb{R}^d}$ is also an <u>iso</u>.

④ <u>Any open $U \subset \mathbb{R}^d$ is good</u>

a) If U is a <u>finite</u> union of open balls, this follows from ① and ③ by an inductive argument.

b) If U is a <u>countable</u> union of balls, follows from a) and ②.

c) Any open subset of $\mathbb{R}^d$ is a countable union of balls

⑤ <u>If M has a countable cover by $\mathbb{R}^d$'s, then it is good</u>

① + ④ gives it for finite covers by $\mathbb{R}^d$'s

② extends to countable covers.

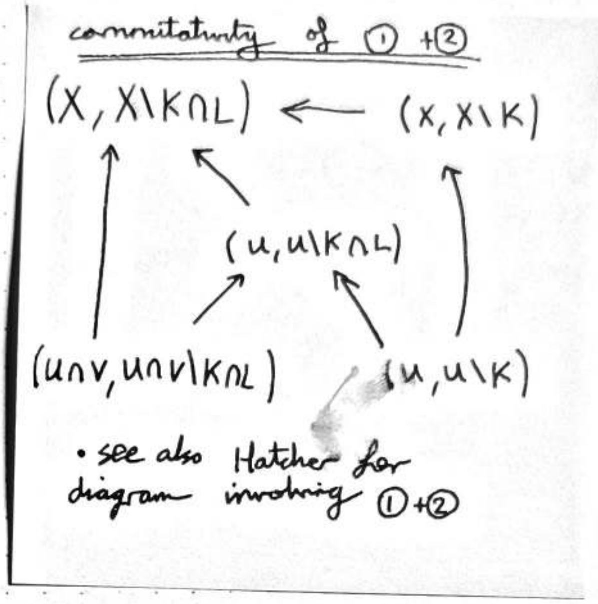
⑥ <u>All M are good</u> Zorn's lemma

□

<u>Corollary</u>: If M is compact and R oriented, then

$$D_M : H^k(M;R) \longrightarrow H_{d-k}(M;R)$$

is an isomorphism.



<u>Corollary</u>: if M is a compact manifold of odd dimension, then

$$\chi(M) = 0$$

<u>Proof</u>: $\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim_{\mathbb{F}_2} H^i(M;\mathbb{F}_2)$, and M is $\mathbb{F}_2$-oriented, so

$$H_i(M;\mathbb{F}_2) \underset{P.D.}{\cong} H^{d-i}(M;\mathbb{F}_2) \underset{UCT}{\cong} H_{d-i}(M;\mathbb{F}_2)^*$$

If d is odd, $d = 2n+1$, then

$$\begin{aligned}(-1)^i \dim_{\mathbb{F}_2} H_i(M;\mathbb{F}_2) &= (-1)^i \dim_{\mathbb{F}_2} H_{2n+1-i}(M;\mathbb{F}_2) \\ &= -(-1)^{2n+1-i} \dim_{\mathbb{F}_2} H_{2n+1-i}(M;\mathbb{F}_2)\end{aligned}$$

So these terms <u>cancel</u> in the formula for $\chi(M)$. ▨

If $\alpha \in C_{k+l}(X;R)$, $\varphi \in C^l(X;R)$, $\psi \in C^k(X;R)$, then if σ is a $(k+l)$-simplex, we have

$$\begin{aligned}\psi(\sigma \frown \varphi) &= \psi\left(\varphi(\sigma|_{[v_0,\dots,v_l]}) \cdot \sigma|_{[v_l,\dots,v_{k+l}]}\right) \\ &= \varphi(\sigma|_{[v_0,\dots,v_l]}) \cdot \psi(\sigma|_{[v_l,\dots,v_{k+l}]}) \\ &= (\varphi \smile \psi)(\sigma)\end{aligned}$$

so we get $\boxed{\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha).}$

In particular

$$\begin{aligned} C_k(X;R) \otimes C^k(X;R) &\xrightarrow{-\frown-} C_0(X;R) \xrightarrow{\varepsilon} R \\ \alpha \otimes \varphi &\longmapsto \varphi(\alpha)\end{aligned}$$

induces the map $H^k(X;R) \xrightarrow{h} \mathrm{Hom}_R(H_k(X;R), R)$ from the UCT.

$$[\varphi] \longmapsto \left([\alpha] \mapsto \varphi(\alpha)\right)$$

So

$$\begin{array}{ccc} H^l(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(X;R), R) \\ \downarrow \varphi\smile - & \circlearrowleft & \downarrow (-\frown\varphi)^* \\ H^{k+l}(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(X;R), R). \end{array}$$

So when the UCT applies, PD should <u>constrain</u> cup products.

For M^d compact and R-oriented, consider

$$\langle -,- \rangle : H^k(M;R) \otimes H^{d-k}(M;R) \xrightarrow[-\smile-]{} H^d(M;R) \xrightarrow[{[M]\frown-}]{} H_0(M;R) \xrightarrow[\varepsilon]{\sim} R$$

<u>Theorem</u>: if $H_*(M;R)$ are free R-modules (e.g. R is a field) then this pairing is non-singular, (i.e. the maps $H^k(M;R) \to \mathrm{Hom}_R(H^{d-k}(M;R), R)$

$H^{d-k}(M;R) \to \mathrm{Hom}_R(H^k(M;R), R)$

are both isomorphisms)

<u>Proof</u>: consider $H^k(M;R) \xrightarrow[h]{\cong \text{UCT}} \mathrm{Hom}_R(H_k(M;R), R) \xrightarrow[\cong]{D_M^*} \mathrm{Hom}_R(H^{d-k}(M;R), R)$

This sends $[\varphi]$ to the map $[\psi] \mapsto [\varphi]([M] \frown [\psi])$

$$= [\psi] \smile [\varphi]([M]). \qquad \square$$

<u>Example</u>: $H_*(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = 2i,\ 0 \le i \le n, \\ 0, & \text{o/w} \end{cases}$ e.g. by cellular homology.

$\mathbb{C}P^n$ is $\mathbb{Z}$-orientable, a manifold

Compute cup product structure by induction on n, suppose

$$H^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n)$$

Have $\mathbb{C}P^{n-1} \overset{i}{\hookrightarrow} \mathbb{C}P^n$ as the $2(n-1)$ skeleton.

So there is a unique $z \in H^2(\mathbb{C}P^n)$ s.t. $i^* z = x$.

Then $i^*(z^{n-1}) = x^{n-1}$ a generator of $H^{2(n-1)}(\mathbb{C}P^{n-1})$.

$\Rightarrow 1, z, \dots, z^{n-1}$ generate $\mathbb{C}P^n$'s cohomology for degrees $< 2n$.

This leaves the question: does $\mathbb{C}P^n$'s top cohomology have generator z^n?

Consider $\langle -,- \rangle : H^2(\mathbb{C}P^n) \otimes H^{2(n-1)}(\mathbb{C}P^n) \to \mathbb{Z}$

$z \otimes z^{n-1} \longmapsto \pm 1$ by non-singularity

$\longmapsto z^n([\mathbb{C}P^n])$

So z^n generates $H^{2n}(\mathbb{C}P^n)$. ⌜not divisible⌟

Hence $H^*(\mathbb{C}P^n) = \mathbb{Z}[z]/(z^{n+1})$.

<u>Middle cohomology</u>: If M is $2n$-dimensional compact, $\mathbb{Z}$-oriented, get

$$\langle -,- \rangle : H^n(M) \otimes H^n(M) \to \mathbb{Z}$$

a bilinear form $[\varphi] \otimes [\psi] \longmapsto [\varphi \smile \psi][M]$, which is not singular (...)

Also, as $\varphi \smile \psi = (-1)^{n^2} \psi \smile \varphi$.

So this $\langle -,- \rangle$ is <u>symmetric</u> if n is even,

<u>skew-symmetric</u> if n is odd.

Degree: If $f: M^d \to N^d$ is a map between $\mathbb{Z}$-oriented d-dimensional connected manifolds,

$$f_*[M] \in H_d(N;\mathbb{Z}) \underset{\cong \, PD}{\xleftarrow{[N] \frown -}} H^0(N;\mathbb{Z}) = \mathbb{Z}1$$
$$\| $$
$$\mathbb{Z}\{[N]\}$$

So $f_*[M] = \deg(f) \cdot [N]$ for some $\deg(f) \in \mathbb{Z}$.

- This generalises degree of maps $S^d \xrightarrow{f} S^d$
- Can calculate $\deg(f)$ by local degrees exactly as for spheres

Corollary: Let $\mathbb{F}$ be a field and $\deg(f) \neq 0$ in $\mathbb{F}$, then
$$f^*: H^*(N;\mathbb{F}) \longrightarrow H^*(M;\mathbb{F}) \quad \text{is injective.}$$

Proof: Let $\alpha \in H^k(N;\mathbb{F})$ satisfy $f^*(\alpha) = 0$, $\alpha \neq 0$.

The pairing $H^k(N;\mathbb{F}) \otimes H^{d-k}(N;\mathbb{F}) \longrightarrow \mathbb{F}$ is non-singular by PD,

i.e. $\exists \beta \in H^{d-k}(N;\mathbb{F})$ s.t. $\langle \alpha, \beta \rangle \neq 0$ i.e. $(\alpha \smile \beta)[N] \neq 0$.

Then
$$\begin{aligned} \deg(f) &= (\alpha \smile \beta)(\deg(f)[N]) \\ &= (\alpha \smile \beta)(f_*[M]) \\ &= f^*(\alpha \smile \beta)[M] \\ &= (\underset{0}{f^*\alpha} \smile f^*\beta)[M] = 0. \quad \#. \end{aligned}$$
□

Submanifolds Let M^d be a smooth compact R-oriented manifold, $N^n \subset M$ a smooth compact R-oriented submanifold, $i: N \hookrightarrow M$.

① Get $i_*[N] \in H_n(M;R)$. Abuse notation and call this $[N]$ too.

② Choosing an inner product on TM, have $i^*TM \cong TN \oplus \nu_{N \subset M}$, and by the tubular nbhd theorem, $\nu_{N\subset M} \cong \mathcal{U}$ a nbhd of N in M.

$N^n \subset M^d$ R-oriented compact manifolds

$$i : N \hookrightarrow M$$

② Choosing an inner product on TM, get

$$i^* TM = TN \oplus \nu_{N\subset M}$$

By the tubular ngbd thm there is a homeo $\nu_{N\subset M} \cong U$ for U an open ngbd of N.

The orientation of M induces one on the open submanifold U.

$$H^i(\nu_{N\subset M}, \nu^\#_{N\subset M}; R) \xrightarrow{\cong} H^i_c(\nu_{N\subset M}; R) \cong H^i_c(U;R)$$

(the first isomorphism: as in the proof of $H^i(\mathbb{R}^d|0) \to H^i_c(\mathbb{R}^d)$)

$$H^i_c(U;R) \xrightarrow[\cong]{D_U} H_{d-i}(U;R) \xleftarrow[\cong]{inc_*} H_{d-i}(N;R) \quad (\text{as } \nu_{N\subset M} \xrightarrow{\simeq} N)$$

$$\varepsilon_{N\subset M} \in H^{d-n}(\nu_{N\subset M}, \nu^\#_{N\subset M}; R) \longleftarrow [N] \in H_n(N;R)$$

Check $\varepsilon_{N\subset M}$ is a Thom class for $\nu_{N\subset M}$ ← HOW!?

So this bundle is R-oriented

③ $$H_n(N;R) \xrightarrow[\cong]{} H_n(U;R) \xleftarrow[\cong]{D_U} H^{d-n}_c(U;R) \xrightarrow{\text{ext by } 0} H^{d-n}(M;R)$$

$$[N] \longmapsto D_M^{-1}([N])$$

so $$H^{d-n}(\nu_{N\subset M}, \nu^\#_{N\subset M}; R) \xrightarrow{\cong} H^{d-n}_c(\nu_{N\subset M}; R) \cong H^{d-n}_c(U;R)$$

$$\downarrow \text{ext by } 0$$

$$\varepsilon_{N\subset M} \longmapsto D_M^{-1}([N]) \in H^{d-n}(M;R)$$

"The Poincaré dual of $[N] \in H_n(M;R)$ is the extension by 0 of the Thom class of $\nu_{N\subset M}$"

Def^n Say submanifolds $N, W \subset M$ intersect transversely if

$$\forall x \in N \cap W, \quad T_xN + T_xW = T_xM$$

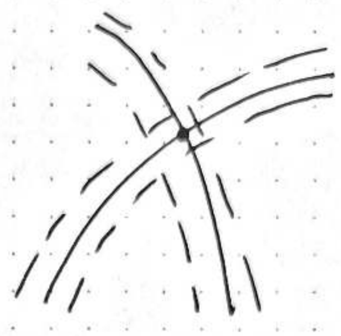
(↑ $+$: not direct in general)

In this case, $N \cap W$ is again a submanifold, and

$$(\nu_{N\cap W\subset M})_x = (\nu_{N\subset M})_x \oplus (\nu_{W\subset M})_x$$

If $i_N : N \cap W \hookrightarrow N$

$i_W : N \cap W \hookrightarrow W$



then $\nu_{N\cap W \subset M} = i_N^* \nu_{N\subset M} \oplus i_W^* \nu_{W \subset M}$,

so $i_N^*(\varepsilon_{N\subset M}) \cup i_W^*(\varepsilon_{W\subset M}) \in H^*(\nu_{N\cap W\subset M}, \nu^{\#}_{N\cap W \subset M})$

is a Thom class.

⌈ If $E, F \to X$ are v. bundles, then $(E\oplus F)^{\#} = E^{\#}\oplus F \cup E \oplus F^{\#} \subset E \oplus F$

so using $E \xleftarrow{\pi_E} E\oplus F \xrightarrow{\pi_F} F$,

$\pi_E^*(\varepsilon_E) \in H^e(E\oplus F, E^{\#}\oplus F)$

$\pi_F^*(\varepsilon_F) \in H^f(E\oplus F, E\oplus F^{\#})$

so $\pi_E^* \varepsilon_E \cup \pi_F^* \varepsilon_F \in H^{e+f}(E\oplus F, (E\oplus F)^{\#})$ is a Thom class ⌋

Then we get

$$D_M(D_M^{-1}[N] \cup D_M^{-1}[W]) = [N\cap W] \in H_*(M;R)$$

(with induced orientation, on $[N\cap W]$)

by interpretation via tubular ngbd theorem.

" Cup product is Poincaré dual to intersection "

<u>Example</u> the classes $y_k \in H^{2k}$ ö

$$[\mathbb{CP}^k] \in H_{2k}(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$$

(with $y_k = [\mathbb{CP}^k]$)

are generators (by cellular H_*) calculation.

If $\mathbb{CP}^{k_1} \subseteq \mathbb{CP}^n$ is given by first k_1+1 hom coords

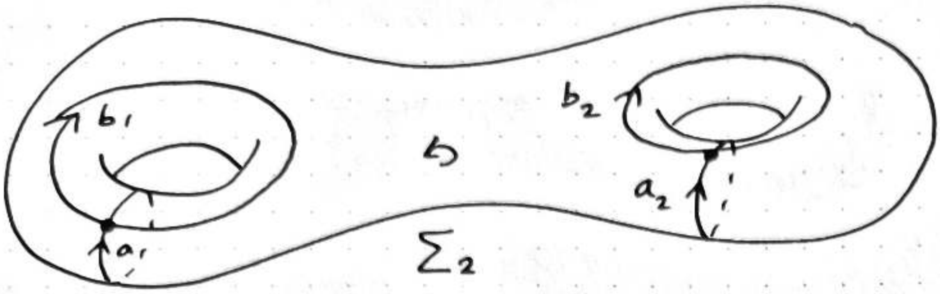
$\mathbb{CP}^{k_2}$ .. last k_2+1 hom coords

Then $\mathbb{CP}^{k_1} \cap \mathbb{CP}^{k_2} \cong \mathbb{CP}^{k_1+k_2-n}$

So $D(y_{k_1}) \cup D(y_{k_2}) = D(y_{k_1+k_2-n})$ determines cup product structure in a different way.

(where $D(y_{k_1}) = x^{n-k_1}$, $D(y_{k_2}) = x^{n-k_2}$, $D(y_{k_1+k_2-n}) = x^{2n-k_1-k_2}$)

Example



This has $a_i \cap b_i = pt_i$ with the correct choice of orientations, $+1 = [a_i \cap b_i] \in H_0(\Sigma_g, \mathbb{Z})$

Visibly, $a_i \cap \binom{a_j}{b_j} = \emptyset$ for $j \neq i$ etc.

So if $\alpha_i = D_{\Sigma_g}(a_i)$, $\beta_i = D_{\Sigma_g}(b_i)$

then $\alpha_i \cup \beta_i = [\Sigma_g]$

$\alpha_i \cup \binom{\alpha_j}{\beta_j} = 0$ for $j \neq i$ $\Rightarrow$ ring structure of $H^*(\Sigma_g; \mathbb{Z})$

Euler class, Euler characteristic and the diagonal

M a closed compact $\mathbb{Z}$-oriented smooth manifold.
(so $\mathbb{Q}$-oriented)

Let $\{a_i\}_{i \in I}$ be a homogeneous basis for $H^*(M; \mathbb{Q})$.

Let $\{b_i\}_{i \in I}$ be the dual basis wrt $\langle -,- \rangle : H^i \otimes H^{d-i} \to \mathbb{Q}$

$$a \otimes b \mapsto (a \cup b)[M]$$

i.e. $\langle a_i, b_j \rangle = \delta_{ij}$.

Consider $\Delta : M \longrightarrow M \times M$.
$x \mapsto (x,x)$

$M \times M$ is also a manifold, compact, $2d$-dim, $\mathbb{Z}$-oriented, inherited from M.

So get a cohomology class

$$\delta := D^{-1}_{M \times M}(\Delta_*[M]) \in H^d(M \times M; \mathbb{Q})$$

Theorem: $\delta = \sum_{i \in I} (-1)^{|a_i|} a_i \times b_i$

$\Delta : M \longrightarrow M \times M$
$x \longmapsto (x, x)$

$\{a_i\}$ basis of $H^*(M;\mathbb{Q})$

$\{b_i\}$ dual basis : $\langle a_i, b_j \rangle = \delta_{ij}$

$\delta := D^{-1}_{M\times M}(\Delta_*[M])$

<u>Theorem</u> : $\delta = \sum_{i\in I} (-1)^{|a_i|} a_i \times b_i$

<u>Proof</u> : $\delta = \sum c_{ij}\, a_i \times b_j$, some c_{ij}

$H^*(M\times M;\mathbb{Q}) \cong H^*(M,\mathbb{Q}) \otimes H^*(M,\mathbb{Q})$ Künneth

$((b_k \times a_l) \cup \delta)[M\times M]$

$\|$

$\sum_{i,j} c_{ij} (b_k \times a_l) \cup (a_i \times b_j)[M\times M]$

$\|$

$\sum_{i,j} c_{ij} (-1)^{|a_i||a_l|} (b_k \cup a_i) \times (a_l \cup b_j)[M\times M]$

$\|$

$\sum_{i,j} c_{ij} (-1)^{|a_i||a_l|} (-1)^{|b_k||a_i|} \underbrace{(a_i \cup b_k)[M]}_{\delta_{ik}} \cdot \underbrace{(a_l \cup b_j)[M]}_{\delta_{lj}}$

what ?! maybe use homology cross product $[M\times M] = [M]\times[M]$

$\|$

$c_{kl} (-1)^{|a_k||a_l|} (-1)^{|b_k||a_k|}$

Also, $((b_k \times a_l)\cup\delta)[M\times M]$

$= (b_k \times a_l)(\delta \cap [M\times M])$

$= (b_k \times a_l)(\Delta_*[M])$

$= (b_k \cup a_l)[M]$

$= (-1)^{|b_k||a_l|} \underbrace{(a_l \cup b_k)[M]}_{\delta_{lk}}$

So $c_{kl} = \delta_{kl} \cdot (-1)^{|a_k||a_k| + \cancel{|a_k||b_k|} + \cancel{|b_k||a_k|}}$

$= \delta_{kl} \cdot (-1)^{|a_k|}$. ∎

<u>Corollary</u>: $\Delta^*(\delta)[M] = \chi(M)$

<u>Proof</u>: $\Delta^*(\delta) = \sum_i (-1)^{|a_i|} a_i \cup b_i$

So $\Delta^*(\delta)[M] = \sum_i (-1)^{|a_i|} = \sum_{j=0}^{d} (-1)^j \dim H^j(M;\mathbb{Q}) = \chi(M)$. ▨

If M is smooth, then $D\Delta : TM \longrightarrow T(M\times M) = \pi_1^* TM \oplus \pi_2^* TM$ is the diagonal map, so $\nu_{\Delta(M)\subset M\times M} \cong TM$

If $\Delta(M) \subset \mathcal{U} \cong \nu_{\Delta(M)\subset M\times M}$ is a tubular ngbd, then can form

$$\delta_{\mathcal{U}} = D_{\mathcal{U}}^{-1}(\Delta_*[M]) \in H_c^d(\mathcal{U}) \cong H_c^d(\nu_{\Delta(M)\subset M\times M})$$
$$\cong H^d(\nu_{\Delta(M)\subset M\times M}, \nu^{\#}_{\Delta(M)\subset M\times M})$$

which is a Thom class for $\nu_{\Delta(M)\subset M\times M} \cong TM$.

Now, δ is the extension by 0 of $\delta_{\mathcal{U}}$ to the whole of $M\times M$

So $\Delta^*(\delta) = \Delta^*(\delta_{\mathcal{U}}) =$ pullback of Thom class of normal bundle along 0-section

$= e(\nu_{\Delta(M)\subset M\times M})$

$= e(TM)$

$\Rightarrow e(TM)[M] = \chi(M)$ whence "Euler class"

<u>Corollary</u>: If M has a nowhere zero vector field, then $\chi(M) = 0$.

<u>Pf</u>: Under the hypothesis, $e(TM) = 0$. ▨

~Poincaré-Hopf

We saw that a Thom class of $E \oplus F \to X$ can be given by the product of Thom classes.

So $e(E\oplus F) = e(E) \cup e(F)$.

<u>Corollary</u>: $TS^{2n} \to S^{2n}$ has no proper sub-bundles.

<u>Pf</u> $e(TS^{2n})[TS^{2n}] = 2$. But if $TS^{2n} = E \oplus F$ then $e(TS^{2n}) = e(E) \cup e(F)$, but there are no non-trivial cup products in $H^*(S^{2n})$. □

Lefschetz fixed point theorem

Let M be a $\mathbb{Z}$-oriented compact manifold,

$f: M \to M$ s.t. $\Gamma_f = \{(x, f(x)) \in M \times M \mid x \in M\}$ is a submanifold of $M \times M$, transverse to the diagonal $\Delta(M) \subset M \times M$.

Then
$$\sum_{x \in Fix(f)} sign(x) = \sum_{i=0}^{d} (-1)^i \, Tr(f^*: H^i(M;\mathbb{Q}) \circlearrowleft)$$

where $sign(x) = \det(I - D_x f: T_x M \to T_x M)$. ⌜sign?⌟

<u>Proof</u>: Compute $[\Gamma_f]\cdot[\Delta(M)] \in H_0(M \times M; \mathbb{Q})$

So
$$\varepsilon([\Gamma_f]\cdot[\Delta(M)]) = \Big(D^{-1}_{M\times M}([\Gamma_f]) \cup \underbrace{D^{-1}_{M\times M}([\Delta(M)])}_{\delta}\Big)[M \times M]$$

$-\cup-$ on H^* translates under PD to a product $-\cdot-$ on H_*

which using $-\cup-$ vs $-\cap-$ is the same as

$$(D^{-1}_{M\times M})(\Delta(M))[\Gamma_f] = \delta((id \times f)_*[M])$$
$$= ((id \times f)^* \delta)[M]$$
$$= \sum (-1)^{|a_i|}(a_i \cup f^* b_i)[M]$$

Let $f^* b_i = \sum C_{ij} b_j$. Then get

$$= \sum_i (-1)^{|a_i|}(a_i \cup \sum_j C_{ij} b_j)[M]$$
$$= \sum_i (-1)^{|a_i|} C_{ii}$$
$$= \sum_r (-1)^r \, Tr(f^*: H^r(M, \mathbb{Q}) \circlearrowleft)$$ ⌜sign?⌟
$$= RHS.$$

On the other hand, as Γ_f and $\Delta(M)$ are transverse, $[\Gamma_f]\cdot[\Delta(M)]$ is represented by the 0-manifold

$$\Gamma_f \cap \Delta(M) = \{(x,x) \in M \times M \mid x = f(x)\} =: Fix(f)$$

The orientations of $\Delta(M) \cong M$ and $\Gamma_f \cong M$ induce an orientation

At $x \in Fix(f)$, this is

$$\det\begin{pmatrix} I & I \\ D_x f & I \end{pmatrix} = \det\begin{pmatrix} I & 0 \\ D_x f & I - D_x f \end{pmatrix}$$

$$= \det(I - D_x f) .$$ ▨

<u>Example</u>. Any $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ has a fixed point.

If not, then RHS in formula is zero.

But $f^* : H^*(\mathbb{C}P^{2n}; \mathbb{Q}) \circlearrowleft$

$$H^*(\mathbb{C}P^{2n}; \mathbb{Q}) = \mathbb{Q}[x]/(x^{2n+1})$$

So $f^* x = \lambda x$ for some $\lambda \in \mathbb{Z}$. [$\mathbb{Z}$-coeffs]

$\therefore f^*(x^i) = \lambda^i x^i$

$\Rightarrow \sum (-1)^i \operatorname{tr}(f^* : H^i \circlearrowleft) = 1 + \lambda^2 + \cdots + \lambda^{2n} = 0$

has no integer solution (2n even). ⋇