Complex Manifolds

A.G.Kovalev@dpmms.cam.ac.uk

- prerequisites : · Differential Geometry e.g. in Mich Term (Part III)
  · basic complex analysis (holomorphic functions)

Remarks Riemann surfaces is useful, but not essential
(1-dim cx mfds)
Algebraic geometry (Part III) is related ; but not really used

Books Huybrechts "Complex geometry"
Griffist & Harris "Principles of algebraic geometry" Ch. 0 & 1
printed notes — later

- 4 example sheets & examples classes

Recall : smooth (real) n-dim mfd M – Hausdorff 2^{nd}-countable top. space with an atlas of charts

homeo. $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$, with $U_\alpha \subset M$ open (connected), $V_\alpha \subset \mathbb{R}^n$

$M = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$

$\varphi_\beta \circ \varphi_\alpha^{-1}$ smooth (C^∞) on its domain $\subset \mathbb{R}^n$

~ smooth structure

- Basic idea : replace $\mathbb{R}^n$ with $\mathbb{C}^n$
  " C^∞ with holomorphic (cx. analytic)

Need some basics of several complex variables.

Recall first about one cx variable.

$U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ smooth in $\mathbb{R}$-sense

f is holomorphic iff · cx analytic $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ valid on $\{|z-a| < \varepsilon\}$

· cx diff'ble ($\mathbb{R}$-smooth + Cauchy-Riemann)

- $z = x + iy$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Cauchy–Riemann : $\dfrac{\partial f}{\partial \bar z} \equiv 0$ on $\mathcal{U}$

- A general smooth function

$$f(z) = f(0) + \frac{\partial f}{\partial z}(0)\, z + \frac{\partial f}{\partial \bar z}(0)\, \bar z + o(|z|)$$

as $|z| \to 0$

the differential $\quad df = \partial f + \bar\partial f$

$$:= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$$

where $\quad dz = dx + i\,dy$

$\quad d\bar z = dx - i\,dy$

- Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{w-z}\, dw \qquad \left(\text{if } \{|w-z| \leq r\} \text{ lies in } \mathcal{U}\right)$$

Now let $\mathcal{U} \subset \mathbb{C}^n$ open, $f : \mathcal{U} \to \mathbb{C}$ of real class C^1

Say f is <u>holomorphic</u> if

$$g_j(z) := f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$$

is holomorphic in z for all $j, z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$

i.e. $\dfrac{\partial f}{\partial \bar z_j} \equiv 0$ for all j

where $\dfrac{\partial}{\partial \bar z_j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} + i \dfrac{\partial}{\partial y_j}\right)$, $\dfrac{\partial}{\partial z_j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} - i\dfrac{\partial}{\partial y_j}\right)$

$z_j = x_j + i y_j$

Short hand : $\bar\partial f = 0$

N.B. often convenient to set $\mathcal{U}$ is a polydisc

$$\Delta_1 \times \dots \times \Delta_n = \{ z \in \mathbb{C}^n : |z_j - a_j| < r_j \ \forall j \}$$

for $a_j \in \mathbb{C}$, $r_j \geq 0$

Cauchy Integral Formula

If $f: \underbrace{\overline{\Delta_1 \times \cdots \times \Delta_n}}_{\Delta} \longrightarrow \mathbb{C}$ holo.

then
$$f(z) = \frac{1}{(2\pi i)^n} \int_{|w_j - a_j| = r_j} \frac{f(w)}{(w_1 - a_1)\cdots(w_n - a_n)} \, dw_1 \cdots dw_n$$

for $z \in \Delta$, suitable r_j

N.B. Integrating over submanifold $\subsetneq \partial\Delta$ (proper when $n > 1$)

Proof (gist.) Can do repeated integration in each w_j treat $w_{j+1}, \dots, w_n$ as parameters. QED

Power series holo $\iff$
$$f(z) = \sum_{i_1, \dots, i_n = 0}^{\infty} \frac{\partial^{i_1 + \cdots + i_n} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}(a) \cdot (z_1 - a_1)^{i_1} \cdots (z_n - a_n)^{i_n} \frac{1}{i_1! \cdots i_n!}$$

Def^n:
$f = (f_1, \dots, f_n) : \underset{\mathbb{C}^n}{\overset{\text{open}}{U}} \longrightarrow \mathbb{C}^m$ ($U \subset \mathbb{C}^n$)

is holomorphic if each f_j is so.

Now biholomorphic means bijective and f, f^{-1} holo

Complex Jacobian of a holo function $f = (f_1, \dots, f_m)$

$$J(f)_z := \left(\frac{\partial f^k}{\partial z_j}(z) \right)_{\substack{k=1,\dots,m \\ j=1,\dots,n}}$$ defines a complex linear map $\mathbb{C}^n \to \mathbb{C}^m$

If $J(f)_z$ surjective then say z is a regular point for f

Say $w \in \mathbb{C}^n$ is a regular value if $\forall z \in f^{-1}(w)$, z is a regular point.

Suppose $m = n = 1$, $f = u + iv$.

Then
$$J_{\mathbb{R}}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \underset{\substack{\text{similar} \\ \text{cx } 2\times 2 \\ \text{matrices}}}{\sim} \begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \bar{f}}{\partial \bar{z}} \end{pmatrix}$$

(f is holo)

If $\alpha, \beta \in \mathbb{R}$ then

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$$

Can extend this to dim's $m, n > 1$

$$J_{\mathbb{R}}(f) \sim \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$$ for any <u>holo</u> f

When $m = n$, taking det gives

$$\det J_{\mathbb{R}}(f) = \det J(f) \cdot \det \overline{J(f)}$$
$$= \left| \det J(f) \right|^2 \geqslant 0$$

Is positive when $J_{\mathbb{R}}(f)$ is non-singular.

(Holo.) Inverse f^n thm

$U, V \subseteq \mathbb{C}^n$ open, $f : U \to V$ holo, $z \in U$ regular

Then $\exists$ ngbd U_0 of z s.t. f maps U_0 biholomorphically onto its image.

Examples Class 1 on 10^{th} Feb, 3:30 – 5 pm

Laurence Mayther <llim32@ cam.ac.uk>

Def A complex n-fold M is a Hausdorff, 2^{nd}-countable topological space equipped with complex coord charts:

homeo $\varphi_i : U_i \longrightarrow \varphi_i(U_i)$; both $U_i, \varphi_i(U_i)$ open and connected
(U_i in M, $\varphi_i(U_i)$ in $\mathbb{C}^n$)

s.t. $\forall i, j$, $\varphi_j \circ \varphi_i^{-1}$ are holomorphic on $\varphi_i(U_i \cap U_j)$

and $M = \bigcup_i U_i$.

$p \in U_i$, $\varphi_i(p) = (z_1, \ldots, z_n)$ cx local coords

N.B. can think of M as a real $2n$-mfd with a choice of holomorphic atlas $\{(\varphi_i, U_i)\}$

Def M, N cx mfds; φ_i, ψ_α respective cx coord charts

Given a cts map $F : M \longrightarrow N$, we say F is holomorphic if each $\psi_\alpha \circ F \circ \varphi_i^{-1}$ is holomorphic on its domain

$$\varphi_i(U_i \cap \underbrace{F^{-1}(W_\alpha)}_{\substack{\uparrow \\ \text{domain} \\ \text{of } \psi_\alpha}}) \subseteq \mathbb{C}^n$$

M, N are bihdomorphic or isomorphic if $\exists$ biholo $F : M \longrightarrow N$
(in fact it suffices if F is a holomorphic bijection) ← harder than in dimension 1, see [H] 1.1.13

Prop^n Let M be a compact cx mfd.

Then any holo. $f : M \longrightarrow \mathbb{C}$ is constant.

Proof $|f| : M \rightarrow \mathbb{R}$ is cont's

$\Rightarrow$ attains a maximum, say at $p \in M$

Let $\varphi : U \rightarrow \Delta \subseteq \mathbb{C}^n$ be a cx coord chart about p.
(Δ ← polydisc)

$f \circ \varphi^{-1}$ satisfies maximum modulus principle on Δ [ES1, Q1]

So f is constant on U.

M is covered by finitely many charts (as compact)

– repeat the above argument for each chart ! connected

$\Rightarrow$ f is const on M Q.E.D.

Examples of cx mfds

0th Trivially, any open subset of $\mathbb{C}^n$

1st 1-dim cx mfd is a Riemann surface

classified by the Uniformization Theorem:

the Riemann sphere $\mathbb{CP}^1 (= S^2)$,

$\mathbb{C}$, $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$, elliptic curves $\mathbb{C}/\Lambda$ $(\cong S^1 \times S^1)$

and Δ/Γ where

Lattice in $\mathbb{C}$

$\Lambda = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$

for $\lambda_1/\lambda_2 \notin \mathbb{R}$

$\Delta = \{ |z| < 1 \} \subset \mathbb{C}$ and

Γ is a subgroup of Möbius

transformations action on Δ properly discontinuously

More generally, the quotient construction of cx mfds

– see Sheet 1, Q2

2nd $\mathbb{CP}^n$ (or just $\mathbb{P}^n$), cx projective spaces

$= \{$ 1-dim cx subspaces of $\mathbb{C}^{n+1} \}$

$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$

Points in $\mathbb{CP}^n$ are $[z_0 : \cdots : z_n]$ $(= [\lambda z_0 : \cdots : \lambda z_n] \ \forall \lambda \in \mathbb{C}^*)$

$U_i = \{ [z_0 : \cdots : z_n] \mid z_i \neq 0 \}$, $i = 0, \dots, n$

$\varphi_i : U_i \to \mathbb{C}^n$, $[z_0 : \cdots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i} \right)$

$\varphi_j \circ \varphi_i^{-1} (w_0, \dots, w_n) = \left(\frac{w_0}{w_j}, \dots, \frac{w_{i-1}}{w_j}, \frac{1}{w_j}, \frac{w_{i+1}}{w_j}, \dots, \widehat{\frac{w_j}{w_j}}, \dots, \frac{w_n}{w_j} \right)$

! $(j > i)$ ⌜skip w_i⌟

Very important cx mfd $\mathbb{CP}^n$:

(1) compact cx mfds never embed (holo) in $\mathbb{C}^n$ but some do embed in $\mathbb{CP}^n$ – they're called projective

__Proof of (1)__ If M is a cpct cx mfd and
$\iota : M \longrightarrow \mathbb{C}^n$ is a holo embedding
then $\forall j$, $\pi_j \circ \iota : M \longrightarrow \mathbb{C}$ is holo, hence constant
projection onto j^{th} coord
So $\iota(M)$ is one point ⋇ Q.E.D.

__Easy example__: making S^2 into a cx mfd $\mathbb{CP}^1$

$S^2 = \{X^2+Y^2+Z^2=1\} \subset \mathbb{R}^3$

$$f : (X,Y,Z) \longmapsto \begin{cases} \left[\frac{X+iY}{1-Z} : 1\right] & \text{if } Z \neq 1 \\ \left[1 : \frac{X-iY}{1+Z}\right] & \text{if } Z \neq -1 \end{cases}$$

Can check $f : S^2 \to \mathbb{CP}^1$ is a diffeo
The induced charts on S^2 are stereographic projection from $(0,0,\pm 1)$

__3rd__ __Complex tori__ $\mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is a lattice in $\mathbb{C}^n$, discrete additive subgroup

quotient topology
(Hausdorff, 2nd countable, and compact)

Charts: local inverses of the __quotient map__

$\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n/\Lambda$
$\underline{z} \longmapsto \underline{z} + \Lambda$

$D_j \subset \mathbb{C}^n$ suff small open ball
$\varphi_i : \pi(D_j) \longrightarrow D_j$

any trans. function is ~~locally~~ $\varphi_j \circ \varphi_i^{-1}(\underline{z}) = \underline{z} + \lambda_{ij}(\underline{z})$
$\lambda_{ij} : V \subset \mathbb{C}^n \longrightarrow \Lambda$ locally const.
cts open

Already for $n=1$, cx tori are generally not biholomorphic.

4th Hopf surface

- $H^2 = \mathbb{C}^2 \setminus \{0,0\} / \mathbb{Z} \sim 2\mathbb{Z}$

a complex manifold ; use Q2 sheet 1

diffeo to $S^3 \times S^1$ as a real manifold

Can show (later) that H^2 is not projective.

Can define H^n for each $n \geq 1$

H^1 is biholomorphic to an elliptic curve (1-dim cx torus)

5th Complex Grassmannians

- Let V be a n-dim cx vector space

$Gr_k(V) := \{ k\text{-dim cx vector subspaces } W \leq V \}$, $k < n$

(e.g. $V = \mathbb{C}^{n+1}$, $k=1$ gives $\mathbb{C}\mathbb{P}^n$)

A $W \in Gr_k(V)$ may be given as a complex $k \times n$ matrix of $\text{rank}_{\mathbb{C}} = k$ (choice of basis)

May diagonalise a non-singular $k \times k$ part

Obtain $k(n-k)$ "free parameters"

Idea of making a holo. atlas

- $Gr_k(V)$ is compact of dimension $k(n-k)$

Also projective — Q6 sheet 1

Proof of compactness

Let $(\mathbb{C}^{k,n})^* = \{ \text{lin. indep. } k\text{-tuples in } \mathbb{C}^n \} \subset \mathbb{C}^{kn}$ open

$\pi : (\mathbb{C}^{k,n})^* \longrightarrow Gr_k(\mathbb{C}^n)$ induces the quotient topology on $Gr_k(\mathbb{C}^n)$

π is a cts map

$(\mathbb{C}^{k,n})^*_u = \{ \text{the orthon. } k\text{-tuples} \} \subset \mathbb{C}^{kn}$ closed & bounded

- $Gr_k(\mathbb{C}^n) = \pi((\mathbb{C}^{k,n})^*_u)$ thus compact

q.e.d.

6th, complex Lie groups G s.t. $G\times G \longrightarrow G$
$(g,h) \longmapsto gh^{-1}$
is a holomorphic map.

E.g. $GL(n,\mathbb{C})$ open set in $\mathbb{C}^{n^2}$

$SO(n,\mathbb{C})$ the proof is similar to the real $SO(n)$

⚠ $U(n)$ is <u>not</u> a cx mfd

Tangent spaces & holo. tangent bundle

Let M be a complex n-manifold.

Then M is a real $2n$-manifold.

Let $p\in M$ and $\{z_j = x_j + iy_j\}$ be local cx coords, $j=1,\dots,n$

So $\{x_j, y_j\}$ give real coords.

The (real) tangent space T_pM has basis

$$\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\}_{j=1}^{n}$$

Set $J_p \in GL_{\mathbb{R}}(T_pM) \subset \mathrm{End}_{\mathbb{R}}(T_pM)$

to be
$$J_p\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$$
$$J_p\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

Then $J_p \circ J_p = -\mathrm{id}_{T_pM}$.

Complexify the tangent space:

$T_pM \otimes_{\mathbb{R}} \mathbb{C}$, cx v.s. with basis $\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\}_{j=1}^{n}$

Consider the cx linear extension $J_p \in GL_{\mathbb{C}}(T_pM \otimes \mathbb{C})$

Then still have $J_p^2 = -1$.

So $T_pM\otimes\mathbb{C}$ splits as the sum of the eigenspaces

$$T_p^{1,0}M = \{v\in T_pM\otimes\mathbb{C} \mid J(v) = iv\}$$ "holomorphic tangent space"

(*) $$T_p^{0,1}M = \{v\in T_pM\otimes\mathbb{C} \mid J(v) = -iv\}$$ "anti-holomorphic tangent space"

Now cx conj is a real linear map $T_pM \otimes \mathbb{C} \to T_pM \otimes \mathbb{C}$

$$v \otimes \lambda \mapsto v \otimes \bar{\lambda},$$

moreover an isomorphism, and interchanges $T_p^{1,0}M$, $T_p^{0,1}M$ (J has real coeffs)

<u>Prop^n</u> (i) $\dim_{\mathbb{C}} T_p^{1,0}M = \dim_{\mathbb{C}} T_p^{0,1}M = n$

(ii) J_p and hence $(*)$ is independent of choice of holomorphic coords.

Moreover $p \mapsto J_p$ defines a smooth section of $\mathrm{End}_{\mathbb{R}}(TM)$.

<u>Proof</u> (i) Change to the basis

$$\left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}_{j=1}^{n} ; \quad \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Then $J\left(\frac{\partial}{\partial z_j}\right) = i\,\frac{\partial}{\partial z_j}$ etc.

So $T_p^{1,0}(M)$ has basis $\left\{\frac{\partial}{\partial z_j}\right\}_{j=1}^{n}$

$T_p^{0,1}(M)$ " $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}_{j=1}^{n}$

In fact $T_p^{1,0}M = \{ v - iJv \mid v \in T_pM \}$

$T_p^{0,1}M = \{ v + iJv \mid v \in T_pM \}$

each is isomorphic to T_pM as a real v.s.

(ii) recall that real tangent vectors $\leftrightarrow$ derivations

$$\left(\sum X^i \frac{\partial}{\partial x^i}\right) f, \quad f \in C^\infty(M)$$

acting on $C^\infty(M, \mathbb{R})$.

cx. tangent vectors - resp. act on $C^\infty(M, \mathbb{C})$

$T^{0,1}$ = derivations vanishing precisely on holomorphic functions on M

$T^{1,0}$ = " antiholomorphic "

$\Rightarrow$ the $\pm i$ eigenspaces of J are invariantly defined

$\Rightarrow$ J is invariantly defined.

(Smoothness follows from expression in local coords) q.e.d.

Lemma On overlapping cx coord ngbds with coords (z_j), (w_j) we have

$$\frac{\partial}{\partial w_j} = \sum_k \frac{\partial z_k}{\partial w_j} \frac{\partial}{\partial z_k}$$

$$\frac{\partial}{\partial \bar{w}_j} = \sum_k \frac{\partial \bar{z}_k}{\partial \bar{w}_j} \frac{\partial}{\partial \bar{z}_k}$$

have seen
$$\frac{\partial}{\partial w_k} = \sum_{j=1}^{n} \frac{\partial z_j}{\partial w_k}\frac{\partial}{\partial z_j}, \qquad \frac{\partial}{\partial \bar{w}_k} = \sum_{j=1}^{n} \frac{\partial \bar{z}_j}{\partial \bar{w}_k}\frac{\partial}{\partial \bar{z}_j}$$

Recall that for a holo. $f : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^n$
the Jacobian $J_{\mathbb{R}}(f)$ wrt $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ is similar,
via $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, to
$$\begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$$

Obtain <u>Prop^n</u> Every real mfd underlying a complex mfd is canonically oriented

<u>Proof</u> Indeed
$$\det J_{\mathbb{R}}(f) = |\det J(f)|^2 > 0$$
holds at every $z = f(w)$ on overlaps of coord nghbds.

Q.E.D.

$$\bigsqcup_{p\in M} T_p^{1,0}M = T^{1,0}M$$
is the <u>holomorphic tangent bundle</u> of M
(a complex sub-bundle of $TM \otimes \mathbb{C}$)
sections of $T^{1,0}M$ act on $C^\infty(M, \mathbb{C})$

<u>Def</u> A section $\xi \in \Gamma(T^{1,0}M)$ is a <u>holomorphic vector field</u> if $\forall f \in C^\infty(M, \mathbb{C})$ holo.
then ξf is again holomorphic

$$\bigsqcup_{p\in M} T_p^{0,1}M = T^{0,1}M$$
is the <u>antiholomorphic tangent bundle</u> of M

Note $f : U \subset M \longrightarrow \mathbb{C}$ is holo. iff $\xi f = 0 \;\; \forall \xi \in \Gamma(T^{0,1}M)$

Recall $J \in \Gamma(\mathrm{End}\, TM)$ (Prop^n(ii) J is well-defined)

<u>Remark</u> Have standard representation of $GL(n, \mathbb{C})$ on $\mathbb{R}^{2n}$
i.e. injective hom $GL(n;\mathbb{C}) \xrightarrow{\Phi} GL(2n, \mathbb{R})$
Each cx entry $a+ib$ becomes a $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\phi(GL(n,\mathbb{C}))$ = subgroup of $GL(2n,\mathbb{R})$ commuting with $J_0 = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}$

$\forall f$ have that a change of cx coords given that Cauchy-Riemann for $f \Rightarrow J_{\mathbb{R}}(f) \in \phi(GL(n,\mathbb{C}))$

Then a holomorphic atlas of M, via J, induces a <u>reduction</u> of the structure group of v.b. TM from $GL(2n,\mathbb{R})$ to $GL(n,\mathbb{C})$.

i.e. make TM into a <u>cx vector bundle</u> ($\operatorname{rank}_{\mathbb{C}} = n$)

This cx v.b. is isomorphic to $T^{1,0}M$ via

$$v \in TM \longmapsto (v - iJv) \in T^{1,0}M$$

locally induced by

$$a\frac{\partial}{\partial x_k} + b\frac{\partial}{\partial y_k} \mapsto 2(a+ib)\frac{\partial}{\partial z_k}$$

Using $-J$ in place of J, get isomorphism of cx vector bundles $TM \longrightarrow T^{0,1}M$.

[⚠ $J \mapsto -J$ changes the cx v.b. structure on TM]

<u>Recall</u>: if $f: M \to N$ is smooth between real manifolds, then $d(f)_p : T_pM \longrightarrow T_{f(p)}N$ linear

Consider cx extension $T_pM \otimes \mathbb{C} \longrightarrow T_{f(p)}N \otimes \mathbb{C}$ still denoted $(df)_p$.

<u>Proposition</u> For a smooth map between cx mfds $f: M \to N$, t.f.a.e.

(i) f is holomorphic

(ii) $df \circ J_M = J_N \circ df$

(iii) $df(T^{1,0}M) \subseteq T^{1,0}N$

(iv) $df(T^{0,1}M) \subseteq T^{0,1}N$

Proof all (i)–(iv) are local $\Rightarrow$ suffices to consider

$f : \underset{\overset{\cap \text{ open}}{\mathbb{C}^n}}{U} \longrightarrow \mathbb{C}^m$ Use real bases $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$

on $\mathbb{C}^n \ (= \mathbb{R}^{2n})$,

$\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_m}, \frac{\partial}{\partial v_m}$

on $\mathbb{C}^m \ (= \mathbb{R}^{2m})$

(i) f holo $\Leftrightarrow$ Cauchy-Riemann on U

$\Leftrightarrow$ $(df)_p$ expressed as $J_{\mathbb{R}}(f)_p$ consisting of blocks of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$

$T^{1,0}_p$ is spanned over $\mathbb{C}$ by $\underline{e}_k = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ -i \\ 0 \\ \vdots \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow (2k-1)^{st} \text{ entry} \\ \leftarrow (2k)^{th} \text{ entry} \\ \\ \\ \end{array}$

$J_{\mathbb{R}}(f)\, \underline{e}_k = \begin{pmatrix} \vdots \\ a_l + i b_l \\ b_l - i a_l \\ \vdots \end{pmatrix} \begin{array}{l} \\ \leftarrow 2l-1 \\ \leftarrow 2l \\ \\ \end{array}$ again of type $(1,0)$ $\Rightarrow$ (iii)

Thus (i) $\Rightarrow$ (iii)

(iii) $\Leftrightarrow$ (iv) by cx conjugation

(iii) & (iv) $\Rightarrow$ (ii) since (df) preserves $(1,0)$ and $(0,1)$ subspaces, but J_M, J_N acts on these as $(\pm i)$ id

(ii) $\Rightarrow$ each (2×2) block $B_{kl} = \begin{pmatrix} c_{2k-1,2l-1} & c_{2k-1,2l} \\ c_{2k,2l-1} & c_{2k,2l} \end{pmatrix}$

of $(df)_p$ commutes with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow B_{kl} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$

$\Rightarrow$ Cauchy-Riemann Q.E.D.

Complexified cotangent space

$$T_p^*M \otimes_{\mathbb{R}} \mathbb{C}$$

The (dual of) J acts : $dx_j \mapsto -dy_j$

$dy_j \mapsto dx_j$ $\quad j = 1, \dots, n$

Let $dz_k = dx_k + i dy_k$; $\Leftarrow$ span $+i$ eigenspace of J

$d\bar{z}_k = dx_k - i dy_k$. $\leftarrow$ " $-i$ "

J ? ✓ thx Hans/Kenneth

$$\langle dz_k, \partial/\partial z_j \rangle = \delta_{jk}$$

$$\langle dz_k, \partial/\partial \bar{z}_j \rangle = 0$$

$$T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$$

$$dw_k = \sum_j \frac{\partial w_k}{\partial z_j} dz_j$$

$(T^*M)^{1,0}$ holomorphic cotangent bundle $\rightsquigarrow$ annihilates $T^{0,1}M$

$(T^*M)^{0,1}$ antiholo. " $\rightsquigarrow$ " $T^{1,0}M$

recall sub-bundles $(T^*M)^{1,0}$, $(T^*M)^{0,1}$ on a cx mfd M, induced by J

$$\Lambda^r(T^*M \otimes \mathbb{C}) = \bigoplus_{\substack{p+q=r \\ p,q \geq 0}} \Lambda^{p,q}(T^*M \otimes \mathbb{C})$$

where $\Lambda^{p,q}(T^*M \otimes \mathbb{C}) = \Lambda^p (T^*M)^{1,0} \otimes \Lambda^q (T^*M)^{0,1}$

complex conj. $\overline{\Lambda^{p,q}} = \Lambda^{q,p} \quad \forall q,p$

sections are $\Omega^{p,q}(M)$, cx differential forms of type (p,q)

in local coords $\sum_{I,J} a_{I,J}\, dz_I \wedge d\bar{z}_J$

where if $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$

then $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$

the induced action of J is

$$J\varphi = i^{p-q}\varphi \quad \text{for } \varphi \in \Omega^{p,q}(M)$$

$\Omega^{p,p}(M) \cap \underset{\text{real}}{\Omega^{2p}(M)} = \Omega^{p,p}_{\mathbb{R}}(M)$ real (p,p) forms (invariant under cx conj)

$K_M = \Lambda^{p,0}(T^*M \otimes \mathbb{C}) = \Lambda^n (T^*M)^{1,0}$ the <u>canonical line bundle</u> of M

$n = \dim_{\mathbb{C}} M$

$$d : \Omega^0(M)^{\mathbb{C}} \longrightarrow \Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

$d = \partial + \bar{\partial}$ where $\partial = \pi^{1,0} \circ d$, $\bar{\partial} = \pi^{0,1} \circ d$

In general $\pi^{p,q} : \Omega^*(M)^{\mathbb{C}} \longrightarrow \Omega^{p,q}(M)$

is projection along all other components of $\oplus$

Locally for a f^n f,

$$\partial f = \sum_k \frac{\partial f}{\partial z_k} dz_k, \quad \bar{\partial} f = \sum_k \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k$$

Generally for $\alpha \in \Omega^{p,q}(M)$ "pure type"

$$\partial \alpha := \pi^{p+1,q} \circ d(\alpha)$$

$$\bar{\partial} \alpha := \pi^{p,q+1} \circ d(\alpha)$$

For $\alpha \in \Omega^{p,0}(M)$, $\bar{\partial}\alpha = 0$ iff locally $\alpha = \sum_I f_I\, dz_I$ for f_I <u>holomorphic functions</u>

Call such an α a <u>holomorphic p-form</u>

holomorphic 1-forms are sometimes called <u>holomorphic differentials</u>

- <u>Lemma</u> On a cx mfd M
  (i) $\forall \eta \in \Omega^{p,q}(M)$, $d\eta = \partial\eta + \bar\partial\eta$
  (ii) $\partial^2 = \bar\partial^2 = 0$, $\partial\bar\partial = -\bar\partial\partial$
  (iii) $\bar\partial(\xi \wedge \eta) = (\bar\partial\xi) \wedge \eta + (-1)^{p+q}\xi \wedge (\bar\partial\eta)$, $\xi \in \Omega^{p,q}(M)$
  and similarly for ∂

<u>Proof</u> the statements are local

(i) easy to check in local coords

$$d(f\, dz_I \wedge d\bar z_J) = (\partial f + \bar\partial f) \wedge dz_I \wedge d\bar z_J$$

- then extend by linearity (N.B. both sides defined indep of choice of coordinates)

(ii) clear from (i) and $d^2 = 0$

(iii) wlog let $\eta \in \Omega^{p',q'}(M)$

take the $(p+p'+1, q+q')$, $(p+p', q+q'+1)$ components of $d(\xi \wedge \eta)$

then extend cx linearly to any η Q.E.D.

<u>Corollary</u> $d(\Omega^{p,q}(M)) \subseteq \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ (from (i))

Also (iii) and (i) hold without the pure type assumption.

- Sometimes convenient to (equivalently) replace $\bar\partial$, ∂ by
  d and $d^c := i(\bar\partial - \partial)$ both act on <u>real diff'l forms</u>
  $d = \partial + \bar\partial$

So $\partial = \frac{1}{2}(d + i d^c)$, $\bar\partial = \frac{1}{2}(d - i d^c)$

$(d^c)^2 = 0$, $d d^c = -d^c d = 2i\partial\bar\partial$

Recall pull-back of diff forms by smooth maps $f : M \to N$

$$f^* : \Omega^1(N)^{\mathbb{C}} \to \Omega^1(M)^{\mathbb{C}}, \quad \langle f^*\alpha, X\rangle = \langle \alpha, (df)X\rangle$$

for all v. fields X

- If f is <u>holomorphic</u> and $\alpha \in \Omega^{1,0}(M)$ (resp. $\Omega^{0,1}(M)$)

then $J\alpha = i\alpha$,

Then
$$\begin{aligned}\langle J f^*\alpha, X\rangle &= \langle f^*\alpha, JX\rangle \\ &= \langle \alpha, (df) JX\rangle \\ &\overset{!}{=} \langle \alpha, J(df)X\rangle \quad \text{by holomorphy} \\ &= \langle J\alpha, (df)X\rangle \\ &= \langle i\alpha, (df)X\rangle \\ &= \langle i f^*\alpha, X\rangle \quad \forall X\end{aligned}$$

Thus $f^*\alpha \in \Omega^{1,0}(M)$.

Similarly for $(0,1)$, further since $f^*(\xi \wedge \eta) = f^*\xi \wedge f^*\eta$

<u>Prop</u> The pullback by a holo. map f preserves the type decomposition. // Further $f^* \circ d = d \circ f^*$ $\forall$ smooth f.

If f is holo, then $\forall \eta \in \Omega^{p,q}(M)$,
$$\begin{aligned}(\bar\partial \circ f^*)\eta &= \pi^{p,q+1} \circ d \circ f^*\eta \\ &= \pi^{p,q+1} \circ f^* \circ d\eta \\ &\overset{!}{=} f^* \circ \pi^{p,q+1} \circ d\eta \\ &= (f^* \circ \bar\partial)\eta\end{aligned}$$

Obtain <u>Prop^n</u> $\bar\partial \circ f^* = f^* \circ \bar\partial$

$\partial \circ f^* = f^* \circ \partial$ whenever f is holomorphic

<u>Def^n</u> (Dolbeault cohomology)
$$H^{p,q}(M) = \frac{\ker \bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)}{\operatorname{im} \bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)}$$

<u>Cor</u> If f^* is pullback then $H^{p,q}(N) \longrightarrow H^{p,q}(M)$ is a well-defined cx lin map for f holo

M biholo. to N implies $H^{p,q}(M) \cong H^{p,q}(N)$ $\forall p,q \geq 0$

recall $H^{p,q}(M) = \dfrac{\ker \bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)}{\operatorname{im} \bar\partial : \Omega^{p,q-1}(M) \longrightarrow \Omega^{p,q}(M)}$

<u>rmks</u>: ① <u>Not</u> true in general that

$$\bigoplus_{p+q=r} H^{p,q}(M) = H^r_{dR}(M)^{\mathbb{C}}$$

② $H^{p,q}(M)$ are <u>not</u> tolpoogical invariants

Notation: for $S \subset \mathbb{R}^n$ or $\mathbb{C}^n$ and $f : S \to \mathbb{R}$,

$f \in C^\infty(S)$ means $\exists$ <u>open</u> $U \supset S$ and smooth (C^∞)

$F : U \to \mathbb{R}$ s.t. $f = F|_S$

<u>$\bar\partial$ - Poincaré lemma</u> in one complex variable

$D = \{ z \in \mathbb{C} : |z-a| < r \}$,

$g \in C^\infty(\bar D)$ ← closed disc

Then $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_D \dfrac{g(w)}{w-z}\, dw \wedge d\bar w$

is in $C^\infty(D)$ and $\dfrac{\partial f}{\partial \bar z} = g$ on D.

Need a <u>Lemma</u> (Extended Cauchy Integral Formula)

$F \in C^\infty\{|z-a| \le r\}$, and $z \in \mathbb{C}$ s.t. $|z-a| < r$

Then
$$F(z) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{F(w)}{w-z}\, dw + \frac{1}{2\pi i} \int_{|w-a|<r} \frac{\partial F}{\partial \bar w}(w) \frac{dw \wedge d\bar w}{w-z}$$

<u>Proof</u> Stokes' thm for the 1-form

$$\eta = \frac{1}{2\pi i} \frac{F(w)\, dw}{w-z}$$

on $D_\varepsilon = \{|w-a| < r\} \setminus \{|w-z| < \varepsilon\}$

$$d\eta = -\frac{1}{2\pi i} \frac{\partial F}{\partial \bar w} \frac{dw \wedge d\bar w}{w-z}$$

$$\int_{D_\varepsilon} d\eta = \int_{|w-a|=r} \eta - \int_{|w-z|=\varepsilon} \eta$$

- Now $\displaystyle\int_{|w-z|=\varepsilon} \eta = \frac{1}{2\pi} \int_0^{2\pi} F(z+\varepsilon e^{i\theta})\, d\theta \longrightarrow F(z)$ as $\varepsilon \to 0$

Pole of ~~order 1~~ is integrable

$$\left| \frac{\partial F}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \right| = \left| \frac{\partial F}{\partial \bar{w}} \cdot \frac{2\, dx \wedge dy}{r} \right| = 2 \left| \frac{\partial F}{\partial \bar{w}}\, dr \wedge d\theta \right|$$

$$dw \wedge d\bar{w} = -2i\, dx \wedge dy = -2i\, r\, dr \wedge d\theta$$

So taking $\varepsilon \to 0$ in 2-dim integral gives the result. $\square$

- <u>Proof of 1-dim Poincaré</u>

Let $z_0 \in D$, set $D_0 = \{|z-z_0| < 2\varepsilon\} \subset D$, $\overline{D_0} \subset D$

$g(z) = g_1(z) + g_2(z)$ s.t. $g_1|_{\{|z-z_0| \geq 2\varepsilon\}} \equiv 0$

both smooth $\quad g_2|_{\{|z-z_0| \leq \varepsilon\}} \equiv 0$

$$f_2(z) = \frac{1}{2\pi i} \int_D \frac{g_2(w)}{w-z}\, dw \wedge d\bar{w}$$

well-def for z near z_0

<u>not</u> an improper integral

$$\frac{\partial f_2}{\partial \bar{z}}(z) = \frac{1}{2\pi i} \int_D \frac{\partial}{\partial \bar{z}} \left(\frac{g_2(w)}{w-z} \right) dw \wedge d\bar{w} = 0$$

- g_1 has compact support so

$$f_1(z) = \frac{1}{2\pi i} \int_D \frac{g_1(w)}{w-z}\, dw \wedge d\bar{w} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w)}{w-z}\, dw \wedge d\bar{w}$$

$\lceil u = w-z \rfloor$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(u+z)}{u}\, du \wedge d\bar{u}$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} \frac{g_1(z+re^{i\theta})}{e^{i\theta}}\, dr \wedge d\theta$$

$\Rightarrow$ well-def & C^∞ in z

- $$\frac{\partial f_1}{\partial \bar{z}} = + \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial g_1}{\partial \bar{z}}(z+re^{i\theta})\, e^{-i\theta} dr \wedge d\theta$$

$$= \frac{1}{2\pi i} \int_D \frac{\partial g_1}{\partial \bar{w}}(w)\, \frac{dw \wedge d\bar{w}}{w-z}$$

From Lemma,

$$g_1(z) = \frac{1}{2\pi i}\int_{|w-a|=r} \xrightarrow{\ \ 0\ \ }\cancel{\frac{g_1(w)\,dw}{w-z}} + \frac{1}{2\pi i}\int_{|w-a|<r} \frac{\partial g_1}{\partial \bar w} \cdot \frac{dw\wedge d\bar w}{w-z}$$

So $\dfrac{\partial f_1}{\partial \bar z}(z) = g_1(z) = g(z)$

and $\dfrac{\partial f}{\partial \bar z}(z) = \dfrac{\partial f_1}{\partial \bar z}(z)$ as $\dfrac{\partial f_2}{\partial \bar z} = 0$ near z_0. $\square$

$\bar\partial$ - Poincaré Lemma

Let $D = \{|z_1 - a_1| < r_1\} \times \cdots \times \{|z_m - a_m| < r_m\} \subseteq \mathbb{C}^m$ polydisc.

(Allow $r_k = \infty$)

Then $H^{p,q}(D) = 0$ for $q \geqslant 1$.

Proof $\varphi \in \Omega^{p,q}(D)$, $\bar\partial\varphi = 0$

WLOG $p=0$ as $\bar\partial(\underset{(0,q)}{\eta} \wedge \underset{(p,0)}{dz_J}) = \bar\partial\eta \wedge dz_J = 0$ iff $\bar\partial\eta = 0$

⌜respected by linear combinations⌟

Claim $\exists\, \psi \in \Omega^{0,q-1}(D_0)$ s.t. $\bar\partial\psi = \varphi$ on D_0,

where D_0 is a polydisc with smaller radii $\varepsilon_k < r_k$, $k = 1, \dots, n$

Proceed by "integrating" $d\bar z_m$, then $d\bar z_{m-1}$ etc

Suppose φ involves only $d\bar z_1, \dots, d\bar z_k$

Then $\varphi = d\bar z_k \wedge \varphi_1 + \varphi_2$ for φ_1, φ_2 w/o $d\bar z_k$

$\bar\partial$ - closed $\Rightarrow$ for $\varphi_1 = \sum_I \varphi_I \, d\bar z_I$, $I \subseteq \{1, \dots, k-1\}$

have $\dfrac{\partial \varphi_I}{\partial \bar z_L} = 0 \quad \forall\ L > k$

Set $\eta_I = \displaystyle\int_{|w_k - a_k| \le \varepsilon_k} \varphi_I(\dots, w_k, \dots) \frac{dw_k \wedge d\bar w_k}{w_k - z_k}$

Then $\dfrac{\partial \eta_I}{\partial \bar z_k} = \varphi_I$.

But $\dfrac{\partial \eta_I}{\partial \bar z_L} = 0 \quad (L > k)$ as $\dfrac{\partial \varphi_I}{\partial \bar z_L} = 0 \quad (L > k)$

$\therefore\ \varphi - \bar\partial\left(\sum \eta_I \, d\bar z_I\right) = \underset{\text{different } \varphi_2}{\varphi_2}$ with no $d\bar z_k$ occurring

N.B. we needed to reduce D to D_0

To now solve the $\bar\partial$ eq^n on all of D, take $\varepsilon_k^{(n)} \nearrow r_k$ as $n \to \infty$ $(k=1,\dots,m)$

$\forall n \geq 0$, $\exists\, \psi_n$ s.t. $\bar\partial \psi_n = \varphi$ on

D_n polydisc with $\varepsilon_k^{(n)}$

Claim ψ_n will converge as $n \to \infty$

Induction on q, assume true for $(0,q-1)$ form φ, $q \geq 2$

$\bar\partial \alpha = \varphi$ on D_{n+1}

$\bar\partial(\alpha - \psi_n) = 0$ on D_n

$\Rightarrow \underset{(0,q-1)}{\psi_n - \alpha} = \underset{(0,q-2)}{\bar\partial \beta}$ on D_{n-1} by hypothesis

define $\psi_{n+1} = \alpha + \bar\partial\beta$ then $\bar\partial \psi_{n+1} = \bar\partial\alpha = \varphi$ on D_{n+1}

now induction on n

#bruh [see 1.3.9 in Huybrechts]

Recall we were proving $\bar\partial$-Poincaré lemma for $(0,q)$-forms ($\Rightarrow$ for (p,q) forms) by induction on q:

[I will mark work in Q2, Q5]

$D_0 \subset \bar D_0 \subset \cdots \subset D_n \subset \bar D_n \subset \cdots$ with $\bigcup_{n \geq 0} D_n = D$

$\forall n \geq 0$, $\exists \psi_n \in \Omega^{0,q-1}(D)$ with $\bar\partial \psi_n = \phi$ on D_n

$\exists \alpha$ s.t. $\bar\partial \alpha = \phi$ on D_{n+1}

$\Rightarrow \bar\partial(\alpha - \psi_n) = 0$ on D_n

Now via inductive assumption $\exists \beta \in \Omega^{0,q-2}(D)$

s.t. $\psi_n - \alpha = \bar\partial \beta$ on D_{n-1}

Set $\psi_{n+1} = \alpha + \bar\partial \beta$.

Then $\bar\partial \psi_{n+1} = \bar\partial \alpha = \phi$ on D_{n+1},

and $\psi_{n+1} = \psi_n$ on D_{n-1}.

$\Rightarrow \psi_n \to \psi$ as $n \to \infty$ with $\bar\partial \psi = \phi$ on D

RTP $\bar\partial$-Poincaré lemma for $(0,1)$-forms

i.e. given $\phi \in \Omega^{0,1}(D)$ with $\bar\partial \phi = 0$

and any open polydisc D_0 with $\bar D_0 \subset D$

$\exists \psi \in C^\infty(D)$ with $\bar\partial \psi = \phi$ on D_0

Then in fact $\exists \psi_0 \in C^\infty(D)$ with $\bar\partial \psi_0 = \phi$ on D_0

$\exists \psi_n \in C^\infty(D)$ s.t. $\bar\partial \psi_n = \phi$ on D_n

$\exists \alpha \in C^\infty(D)$ s.t. $\bar\partial \alpha = \phi$ on D_{n+1}

Now $\bar\partial(\psi_n - \alpha) = 0$ on D_n.

So $\psi_n - \alpha$ admits a power series expansion on D_n, converging uniformly on compact subsets of D_n, e.g. $\overline{D_{n-1}}$

So $\exists$ partial sum (a holo. poly) β s.t.

$$\sup_{D_{n-1}} |(\psi_n - \alpha) - \beta| < \frac{1}{2^n}$$

Set $\psi_{n+1} = \alpha + \beta$.

Then $\bar\partial \psi_{n+1} = \bar\partial \alpha = \phi$ on D_{n+1}

$\psi_{n+1} - \psi_n$ is holomorphic on D_n

with $\sup_{D_{n-1}} |\psi_{n+1} - \psi_n| < \frac{1}{2^n}$

$\Rightarrow$ obtain $(\psi_n)_{n\geq 0}$ in $C^\infty(D)$ with $\psi_n \to \psi$ uniformly on compact subsets of D

$\Rightarrow \lim_{l\to\infty} (\psi_l - \psi_n)$ is holo. on D_{n-1} for each n

and $\bar\partial \psi = \bar\partial \psi_n = \phi$ on D_{n-1} "

So $\bar\partial\psi = \phi$ on D outright. $\square$

Remark $H^{p,0}(\mathbb{C}^n) = \{$ holomorphic p-forms $\}$ is infinite-dimensional when non-zero.

$H^{0,0}(M) \cong \mathbb{C}$ for any compact cx mfd M
(shall later see $H^{p,0}(M)$ is f.dim for compact M, if Kähler)

Almost-Complex Manifolds

Def^n A smooth real mfd M is called almost-complex if it carries $J \in \Gamma(\mathrm{End}\, TM)$ with $J^2 = -1$.
We call J an almost-complex structure.

Lemma (from lin. algebra)
Let $J \in \mathrm{End}(\mathbb{R}^m)$, $J^2 = -1$. Then
0) $J \in GL(m, \mathbb{R})$
1) $m = 2n$, $n \in \mathbb{N}$
2) $\{A \in GL(m,\mathbb{R}) : AJA^{-1} = J\} \cong GL(n, \mathbb{C})$ $\square$

Thus $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ parametrises almost-complex structures via $[S] \mapsto SJ_0S^{-1}$
for $J_0 = \mathrm{diag}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \dots, \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)$.

<u>sketch-proof of Lemma</u>

$\forall v \neq 0$, v, Jv are linearly independent

etc. can get basis $e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n$

Then $J = J_0$ in this basis

<u>Cor</u>: An almost-complex structure on M is equivalent to a $GL(n, \mathbb{C})$-structure on M.

Thus every almost cx mfd is even-dimensional and orientable (oriented even).

Can extend $e_1, Je_1, \dots, e_n, Je_n$ to a local frame field around $p \in M$.

Let $e_1^*, Je_1^*, \dots, e_n^*, Je_n^*$ the 'dual' coframe field $\leadsto$

$\varepsilon = Je_1^* \wedge e_1^* \wedge \dots \wedge Je_n^* \wedge e_n^*$.

E.g. if M is a cx mfd with local cx coords (z_j) then

$$\varepsilon = \frac{i^n}{2^n} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$$

$$= dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n \qquad \left(\begin{array}{l}\text{recall } J\,dx = -dy \\ \quad J\,dy = dx\end{array}\right)$$

<u>Remark</u>: $(-J)$ is another almost cx structure

It gives the same orientation if n is even

" opposite " odd

<u>Def</u> The <u>torsion</u> of the almost cx structure J is a tensor $N_J \in \Gamma(\mathrm{Hom}(\Lambda^2 TM, TM))$ given by

$$N_J(X, Y) = 2\big([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\big)$$

If $N_J = 0$ then J is called <u>torsion-free</u> or <u>integrable</u>.

<u>Fact</u> N_J is a $C^\infty(M)$-linear map (so actually a tensor)
(direct computation)

Coefs of N_J depend on J and 1st derivatives

[makes sense to submit work on an example sheet before the resp. example class]

Sheet 1
minor changes in Q2/3

have defined the torsion tensor

N_J (a.k.a. Nijenhuis tensor)

of an almost-complex structure J on a smooth manifold M

for X, Y (loc.) vector fields on M,

$N_J(X,Y)$ is again a vector field

$N(X,Y) = -N(Y,X) \quad \forall X, Y$

$N(fX, Y) = f N(X,Y) \quad \forall f \in C^\infty(M)$

in loc. coordinates

$N_J(\partial_i, \partial_j) = \sum_k N_{ij}^k \partial_k$ where $\partial_i = \frac{\partial}{\partial u_i}$ (u_i) real coords

Newlander-Nirenberg thm

An almost-complex structure J on M is torsion-free (integrable) if and only if it arises from an atlas of loc. cx coords

Remark on the proof

"<=" Let $z_\alpha = x_\alpha + i y_\alpha$ be loc. cx coords.

Then $\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}, J\frac{\partial}{\partial x_\alpha}, J\frac{\partial}{\partial y_\alpha}$ all have constant coeffs wrt $\{\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}\}$ so all Lie brackets involved in N_J vanish.

That $N_J \equiv 0$ follows from it being a tensor.

"=>" is difficult (see Kobayashi-Nomizu for real-analytic mfd M)

J suffices for definition of $T^{1,0}M, T^{0,1}M, \Lambda^{p,q}(T^*M)^{\mathbb{C}}$

$\therefore$ also define $\partial, \bar\partial$ (on $\Omega^{p,q}$)

Prop If M is an almost-complex structure, then

$$d(\Omega^{p,q}(M)) \subseteq \Omega^{p-1,q+2}(M) + \Omega^{p,q+1}(M) + \Omega^{p+1,q}(M) + \Omega^{p+2,q-1}(M)$$

Proof Obviously

$$\left.\begin{aligned} d(\Omega^{0,1}) &\subseteq \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0} \\ d(\Omega^{1,0}) &\subseteq \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0} \end{aligned}\right\} \quad (*)$$

For an arbitrary (p,q)-form, write as

$$\sum_{i=1}^{N} \varepsilon_1^{(i)} \wedge \cdots \wedge \varepsilon_{p+q}^{(i)} \quad \text{for } \varepsilon_j^{(i)} \in \Omega^{1,0} \text{ or } \Omega^{0,1}$$

Now apply Leibniz rule and $(*)$ to get the result. Q.E.D.

Theorem If M is an almost-complex manifold, TFAE

(a) $Z, W \in \Gamma(T^{1,0}(M)) \Rightarrow [Z,W] \in \Gamma(T^{1,0}(M))$

(b) $Z, W \in \Gamma(T^{0,1}(M)) \Rightarrow [Z,W] \in \Gamma(T^{0,1}(M))$

(c) $d(\Omega^{1,0}(M)) \subset \Omega^{1,1}(M) + \Omega^{2,0}(M)$
& $d(\Omega^{0,1}(M)) \subset \Omega^{1,1}(M) + \Omega^{0,2}(M)$

(d) $d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) + \Omega^{p,q+1}(M)$

(e) $N_J \equiv 0$ i.e. J is integrable

Proof (a) $\Leftrightarrow$ (b) via cx conjugation

Indeed $\overline{[Z,W]} = [\bar{Z}, \bar{W}]$

and $Z \in T^{1,0} \Leftrightarrow \bar{Z} \in T^{0,1}$

(a), (b) $\Rightarrow$ (c)

Recall $(d\omega)(Z,W) = Z\,\omega(W) - W\omega(Z) - \omega([Z,W])$. $(\dagger)$

Then if $\Omega^{1,0} \ni \omega$ and $Z, W \in T^{0,1}$ the RHS vanishes.

So $d\omega$ has no $(0,2)$ component as desired.

Second statement is similar (or use cx conj)

(c) $\Rightarrow$ (a) Use relation $(\dagger)$. For $Z, W \in T^{1,0}$ and $\omega \in \Omega^{0,1}$
then deduce $\omega([Z,W]) = 0$. So $[Z,W] \in T^{1,0}$.

(c) $\Rightarrow$ (d) Similar to proof of previous Prop^{n}.

N.B. suffices to work locally

(d) $\Rightarrow$ (c) Trivial

(a) $\Leftrightarrow$ (e) A general $(1,0)$ vector field is $X - iJX$ for X a real vector field.

Let $Z = [X - iJX, Y - iJY]$

$$= -[JX, JY] + [X,Y] - i[JX, Y] - i[X, JY] \quad (?)$$

Direct calculation gives

$$2(Z + iJZ) = -N_J(X,Y) - \overset{J}{\downarrow}\!\!i N_J(X,Y) \stackrel{!}{=} 0 \quad \text{iff } Z \in T^{1,0}$$

Q.E.D.

Remarks
- existence of J is a topological question
  largely understood (via characteristic classes)
- integrability of J — non-linear PDE for J
  (harder)
- real surfaces (orientable) $\dim_{\mathbb{R}} M = 2$, no $(0,2)$-forms, $(2,0)$-forms

  $\therefore$ J always integrable

Kodaira deformation

Submanifolds & Subvarieties

Recall: $Y \subset X$ is a (embedded) C^∞ submfd of a mfd X

means the inclusion $\iota : Y \to X$ is smooth

with $(d\iota)_y : T_y Y \to T_y X$ injective $\forall y \in Y$

and ι is a homeo onto the image.

Then (and only then) locally Y around each $y \in Y$ is the inverse image of a regular value

<u>Def^n</u> Let X be a cx n-fold, and $Y \subset X$ a C^∞ submfd, $\dim_{\mathbb{R}} Y = 2k$. We say Y is a <u>cx k-dim submfd</u> if $\forall y \in Y$

(*) [$\exists$ a cx coord chart $\varphi_y : \underset{\substack{\text{open} \\ \cap \\ X}}{U_y} \longrightarrow \mathbb{C}^n$, $y \in U_y$

s.t. $\varphi_y(U_y \cap Y) = \varphi_y(U_y) \cap (\mathbb{C}^k \times \{0\})$

i.e. last $n-k$ complex coords vanish

<u>Remarks</u> • thus Y is a cx k-fold with holo. atlas $\{(U_y \cap Y, \varphi_y)\}_y$

$\operatorname{codim}_{\mathbb{C}} Y/X = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y = n-k$

• (*) <=> $\forall y \in Y$ $\exists$ holo $F : \underset{\substack{\text{open} \\ \cap \\ X}}{W_y} \longrightarrow \mathbb{C}^{n-k}$, $y \in W_y$

s.t. $\operatorname{rk}_{\mathbb{C}} \left(\frac{\partial F_i}{\partial w_j} \right) = n-k$ on W_y, $F^{-1}(0) = Y \cap W_y$

• then $\iota : Y \hookrightarrow X$ is a holo. map <=> $T_y Y \subset T_y X$ <u>cx</u> vector subspace

<=> $T^{1,0}_y Y \subset T^{1,0}_y X$ (by prev. Thm.)

• recall: if $X = \mathbb{CP}^n$ and Y is compact then Y is a projective mfd

<u>Def^n</u> $Y \subset X$ is an <u>(analytic) subvariety</u> if $Y \subset X$ is closed and $\forall p \in Y$ $\exists$ ngbd U_p s.t. $U_p \cap Y = f^{-1}(0)$ for some holomorphic $f : U_p \to \mathbb{C}^m$

Say p is a <u>smooth point</u> of Y if $\exists$ such f with $\operatorname{rank}_{\mathbb{C}} J(f)_p = m$ i.e. $(df)_p$ is surjective

Say p is a singular point o/w.

- The singular locus of Y is $Y^s = \{$ all singular points of $Y \}$

Say Y is smooth ~~or~~ non-singular if $Y^s = \emptyset$.

Implicit F^n Thm $\Rightarrow$ $Y^* := Y \setminus Y^s$ is a complex submfd

every connected component

Y is irreducible if one cannot write $Y = Y_1 \cup Y_2$ with Y_1, Y_2 subvarieties properly contained in Y.

One can show (sheet 2): Y irred. $\Rightarrow$ Y^* connected

- Suppose Y is irred. Then $\operatorname{codim} Y/X = \operatorname{codim} Y^*/X$

Fact: Y^s is a subvariety and $\operatorname{codim} Y^s/X > \operatorname{codim} Y/X$

Can check weaker statements: ~~If~~ $Y^* \neq \emptyset$ and dense open in Y

$Y^s \subset$ subvariety of X, this subvar does not contain Y

If $\operatorname{codim}_{\mathbb{C}} Y/X = 1$ then Y is called a hypersurface

II Holomorphic Geometry

- ## Holomorphic Vector Bundles

Let X be a complex mfd.

Def^n A holo. v.b. of rank k over the base X is a cx mfd E (the total space) equipped with a holo. submersion $\pi : E \to X$ onto X, and for each $x \in X$ the structure of a k-dim cx vector space on the fiber $\pi^{-1}(x)$, such that:

- $\forall y \in X$ $\exists$ ngbd U of y and a biholomorphism ϕ_U, called a holo. local triv^n

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{C}^k \\ \pi \downarrow & & \downarrow \text{proj}_1 \\ U & \xrightarrow{=} & U \end{array}$$

s.t. the diagram commutes and $\phi_U|_{E_x = \pi^{-1}(x)} \longrightarrow \mathbb{C}^k$ a complex linear isomorphism for all $x \in U$.

If U_α, U_β overlapping trivialising ngbds with holo. loc triv^{n}s ϕ_α, ϕ_β then $\phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^k$

$$(x, v) \longmapsto (x, \psi_{\beta\alpha}(x) v)$$

for some $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{C})$ holomorphic map.

It makes sense to speak of holomorphic local sections of E, namely $s : U \longrightarrow E$ holomorphic s.t. $\pi \circ s = id_U$, $U \subset X$.

Properties (c.f. Sheet 2)

If $E, \tilde{E}$ are holo. v.b. then so are $E \oplus \tilde{E}$, $E \otimes \tilde{E}$, $\Lambda^r E$, $\mathrm{End}\, E = E \otimes E^*$, E^*, $\det E = \Lambda^{\mathrm{rk} E} E$

recap: holo. v. bundles are distinguished by holo. trans. f's

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$$

- the data $\{ U_\alpha, \psi_{\beta\alpha} : \alpha, \beta \in A \}$ determines the holo. v.b. E up to isomorphism, ~~i.e.~~ where

two holo. v.b. $E, \tilde{E}$ are <u>isomorphic</u> if $\exists$ biholo $F : E \to \tilde{E}$ s.t.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi_E"'] & \tilde{E} \arrow[d, "\pi_{\tilde{E}}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes and each $F|_{E_x} : E_x \to \tilde{E}_x$ is a cx lin isomorphism

<u>pullback</u> of E via holo. map $f : Y \to X$

[fix notation X – cx mfd, E – holo v.b. over X]

- is a holo. v.b. f^*E over Y s.t. $\exists$ holo map F with

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "\pi_Y"'] & E \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

commuting and given in each holo. trivialisation over $U \subset X$ say, by

$$\begin{array}{ccc} (b, v) & \longmapsto & (f(b), v) \\ \cap & & \cap \\ f^{-1}(U) \times \mathbb{C}^k & & U \times \mathbb{C}^k \end{array}$$

- transition functions of f^*E are

$$\psi_{\beta\alpha} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to GL(k, \mathbb{C})$$

Thus f^*E is a well-defined holo. v.b.

<u>Examples</u> 1. $T^{1,0}X$, $(T^*X)^{1,0}$, $\Lambda^p (T^*X)^{1,0}$, K_X are holo. v.b., transition functions are composition of cx Jacobians for loc. coords with holo. f^ns (in fact algebraic)

2. $Y \subset X$ cx. submanifold, then inclusion $Y \to X$ is holo. and $\iota^* E \to Y$ is holo v.b. – the restriction $E|_Y$

- Shall mostly consider holo. v.b. of rank 1 (line bundles)

<u>Propⁿ / Defⁿ</u> Holo. line bundles over X form an abelian group under $\otimes$, called the <u>Picard group</u> of X, $\mathrm{Pic}(X)$

<u>Proof</u> let L have the trans fⁿs $f_{\beta\alpha}$ (valued in $\mathbb{C}^*$)
$\tilde{L}$ " $\tilde{f}_{\beta\alpha}$

Then $L \otimes \tilde{L}$ has " $f_{\beta\alpha} \cdot \tilde{f}_{\beta\alpha}$

The inverse of L is L^*, noting $z \in X$, $f(z) : V \to W$ cx iso

$f(z)^* : W^* \to V^*$ given by $f(z)^T$ w.r.t. dual basis

$(f(z)^*)^{-1} : V^* \to W^*$

If $\dim V = \dim W = 1$, get $f(z)^{-1}$

trans f's are trivial so get $X \times \mathbb{C}$ trivial v.b.

Q.E.D.

<u>Cor</u> If $f : Y \to X$ is holo. then

$f^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is a group hom.

<u>Example</u> (tautological line b. $\mathcal{O}(-1)$ over $\mathbb{CP}^n$)

start with $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$

$$\downarrow$$

$$[z_0 : \dots : z_n] \in \mathbb{CP}^n$$

want to exhibit $\mathbb{C}^{n+1} \setminus \{0\}$ as a line b. $E = \mathcal{O}(-1)$ with zero section removed

this suffices to work out the transition functions

$U_\alpha = \{z_\alpha \neq 0\} \subset \mathbb{CP}^n$ coord patch, $\alpha \in \{0, \dots, n\}$

$\phi_\alpha^{-1}([z_0 : \dots : z_n], w) = \left(\frac{z_0}{z_\alpha} w, \dots, \frac{z_n}{z_\alpha} w\right)$ assume $w \neq 0$

$\phi_{\beta}(\zeta_0, \dots, \zeta_n) = \left(\left[\frac{\zeta_0}{\zeta_\beta} : \dots : \underset{\substack{\uparrow \\ \beta^{th}}}{1} : \dots : \frac{\zeta_n}{\zeta_\beta}\right], \zeta_\beta\right)$ $\quad \zeta_\beta = \frac{z_\beta}{z_\alpha} w$

So $\phi_\beta \circ \phi_\alpha^{-1}(p, w) = (p, \psi_{\beta\alpha}(p) w)$

with $\psi_{\beta\alpha}(p) = \psi_{\beta\alpha}([z_0 : \dots : z_n]) = \frac{z_\beta}{z_\alpha}$ on $U_\alpha \cap U_\beta$

Thus $\mathcal{O}(-1)$ well-defined.

$\mathcal{O}(1) := \mathcal{O}(-1)^*$ is the hyperplane bundle

$$\mathcal{O}(n) := \underbrace{\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)}_{n \text{ times}} = \mathcal{O}(n-1) \otimes \mathcal{O}(1)$$

$$\mathcal{O}(-n) := \underbrace{\mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1)}_{n \text{ times}} = \mathcal{O}(-n+1) \otimes \mathcal{O}(-1)$$

$\mathcal{O}(0) :=$ trivial line bundle

Thus $\mathbb{Z} \longrightarrow \mathrm{Pic}(\mathbb{CP}^n)$ is hom.

In fact $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$ and this is an iso^m.

Divisors

Need some facts from commutative algebra

Local rings

Consider $p \in X$.

$\mathcal{O}_{X,p} = \{$ holo. f's f defined on an open nghbd $U_f \ni p \}/\sim$

where $f \sim f'$ if $\exists$ open $p \in W \subset U_f \cap U_{f'}$

s.t. $f|_W = f'|_W$ — local ring at p

$f \in \mathcal{O}_{X,p}$ is an element

f is an invertible element at p $\iff$ $f(p) \neq 0$

f " irreducible element " $\iff$ whenever $f = uv$ then one of u, v is invertible

f divides g means $g = uf$ for some $u \in \mathcal{O}_{X,p}$

f, g are coprime $\iff$ whenever u divides both f, g we have u invertible

(weak) Nullstellensatz Let f be an irreducible elt at $0 \in \mathbb{C}^n$ and let elt h vanish on $f^{-1}(0) \cap$ (domain of h).
Then f divides h.

X is a cx n-fold (throughout the lecture)

recall <u>Nullstellensatz</u> if $f : \underset{\mathbb{C}^n}{D} \to \mathbb{C}$, $0 \in D$, f holo, $D \subset \mathbb{C}^n$

$h|_{f^{-1}(0)} \equiv 0$ holo ⌜need f irred in $\mathcal{O}_{\mathbb{C}^n,0}$⌟

then $f | h$ in $\mathcal{O}_{\mathbb{C}^n,0}$ "$|$" means "divides"

e.g. $n=1$, an isolated zero $f(0) = 0$

irred $\Rightarrow$ zero order 1

Then $f(z) = z f_1(z)$ for $f_1(0) \neq 0$

And if $h(0) = 0$ so $h(z) = z^d h_1(z)$, $h_1(0) \neq 0$

then $h(z) = \left(z^{d-1} \dfrac{h_1(z)}{f_1(z)} \right) f(z)$

shall also need

<u>Theorem</u> (U.F.D.) $\mathcal{O}_{\mathbb{C}^n,0}$ is a unique factorisation domain

i.e. given $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $f \neq 0$, we can write $f = f_1 \cdots f_m$

where each $f_j \in \mathcal{O}_{\mathbb{C}^n,0}$ is irreducible; this factorisation

is unique up to reordering and mult^n by a unit in $\mathcal{O}_{\mathbb{C}^n,0}$

<u>Proposition</u> Let $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$

If f, g are coprime, then $\exists\, \varepsilon > 0$ s.t. f, g are coprime

in $\mathcal{O}_{\mathbb{C}^n,z}$ whenever $\|z\| < \varepsilon$ ⌜use WPT & Bézout -type idea⌟

Useful for study of hypersurfaces $Y \subset X$.

Recall: if $p \in Y^*$ then $\exists$ holo $f : U_p \longrightarrow \mathbb{C}$, $U_p \subset X$, $p \in U_p$

s.t. $Y \cap U_p = f^{-1}(0)$ and $(df)_p \neq 0$

Then $\exists$ loc cx coords $z_1 = f, z_2, \ldots, z_n$ around p

$\therefore \forall$ holo $g : U_p \to \mathbb{C}$ s.t. $g|_{Y \cap U_p} \equiv 0$ then $g = fu$

– consider power series at p; induction on $\dim X$

Thus have irred elt $f \in \mathcal{O}_{X,p}$ s.t. $\forall g$ vanishing on Y ⌜f <u>is</u> irred⌟

near p, have $f | g$

Such f is unique up to an invertible elt

Def^n A subvariety $Y \subset X$ is (loc.) irred at $p \in Y$ if $\exists$ small polydisc around p s.t. $Y \cap \mathcal{U}$ is irreducible
$\mathcal{U}$

Suppose $p \in Y^s$ and Y is irred at p (Y is a hypersurface)

Claim $\exists\, \mathcal{U}_p \subset X$ and $f : \mathcal{U}_p \to \mathbb{C}$ holo s.t. $Y \cap \mathcal{U}_p = f^{-1}(0)$

Pf Suppose not — we know $Y \cap \mathcal{U}_p \subseteq \{q : f_1(q) = f_2(q) = 0\}$ for some holo: $f_i : \mathcal{U}_p \to \mathbb{C}$

Wlog f_1, f_2 irred at p $\Rightarrow$ coprime at p

(since ideal is prime) $\Rightarrow$ coprime at some $q \in Y^*$ (since Y^* dense open)

$\Rightarrow f_1 = f \circ u,\ f_2 = f \circ u'$ in $\mathcal{O}_{X,q}$

$\divideontimes$ to coprime assumption $\square$

If Y is not irreducible, apply to each component, then multiply out.

obtained

Def^n / Prop^n Let $Y \subset X$ be a hypersurface, $p \in Y$

Then $\exists\, f \in \mathcal{O}_{X,p}$ s.t. $Z(f) = Y|_p$

and any other $g \in \mathcal{O}_{X,p}$ with $g|_Y \equiv 0$ has $f \mid g$. $\divideontimes$

This f is a locally defining function for Y at p.

For $p \notin Y$ formally set f to be a unit at p.

Lemma Y is irred at p iff the locally defining function for Y is irred in $\mathcal{O}_{X,p}$

Proof Suppose f is irred in $\mathcal{O}_{X,p}$, and that $Y \cap \mathcal{U}_p = Y_1 \cup Y_2$ is a non-trivial decomposition

Then $\exists\, f_j$, $j \in \{1,2\}$, locally defining f^n for Y_i at p

Nullstellensatz $\Rightarrow f \mid f_1 f_2 \Rightarrow f \mid f_1$ or $f \mid f_2$ (f irred)

$\divideontimes$ to Y_i being proper subsets of $Y \cap \mathcal{U}_p$

Now suppose $f = f_1 f_2$ where f_1, f_2 coprime

$\Rightarrow Y_{\pm} \cap U_p = \{f_1 = 0\} \cup \{f_2 = 0\} = Y_1 \cup Y_2, \quad Y_i \neq Y \cap U_p$ Q.E.D.

In general, using loc. def^n f's and UFD property, obtain $\forall p \in X$, $\exists$ open $U_p \subset X$ s.t. $Y \cap U_p = Y_{p,1} \cup \dots \cup Y_{p,m}$ for $Y_{p,j} = V(f_{p,j})$ irred.

If X is compact, then $\{U_p\}_{p \in X}$ becomes a finite cover and we can patch the $Y_{p,j}$ and obtain

(*) $Y = Y_1 \cup \dots \cup Y_N$, Y_j globally irreducible analytic hypersurface

Def^n A divisor on X is a locally finite formal sum

$$D = \sum_i a_i Y_i$$

where each Y_i is an irred. hypersurface in X and each $a_i \in \mathbb{Z}$.

{ loc. finite: $\forall p \in X$, nghd U_p s.t. U_p intersects finitely-many Y_i }

$\mathrm{Div}(X)$ = additive group of "all" divisors on X

Let X be compact

Then $\exists$ finite open cover by U_p's say $\{U_\alpha\}$ and for each α a well-defined holo. loc. def^n f^n $f_{\alpha,j} : U_\alpha \to \mathbb{C}$ for Y_j c.f. (*)

Then can assign to any $D \in \mathrm{Div}(X)$ the data $\{(U_\alpha, f_\alpha)\}$ where $f_\alpha = \prod_{i=1}^{N} f_{\alpha,i}^{a_i} : U_\alpha \to \mathbb{C}$ "loc. def^n f^n" for D at p

We say a divisor D is effective if all coeffs a_i are ≥ 0

Then f_α is holomorphic on U_α

Def^n f is a meromorphic f^n on X if f is locally a quotient of two holomorphic functions

$X = \bigcup U_i$ open cover

$\exists$ coprime holo. ~~f_i~~ $g_i, h_i : U_i \to \mathbb{C}$, $h_i \neq 0$

s.t. $f|_{U_i} = \dfrac{g_i}{h_i}$ (wherever this makes sense).

In particular $g_i h_j = g_j h_i$ on $U_i \cap U_j$

Warning If $\dim X = 1$ then any meromorphic f^n is equivalent to a holo. to $\mathbb{CP}^1$

But for $\dim X > 1$ this is no longer true

e.g. $X = \mathbb{C}^n$, $g(z) = z_\alpha$

$h(z) = z_\beta$, $\alpha \neq \beta$ $\quad \dfrac{g}{h}$ undefined where $z_\alpha = z_\beta = 0$

Basic example of a meromorphic function

[X is a cx n-fold throughout today's lecture]

(in $\dim_{\mathbb{C}} > 1$)

$X = \mathbb{C}^n$, $\quad g(\underline{z}) = z_\alpha$
$\quad\quad h(\underline{z}) = z_\beta \quad \alpha \neq \beta$

$\dfrac{g(\underline{z})}{h(\underline{z})}$ undefined on $\{ z_\alpha = z_\beta = 0 \}$ $\operatorname{codim}_{\mathbb{C}} = 2$ subspace

(unlike $\dim_{\mathbb{C}} = 1$ where mero. f^ns $<=>$ holo maps to $\mathbb{CP}^1$)

Let $Y \subset X$ be an irreducible analytic hypersurface

$p \in Y$, $\quad f$ a locally defining f^n at p

If g is a holo. f^n around p ; $g \not\equiv 0$,

$g = \underbrace{g_1 \cdots g_L}_{\text{irred factors}}$ ($\exists$ by UFD)

<u>Def^n</u> $\operatorname{ord}_{Y,p}(g) := \max\{ a \in \mathbb{Z} : g = f^a h, \ h \text{ holo at } p \}$

i.e. $f^a | g$ in $\mathcal{O}_{X,p}$

This is indep. of the choice of f.

<u>Recall</u> $Y^* = Y \setminus Y^s$ is connected, open, dense in Y

<u>Claim</u> $\operatorname{ord}_{Y,p}(g)$ is locally constant for $p \in Y^*$

Thus independent of $p \in Y^*$ and define $\operatorname{ord}_Y(g)$.

<u>Pf</u> wlog $p = 0 \in \mathbb{C}^n = \mathbb{C}_w \times \mathbb{C}_z^{n-1}$

X a polydisc around 0, $\quad Y = \{ w = 0 \}$ (use smoothness)

$\operatorname{ord}_{Y,0}(g) = a \quad <=> \quad g(w,z) = w^a h(w,z)$, $\quad w \nmid h$ in $\mathcal{O}_{\mathbb{C}^n,0}$

so $h = w h_0 + h_1$ for h_0, h_1 holo. near 0

$\dfrac{\partial h_1}{\partial w} \equiv 0$, $\quad h_1(0,0) \neq 0$ rep. by non-trivial power series ii

what?

$h_1(0,z) \neq 0$ for small $|z|$ $\quad$ for "a lot" of z

Re-expand h_1 at $(0,z)$; still non-trivial power series

$\Rightarrow w \nmid h$ at $(0,z)$ if $|z|$ small

Q.E.D.

easy to see

$$\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h)$$

by UFD and noting f is "irreducible" ← define ord via smooth points

If $F \not\equiv 0$ meromorphic then locally $F = \frac{g}{h}$, g, h holo

Define $\mathrm{ord}_Y F = \mathrm{ord}_Y g - \mathrm{ord}_Y h$

If $d = \mathrm{ord}_Y(F) > 0$ then a zero order d along Y

If $d = \mathrm{ord}_Y(F) < 0$ then a pole order $(-d)$ along Y

The divisor of a meromorphic f^n $F \neq 0$ on X is

$$(F) = \sum_{Y \text{ irred hypsfc}} \mathrm{ord}_Y(F) \cdot Y \qquad (*)$$

well-defined by the UFD property ⌈as in, locally finite⌋

any such divisor called a principal divisor

- $D \sim D'$ if $D - D' = (f)$ for some meromorphic f

Say D, D' are linearly equivalent

- the sum $(*)$ is finite when X is compact
- $(f) \geqslant 0$ i.e. (f) effective iff f holomorphic
- $(fg) = (f) + (g)$, $\left(\frac{f}{g}\right) = (f) - (g)$

so long as $g \not\equiv 0$

N.B. if $\dim_{\mathbb{C}} X = 1$ i.e. X Riemann surface,

then $\mathrm{Div}(X) = \{ \sum_i n_i P_i \,,\; n_i \in \mathbb{Z} \,,\; P_i \in X \}$

if $\dim_{\mathbb{C}} X > 1$ then needn't exist any divisors on X (in general)

but divisors exist when X is projective

Suppose $F: Z \to X$ is a holo. map of cx mfds, with Z, X compact connected.

Let $D \in \mathrm{Div}(X)$, $D = \sum_i a_i Y_i$, and assume $F(Z) \not\subset Y_i$ for all i s.t. $a_i \neq 0$

Then $F^*D \in \mathrm{Div}(Z)$ is well-defined

Recall $X = \bigcup_\alpha U_\alpha$ and $\forall i, \alpha$ have $f_{\alpha,i} : U_\alpha \to \mathbb{C}$

locally defining f^n for Y_i

"data" of D $\iff$ $\{(U_\alpha, f_{\alpha, \forall i})\}$ $\quad f_\alpha = \prod_i f_{\alpha,i}^{a_i}$ for D

"Cartier divisor"

Then ~~X~~ F^*D corresponds to $\{(F^{-1}(U_\alpha), f_\alpha \circ F)\}$

N.B. when $D = Y$ is an irred hypersurface in X, then F^*D needn't be irreducible and may have "multiplicities"

D given by $\{(U_\alpha, f_\alpha)\}$, define

$$\psi_{\beta\alpha} = \frac{f_\beta}{f_\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}$$

Note $\psi_{\beta\alpha}$ is holo. non-zero well-def on $U_\alpha \cap U_\beta$

And $\psi_{\beta\alpha}\psi_{\gamma\beta}\psi_{\alpha\gamma} \equiv 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (cocycle)

$\Rightarrow$ determine a holomorphic line bundle over X denoted by $[D] \in \mathrm{Pic}(X)$ called an associated line b. to D, $D \in \mathrm{Div}(X)$

Rmk $[D]$ is well-defined:

ambiguity in $\tilde{f}_\alpha = h_\alpha f_\alpha$, h_α holo never zero

$$\tilde{\psi}_{\beta\alpha} = \psi_{\beta\alpha} \cdot \frac{h_\beta}{h_\alpha}, \quad [D]^\sim = [D] \otimes L = [D]$$

as L admits a nowhere zero holo. section

Indeed, $h : X \to L$ given by $h|_{U_\alpha} = h_\alpha : U_\alpha \to \mathbb{C}$

$$h_\beta = \underbrace{\frac{h_\beta}{h_\alpha}}_{\text{cocycle for } L} \cdot h_\alpha : U_\beta \longrightarrow \mathbb{C}$$

If $D = (f)$ then $f_\alpha = f|_{U_\alpha} = \dfrac{\prod g_{\alpha,i}}{\prod h_{\alpha,i}}$

g, h holo irred
$f_\alpha = g/h$ $\Rightarrow \psi_{\beta\alpha} \equiv 1$

so $[D]$ is holo. trivial

Conversely, if $[D]$ holo. trivial then get a nowhere zero holo. section s

$D \to \{(U_\alpha, f_\alpha)\}$ $\quad \underset{s_\alpha}{\overset{\shortparallel}{s|_{U_\alpha}}}$ is a holo nowhere zero s.t.

$$s_\beta = \psi_{\beta\alpha} \cdot s_\alpha = \frac{f_\beta}{f_\alpha} \cdot s_\alpha \quad \text{i.e.} \quad \frac{s_\beta}{s_\alpha} = \frac{f_\beta}{f_\alpha}$$

so $\dfrac{f_\alpha}{s_\alpha} = \dfrac{f_\beta}{s_\beta}$,

patches to a global meromorphic function

$$H^0(\mathcal{K}_X^* / \mathcal{O}_X^*)$$

- if $D=(f)$, then $f_\alpha = f|_{U_\alpha}$

$\psi_{\beta\alpha} = 1$ so $[D]$ is holo. trivial

- if $[D]$ is holo. trivial, then it has a nowhere-zero holo. section s

then over U_α, s is represented by $s_\alpha : U_\alpha \to \mathbb{C}^* \subset \mathbb{C}$

and with $s_\beta = \psi_{\beta\alpha} s_\alpha$ on $U_\alpha \cap U_\beta$

i.e. $\frac{s_\alpha}{s_\beta} = \psi_{\beta\alpha}^{-1} = \frac{f_\alpha}{f_\beta}$ by def^n of $[D]$

now $\frac{f_\alpha}{s_\alpha} = \frac{f_\beta}{s_\beta}$ so glue to a global meromorphic f

and $D = (f)$

Therefore, $[D] = [\tilde{D}]$ in $\mathrm{Pic}(X)$ iff $D \sim \tilde{D}$

- $D + \tilde{D}$ has loc def^n functions $f_\alpha \cdot \tilde{f}_\alpha$ $\left(\begin{array}{l} f_\alpha \text{ for } D \\ \tilde{f}_\alpha \text{ for } \tilde{D} \end{array} \right)$

$\therefore [D + \tilde{D}] = [D] \otimes [\tilde{D}]$

We have thus proved

<u>Prop^n</u>
$$\begin{aligned} \mathrm{Div}(X) &\longrightarrow \mathrm{Pic}(X) \\ D &\longmapsto [D] \end{aligned}$$

is a group hom, whose kernel is the subgroup of principal divisors

- if $F: Z \to X$ holo map of mfds and $F^*D \in \mathrm{Div}(Z)$ is well-defined, then

$$F^*[D] = [F^*D]$$

by considering (pull-back of) the ~~trans~~ loc. def functions

Recall: a sec^n of holo line bundle $L \to X$ is holo iff it is locally expressed by holo f^ns in each holo trivialisation

We can similarly define <u>meromorphic sections</u> of L

(i.e. loc. expressed as a mero. f^n in each holo. triv^n)

<u>Basic properties</u> : • if $s_0 \not\equiv 0$ and s are meromorphic sections, of L say, then $\cancel{A}$ $s = f s_0$ for some mero. f^n f

• Conversely, if s mero. sec^n, f mero f^n, then fs is again a meromorphic section

$\therefore$ choosing $s_0 \not\equiv 0$, obtain $f \longmapsto f s_0$;

an isomorphism b/w $\{$meromorphic f^ns on $X\}$, $\{$meromorphic sec^{n}s of $L\}$

• Let $s \not\equiv 0$ be a mero. section

write $s_\alpha = s|_{U_\alpha}$ for U_α triv nghbd, s_α mero f^n on U_α

Then $\dfrac{s_\alpha}{s_\beta} = \psi_{\alpha\beta}$ holo. never zero

$\Rightarrow$ $\forall$ irred hypersurface $Y \subset X$, $\operatorname{ord}_Y s_\alpha = \operatorname{ord}_Y s_\beta$ (on $U_\alpha \cap U_\beta$)

Thus $\operatorname{ord}_Y s$ is well defined

$\therefore$ $(s) \in \operatorname{Div}(X)$ well-def by $(s) = \sum\limits_{\substack{Y \text{ irred} \\ \text{hyp}}} \operatorname{ord}_Y(s) \{Y\}$

generalises divisor of mero f^ns

• $(s) \geqslant 0$ means s is a holo. section

• D corresponds to $\{(U_\alpha, f_\alpha)\}$ "Cartier divisor"

$f_\beta = \psi_{\beta\alpha} f_\alpha$ (recall defn of $[D]$)

$\Rightarrow$ $[D]$ has a meromorphic section s, with

$s|_{U_\alpha} = f_\alpha$

so $(s) = D$ (reversing the earlier remarks)

$\Rightarrow$ $[(s)] = [D]$ in $\operatorname{Pic}(X)$

Furthermore, we obtain

$\forall$ mero. sec^n of L have $L = [(s)]$

Conversely, $\forall$ $L \to X$ holo line bundle,

(1) $\{ D \in \mathrm{Div}\, X : [D] = L \} \cong \left\{ \begin{matrix} \text{nontrivial mero.} \\ \text{sections of } L \end{matrix} \right\} / \mathbb{C}^*$

[i.e. up to mult^n by nowhere zero holo f^n]

Thus, the image of $\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$ is the subgroup of $\mathrm{Pic}(X)$ of line b. admitting a non-trivial meromorphic section

(2) $\mathcal{L}(D) := \{ f \text{ mero } f^n\text{'s on } X : D + (f) \geq 0 \} \cup \{0\}$

$\cong \left\{ \begin{matrix} \text{v. space of } \cancel{\text{mer}} \text{ holo. secn's} \\ \text{of } [D] \end{matrix} \right\}$

<u>Fact</u>: $\dim_{\mathbb{C}} \mathcal{L}(D) < \infty$ whenever X is compact

<u>Remark</u>: $\exists$ cx mfds ($\dim X \geq 2$) without divisors but with holo bundles

<u>The first Chern class</u>

$L \to X$ a (smooth) complex line bundle over X a (compact) complex manifold

d_A a covariant derivative (corresponding to a connection A)

$d_A : \Gamma(L) \longrightarrow \Gamma(T^*X \otimes L) = \Omega^1_X(L)$

more generally

$d_A : \Omega^r_X(L) \longrightarrow \Omega^{r+1}_X(L)$

locally $d_A s_\alpha = ds_\alpha + A_\alpha s_\alpha$ in triv nghd U_α for L ~~and A~~

$A_\alpha \in \Omega^1(U_\alpha)$

recall transformation rule

$A_\beta = A_\alpha + \varphi_{\beta\alpha} d \varphi_{\beta\alpha}^{-1}$ ($\varphi_{\beta\alpha}$ trans f^n's for L)

recall curvature

$$d_A d_A s = F(A) s \quad , \quad F(A) \in \Omega^2_X(\mathrm{End}\, L)$$
$$\cong \Omega^2_X(\mathbb{C}) \qquad \text{since } rk_{\mathbb{C}} L = 1$$

$$F(A)|_{U_\alpha} = dA_\alpha \qquad ([A,A]=0 \text{ in this case})$$

Then $dF(A) = 0$ but $F(A)$ need not be exact since $\{A_\alpha\}$ not global 1-form

Any other connection on L is $A + a$ where $a \in \Omega^1(X)$

So $F(A+a) = F(A) + \underbrace{da}_{\text{genuinely exact}}$

So $[F(A)] \in H^2(X, \mathbb{C})$ is well def de Rahm class, depends only on L

$$H^2(X,\mathbb{C}) \cong H^2_{dR}(X) \otimes_{\mathbb{R}} \mathbb{C}$$

X is complex n-fold

$L \to X$ a complex line bundle

A is a connection on L

$[F(A)] \in H^2(X, \mathbb{C}) \cong H^2_{dR}(X) \otimes \mathbb{C}$

depends only on L (but not on A)

can choose a <u>Hermitian inner product</u> $\langle \cdot , \cdot \rangle^{h}$ on the fibers of L

suppose the connection A is <u>unitary</u>

$$d\langle s_1, s_2 \rangle = \langle d_A s_1, s_2 \rangle + \langle s_1, d_A s_2 \rangle \quad \forall s_1, s_2 \in \Gamma(L)$$

then in a <u>unitary local triv^n</u> A_α are skew-Hermitian

(i.e. h repr by $\equiv 1$)

$\Downarrow$

A_α is pure imaginary

($\mathrm{rk}\, L = 1$)

$$\therefore \left[\frac{i F(A)}{2\pi}\right] \in H^2_{dR}(X) \quad \text{denoted} \quad c_1(L)$$

Defines the <u>1^{st} Chern class of L</u>

<u>Prop^n</u> $c_1(L \otimes \tilde{L}) = c_1(L) + c_1(\tilde{L})$

in particular $c_1(L^*) = -c_1(L)$

<u>Proof</u> consider $s \in \Gamma(L)$, $\tilde{s} \in \Gamma(\tilde{L})$

$s \otimes \tilde{s}$ rep over each triv nghbd U_α (for L and $\tilde{L}$)

by the product $s_\alpha \cdot \tilde{s}_\alpha$ (locally)

thus $s \otimes \tilde{s} \in \Gamma(L \otimes \tilde{L})$ is well-defined

Let $A, \tilde{A}$ be connections of $L, \tilde{L}$.

Set $d_{A \otimes \tilde{A}}(s \otimes \tilde{s}) := (d_A s) \otimes \tilde{s} + s \otimes (d_{\tilde{A}} \tilde{s})$ [this is a connection]

In a triv^n, do $d(s_\alpha \tilde{s}_\alpha) + (A_\alpha + \tilde{A}_\alpha) \cdot s_\alpha \tilde{s}_\alpha$

Then $d_{A \otimes \tilde{A}}(d_{A \otimes \tilde{A}}(s \otimes \tilde{s})) = (F(A) + F(\tilde{A})) \cdot (s \otimes \tilde{s})$

locally, $d(A_\alpha + \tilde{A}_\alpha) \cdot s_\alpha \tilde{s}_\alpha$

Thus on $L \otimes \tilde{L}$ obtain $\left[\frac{i}{2\pi}(F(A) + F(\tilde{A}))\right]$.

For the last part, note that for the trivial line bundle $L^* \otimes L$

we can choose a flat connection / ~~[illegible]~~ global 1-form A. $\square$

<u>Propⁿ</u> (Chern connection – special case of cx line b.)

Suppose L is a holomorphic line b. with a Hermitian inner product on fibres :

Then $\exists!$ conn. A on L s.t. (i) A is unitary

(ii) in any <u>holo</u> triv^n of L over say $\mathcal{U}_\alpha$, $A_\alpha \in \Omega^{1,0}(\mathcal{U}_\alpha)$

<u>Proof</u> wlog $\mathcal{U}_\alpha$ is also a coord ngbd

consider (loc.) holo section $e : \mathcal{U}_\alpha \longrightarrow \mathbb{C}$, $e(z) \equiv 1$ coords $z \in \mathbb{C}^n$

the Hermitian product $h_\alpha(z) = |e(z)|^2 = \langle e(z), e(z) \rangle$

any section over $\mathcal{U}$ is λe { drop α subscript }

for some $\lambda : \mathcal{U} \longrightarrow \mathbb{C}$

for (i) we require

$$d|s|^2 = \langle d_A s, s \rangle + \langle s, d_A s \rangle$$

$$\underset{\text{over } \mathcal{U}}{=} \langle (d\lambda + A\lambda)e, \lambda e \rangle + \langle \lambda e, (d\lambda + A\lambda)e \rangle$$

$$= h\bar{\lambda}\, d\lambda + h\lambda\, d\bar{\lambda} + h|\lambda|^2(A + \bar{A})$$

also have

$$d|s|^2 = h\bar{\lambda}\, d\lambda + h\lambda\, d\bar{\lambda} + |\lambda|^2 dh$$

$\therefore$ we must have

$$A + \bar{A} = \frac{dh}{h}$$

(ii) requires $(1,0)$ $(0,1)$ and so $A = A^{1,0} = \dfrac{\partial h}{h} = \partial \log h$ (on $\mathcal{U}$)

for any other holo. loc. triv^n, say $\mathcal{U}_\beta$ with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$,

with transition function $\psi_{\alpha\beta}$ s.t. $\bar{\partial}\psi_{\alpha\beta} = 0$

then $d\psi_{\alpha\beta} = \partial\psi_{\alpha\beta}$, $\partial\bar{\psi}_{\alpha\beta} = 0$

Then $h_\beta = \psi_{\alpha\beta}\bar{\psi}_{\alpha\beta} h_\alpha$

Thus $A_\beta = \partial \log h_\beta = \partial \log h_\alpha + \dfrac{\partial(\psi_{\beta\alpha}^{-1}\overline{\psi_{\beta\alpha}^{-1}})}{\psi_{\beta\alpha}^{-1}\overline{\psi_{\beta\alpha}^{-1}}}$

$$\therefore\ A_\beta = A_\alpha + \psi_{\beta\alpha}\, d(\psi_{\beta\alpha}^{-1})$$

which is the correct transformation law. Q.E.D.

<u>Cor 1</u> <u>The curvature of the Chern connection</u>

$$\boxed{F(A) = \bar\partial \partial \log |e|^2 = \frac{i}{2}\, d d^c \log |e|^2}$$

where e is some/any local holo section of L w/o zeroes

<u>Exercise</u> 1. Prove Cor 1

2. Explain why $F(A)$ need not be $\bar\partial$ exact

<u>Cor. 2</u> $\boxed{\frac{i}{2\pi}[F(A)] = c_1(L) \in H^{1,1}(X)}$

from Cor. 1

<u>Remark</u> In topology, c_1 of cx vect b is defined as a class in $H^2(X;\mathbb{Z})$

$H^*_{dR}(X) \cong H^*(X;\mathbb{R})$, $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces hom

$$\begin{array}{ccc} H^2(X;\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{R}) \\ & & \parallel \\ & & H^2_{dR}(X) \end{array}$$

"Our" $c_1(L)$ is the image of "topologists'" $c_1(L)$ under the above map.

X a compact connected cx n-fold

[ES3 handed out
EC2 on Thursday
nominated sheet 2
q's are 1, 2]

$Y \subset X$ analytic hypersurface ($\Rightarrow Y$ also cpct)

$\forall\, \varphi \in \Omega^{2n-2}(X)$ with $d\varphi = 0$, consider

lin. functional $[\varphi] \in H^{2n-2}_{dR}(X) \longrightarrow \int_{Y^*} \varphi \in \mathbb{R}$

use orientation on Y^* coming from cx structure

By the Poincaré duality

$$\exists!\ \eta_Y \in H^2_{dR}(X) \text{ s.t. } \int_{Y^*} \varphi = \int_X \eta_Y \wedge \varphi$$

i.e. $\eta_Y = \text{P.D.}[Y]$, $[Y] \in H_{2n-2}(X, \mathbb{R})$ image of $[Y] \in H_{2n-2}(X;\mathbb{Z})$

$\forall\, D \in \mathrm{Div}(X)$, $D = \sum a_i Y_i$,

define $\eta_D = \sum a_i \eta_{Y_i} \in H^2_{dR}(X)$

<u>Prop^n</u> $\eta_D = c_1([D])$

<u>Cor</u> $c_1([D])$ is in the image of the natural hom

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}) \cong H^2_{dR}(X)$$

<u>Proof of Prop^n</u> We are to show that $\forall\, \varphi \in \Omega^{2n-2}(X)$ s.t. $d\varphi = 0$, have

$$\frac{i}{2\pi} \int_X F(A) \wedge \varphi = \sum_i a_i \int_{Y_i} \varphi$$

where $D = \sum a_i Y_i$ and A is a connection on $[D]$

Let A be the Chern connection for some choice of Hermitian metric on the fibres of $[D]$.

Wlog $D = Y_i$ ("one" hypersurface)

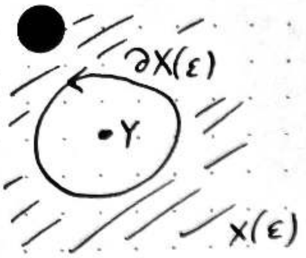
$X = \bigcup_{\alpha=1}^N U_\alpha$, f_α loc. def f^n for Y on U_α

i.e. $Y = (s)$, s is mero. section of $[D] = [Y]$

where $s|_{U_\alpha} = s_\alpha = f_\alpha$

In fact s is holo. section

Put $X(\varepsilon) = \{ p \in X : |s(p)| > \varepsilon \}$ for $\varepsilon > 0$



"cross-section"

Then $\int_X F(A) \wedge \varphi$

$$= \frac{i}{2} \lim_{\varepsilon \to 0} \int_{X(\varepsilon)} (d d^c \log |s|^2) \wedge \varphi$$

$$\overset{\text{Stokes}}{=} -\frac{i}{2} \lim_{\varepsilon \to 0} \int_{\partial X(\varepsilon)} (d^c \log |s|^2) \wedge \varphi$$

[note the choice of orientation on $\partial X(\varepsilon)$]

$$|s|^2 \big|_{U_\alpha \cap (X \setminus X(\varepsilon))} = |f_\alpha|^2 h_\alpha = f_\alpha \overline{f_\alpha} h_\alpha$$

$h_\alpha > 0$ is local expression for Hermitian norm on fibres of $[D]$ over U_α

$$d^c \log |s|^2 = i (\bar\partial - \partial) \log (f_\alpha \bar f_\alpha h_\alpha)$$
$$= i \left[\bar\partial \log \bar f_\alpha - \partial \log f_\alpha + (\bar\partial - \partial) \log h_\alpha \right]$$

$\mathrm{vol}(\partial X(\varepsilon)) \longrightarrow 0$ as $\varepsilon \to 0$

h_α bounded away from 0 (good choice of U_α)

∇h_α bounded

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{\partial X(\varepsilon) \cap U_\alpha} (d^c \log h_\alpha) \wedge \varphi = 0$$

φ is a real diff'l form, so

$$\int_{\partial X(\varepsilon) \cap U_\alpha} (\bar\partial \log \bar f_\alpha) \wedge \varphi = \overline{\int_{\partial X(\varepsilon) \cap U_\alpha} (\partial \log f_\alpha) \wedge \varphi}$$

$$\therefore \lim_{\varepsilon \to 0} -\frac{i}{2} \int_{\partial X(\varepsilon) \cap U_\alpha} (d^c \log |s|^2) \wedge \varphi$$

$$= \lim_{\varepsilon \to 0} -i \,\mathrm{Im} \int_{\partial X(\varepsilon) \cap U_\alpha} (\partial \log f_\alpha) \wedge \varphi \qquad (*)$$

choose loc. coords on U_α s.t. $f_\alpha(z) = z_1$

can assume $U_\alpha \cong$ coord polydisc in $\mathbb{C}^n$

write $\varphi = \tilde{\varphi} + \varphi_1$

where $\tilde{\varphi}$ is all summands containing dz_1 or $d\bar{z}_1$

So $-i\,\mathrm{Im} \lim_{\varepsilon \to 0} \int_{\substack{\partial X(\varepsilon) \cap U_\alpha \\ = \{|z_1| = \frac{\varepsilon}{\sqrt{h_\alpha}}\}}} \frac{dz_1}{z_1} \wedge \varphi_1(z_1, \dots)$

[integral of $dz_1 \wedge d\bar{z}_1$ doesn't contribute]

$= -i\,2\pi \int_{z_1 = 0} \varphi_1(0, \dots)$ i.e. the residue

[business w/ behaviour of h_α, all ok locally enough?]

Thus, patch over U_α's

$$\lim_{\varepsilon \to 0} -\frac{i}{2} \int_{\partial X(\varepsilon)} (d^c \log |s|^2) \wedge \varphi = -2\pi i \int_Y \varphi$$

i.e. $\int_X F(A) \wedge \varphi = -2\pi i \int_Y \varphi$ Q.E.D.

<u>Examples</u> $X = S$ a Riemann surface, cpct, connected

$D = \sum a_i P_i$, P_i points of S, $D \in \mathrm{Div}(S)$

$\forall P \in S$, generate $H_0(S;\mathbb{Z})$ with $[P]$

have a hom $\mathrm{Div}(S) \xrightarrow[\deg]{} \mathbb{Z}$

where $\deg D = \sum a_i$

$\forall P \in S$, $\eta_P = \text{P.D.}\,[P] \in H^2(S, \mathbb{Z}) \cong \mathbb{Z}$

generated by any $\eta_P = [\varphi]$, $\varphi \in \Omega^2(S)$, $\int_S \varphi = 1$

If L is cx line bundle over S, define $\deg L := \langle c_1(L), [S] \rangle$

element of $\mathbb{Z}$ since $c_1(L) \in H^2(S;\mathbb{Z})$

If $L = [D]$ then the definitions agree:

$-\frac{1}{2\pi i} \int_S F(A) = \deg D$ by the previous prop^n

As $\deg : \mathrm{Div}\, D \to \mathbb{Z}$ is surjective, $\exists$ holo line b. with meromorphic sections for each value of c_1

Recall, for $L \in \mathrm{Pic}(S)$, S a compact Riemann surface

[Reminder Examples Class 2 today at 3:30pm MR4

Copies of sheet 3 available

I'll mark q's 1, 2]

$$\deg L := \langle c_1(L), [S] \rangle \in \mathbb{Z}$$

[$c_1(L)$ is always integral, see 4.4.12]

(= 1st Chern number in topology)

$[S] \in H_2(S, \mathbb{Z})$

Our Prop^n asserts that if $L = [D]$,

then $\underset{\in \mathrm{Div}(S)}{\deg D} = -\frac{1}{2\pi i}\int_S F(A) = \underset{\in \mathrm{Pic}(S)}{\deg L}$

$\deg : \mathrm{Div}(S) \to \mathbb{Z}$ is clearly surjective

$\therefore \exists$ holo line b. with $\neq 0$ mero. sec^n over S and for any value of $c_1(S)$ (i.e. any degree)

Now let $S = \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$

- non-constant holo. maps $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ are precisely the rational functions
- every rat^l f^n has the <u>same</u> # zeros and # poles in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ (w/ multiplicities of course)
- $\sum_{\text{finite}} a_i P_i \sim \sum_{\text{finite}} b_j Q_j \iff \sum a_i = \sum b_j$

  linear eq. in $\mathrm{Div}(\mathbb{CP}^1)$

Consider $\mathbb{C}^2 \setminus \{(0,0)\} \xrightarrow{\pi} \mathbb{CP}^1$ the Hopf b. $\mathcal{O}(-1)$

Then the map $[z_1 : z_2] \longmapsto (1, z_2/z_1)$

$$\mathbb{CP}^1 \longrightarrow \mathbb{C}^2 \setminus \{(0,0)\}$$

induces a mero. section of $\mathcal{O}(-1)$.

Locally $s_1 \equiv 1$ on $U_1 = \{z_1 \neq 0\} \subset \mathbb{CP}^1$

$s_2([z:1]) = \frac{1}{z}$ on $U_2 = \{z_2 \neq 0\}$

Thus $\exists!$ pole at $[0:1]$, of order 1.

Thus $\deg \mathcal{O}(-1) = -1$

Generally $\mathcal{O}(k)$ has degree k.

<u>Prop^n</u> Let $L \to \mathbb{CP}^1$ be holo. line b with $c_1(L) = 0$

Then L is holomorphically trivial.

<u>Cor</u> $\mathrm{Pic}(\mathbb{CP}^1) = \{\mathcal{O}(n) : n \in \mathbb{Z}\} \cong \mathbb{Z}$

<u>Proof of Prop^n</u> If $c_1(L) = 0 \Rightarrow L$ is C^∞-trivial as $F(A)$ is exact and A can be represented by a global 1-form; thus get a trivialisation.

Fix a nowhere-zero C^∞ section s of L.

$L \xrightarrow{\cong} \mathbb{CP}^1 \times \mathbb{C}$ choose a Hermitian metric on fibres and let A be the Chern connection

$$d_A = \underbrace{\partial_A}_{(1,0)\text{ component}} + \underbrace{\bar\partial_A}_{(0,1)\text{ component}}$$

A section is holomorphic iff $\bar\partial_A s = 0$

Want a global section w/o zeros and $\bar\partial_A s = 0$.

Consider $s = e^f : \mathbb{CP}^1 \to \mathbb{C}$, f smooth

$$\bar\partial_A s = \bar\partial s + A'' s = 0 \iff \bar\partial f = -A''$$

$$A'' \in \Omega^{0,1}(\mathbb{CP}^1)$$

Consider $\mathbb{CP}^1 = U_1 \cup U_2$, $U_1 = \mathbb{C}$, $U_2 = \mathbb{C}^* \cup \{\infty\}$

$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, coord z on U_1, coord $\zeta = \frac{1}{z}$ on U_2

By $\bar\partial$-Poincaré lemma, $\exists f_j : U_j \to \mathbb{C}$ with $\bar\partial f_j = -A''|_{U_j}$, $j = 1, 2$

So $\bar\partial(f_1 - f_2) = 0$ on $\mathbb{C}^*$

So $f_1(z) - f_2(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{C}^*$

Then $f = \begin{cases} f_1 - \sum_{n=0}^{\infty} c_n z^n & \text{on } U_1 \\ f_2 + \sum_{n=-\infty}^{-1} c_n z^n & \text{on } U_2 \end{cases}$ well-def on $\mathbb{CP}^1$

And $\bar\partial f = -A''$. QED

<u>Remarks</u> · In fact $\mathrm{Pic}(\mathbb{CP}^n) = \{\mathcal{O}(k)\} \cong \mathbb{Z}$ for all $n \geq 1$

· In particular $\mathcal{O}(1)$ has holo section s with
$(s) = H_0 = \{ \underline{z} \in \mathbb{CP}^n : z_0 \cancel{\neq} = 0 \} \cong \mathbb{CP}^{n-1}$ a hyperplane

$H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$, then $c_1(\mathcal{O}(k)) = k$ via this iso

· Ex Sheet 3, Q9, compute $E = \mathbb{C}/\Lambda$ has $\mathrm{Pic}(E) \cong \mathbb{Z} \oplus E$

<u>Def^n</u> Let $Y \subset X$ be a non-singular (smooth) analytic hypersurface. Then the <u>normal bundle</u> of Y is

$$N_{Y/X} = \frac{(T^{1,0}X)|_Y}{T^{1,0}Y} \quad \text{the quotient b.}$$

fibre of $N_{Y/X}$ is $T_p^{1,0}X / T_p^{1,0}Y$, $p \in Y$

The <u>conormal bundle</u> $N^*_{Y/X}$ is the dual of $N_{Y/X}$
The fibre at $p \in Y$ is

$$\{ \alpha \in (T_p^*X)^{1,0} : \alpha|_{T_p^{1,0}Y} = 0 \}$$

$N^*_{Y/X}$ is thus a sub-bundle of $(T^*X)^{1,0}|_Y$
(exercise – check this!)

Let f_α be loc. def f^n on $U_\alpha \subset X$.
Then $df_\alpha|_{Y \cap U_\alpha} = 0$ but $(df_\alpha)_p \neq 0$
in $(T_p^*X)^{1,0}$ $\forall p \in Y \cap U_\alpha$

$\therefore$ df_α defines a local nowhere zero holo section of $N^*_{Y/X}$

Recall $\psi_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ are transition functions of $[Y] \in \mathrm{Pic}(X)$

Transition f's for $N^*_{Y/X}$ are

$$\tilde{\psi}_{\alpha\beta} = \psi_{\beta\alpha}\big|_{Y \cap \mathcal{U}_{\alpha\beta}} = \psi^{-1}_{\alpha\beta}\big|_{Y \cap \mathcal{U}_{\alpha\beta}}$$

as $df_\alpha = d(\psi_{\alpha\beta} f_\beta) = (d\psi_{\alpha\beta}) \underbrace{f_\beta}_{\substack{\text{zero}\\ \text{on } Y}} + \psi_{\alpha\beta}\, df_\beta$

Indeed $s_\beta\, df_\beta = s_\alpha\, df_\alpha$

iff $s_\alpha = \psi^{-1}_{\alpha\beta}\, s_\beta$

So $[Y]\big|_Y \otimes N^*_{Y/X}$ is holo. trivial

$$\boxed{N^*_{Y/X} = [-Y]\big|_Y}$$

Adjunction formula 1

have proved

Adjunction formula I $\boxed{N^*_{Y/X} = [-Y]|_Y}$

f_α a loc. def f^n of Y on U_α

f_α extends to loc. cx coords $f_\alpha, \underbrace{\xi_2, \dots, \xi_n}_{\text{loc. coords on } Y}$ on $U_\alpha \subset X$

any holo loc sec of K_X on U_α is $h \, df_\alpha \wedge \omega_{Y,\alpha}$

loc. sec of K_Y pulled back via $(f, \xi) \mapsto \xi$

then on $U_\alpha \cap U_\beta$, have $\xi^{(\beta)} = F_{\beta\alpha}(\xi^{(\alpha)})$

$$f_\beta = G_{\beta\alpha}(f_\alpha, \xi^{(\alpha)}) f_\alpha$$

with $G_{\beta\alpha}(0, \xi^{(\alpha)}) = \psi_{\beta\alpha}(\xi^{(\alpha)})$

we find $\boxed{K_X|_Y = N^*_{Y/X} \otimes K_Y}$

Adjunction formula II $\boxed{K_Y = (K_X \otimes [Y])|_Y}$

can be used to determine K_Y for hypersurfaces in $\mathbb{CP}^n$

more generally, any nice projective manifold

The canonical bundel of $\mathbb{CP}^n$

$[z_0 : \dots : z_n]$

$U_i = \{z_i \neq 0\} \subset \mathbb{CP}^n$, $i = 0, \dots, n$

cx coords on U_0 are $w_j = \frac{z_j}{z_0}$, $j \neq 0$

on U_0, $\omega = \frac{dw_1}{w_1} \wedge \dots \wedge \frac{dw_n}{w_n}$ mero. section of $K_{\mathbb{CP}^n}$

$H_j = \{z_j = 0\} \subset \mathbb{CP}^n$

$\mathrm{ord}_{H_j}(\omega) = -1$ for $j = 1, \dots, n$

On U_j, coords $\tilde{w}_k = \frac{z_k}{z_j} = \frac{w_k}{w_j}$ if $k \neq 0$

$\frac{1}{w_j}$ if $k = 0$

$w_i = \frac{\tilde{w}_i}{\tilde{w}_0}$ if $i \neq j$

$w_j = \frac{1}{\tilde{w}_0}$ so $\frac{dw_i}{w_i} = \frac{d\tilde{w}_i}{\tilde{w}_i} - \frac{d\tilde{w}_0}{\tilde{w}_0}$, $i \neq j$, $i \neq 0$

$$\frac{dw_j}{w_j} = -\frac{d\tilde{w}_0}{\tilde{w}_0}$$

and $\omega = (-1)^j \frac{d\tilde{w}_0}{\tilde{w}_0} \wedge \cdots \wedge \widehat{\frac{d\tilde{w}_j}{\tilde{w}_j}} \wedge \cdots \wedge \frac{d\tilde{w}_n}{\tilde{w}_n}$

Thus we get $n+1$ simple poles, one along each hyperplane H_j. Since $\frac{z_i}{z_j}$ is meromorphic on $\mathbb{CP}^n$, have $H_i \sim H_j$ in $\mathrm{Div}\, \mathbb{CP}^n$ $\forall i, j$.

Hence $\boxed{K_{\mathbb{CP}^n} = [(\omega)] = [-(n+1) H] = \mathcal{O}(-n-1)}$

c.f. Q6 in Sheet 2.

As $\mathcal{O}(-1) = [-H_0]$, mero. section locally $s|_{U_0} \equiv 1$

$s|_{U_j} = \frac{z_j}{z_0}$

Blow-up

$\Delta \subset \mathbb{C}^n$ polydisc about 0

$\tilde{\Delta} = \{ (z, w) \in \Delta \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \ \forall i, j \}$ cx mfd

i.e. $z \in$ line given by w

charts: for each std $h_j : \underset{\mathbb{CP}^{n-1}}{\underset{\cap}{U_j}} \longrightarrow \mathbb{C}^{n-1}$, put

$$\underset{(\Delta \times U_j) \cap \tilde{\Delta}}{\underset{\Cap}{\hat{h}_j (z, w)}} = \underset{\mathbb{C}^n}{\underset{\Cap}{(h_j(w), z_j)}}$$

Def $\sigma : \tilde{\Delta} \to \Delta$

$(z,w) \mapsto z$ is the blow-up of Δ at 0

$\tilde{\Delta} \setminus \sigma^{-1}(0)$ is mapped biholomorphically onto $\Delta \setminus \{0\}$

$\sigma^{-1}(0) \cong \mathbb{CP}^{n-1}$

Informally $\tilde{\Delta}$ means "lines through 0 in Δ are made distinct"

- The blow-up is trivial in case $n = 1$
- Let $\Delta = \mathbb{C}^n$. Then the 2^{nd} proj $\tilde{\mathbb{C}}^n \longrightarrow \mathbb{CP}^{n-1}$ really $\mathcal{O}(-1)$

  and charts of $\tilde{\mathbb{C}}^n$ correspond to loc. triv's of $\mathcal{O}(-1)$
- Can generalise to any cx n-fold X.

  Let $x \in X$, $U \subset X$ a coord polydisc with chart φ

$$\varphi : U \xrightarrow{\cong} \Delta \subset \mathbb{C}^n$$
$$x \longmapsto 0$$

Put $\tilde{X} := (X \setminus \{x\}) \cup_{\varphi^{-1} \circ \sigma} \tilde{\Delta}$

identify via

$\tilde{\Delta} \setminus \sigma^{-1}(0) \cong U \setminus \{x\}$

We obtain a holo. map $\pi : \tilde{X} \to X$, the blow-up of X at x

$\pi^{-1}(x) = E$ is the exceptional divisor

$E \cong \mathbb{CP}^{n-1}$, $E \in \mathrm{Div}\, \tilde{X}$

Prop^n $[E]|_E = \mathcal{O}(-1)$

Pf In loc. coords near E, $E \cap (\Delta \times U_j) = \{(z,w) \mid z = 0\}$

$\psi_{ij}(w) = \dfrac{z_i}{z_j} = \dfrac{w_i}{w_j}$

Q.E.D.

<u>Lemma</u> $\tilde{X}$ is independent of the choice of coord chart φ

<u>Pf</u> $z_j' = f_j(z)$ new cx coords

$$w_j' = \sum_i \frac{\partial f_j}{\partial z_i}(0)\, w_i$$

$$\begin{tikzcd} \tilde{\Delta} \arrow[r, "F"] \arrow[d, "\sigma"'] & \tilde{\Delta} \arrow[d, "\sigma'"] \\ \Delta \arrow[r, "f"'] & \Delta \end{tikzcd}$$

claim $F(z, w) = (z', w')$ as above is biholomorphic, makes diagram commute

If f given by $(A^i_j) \in G(n, \mathbb{C})$ then

$$z_i' w_j' = \sum_k A^k_i z_k \sum_L A^L_j w_L$$

$$\overset{!}{=} \sum_{kL} A^k_i A^L_j z_L w_k = z_j' w_i'$$

completing the proof of Lemma

$$\begin{tikzcd} \tilde{\Delta} \arrow[r, "F"] \arrow[d, "\sigma"'] & \tilde{\Delta}' \arrow[d, "\sigma'"] \\ \Delta \arrow[r, "f"] & \Delta \end{tikzcd}$$

recall : $F(z,w) = (z', w')$

where $z' = f(z)$ biholo change of cx coords

$$w'_j = \sum_i \frac{\partial f_j}{\partial z_i}(0)\, w_i$$

[this is not correct in general]

suppose f is (cx) linear, given by $A^i_j \in GL(n, \mathbb{C})$

then $z'_i w'_j = \sum_K A^K_i z_K \sum_L A^L_j w_L$

$$= \sum_{K,L} A^K_i A^L_j z_L w_K = z'_j w'_i$$

thus F biholo & diagram commutes

$\therefore$ wlog $\frac{\partial f_j}{\partial z_i}(0) = \delta_{ij}$ so $w'_j = w_j \;\; \forall j$

in loc. coords $\tilde{\Delta}$ (as considered before) have

$w_1, \dots, \hat{w}_j, \dots, w_n$, $f_j(z) = z_j$ + higher order terms [what is this spinach?]

thus $(dF)_p$ = identity of $T_p\tilde{\Delta}$ $\forall p \in \tilde{\Delta}$

$\Rightarrow F$ biholo by IFT Q.E.D.

<u>Prop^n</u> $\sigma : \tilde{X} \to X$ the blow-up of X at $x \in X$

Then $K_{\tilde{X}} = \sigma^* K_X \otimes [(n-1)E]$, where $n = \dim_{\mathbb{C}} X$.

<u>Proof</u> – assuming K_X admits non-trivial mero. sections

i.e. let ω be a mero. $(n,0)$-form on X

zeros & poles of $\sigma^*\omega$ away from $E \subset \tilde{X}$ are biholo to those of ω, with the same orders

near $x \in X$, have $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$, $\bar{\partial} f = 0$

σ in coords is

$$\sigma|_{U_j} : (\underbrace{v_1, \dots, \hat{v}_j, \dots, v_n}_{\mathbb{CP}^{n-1}}, z) \longmapsto (zv_1, \dots, \underset{\substack{\uparrow \\ j^{th}\text{ position}}}{z}, \dots, zv_n)$$

$$\Rightarrow \sigma^*\omega = (f \circ \sigma)\, d(zv_1) \wedge \cdots \wedge dz \wedge \cdots \wedge d(zv_n)$$

$$= (f \circ \sigma)\, z^{n-1}\, dv_1 \wedge \cdots \wedge \underset{\substack{\uparrow \\ j^{th}}}{dz} \wedge \cdots \wedge dv_n$$

$\Rightarrow$ thus an "extra" zero, order $n-1$,
along $E \cap U_j = \{z = 0\}$ Q.E.D.

- <u>Def^n</u> $\forall$ cx manifold X, define $c_1(X) = -c_1(K_X)$ is the 1st Chern class of X

<u>Cor</u> $c_1(\tilde{X}) = \sigma^* c_1(X) - (n-1)\, P.D.[E]$

<u>Remark</u> (for topologists) When $\dim_{\mathbb{C}} X = 2$,

$$\deg([E]|_E) = -1 = \int_E c_1([E]) = E \cdot E$$

self-intersection number

- <u>Blow-up as a connected sum</u>

Let M_1, M_2 be smooth real mfds, $\dim M_1 = \dim M_2 = m$

Choose $p_1 \in M_1$, $p_2 \in M_2$

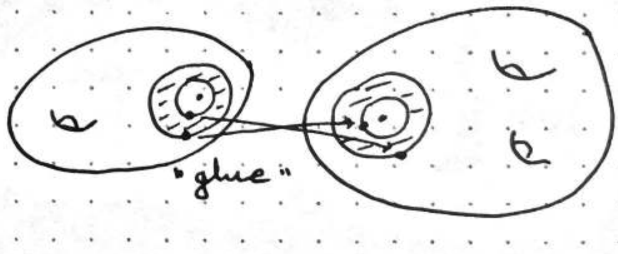
$\varphi_i : U_i \subset M_i \longrightarrow \mathbb{R}^m$ charts near p_i

s.t. $\varphi_i(U_i) = B_0 = \{x \in \mathbb{R}^m : \|x\| \leq 3\}$

$$\xi : \{\tfrac{1}{2} < \|x\| < 2\} \longrightarrow \{\tfrac{1}{2} < \|x\| < 2\}$$

$$x \longmapsto \frac{x}{\|x\|^2}$$

diffeo of a spherical shell



- The the <u>connected sum of M_1 and M_2 at p_1, p_2</u> is

$$M_1 \# M_2 = \left[M_1 \setminus \varphi_1^{-1}(\|x\| \leq \tfrac{1}{2})\right] \bigcup_{\varphi_2^{-1} \circ \xi \circ \varphi_1} \left[M_2 \setminus \varphi_2^{-1}(\|x\| \leq \tfrac{1}{2})\right]$$

$M_1 \# M_2$ is independent (iii) of charts φ_1, φ_2

If M_1, M_2 both oriented & φ_1 preserves the orientation, φ_2 reverses orientation, then since ξ is orientation-reversing,

- obtain $M_1 \# M_2$ oriented with appropriate choice of oriented atlases of M_1, M_2

<u>Prop^n</u> Let X be a complex n-fold.

Then the blow-up <u>at</u> x is diffeo (as an oriented real mfd) to $X \# \overline{\mathbb{CP}^n}$ at x and any pt in $\overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ is the underlying real mfd for $\mathbb{CP}^n$ but with orientation reversed from that of cx structure.

<u>Proof</u> Wlog $X = \Delta$ polydisc in $\mathbb{C}^n$, $x = 0$

to show: $\tilde{\Delta}$ is orientation-preserving diffeo to $\overline{\mathbb{CP}^n} \setminus$ (coord ball)

$$\tilde{\Delta} = \{ (z, w) \in \Delta \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \ \forall i, j \}$$

$$\overline{\mathbb{CP}^n} = \{ \overline{z}_0 : z \mid z_0 \in \mathbb{C}, z \in \mathbb{C}^n, |z_0|^2 + \|z\|^2 \neq 0 \} / \sim$$

$$\varphi : U = \{ 1 : z \} \longrightarrow \mathbb{C}^n$$

$$1 : z \longmapsto z$$

(holo) orientation reversing chart

$$\overline{\mathbb{CP}^n} \setminus \overbrace{\varphi^{-1}(\|z\| < \tfrac{1}{2})}^{K} = \{ \overline{z}_0 : z \mid \|z\| > \tfrac{1}{2} |z_0| \}$$

gluing map for $\#$

$$\psi : \overline{\mathbb{CP}^n} \setminus K \longrightarrow \tilde{\Delta}$$

$$\overline{z}_0 : z \longmapsto \Big(\underset{\in \mathbb{C}^n}{\frac{z_0}{\|z\|^2} z}, \underset{\in \mathbb{CP}^{n-1}}{\pi(z)} \Big)$$

a diffeo onto $\sigma^{-1}(\|z\| < 2) \subset \tilde{\Delta}$

where $\sigma : \tilde{\Delta} \to \Delta$ is blow-up

$$\pi : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{CP}^{n-1}$$

Then for $\frac{|z_0|}{2} < \|z\| < 2|z_0|$ have $\sigma \circ \psi(1 : z) = \frac{z}{\|z\|^2}$

Use the id. chart on Δ, to obtain that as $\sigma \circ \psi$ reverses orientation, σ is holo., ψ reverses orientation wrt complex structures.

So ψ is o. preserving for $\overline{\mathbb{CP}^n}$

Q.E.D.

<u>Addendum</u>: Fixing the proof at the start for $\frac{\partial f_j}{\partial z_i}(0) = \delta_{ij}$

First off, the map F is forced at $z \neq 0$, since there is a unique line containing $f(z) \neq 0$.

We do set $F(0, w) = (0, w)$ for $w \in \mathbb{CP}^{n-1}$.

Then F is bijective and it remains to check it is a biholo^{m}.

Say $(z, w) \in \tilde{\Delta}$ has $w_j \neq 0$, $z = 0$.

Then we can find a ngbd contained in $\{w_j \neq 0\}$ whose image under F is again in $\{w_j' \neq 0\} \subset \tilde{\Delta}'$.

Locally $\tilde{\Delta}$ has holo coords $w_1, \dots, \hat{w}_j, \dots, w_n, z_j$ and similarly for $\tilde{\Delta}'$.

Then F has the form

$$(w_1, \dots, \hat{w}_j, \dots, w_n, z_j) \mapsto \left(\frac{f_1}{f_j}, \dots, \widehat{\frac{f_j}{f_j}}, \dots, \frac{f_n}{f_j}, f_j \right)$$

where the f_i are evaluated at $(z_1, \dots, z_n) = (w_1 z_j, \dots, z_j, \dots, w_n z_j)$

By assumption $f_i(z_1, \dots, z_n) = z_i + \text{h.o.t.}$

In terms of the $(w_1, \dots, \hat{w}_j, \dots, w_n, z_j)$ get

$$\frac{f_i}{f_j} = \frac{w_i z_j + z_j^2 \cdot (\text{holo in } \uparrow)}{z_j + z_j^2 \cdot (\text{holo in } \uparrow)} = \frac{w_i + z_j(-)}{1 + z_j(-)}$$

which is holo. and extends over $(0, w)$ with $w_i' = w_i$ as desired.

III HERMITIAN AND KÄHLER GEOMETRY

<u>Def^n</u> A <u>Hermitian metric</u> on a cx mfd X is a (positive-definite) Hermitian inner product h on the holo. tangent b.

$$h(p): T_p^{1,0} \times T_p^{1,0} \longrightarrow \mathbb{C} \quad \text{smooth in } p \in X$$

i.e. $\forall$ smooth sec's A, B of $T^{1,0}X$ we have $h(A,B) \in C^\infty(X)$

In loc coords, $h = \sum_{i,j} h_{i\bar{j}}(z)\, dz_i\, d\bar{z}_j$,
smooth coefs $h_{i\bar{j}}(z)$, so if

$$A = \sum_i A_i \frac{\partial}{\partial z_i}, \quad B = \sum_j B_j \frac{\partial}{\partial z_j}$$

then $h(A,B) = \sum_{i,j} h_{i\bar{j}} A_i \overline{B_j}$

<u>Prop^n</u> There is a natural equivalence between

- Hermitian metrics on X, and
- J-invariant Riemannian metrics g on the underlying real mfd $X^{\mathbb{R}}$, i.e. $g(JA, JB) = g(A,B)$ where $J \in \Gamma(\operatorname{End} TX^{\mathbb{R}})$ is the almost-cx structure

<u>Proof</u> Recall $\gamma: T_x X \longrightarrow T_x^{1,0} X$
$$e \longmapsto e - iJe, \quad \text{isom. of real v.s.}$$

$\gamma(Je) = i\gamma(e)$

"$\Rightarrow$" given h, define $g(u,v) := \frac{1}{2}\operatorname{Re} h(u - iJu, v - iJv)$

$h(iA, iB) = h(A,B) \Rightarrow g(Ju, Jv) = g(u,v)$

"$\Leftarrow$" given g a J-invariant Riem metric, extend to a Hermitian metric h on $TX \otimes_{\mathbb{R}} \mathbb{C}$, i.e.

$$h(\lambda u, \mu v) := \lambda\bar{\mu}\, g(u,v) \quad \forall u,v \in TX,\ \forall \lambda,\mu \in \mathbb{C}$$

Restrict to the subspaces $T_x^{1,0} \subset T_x X \otimes \mathbb{C}$

Check this is inverse to the above. Q.E.D.

In coords, $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ so

- $g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = g\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right) = 2\,\mathrm{Re}\ h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)$

$\therefore$ we can use the concepts of Riem. geometry for <u>Hermitian mfds</u> (X, h) via J-invt Riem. metric (the "Re" of h)

<u>Prop^n – defn</u> $\omega(u,v) = -\frac{1}{2}\,\mathrm{Im}\ h(u - iJu, v - iJv)$ (*)
defines a real $(1,1)$-form, the <u>fundamental form</u> of h.
Furthermore, $\boxed{\omega(u,v) = g(Ju, v)}$ and

- any two of ω, g, J determine the third.

<u>Proof</u> $\omega \in \Omega^{1,1} \iff \omega(Ju, Jv) = \omega(u,v)$
In (*) the action of J on u,v in LHS becomes mult^n by i on $\gamma(u), \gamma(v)$ in RHS.
h Hermitian $\Rightarrow$ "i-invariant"
Also, $-\frac{1}{2}\,\mathrm{Im}\{h(u - iJu, v - iJv)$
$= \frac{1}{2}\,\mathrm{Re}\ h(i(u - iJv), v - iJv)$
$= g(Ju, v)$.

- Last part left as an exercise (easy) Q.E.D.

In coords, $g = 2\sum_{i,j}\left[\begin{array}{l}(\mathrm{Re}\ h_{i\bar{j}})(dx_i dx_j + dy_i dy_j) \\ + (\mathrm{Im}\ h_{i\bar{j}})(dx_i dy_j - \underbrace{\cancel{dx_j dy_i}}_{dy_i dx_j})\end{array}\right]$,

as $g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 2\,\mathrm{Re}\ h\left(\cancel{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial z_i}, i\frac{\partial}{\partial z_j}\right)$

$= 2\,\mathrm{Im}\ h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$

$= -\omega\left(\frac{\partial}{\partial x_i}, \cancel{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial x_j}\right) = -g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right)$

- <u>Lemma</u> In cx. loc. coords

$$\omega = i\sum_{i,j} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

Proof $\quad i\,dz_i \wedge d\bar{z}_j = i(dx_i + i\,dy_i) \wedge (dx_j - i\,dy_j)$
$$= i(dx_i \wedge dx_j + dy_i \wedge dy_j) + (dx_i \wedge dy_j - dy_i \wedge dx_j)$$

Then $\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = 2\,\mathrm{Re}\, h_{i\bar{j}}$

$$\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right) = -2\,\mathrm{Im}\, h_{i\bar{j}}$$
$$= \omega\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$$

Thus $\sum_{i,j} h_{i\bar{j}}\, i\,dz_i \wedge d\bar{z}_j$

$$= \sum_i h_{i\bar{i}}\, i\,dz_i \wedge d\bar{z}_i + \sum_{i<j} 2\,\mathrm{Re}\left(h_{i\bar{j}}\, i\,dz_i \wedge d\bar{z}_j\right)$$

$$= \sum_{i,j} 2\,\mathrm{Re}\, h_{i\bar{j}}\, dx_i \wedge dy_j - \sum_{i<j} 2\,\mathrm{Im}\, h_{i\bar{j}} (dx_i \wedge dx_j + dy_i \wedge dy_j)$$

$$= \omega .$$

QED

It follows: $\forall\, u \in T^{1,0}X$, have $-i\omega(u, \bar{u}) > 0$ if $u \neq 0$

Call any real $(1,1)$-form σ s.t. $-i\sigma(a, \bar{a}) > 0$ for $a \neq 0$, $a \in T^{1,0}X$ a <u>positive $(1,1)$-form</u>, denoted $\sigma > 0$.

Further, 1st Chern class of a cx line b. L say, $c_1(L) > 0$ iff $c_1(L)$ is rep. by (closed) positive $(1,1)$-form

e.g. if X has $c_1(X) > 0$ then X is called a <u>Fano manifold</u>, if $c_1(X) = 0$ then X is <u>Calabi-Yau</u>

Any positive $(1,1)$-form determines a Hermitian metric on X.

If $f: Y \to X$ is a holo. immersion then f^*g is a well-defined Riem. metric on Y (i.e. $(df)^{\mathbb{C}} : T^{1,0}_y Y \to T^{1,0}_{f(y)} X$ injection)

and g is J-invariant as $df \circ J_Y = J_X \circ df$

$\Rightarrow$ Hermitian metric is induced on Y

Y locally is $\{z_{k+1} = \cdots = z_n = 0\}$

f " $(z_1, \dots, z_k) \longmapsto (z_1, \dots, z_k, 0, \dots, 0)$

$\Rightarrow$ get an expression for f^*h, $f^*\omega$

If (X, h) is a Hermitian mfd, $Y \subset X$ cx submfd,

then Y inherits a Hermitian metric by pulling back via immersion.

Locally, Y is given by $\{ z_{k+1} = \cdots = z_n = 0 \}$, $k = \dim Y$, $n = \dim X$

and the immersion $f : Y \longrightarrow X$

$$(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$$

Therefore,

<u>Lemma</u> The fundamental form of $f^* h$ is $f^* \omega$

Can equivalently give a Hermitian mfd as (X, ω) using fund. form $\left(i \sum_{i,j=1}^{k} h_{ij} \, dz_i \wedge d\bar{z}_j \right)$

<u>Def</u> A Hermitian mfd (X, ω) with $d\omega = 0$ is called a <u>Kähler manifold</u>.

Then ω is a <u>Kähler form on X</u>, h is a <u>Kähler metric</u>.

<u>Examples</u> 0. $\mathbb{C}^n$, $h = \frac{1}{2} \sum_j dz_j \, d\bar{z}_j$ standard Hermitian metric (Euclidean)

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$$

$$= \sum_j dx_j \wedge dy_j \qquad \left(\text{standard symplectic form on } \mathbb{R}^{2n}\right)$$

1. A) The metric in 0. descends to any cx torus $\mathbb{C}^n / \Lambda$, $\Lambda \cong \mathbb{Z}^{2n}$ discrete lattice in $\mathbb{C}^n$

B) on a Riemann surface, any non-vanishing 2-form (compatible with the orientation) is automatically closed, $(1-1)$, > 0

Thus every Riemann surface is Kähler (with any Hermitian metric)

2. $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$, coords $[z_0 : \cdots : z_n]$

$V_j = \{ z \in \mathbb{C}^{n+1} \mid z_j = 1 \}$ affine hyperplane

$\pi(V_j) \underset{\text{biholomorphic}}{\overset{}{=}} U_j \subset \mathbb{CP}^n$

$$\omega = \frac{i}{2\pi} \partial \bar{\partial} \log \| z \|^2 \in \Omega^{1,1}(V_j)$$

defines a real $(1,1)$ form on U_j

Change of local coords $V_j \longrightarrow V_k$

$$z \longmapsto fz \;, \quad f = \frac{z_j}{z_k} \quad \text{holo non-zero on } \pi^{-1}(U_j \cap U_k)$$

$$\frac{i}{2\pi} \partial \bar\partial \log \|fz\|^2 = \frac{i}{2\pi} \partial\bar\partial \left(\log \|z\|^2 + \log f\bar f \right)$$

$$= \omega + \frac{i}{2\pi} \partial \frac{\bar\partial (f\bar f)}{f \bar f} \overset{!}{=} \omega$$

as $\partial \dfrac{\bar\partial(f\bar f)}{f\bar f} = \partial \dfrac{\bar\partial \bar f}{\bar f} = - \dfrac{\bar\partial \partial \bar f}{\bar f} = 0$

Thus ω is well-defined on $\mathbb{CP}^n$.

Also $\forall\, T \in U(n+1)$, inducing $T : \mathbb{CP}^n \to \mathbb{CP}^n$ a projective transformation; get $T^*\omega = \omega$. [check using link w/ Chern class]

<u>Lemma</u> (& def^n) $\omega > 0$ thus ω is the fund. form of a Kähler metric on $\mathbb{CP}^n$. This is the <u>Fubini-Study metric</u>.

<u>Pf</u> By $U(n+1)$-symmetry, it suffices to check $\omega > 0$ at one point. Now

$$\omega|_{U_0} = \frac{i}{2\pi} \partial\bar\partial \log\left(1 + \sum_{j=1}^n z_j \bar z_j\right)$$

$$= \frac{i}{2\pi} \partial \frac{\sum z_j \, d\bar z_j}{1 + \sum z_j \bar z_j}$$

$$= \frac{i}{2\pi} \left(\frac{\sum dz_j \wedge d\bar z_j}{1 + \sum z_j \bar z_j} - \frac{\sum \bar z_k dz_k \wedge \sum z_j d\bar z_j}{(1 + \sum z_j \bar z_j)^2} \right)$$

At $[1 : 0 : \cdots : 0]$, i.e. $z_1 = \cdots = z_n = 0$, get

$$\frac{i}{2\pi} \sum dz_j \wedge d\bar z_j$$ which is > 0. (c.f. Example 0) $\square$

3. A positive $(1,1)$-form rep. $c_1(X)$ for a Fano mfd X makes it into a Kähler mfd. In particular $\mathbb{CP}^n$ is Fano – Sheet 4.

4. <u>Prop^n</u> Every complex submanifold of a Kähler mfd is Kähler. (By pulling back the Kähler form)

Cor Every projective mfd is Kähler

Detour to Riem. geom.

If (M,g) is oriented Riem mfd, $\dim_{\mathbb{R}} M = n$,
then $\forall x \in M$ can use Gram–Schmidt to construct a local orthonormal coframe field $(\omega_1, \dots, \omega_n)$ ⌜oriented⌟

Then $dvol_g = \omega_1 \wedge \cdots \wedge \omega_n$ is indep of choice of ω_i's and well-def $dvol_g \in \Omega^n(M)$

$dvol_g$ is called the volume form of (M, g)

Now let (X, h) be a Hermitian mfd, $h = \sum_{i,j} h_{i\bar{j}} \, dz_i \, d\bar{z}_j$

g the induced Riem. metric,

$$g = 2 \operatorname{Re} h = 2 \sum \Big[(\operatorname{Re} h_{i\bar{j}})(\operatorname{Re} dz_i \, d\bar{z}_j) - (\operatorname{Im} h_{i\bar{j}})(\operatorname{Im} dz_i \, d\bar{z}_j) \Big]$$

$\omega = g(J\cdot, \cdot)$ the fund. form

Near each $x \in X$ can find "adapted" loc. orthon. coframe field $\omega_1, \varepsilon_1, \dots, \omega_n, \varepsilon_n$ for T^*X wrt g,

and where $\varepsilon_k = -J\omega_k$, $\omega_k = J\varepsilon_k$ $\forall k$ ⌜spicy Gram-Schmidt⌟

$\omega_1 + i\varepsilon_1, \dots, \omega_n + i\varepsilon_n$ orthon. frame for $(T^*X)^{1,0}$ wrt h

$$h = \frac{1}{2} \sum_k (\omega_k + i\varepsilon_k) \otimes (\omega_k - i\varepsilon_k)$$ ⌜up to factor of 2 ?⌟ yes

$$g = \sum_k (\omega_k \otimes \omega_k + \varepsilon_k \otimes \varepsilon_k)$$

$$\omega = \sum_k (\omega_k \otimes \varepsilon_k - \varepsilon_k \otimes \omega_k) = \sum_k \omega_k \wedge \varepsilon_k$$

$$\omega^n = n! \, \omega_1 \wedge \varepsilon_1 \wedge \cdots \wedge \omega_k \wedge \varepsilon_k$$

Prop^n $\boxed{dvol_g = \frac{1}{n!} \omega^n}$ is the volume form of (X, h)

recap on local orthon. frame on a Hermitian mfd

$\omega_1, \varepsilon_1, \dots, \omega_n, \varepsilon_n$ adapted to the almost-cx str.

$$g = \sum_k (\omega_k \otimes \omega_k + \varepsilon_k \otimes \varepsilon_k)$$

$$\omega = \sum_k (\omega_k \otimes \varepsilon_k - \varepsilon_k \otimes \omega_k) = \sum_k \omega_k \wedge \varepsilon_k$$

$$d\mathrm{vol}_g = \omega_1 \wedge \varepsilon_1 \wedge \dots \wedge \omega_n \wedge \varepsilon_n$$

then $\omega^n = n!\, \omega_1 \wedge \varepsilon_1 \wedge \dots \wedge \omega_n \wedge \varepsilon_n$

Thus obtain <u>Prop</u> $\boxed{d\mathrm{vol}_g = \frac{1}{n!}\omega^n}$ is the volume form of Hermitian cx mfd (X, ω)

N.B. $J(\omega_k + i\varepsilon_k) = -\varepsilon_k + i\omega_k$

$$= i(\omega_k + i\varepsilon_k)$$

thus $\omega_k + i\varepsilon_k$ is a $(1,0)$-form

Let $Y \subset X$ be a cx submanifold, $\dim_{\mathbb{C}} Y = d$.

Then $\omega|_Y$ is the fund. form of the induced metric on Y.

So $\frac{1}{d!}\omega^d\Big|_Y$ is the volume form of Y.

We thus obtain

<u>Theorem</u> (Wirtinger)

$\mathrm{vol}(Y) = \frac{1}{d!}\int_Y \omega^d$ if Y is compact cx submfd of a Hermitian mfd (X, ω)

Now suppose X is <u>compact Kähler</u> : $d\omega = 0$

$[\omega] \in H^2_{dR}(X)$

$$\int_X \omega^n = n!\,\mathrm{vol}(X) \neq 0$$

So $[\omega] \neq 0$ and $[\omega^k] \neq 0$ for all $k = 1, \dots, n$

(so $\langle [\omega]^{\cup n}, [X] \rangle \neq 0$, $[\omega]^{\cup k} \neq 0$ in $H^{2k}(X, \mathbb{R})$)

If $Y \subset X$ is a compact cx submfd, ($\Rightarrow Y^{\mathbb{R}}$ closed submfd)

then $[Y] \in H_{2d}(X, \mathbb{R})$

$\int_Y \omega^d \neq 0$ so $[Y] \neq 0$

So Y is not a boundary of $(2d+1)$-submfd.

In particular, the cycle of each projective mfd in $\mathbb{CP}^n$ is never zero in $H_*(X, \mathbb{R})$.

Hodge Theory

Let M be an oriented Riemannian manifold; $m = \dim_{\mathbb{R}} M$
The inner product on T^*M extends to $\Lambda^r T^*_x M$ $\forall x \in M$ so that $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq m\}$ is orthon. basis
In particular, $dvol_g = \omega_1 \wedge \cdots \wedge \omega_n$ has unit norm at each $x \in X$.

Def The Hodge star $* : \Lambda^r T^*_x M \to \Lambda^{m-r} T^*_x M$
is a linear map satisfying

$$\boxed{\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g \, dvol_g} \quad \forall \alpha, \beta \in \Lambda^r T^*_x M$$

$*$ is uniquely by

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_r}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{m-r}}$$

s.t. $(i_1, \dots, i_r, j_1, \dots, j_{m-r})$ is an even permutation of $(1, 2, \dots, m)$
So $*^2 \alpha = (-1)^{r(m-r)} \alpha$ for $\alpha \in \Lambda^r T^*_x M$

$$* : \Omega^r(M) \to \Omega^{m-r}(M)$$

Now let (X, h) be a Hermitian cx mfd, g the corresponding J-invariant Riem. metric
If $n = \dim_{\mathbb{C}} X$, so $m = 2n$, then

$$* : \Omega^r(X) \to \Omega^{2n-r}(X)$$

If $\omega_1, \varepsilon_1, \dots, \omega_n, \varepsilon_n$ is adapted local coframe field
$(J\omega_k = -\varepsilon_k,\ J\varepsilon_k = \omega_k \quad \forall k)$

h is the std. Hermitian ext^n of g $\therefore$

So $|(\omega_{k_1}+i\varepsilon_{k_1})\wedge\cdots\wedge(\omega_{k_p}+i\varepsilon_{k_p})\wedge(\omega_{\ell_1}-i\varepsilon_{\ell_1})\wedge\cdots\wedge(\omega_{\ell_q}-i\varepsilon_{\ell_q})|_h^2$

$$= 2^{p+q} \quad \text{if } \begin{array}{l} 1\le k_1<\cdots<k_p\le n \\ 1\le \ell_1<\cdots<\ell_q\le n \end{array}$$

and g induces a Hermitian inner product on each $\Lambda^{p,q}T^*X$

Extend $*$ $\mathbb{C}$-linearly

$$*: (\Lambda^r T^*X)^{\mathbb{C}} \longrightarrow (\Lambda^{2n-r}T^*X)^{\mathbb{C}}$$

Lemma $\alpha\wedge *\bar\beta = \langle\alpha,\beta\rangle_h \, d\mathrm{vol}_g$
for all cx diff forms of the same degree

Pf If α,β are real diff forms and $\lambda,\mu\in\mathbb{C}$ then

$$\begin{aligned}\langle\lambda\alpha,\mu\beta\rangle_h \, d\mathrm{vol}_g &= \lambda\bar\mu\,\langle\alpha,\beta\rangle_g\, d\mathrm{vol}_g \\ &= \lambda\bar\mu\ \alpha\wedge*\beta \\ &= (\lambda\alpha)\wedge*(\bar\mu\beta)\end{aligned}$$

noting $*$ is cx linear, real operator

Q.E.D.

Cor. $*: \Omega^{p,q}(X) \longrightarrow \Omega^{n-q,n-p}(X)$

$$*^2|_{\Omega^{p,q}(X)} = (-1)^{p+q}$$

Def^n $d^* = -*d*: \Omega^r(X)\to\Omega^{r-1}(X)$

the Laplacian is $\Delta = dd^*+d^*d: \Omega^r(X)\to\Omega^r(X)$

Both d^*, Δ extend to $\Omega^r(X)^{\mathbb{C}}$

Remark If X is the Euclidean $\mathbb{C}^n$ with coords $z_1,\dots,z_n$
then (for $r=0$) $\Delta = -4\sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial\bar z_j}$

$$= -\sum_{k=1}^n\left(\frac{\partial^2}{\partial x_k^2}+\frac{\partial^2}{\partial y_k^2}\right)$$

$$\partial^* = -*\bar\partial * : \Omega^{p,q}(X) \longrightarrow \Omega^{p-1,q}(X)$$

$$\bar\partial^* = -*\partial * : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q-1}(X)$$

and $d^*|_{\Omega^{p,q}(X)} = \bar\partial^* + \partial^*$

Then $(\partial^*)^2 = 0$, $(\bar\partial^*)^2 = 0$, $(d^*)^2 = 0$

$$\partial^*\bar\partial^* = -\bar\partial^*\partial^*$$

Def^n The L^2-inner product

$$\langle \xi, \eta \rangle_{X,h} := \int_X \langle \xi, \eta \rangle_h \, dvol_g$$

$$= \int_X \xi \wedge *\bar\eta$$

makes $\Omega^r(X)$, $\Omega^{p,q}$ into pre-Hilbert spaces
(smooth forms not complete in L^2 norm)

Observe $\Omega^r(X)^{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}(X)$

orthogonal direct sum wrt norm at a point in X
and also in the L^2 norm.

Prop^n ∂^*, $\bar\partial^*$ are formal adjoints of ∂, $\bar\partial$
wrt the L^2 inner product i.e.

$$\int_X \langle \partial\alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, \partial^*\beta \rangle \, dvol_g$$

$$\int_X \langle \bar\partial\alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, \bar\partial^*\beta \rangle \, dvol_g$$

$\forall$ compactly supported $\alpha \in \Omega^{p-1,q}(X)$ resp. $\Omega^{p,q-1}(X)$

$$\beta \in \Omega^{p,q}(X)$$

Proof We use Stokes' theorem

$$\int_X \langle \bar\partial\alpha, \beta \rangle \, dvol_g = \int_X \bar\partial\alpha \wedge *\bar\beta$$

$$= \int_X \Big(\underbrace{\bar\partial(\underbrace{\alpha \wedge *\bar\beta}_{\text{type } (n,n-1)})}_{\therefore = d(\alpha\wedge *\bar\beta)} - (-1)^{p+q-1} \alpha \wedge \bar\partial * \bar\beta \Big)$$

$$= (-1)^{p+q} \int_X \alpha \wedge \bar{\partial} * \bar{\beta}$$

$$= (-1)^{p+q} \int_X \alpha \wedge \overline{\partial * \beta}$$

check ✓

$$= \int_X \alpha \wedge * \left(\overline{- * \partial * \beta} \right)$$

$$= \int_X \langle \alpha, \bar{\partial}^* \beta \rangle \, dvol_g \qquad \text{Q.E.D.}$$

<u>Cor 1</u> 1) $\int_X \langle d\alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, d^* \beta \rangle \, dvol_g$

2) $\int_X \langle d^c \alpha, \beta \rangle \, dvol_g = \int_X \langle \alpha, (d^c)^* \beta \rangle \, dvol_g$

where $(d^c)^* = -i (\bar{\partial}^* - \partial^*) = - * d^c *$

<u>Def^n</u> $\Delta_\partial = \partial \partial^* + \partial^* \partial$

$\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$

Then $\Delta_\partial, \Delta_{\bar{\partial}} : \Omega^{p,q}(X) \to \Omega^{p,q}(X)$

$\Delta, \Delta_\partial, \Delta_{\bar{\partial}}$ are formally self-adjoint

N.B. Δ in general does <u>not</u> act on (p,q) forms

<u>Def^n</u> An r-form α is (d-) <u>harmonic</u>, $\alpha \in \mathcal{H}^r(X)$ if $\Delta \alpha = 0$.

A (p,q)-form α is <u>$\bar{\partial}$-harmonic</u>, $\alpha \in \mathcal{H}^{p,q}_{\bar{\partial}}(X)$ if $\Delta_{\bar{\partial}} \alpha = 0$.

Similarly define Δ_∂-harmonic (p,q)-forms.

<u>Prop^n</u> Suppose a Hermitian mfd X is compact. Then

$\Delta_{\bar{\partial}} \alpha = 0$ iff $\bar{\partial} \alpha = 0, \ \bar{\partial}^* \alpha = 0$

$\Delta_\partial \alpha = 0$ iff $\partial \alpha = 0, \ \partial^* \alpha = 0$

$\Delta \alpha = 0$ iff $d\alpha = 0, \ d^* \alpha = 0$

Proof $0 = \int_X \langle \Delta_{\bar\partial} \alpha, \alpha \rangle \, dvol_g$

$$\overset{\text{Prop}}{=} \int_X \left(\langle \bar\partial \alpha, \bar\partial \alpha \rangle + \langle \bar\partial^* \alpha, \bar\partial^* \alpha \rangle \right) dvol_g$$

$\Longrightarrow \bar\partial \alpha = 0, \; \bar\partial^* \alpha = 0$ Q.E.D.

Hodge Theorem

Let (X, h) be a compact Hermitian manifold. Then $\forall r$, $\mathcal{H}^r(X)$, $\forall p,q$, $\mathcal{H}^{p,q}_{\bar\partial}(X)$ are finite dimensional and there are L^2-orthogonal decompositions

$$\Omega^r(X) = \mathcal{H}^r(X) \oplus d\,\Omega^{r-1}(X) \oplus d^* \Omega^{r+1}(X)$$
$$\Omega^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \bar\partial \Omega^{p,q-1}(X) \oplus \bar\partial^* \Omega^{p,q+1}(X)$$

Assume without proof.
(Requires analysis of PDEs)

$\Delta_{\bar\partial} \alpha = \overline{\Delta_\partial \bar\alpha}$ $\left(\begin{array}{c} \text{hence suffices to} \\ \text{look at } \Delta_{\bar\partial} \end{array} \right)$

Applications: Prop 1 Assume X is compact Hermitian

(1)
$$\begin{array}{ccc} \alpha & \in & \mathcal{H}^r(X) \\ \downarrow & & \downarrow \\ [\alpha] & \in & H^r_{dR}(X) \end{array}$$
is an isomorphism of $\mathbb{R}$ - v. spaces

(2)
$$\begin{array}{ccc} \alpha & \in & \mathcal{H}^{p,q}_{\bar\partial}(X) \\ \downarrow & & \downarrow \\ [\alpha] & \in & H^{p,q}(X) \end{array}$$
is an isomorphism of $\mathbb{C}$ - v. spaces

Exercise - or see my online notes on Pt. III diff geo

Define $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ the Hodge numbers of X.

In particular,

$$h^{p,0}(X) = \dim(\text{holo. } p\text{-forms on } X)$$

$$h^{n,0}(X) = \dim(\text{holo. sections of } K_X) \qquad n = \dim_{\mathbb{C}} X$$

$$= P_g(X) \quad \text{is the } \underline{\text{geometric genus}} \text{ of } X$$

$\underline{2}$ $\quad \alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X) \iff *\alpha \in \mathcal{H}^{n-q,n-p}_{\cancel{\bar\partial}\partial}(X)$

$$\iff \overline{*\alpha} = *\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$$

So get $\mathbb{R}$-lin isomorphisms, and

$$\boxed{h^{p,q}(X) = h^{n-p,n-q}(X)}$$

the $\underline{\text{Kodaira-serre duality}}$

the Kodaira–Serre duality $h^{p,q}(X) = h^{n-p,n-q}(X)$

for a Hermitian cx n-fold X is induced by an $\mathbb{R}$-linear iso.

$$\mathcal{H}_{\bar\partial}^{p,q}(X) \longrightarrow \mathcal{H}_{\bar\partial}^{n-p,n-q}(X)$$
$$\alpha \longmapsto *\bar\alpha$$

In ptic this implies

$$\mathcal{H}_{\bar\partial}^{p,q} \otimes \mathcal{H}_{\bar\partial}^{n-p,n-q} \longrightarrow \mathbb{C}$$
$$\alpha \otimes \beta \mapsto \int_X \alpha \wedge \beta$$

is a non-degenerate bilinear pairing.

If X is also connected,

$$h^{n,n} = 1$$

given by $H^{n,n}(X) \longrightarrow \mathbb{C}$

$$\alpha \longmapsto \int_X \alpha$$

Let now ω be a Kähler form on X ($d\omega = 0$).

<u>Def^n</u> The <u>Lefschetz operator</u> is

$$L : \Omega^r \to \Omega^{r+2}$$
$$\alpha \mapsto \alpha \wedge \omega$$

and its adjoint is $\Lambda : \Omega^{r+2} \longrightarrow \Omega^r$ (pointwise wrt $\langle\cdot,\cdot\rangle_h$)

i.e. $\langle \Lambda\alpha, \beta\rangle_g = \langle \alpha, L\beta\rangle_g \quad \forall\, \alpha \in \Omega^{r+2}, \beta \in \Omega^r$

N.B. L and Λ extend cx linearly to $\Omega^\bullet(X)^{\mathbb{C}}$

<u>Lemma</u> $\Lambda = *^{-1} \circ L \circ *$

So if $\alpha \in \Omega^r(X)$, then $\Lambda\alpha = (-1)^r * L * \alpha$

<u>Proof</u> $\langle \alpha, L\beta\rangle_h \, d\mathrm{vol}_g = \overline{\langle L\beta, \alpha\rangle_h} \, d\mathrm{vol}_g$

$$= \bar\beta \wedge \omega \wedge *\alpha$$
$$= \bar\beta \wedge * *^{-1}(\omega \wedge *\alpha)$$
$$= \overline{\langle \beta, *^{-1} L * \alpha\rangle_h} \, d\mathrm{vol}_g$$

$= \langle \Lambda\alpha, \beta \rangle \, dvol_g$ Q.E.D.

Theorem (Kähler identities) If (X, ω) is Kähler, then

(i) $\boxed{[\Lambda, \bar\partial] = -i\partial^*}$, $[\Lambda, \partial] = i\bar\partial^*$

(ii) $[\bar\partial, L] = \boxed{[\partial, L] = 0}$

$[\partial^*, \Lambda] = [\bar\partial^*, \Lambda] = 0$

(iii) $[\Lambda, d] = -d^{c*}$, $[L, d^*] = d^c$

(iv) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$

N.B. The two relations in (i) are equivalent via cx conj^n

the (ii) is $\partial(\omega \wedge \alpha) = \omega \wedge \partial\alpha$ as $\partial\omega = 0$

The (iii), (iv) are linear combinations and cx conj and formal adjoints of (i).

Won't prove (i), the proof is messy.

2 ways: either theory of $sl(2, \mathbb{R})$ and Q9 Sheet 4

or prove (i) on $(\mathbb{C}^n, \text{Eucl})$ and check argument works to extend to any Kahler mfd

Theorem On a Kähler manifold X, we have

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$$

In particular, $\Delta(\Omega^{p,q}(X)) \subseteq \Omega^{p,q}(X)$

Proof $i(\partial\bar\partial^* + \bar\partial^*\partial)$

$$= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = 0$$

$$\Delta = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial)$$

$$= \Delta_\partial + \partial\bar\partial^* + \bar\partial\partial^* + \partial^*\bar\partial + \bar\partial^*\partial + \Delta_{\bar\partial}$$

$$= \Delta_\partial + \Delta_{\bar\partial}$$ from above

$$-i\Delta_\partial = \partial(\overset{1}{\Lambda}\bar\partial - \overset{2}{\bar\partial}\Lambda) + (\overset{4}{\Lambda}\bar\partial - \overset{3}{\bar\partial}\Lambda)\partial$$

$$-i\Delta_{\bar\partial} = -\bar\partial(\underset{3}{\Lambda}\partial - \underset{2}{\partial}\Lambda) - (\underset{4}{\Lambda}\partial - \underset{1}{\partial}\Lambda)\bar\partial \qquad \therefore\ \Delta_\partial = \Delta_{\bar\partial}$$

$\square$

<u>Hodge decomposition</u> Let X be compact Kähler. Then

$$H^r_{dR}(X)\otimes\mathbb{C} \cong \bigoplus_{p+q=r} H^{p,q}(X)$$

$$\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$$

<u>Proof</u> Consider harmonic forms

$$\begin{aligned} H^r_{dR}(X)\otimes\mathbb{C} &\cong \mathcal{H}^r(X)\otimes\mathbb{C} \\ &\overset{!}{=} \bigoplus_{p+q=r} (\mathcal{H}^r(X)\otimes\mathbb{C})\cap\Omega^{p,q}(X) \\ &\overset{!}{=} \bigoplus_{p+q=r} \mathcal{H}^{p,q}_{\bar\partial}(X) \\ &\cong \bigoplus_{p+q=r} H^{p,q}(X) \end{aligned}$$

using Δ respects type decomposition

using $2\Delta_{\bar\partial} = \Delta$

<u>The 2^{nd} relation</u>

$$(\mathcal{H}^r(X)\otimes\mathbb{C})\cap\Omega^{p,q}(X) = \overline{(\mathcal{H}^r(X)\otimes\mathbb{C})\cap\Omega^{q,p}(X)}$$

Q.E.D.

<u>Cor 1</u> Every holo. non-zero p-form on X cpct Kähler is d-closed, and d-exact iff zero.

<u>Pf</u> $\{\alpha\in\Omega^{p,0}(X) : \bar\partial\alpha = 0\} = \mathcal{H}^{p,0}_{\bar\partial}$

$$= (\mathcal{H}^p(X)\otimes\mathbb{C})\cap\Omega^{p,0}(X)$$

Q.E.D.

<u>Cor 2</u> $b_{2k-1}(X)\in 2\mathbb{Z}$, $b_{2k}(X)>0$ $\forall\, k=1,\dots,n$ (Betti numbers)

<u>Pf</u> $b_{2k-1} = \sum_{p+q=2k-1} h^{p,q} = 2\sum_{\substack{p+q=2k-1\\ p<q}} h^{p,q}$

second part from ω^k

From Cor. 2 we see that since the Hopf manifolds

$$\mathbb{C}^n \setminus \{0\} / _{\mathbb{Z} \sim 2\mathbb{Z}} \cong S^{2n-1} \times S^1$$

have $b^1 = 1$ is odd $\Rightarrow$ Hopf mfds do not admit Kähler metrics (for $n \geq 2$), hence also not projective

The Hodge diamond

$$\begin{array}{ccccccc} & & & h^{n,n} & & & \\ & & h^{n,n-1} & & h^{n-1,n} & & \\ & \dots & & \vdots & & \dots & \\ h^{n,0} & h^{n-1,1} & & \dots & & h^{1,n-1} & h^{0,n} \\ & \dots & & \vdots & & \dots & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

sum of horizontal line is Betti no.

central symmetry, Kodaira-Serre duality

$\updownarrow$ reflection symmetry, hodge star $*$

$\leftrightarrow$ reflection symmetry, complex conj^n

<u>$\partial\bar\partial$ - lemma</u> Let X be a compact Kähler manifold.

Let $\alpha \in \Omega^{p,q}(X)$, $d\alpha = 0$. Then TFAE:

(a) $\alpha = d\beta$ is d-exact, $\beta \in \Omega^{p+q-1}(X)$

(b) $\alpha = \partial\beta$ is ∂-exact, $\beta \in \Omega^{p-1,q}(X)$

(c) $\alpha = \bar\partial\beta$ is $\bar\partial$-exact, $\beta \in \Omega^{p,q-1}(X)$

(d) $\alpha = \partial\bar\partial\beta$ is $\partial\bar\partial$-exact, $\beta \in \Omega^{p-1,q-1}(X)$

<u>Proof</u> (d) $\Rightarrow$ (a), (b), (c) noting $\partial\bar\partial = -\bar\partial\partial$ and $d\bar\partial = \partial\bar\partial$

Each of (a), (b), (c), (d) implies α is L^2-orthogonal to the space of harmonic forms $\mathcal{H}^{p,q}_{\bar\partial} (= \mathcal{H}^{p,q}_{\partial} = \mathcal{H}^{p+q} \cap \Omega^{p,q})$ by the Hodge theorem.

[do we really need the full Hodge thm?]

So ETS latter statement implies (d).

Let α be L^2-orthogonal to $\mathcal{H}^{p,q}_{\bar\partial}$.

Now $d\alpha = 0$ implies $\partial\alpha = 0$, $\bar\partial\alpha = 0$.

But also $\bar{\alpha} \perp \mathcal{H}^{q,p}_{\partial} = \mathcal{H}^{q,p}_{\bar{\partial}}$

So $\bar{\alpha} = \bar{\partial}\beta$, $\alpha = \partial\bar{\beta}$ by $\bar{\partial}$-Hodge theorem.

And $\bar{\beta} = \gamma_0 + \bar{\partial}\gamma_1 + \bar{\partial}^*\gamma_2$ for $\gamma_0 \in \mathcal{H}^{p-1,q}_{\bar{\partial}}$

$$\Rightarrow \alpha = \partial\bar{\partial}\gamma_1 + \partial\bar{\partial}^*\gamma_2$$
$$= \partial\bar{\partial}\gamma_1 - \bar{\partial}^*\partial\gamma_2 \qquad (\text{noting } \partial\bar{\partial}^* + \bar{\partial}^*\partial = 0)$$

As $\bar{\partial}\alpha = 0$, get $\bar{\partial}\bar{\partial}^*\partial\gamma_2 = 0$

And now $\langle \bar{\partial}\bar{\partial}^*\partial\gamma_2, \partial\gamma_2 \rangle_{L^2} = \langle \bar{\partial}^*\partial\gamma_2, \bar{\partial}^*\partial\gamma_2 \rangle_{L^2}$

implies $\bar{\partial}^*\partial\gamma_2 = 0$.

Hence $\alpha = \partial\bar{\partial}\gamma_1$. Q.E.D.

N.B. When X is a polydisc in $\mathbb{C}^n$, we can deduce $\partial\bar{\partial}$-lemma from the $\bar{\partial}$-Poincaré Lemma

When $q=0$, the hypothesis means α is a holomorphic p-form: (!)

If $\alpha \neq 0$ then α is <u>not</u> d-exact, equivalently α is never ∂-exact

There is an equivalent version known as the dd^c-lemma.

<u>Another view on Kähler manifolds</u>

Let (X, h) be a Hermitian manifold, and g be the corresponding Riem. ~~connection~~ metric on X.

$\exists!$ connection ∇ on TX (the Levi-Civita)

s.t. $d\langle u, v\rangle_g = \langle \nabla u, v\rangle_g + \langle u, \nabla v\rangle_g \quad \forall u, v \in \mathfrak{X}(X)$

$$\Leftrightarrow \nabla g = 0 \quad \text{since}$$
$$d(g(u,v)) = (\nabla g)(u,v) + g(\nabla u, v) + g(u, \nabla v)$$

<u>and</u> $\nabla_u v - \nabla_v u = [u, v]$ (**)

∇ on $\Lambda^2 T^* X$ is given by [not easy to prove?]

$$(d\alpha)(u,v,w) = (\nabla_u \alpha)(v,w) - (\nabla_v \alpha)(u,w) + (\nabla_w \alpha)(u,v)$$

$\forall \alpha \in \Omega^2(X)$, $u, v, w \in \mathfrak{X}(X)$.

equivalently

$$d(\alpha(u,v)) = (\nabla\alpha)(u,v) + \alpha(\nabla u, v) + \alpha(u, \nabla v)$$

Recall the v.b. isomorphism

$$TX \longrightarrow T^{1,0}X \qquad (\ddagger)$$
$$u \longmapsto \tfrac{1}{2}(u - iJu)$$

$$\langle u,v\rangle_h := \tfrac{1}{2} h(u - iJu, v - iJv) = (g - i\omega)(u,v) = \langle u,v\rangle_{g-i\omega}$$

If d_A is a unitary connection on $T^{1,0}X$

i.e. $d\langle u,v\rangle_h = \langle d_A u, v\rangle_h + \langle u, d_A v\rangle_h \qquad (\dagger)$

then d_A induces connection ∇_A on TX

$$u - iJu \mapsto d_A u := \nabla_A u - iJ\nabla_A u$$

Then the real part of $(\dagger)$ says

$$d\langle u,v\rangle_g = \mathrm{Re}\left(\langle d_A u, v\rangle_h + \langle u, d_A v\rangle_h\right)$$
$$= \langle \nabla_A u, v\rangle + \langle u, \nabla_A v\rangle$$

<u>Prop^n</u> Assume ∇_A (as above) is the Levi-Civita connection i.e. $(**)$ holds. Then (i) d_A is the Chern connection c.f. ES3

(ii) h is a Kähler metric

<u>Proof of (ii)</u> Suffices to show $\nabla\omega = 0$ (then $d\omega = 0$)

$$(\nabla\omega)(u,v) = d(\omega(u,v)) - \omega(\nabla u, v) - \omega(u, \nabla v)$$
$$= d(g(Ju,v)) - g(\nabla Ju, v) - g(Ju, \nabla v)$$
$$= 0 \quad \text{by } (*)$$

Q.E.D.

$(\ddagger)$ converts J on TX into $i\cdot$ on $T^{1,0}X$,

also d_A is $\mathbb{C}$-linear

$\therefore \nabla J = J\nabla$ (used in above proof)