Coxeter Groups

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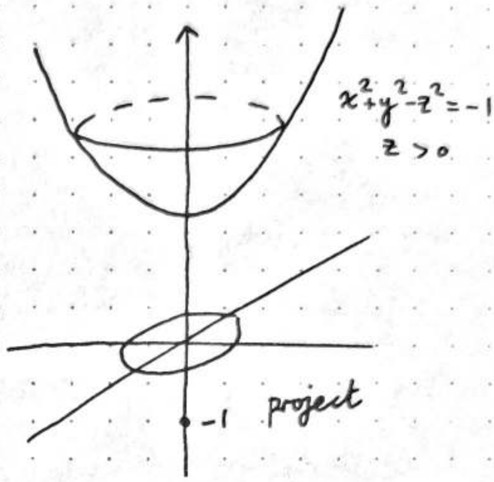
M. Davis – Geometry & Topology of Coxeter Groups

A. Thomas – Geometric & Topological Aspects of Coxeter Groups and Buildings

Course outline



Poincaré disk model

§1. Geometric reflection groups

§2. Defining abstract reflection groups

§3. Combinatorics of Coxeter groups

§4. The Tits representation

§5. Finite Coxeter groups

§6. The basic construction

§7. The Davis Complex

§ Geometric reflection groups

Coxeter groups are discrete groups generated by 'reflections'. In §2, 3 we will make this precise. In this section we will see some examples.

Recall a Riemannian manifold is a smooth manifold M with a positive definite inner product on $T_x M$ $\forall x \in M$.

This IP allows us to define

- Isometries : IP – preserving (self–)diffeos
- a metric
- geodesics on M
- sectional curvature of M

1.1 Notation

- $\mathbb{S}^n$ – n-dim sphere $\subseteq \mathbb{R}^{n+1}$, centred at origin with the round metric
- $\mathbb{E}^n$ – n-dim Euclidean space $\cong (\mathbb{R}^n, \cdot)$
- $\mathbb{H}^n$ – n-dim real hyperbolic space

$\mathbb{X}^n$ is one of $\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n$

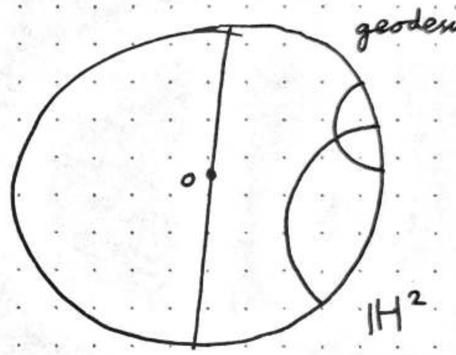
$\mathrm{Isom}(\mathbb{X}^n)$ – isometry gp of $\mathbb{X}^n$

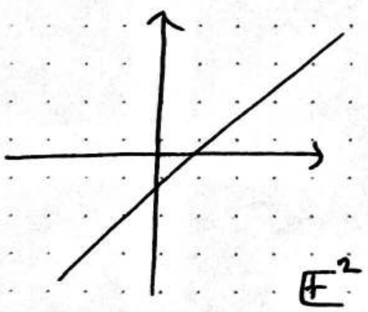
1.1 Remark

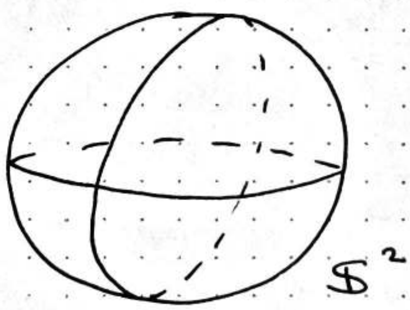
$\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n$ are all* Riemannian mfds with constant sectional curvature
$1 \quad 0 \quad -1$

1.3 Definition

A hyperplane $\mathcal{H} \subseteq \mathbb{X}^n$ is a totally geodesic codimension-1 submanifold of $\mathbb{X}^n$. $\mathcal{H}$ separates $\mathbb{X}^n$ into two connected components, called half-spaces.







For each $\mathcal{H} \subset \mathbb{X}^n$, $\exists$ a reflection in $\mathrm{Isom}(\mathbb{X}^n)$ which fixes $\mathcal{H}$ (pointwise) and exchanges the associated half-spaces.

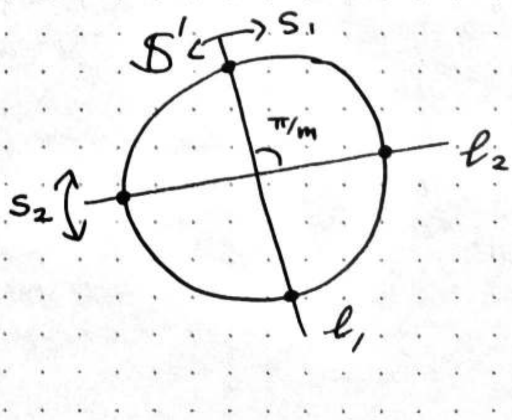
1.4 Example - finite dihedral groups

$\mathbb{S}^1$ with 'hyperplanes' l_1 and l_2 meeting at angle π/m.

Let s_i be reflection across l_i.

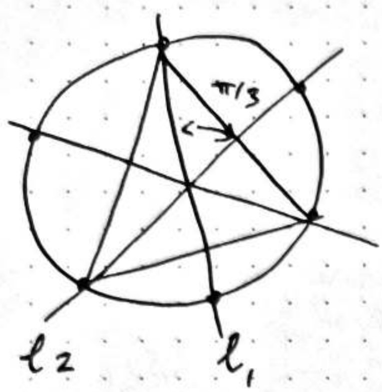
Then $s_1 s_2$ is rotation by $\frac{2\pi}{m}$,

i.e. $\langle s_1 s_2 \rangle \cong C_m$.



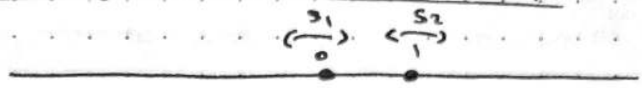
$$W = \langle s_1, s_2 \rangle \cong D_{2m} \overset{!}{=} \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = \underset{\substack{\uparrow \\ \text{identity}}}{e} \rangle$$

is the dihedral group of order $2m$ (symmetries of the m-gon)



1.5 <u>Example - Infinite Dihedral group</u>

$\mathbb{E}^1 \cong \mathbb{R}^1$



& hyperplanes the points $0, 1$, giving reflections s_1, s_2

i.e. $s_1(t) = -t$, $s_2(t) = 2 - t$ $\forall t \in \mathbb{R}$

Then $s_1 s_2$ is translation by 2 i.e. $\langle s_1 s_2 \rangle \cong \mathbb{Z}$.

$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e \rangle = D_\infty$ is the infinite dihedral group.

<u>Notation</u>: if $\langle s_1 s_2 \rangle \cong \mathbb{Z}$ i.e. $s_1 s_2$ has infinite order, we write $(s_1 s_2)^\infty = e$

So $D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = e \rangle$

<u>1.7 Definition</u>

Let X be a topological space and $G \curvearrowright X$ by homeomorphisms. Let Gx be the orbit of $x \in X$.

Then a <u>fundamental domain</u> for the action of G is a subset $K \subset X$ s.t.

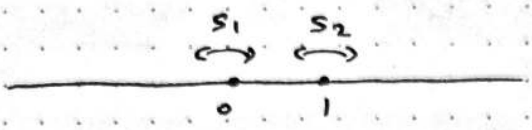
- K is closed and connected
- $\forall x \in X$, $Gx \cap K \neq \emptyset$
- $\forall x \in K^\circ$, $Gx \cap K = \{x\}$

K is a <u>strict fundamental domain</u> if

$\forall x \in K$, $Gx \cap K = \{x\}$

i.e. K contains exactly one point from each orbit.

1.8 Example



$D_\infty \curvearrowright \mathbb{R}$

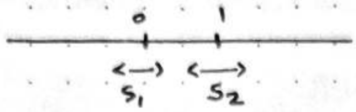
The closed interval $[0,1]$ is a strict fundamental domain as is any $[t, t+1]$, $t \in \mathbb{R}$. (?!) no...

$[t, t+2]$ is a fundamental domain, but is not strict (?!)

<u>Last time</u> $\mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ hyperplanes $\mathcal{H}$

- Fundamental domain $K \subset X$, X top space, $G \curvearrowright X$

<u>Example</u> $D_\infty \curvearrowright \mathbb{R}$

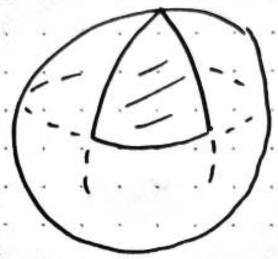
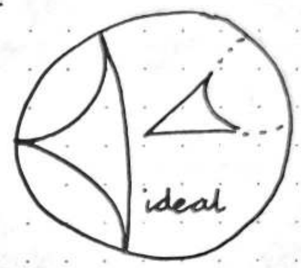


<u>Correction</u> $[t, t+2]$ fundamental domain for $\langle s_1 s_2 \rangle \cong \mathbb{Z} \curvearrowright \mathbb{R}$

<u>Recall</u> simplex Δ^k convex hull of $k+1$ points

Regular if any perm of vertices can be realised by an isometry of $\mathbb{X}^n$

<u>e.g.</u> $n=2$, $k=2$

- $\mathbb{E}^2$  $\mathbb{S}^2$  $\mathbb{H}^2$ 

1.9 <u>Definition</u>

A <u>convex polytope</u> $P \subseteq \mathbb{X}^n$ is a (convex), compact intersection of a finite number of closed half-spaces with non-empty interior

The <u>link</u> of a vertex v of P is

$$\operatorname{link}_P(v) = P \cap \text{unit } (n-1)\text{-sphere centred at } v$$

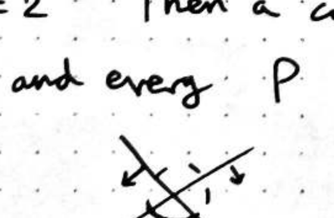
- $\left(\text{sphere in } T_v \mathbb{X}^n\right)$

This is spherical $(n-1)$-dim polytope.

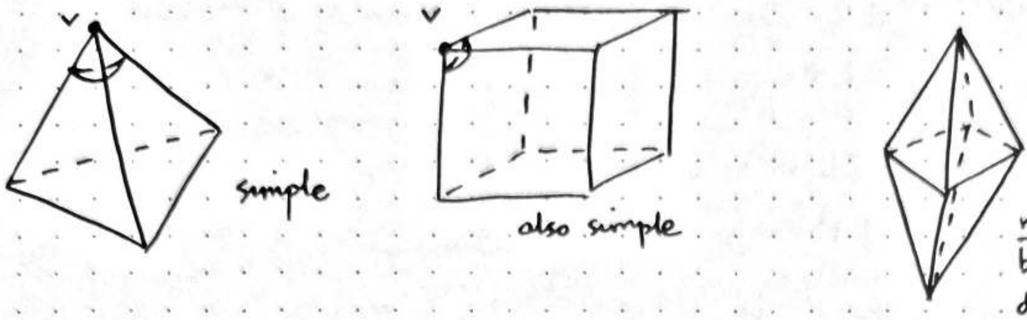
P is <u>simple</u> if $\forall v \in P$ vertices, $\operatorname{link}_P(v)$ is a spherical simplex.

1.10 <u>Examples</u>

$n=2$ Then a convex polytope in $\mathbb{X}^2$ is a convex polygon, and every P is simple

- 

$n = 3 \quad P \subset \mathbb{E}^3$



simple

also simple

not simple because 4 points do not give a simplex on $\mathbb{S}^2$

1.11 Theorem (to be proven later)

Let $P \subseteq \mathbb{X}^n$ be a simple convex polytope, $n \geq 2$

$\{F_i\}_{i \in I}$ set of codimension-1 faces of P

Then each F_i lies in a hyperplane $\mathcal{H}_i \subset \mathbb{X}^n$

Suppose $\forall i \neq j$, if $F_i \cap F_j \neq \emptyset$ then $\mathcal{H}_i, \mathcal{H}_j$ intersect at an angle of π/m_{ij} where $m_{ij} \geq 2$ is in $\mathbb{Z}$.

Set $m_{ii} = 1$ and $m_{ij} = \infty$ when $F_i \cap F_j = \emptyset$.

Let $s_i \in \mathrm{Isom}(\mathbb{X}^n)$ be reflection across $\mathcal{H}_i$.

Let W be the group generated by $\{s_i\}_{i \in I}$. Then

(i) W has presentation

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = e \quad \forall i, j \in I \rangle$$

(ii) W is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$

(iii) P is a strict fundamental domain for $W \curvearrowright \mathbb{X}^n$ and the action induces a tesselation of $\mathbb{X}^n$ by copies of P.

tiling

Rmk 1.12 setting $m_{ii} = 1$ gives $(s_i s_i)^1 = e$ i.e. $s_i^2 = e \ \forall i$

1.13 Definition A group W is a <u>geometric reflection group</u> if it is D_{2m}, D_∞ or a group arising as in Thm 1.11.

Say W is <u>spherical</u> if $\mathbb{X}^n = \mathbb{S}^n$

<u>euclidean</u> if $\mathbb{X}^n = \mathbb{E}^n$

<u>hyperbolic</u> if $\mathbb{X}^n = \mathbb{H}^n$

1.14 Remark Geometric reflection groups are our first examples of Coxeter groups. Coxeter classified all spherical and euclidean groups (~1930) Hyperbolic reflection groups are still not classified.

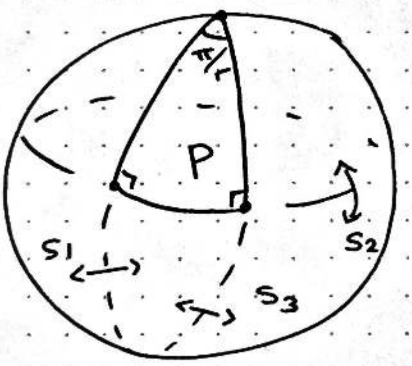
1.15 Example (Triangle groups)

$\forall p,q,r \in \mathbb{Z}$ s.t. $2 \leq p \leq q \leq r$ there exists a triangle P in some $\mathbb{X}^2$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. Then we get

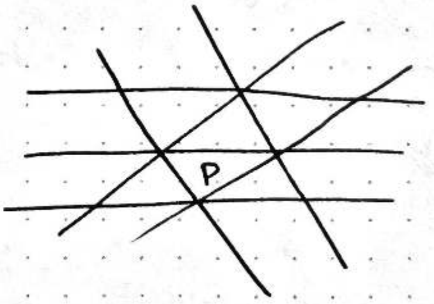
$$W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^p = (s_2 s_3)^q = (s_3 s_1)^r = 1 \rangle$$

$\mathbb{X}^2 = \mathbb{S}^2$ then $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$,

and the possible triples are
$(2,2,r)$ $\forall r \geq 2$
$(2,3,3)$ ← c.f. tetrahedron + construction 1.16
$(2,3,4)$ ← subdivide octahedron
$(2,3,5)$ ← " icosahedron



$\mathbb{X}^2 = \mathbb{E}^2$ then
$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$$
e.g. $(3,3,3)$



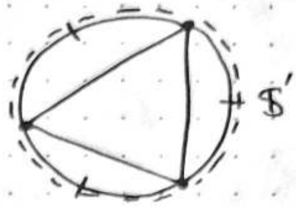
$(2,3,6)$
$(2,4,4)$

$\mathbb{X}^2 = \mathbb{H}^2$ then
$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$$

Example 1.16 (Symmetric group)

For $n \geq 2$ and P the regular Euclidean simplex $\Delta^n \subset \mathbb{E}^n$
Labelling the vertices in the set $\{1, \dots, n+1\}$
Then $\mathrm{Isom}(\Delta^n) \cong S_{n+1}$ is the symmetric group on $n+1$ letters.

Embed $\Delta^n \subset \mathbb{E}^n$ s.t. vertices lie on $\mathbb{S}^{n-1}$, and then 'puff out' Δ^n to lie on $\mathbb{S}^{n-1}$, we get a tesellation for $\mathbb{S}^{n-1}$ by $\partial\Delta^n$.



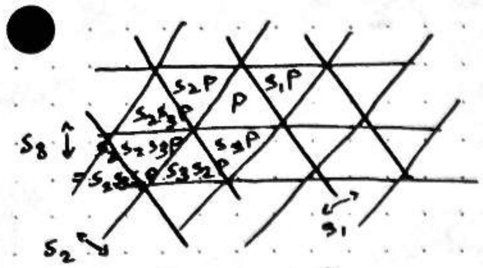
Take barycentric subdivision of $\partial\Delta^n$.

A maximal simplex P in this subdivision gives a Coxeter group

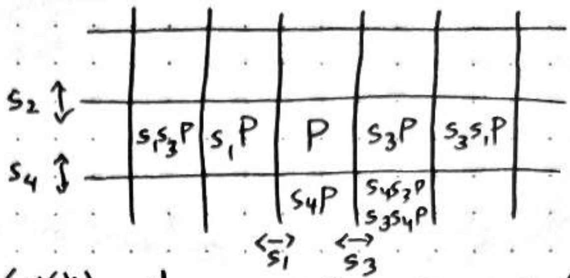
$$W = S_{n+1} = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^2 = 1 \text{ for } |i-j| \geq 2, \; (s_i s_{i+1})^3 = 1 \text{ for } 1 \leq i \leq n-1 \rangle$$

Let $s_i = (i \;\; i+1)$ to get usual description as a perm^n group

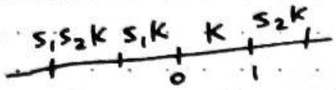
Triangle group $(3,3,3)$



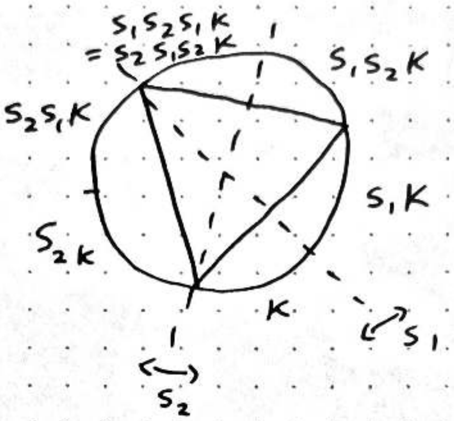
Ex 1.17 Tiling of $\mathbb{E}^n$ by n-cubes



<u>Correction</u> If $s_1 s_2 \in \mathrm{Isom}(\mathbb{X}^n)$ then $s_1 s_2 \cdot x = s_1(s_2(x))$



$D_\infty \curvearrowright \mathbb{R}$



<u>Clarification</u> <u>Def^n</u> A simplex $\sigma_k \subseteq \mathbb{X}^n$ for $n \geq k$ is the convex hull of $k+1$ basis vectors in $\mathbb{X}^n$. It is k-dimensional.

A simplex is regular if all edges have the same length.

Δ^k is the regular Euclidean k-simplex.

§2 Defining abstract reflection groups

<u>2.1 Definition</u> (Tits 1950s)

Let $S = \{s_i\}_{i \in I}$, I finite indexing set.

A <u>Coxeter matrix</u> is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ such that:
- $m_{ii} = 1 \quad \forall i \in I$
- $m_{ij} = m_{ji} \in \{2,3,4,\dots\} \cup \{\infty\} \quad \forall i \neq j \in I$

The <u>Coxeter group</u> W associated to S, M is

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = e \quad \forall i,j \in I \rangle$$

and the pair (W,S) is a <u>Coxeter system</u>.

<u>2.2 Remark</u>
- Note that $m_{ii} = 1 \Rightarrow s_i^2 = e \quad \forall i \in I$
- Also $(s_i s_j)^{m_{ij}}$ can be rewritten $\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij} \text{ generators}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ generators}}$

- Geometric refl gps are Coxeter groups. But not all Coxeter gps are geom refl gps.
- A Coxeter group W can correspond to multiple Cox systems see 'Iso. problem for Cox gps'
- One can define (W, S) with $|S|$ infinite. We restrict ourselves to finite generating sets in this course.

Next two Corollaries follow from the Tits representation.

2.3 Corollary If (W, S) is a Coxeter system, then the elements of S are • pairwise distinct non-identity elts
(• involutions)

2.~~4~~ Corollary If (W, S) is a Coxeter system, then $\forall i \neq j$ the element $s_i s_j$ has order m_{ij}.

Let G be a group with generating set $S \not\ni e$.

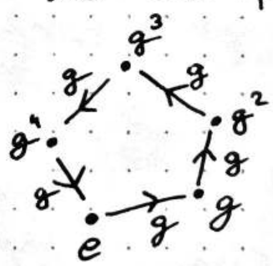
2.4 Definition The Cayley graph of G wrt S $Cay_S(G)$ is the graph with vertex set G

It has directed edge set $\{(g, gs) : g \in G, s \in S, s^2 \neq e\}$
and undirected edge set $\{\{g, gs\} : g \in G, s \in S, s^2 = e\}$.

All edges are labelled by the corresponding generator $s \in S$.

e.g. $G = C_5 = \langle g \rangle$

$Cay_{\{g\}}(C_5)$ is

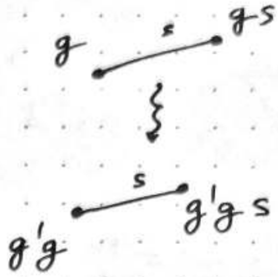


In our examples, S is always a set of involutions, so all edges in $Cay_S(G)$ are undirected.

2.5 Remark Since S generates G, $Cay_S(G)$ is connected.
From Cor 2.3 we also know that for (W, S) a Coxeter system $Cay_S(W)$ is simple
- no loops at v
- no double edges

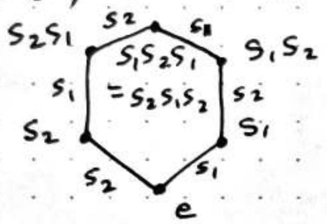
2.6 Lemma

G acts on $\mathrm{Cay}_S(G)$ via multiplication, on the left.
This action preserves edge labels
Under this action, if $s^2 = e$
then $g s g^{-1}$ is the unique group
element which flips the
undirected edge $\{g, gs\}$



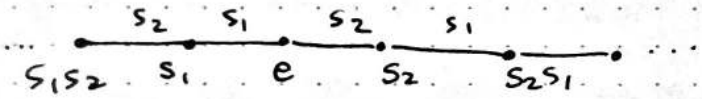
2.7 Example D_{2m}

e.g. D_6, $S = \{s_1, s_2\}$



c.f. translates of K

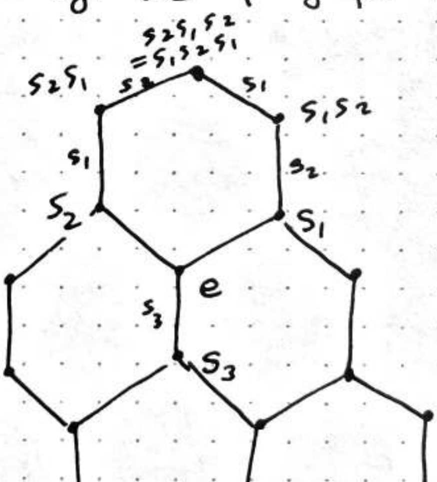
$D_\infty \curvearrowright \mathbb{R}$, $S = \{s_1, s_2\}$



2.8 Remark

In case of geom. refl. groups, $\mathrm{Cay}_S(W)$ is dual to the tessellation of $\mathbb{X}^n$ by the polytope P.

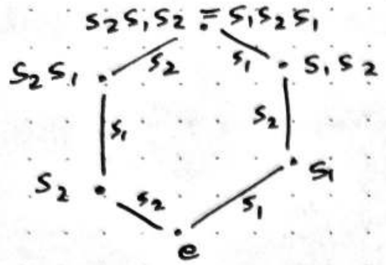
Triangle group $(3,3,3)$
$S = \{s_1, s_2, s_3\}$



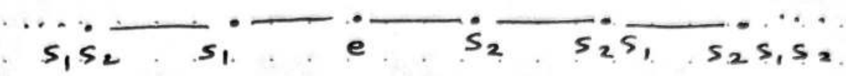
Last time $Cay_S(G)$, S involutions

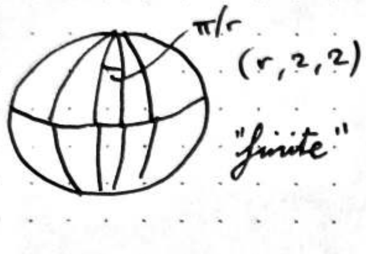
vertices G, edges $\{g, gs\}$ $s \in S$

$W = D_6 = \{s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e\}$, $S = \{s_1, s_2\}$



$W = D_\infty = \{s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = e\}$





Coxeter System (W, S)

Def^n 2.10 Given a group G with generating set S of involutions, an element is a product of generators $s_1 \cdots s_n$, $s_i \in S$ and a word is a finite sequence of generators $(s_1, \ldots, s_n)$, $s_i \in S$

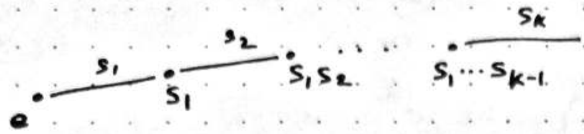
The word length of $g \in G$ wrt S is the
$$\ell_S(g) = \min\{n \in \mathbb{N} \mid \exists s_1, \ldots, s_n \in S \text{ s.t. } g = s_1 \cdots s_n\}$$
By convention $\ell_S(e) = 0$.

If $\ell_S(g) = n \geq 1$ and $g = s_1 \cdots s_n$ then we call $(s_1, \ldots, s_n)$ a reduced word for g.

e.g. in D_6, $s_1 s_2 s_1$ element $= s_2 s_1 s_2$, (s_1, s_2, s_1) and (s_2, s_1, s_2) are reduced words for this element

2.11 Def^n The word metric on G is given by
$$d_S(g, h) = \ell_S(g^{-1} h) \quad \text{for } g, h \in G.$$
This extends to a path metric on $Cay_S(G, S)$. Each edge is given length 1. The distance between two vertices is the shortest path between them.

2.12 Example $d_S(e, g) = \ell_S(g) = k$ and if $(s_1, \ldots, s_k)$ is a reduced word for g, then we get a path of length k from e to g in $Cay_S(G)$.



$g = s_1 \cdots s_k$ "minimal length path"

<u>2.13 Definition</u> A <u>pre-reflection system</u> for a group G is a pair (X, R) s.t.
- X is a connected simple graph
- G acts on X by graph automorphisms (on left)
- R is a subset of G, R generates G, and

  a) every $r \in R$ is an involution

  b) R is closed under conjugation, i.e.

  $\forall g \in G, r \in R, \ grg^{-1} \in R$

  c) R generates G

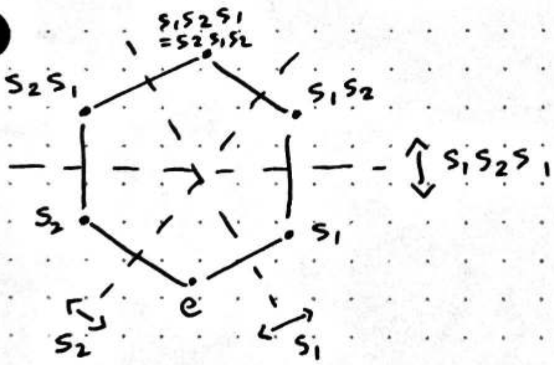
  d) $\forall \{v, w\} \in E(X)$, $\exists!$ $r \in R$ which flips $\{v, w\}$ i.e. interchanges v and w

  e) each $r \in R$ flips at least one edge of X

For $r \in R$, let $H_r = \{$ midpoints of edges flipped by $r \}$

<u>2.14 Example</u>

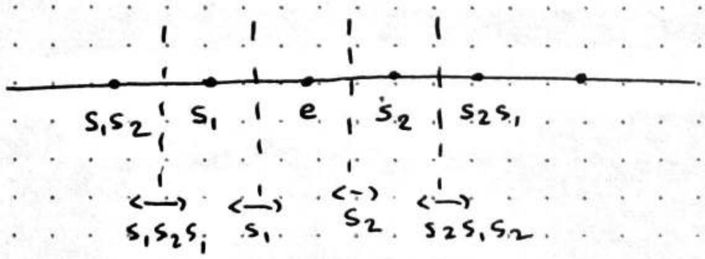
$W = D_6$, $X = Cay_S(W)$, $R = \{s_1, s_2, s_1 s_2 s_1 = s_2 s_1 s_2\}$



Then (X, R) is a pre-reflection system for $W = D_6$

$W = D_\infty$, $X = Cay_S(W)$,

$R = \{ w s_i w^{-1} \mid i = 1, 2, \ w \in W \}$



<u>2.15 Lemma</u> If (X, R) is a pre-reflection system for G, then G acts transitively on $V(X)$.

<u>Proof</u> X is connected, so $\exists$ path between any two vertices v and w, say $(v = v_0, v_1, \dots, v_k = w)$

Let r_i be the unique element of R which flips $\{v_i, v_{i+1}\}$

Then $r_{k-1} \cdots r_1 r_0 \cdot v = w$. $\square$

<u>2.16 Lemma</u> Let (W, S) be a Coxeter system, and set $R = \{ wsw^{-1} \mid s \in S, w \in W \}$

Then $(\mathrm{Cay}_S(W), R)$ is a pre-reflection system for W.

<u>Proof</u> From Rmk 2.5, $\mathrm{Cay}_S(W)$ is always a connected simple graph. Next $(wsw^{-1})^2 = e \ \forall w \in W, s \in S$, and is the unique reflection which flips the edge $\{w, ws\} \in \mathrm{Cay}_S(W)$. $\square$

<u>2.17</u> Let (X, R) be a pre-reflection system for G. Then (X, R) is a <u>reflection system</u> if in addition it satisfies

f) for each $r \in R$, $X \setminus H_r$ has two connected components

§3 <u>Combinatorics of Coxeter groups</u>

In this section we will prove

<u>3.1 Theorem</u> Let W a group generated by a set S of distinct involutions. Then TFAE:

(1) (W, S) is a Coxeter system

(2) Let $X = \mathrm{Cay}_S(W)$, $R = \{ wsw^{-1} \mid s \in S, w \in W \}$.

Then (X, R) is a reflection system.

(3) (W, S) satisfies the 'deletion' condition

(4) " 'exchange' "

<u>3.2 Def^n</u> The pair (W, S) is said to satisfy the <u>deletion condition</u> if the following holds:

(D) If $w = (s_1, \dots, s_k)$ is a word in S with $\ell_S(s_1 \cdots s_k) < k$ then $\exists$ indices $i < j$ s.t. $s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$

3.3 Def^{n} (W,S) satisfies the exchange condition if:

(E) If $(s_1,\dots,s_k)$ reduced word then for any $s \in S$,

either $\ell_S(ss_1\cdots s_k) = k+1$

or $s_1\cdots s_k = ss_1\cdots\hat{s}_i\cdots s_k$ for some $i \in \{1,\dots,k\}$

Proof of Thm 3.1 (3) $\Rightarrow$ (4)

Suppose $(s_1,\dots,s_k)$ (reduced) for $w = s_1\cdots s_k$ and $s_0 \in S$.

Then $\ell_S(s_0\cdots s_k) = \ell_S(s_0 w) \le \ell_S(s_0) + \ell_S(w) = k+1$.

If $\ell_S(s_0 w) = k+1$ done, so suppose $\ell_S(s_0 w) < k+1$.

Then by (D), $\exists\ 0 \le i < j \le k$ s.t.

$$s_0 w = s_0 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

Since $(s_1,\dots,s_k)$ reduced, we must have $i = 0$; otherwise left-multiply by s_0 to get ※

So $s_0 w = s_1 \cdots \hat{s}_j \cdots s_k$

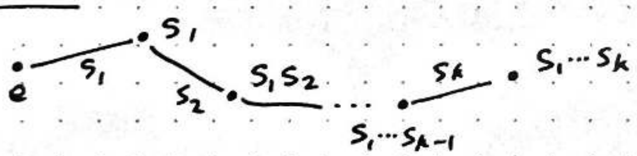
So $w = s_0^2 w = s_0 s_1 \cdots \hat{s}_j \cdots s_k$ as desired. □

Discussion Let W be gen. by S as in Thm 3.1

Then $\exists$ bijection

$$\{\text{words in } S\} \longleftrightarrow \left\{\begin{matrix}\text{paths in } Cay_S(W) \\ \text{starting at } e\end{matrix}\right\}$$

Ex 2.12



Let $R = \{ w s w^{-1} \mid w \in W, s \in S\}$.

From Lemma 2.6, $\exists$ unique $r \in R$ flipping an edge

$s_1\cdots s_{j-1}$ —— $s_1 \cdots s_j$, namely $r_j = s_1 \cdots s_j s_{j-1} \cdots s_1$

e.g. $r_1 = s_1$, $r_2 = s_1 s_2 s_1$, $r_3 = s_1 s_2 s_3 s_2 s_1$.

$\rightsquigarrow$ a reflection sequence $(r_1,\dots,r_k)$ for a word $(s_1,\dots,s_k)$

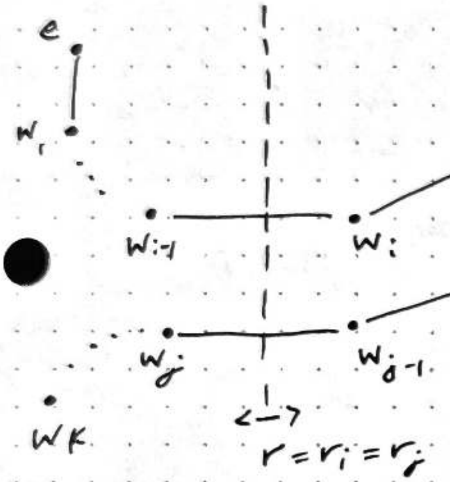
$$H_r = \{ \text{midpoint } \{v,w\} \mid r \text{ flips } \{v,w\} \}$$

We say the word $(s_1,\dots,s_k)$ crosses H_r if the associated path does, i.e. $(s_1,\dots,s_k)$ crosses $H_{r_1},\dots,H_{r_k}$

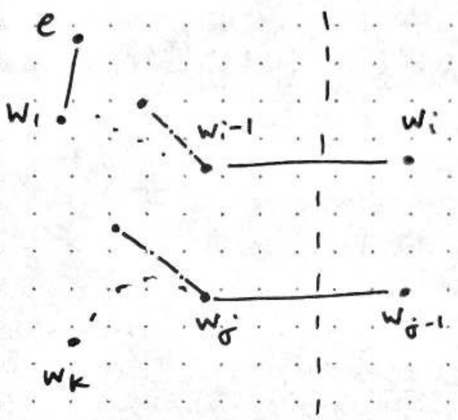
3.4 Lemma Let W, S, R as above, and $(s_1, \dots, s_k)$ a word in S with associated refl. seq $(r_1, \dots, r_k)$ s.t. $r_i = r_j$ for some $1 \le i \lneq j \le k$. Then in W,

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

Proof Let $r_i = r_j$ and $w_p = s_1 \cdots s_p$.
Then in $\mathrm{Cay}_S(W)$ we have



Apply r to $w_i \cdots w_{j-1}$



$W \curvearrowright \mathrm{Cay}_S(W)$ preserves edge labels
So get a new path to w_k

$$(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k)$$

As desired. $\square$

3.5 Lemma Let W, S, R as above. Then for each $r \in R$,
$\mathrm{Cay}_S(W) \setminus H_r$ has at most two connected components.

Proof $r = wsw^{-1}$ for some $w \in W$, $s \in S$
We claim $w \cdot H_s = H_{wsw^{-1}} = H_r$

sketch s flips edge $\qquad\qquad$ wsw^{-1} flips

$$g \overset{s'}{\text{———}} gs' \quad \Rightarrow \quad wg \overset{s'}{\text{———}} wgs'$$

Then wlog can prove for H_s (because $W \curvearrowright \mathrm{Cay}_S(W)$ by isometries)

First, we show for all $v \in V(\mathrm{Cay}_S(W)) = W$ that either v or ~~v~~ sv is in the same component of $\mathrm{Cay}_S(W) \setminus H_s$ as e.

Let $(s_1, \ldots, s_k)$ be a reduced word for v

$\rightsquigarrow$ path in $\mathrm{Cay}_S(W)$ from e to v, with associated refl. sequence $(r_1, \ldots, r_k)$.

If $s \neq r_i$ $\forall i$ then e, v are in the same component of $\mathrm{Cay}_S(W) \setminus H_s$.

So suppose $s = r_i$ for some i.

Then by Lemma 3.4, $(s_1, \ldots, s_k)$ is reduced, i is the unique index s.t. $s = r_j$.

Then the word $(s, s_1, \ldots, s_k)$ has refl sequence

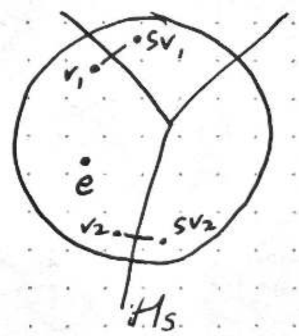
$(s, r_1', \ldots, r_k')$ where $r_j' = s r_j s$

Then $r_i' = sss = s$ and $r_j' \neq s$ for $j \neq i$

So $(s, r_1', \ldots, r_j', \ldots, r_k')$ has exactly two instances of s.

By the previous lemma, may delete to get a word for sv which $\rightsquigarrow$ path from e to sv not crossing H_s. $\square$

<u>N.B.</u> We are done because if $\mathrm{Cay}_S(W) \setminus H_s$ has more than 2 components. Then $\exists\, v_1 \neq v_2$ in the same component as e s.t. sv_1, sv_2 are in 2 other components.



Now every path from sv_1 to sv_2 has to cross H_s $\Rightarrow$ every path from v_1 to v_2 crosses $sH_s = H_s$ ⨳

3.6 Lemma For any word $(s_1,\ldots,s_k) = \underline{s}$, where (W,S) is a Coxeter system, Let $n(r,\underline{s})$ be the no. of times the corresponding path crosses H_r in $\mathrm{Cay}_S(W)$.

(i) Then for any word $(s_1,\ldots,s_k) = \underline{s}$ with $w = s_1 \cdots s_k$, any $r \in R$, $(-1)^{n(r,\underline{s})} \in \{\pm 1\}$ depends only on w.

(ii) $\exists$ group hom. $W \longrightarrow \mathrm{Sym}(R \times \{\pm 1\})$

$$w \longmapsto \phi_w,$$

$$\phi_w(r,\varepsilon) = (wrw^{-1}, (-1)^{n(r,\underline{s})}\varepsilon)$$ where $\underline{s}$ is any word for w

Chap 4 The Tits representation

Thm (Tits) Let I be a finite indexing set, let $S = \{s_i\}_{i \in I}$ and let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix.
Then there's a faithful representation $\rho : W \longrightarrow GL_n(\mathbb{R})$ where $W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ and $n = |S| = |I|$ s.t.

- for all i, $\rho(s_i) = \sigma_i$ is a linear involution with fixed point set a hyperplane
- for all i, j, the product $\sigma_i \sigma_j$ has order m_{ij}

N.B. $\sigma_i \in GL(n, \mathbb{R})$ won't usually be an o.n. reflection

Construction of the Tits representation

(W, S) as above. Wlog $I = \{1, \dots, n\}$.

V = n-dim real vector space, basis $e_1, \dots, e_n$

Define a symmetric bilinear form B on V by

$$B(e_i, e_j) = \begin{cases} -\cos(\pi/m_{ij}) & \text{if } m_{ij} \text{ finite} \\ -1 & \text{if } m_{ij} = \infty \end{cases}$$

Note $B(e_i, e_i) = 1$ & $B(e_i, e_j) \leq 0$ for $i \neq j$

Define $\sigma_i : V \longrightarrow V$ by $\sigma_i(v) = v - 2B(e_i, v)e_i$

First properties:
- σ_i linear map
- $\sigma_i(e_i) = -e_i$
- fixed points of σ_i; $\mathrm{Fix}(\sigma_i) = \{v \in V \mid B(e_i, v) = 0\} =: H_i$ is a hyperplane
- σ_i preserves the bilinear form
  [$B(\sigma_i(e_j), \sigma_i(e_k)) = B(e_j, e_k)$]

Proposition 4.2 $\sigma_i \sigma_j$ has order m_{ij} for all $i, j \in I$

Corollary 4.3 The map $s_i \longmapsto \sigma_i$ extends to a homomorphism $\rho : W \rightarrow GL(n, \mathbb{R})$

Proof of prop^n 4.2
- $i = j$ ✓
- Assume $i \neq j$. Let $V = \mathrm{span}(e_i, e_j)$

$\sigma_i(V_{ij}) = V_{ij} = \sigma_j(V_{ij})$

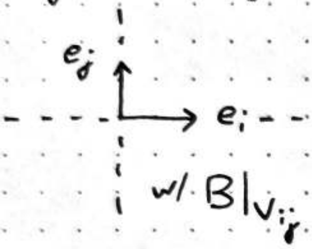
so can consider $\sigma_i, \sigma_j |_{V_{ij}}$

- **Case (a): m_{ij} finite**

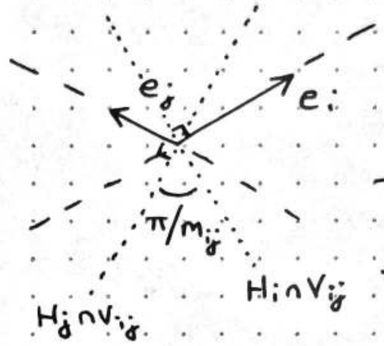
Matrix rep^n of $B|_{V_{ij}}$ wrt (e_i, e_j): $\begin{pmatrix} 1 & -\cos(\frac{\pi}{m_{ij}}) \\ -\cos(\frac{\pi}{m_{ij}}) & 1 \end{pmatrix}$

det > 0, tr > 0, so $B|_{V_{ij}}$ +ve def!

After change of basis, get standard IP on $\mathbb{R}^2$



change basis to $\rightsquigarrow$ standard inner product



Angle b/w e_i, e_j?

$-\cos(\frac{\pi}{m_{ij}}) = \cos(\pi - \frac{\pi}{m_{ij}})$

$\therefore$ angle b/w e_i, e_j

$\sigma_i|_{V_{ij}}$: (o.n.) reflection in $H_i \cap V_{ij}$

$\sigma_j|_{V_{ij}}$ similar

Upshot $\sigma_i \sigma_j |_{V_{ij}}$ is rotation by $\frac{2\pi}{m_{ij}}$ ($\Rightarrow$ of order m_{ij})

$V_{ij}^{\perp} = \{ w \in V : B(w,v) = 0 \ \forall v \in V_{ij} \}$

Note $V = V_{ij} \oplus V_{ij}^{\perp}$

use $B|_{V_{ij}}$ non-deg

$\sigma_{ij}|_{V_{ij}^{\perp}} = Id$

Hence $\sigma_i \sigma_j$ has order m_{ij} on V as required.

Case (b): m_{ij} infinite

Matrix rep^n of $B|_{V_{ij}}$ wrt (e_i, e_j): $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

+ve semi-def but not def. Calculate

$\sigma_i \sigma_j (e_i) = \sigma_i (e_i + 2e_j) = e_i + 2(e_i + e_j)$ $\quad *$

$\Rightarrow (\sigma_i \sigma_j)^k (e_i) = e_i + 2k(e_i + e_j)$ $\quad$ [$e_i + e_j$ fixed by σ_i, σ_j]

$\Rightarrow \sigma_i \sigma_j$ has infinite order $\square$

Cor 4.4 Let (W, S) be a Coxeter system.
Then elts of S are pairwise distinct (in W)

Pf $\sigma_i \neq \sigma_j$ (use $\sigma_i \sigma_j$ has order m_{ij}
or just notice diff linear maps) $\square$

Cor 4.5 $s_i s_j$ has order m_{ij} in W

Pf Immediate as $\sigma_i \sigma_j$ has order m_{ij}. $\square$

Geometry when $m_{ij} = \infty$ Matrix repⁿ: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Null space of $B|_{V_{ij}}$ spanned by $e_i + e_j$

$N := \langle e_i + e_j \rangle$

$B|_{V_{ij}}$ induces a +ve def form on V_{ij}/N ← 1-dim

$W_{ij} = \langle s_i, s_j \rangle \leq W$. Note: $W_{ij} \cong D_\infty$ we'll recover action from before

Note: $H_i|_{V_{ij}} = N = H_j|_{V_{ij}}$ so ... no fruit

Idea: Consider dual vector space $V_{ij}^* = \mathrm{Hom}_{\mathbb{R}}(V_{ij}, \mathbb{R})$

Dual repⁿ $(w \cdot \varphi)(v) = \varphi(w^{-1} \cdot v)$

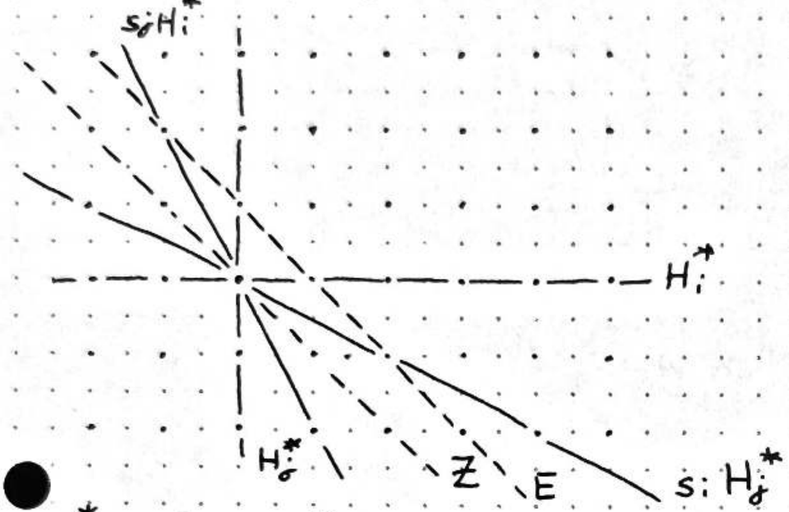
where $w \in W_{ij}$, $\varphi \in V_{ij}^*$, $v \in V_{ij}$

This is faithful as dual of faithful

(Last time) <u>Geometry when $m_{ij} = \infty$</u>

$V_{ij}^* = \text{span}(e_i, e_j)^* \circlearrowleft_{\text{dual rep}^n} W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$

Lecture "9" starts



$H_i^* = \{ \varphi \in V_{ij}^* \mid \varphi(e_i) = 0 \}$ ⌜$\varphi \in \text{fix}(s_i^*)$ iff $\text{im}(s_i - 1) \subset \ker \varphi$ ⌟
$ \overset{\shortparallel}{\langle e_i \rangle}$

$Z = \{ \varphi \mid \varphi(e_i + e_j) = 0 \}$

$(\simeq V_{ij}/N)^*$

⌜both s_i, s_j flip Z⌟

$\rho^*(s_i) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ wrt dbv basis e_j^*, e_i^* (e_j^* spans H_i^*, e_i^* spans H_j^*)

$\rho^*(s_j) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$

$E = Z + 1$

$\circlearrowleft$ std act^n of D_∞

<u>Faithfulness of the Tits representation</u> Dual rep^n $\rho^* : W \to GL(V^*)$

$$(\underbrace{\rho^*(w)}_{\in W}\,(\underbrace{\varphi}_{\in V^*})\,)(\underbrace{v}_{\in V}) = \varphi(\rho(w^{-1})(v))$$

Goal: ρ^* faithful ($\Leftrightarrow$ ρ is too)

Define $\varphi_i \in V^*$ by $\varphi_i(v) = B(e_i, v)$. ⌜$\varphi_i(e_i) = 1$⌟

Then if $\sigma_i^* = \rho^*(s_i)$,

$$\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$$

$H_i^* = \{ \varphi \in V^* \mid \varphi(e_i) = 0 \}$ hyperplane ⌜fixed point set of σ_i^*⌟

$C_i = \{ \varphi \in V^* \mid \varphi(e_i) > 0 \}$ half-space

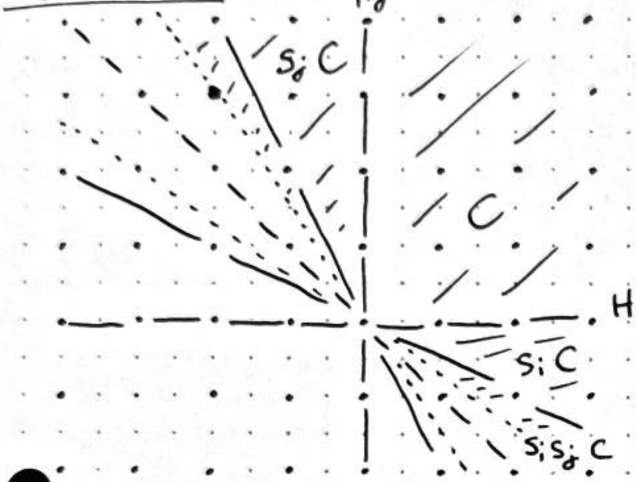
$C = \bigcap_{i \in I} C_i$, its closure $\overline{C}$ is the chamber associated to the rep^n

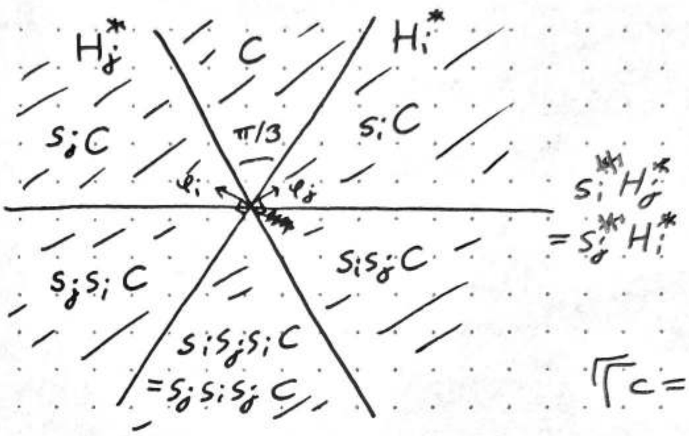
- $C_{ij} = C_i \cap C_j$

m_{ij} infinite

Ex m_{ij} finite : span(φ_i, φ_j)

$\mathbb{E}^2$ w/ std inner product





⌜$C = C_{ij}$⌟

- 2d containing φ_i, φ_j ⌜$\varphi_j = -\varphi_i$⌟ · span $\mathbb{Z}$ ·

$\oplus \bigcap_{k=i,j} H_k^*$ ⟲ s_i, s_j trivial ⌜actually above $\varphi_i = \binom{-1}{1}$⌟

$\oplus \bigcap_{k=i,j} H_k^*$ ⟲ s_i, s_j fix

Def Let G be a group acting on a set Π. $\mathcal{C} \subset \Pi$ is prefundamental for G if $\forall g \in G$, $g\mathcal{C} \cap \mathcal{C} \neq \emptyset \Rightarrow g = 1 \in G$

Ex C_i is prefundamental for $W_i = \langle s_i \rangle$ acting on V^*

C_{ij} is prefundamental for W_{ij}

- Theorem 4.6 $(W, S, \{C_i\})$ satisfies property P:

for any $w \in W$, $i \in I$, either $wC \subset C_i$

or $wC \subset s_i C_i$

Moreover, in 2nd case, $\ell(s_i w) = \ell(w) - 1$

(word length on W wrt S

⌜faith intensifies⌟

Cor 4.7 C is prefundamental for W ($\Rightarrow \rho^*$ faithful)

Key : We already have that $(W_{ij}, \{s_i, s_j\}, \{C_i, C_j\})$

- satisfies property P

Strategy P_n : (P true for all w with $l(w)=n$)

Q_n : ($\forall w \in W$ w/ $l(w)=n$, $i \neq j$, $\exists u \in W_{ij}$ s.t. $wC \subset uC_{ij}$

& $l(u^{-1}w) = l(w) - \underset{\uparrow}{l'}(u)$)

length wrt $\{s_i, s_j\}$

P_0, Q_0 ✓

Show a) $(P_n \text{ \& } Q_n) \Rightarrow P_{n+1}$

b) $(P_{n+1} \text{ \& } Q_n) \Rightarrow Q_{n+1}$

a) Suppose $l(w) = n+1$, $s_i \in S$. Then $w = s_j w'$ for some w', j w/ $l(w') = n$

- If $i = j$: apply P_n to w'. Must have $w'C \subset C_i$ [else $l_S(w) = n-1$]

$\Rightarrow s_i w' C \subset s_i C_i$

& $l(s_i w) = n$ ✓

- If $i \neq j$: apply Q_n to w'. $\exists u \in W_{ij}$ s.t. $w'C \subset uC_{ij}$

& $l(u^{-1}w') = l(w') - l'(u)$

(i) $s_j u C_{ij} \subset C_i \Rightarrow wC \subset C_i$ ~~$\Rightarrow w$~~ OR (ii) $s_j u C_{ij} \subset s_i C_i$ [c.f. (Key)]

$\Rightarrow wC \subset s_i C_i$

Word length for (ii) ? $l(s_i w) = l(s_i s_j w') \leq l'(s_i s_j u) + l(u^{-1}w')$

$= l'(s_j u) \underset{-1}{\cancel{\ }} + l(w') - l'(u)$

$\underset{\text{must be } =}{\leq} l(w) - 1$

b) Suppose $l(w) = n+1$, $i \neq j$. If $wC \subset C_{ij}$, done $(u = 1)$

Assume not.

Wlog $wC \not\subset C_i$. By P_{n+1}, $wC \subset s_i C_i$, $l(s_i w) = l(w) - 1$

By Q_n, $\exists v \in W_{ij}$ s.t. $s_i w C \subset v C_{ij}$ & $l(s_i w) = l'(v) + l(v^{-1} s_i w)$

Then:

$wC \subset s_i v C_{ij}$ & $l(w) = 1 + l(s_i w)$

$= 1 + l'(v) + l(v^{-1} s_i w)$

so equality throughout, & Q_{n+1} holds, w/ $u = s_i v$ $\geq l'(s_i v) + l((s_i v)^{-1} w) \geq l(w)$ $\square$

§4 The Tits representation (cont.)

Recall $B(e_i, e_j) = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & m_{ij} \text{ finite} \\ -1 & m_{ij} = \infty \end{cases}$

Use this to define
- Tits representation
- Dual rep^n

$\varphi_i \in V^*$, $\varphi_i(v) = B(e_i, v)$

Then $\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$

$H_i^* = \{\varphi \in V^* : \varphi(e_i) = 0\}$ hyperplane

$C_i = \{\varphi \in V^* : \varphi(e_i) > 0\}$ $\frac{1}{2}$ space

$C_{ij} = C_i \cap C_j$, $C = \bigcap_{i \in I} C_i$ $\quad \bar{C}$ chamber associated to Tits representation

4.8 Notation We replace $C \rightsquigarrow C^\circ$

$\bar{C} \rightsquigarrow C$

to agree with the notation in the literature.

4.9 Definition The Tits cone of (W, S) is

$$\bigcup_{w \in W} wC \subset V^*$$

4.10 Examples

1) D_{2n}, n finite, then $V^* \leftrightsquigarrow \mathbb{E}^2$ and the Tits cone is all of $\mathbb{E}^2$

2) D_∞, $V^* = V_{ij}^*$, Tits cone is

$$\{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) > 0\} \cup \{0\}$$

and the interior is the open half space bound by Z and containing Z (?)

§ 5 Finite Coxeter Groups

5.1 Definition

The Coxeter system (W, S) is <u>reducible</u> if $S = S' \sqcup S''$ such that $m_{ij} = 2 \quad \forall s_i \in S', s_j \in S''$

i.e. $s_i s_j = s_j s_i$ is a rel^n in W for $s_i \in S', s_j \in S''$

(W, S) is <u>irreducible</u> o/w

5.2 Remark

If (W, S) is reducible then

$$W \cong \langle S' \rangle \times \langle S'' \rangle$$

⌜ $\langle S' \rangle \cap \langle S'' \rangle = \{1\}$ needs some justifying ⌟

But (W, S) can be irreducible and still split as a product

e.g. for k odd, $D_{2(2k)} \cong D_{2k} \times C_2$

but usual $S = \{s_i, s_j\}$ not reducible

5.3 Theorem

Let (W, S) be irreducible and $|S| = n$. Then TFAE

(i) W is a geometric reflection group on $\mathbb{S}^{n-1}$ generated by $S = \{s_i\}_{i \in I}$ the set of reflections in codimension one faces $\{F_i\}_{i \in I}$ of a simplex in $\mathbb{S}^{n-1}$ s.t. F_i, F_j intersect at an angle $\frac{\pi}{m_{ij}}$

(ii) B is positive definite

(iii) W is finite

<u>Proof</u> see Davis § 6, uses Thm 1.11 □

Aside — Similar theorems for Euclidean (B positive semi-definite of corank 1) and hyperbolic

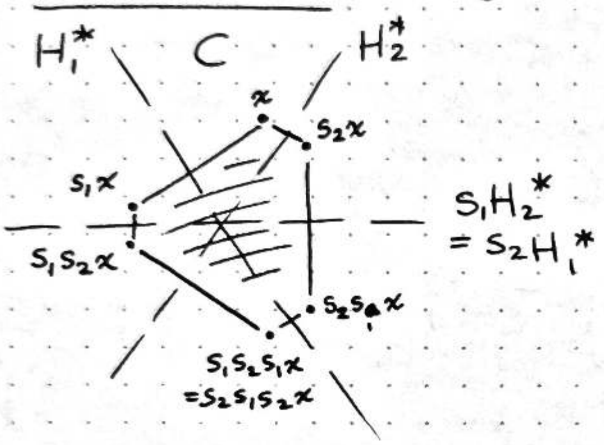
If W finite, $|S| = n$, then $V^* \leftrightsquigarrow \mathbb{E}^n$ and C is a closed Euclidean simplicial cone with boundary given by hyperplanes.

Recall Corollary 4.7 (ES2) which says: for $w \in W$ if $wC^\circ \cap C^\circ \neq \emptyset$ then $w = e$.

$\Rightarrow$ if $x \in C^\circ$ then the orbit Wx has $|W|$ points

5.4 Definition Let (W,S) be finite.
The Coxeter polytope for W is the convex hull of the W-orbit on V^* of a point $x \in C^\circ$.
These are convex Euclidean polytopes but are not in general regular.

5.5 Example D_6



Rmk 1-skeleton is iso. (as a non-metric graph) to $\mathrm{Cay}_S(W)$

Forms $\mathcal{B}$ associated to irreducible Coxeter systems can be classified by graphs $\rightsquigarrow$ Coxeter classification of finite Cox. groups

5.6 Definition
A Coxeter-Dynkin diagram Γ is a simple labelled graph with finite vertex set $V(\Gamma) = S = \{s_i\}_{i\in I}$ and edge labels

$s_i \overset{m_{ij}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} s_j$ where $m_{ij} \geq 3$ (or $m_{ij} = \infty$)

5.7 Lemma There is a one-to-one correspondence between Coxeter systems (W,S) and Coxeter diagrams Γ

Proof We give a bijection

Cox system $\longleftrightarrow$ Coxeter diag

$S \longleftrightarrow V(\Gamma)$

$m_{ij} = m_{ji} = 2 \longleftrightarrow$ no edge b/w s_i, s_j

$m_{ij} = m_{ji} \geq 3 \longleftrightarrow$ edge labelled m_{ij} b/w s_i, s_j $\square$

5.8 Notation We omit edge labels $m_{ij} = 3$ for the rest of the course

i.e. $\underset{s_i}{\bullet} \overset{3}{\text{——}} \underset{s_j}{\bullet} \quad \longleftrightarrow \quad \underset{s_i}{\bullet} \text{——} \underset{s_j}{\bullet}$

Under the above bijection, denote the image of Γ by $(W(\Gamma), V(\Gamma))$ or $(W(\Gamma), S)$

5.9 Remark Many mathematicians use a different convention where $\overset{\bullet}{s_i} \quad \overset{\bullet}{s_j} \Rightarrow m_{ij} = \infty$

5.10 Theorem (Coxeter 1930s)

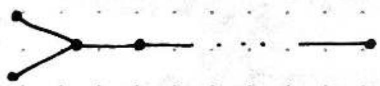
(W, S) gives rise to a finite Coxeter group

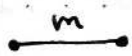
$$\Updownarrow$$

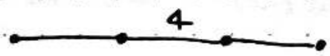
$(W,S) = (W(\Gamma), V(\Gamma))$ for Γ a disjoint union of a finite number of the following graphs:

A_n $(n \geq 1)$  n vertices

B_n $(n \geq 2)$  n vertices

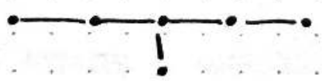
D_n $(n \geq 4)$  n vertices

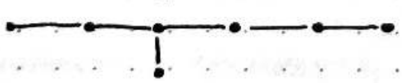
$I_2(m)$  $m \geq 5$

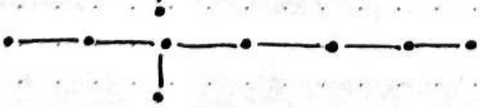
F_4 

H_3 

H_4 

E_6 

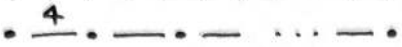
E_7 

E_8 

Example $S_n \cong W(A_{n-1})$

Classification of Finite Coxeter Groups

A_n $(n \geq 1)$ 

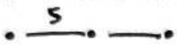
B_n $(n \geq 2)$ 

D_n $(n \geq 4)$ 

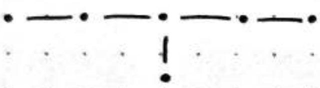
n vertices

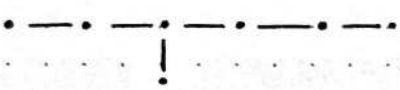
$I_2(m)$ $(m \geq 5)$ 

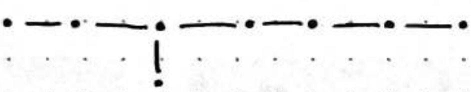
F_4 

H_3 

H_4 

E_6 

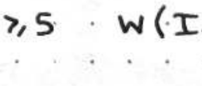
E_7 

E_8 

5.11 Remark If $\Gamma = \Gamma_1 \sqcup \Gamma_2$

then $(W(\Gamma), V(\Gamma))$ is reducible, only if

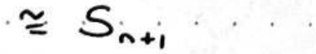
5.12 Examples

D_{2m} $\quad W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = e \rangle$ 

$m = 3$ $\quad W(A_2)$

$m = 4$ $\quad W(B_2)$

$m \geq 5$ $\quad W(I_2(m))$

A_n $(n \geq 1)$  n vertices

$W(\Gamma) \cong S_{n+1}$

$W(A_n) = \langle s_1, \dots, s_n \mid s_i^2 = e,\ (s_i s_{i+1})^3 = e,\ (s_i s_j)^2 = e \text{ for } |i-j| \geq 2 \rangle$

Given a Cox. diagram Γ, let $S = V(\Gamma)$.

Let Γ_T be the <u>full subgraph</u> of Γ spanned by $T \subseteq S$

↳ if $t_1, t_2 \in T$ and in Γ $\underset{t_1}{\bullet}\overset{m}{\text{———}}\underset{t_2}{\bullet}$ then so too in Γ_T

Then $(W(\Gamma_T), T)$ is a Coxeter system

e.g. $\Gamma = \underset{s}{\bullet}\overset{4}{\text{——}}\underset{t}{\bullet}\text{——}\underset{u}{\bullet}$ $\quad S = \{s, t, u\}$, $T = \{s, t\}$ then $\Gamma_T = \underset{s}{\bullet}\overset{4}{\text{——}}\underset{t}{\bullet}$

<u>5.13 Definition</u> (W, S) Cox system, $T \subseteq S$

The <u>parabolic subgroup</u> W_T of W is $W_T = \langle T \rangle$.

If $T = \emptyset$ then $W_T = \{e\}$ is the trivial group.

<u>5.14 Lemma</u> (W, S) Cox system and W_T, $(W(\Gamma_T), T)$ defined as above.

Then $W(\Gamma_T) \cong W_T$.

<u>Proof</u> Let $|S| = n$ and V an n-dim v. space with basis $\{e_s\}_{s \in S}$, and $\rho : W \to GL(V)$ the Tits representation with form B. Let G_T be the subgroup of $GL(V)$ which stabilises (read: acts on) the subspace $V_T = \langle e_t \mid t \in T \rangle$

Now $(W(\Gamma_T), T)$ has its own Tits rep^n with form B_T, vector space $V' = \langle e'_t \mid t \in T \rangle$. Then

$$\begin{aligned} V' &\longrightarrow V \\ e'_t &\longmapsto e_t \end{aligned}$$

is a v. space inclusion.

By naturality of the Tits representation (i.e. $B|_T = B_T$) we get a comm. diagram

$$\begin{tikzcd} W(\Gamma_T) \arrow[r, hook, "\rho_T"', "4.7"] \arrow[d, two heads, "\text{univ prop of group presentation}"'] & GL(V') \\ W_T \arrow[r, "\rho|_{V_T}"'] & G_T \arrow[u, "\text{restricting } V \text{ to } V_T \cong V'"'] \end{tikzcd}$$

Top arrow injection (Cor 4.7)

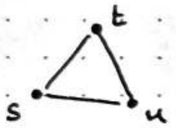
$\Rightarrow$ Left arrow injection

$\Rightarrow$ $W(\Gamma_T) \cong W_T$ as required. $\square$

<u>5.15 Definition</u> If a parabolic subgroup is finite, we call it a <u>spherical subgroup</u>.

<u>5.16 Corollary</u> Combining Theorem 5.10 with Lemma 5.14, we see that all spherical subgroups can be obtained by observing Γ for (W, S)

<u>5.17 Example</u> Example of the $(3,3,3)$ triangle group



(W, S) has spherical subgroups

$$W_\emptyset, \quad \underbrace{W_s, W_t, W_u}_{\text{type } A_1}, \quad \underbrace{W_{\{s,t\}}, W_{\{t,u\}}, W_{\{u,s\}}}_{\text{type } A_2}$$

$S = \{s, t, u\}$

<u>5.18 Theorem</u> (W, S) Cox. system. Then

(a) (W_T, T) is also a Cox. system $\forall\, T \subseteq S$

(b) For all $T \subseteq S$, $w \in W_T$, $\ell_T(w) = \ell_S(w)$ and any reduced word for w in S, $(s_1, \dots, s_k)$ satisfies $\forall i,\ s_i \in T$

(c) If $T, T' \subseteq S$ then
$$W_T \cap W_{T'} = W_{T \cap T'}$$
$$\langle W_T, W_{T'} \rangle = W_{T \cup T'}$$

(d) The bijection
$$\left\{ \begin{matrix} \text{subsets of} \\ S \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{parabolic subgroups} \\ \text{of } W \end{matrix} \right\}$$
$$T \longmapsto W_T$$
preserves the partial order given by inclusion.

<u>5.19 Lemma</u> For (W,S) a Cox. system, $w \in W$,
$\exists$ subset $S(w) \subset S$ s.t. given any reduced word $(s_1, \dots, s_k)$ for w, $S(w) = \{s_1, \dots, s_k\}$

⌜i.e. depends only on w⌟

<u>Proof</u> Let w be a minimal length counter-example
i.e. $w = s_1 \cdots s_k = t_1 \cdots t_k$ s.t. $\{s_1, \dots, s_k\} \neq \{t_1, \dots, t_k\}$

Then $w = s_1 v$ where $(s_2, \dots, s_k)$ is reduced for v.

By the exchange condition, since $\ell(s_1 w) < \ell(w)$, $\exists i$ s.t.

$$w = s_1 t_1 \cdots \hat{t}_i \cdots t_k$$

So v satisfies $S(v) \subseteq \{t_1, \dots, t_k\}$.

Since $\ell_S(v) < \ell_S(w)$, by induction

$$\{s_2, \dots, s_k\} \subseteq \{t_1, \dots, t_k\}$$

By same argument on $w^{-1} = s_k \cdots s_1$
we get $\{s_{k-1}, \dots, s_1\} \subseteq \{t_1, \dots, t_k\}$.

So $\{s_1, \dots, s_k\} \subseteq \{t_1, \dots, t_k\}$.

By symmetry $\{s_1, \dots, s_k\} = \{t_1, \dots, t_k\}$ $\#$ $\square$

5.18 Theorem (W,S) a Cox. sys.

(a) (W_T, T) Cox system $\forall\ T \subseteq S$

(b) $\forall\ T \subseteq S$, $w \in W_T$, $\ell_T(w) = \ell_S(w)$ and any reduced word for w in S $(s_1, \ldots, s_k)$ satisfies $\forall i$, $s_i \in T$

(c) If $T, T' \subset S$, then $W_T \cap W_{T'} = W_{T \cap T'}$

$$\langle W_T, W_{T'} \rangle = W_{T \cup T'}$$

(d) $\{$ subsets of $\cancel{X}^{S}$ $\} \longrightarrow \{$ parabolic subgroups of $W \}$

$$T \longmapsto W_T$$

preserves partial order by inclusion.

5.19 Lemma (W,S) Cox syst. $w \in W$, $\exists\, S(w) \subseteq S$ s.t. given any reduced word $(s_1, \ldots, s_k)$ for w, $S(w) = \{s_1, \ldots, s_k\}$

Proof of Thm 5.18

(a) Follows from Lemma 5.14 $(W_T \cong W(\Gamma_T))$

deletion condition

(b) Used Lemma 5.19. Let $w \in W_T$, then $S(w) \subset T$. Then if $(s_1, \ldots, s_k)$ is a reduced word for $w \in W$ then each $s_i \in T$ so $\ell_S(w) = \ell_T(w)$

(c) Clearly $W_{T \cap T'} \subset W_T \cap W_{T'}$

To show $W_T \cap W_{T'} \subset W_{T \cap T'}$, use L. 5.19:

If $w \in W_T \cap W_{T'}$ then $S(w) \subset T$ and $S(w) \subset T'$.

So $S(w) \subset T \cap T'$, and $w \in W_{T \cap T'}$

$\langle W_T, W_{T'} \rangle = W_{T \cup T'}$ exercise

(d) $T' \subsetneq T$, want to show $W_{T'} \subsetneq W_T$.

Then from (c) $W_{T' \cap T} = W_{T'} = W_{T'} \cap W_T$ so $W_{T'} \subseteq W_T$.

Let $s \in T \setminus T'$. By L 5.19, $S(s) = \{s\}$ so any reduced word representing s only involves s. Therefore $s \notin W_{T'}$, whence $W_{T'} \subsetneq W_T$. □

5.20 Definition Given a Cox. sys (W, S). Let

$\mathcal{S} = (T \subseteq S \mid W_T$ is spherical $)$

5.21 Remark $\mathcal{S}$ depends on (W, S) but omit from notation

§6 The basic construction

6.1 Definition An (abstract) simplicial complex is a (possibly infinite) set V – the vertex set, and a collection X of finite subsets of V s.t:

(1) $\forall v \in V$, $\{v\} \in X$

(2) if $\Delta \in X$ and $\Delta' \subseteq \Delta$ then $\Delta' \in X$

An element $\Delta \in X$ is called an (abstract) simplex.

If $\Delta' \subsetneq \Delta$ then Δ' is a face of Δ.

$\dim(\Delta) = |\Delta| - 1$

And Δ is a k-simplex if $\dim(\Delta) = k$.

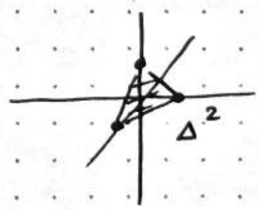
A 0-simplex $\leftrightsquigarrow$ vertex   1-simplex $\leftrightsquigarrow$ edge

The k-skeleton is $X^{(k)} = \bigcup_{\substack{\Delta \in X \\ \dim(\Delta) \leq k}} \Delta$

$\dim(X) = \sup\{\dim(\Delta) \mid \Delta \in X\}$

If $\dim(X) < \infty$ then X is finite dim'l

The standard n-simplex Δ^n is the convex hull of $e_1, \dots, e_{n+1}$ in $\mathbb{R}^{n+1}$.



abstract simplicial complex $\leftrightsquigarrow$ simplicial cell complex

n-simplex $\Delta \longmapsto$ standard n-simplex

Δ' a face of $\Delta \longmapsto$ glue accordingly

⌜totally order V⌟

$V = V(X) = X^0 \longleftarrow\!\!| \; X$

vertex set of X

$\Delta \subseteq V$ abstract simplex if Δ spans a Δ^n in X

Aim of §6: to define basic construction ~~$\mathcal{U}(S)$~~

$\mathcal{U}(W,S)$ for (W,S) a Cox. system.

6.2 Definition (W,S) Cox. syst., X a connected Hausdorff topological space. A mirror structure on X over S is a family $(X_s)_{s\in S}$ of closed non-empty subsets of X.
X is called a mirrored space over S and X_s is the s-mirror of X.

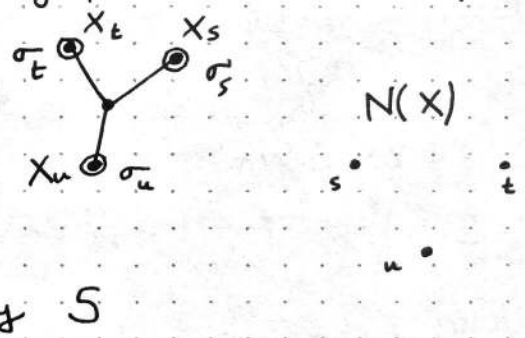
6.3 Remark There is a more general definition for G any group and S indexing a family of subgroups $\rightsquigarrow$ Davis §5.1

$\mathcal{U}(W,X)$ obtained by gluing $|W|$ copies of X along mirrors

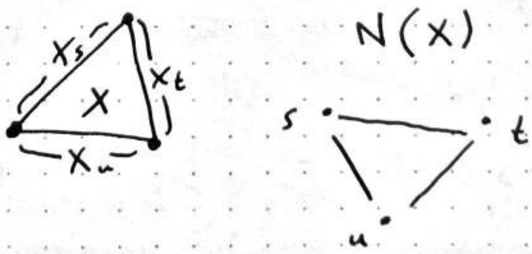
6.4 Definition (W,S) Cox sys., X mirrored space over S
Then the nerve of X, $N(X)$, is an abstract simplicial complex with vertex set S, and $T\subseteq S$ a simplex if $\bigcap_{t\in T} X_t \neq \emptyset$

6.5 Examples

(1) $X = \text{Cone}\{\sigma_s \mid s\in S\}$ i.e. star graph with valence $|S|$
$X_s = \{\sigma_s\}$ e.g. $S = \{s,t,u\}$



(2) $X = \Delta^n$, when $|S| = n+1$
$|S|$ codimension 1 faces, label by S
$\{\Delta_s \mid s\in S\}$, $X_s = \Delta_s$ e.g. $S = \{s,t,u\}$



(3) P^n convex polytope in $\mathbb{X}^n$, $n \geq 2$,

$\{F_i\}_{i \in T}$ faces $i \neq j$, $F_i \cap F_j = \phi$ so $m_{ij} = \infty$

or angle between $\frac{\pi}{m_{ij}}$, $m_{ij} \geq 2$

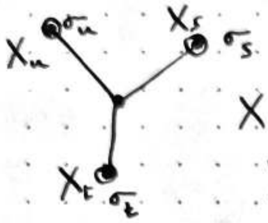
(W, S) Coxeter system with matrix (m_{ij})

Then take ~~$\mathbb{X}$~~ $X = P^n$ and $X_{s_i} = F_i$

Last Time (W,S) Cox system, X connected, Hausdorff top space

Mirror structure on X over S is $(X_s)_{s\in S}$ of closed non-empty subsets. X_s - s mirror

6.5 (i)



$S=\{s,t,u\}$

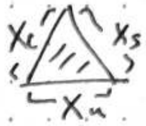
$X = \text{cone}\{\sigma_s \mid s\in S\}$

$X_s = \sigma_s$

(ii) $X = \Delta^n$ $|S| = n+1$

$X_s = \Delta_s$ codim 1 face

e.g. $S=\{s,t,u\}$



(iv) $C \subseteq V^*$ chamber associated to Tits rep^n

H_i^* dual HP fixed by $\sigma_i = \rho^*(s_i)$.

Take $X = C$, $X_{s_i} = C \cap H_i^*$.

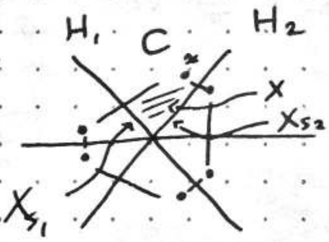
(v) W finite $V^* \leftrightsquigarrow E^n$, $C = \{v\in\mathbb{R}^n \mid \langle v, e_i\rangle \geq 0 \ \forall i\}$

$x\in C^\circ$, Coxeter polytope given by $W\cdot x$

$X = C \cap$ Coxeter polytope

$X_{s_i} = X \cap H$

e.g. D_6



<u>From now</u> (W,S) Cox system, X mirrored space over S

$\exists x\in X$ s.t. $x \notin \bigcup_{s\in S} X_s$

$\forall x\in X$, let $S(x) = \{s\in S \mid x\in X_s\}$ (don't confuse with $S(w)$ from §5)

<u>Examples 6.6</u>

In 6.5 (i), $S(x) = \begin{cases} \varnothing & \text{if } x\notin\{\sigma_s \mid s\in S\} \\ \{s\} & \text{if } x = \sigma_s \end{cases}$

6.5 (ii), $S(x) = \begin{cases} \varnothing & \text{if } x\in X^\circ \\ T\subsetneq S & \text{if } x\in \bigcap_{t\in T}\Delta_t \text{ (and } x \text{ isn't in a smaller intersection)} \end{cases}$

<u>Definition</u> 6.7 Consider W with the discrete topology

$W\times X$ with the product topology

The <u>basic construction</u> is the top space $\mathcal{U}(W,X) = W\times X/\sim$

with the quotient topology and where

$$(w,x) \sim (w',x') \quad \text{iff} \quad x = x',\ w^{-1}w' \in W_{S(x)}$$

Write $[w,x]$ for equiv class of (w,x) in $\mathcal{U}(W,X)$.

E.g. If $x \in X_s$ then $s \in S(x)$, so $(w,x) \sim (ws,x)$

$\Rightarrow$ $[w,x]$ contains at least (w,x) and (ws,x)

Definition 6.8 Write wX for $\{w\} \times X$ in $\mathcal{U}(W,X)$, any $w \in W$

wX is called a chamber of $\mathcal{U}(W,X)$

The fundamental chamber is eX which we identify with X

wX and wsX are identified along X_s

Lemma 6.9 (The Cayley graph)

For X as in 6.5 (i), up to subdivision, $\mathcal{U}(W,X)$ is $Cay_S(W)$

Proof Let $x \in X$. Then if $x \notin \{\sigma_s \mid s \in S\}$ we have $W_{S(x)} = W_\emptyset = \{e\}$

So $(W,x) \sim (w',x) \iff w^{-1}w' \in \{e\}$

$$\iff w = w'.$$

So $[w,x] = \{(w,x)\}$.

Otherwise, $x = \sigma_s$, some $s \in S$, so $W_{S(x)} = W_{\{s\}} = \{e,s\}$

$$(w,x) \sim (w',x) \iff w^{-1}w' \in \{e,s\}$$

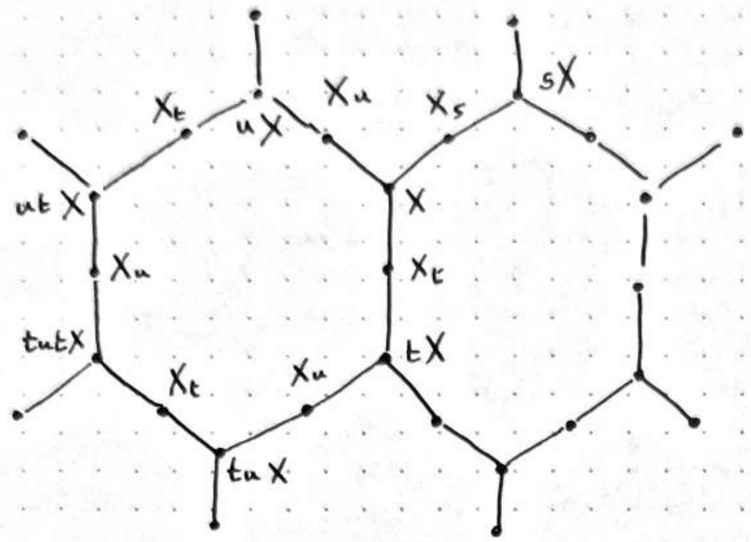
$$\iff w' = w \text{ or } w' = ws$$

i.e. $[w,x] = \{(w,x),(ws,x)\}$

Therefore in $\mathcal{U}(W,X)$ we glue wX and wsX along $X_s = \{\sigma_s\}$ and these are all the gluings.

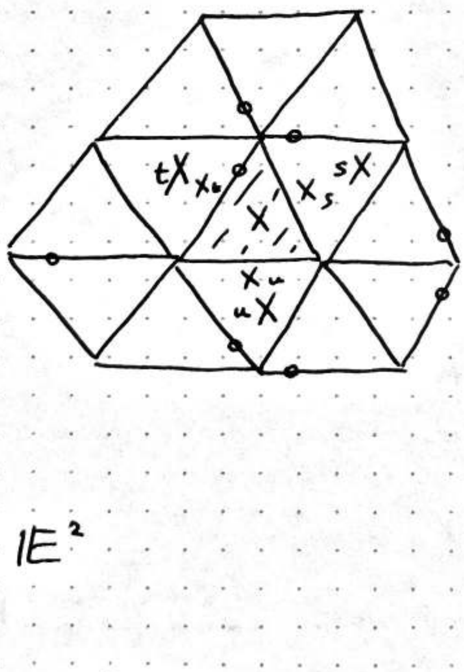
If we label the star point of wX by w, get a map to $Cay_S(W)$ with edges divided by σ_s and mirror labelled with the edge label in $Cay_S(W)$. $\square$

e.g. $\Gamma = {}_t\Delta^s_u$ (3,3,3) triangle group
X as in 6.5 (i)



6.10 Definition For X the mirrored space in 6.5 (ii), i.e. X a simplex with codim 1 faces $\{\Delta_s \mid s \in S\}$, the $\mathcal{U}(W, X)$ is called the Coxeter complex

6.12 Example Cox. complex for (3,3,3) $s \triangle^t u$



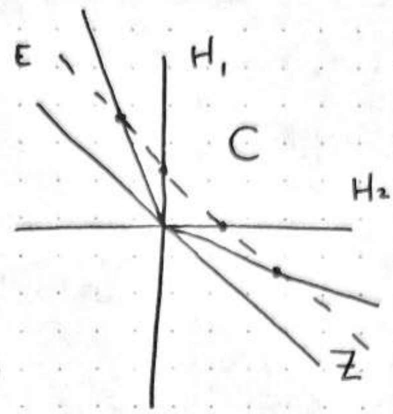
If $x \in X_s \cap X_t$ then $W_{S(x)} = W_{\{s,t\}} \cong D_6$

So $\forall w \in W$, wX is glued to wsX, wtX, $wstX$, $wtsX$, $wstsX = wtstX$ at $x \in X_s \cap X_t$

$\mathcal{U}(W, X)$
tesselation of $\mathbb{E}^2$
by triangles

6.11 Remark If W is an irreducible finite Coxeter group, then the Coxeter complex can be identified with the tessellation of the sphere by spherical simplices induced by W.

If W is affine then $\exists$ affine subspace $E \subset V^*$ given by slicing along the interior of Tits cone.



Then $W \curvearrowright E$ by isometries and Coxeter complex $\leadsto$ tessellation of $\mathbb{E}^n$ by intersecting E with interior (Tits cone)

<u>6.13 Lemma</u> $\mathcal{U}(W, X)$ is a connected top space

<u>Proof</u> $\mathcal{U}(W, X)$ has quotient topology

$\Rightarrow A \subseteq \mathcal{U}(W, X)$ open (resp. closed)

iff $A \cap wX$ open (resp. closed) in $wX \cong X$

6.13 Lemma $\mathcal{U}(W, X)$ connected

Proof $\mathcal{U}(W,X)$ has quotient topology

$\Rightarrow$ $A \subseteq \mathcal{U}(W,X)$ open (resp. closed)

$\Leftrightarrow$ $A \cap wX$ open (resp. closed) for each $w \in W$

Let $\emptyset \neq A \subseteq \mathcal{U}(W,X)$ and assume A is open & closed.

Then $A \cap wX$ open and closed, but X connected

$\Rightarrow$ $A \cap wX$ is $\emptyset$ or wX for each $w \in W$

$\Rightarrow$ $A = \bigcup_{v \in V} vX$ for some $\emptyset \neq V \subseteq W$

For $v \in V, s \in S$, $\exists\, x \in X_s$ ($X_s \neq \emptyset$)

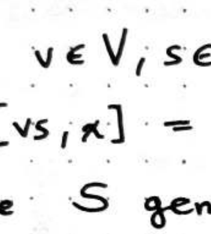
and $[vs, x] = [v, x]$ $\Rightarrow$ $v \in V$ implies $vs \in V$

Since S generates W, get $V = W$.

$\Rightarrow$ $A = \mathcal{U}(W, X)$. $\square$

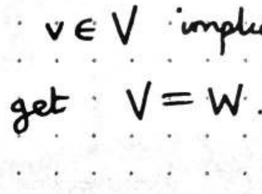
6.14 Definition $\mathcal{U}(W,X)$ is <u>locally finite</u> if for all $[w, x] \in \mathcal{U}(W,X)$ there is an open nghd of $[w, x]$ which meets only finitely many chambers

6.15 Example Ex 6.9 – Cayley graph as basic construction is



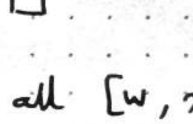
Def 6.11 Coxeter complex is not always

Ex 6.12



is

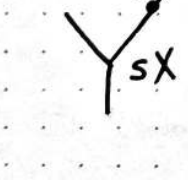
But Cox. complex for Γ



$\exists\, x \in X_s \cap X_u$ with $W_{S(x)} \cong D_\infty$

so infinitely many chambers glued at (e, x)

$\Rightarrow$ not locally finite



$\leftarrow$ infinitely many chambers

6.16 Lemma TFAE:

(i) $\mathcal{U}(W, X)$ is locally finite

(ii) $\forall x \in X$, $W_{S(x)}$ is finite

(iii) $\forall T \subseteq S$ s.t. W_T is infinite, $\bigcap_{t \in T} X_t = \emptyset$

Proof (i) ⇒ (iii) If (iii) fails let $x \in X$ have $W_{S(x)}$ infinite. Then an infinite number of chambers are identified at $[e, x]$. So (i) fails ✓

(ii) <=> (iii) immediate

(ii) => (i) For each $[w, x] \in \mathcal{U}(W, X)$ we can find an open ngbd $\mathcal{U}$ which only intersects chambers $w'X$ with $w^{-1}w' \in W_{S(x)}$ ⌜SINCE EACH S_x IS CLOSED !!⌟

Then $W_{S(x)}$ finite => $\mathcal{U}$ intersects a finite no. of chambers. So $\mathcal{U}(W, X)$ is locally finite. □

W acts on $\mathcal{U}(W, X)$ by homeomorphisms by acting on left of $W \times X$ i.e. $w' \cdot [w, x] = [w'w, x]$

Check this is well-defined, continuous

6.17 Lemma X is a strict fundamental domain (Def. 1.7) for $W \curvearrowright \mathcal{U}(W, X)$

Proof Example Sheet 3. □

So $\mathcal{U}(W, X)/W$ is X.

Moreover, we have $w' \cdot (wX) = w'wX$ gives a transitive free action on the set of chambers of $\mathcal{U}(W, X)$.

6.18 Lemma $\mathrm{Stab}_W([w, x]) = \{ w' \in W \mid w^{-1}w'w \in W_{S(x)} \}$

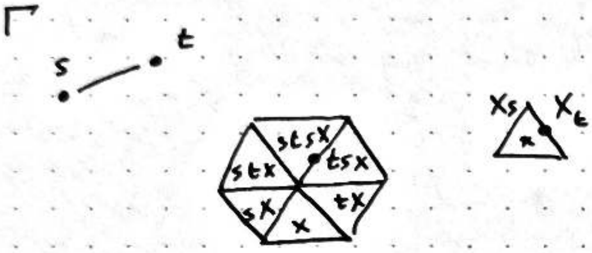
$$= w W_{S(x)} w^{-1}$$

6.19 Lemma $\mathcal{U}(W,X)$ is Hausdorff

Proof Let $y=[w,x]\in\mathcal{U}(W,X)$. Set $W_y=\mathrm{Stab}_W(y)$.
Then for $x\in U_x\subset X$ an open ngbd,

$$V_y = W_y\, w\Big(U_x \setminus \bigcup_{x\notin X_s} X_s\Big) \text{ is open in } \mathcal{U}(W,X)$$

If $y'=[w';x']$ is s.t. $y\neq y'$ then U_x and $U_{x'}$ can be chosen small enough s.t. $V_y\cap V_{y'}=\emptyset$. $\square$



$y=[ts,x]$

$W_y = tsW_t st$

$\quad = \{e,s\}$

6.20 Definition If G is a discrete group, Y a Hausdorff space, then an action by homeo^s $G \curvearrowright Y$ is *properly discontinuous* if:
(i) Y/G is Hausdorff
(ii) for all $y\in Y$, $G_y=\mathrm{Stab}_G(y)$ finite
(iii) $\forall y\in Y$ $\exists$ ngbd U_y of y s.t. $G_y U_y = U_y$ and if $g\notin G_y$ then $gU_y\cap U_y=\emptyset$

6.21 Lemma The W action on $\mathcal{U}(W,X)$ is properly discontinuous iff $W_{S(x)}$ is finite $\forall x\in X$

Proof "$\Rightarrow$" easy
"$\Leftarrow$" (i), (ii) follow from Lemma 6.18, (6.19) ⌜moreso that $\mathcal{U}(W,X)/W\equiv X$⌟
For (iii), wlog use $[e,x]$
Then $V_y = wW_{S(x)}\Big(X\setminus\bigcup_{x\notin X_s}X_s\Big)$ from 6.19 satisfies

$W_y\cdot V_y = V_y$ and $w\cdot V_y\cap V_y=\emptyset$ for $w\in W\setminus W_{S(x)}$. $\square$

§7. The Davis Complex

- Recall $\mathcal{S} = \{ T \subseteq S \mid W_T \text{ is spherical (finite)} \} \ni \phi$
  $W_\phi = \{id\}$

7.0 Remark In this section, abstract simp. complexes do not have ϕ simplex (-1 dim'l)

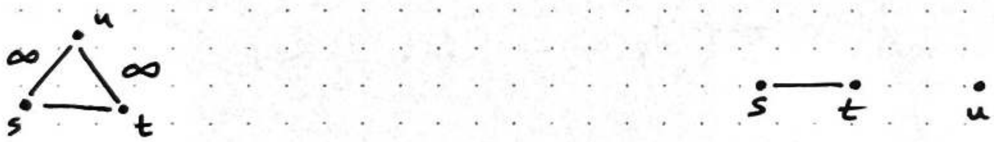
7.1 Definition The nerve of (W,S) denoted $L(W,S)$ is an abstract simplicial complex with vertex set S and simplex set $\mathcal{S} \setminus \{\phi\}$.

7.2 Examples

- Γ $\qquad\qquad$ $L(W(\Gamma), S(\Gamma))$

① 

② 

③ 

④ $S = S_1 \sqcup S_2$

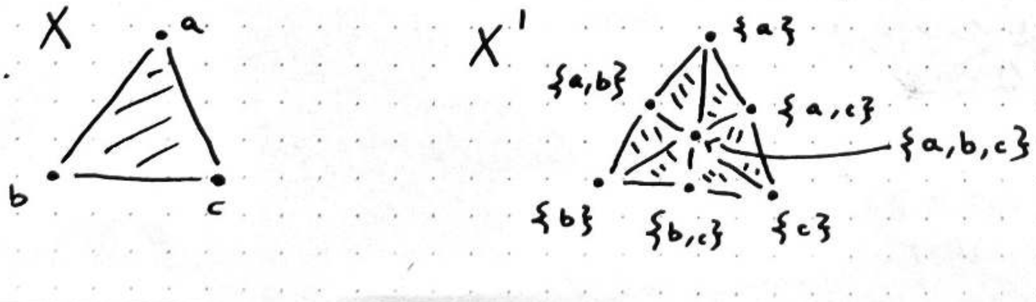
- with $m_{ij} = \infty$ $\qquad$ $L(W,S) = L(W_1,S_1) \sqcup L(W_2,S_2)$

  $\forall s_i \in S_1,\ s_j \in S_2$

  Let $W_i = \langle S_i \rangle$

7.3 Definition

Given an abstract simplicial complex X, its barycentric subdivision is an a.s.c. X' with vertex set X and ~~vertex~~ simplex set $X' = \{ \{\Delta_0, \dots, \Delta_p\} \mid \Delta_0 \subsetneq \cdots \subsetneq \Delta_p \}$

- e.g. 

7.4 Definition

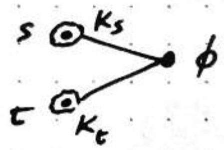
The chamber K of (W,S) is the cone on the barycentric subdivision L' of the nerve $L = L(W,S)$.
Let $K_s \subset K$ be the star in L' of the vertex s,

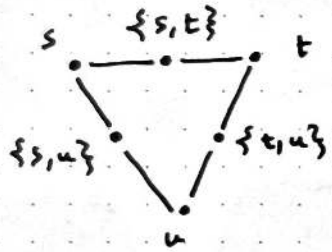
$$K_s = \bigcup_{\substack{\sigma_s \in L' \\ s \in \sigma_s}} \sigma_s$$

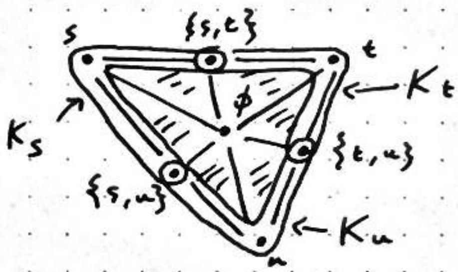
We label the cone point ϕ

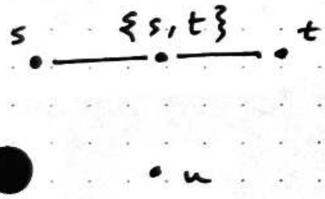
7.2 (cont.)

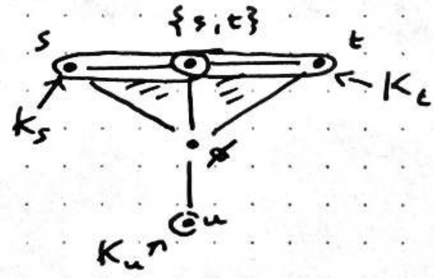
$L'(W,S)$ | K











7.6 Remarks

- K is connected and Hausdorff, so $\{K_s\}_{s \in S}$ is a mirror structure on K
- K is the simplicial complex $\text{Flag}(\mathcal{S})$ – $\mathcal{S}$ poset with inclusion
- ϕ is contained in no mirror
- In all examples above K is one-dim; this is not always the case.

7.8 Lemma A mirrored space X for (W,S) satisfies $W_{S(x)}$ finite $\forall x \in X$ iff $N(X) \subseteq L(W,S)$

Proof ($\Rightarrow$) Let $\{t_1, \dots, t_k\}$ be a simplex in $N(X)$.
Then $\exists\, x \in \bigcap_{1 \le i \le k} X_{t_i}$ $\Rightarrow$ $W_{\{t_1,\dots,t_k\}}$ is finite
$\Rightarrow \{t_1, \dots, t_k\} \in \mathcal{S}$
$\Rightarrow \{t_1, \dots, t_k\}$ simplex in $L(W,S)$

($\Leftarrow$) Let $x \in \bigcap_{t \in T} X_t$ $\Rightarrow$ T is a simplex in $N(X)$
$\Rightarrow$ T simplex in $L(W,S)$
$\Rightarrow$ $T \in \mathcal{S}$
$\Rightarrow$ W_T finite $\quad$ $\ulcorner \forall\, T$ s.t. $x \in \bigcap_{t \in T} X_t \lrcorner$
$\overset{!}{\Rightarrow}$ $W_{S(x)}$ finite $\square$

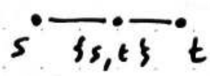
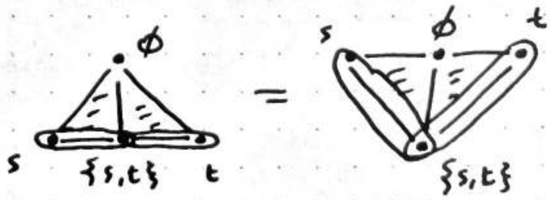
7.9 K satisfies $N(K) = L(W,S)$
So $W_{S(x)}$ is finite $\forall\, x \in K$. (ES3)

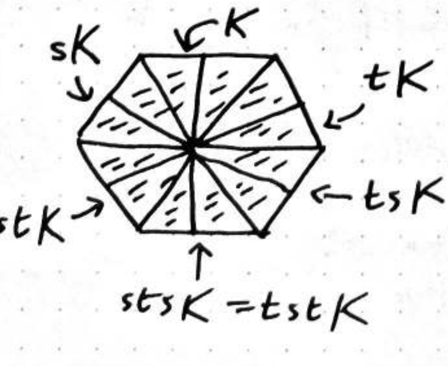
7.10 The Davis complex $\Sigma(W,S) = \mathcal{U}(W,K)$ is. (Def^n!)

7.11 Corollary $\Sigma(W,S)$ is connected, Hausdorff, locally finite. W acts properly discontinuously on Σ with quotient K. All point stabilisers are conjugates of spherical subgroups of W.

EG :)

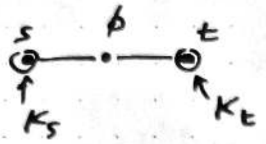
7.12 Examples D_6 

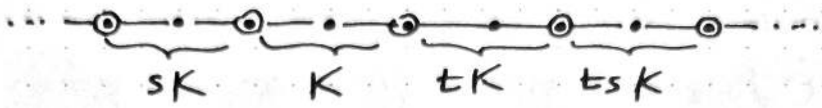
L  $\quad L'$  $\quad K$ 

$\Sigma(W,S)$ 

"our favorite hexagon, but now it's subdivided"

D_∞ $\overset{\infty}{\underset{s}{\bullet}\!\!-\!\!-\!\!\underset{t}{\bullet}}$ L : $\underset{s}{\bullet}$ $\underset{t}{\bullet}$ L' $\bullet s$ $\bullet t$

K 



$\Sigma(W,S)$ is barycentric subdivision of tessellation of the line $\mathbb{R}$ by intervals

If W is the $(3,3,3)$ triangle group, then $\Sigma(W,S)$ is the barycentric subdivision of the tiling of $\mathbb{E}^2$ by triangles (equilateral).

Remark 7.13 If W is a Euclidean or hyperbolic geom. reflection group, then $\Sigma(W,S)$ is the barycentric subdivision of the corresponding tessellation of $\mathbb{E}^n$ or $\mathbb{H}^n$ by P.

If W is spherical, $\Sigma(W,S)$ can be identified with the barycentric subdivision of the associated Coxeter polytope.

Recall $L(W,S) \rightsquigarrow L'$ barycentric subdivision of L

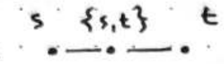
$\rightsquigarrow K$ cone on L'

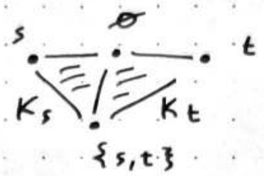
$K_s = \text{star}(\{s\})$ in $L' \subset K$

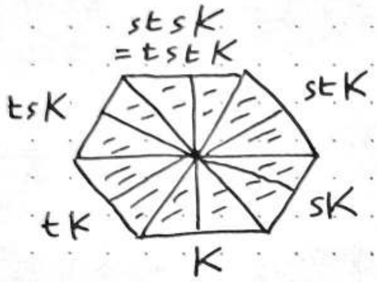
$\rightsquigarrow \Sigma(W,S) = \mathcal{U}(W,K)$

E.g. D_6 

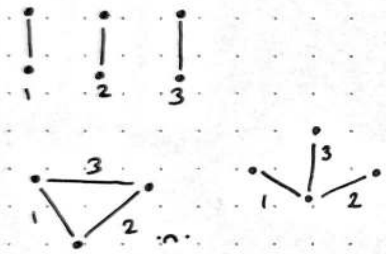
$L =$ 

$L' =$ 

$K =$ 

$\Sigma(W,K)$ 

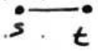
Thm 7.14 The Davis complex is contractible



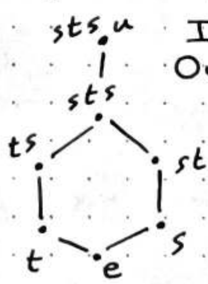
Def^n 7.15 For $w \in W$ define

$$\mathrm{In}(w) = \{ s \in S \mid \ell_S(ws) < \ell_S(w) \}$$

$$\mathrm{Out}(w) = \{ s \in S \mid \ell_S(ws) > \ell_S(w) \}$$

Ex $w = sts$ in 

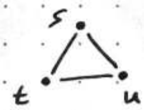
$\mathrm{In}(w) = \{s, t\}$



$w = t$

$\mathrm{In}(w) = \{t\}$

$\mathrm{Out}(w) = \{s\}$



$w = sts$

$\mathrm{In}(w) = \{s,t\}$

$\mathrm{Out}(w) = \{u\}$

Remark 7.16 $\ell_S(ws) = \ell_S(w) \pm 1$

So $S = \mathrm{In}(w) \sqcup \mathrm{Out}(w)$.

- If $\ell_S(ws) < \ell_S(w)$ then if $(s_1, \dots, s_k)$ reduced for w, by (E) on $(ws)^{-1} \Rightarrow w^{-1} = s s_k \cdots \hat{s}_i \cdots s_1 \Rightarrow w = s_1 \cdots \hat{s}_i \cdots s_k s$

So $\mathrm{In}(w) = \{ s \in S \mid \exists$ reduced word for w ending in $s \}$

Lemma 7.17 (W, S) Coxeter system

Suppose $\exists\, w_0 \in W$ s.t. $\forall s \in S$, $\ell_S(sw_0) < \ell_S(w_0)$.

Then W is finite. (wd)

Pf ES 3

Lemma 7.18 For $T \subset S$, there is a unique element w of minimal length in the coset wW_T, such that all $w' \in wW_T$ can be written in the form $w' = wa$, some $a \in W_T$, with

$$\ell_S(w') = \ell_S(w) + \ell_S(a).$$

Pf ES 3

Proposition 7.19 For all $w \in W$, $\mathrm{In}(w) \in \mathcal{S}$ i.e. $W_{\mathrm{In}(w)}$ is finite

Proof Consider the coset $wW_{\mathrm{In}(w)}$. Let u be the unique elt of minimal length in $wW_{\mathrm{In}(w)}$.

By L. 7.18, $w \in wW_{\mathrm{In}(w)}$ so $w = ua$, some $a \in W_{\mathrm{In}(w)}$,

$$\ell_S(w) = \ell_S(u) + \ell_S(a)$$

Now $\forall s \in \mathrm{In}(w)$, $\ell_S(ws) < \ell_S(w)$

~~$\forall a$~~ $\forall s \in \mathrm{In}(w)$, $as \in W_{\mathrm{In}(w)}$ so $ws = uas$

satisfies $\ell_S(ws) = \ell_S(u) + \ell_S(as)$

$\Rightarrow \forall s \in \mathrm{In}(w)$, $\ell_S(as) < \ell_S(a)$

Apply L 7.17 to a^{-1} $\Rightarrow$ $\mathrm{In}(w) \in \mathcal{S}$. $\square$

Lemma 7.20 K is contractible, and if $T \in \mathcal{S}$, then

$$K^T = \bigcup_{t \in T} K_t \text{ is contractible.}$$

Proof K is contractible since it is a cone – contract to cone point $\emptyset$.

For $\emptyset \neq T \in \mathcal{S}$, T spans a simplex in L, σ_T.

Let σ_T' be the barycentric subdivision of σ_T, in L'.

Then σ_T' is contractible, so it's enough to construct
$r : K^T \to \sigma_T'$ a deformation retract.

A vertex in $K^T \rightsquigarrow T' \in \mathcal{S}$ s.t. $t \in T'$ some $t \in T$
$\Rightarrow T \cap T' \neq \phi$

So map this vertex x to vertex of σ_T' corresponding to $T \cap T'$.

Extend to simplices by mapping σ_V with vertices $\{v_0, \dots, v_k\}$ to simplex $\{T \cap v_0, \dots, T \cap v_k\}$.

(Note $\sigma_V \in K^T \iff \exists t \in T$ s.t. $\forall i,\ t \in V_i$) $\square$

<u>Proof of Thm 7.14</u>

List the elements of W as $w_1, w_2, \dots$ s.t. $\ell_S(w_n) \leq \ell_S(w_{n+1})$ $\forall n \geqslant 1$.

If W finite, repeat last element to get infinite list.

Let $U_n = \{w_1, \dots, w_n\} \subseteq W$, so $W = \bigcup_{n \geqslant 1} U_n$.

Let $P_n = \bigcup_{w \in U_n} wK = \bigcup_{i=1}^{n} w_i K \subseteq \Sigma(W, S)$.

So $P_i \subseteq P_{i+1}$ and $\Sigma(W, S) = \bigcup_{i \geqslant 1} P_i$.

$P_1 = K$ which is contractible

$P_n = P_{n-1} \cup w_n K$

glue along mirrors

Which mirrors do we glue along? $\{K_s \mid \ell_S(w_n s) < \ell_S(w_n)\}$
$= \{K_s \mid s \in In(w_n)\}$

so we glue along $K^{In(w_n)}$.

By P7.19, $In(w_n) \in \mathcal{S}$ so by L7.20, $K^{In(w_n)}$ is contractible.

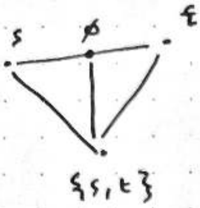
We have $P_1 \simeq *$ and $P_{n-1} \rightsquigarrow P_n$ we glue on $w_n K$ (contractible) along $K^{In(w)}$ (contractible)

// not convinced but alternative argument not too hard to do

$\Rightarrow$ Each stage P_n contractible

$\overset{!}{\Rightarrow} \Sigma(W, S)$ is contractible. $\square$

Ex

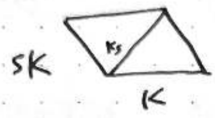
$K =$ 

D_6 list elements

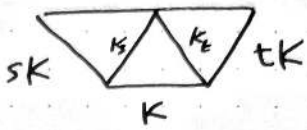
e, s, t, st, ts, sts

P_1

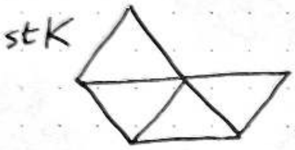
P_2 $In(s) = \{s\}$



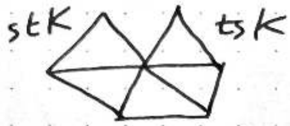
P_3 $In(t) = \{t\}$



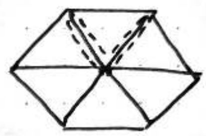
P_4 $In(st) = \{t\}$



P_5 $In(ts) = \{s\}$



P_6 $In(sts) = \{s, t\}$



Recall : 3.1 Theorem Let W be a group gen. by a finite set of distinct involutions S. Then TFAE

(1) (W,S) is a Coxeter system

(2) Let $X = Cay_S(W)$, $R = \{ wsw^{-1} \mid w \in W, s \in S \}$

Then (X,R) is a reflection system

(3) (W,S) satisfies the deletion condition

(4) (W,S) satisfies the exchange condition

3.4 Lemma W, S, R as above, $(s_1, \dots, s_k)$ word in S with associated ~~element~~ reflection ~~system~~ sequence $(r_1, \dots, r_k)$ s.t. $r_i = r_j$ for some $1 \le i < j \le k$. Then in W

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

3.5 Lemma W, S, R as above. Then for each $r \in R$, $Cay_S(W) \setminus H_r$ has at most two connected components

Let $(s_1, \dots, s_k) = \underline{s}$ word in S and $n(r, \underline{s})$ the number of times the associated path crosses the wall H_r in $Cay_S(W)$.

3.6 Lemma (i) For any word $(s_1, \dots, s_k) = \underline{s}$ with $w = s_1 \cdots s_k$, for any $r \in R$, $(-1)^{n(r, \underline{s})}$ depends only on w. ⌜in a Coxeter group!⌟

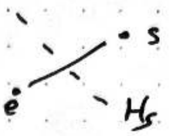
3.10 Thm (Tits) W group generated by a set of distinct involutions and (W,S) satisfies (E). Then:

(1) A word $(s_1, \dots, s_k)$ is reduced iff it cannot be shortened by a sequence of
- deleting (s,s) subwords, $s \in S$
- braid moves

Proof of Thm 3.1 "(1) ⇒ (2)" By Lemma 2.16, (X,R) is a pre-reflection system. So by Lemma 3.5 it suffices to prove each $Cay_S(W) \setminus H_r$ has two connected components.

Wlog show for H_s, $s \in S$ $(w \cdot H_s = H_{wsw^{-1}})$

Lemma 3.6 (i) implies that, since $n(s,s) = 1$, there is no path from e to s which avoids H_s.



This proves (1) ⇒ (2).

"(2) ⇒ (3)" We show that if $\underline{s}$ is a reduced word then the r_i in refl sequence are all distinct, and only if.

Then (D) follows from L 3.4.

(⇒) Actually trivial.

(⇐) Assume r_i are all distinct, let $w = s_1 \cdots s_k$.

Then if $R(e,w) = \{ r \in R \mid$ e and w lie in different components of $X \setminus H_r \}$

we have that for $r \in R(e,w)$ any word for w must cross H_r at least once.

Now $r_1, \dots, r_k \in R(e,w)$ ~~by Lemma 3.6 (i)~~ using that (X,R) is a refl. system (!)

Hence $\ell_S(w) \geq k$ and we are done.

"(4) ⇒ (1)" Let (W,S) satisfy (E).

WTS (W,S) is a Coxeter system. $S = \{s_i\}_{i \in I}$

Let m_{ij} be the order of $s_i s_j$ in W.

Let (W', S') be the Coxeter system with matrix (m_{ij}) and gen set $S' = \{s_i'\}_{i \in I}$.

$$\phi : W' \longrightarrow W$$
$$s_i' \longmapsto s_i$$

is a surjective group hom from universal property of group presentations.

RTP ϕ is injective

Let $w' \in \ker\phi$, $w' \neq e$ in W'.

Then w' rep by reduced word in W', $(s_1', \dots, s_k')$

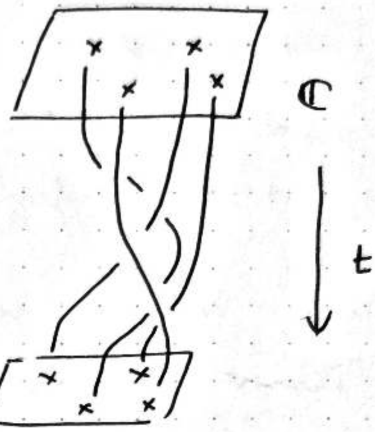
Since $\phi(w') = e$ in W, have $(s_1, \dots, s_k)$ not reduced.

By Thm 3.10, $(s_1, \dots, s_k)$ can be ~~reduced~~ shortened using only braid moves and deleting (s,s) subwords.

But these moves work equally well on $(s'_1, \dots, s'_k)$.
So we have $w' = e$ after all. $\square$

— — — NON - EXAMINABLE — — —

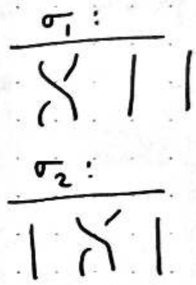
$\mathrm{Conf}_n(\mathbb{C}) = \mathbb{C}^n \setminus \{ z_i \neq z_j \ \forall i \neq j \}$ ≈



$\mathrm{Conf}_n(\mathbb{C}) / _{W(A_n)} = \mathrm{UConf}_n(\mathbb{C})$

$\pi_1(\mathrm{Conf}_n(\mathbb{C}) / W(A_n))$ braid group
$= A(A_n)$

$$A(A_n) \overset{!}{=} \left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \ |i-j| = 1 \end{array} \right\rangle$$



generate

σ_1 — σ_2 — ⋯ — σ_n

$W(\Gamma) \xrightarrow[\text{complement}]{\text{hyperplane}} \mathcal{M}(\Gamma) / _{W(\Gamma)}$

Then $\pi_1(\mathcal{M}(\Gamma) / _{W(\Gamma)}) \cong A(\Gamma)$ ← Artin group

For many Γ can show $\pi_i(\mathcal{M}(\Gamma) / W(\Gamma)) = 0$ for $i \geq 2$

Conjecture ($K(\pi, 1)$ conjecture)
For any Γ, $\mathcal{M}(\Gamma) / _{W(\Gamma)}$ is a $K(A(\Gamma), 1)$.