III Differential Geometry

~Jack Smith
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Nicolaescu, Lectures on the geometry of manifolds

Lee, Intro to smooth manifolds

Sontz, Principal bundles. The classical case

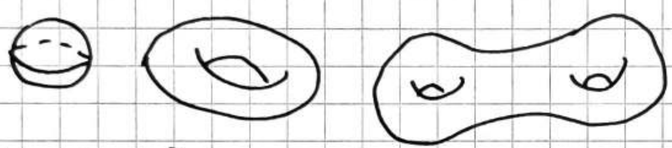
Nakahara, Geometry, topology and physics

What is differential geometry?

The study of smooth manifolds: spaces that locally look like $\mathbb{R}^n$ in a smooth way

Two ways to define manifolds:

- Embedded manifolds; smoothly embedded subspaces in $\mathbb{R}^N$

e.g.  $\subset \mathbb{R}^3$

⌜Extrinsic⌟

$\{x^2 = y^2 + 1\} \subset \mathbb{R}^2$

$SO(n) \subset \mathbb{R}^{n^2}$ $\quad M^T M = I$, $\det M = 1$

- Abstract manifolds

(reasonable) topological space s.t. about each point p there are local coordinates, s.t. coord transformations are smooth (i.e. C^∞)

⌜Intrinsic⌟

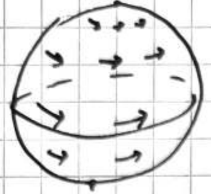
In fact, two definitions are equivalent.

Basic constructions with manifolds

- Tangent space: linear approximation to manifold at a point

- Smooth maps, their derivatives

· Vector fields and flows



· Submanifolds (Embedded manifold will be a submanifold of $\mathbb{R}^N$)

could give your manifold more structure, e.g.

a smooth group structure (a Lie group)

· tangent space at id becomes a Lie algebra

· there's a map Lie algebra $\to$ Lie group

the exponential map

Ex For $GL(n, \mathbb{R})$, tangent space at id

$Mat_{n\times n}(\mathbb{R})$ Lie algebra structure is

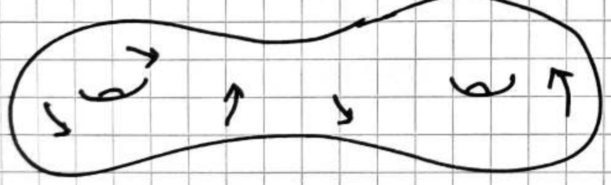
$$[A, B] = AB - BA$$

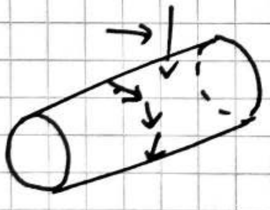
exponential map is $A \mapsto I + A + \frac{A^2}{2!} + \cdots$

How to differentiate a vector field?

vector field on $\mathbb{R}^N$ — easy

what about a vector field on an surface Σ in $\mathbb{R}^3$





(i) can't differentiate in directions out of Σ

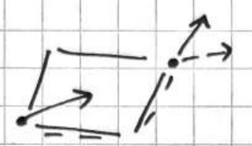
(ii) if you differentiate along directions in the surface, the answer may point out!

(iii) orthogonally project the answer back onto the surface

How does this depend on the embedding?

To answer this question, we'll use

- tensors and differential forms
- connections
- parallel transport — moving a vector along a path so it's derivative is zero
- curvature



A more abstract example

Spacetime = manifold X

quantum particle described by a <u>wavefunction</u> ψ $\left(\begin{smallmatrix}\text{about}\\ X \to \mathbb{C}\end{smallmatrix}\right)$

what matters is $|\psi|$ and relative phases of ψ_1 and ψ_2

What is ψ ? It's a section of a complex line bundle

Want to differentiate it. How? Answer: introduce a $U(1)$- connection!

It turns out that the connection is the EM potential and its curvature is EM field strength

<u>Course plan</u>

- Manifolds + smooth maps
- (Co)tangent bundles + tensors
- Differential forms
- Submanifolds + foliations
- Flows + Lie derivatives
- Lie groups + Lie algebras
- Principal bundles, connections + curvature
- Intro to Riemannian geometry

<u>Where next?</u>

- Geometric topology
  - low-dimensional topology (3 or 4)
  - knot theory
- Geometry of manifolds with extra structure
  - (Pseudo) Riemannian geometry

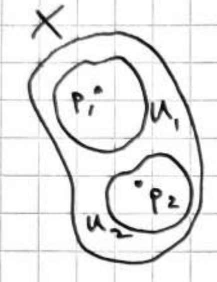
- Symplectic geometry + topology
- Complex manifolds, Kähler geometry

§1 Manifolds & Smooth Maps

Ex Classes
Weds 27^{th} Oct
10^{th} Nov
24^{th} Nov
1:30 – 3 pm

§1.1 Manifolds

Def 1.1 A topological n-manifold is a topological space X s.t. $\forall p \in X$,
$\exists$ open ngbd $U \ni p$ in X s.t.
$\exists V \subset \mathbb{R}^n$ open with $\varphi : U \xrightarrow{\sim} V$ homeo.
We also require X to be Hausdorff and 2^{nd}-countable.



Hausdorff: for distinct points $p_1, p_2 \in X$ there exist disjoint open sets U_1, U_2 s.t. $p_i \in U_i$

2^{nd}-countable: $\exists$ countable basis for the topology i.e. $\exists$ countable collection of open sets U_i s.t. any open set is some union of U_i

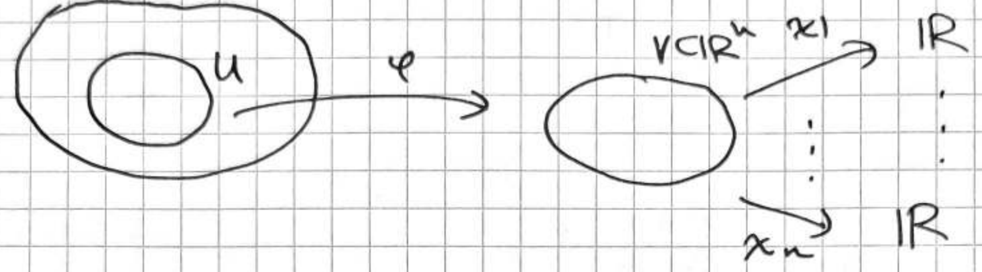
Example 1.2 $\mathbb{R}^n$ is a topological n-manifold
- For any $p \in \mathbb{R}^n$, take $U = V = \mathbb{R}^n$, $\varphi = id$
- $\mathbb{R}^n$ is Hausdorff, e.g. because it's metrisable
- A countable basis is given by the open balls of rational centre and rational radius

Remark 1.3 (i) "Hausdorff and 2^{nd}-countable" is important but not restrictive in practice.
(ii) For a space locally homeo to $\mathbb{R}^n$, it is equivalent to "X is metrisable and has countably many components"
(iii) The two conditions are inherited by subspaces

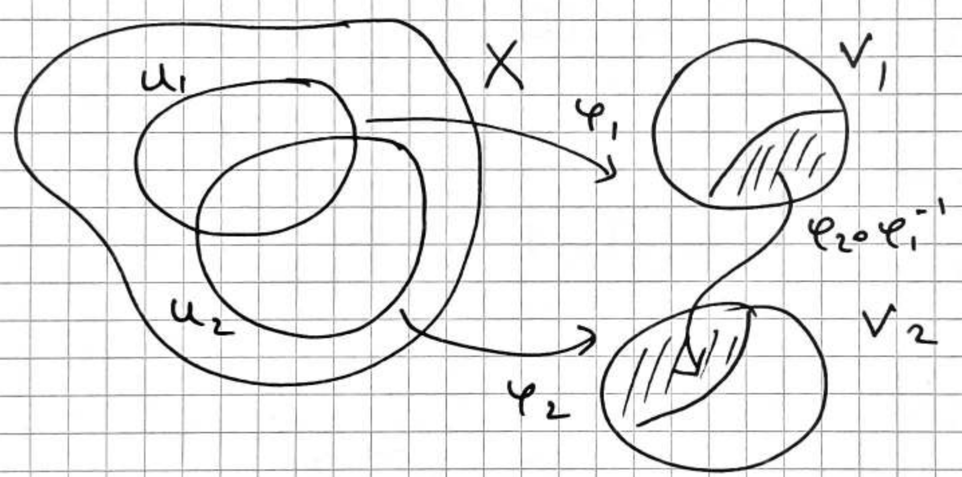
Example 1.4 If X is a top. n-manifold and $U \subset X$ is open, then U is a top. n-manifold.
Given $p \in U$, pick $\varphi : \underset{\subset X}{W} \xrightarrow{\sim} \underset{\subset \mathbb{R}^n}{V}$.
and restrict to $\varphi|_{U \cap W} : U \cap W \xrightarrow{\sim} \varphi(U \cap W)$.

Terminology • φ is a chart about p

- U is a coordinate patch
- If $x_1, \dots, x_n$ are the standard coords on $\mathbb{R}^n$, then $x_1 \circ \varphi, \dots, x_n \circ \varphi$ are local coords on X



- The inverse of a chart is a parametrisation
- If $\varphi_1 : U_1 \xrightarrow{\sim} V_1$, $\varphi_2 : U_2 \xrightarrow{\sim} V_2$ are two charts, the coordinates are related by the transition function $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$



$\rightsquigarrow$ translate b/w coords $x_1, \dots, x_n$ and $y_1, \dots, y_n$

Definition 1.5 A map from an open subset of $\mathbb{R}^a$ to $\mathbb{R}^b$ is smooth if it has all partial derivatives of all orders.

—

Given $f : X \longrightarrow \mathbb{R}$.

Preliminary def^n : f is smooth if $f \circ \varphi^{-1}$ is smooth for all charts φ, i.e. $f(x_1, \dots, x_n)$ is smooth as a function of local coords.

Def 1.6 · An atlas on a topological n-manifold is a collection $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ of charts that cover X. ($\bigcup_\alpha U_\alpha = X$)

· An atlas is smooth if all of its transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth (as in Def 1.5)

· Given a smooth atlas $\mathbb{A}$, a function $f : X \to \mathbb{R}$ is smooth wrt $\mathbb{A}$ if for all $\alpha \in \mathcal{A}$, $f \circ \varphi_\alpha^{-1}$ is smooth.

Lemma 1.7 f is smooth wrt $\mathbb{A}$ iff for all $p \in X$ there is a chart φ_α about p s.t. $f \circ \varphi_\alpha^{-1}$ is smooth.

Pf Only if ✓

Conversely, take a chart $\varphi_\beta : U_\beta \to V_\beta$.

Want to show $f \circ \varphi_\beta^{-1}$ is smooth.

Know that for all $p \in U_\beta$, $\exists\, \varphi_\alpha$ s.t. $f \circ \varphi_\alpha^{-1}$ is smooth.

Then write, near $\varphi_\beta(p)$,

$$f \circ \varphi_\beta^{-1} = \underbrace{(f \circ \varphi_\alpha^{-1})}_{\text{smooth}} \circ \underbrace{(\varphi_\alpha \circ \varphi_\beta^{-1})}_{\text{smooth}}$$

□

Corollary 1.8 Given a smooth atlas, all local coordinate functions are smooth. □

Def 1.9 · Two smooth atlases $\mathbb{A}, \mathbb{B}$ are smoothly equivalent if $\mathbb{A} \cup \mathbb{B}$ is smooth.

· A smooth structure on X is an equivalence class of smooth atlases under $\sim$ above.

· A smooth n-manifold is a topological n-manifold equipped with a smooth structure.

We'll abbreviate to "n-manifold" or "manifold".

Lemma 1.10 If $\mathbb{A}$, $\mathbb{B}$ are smoothly equivalent atlases, then f is smooth wrt $\mathbb{A}$ iff it is so wrt $\mathbb{B}$

Pf Exercise. $\square$.

Def^n 1.11 If X is a smooth manifold then $f : X \to \mathbb{R}$ is smooth if it's smooth wrt some (equiv, all) atlas on X.

Example 1.12 • $\mathbb{R}^n$ is a smooth n-manifold, smooth atlas containing one chart, $id : \mathbb{R}^n \to \mathbb{R}^n$.

- Open subsets, as before.
- If X, Y are smooth m-manifold, n-manifold, then $X \times Y$ is a smooth $(m+n)$-manifold defined via product charts

Remark 1.13 (i) Being a topological n-manifold is a property of a top. space.

(ii) Being a smooth n-manifold is a property (being a top mfd admitting a smooth atlas) plus a choice of smooth structure.

(iii) For $n \leq 3$ every topological n-manifold admits a unique (up to ...) smooth structure

(iv) For $n \geq 4$, a top n-manifold may admit no smooth structure (e.g. the E_8 4-manifold) or multiple different (i.e. not ...) smooth structures. (e.g. exotic S^7, exotic $\mathbb{R}^4$) But these are hard results.

<u>Def^n 1.14</u> For a smooth n-manifold X, the integer n is the <u>dimension of X</u>, $\dim X$.

You're free to add charts (or equivalently, local coords) to your atlas, as long as they preserve smoothness.

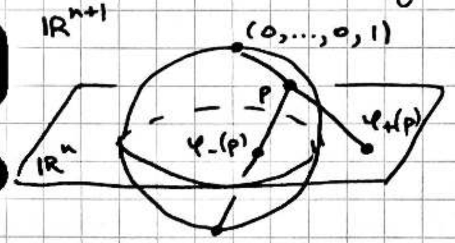
<u>Example 1.15</u> (i) Given a point $p \in X$, and an open nghbd W of p, we can always take / add a chart about p contained in W

(ii) Can choose / add local coords about p s.t. p corresponds to the origin in these coords: take any chart φ about p and consider

$$\varphi - \varphi(p)$$

$$\ulcorner \quad \begin{matrix} \varphi : & U & \xrightarrow{\sim} & V \\ & \cap \; \omega_p & & \cap \\ & X & & \mathbb{R}^n \end{matrix} \quad , \quad \varphi - \varphi(p) \text{ means do the subtraction on } \mathbb{R}^n \quad \lrcorner$$

<u>Example 1.16</u> The <u>n-sphere</u>, S^n, is the n-manifold whose underlying top space is $\{ y \in \mathbb{R}^{n+1} : \|y\|^2 = 1 \} \subset \mathbb{R}^{n+1}$ with the subspace topology. Smooth structure is defined by the following atlas. There are two charts $\varphi_\pm : U_\pm \xrightarrow{\sim} \mathbb{R}^n$ where $U_\pm = S^n \setminus \{ (0, \dots, 0, \pm 1) \}$ and $\varphi_\pm$ is stereographic projection:



$$\varphi_\pm(y_1, \dots, y_{n+1}) = \frac{1}{1 \mp y_{n+1}} (y_1, \dots, y_n)$$

Local coords $x^\pm$ satisfy $\quad x_i^\pm = \dfrac{y_i}{1 \mp y_{n+1}}$

The <u>height function</u> y_{n+1} is smooth since it's given by

$$y_{n+1} = \pm \frac{\|x^\pm\|^2 - 1}{\|x^\pm\|^2 + 1} \quad \text{on } U_\pm$$

§1.2 Manifolds from sets

Observe: if X is a smooth manifold, the charts "know the topology": a set $W \subset X$ is open iff $\varphi_\alpha(W)$ is open in $\mathbb{R}^n$ for all charts φ_α.

Suppose we're given a set X,

a collection $\{U_\alpha\}_{\alpha \in A}$ of sets covering X

for each α, an open set $V_\alpha \subset \mathbb{R}^n$ with

a bijection $\varphi_\alpha : U_\alpha \to V_\alpha$

Suppose that $\forall \alpha, \beta$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (or $\mathbb{R}^n$) and the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

<u>Def^n 1.17</u> (Non-standard) Call such data a <u>smooth pseudo-atlas</u> on X, and the φ_α <u>pseudo-charts</u>

Declare a set $W \subset X$ to be open iff $\forall \alpha$, the set $\varphi_\alpha(W \cap U_\alpha)$ is open in $\mathbb{R}^n$.

<u>Lemma 1.18</u> This defines a topology on X. □

<u>Proposition 1.19</u> Apart from the possible failure of "Hausdorff and second countable", the resulting space is a topological manifold, and the pseudo-atlas is a smooth atlas (hence it defines a smooth structure)

<u>Proof</u> We need to check that each U_α is open and that each φ_α is a homeomorphism, i.e. that $\forall\, W \subset U_\alpha$, W open in X if and only if $\varphi_\alpha(W)$ is open in V_α.

"$\Rightarrow$" Obvious

"$\Leftarrow$" Suppose $\varphi_\alpha(W)$ is open. Take any β. Want to show $\varphi_\beta(W \cap U_\beta)$ is open.

We have
$$\varphi_\beta(W \cap U_\beta) = (\varphi_\beta \circ \varphi_\alpha^{-1})\big(\varphi_\alpha(W) \cap \varphi_\alpha(U_\alpha \cap U_\beta)\big)$$
$$= \underset{\uparrow \text{cts}}{(\varphi_\alpha \circ \varphi_\beta^{-1})^{-1}}\big(\underset{\uparrow \text{open}}{\varphi_\alpha(W)} \cap \underset{\uparrow \text{open}}{\varphi_\alpha(U_\alpha \cap U_\beta)}\big) \quad □$$

Say two smooth pseudo atlases are equivalent if their union is a smooth pseudo atlas.

<u>Lemma 1.20</u> Equivalent smooth pseudo-atlases define the same manifold structure. $\square$ ← only really need to check topology unchanged

<u>Example 1.21</u> The n-dimensional <u>real-projective space</u> $\mathbb{R}\mathbb{P}^n$ is the space of lines (1-dim linear subspaces) in $\mathbb{R}^{n+1}$.

- Any non-zero $x \in \mathbb{R}^{n+1}$ defines a point $\langle x \rangle \in \mathbb{R}\mathbb{P}^n$
- All lines aries in this way
- Two points define the same line iff they differ by rescaling.

So we can label points of $\mathbb{R}\mathbb{P}^n$ by the ratios

$[x_0 : \cdots : x_n]$ called <u>homogereous coordinates</u>

(Explicitly, $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$

iff $\exists \lambda \in \mathbb{R}^\times$ s.t. $y = \lambda x$)

Define the following pseudo-charts.

For $i = 0, \dots, n$, let

$$U_i = \{ [x_0 : \cdots : x_n] : x_i \neq 0 \}$$

and define a bijection

$$\varphi_i : U_i \to \mathbb{R}^n$$

$$[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right)$$

This is a smooth pseudo-atlas and makes $\mathbb{R}\mathbb{P}^n$ into a smooth n-manifold (Example Sheet 1)

Can put $\mathbb{C}$ everywhere and get <u>complex projective space</u> $\mathbb{C}\mathbb{P}^n$, smooth $2n$-manifold.

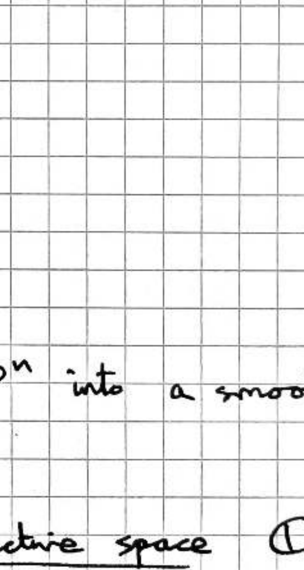
<u>Example 1.22</u> Take $X = \mathbb{R} \times \{1, 2\} / \sim$

where $(x,1) \sim (x,2)$ if $x < 0$.

Pseudo-atlases given by

$$\mathbb{R} \times \{i\} \xrightarrow{\sim} \mathbb{R}$$

But X is <u>not</u> Hausdorff.



Remark 1.23 Need not start with a set X. Could start with V_α in $\mathbb{R}^n$ and specify how to glue them together.

§1.3 Smooth maps

Fix two manifolds X, Y with atlases $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ on X, and $\{\psi_\beta : S_\beta \to T_\beta\}_{\beta \in B}$ on Y.

Def^n 1.24 A map $f : X \to Y$ is *smooth* if it is continuous and for all α, β,

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(f^{-1}(S_\beta) \cap U_\alpha) \to T_\beta \quad \text{is smooth,}$$

as a map between open subsets of $\mathbb{R}^{\dim X} \to \mathbb{R}^{\dim Y}$.

Remark 1.25 We ask that f be continuous so $f^{-1}(S_\beta)$, and hence $\varphi_\alpha(f^{-1}(S_\beta) \cap U_\alpha)$ is open and smoothness makes sense.

Example 1.26 1) id_X is smooth

2) any constant map is smooth

3) the projections onto $X \times Y \to X$,
$X \times Y \to Y$ are smooth

4) the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth

Lemma 1.27 We have the following basic propeties

i) A map $f : X \to \mathbb{R}$ is smooth iff it is so in the sense of §1.1

ii) A map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ is smooth iff it is so in the multivariable calculus sense

iii) Smoothness is local in the source: it's enough to check it locally near each $p \in X$

iv) Composition of smooth maps is smooth $\square$

Example 1.28 Viewing $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2(n+1)}$, can think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$.

Any point $x \in S^{2n+1}$ then defines a point $\mathbb{C}x \in \mathbb{CP}^n$

This gives a map $H : S^{2n+1} \to \mathbb{CP}^n$ called the <u>Hopf map</u>. This is smooth (Ex Sheet 1)

Def^n 1.29: A *diffeomorphism* $X \to Y$ is a smooth map with a smooth two-sided inverse.

Example: $\mathbb{CP}^1$ is diffeomorphic to S^2

So it makes sense to think of $\mathbb{CP}^1$ as a sphere, the *Riemann sphere*.

Lemma 1.31: If X, Y are diffeomorphic (non-empty) manifolds then $\dim X = \dim Y$

Pf Pick a point $p \in X$ and a diffeo $f : X \to Y$.

Pick charts $\varphi : U \to V$ about p

$\psi : S \to T$ about $f(p)$

Shrink charts so that $f(U) = S$

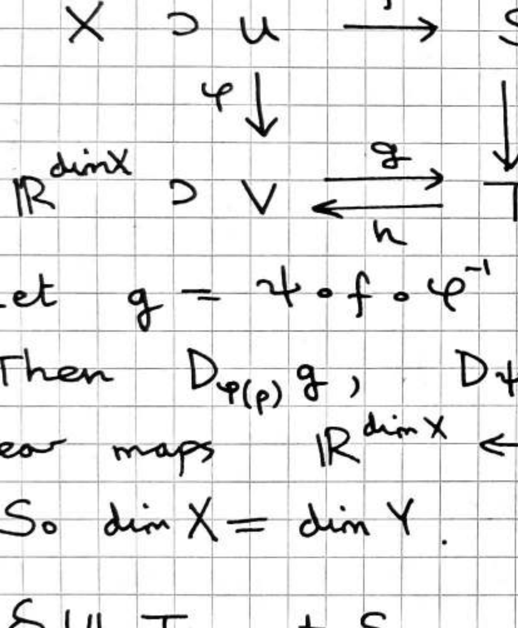
$$\begin{tikzcd} X \supset U \arrow[r, "f"] \arrow[d, "\varphi"'] & S \subset Y \arrow[d, "\psi"] \\ \mathbb{R}^{\dim X} \supset V \arrow[r, shift left, "g"] & T \subset \mathbb{R}^{\dim Y} \arrow[l, shift left, "h"] \end{tikzcd}$$

Let $g = \psi \circ f \circ \varphi^{-1}$, $h = \varphi \circ f^{-1} \circ \psi^{-1}$

Then $D_{\varphi(p)} g$, $D_{\psi(f(p))} h$ are mutually inverse linear maps $\mathbb{R}^{\dim X} \longleftrightarrow \mathbb{R}^{\dim Y}$.

So $\dim X = \dim Y$. $\square$

§1.4 Tangent Spaces



Fix an n-manifold X, $p \in X$.

Definition 1.32 A *curve based at p* is a smooth map

$$\gamma : I \to X$$

where I is an open ngbd of 0 in $\mathbb{R}$, s.t. $\gamma(0) = p$.

Say that curves γ_1, γ_2 <u>agree to first order</u> at p if there exists a chart φ about p s.t.

$$(\varphi \circ \gamma_1)^{\cdot}(0) = (\varphi \circ \gamma_2)^{\cdot}(0) \qquad (*)$$

as vectors in $T_{\varphi(p)} \mathbb{R}^n$ ← (just $\mathbb{R}^n$, but keep point in mind)

<u>Lemma 1.33</u> If $(*)$ holds for some chart φ then it holds for any chart about p.

<u>Pf</u> Given a chart φ about p, write π_p^{φ} for the map

$$\{\text{curves based at } p\} \longrightarrow T_{\varphi(p)}\mathbb{R}^n$$
$$\gamma \longmapsto (\varphi \circ \gamma)^{\cdot}(0).$$

Now suppose φ_1, φ_2 are different charts about p. Then by the chain rule

$$\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$$

where $A = D(\varphi_2 \circ \varphi_1^{-1})_{\varphi_1(p)}$

Note A is an isomorphism.

So for curves γ_1, γ_2, we have

$$\pi_p^{\varphi_1}(\gamma_1) = \pi_p^{\varphi_1}(\gamma_2) \quad \text{iff} \quad \pi_p^{\varphi_2}(\gamma_1) = \pi_p^{\varphi_2}(\gamma_2). \qquad \square$$

<u>Corollary 1.34</u> Agreement to first order is an equivalence relation on curves based at p.

<u>Definition 1.35</u> The tangent space to X at p, denoted T_pX is $\{\text{curves based at } p\}/\sim$ where $\sim$ identifies curves agreeing to first order

We'll write $[\gamma]$ for the tangent vector represented by the curve γ.

<u>Proposition 1.36</u> T_pX naturally carries the structure of an n-dim real vector space

<u>Pf</u> For each chart φ about p, π_p^{φ} induces a map

$$T_pX \xrightarrow{\pi} \mathbb{R}^n = T_{\varphi(p)}\mathbb{R}^n$$

tautologically injective.

We claim it's surjective.
Thus π_p^φ identifies T_pX, with $T_{\varphi(p)}\mathbb{R}^n$ and the identifications for different φ differ by a linear isomorphism (the map A).
So the induced vector space structure on T_pX is well-defined.

It remains to prove π_p^φ is surjective.
Take $v \in \mathbb{R}^n$, and consider the curve γ_v

$$t \mapsto \varphi^{-1}(\varphi(p) + tv) \quad \text{defined on some nbhd of } 0$$

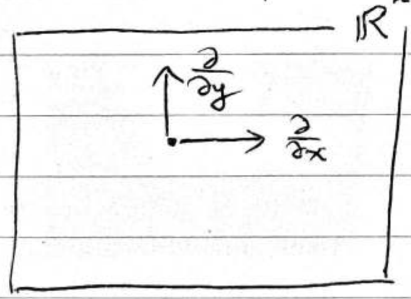
This curve satisfies $\pi_p^\varphi(\gamma_v) = v$. $\square$

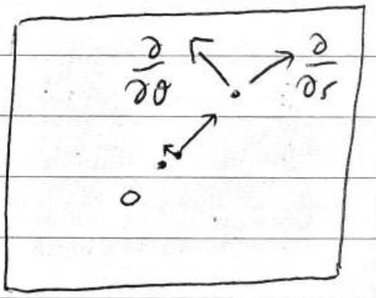
<u>Definition 1.37</u> If $x_1, \dots, x_n$ are the local coordinates defined by φ, and $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$, then write

$$\frac{\partial}{\partial x_i} \text{ or } \partial_{x_i} \text{ or } \partial_i$$

for the tangent vector $(\pi_p^\varphi)^{-1}(e_i)$.

<u>Intuitively</u>, ∂_{x_i} is the direction obtained by moving along the x_i axis, keeping other x_j constant and increasing x_i at unit speed.





⚠ Warning: $\frac{\partial}{\partial x_i}$ depends on the whole chart φ, not just x_i

e.g. if $y_1, \dots, y_n$ are local coords s.t. $y_i = x_i$
then it need not be true that $\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i}$

<u>Lemma 1.38</u> Given two sets of local coords about p, $\{x_i\}, \{y_i\}$, we have

$$\underbrace{\frac{\partial}{\partial y_i}}_{\text{tangent vectors}} = \sum_j \frac{\partial x_j}{\partial y_i} \underbrace{\frac{\partial}{\partial x_j}}$$

comes from transition function

$$\frac{\partial}{\partial y_i} = \sum_j D_i(x_j \circ \varphi_1 \circ \varphi_2^{-1}) \frac{\partial}{\partial x_j}$$

Last time:

$$\begin{tikzcd} & \{\text{curves based at } p\} \arrow[dl] \arrow[d, "\pi_p^{\varphi}"] \\ T_pX \arrow[r, leftrightarrow, "\sim"] & \mathbb{R}^n \end{tikzcd}$$

<u>Lemma 1.38</u> $\displaystyle \frac{\partial}{\partial y_i} = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$

<u>Proof</u> Let φ_1, φ_2 be the charts defining x, y. By definition,

$$\frac{\partial}{\partial y_i} = (\pi_p^{\varphi_2})^{-1}(e_i)$$

Let $A = D(\varphi_2 \circ \varphi_1^{-1})$, so $\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$.

Get $\partial_{y_i} = (\pi_p^{\varphi_1})^{-1}(A^{-1}e_i)$

Note $A^{-1} = D(\varphi_1 \circ \varphi_2^{-1})$

So $A^{-1}e_i = \sum_j \frac{\partial x_j}{\partial y_i} e_j$

$$\begin{aligned} \text{So } \partial_{y_i} &= (\pi_p^{\varphi_1})^{-1}\Big(\sum_j \frac{\partial x_j}{\partial y_i} e_j\Big) \\ &= \sum_j \frac{\partial x_j}{\partial y_i} (\pi_p^{\varphi_1})^{-1} e_j \\ &= \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}. \end{aligned}$$

□

<u>Remark 1.39</u> If $[\gamma] = \sum a_i \partial_{x_i}$, then $\underbrace{(\varphi \circ \gamma)^{\cdot}(0)}_{i^{th} \text{ component is } (x_i \circ \gamma)^{\cdot}(0)} = \pi_p^{\varphi}([\gamma]) = \sum a_i e_i$

i.e. $a_i = (x_i \circ \gamma)^{\cdot}(0)$.

<u>§1.5 Derivatives</u>

Fix manifolds X, Y and a smooth map $F : X \to Y$.

<u>Definition 1.40</u> The <u>derivative</u> of F at p, written D_pF, is the map $T_pX \to T_{F(p)}Y$

$$[\gamma] \mapsto [F \circ \gamma]$$

We sometimes write D_pF as F_*, <u>pushforward</u>.

<u>Lemma 1.41</u> The map D_pF is well-defined and linear

<u>Proof</u> Fix charts φ about p,
ψ about $F(p)$.

We have
$$\pi^{\psi}_{F(p)}(F\circ\gamma) = (\psi\circ F\circ\gamma)^{\cdot}(0)$$
$$= \left[(\psi\circ F\circ\varphi^{-1})\circ(\varphi\circ\gamma)\right]^{\cdot}(0)$$
$$= T\,\pi^{\varphi}_p(\gamma)$$
where $T = D(\psi\circ F\circ\varphi^{-1})_{\varphi(p)}$

So if $\cancel{\varphi}$ γ_1, γ_2 are curves based at p representing the same tangent vector, i.e. $[\gamma_1] = [\gamma_2]$, then $[F\circ\gamma_1] = [F\circ\gamma_2]$.

So D_pF is well-defined, and fits into the commutative diagram

$$\begin{tikzcd} T_pX \arrow[r, "D_pF"] \arrow[d, "\pi^{\varphi}_p"', "\wr"] & T_pY \arrow[d, "\wr"', "\pi^{\psi}_{F(p)}"] \\ \mathbb{R}^{\dim X} \arrow[r, "T"] & \mathbb{R}^{\dim Y} \end{tikzcd}$$

So $D_pF = (\pi^{\psi}_{F(p)})^{-1}\circ T\circ\pi^{\varphi}_p$ is linear. $\square$

If x, y are local coords associated to φ, ψ, then $\psi\circ F\circ\varphi^{-1}$ expresses F as giving the y's in terms of the x's.

So T is $\left(\dfrac{\partial y_i}{\partial x_j}\right)$.

Hence $$D_pF(\partial_{x_i}) = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

<u>Remark 1.42</u> (i) The new notion of derivative coincides with the usual one for maps $F: \mathbb{R}^m \to \mathbb{R}^n$.

(ii) If f is a function $X \to \mathbb{R}$, then
$$Df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$$

t coord on $\mathbb{R}$

(iii) For a curve γ based at p, write $[\gamma]$ as $D_0\gamma(\partial_t)$.

Prop 1.43 (chain rule)
For smooth maps $X \xrightarrow{F} Y \xrightarrow{G} Z$
we have $D_p(G \circ F) = D_{F(p)}G \circ D_p F$

Proof For $[\gamma]$ in T_pX, both sides give $[G \circ F \circ \gamma]$. □

§2. VECTOR BUNDLES & TENSORS

§2.1 The tangent bundle

Given local coords $x_1, \dots, x_n$ on an open set $U \subset X$, write $a_1, \dots, a_n$ for the components of a tangent vector w.r.t the ∂_{x_i}.
This gives us coordinates on

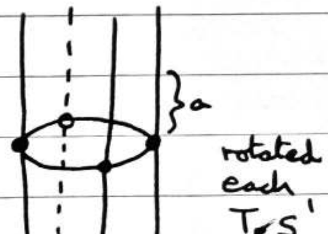
$$(x_1, \dots, x_n, a_1, \dots, a_n) : \coprod_{p \in U} T_pX \longrightarrow \mathbb{R}^{2n}$$

Doing this for all coordinate patches defines a smooth pseudo-atlas on $TX := \coprod_{p \in X} T_pX$.
This inherits
"Hausdorff & 2nd countable" from X.

Def^n 2.1: The tangent bundle of X is TX equipped with the above manifold structure.

Example 2.2: Let $X = S^1 = \{e^{i\theta}\} \subset \mathbb{C} = \mathbb{R}^2$
The vector ∂_θ makes sense everywhere and forms a basis in each tangent space.
Get a diffeo $TS^1 \to S^1 \times \mathbb{R}$

$$(\underset{\in S^1}{p}, \underset{\in T_pS^1}{a\tfrac{\partial}{\partial\theta}}) \mapsto (p, a)$$

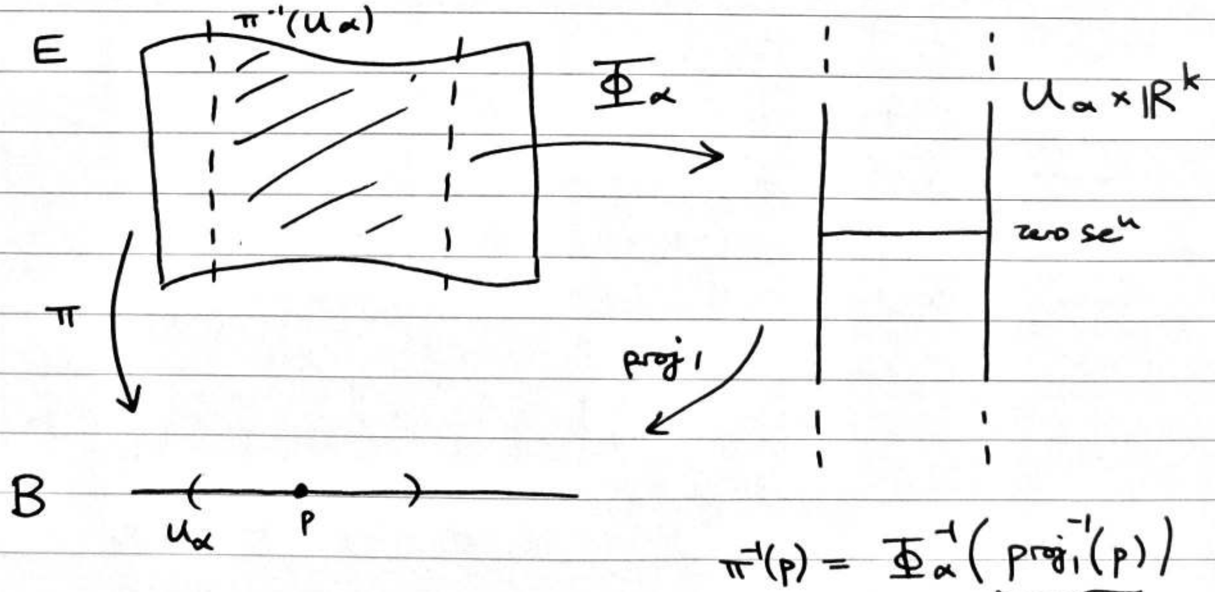


Def^n 2.3 A vector field on X is a smooth map
$$v : X \longrightarrow TX$$
such that $v(p) \in T_pX \quad \forall p$.

§2.2 Vector bundles

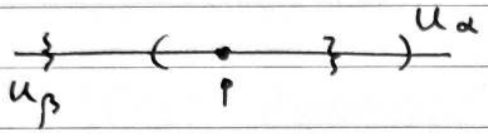
Def^n 2.4 A vector bundle of rank k over a manifold B is a manifold E equipped with

- a smooth surjection $\pi : E \to B$
- an open cover $\{U_\alpha\}$ of B and for each α a diffeo $\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ such that:
  - (a)
$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times \mathbb{R}^k \\ \pi \downarrow & & \downarrow \text{proj}_1 \\ U_\alpha & \xrightarrow{id} & U_\alpha \end{array}$$
commutes
  - (b) for all α, β the map
$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$
is of the form $(p, x) \mapsto (p, g_{\beta\alpha}(p)(x))$
for some smooth map
$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$$



$$\pi^{-1}(p) = \Phi_\alpha^{-1}\big(\underbrace{\text{proj}_1^{-1}(p)}_{\mathbb{R}^k}\big)$$

$$\begin{tikzcd} & \mathbb{R}^k \arrow[dd, bend left, "g_{\beta\alpha}(p) \in GL(k,\mathbb{R})"] \\ \pi^{-1}(p) \arrow[ur, "\mathrm{proj}_2 \circ \Phi_\alpha|"] \arrow[dr, "\mathrm{proj}_2 \circ \Phi_\beta|"'] & \\ & \mathbb{R}^k \end{tikzcd}$$



These $g_{\beta\alpha}(p)$ depend smoothly on $p \in U_\alpha \cap U_\beta$.

⌜Each fiber gets a natural vector space structure⌟

The manifold E is the <u>total space</u>

B is the <u>base</u>

map π is the <u>projection</u>

$\pi^{-1}(p)$ are the <u>fibres</u> (denoted E_p)

Φ_α are <u>local trivialisations</u>

$g_{\beta\alpha}$ are <u>transition functions</u>

<u>Recap</u> Vector bundle over B

$$E \overset{\pi}{\twoheadrightarrow} B \quad \text{smooth}$$

$$\{U_\alpha\}$$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha", "\sim"'] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{R}^k \arrow[d, "\mathrm{proj}_1"] \\ & U_\alpha \end{tikzcd} \quad \text{diffeo}$$

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (p, x) \mapsto (p, g_{\beta\alpha}(p)(x))$$

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R}) \subset \mathrm{Mat}_{k\times k}(\mathbb{R}) \equiv \mathbb{R}^{k^2}$$

<u>Remark 2.5</u> Each fiber E_p has the structure of a k-dimensional real vector space via

$$\Phi_\alpha : E_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^k \xrightarrow[\mathrm{proj}_2]{\sim} \mathbb{R}^k$$

⌜I'd say: give E_p explicit v. space structure, demand Φ_α respect⌟

<u>Remark 2.6</u> Really each trivialisation Φ_α is like a chart and the collection $\{\Phi_\alpha\}_\alpha$ is like an atlas.
There's an obvious notion of equivalence between two collections, and only the equivalence class is what we care about.

<u>Remark 2.7</u> Can similarly define complex vector bundles

<u>Example 2.8</u> Take $E = B \times \mathbb{R}^k$, $\pi = \mathrm{proj}_1$

$$\{U_\alpha\} = \{B\}$$

$$\Phi : E \to B \times \mathbb{R}^k \text{ the identity}$$

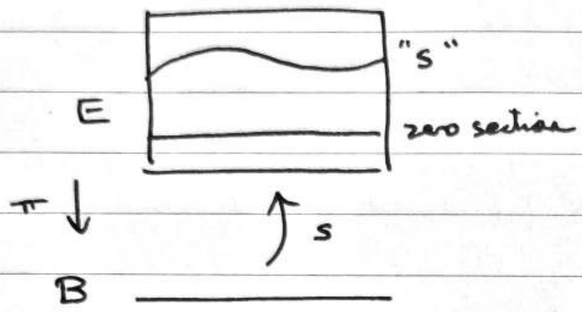
This is the <u>trivial</u> vector bundle of rank k over B, denoted by $\underline{\mathbb{R}^k}$. The map Φ is a <u>global</u> trivialisation.

<u>Example 2.9</u> TX is a rank n vector bundle over X ($n = \dim X$)

<u>Definition 2.10</u> A <u>section</u> of a vector bundle $\pi : E \to B$ is a smooth map $s : B \to E$ s.t. $\pi \circ s = \mathrm{id}_B$.

<u>Example 2.11</u> The <u>zero section</u> is given by
$$s(p) = (p, 0) \quad \forall p$$
<u>Example 2.12</u> A vector field is a section of TX

<u>Picture</u>



<u>Definition 2.13</u> Given a smooth map $F: B_1 \to B_2$ and vector bundles $\pi_i : E_i \to B_i$, a <u>morphism</u> of vector bundles $E_1 \to E_2$ <u>covering F</u> is a smooth map $G: E_1 \to E_2$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{G} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{F} & B_2 \end{array}$$
commutes,

and $\forall p$ the induced map $(E_1)_p \longrightarrow (E_2)_{F(p)}$ is linear.

An <u>isomorphism</u> between vector bundles over B is a morphism covering id_B with a two-sided inverse.
A bundle isomorphic to the trivial bundle is called <u>trivial</u>.

<u>Example 2.14</u> Last time we saw that TS^1 is trivial
$$TS^1 \longrightarrow S^1 \times \mathbb{R}$$
$$\left(p, a\frac{\partial}{\partial \theta}\right) \mapsto (p, a)$$

<u>Remark 2.15</u> A morphism $\underline{\mathbb{R}} \to E$ covering the identity
$$\begin{array}{ccc} \underline{\mathbb{R}} & \longrightarrow & E \\ & \searrow \; B \; \swarrow & \end{array}$$
is the same as a global section s.

$G \rightsquigarrow s(p) := G(p,1)$

$s \rightsquigarrow G(p,t) := t\,s(p)$

More generally, morphisms $\underline{\mathbb{R}^k} \to E$ correspond to k-tuples of sections of E.
This morphism is an isomorphism iff the k-tuple forms a basis in each fibre.

Definition 2.16 Given a rank K vector bundle E, a rank L subbundle is a submanifold(!) $F \subset E$ such that $\forall p \in B$ $\exists$ trivialisation $\Phi : \pi_E^{-1}(U) \to U \times \mathbb{R}^k$ under which $\pi_F^{-1}(U)$ gets sent to $U \times (\mathbb{R}^L \oplus 0)$

Can then define E/F and get morphisms

$F \to E \to E/F$.

§2.3 Constructing vector bundles by gluing

To define a vector bundle over B it's enough to give:

- a set E
- a map $\pi : E \to B$
- an open cover $\{U_\alpha\}$ of B
- for each α, a bijection $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$

  s.t.
  $$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r] \arrow[dr] & U_\alpha \times \mathbb{R}^k \arrow[d] \\ & U_\alpha \end{tikzcd}$$
  commutes

  and on overlaps $\Phi_\beta \circ \Phi_\alpha^{-1} : (p,x) \mapsto (p, g_{\beta\alpha}(p)(x))$

  for $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$ smooth

Then our pseudo-atlas construction makes E into a manifold (automatically Hausdorff and 2nd countable) and the Φ_α become trivialisations.

Example 2.17 Take $B = \mathbb{RP}^n = \{\text{lines in } \mathbb{R}^{n+1}\}$

Let $E = \{(p,x) : p \in \mathbb{RP}^n, x \in \mathbb{R}^{n+1} \text{ lies in line labelled by } p\}$

Define $\pi(p,x) = p$.
Open cover $\{U_i\}_{i=0}^n$ with $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$.
Define $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$

$$([x_0 : \cdots : x_n], \lambda(x_0, \ldots, x_n)) \mapsto$$

$$\mapsto ([x_0 : \cdots : x_n], \lambda x_i)$$

Then we have $\mathrm{proj}_1 \circ \Phi_i = \pi$,
and $\Phi_j \circ \Phi_i^{-1}([x_0 : \cdots : x_n], t)$

$$= \left([x_0 : \cdots : x_n], \frac{t x_j}{x_i}\right)$$

This is of the required form with

$$g_{ji} : U_i \cap U_j \longrightarrow GL(1, \mathbb{R}) = \mathbb{R}^*$$
$$[x_0 : \cdots : x_n] \longmapsto \frac{x_j}{x_i}$$

This is the <u>tautological bundle</u> over $\mathbb{RP}^n$

In fact we can drop the set E and just specify an open cover $\{U_\alpha\}$ of B and smooth maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$$

s.t. • $g_{\alpha\alpha}(p) = id_{\mathbb{R}^k}$ for all α, p
• $g_{\gamma\alpha} = g_{\gamma\beta}\, g_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ "cocycle condition"

We define

$$E = \coprod_\alpha U_\alpha \times \mathbb{R}^k / \sim$$

where

$$(p \in U_\alpha, x \in \mathbb{R}^k) \sim (p \in U_\beta, g_{\beta\alpha}(p)(x) \in \mathbb{R}^k)$$

The conditions above make $\sim$ an equivalence relation.

<u>Example 2.18</u> For any $r \in \mathbb{Z}$ we can define a line bundle (rank 1 v.b.) over $\mathbb{RP}^n$ trivialised over the U_i, with $g_{ji} = \left(\frac{x_j}{x_i}\right)^{-r}$

This is denoted $\mathcal{O}_{\mathbb{RP}^n}(r)$.
(The tautological bundle is $\mathcal{O}(-1)$.)

Lemma 2.19 If $\pi: E \to B$ is a rank k vector bundle trivialised over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then it is isomorphic to the output of the above construction. $\square$

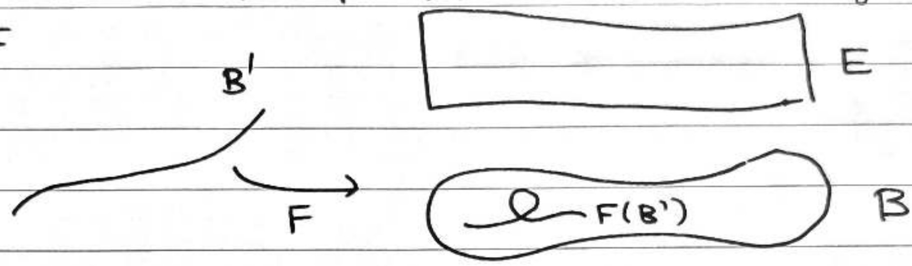
Corollary 2.20 Two bundles are isomorphic if they have trivialisations over an open cover $\{U_\alpha\}$ with the same transition functions.

Definition 2.21: Given a bundle $\pi : E \to B$ and a smooth map $F : B' \to B$, the <u>pullback bundle</u> F^*E has total space

$$\coprod_{p \in B'} E_{F(p)}$$

with the following bundle structure. Suppose E is trivialised over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$. Then F^*E is trivialised over $\{F^{-1}(U_\alpha)\}$ with transition functions

$g_{\beta\alpha} \circ F$



Definition 2.22: The <u>dual bundle</u> $E^\vee$ is the bundle over B with total space

$$\coprod_{p \in B} (E_p)^\vee,$$

trivialised over $\{U_\alpha\}$ with transition functions $(g_{\beta\alpha}^\vee)^{-1}$.
(c.f. dual representation)

Example 2.23: If E is locally trivialised by smooth sections $s_1, \dots, s_k$ over $U \subset B$, then the fiberwise dual basis defines smooth sections $\sigma_1, \dots, \sigma_k$ of $E^\vee$ which trivialise it.

should have specified trivialisations, not just $g_{\beta\alpha}$!!!

§2.4 The cotangent bundle

Fix an n-manifold X.

Definition 2.24: The <u>cotangent bundle</u> of X is the dual of the tangent bundle X. Denoted T^*X. The fibre over $p \in X$ is T_p^*X, the <u>cotangent space</u> of X at p.

Consider

$$\{\text{functions at } p\} = \{(U, f) : U \text{ open nghd of } p,\ f : U \to \mathbb{R} \text{ smooth}\}$$

Say f_1 and f_2 agree to 1st order at p if $D_p f_1 = D_p f_2$

Proposition 2.25: There's a canonical isomorphism

$$\{\text{functions at } p\}/\sim \longrightarrow T_p^* X$$

Proof: There's a pairing

$$\{\text{functions at } p\} \times \{\text{curves based at } p\} \longrightarrow \mathbb{R}$$

$$(f, \gamma) \longmapsto (f \circ \gamma)'(0)$$

This induces a map

$$\{\text{functions at } p\} \xrightarrow{\Theta} T_p^* X$$

$$f \longmapsto \left([\gamma] \longmapsto (f \circ \gamma)'(0) \right)$$

In coordinates, this map is

$$f \longmapsto \left(\sum a_i \partial_{x_i} \longmapsto \sum a_i \left.\frac{\partial f}{\partial x_i}\right|_{x(p)} \right) \qquad (*)$$

We want to show Θ is onto and that

$$\Theta(f_1) = \Theta(f_2) \iff f_1 \sim f_2$$

Surjective The coordinate functions $x_1, \ldots, x_n$ are sent to the dual basis to $\partial_{x_1}, \ldots, \partial_{x_n}$:

$$\Theta(x_i)\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

Last part Observe $\Theta(f_1) = \Theta(f_2)$

iff $\frac{\partial f_1}{\partial x_i} = \frac{\partial f_2}{\partial x_i}$ $\forall i$ (at p)

i.e. $D_p f_1 = D_p f_2$. $\square$

Notice that if

$f : \mathcal{U} \to \mathbb{R}$

is a smooth function, by the Proposition f defines an element of $T_p^* X$ for each $p \in \mathcal{U}$.

<u>Lemma 2.26</u>: This defines a (smooth) section of T^*X over $\mathcal{U}$. We denote it by df.

<u>Proof</u>: We saw in the proof of surjectivity above that

$dx_1, \dots, dx_n$

are fiberwise dual to

$\partial_{x_1}, \dots, \partial_{x_n}$.

Hence by the construction of T^*X, $dx_1, \dots, dx_n$ is a smooth basis of sections.

By $(*)$, $$df = \sum_i \underbrace{\frac{\partial f}{\partial x_i}}_{\text{smooth}} \underbrace{dx_i}_{\text{smooth}}.$$

So df is smooth. $\square$

<u>Lemma 2.27</u>: A section of T^*X is called a <u>1-form</u>. The 1-form df is the <u>differential</u> of f.

By construction,

$df(v)$ = derivative of f in the dir^n of v

<u>Remark 2.28</u>: Each dx_i depends only on x_i.

<u>Definition 2.29</u>: Given a smooth map $F : X \to Y$, the map $D_pF^\vee : T^*_{F(p)} Y \to T_p X$ is called <u>pullback</u> by F, denoted F^*.

<u>Lemma 2.30</u>: If $g : Y \to \mathbb{R}$ is smooth then

$$F^* dg = d(g \circ F)$$

<u>Proof</u>: Given a vector $[\gamma] \in T_p X$ we have

$$\begin{aligned} F^* dg([\gamma]) &= dg(D_pF([\gamma])) \\ &= dg([F \circ \gamma]) \end{aligned}$$

$$= (g \circ F \circ \gamma)'(0)$$
$$= ((g \circ F) \circ \gamma)'(0)$$
$$= d(g \circ F)([\gamma]) \ . \qquad \square$$

§ 2.5 Multilinear algebra

Fix U, V finite dimensional vector spaces over $\mathbb{K}$.

Definition 2.31: The tensor product

$$U \otimes V \ (\text{or } U \otimes_{\mathbb{K}} V) \text{ is}$$

a $\mathbb{K}$-vector space generated by symbols

$$u \otimes v \quad \text{for } u \in U, v \in V$$

modulo the relations

$$(\lambda_1 u_1 + \lambda_2 u_2) \otimes v = \lambda_1 (u_1 \otimes v) + \lambda_2 (u_2 \otimes v),$$
$$u \otimes (\mu_1 v_1 + \mu_2 v_2) = \mu_1 (u \otimes v_1) + \mu_2 (u \otimes v_2).$$

Lemma 2.32: If $e_1, \dots, e_m$ is a basis for U,
and $f_1, \dots, f_n$ is a basis for V,
then $\{e_i \otimes f_j\}_{i=1, j=1}^{m, n}$ form a basis for $U \otimes V$.
So $\dim(U \otimes V) = (\dim U)(\dim V)$. $\square$

Warning! A general element of $U \otimes V$ is of the form $\sum \lambda_i u_i \otimes v_i$, not necessarily $u \otimes v$.

Lemma 2.33: Tensor product is functorial:
if $\alpha : U \to U'$, $\beta : V \to V'$
are linear then there's an induced map

$$U \otimes V \to U' \otimes V' \quad \text{denoted } \alpha \otimes \beta$$

defined by

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v)$$

and extending linearly. $\square$

Lemma 2.34 (Universal property of $\otimes$)
A linear map $U \otimes V \to W$ is the same as a bilinear map $U \times V \to W$.

<u>Example 2.35</u> Fix U, V, W. Composition defines a bilinear map

$$\mathcal{L}(U,V) \times \mathcal{L}(V,W) \longrightarrow \mathcal{L}(U,W)$$

Get an induced linear map

$$\begin{aligned} \mathcal{L}(U,V) \otimes \mathcal{L}(V,W) &\longrightarrow \mathcal{L}(U,W) \\ \beta \otimes \alpha &\longmapsto \cancel{\beta\alpha}\ \alpha \circ \beta \end{aligned}$$

Now take $U = W = \mathbb{K}$. Get

$$V^\vee \otimes V \longrightarrow \mathbb{K}$$

This is called <u>contraction</u>. Other tensor factors "come along for the ride", e.g.

$$A \otimes V^\vee \otimes V \otimes B \longrightarrow A \otimes \mathbb{K} \otimes B = A \otimes B.$$

Last time $\{\text{functions at } p\}/\sim \;\cong\; T_p^* X \ni d_p f$

$\{\text{curves based at } p\}/\sim \;\cong\; T_p X \ni [\gamma]$

$f = \text{const.} + \underbrace{\text{linear piece}}_{\text{determines } df} + \text{higher order}$

Contraction $V^\vee \otimes V \longrightarrow \mathbb{K}$

induced by $V^\vee \times V \longrightarrow \mathbb{K}$

$(\theta, v) \longmapsto \theta(v)$

If $e_1, \dots, e_n$ is a basis for V,

$\varepsilon_1, \dots, \varepsilon_n$ is the dual basis,

then $\varepsilon_i \otimes e_j \mapsto \delta_{ij}$

$\therefore \sum_{ij} a_{ij}\, \varepsilon_i \otimes e_j \mapsto \sum_i a_{ii}$

$$V^\vee \otimes V \otimes W \longrightarrow \mathbb{K} \otimes W = W$$
$$1 \otimes w \leftarrow\!\shortmid\, w$$

Definition 2.36 The tensor algebra on V is

$$TV := \bigoplus_{r=0}^{\infty} V^{\otimes r}$$
$$= \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots$$

This is a $\mathbb{K}$-algebra with multiplication

$$V^{\otimes r_1} \times V^{\otimes r_2} \longrightarrow V^{\otimes (r_1 + r_2)}$$
$$(P, \quad Q) \longmapsto P \otimes Q$$

e.g. $(\lambda + v_1 \otimes v_3) \times v_2 \longmapsto \lambda v_2 + v_1 \otimes v_3 \otimes v_2$

Associative, unital, non-commutative.

The exterior algebra $\Lambda^\bullet V$ is the quotient of TV by the ideal (two-sided) generated by elements of the form $v \otimes v$. [The subspace of TV containing each $v \otimes v$ and closed under mult^n on both sides]

elt of V

e.g. $v_1 \otimes v_2 \otimes v_2 \otimes v_3 \longmapsto 0$ in the quotient.
This is an associative unital algebra.
Write $\Lambda^r V$ for the image of $V^{\otimes r}$ in $\Lambda^\bullet V$.
Called the r^{th} exterior power of V.
Represents "signed r-dimensional volumes inside V".

We write $\wedge$ for the product on $\Lambda^\bullet V$ coming from $\otimes$ on TV.

「well def」

e.g.
$$\begin{array}{ccc} v_1 \otimes v_2 & \longmapsto & v_1 \wedge v_2 \\ \cap & & \\ TV & \longrightarrow & \Lambda^\bullet V \end{array}$$

Note $v \wedge v = 0$ for all $v \in V$.

<u>Lemma 2.37</u> ΛV is graded commutative:
if $P \in \Lambda^r V$, $Q \in \Lambda^s V$
then $P \wedge Q = (-1)^{rs} Q \wedge P$.

<u>Proof</u> For $r = s = 1$, i.e. $v, w \in V$, we have
$$0 = (v+w) \wedge (v+w) = \underbrace{v \wedge v}_{\text{zero}} + v \wedge w + w \wedge v + \underbrace{w \wedge w}_{\text{zero}} \quad \checkmark$$

So $v \wedge w = - w \wedge v$.
The general case follows by associativity.
e.g. $(v_1 \wedge v_2) \wedge (v_3 \wedge v_4 \wedge v_5)$

pick up rs minus signs $\square$

<u>Definition 2.38</u> By a multi-index I we mean a tuple $(i_1, \dots, i_r)$ of elements in $\{1, \dots, n\}$ in strictly increasing order.
For a basis $e_1, \dots, e_n$ of V, write
e_I for $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r}$.
Similarly write $\varepsilon_I = \varepsilon_{i_1} \wedge \dots \wedge \varepsilon_{i_r}$ for dual basis $\varepsilon_1, \dots, \varepsilon_n$

Lemma 2.39 The elements e_I, where $|I| = r$, form a basis for $\Lambda^r V$. So
$$\dim \Lambda^r V = \binom{n}{r}. \qquad \square$$

Lemma 2.40 There's a natural isomorphism
$$(\Lambda^r V)^\vee = \Lambda^r (V^\vee)$$
induced by the pairing

very important, → here hide details

$$\Lambda^r(V^\vee) \times \Lambda^r V$$
$$(\theta_1 \wedge \cdots \wedge \theta_r), (v_1 \wedge \cdots \wedge v_r) \mapsto \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\, \theta_{\sigma(1)}(v_1) \cdots \theta_{\sigma(r)}(v_r) \qquad \square$$

Note e_I becomes dual to ε_I under this pairing.

Lemma 2.41 Λ^r is functorial, i.e. for any linear map $\alpha : V \to W$, get
$$\Lambda^r V \longrightarrow \Lambda^r W$$
$$v_1 \wedge \cdots \wedge v_r \mapsto \alpha(v_1) \wedge \cdots \wedge \alpha(v_r) \qquad \square$$

E.g. $\Lambda^n V$ is 1-dimensional and the induced map $\Lambda^n V \to \Lambda^n V$ is the scalar $\det(\alpha)$.

§2.6 Tensors and Forms

Just as for the dual bundle, you can upgrade functorial algebraic operations from vector spaces to vector bundles.

Example 2.42 Given vector bundles
$$E, F \longrightarrow \mathbb{B}, \quad \text{trivialised over } \{U_\alpha\}$$
with transition functions $g_{\beta\alpha}$, $h_{\beta\alpha}$ resp.
Then $E \oplus F$ is the bundle over $\mathbb{B}$ with fibre $E_p \oplus F_p$ over $p \in \mathbb{B}$, and transition functions
$$g_{\beta\alpha} \oplus h_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(\cancel{\dim}\ \mathrm{rk}E, \mathbb{R}) \times GL(\mathrm{rk}F, \mathbb{R})$$
$$\underset{\text{block diagonal}}{\cap} GL(\mathrm{rk}E + \mathrm{rk}F, \mathbb{R})$$

Can similarly define

$$E \otimes F, \ E^{\otimes r}, \ \Lambda^r E$$

<u>Example 2.43</u> Given a smooth map $F: X \to Y$, DF is naturally a section of $T^*X \otimes F^*TY$

⌈ For each $p \in X$, have

$$(T^*X \otimes F^*TY)_p = (T_p^*X) \otimes (T_{F(p)}Y)$$

$$\underset{\text{natural}}{\cong} \mathcal{L}(T_pX, T_pY)$$

↓ sheet 2

<u>Definition 2.44</u> A <u>tensor</u> (<u>field</u>) of <u>type</u> (p,q) is a section of $TX^{\otimes p} \otimes T^*X^{\otimes q}$

A <u>r-form</u> is a section of $\Lambda^r T^*X$

(Note this coincides with our earlier definition for a 1-form)

<u>Example 2.45</u> A tensor field of type $(0,0)$ is a section of <u>$\mathbb{R}$</u> i.e. a smooth function.
A tensor field of type $(1,0)$ is a vector field.
Type $(0,1) \to$ 1-form

In coordinates $x_1, \dots, x_n$, an r-form α looks like

$$\sum_I \alpha_I \underbrace{dx_{i_1} \wedge \dots \wedge dx_{i_r}}_{dx_I}$$

smooth functions

(sum over I s.t. $|I| = r$)

Can view this as a tensor of type $(0,r)$ via

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r} \mapsto \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, dx_{i_{\sigma(1)}} \otimes \cdots \otimes dx_{i_{\sigma(r)}} \quad (*)$$

<u>Example 2.46</u> On $\mathbb{R}^2$, a 2-form looks like

$$f\,dx \wedge dy \quad \text{for some smooth function } f$$

We can view this as

$$f(dx \otimes dy - dy \otimes dx)$$

<u>Warning!</u> Some authors divide by $r!$ in $(*)$ [physicists]

$\Lambda^r V$ generated by $v_1 \wedge \cdots \wedge v_r$

modulo • $v_1 \wedge \cdots \wedge (v_i + \lambda v_i') \wedge \cdots \wedge v_r$

$= v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r + \lambda (v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_r)$

• $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots v_r$

$= (-1)\, v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r$

Tensor of type (p,q)

A section of $(TX)^{\otimes p} \otimes (T^*X)^{\otimes q}$

Locally

$$\sum_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_q}} \underbrace{T^{i_1\cdots i_p}_{ j_1\cdots j_q}}_{\substack{\text{smooth} \\ \text{function(s)}}} \partial x_{i_1} \otimes \cdots \otimes \partial x_{i_p} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_q}$$

r - form

$$\sum_I \alpha_I \, dx_I = \sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

<u>E.g.</u> $\mathbb{R}^2$, coords (x,y)

Tensors of type $(0,2)$

$$f_{11} \underbrace{dx \otimes dx}_{\mapsto 0} + f_{12} \underbrace{dx \otimes dy}_{\mapsto dx \wedge dy} + f_{21} \underbrace{dy \otimes dx}_{\mapsto -dx \wedge dy} + f_{22} \underbrace{dy \otimes dy}_{\mapsto 0}$$

natural map

2 - form

$g \, dx \wedge dy$

so get

$(f_{12} - f_{21})\, dx \wedge dy$

To go from r-forms to $(0,r)$ tensors, could try

$dx \wedge dy \mapsto dx \otimes dy$ not defined

$\; \shortparallel$

but $-dy \wedge dx \mapsto -dy \otimes dx$

Need to antisymmetrise:

$$dx \wedge dy \longmapsto dx \otimes dy - dy \otimes dx$$

If $F : X \to Y$ is a diffeomorphism, then for any tensor T on X, there's an induced tensor $F_* T$ on Y of the same type — the <u>pushforward</u> by F.

$$(F_*T)_y = \text{image of } \underset{\substack{\Large\cap \\ (T_{F^{-1}(y)}X)^{\otimes p} \otimes (T^*_{F^{-1}(y)}X)^{\otimes q}}}{T_{F^{-1}(y)}} \text{ under } \underbrace{(D_{F^{-1}(y)}F)^{\otimes p}}_{\text{on } T \text{ factors}} \otimes \underbrace{((D_{F^{-1}(y)}F)^{-1})^{*\otimes q}}_{\text{on } T^* \text{ factors}}$$

Similarly, can turn a tensor T on Y into a tensor F^*T on X; the pullback by F.
Can do the same for forms.

If $F: X \to Y$ is an arbitrary smooth map, can no longer pushforward, and can only pull back $(0,q)$ Tensors or forms.
So given r-form α on Y,
$F^*\alpha$ is an r-form on X.

§ 2.7 Abstract index notation

(Boo!) (Poisons the mind)

A Tensor of type (p,q) is written with p up-indices and q down-indices.

Example 2.47 : T^a denotes a vector field
T_a denotes a 1-form = covector field

A tensor of type $(2,1)$ is written either of
$T^{ab}{}_c$, $T^a{}_b{}^c$, $T_a{}^{bc}$
depending on whethere we're thinking of it as a section of $TX \otimes TX \otimes T^*X$, $TX \otimes T^*X \otimes TX$, $T^*X \otimes TX \otimes TX$

Tensor product is expressed via concatenation
Example 2.48 $S_a T^b$ is a tensor of type $(1,1)$ given by $S \otimes T$
Contraction is expressed by a repeated index, 1 up, 1 down.
Example 2.49 $S_a T^a$ represents the 1-form S contracted with the vector field T.

Similarly
$$S_{ab}{}^{c}\, T_d{}^{b}$$
represents contracting the second T^*X factor of S with the TX factor of T.
The specific choice of labels for the indices doesn't matter, but for an equality to make sense you must have the same uncontracted indices on both sides.
Reordering indices corresponds to permuting tensor factors.
e.g. $\quad g_{ab} = g_{ba}$
Warning! This notation is independent of any choice of coordinates, $T^a{}_b$ does not represent components.
However, it's easy to turn these into coordinate expressions. E.g. write a vector field T as $T^i \frac{\partial}{\partial x^i}$ where T^i are the components of T wrt $\{\frac{\partial}{\partial x^i}\}$. (Note x_i has become x^i) Similarly $\alpha = \alpha_i dx^i$.

We implicitly sum over repeated indices (one up, one down)
The expressions for $\otimes$ and contraction in components look exactly like they did in abstract index notation.

§3 DIFFERENTIAL FORMS

§3.1 The Exterior Derivative

Suppose α is a 1-form on X.

In local coords, $\alpha = \sum_i \alpha_i dx^i$

Naïvely try to differentiate. Get

$$\sum_{i,j} \frac{\partial \alpha_i}{\partial x^j} dx^j \otimes dx^i$$

In different coordinates y_i we have

$$\alpha = \sum_i \alpha'_i dy^i$$

where
$$\alpha'_i = \alpha\left(\frac{\partial}{\partial y^i}\right) = \sum_j \frac{\partial x^j}{\partial y^i} \alpha_j$$

Then the naïve derivative is

$$\sum_{i,j} \frac{\partial \alpha_i'}{\partial y^j} dy^j \otimes dy^i = \sum_{i,j,k} \frac{\partial}{\partial y^j}\left(\frac{\partial x^k}{\partial y^i}\alpha_k\right) dy^j \otimes dy^i$$

$$= \sum_{i,j,k} \left(\frac{\partial \alpha_k}{\partial y^j} \cdot \frac{\partial x^k}{\partial y^i} dy^j \otimes dy^i + \alpha_k \cdot \frac{\partial^2 x^k}{\partial y^j \partial y^i} dy^j \otimes dy^i\right)$$

$$= \sum_{k,l} \frac{\partial \alpha_k}{\partial x^l} dx^l \otimes dx^k + \sum_{i,j,k} \alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i} dy^j \otimes dy^i$$

<u>Problem</u>: The answer depends on which local coordinates we use. <u>But</u> the "error" term is symmetric so we can kill it by replacing $\otimes$ with $\wedge$.

<u>Definition 3.1</u> The <u>exterior derivative</u> of α, denoted $d\alpha$, is defined in local coordinates by

$$d\alpha = \sum_{i,j} \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i$$

By the above calculation this is coordinate independant.

<u>Warning</u>! This doesn't work for vector fields.

<u>Definition 3.2</u> For an r-form $\alpha = \sum_I \alpha_I \, dx^I$, its exterior derivative is

$$d\alpha := \sum_{I,j} \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I \quad , \text{ an } (r+1)\text{-form}$$

Can check this is coordinate independent.

<u>Lemma 3.3</u> d is $\mathbb{R}$-linear, and on 0-forms (i.e. functions) it agrees with the differential. $\square$

<u>Proposition 3.4</u> d has the following properties:

(i) $d^2 = 0$ i.e. $d(d\alpha) = 0 \ \forall \alpha$

(ii) For α a p-form, β a q-form,
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad \text{(graded Leibniz)}$$

(iii) $d(F^*\alpha) = F^*(d\alpha)$ for any smooth map F

Recall: α r-form, $\alpha = \alpha_I dx^I$

$$d\alpha := \sum \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I$$
$$= \sum d\alpha_I \wedge dx^I$$

Prop 3.4: (i) $d^2 = 0$

(ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$

(iii) $F^*(d\alpha) = d(F^*\alpha)$

Proof: (i) Take $\alpha = \alpha_I dx^I$ locally.

Then $d^2\alpha = d\left(\frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I\right)$

$$= \frac{\partial^2 \alpha_I}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^I$$

$= 0$ since $\frac{\partial^2 \alpha_I}{\partial x^j \partial x^k}$ is symmetric in j,k but $dx^j \wedge dx^k$ is antisymmetric

⌜Aside: If a 1-form $\alpha = df$ then $d\alpha = 0$
So to find a 1-form that's not a differential of some function, it's enough to find one, say α, s.t. $d\alpha \neq 0$
e.g. $\alpha = x\,dy$ in $\mathbb{R}^2$ ⌟

(ii) Write $\alpha = \alpha_I dx^I$, $\beta = \beta_J dx^J$

Then $d(\alpha \wedge \beta) = d(\alpha_I \beta_J dx^I \wedge dx^J)$

$$= \frac{\partial \alpha_I}{\partial x^k} \beta_J dx^k \wedge dx^I \wedge dx^J$$
$$+ \alpha_I \frac{\partial \beta_J}{\partial x^k} dx^k \wedge dx^I \wedge dx^J$$
$$= \frac{\partial \alpha_I}{\partial x^k} dx^k \wedge dx^I \wedge \beta_J dx^J$$
$$+ (-1)^p \alpha_I dx^I \wedge \frac{\partial \beta_J}{\partial x^k} dx^k \wedge dx^J$$
$$= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta .$$

(iii) Suppose $F : X \longrightarrow Y$ is a smooth map, $\alpha \in \Omega^r(Y)$.

Let $\alpha = \alpha_I \, dy^I$.

Then
$$
\begin{aligned}
d(F^*\alpha) &= d\left(F^*(\alpha_I \, dy^{i_1} \wedge \dots \wedge dy^{i_r})\right) \\
&= d\left((\alpha_I \circ F)(F^* dy^{i_1}) \wedge \dots \wedge (F^* dy^{i_r})\right) \\
&= d\left((\alpha_I \circ F)\, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_r} \circ F)\right) && \text{by §2} \\
&= d(\alpha_I \circ F) \wedge d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_r} \circ F) && \text{using (i),(ii)} \\
&= F^*(d\alpha_I) \wedge F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_r}) && \text{by §2} \\
&= F^*(d\alpha_I \wedge dy^{i_1} \wedge \dots \wedge dy^{i_r}) \\
&= F^* d\alpha . \qquad \square
\end{aligned}
$$

In fact, those three properties, characterise d among all $\mathbb{R}$-linear maps $\Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ which send $f \mapsto df$ for $f \in \Omega^0(X)$.

An r-form α is <u>closed</u> if $d\alpha = 0$

<u>exact</u> if $\exists \beta$ s.t. $\alpha = d\beta$

By (i) above, exact forms are closed.

§3.2 de Rahm cohomology

Fix a manifold X, write $Z^r(X) = \{\text{closed } r\text{-forms}\}$
$B^r(X) = \{\text{exact } r\text{-forms}\}$

We saw that $B^r(X) \subset Z^r(X)$ (since $d^2=0$).

<u>Def^n 3.5</u> The r^{th} <u>de Rahm cohomology</u> group of X is

$$H^r_{dR}(X) = Z^r(X)/B^r(X)$$

(An $\mathbb{R}$-vector space)

Note $H^r_{dR}(X) = 0$ for $r > n$.

By convention / definition $H^r_{dR}(X) = 0$ for $r < 0$.

<u>Example 3.6</u> We have

$$H^0_{dR}(X) = Z^0(X)/\underbrace{B^0(X)}_{\text{zero}}$$

$$= \{\text{functions } f: X \to \mathbb{R} \text{ s.t. } df = 0\}$$
$$= \{\text{locally constant functions}\}$$
$$\cong \mathbb{R}^{\{\text{connected components of } X\}}$$

So $\dim H^0_{dR}(X) = \#$ connected components

<u>Example 3.7</u> We have $H^r_{dR}(pt) = 0$ unless $r=0$,
since $\dim pt = 0$.

By previous example, $H^0_{dR}(pt) \cong \mathbb{R}$.

For a closed form α, write $[\alpha]$ for its class in H^r_{dR}
— called the <u>cohomology class</u> of α.

Say α, β are <u>cohomologous</u> if $[\alpha] = [\beta]$

<u>Example 3.8</u> We know

$$H^r_{dR}(S^1) = \begin{cases} 0 & \text{if } r \neq 0,1, \\ \mathbb{R} & \text{if } r = 0, \\ ? & \text{if } r = 1. \end{cases}$$

We have $H^1 = Z^1/B^1$
$= \{1\text{-forms on } S^1\} / \text{differentials}$

A general 1-form α looks like $f(\theta)\,d\theta$,
whilst a general differential looks like $\frac{\partial g}{\partial \theta}(\theta)\,d\theta$

(f, g are 2π-periodic)

Note that $\int_0^{2\pi} \frac{\partial g}{\partial \theta} = 0$ by FTC.

This means that the map

$$\Omega^1(S^1) \longrightarrow \mathbb{R}$$
$$f(\theta)\,d\theta \longmapsto \int_0^{2\pi} f(\theta)\,d\theta$$

induces a well-defined map

$$I : H^1_{dR}(S^1) \longrightarrow \mathbb{R}.$$

Obviously linear and surjective (take $f = 1$).

<u>Claim</u>: I is an isomorphism

<u>Pf</u>: Just need to check injectivity.

Suppose $I(f\,d\theta) = 0$.

We want g s.t. $f = \frac{\partial g}{\partial \theta}$.

Define $g(\theta) = \int_0^\theta f(\tau)\,d\tau$.

This g is 2π-periodic since $I(f\,d\theta) = 0$. $\square$

<u>Lemma 3.9</u> (Cohomology is contravariant)

If $F : X \to Y$ is smooth,
then $F^* : \Omega^r(Y) \to \Omega^r(X)$
induces a map $F^* : H^r(Y) \longrightarrow H^r(X)$.

<u>Proof</u>: We need to show that if $\alpha \in Z^r(Y)$ then $F^*\alpha \in Z^r(X)$, and that if $\alpha' \in B^r(Y)$ then $F^*(\alpha') \in B^r(X)$.

Both follow from $F^* d = d F^*$:

if $d\alpha = 0$ then $dF^*\alpha = F^* d\alpha = 0$,

if $\alpha = d\beta$ then $F^*\alpha = F^* d\beta = dF^*\beta$. $\square$

Lemma 3.10 Wedge product of forms induces a product on cohomology groups. This is associative and graded commutative.

Proof: Suppose α, β are closed.

Then the Leibniz rule gives

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$
$$= 0$$

So $\alpha \wedge \beta$ is closed.

Now suppose α is closed, β is exact $\beta = d\gamma$.

Then $\alpha \wedge \beta = \alpha \wedge d\gamma$
$$= \pm\, d(\alpha \wedge \gamma)$$
is exact.

Similarly if α is exact, β is closed. $\square$

<u>Proposition 3.11</u> If $F_0, F_1 : X \to Y$ are (smoothly) homotopic, then they induce the same map

$$H^\bullet_{dR}(Y) \to H^\bullet_{dR}(X).$$

<u>Proof</u> Section 4. $\square$

Say F_0, F_1 are <u>homotopic</u> if there exists a homotopy between them, i.e. a smooth map $F : X \times [0,1] \to Y$ s.t. $F(-,0) = F_0$, $F(-,1) = F_1$.

<u>Corollary 3.12</u> If $F : X \to Y$ is a homotopy equivalence (i.e. $\exists\, G : Y \to X$ s.t. $G \circ F$ homotopic to id_X, $F \circ G$ " id_Y) then $F^* : H^\bullet_{dR}(Y) \to H^\bullet_{dR}(X)$ is an isomorphism.

<u>Proof</u> If such a G exists, then Prop 3.11 says that

$$G^* \circ F^* = (F \circ G)^* = (id_Y)^* = id_{H^\bullet(Y)}$$

$$F^* \circ G^* = (G \circ F)^* = (id_X)^* = id_{H^\bullet(X)}$$

so F^* is an isomorphism with inverse G^*. $\square$

<u>Example 3.13</u> (Poincaré Lemma)

For all n, $H^\bullet_{dR}(\mathbb{R}^n) \cong H^\bullet_{dR}(pt)$

§ 3.3 Integration

Want to define $\int_X \omega$ for X an n-manifold, ω a compactly supported n-form on X.

We need two technical ingredients: orientations and partitions of unity.

<u>Orientations</u> $\int_{\mathbb{R}} f\,dx$ could mean $\int_{-\infty}^{\infty}$ or $\int_{\infty}^{-\infty}$

Need to specify which one we mean. Need an <u>orientation</u> on X.

Definition 3.14 An orientation of an n-dimensional real vector space V is a non-zero element of $\Lambda^n V$, modulo positive rescalings. An ordered basis $e_1, \ldots, e_n$ induces an orientation $[e_1 \wedge \cdots \wedge e_n]$.

An orientation of $E \to B$ is a nowhere-zero section of $\Lambda^{top} E$ modulo rescaling by positive smooth functions.

Say E is orientalible if it admits an orientation (equivalent to $\Lambda^{top} E$ being trivial), and it's oriented if it's equipped with a choice of orientation.

Example 3.15 Any trivial bundle is orientable. But $\mathcal{O}_{\mathbb{RP}^1}(-1)$ is not orientable (sheet 2).

Definition 3.16 A manifold X is orientable / oriented if its tangent bundle TX is orientable / oriented.

Example 3.16 S^n is orientable is $\forall n$ (it's the boundary of the $(n+1)$-ball)

$\mathbb{RP}^n$ is not always orientable (sheet 2)

Sending a basis for V to its dual basis induces a map $V \longrightarrow V^\vee$. This induces a map $\Lambda^n V \longrightarrow \Lambda^n V^\vee$ which becomes canonical after quotienting by positive rescalings. So orientations of V are equivalent to orientations of $V^\vee$.

Definition 3.18 A nowhere-zero n-form on an n-manifold X is called a volume form. An orientation of X is equivalent to a volume form up to positive rescaling.

Partitions of unity These allow us to patch together local constructions.

<u>Definition 3.19</u> Given an open cover $\{\mathcal{U}_\alpha\}$ of a manifold X, a <u>partition of unity</u> subordinate to this cover is a collection of smooth functions $\rho_\alpha : X \longrightarrow [0,1]$ satisfying

- $\forall \alpha$, $\operatorname{supp} \rho_\alpha \subset \mathcal{U}_\alpha$
  $$\operatorname{supp} \rho_\alpha = \operatorname{closure}(\rho_\alpha^{-1}(\mathbb{R}^*))$$
- $\forall p \in X$, $\exists$ open ngbd $\mathcal{U} \ni p$ s.t. all but finitely many ρ_α vanish on $\mathcal{U}$ (local finiteness)
- $\sum \rho_\alpha = 1$

<u>Lemma 3.20</u> Given any open cover $\{\mathcal{U}_\alpha\}$ of X, there exists a partition of unity subordinate to it.

<u>Proof</u> See Lee (Theorem 2.23) $\square$

Now fix an oriented n-manifold X and a compactly supported n-form ω on X.

<u>Definition 3.21</u> The integral of ω over X, denoted $\int_X \omega$, is defined as follows.

- Cover X by coordinate patches $\{\mathcal{U}_\alpha\}$ s.t. WLOG the local coordinates are all positively oriented, i.e. $[\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}]$ is the orientation on X
- Pick a partition of unity $\{\rho_\alpha\}$ subordinate to this cover. Each $\rho_\alpha \omega$ has compact support contained in $\mathcal{U}_\alpha$. Write it in coordinates as
  $$(\rho_\alpha \omega)_{\underbrace{12\cdots n}_{\text{multi-index}}} \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$
- Define
  $$\int_X \omega = \sum_\alpha \int_{\mathbb{R}^n} (\rho_\alpha \omega)_{12\cdots n} \, dx^1 \cdots dx^n$$
  ↖ usual integral of a compactly supported function on $\mathbb{R}^n$

Lemma 3.22 This is independent of choices

Pf Suppose $\{V_\beta\}$ is another cover by coordinate patches with coordinates y_β, and that $\{\sigma_\beta\}$ is a partition of unity subordinate to this cover. ⌜and respecting orientation⌟

WTS

$$\sum_\alpha \int_{\mathbb{R}^n} (\rho_\alpha \omega)_{12\cdots n}\, dx^1 \cdots dx^n = \sum_\beta \int_{\mathbb{R}^n} (\sigma_\beta \omega)_{12\cdots n}\, dy^1 \cdots dy^n$$

Have

$$\text{LHS} = \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\rho_\alpha \sigma_\beta \omega)_{\underset{\uparrow\, x\text{-comps}}{12\cdots n}}\, dx^1 \cdots dx^n$$

⌜mega-cursed notation⌟

$$= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\rho_\alpha \sigma_\beta \omega)_{\underset{\uparrow\, y\text{-comps}}{12\cdots n}} \det \frac{\partial y^i_\beta}{\partial x^j_\alpha}\, dx^1 \cdots dx^n$$

⌜pullback of top forms⌟

$$\overset{!}{=} \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\rho_\alpha \sigma_\beta \omega)_{12\cdots n}\, dy^1 \cdots dy^n$$

by change of variables formula

⌜use that y^β_i, x^α_j have same orientation⌟

$$= \text{RHS} . \quad \square$$

Lemma 5.22 $\int_X \omega$ is indep of choices

Pf: On $U_\alpha \cap V_\beta$ we have

$$\rho_\alpha \sigma_\beta \omega = \sigma_\beta (\rho_\alpha \omega)_{12\cdots n} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

$$\shortparallel$$

$$\rho_\alpha (\sigma_\beta \omega)\, dy_\beta^1 \wedge \cdots \wedge dy_\beta^n$$

So $\rho_\alpha \cdot (\sigma_\beta \omega) = \det\left(\frac{\partial y_\beta^i}{\partial x_\alpha^j}\right) \cdot \sigma_\beta (\rho_\alpha \omega)_{12\cdots n}$

So

$$\sum_\alpha \int (\rho_\alpha \omega)_{12\cdots n} dx^1 \cdots dx^n$$

$$= \sum_{\alpha,\beta} \int \sigma_\beta \cdot (\rho_\alpha \omega)_{12\cdots n} dx^1 \cdots dx^n$$

⌜WTF is this notation⌟

$$= \sum_{\alpha,\beta} \int \rho_\alpha (\sigma_\beta \omega)_{12\cdots n} \det \frac{\partial y_\beta^i}{\partial x_\alpha^j} dx^1 \cdots dx^n$$

$$= \sum_{\alpha,\beta} \int \rho_\alpha \cdot (\sigma_\beta \omega)_{12\cdots n} dy^1 \cdots dy^n$$

$$= \sum_\beta \int (\sigma_\beta \omega)_{12\cdots n} dy^1 \cdots dy^n \quad \square$$

Remark 3.23 (i) All the sums are finite (all but finitely many terms are zero). For all p in $\text{supp}(\omega)$, $\exists$ open $U_p \ni p$ on which only finitely many ρ_α are non-zero.

U_p cover $\text{supp}(\omega)$

by compactness, have finite subcover

Hence only finitely many of the $\rho_\alpha \omega$ are zero.

Similarly for $\sigma_\beta \omega$.

(ii) We use the orientation hypothesis to ensure that all Jacobians are positive.

§3.4 Stokes's Theorem

The fundamental theorem of calculus says that for a smooth function f on $[a,b]$, we have

$$\int_{[a,b]} \frac{df}{dx}\, dx = f(b) - f(a)$$

Setting $X = [a,b]$ we can write this as

$$\int_X df = \int_{\partial X} f \qquad \text{(give } X \longrightarrow \text{orientation)}$$

<u>Definition 3.24</u>: A <u>(smooth) n-manifold with boundary</u> is defined in exactly the same way as an ordinary n-manifold except that codomains of charts are now open subsets of $\mathbb{R}^n$ <u>or</u> $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

[A function f on an open subset W of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ is smooth if there exists an open set W' in $\mathbb{R}^n$ containing W s.t. f extends to a smooth function on W'.]

Smooth maps between manifolds with boundary are defined in the obvious way. A point p in X a mfd with boundary lies in the <u>boundary of X</u>, denoted ∂X, if its image under some chart (equiv all charts) $\varphi : U \to V$, $U \ni p$, lies in $\{0\} \times \mathbb{R}^{n-1}$

The interior of X, denoted $\mathring{X}$, is $X \setminus \partial X$

<u>Example 3.25</u>: (i) An ordinary n-manifold X is an n-manifold-with-boundary, with $\partial X = \emptyset$.

(ii) The interval $[a,b]$ is a manifold-with-boundary.

(iii) The closed unit ball

$$D^n := \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$$

is an n-manifold with boundary.

This has $\mathring{D}^n$ = open unit ball

$$\partial D^n = S^{n-1}$$

(iv) If X is a manifold-with-boundary and Y is a manifold, then $X \times Y$ is a manifold-with-boundary.

It has boundary $\partial X \times Y$.

[If X, Y are both manifolds with-boundary, then $X \times Y$ need not be]

it has corners at $\partial X \times \partial Y$

<u>Proposition 3.26</u>: If X is an n-manifold-with-boundary, then $\mathring{X}$ is an ordinary n-manifold

∂X is an ordinary $(n-1)$-manifold

<u>Proof</u>: For $\mathring{X}$ it's immediate.

For ∂X, for each point $p \in \partial X$ and each chart $\varphi : U \to V$ about p, let

$$\partial U = U \cap \partial X = \varphi^{-1}\big(\underbrace{(\{0\} \times \mathbb{R}^{n-1}) \cap V}_{\partial V}\big)$$

Then ∂U is an open ngbd of p in ∂X and ∂V is an open set in $\{0\} \times \mathbb{R}^{n-1} = \mathbb{R}^{n-1}$

And $\varphi|_{\partial U} : \partial U \to \partial V$ is a chart on ∂X about p. $\square$

<u>Theorem 3.27</u> (Stokes's Theorem)

If X is an oriented n-manifold-with-boundary and ω is a compactly supported $(n-1)$-form on X then

$$\int_{\partial X} \omega := \int_{\partial X} \imath^* \omega = \int_X d\omega$$

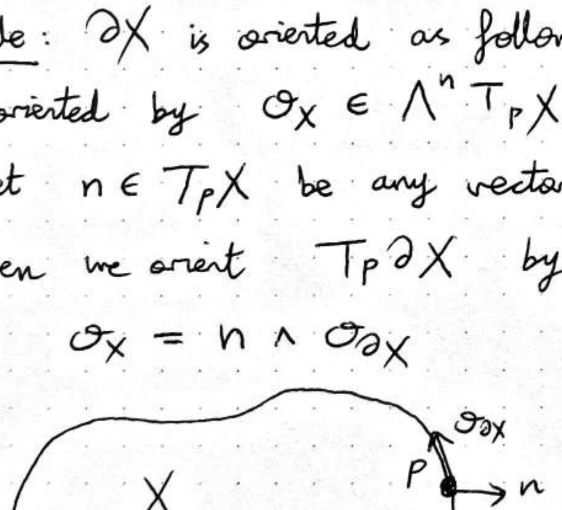
where $\imath : \partial X \to X$ is inclusion, and ∂X is given the induced orientation.

<u>Aside</u>: ∂X is oriented as follows. Suppose $p \in \partial X$ and T_pX is oriented by $\sigma_X \in \Lambda^n T_pX$.

Let $n \in T_pX$ be any vector in $T_pX \setminus T_p(\partial X)$ pointing outwards.

Then we orient $T_p \partial X$ by $\sigma_{\partial X}$ defined by

$$\sigma_X = n \wedge \sigma_{\partial X}$$



<u>Example 3.28</u> On $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$,
oriented by $\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}$,
the vector $-\partial_{x_1}$ is outward pointing,
so the induced orientation on $\{0\} \times \mathbb{R}^{n-1}$ is

$$\left[-\partial_{x_2} \wedge \cdots \wedge \partial x_n \right].$$

<u>Proof of Stokes</u>:

<u>Step 1</u>: Reduce to a coordinate patch

Cover X by coordinate patches $\{U_\alpha\}$ and take a partition of unity $\{\rho_\alpha\}$ subordinate to the cover.

Then $\int_X d\omega = \int_X d\left(\sum_\alpha \rho_\alpha \omega\right)$

$$= \sum_\alpha \int_X d(\rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega)$$

and $\int_{\partial X} \iota^* \omega = \int_{\partial X} \iota^* \left(\Sigma_\alpha \rho_\alpha \omega \right)$

$$= \sum_\alpha \int_{\partial X} \iota^*(\rho_\alpha \omega) = \sum_\alpha \int_{\partial U_\alpha} \iota^*(\rho_\alpha \omega)$$

STP $\int_{U_\alpha} d(\rho_\alpha \omega) = \int_{\partial U_\alpha} \iota^*(\rho_\alpha \omega)$

<u>Step 2</u>: Compute both sides

By Step 1, wlog U_α is some half-space $X = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

For a compactly supported $(n-1)$ form ω on this half-space, write

$$\omega = \Sigma_i \, \omega_i \, dx^1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^n$$

Then have

$$\int_{\partial X} \iota^* \omega = - \int_{\partial X} \omega_1 \, dx_2 \cdots dx_n \qquad \text{「}dx^1 \text{ pulls back to zero」}$$

And $\int_X d\omega = \int_X \sum_i \frac{\partial \omega_i}{\partial x^i} \cdot (-1)^{i-1} dx^1 \wedge \cdots \wedge dx^n$ 「put dx^i, move along $(i-1)$ places」

$$\ldots \int_X d\omega = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\geq 0}} \frac{\partial \omega_1}{\partial x^1} dx^1 \right) dx^2 \cdots dx^n$$

$$+ \sum_{i \geq 2} (-1)^{i-1} \int_{\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-2}} \left(\int_{\mathbb{R}} \left(\frac{\partial \omega_i}{\partial x^i} dx^i \right) \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

by FTC, we get

$$\int_{\mathbb{R}_{\geq 0}} \frac{\partial \omega_1}{\partial x^1} dx^1 = - \omega_1 \Big|_{x^1 = 0}$$

$$\int_{\mathbb{R}} \frac{\partial \omega_i}{\partial x^i} dx^i = 0$$

Both sides reduce to

$$- \int_{\{0\} \times \mathbb{R}^{n-1}} \omega_1 \, dx^2 \cdots dx^n$$

$\square$

Stokes : $\int_X d\omega = \int_{\partial X} \omega$

Rmk : On a 3-manifold (with a metric, orientation)

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla\times} \text{vector fields} \xrightarrow{\nabla\cdot} \text{functions}$$

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3$$

§3.5 Applications of Stokes'

Corollary 3.29 (Integration by parts):
Let X be an oriented n-manifold.
Let α, β be a $(p-1)$-form and and $(n-p)$-form on X, at least one of which is compactly supported.
Then
$$\int_X d\alpha \wedge \beta = \int_{\partial X} \alpha \wedge \beta + (-1)^p \int_X \alpha \wedge d\beta$$

Proof: By Stokes',
$$\int_X d(\alpha \wedge \beta) = \int_{\partial X} \alpha \wedge \beta$$

By Leibniz, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{p-1} \alpha \wedge d\beta$
Put these together to get the result. $\square$

Proposition 3.30 : If X is a compact oriented n-manifold, i.e. 'closed'
then $\int_X : \Omega^n(X) \longrightarrow \mathbb{R}$
induces a map $H^n_{dR}(X) \longrightarrow \mathbb{R}$

Pf : Suppose α, β are n-forms on X such that $\alpha = \beta + d\gamma$ for some $(n-1)$-form γ. Then
$$\int_X \alpha = \int_X \beta + \underbrace{\int_X d\gamma}_{= \int_{\partial X} \gamma = 0}$$
$\square$

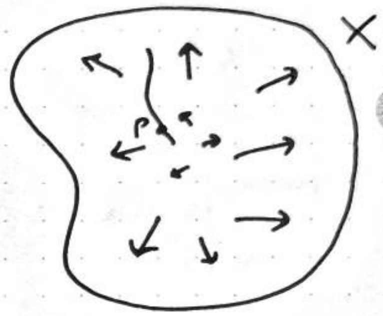
Corollary 3.31 If X is a compact oriented n-manifold then $H^n_{dR}(X) \neq 0$.

Pf: Let ω be a volume form on X. This is automatically closed so defines a class $[\omega] \in H^n_{dR}(X)$.
And we have $\int_X \omega > 0$. So $[\omega] \neq 0$. $\square$

§4. Flows & Lie Derivatives

§4.1 Flows Fix an n-manifold X and a vector field v on X.
Given a point $p \in X$ we can flow along v from p, i.e. solve the ODE

$$\begin{cases} \dot{\gamma}(t) = v(\gamma(t)) \\ \gamma(0) = p \end{cases}$$



By standard ODE theory, this equation has a solution defined on $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ suff small.
Moreover, the solution is unique and depends smoothly on p.
Solutions γ are called integral curves of v.

Definition 4.1 (Non-standard):
A flow domain is an open ngbd $\mathcal{U}$ of $\{0\} \times X \subseteq \mathbb{R} \times X$ such that $\forall p \in X$, the set $\mathcal{U} \cap (\mathbb{R} \times \{p\})$ is connected (i.e. is an open interval around 0)

Definition 4.2: A local flow of v comprises a flow domain $\mathcal{U}$ and a smooth map $\Phi : \mathcal{U} \to X$ such that:

- $\Phi(0, p) = p \quad \forall p$
- $\frac{d}{dt} \Phi(t, p) = v(\Phi(t, p)) \quad \forall (t, p) \in \mathcal{U}$

It's called a global flow if $\mathcal{U} = \mathbb{R} \times X$.
Write $\Phi^t = \Phi(t, \cdot)$.

By ODEs discussion, local flows always exist and are unique in the sense that if

- $\Phi : U \to X$, $\Psi : U' \to X$

are two local flows, then $\Phi = \Psi$ on $U \cap U'$.

<u>Proposition 4.3</u> : If Φ is a local flow of v, then
$$\Phi^s \circ \Phi^t = \Phi^{s+t}$$
wherever this makes sense.

So in particular $\Phi^{-t} = (\Phi^t)^{-1}$ wherever this makes sense.

<u>Proof</u> Fix $p \in X$ s.t. $\Phi^t(p)$, $\Phi^{s+t}(p)$ and $\Phi^s \circ \Phi^t(p)$ are defined, let $q = \Phi^t(p)$.

- Consider the curves
$$\gamma_1(\lambda) = \Phi^{\lambda s}(q)$$
$$\gamma_2(\lambda) = \Phi^{\lambda s + t}(p)$$

Our conditions ensure that γ_1, γ_2 are defined on $[0,1]$.
Moreover, they satisfy
$$\gamma_1(0) = \Phi^0(q) = q$$
$$\gamma_2(0) = \Phi^t(p) = q$$
and

- $\dot{\gamma}_1(\lambda) = s \cdot v(\gamma_1(\lambda))$
  $\dot{\gamma}_2(\lambda) = s \cdot v(\gamma_2(\lambda))$

So γ_1, γ_2 are integral curves of sv with the same initial condition.

$\therefore \gamma_1 = \gamma_2$

Hence $\Phi^s \circ \Phi^t(p) = \gamma_1(1) = \gamma_2(1) = \Phi^{s+t}(p)$. $\square$

A vector field is <u>complete</u> if it admits a global flow.

- Not all vector fields are complete (e.g. $x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$) but compactly supported vector fields <u>are</u> complete.

(construct a local flow on $(-\varepsilon, \varepsilon) \times X$)
(then define $\Phi^t = (\Phi^{t/N})^N$ for N large)
(this is well-defined by Prop 4.3)

§4.2 The Lie Derivative

Again for X and v. Let Φ be a local flow of v.

Definition 4.5 The Lie derivative of a tensor T on X is

$$\mathcal{L}_v T := \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T$$

It measures how T changes along the flow.
It's independent of the choice of local flow Φ.

For an arbitrary t we have

$$\frac{d}{dt}(\Phi^t)^* T = \frac{d}{dh}\Big|_{h=0} (\Phi^{t+h})^* T$$

⌜this notation ticks me off⌟

$$= \frac{d}{dh}\Big|_{h=0} (\Phi^t)^* (\Phi^h)^* T$$

$$= (\Phi^t)^* \mathcal{L}_v T$$

Lemma 4.6 For a function f, $\mathcal{L}_v f = df(v)$

For a vector field w,

$$\mathcal{L}_v w = \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$ in local coords.

Proof For a function f we have

$$\mathcal{L}_v f = \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* f$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ \Phi^t)$$

$$= df(v).$$

<u>Goal</u>: $\mathcal{L}_v w = \sum_{i,j} \left(v_i \frac{\partial w_j}{\partial x^i} - w_i \frac{\partial v_j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$

- Let Ψ be a local flow of w.

At a point p in our coordinate patch

$$\mathcal{L}_v w = \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* w \ (p)$$

$$= \frac{d}{dt}\Big|_{t=0} (D_p\Phi^t)^{-1} w(\Phi^t(p))$$

$$= \frac{d}{dt}\Big|_{t=0} (D_p\Phi^t)^{-1} \frac{d}{du}\Big|_{u=0} \Psi^u \circ \Phi^t(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{du}\Big|_{u=0} \Phi^{-t} \circ \Psi^u \circ \Phi^t(p)$$

- Let p have coords (x^i).

Then have $\Phi^t(p) = x^i + tv^i + o(t)$.

So $\Psi^u \circ \Phi^t(p) = x^i + tv^i + u\left(w^i + tv^j \frac{\partial w^i}{\partial x^j}\right) + o(t,u)$

Hence $\Phi^{-t} \circ \Psi^u \circ \Phi^t(p) = x + tv^i + uw^i + tuv^j \frac{\partial w^i}{\partial x^j}$

$$- t\left(v^i + uw^j \frac{\partial v^i}{\partial x^j}\right) + o(t,u)$$

So $\frac{d}{du}\Big|_{u=0}$ gives $w^i + tv^j \frac{\partial w^i}{\partial x^j} - tw^j \frac{\partial v^i}{\partial x^j} + o(t)$

And $\frac{d}{dt}\Big|_{t=0}$ gives $v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}$ as desired. $\square$

- <u>Lemma 4.7</u> (i) For a 1-form S and a vector field T,

$$\mathcal{L}_v(S_a T^a) = (\mathcal{L}_v S)_a T^a + S_a (\mathcal{L}_v T)^a$$

(ii) For any tensors S and T,

$$\mathcal{L}_v(S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T)$$

<u>Proof</u> Pullback commutes with contraction and $\otimes$.

So just use the standard product rule. $\square$

$$\ulcorner \mathcal{L}_v(S \otimes T) = \frac{d}{dt}\Big|_{t=0} (\Phi^t)^*(S \otimes T)$$

$$= \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* S \otimes (\Phi^t)^* T \lrcorner$$

Note : $\mathcal{L}_v w = -\mathcal{L}_w v$,
which was not obvious from the definition !

Definition 4.8 : The Lie bracket of vector fields is

$$[v, w] := \mathcal{L}_v w = -\mathcal{L}_w v$$

This makes the space of vector fields on X into a Lie algebra : a vector space equipped with a ($\mathbb{R}$-) bilinear bracket operation that is:
- alternating i.e. $[v,v] = 0$
- Jacobi i.e. $[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0$

Lemma 4.9 : The Lie derivative is diffeomorphism invariant, i.e if $F : X \to Y$ is a diffeo , v, w are vector fields on X, then $F_*[v,w] = [F_* v, F_* w]$, $F^*(\mathcal{L}_v T) = \mathcal{L}_{F^* v} F^* T$.

Proof We have

$$F^*(\mathcal{L}_v T) = F^* \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T$$

$$= \frac{d}{dt}\Big|_{t=0} F^*(\Phi^t)^* T$$

$$= \frac{d}{dt}\Big|_{t=0} \underbrace{(F^{-1} \circ \Phi^t \circ F)}_{\text{flow of } F^* v}{}^* (F^* T)$$

[also with pushforward]

$\square$

§4.3 Homotopy invariance of de Rahm cohomology

Theorem 4.11 (Cartan)

$$\mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega$$

Definition 4.10 Given an r-form α and a vector field v, write $\iota_v \alpha$ or $v \lrcorner \alpha$ for the $(r-1)$-form

$$\iota_v \alpha (w_2, \dots, w_r) = \alpha(v, w_2, \dots, w_r),$$

called the interior product of α with v

Proof of Theorem 4.11 : Ex Sheet 3 $\square$

Recall Proposition 3.11 : if $F_0, F_1 : X \to Y$ are homotopic then $F_0^* = F_1^*$ on cohomology.

Proof of Proposition 3.11 : Suppose $F : [0,1] \times X \to Y$ is a homotopy between F_0, F_1.

Let Φ^t be the flow of $\frac{\partial}{\partial t}$ in $[0,1] \times X$ (i.e. translation by t in $[0,1]$-direction).

Let i_t be the inclusion
$$X \longrightarrow [0,1] \times X$$
$$x \mapsto (t, x).$$

So $i_t = \Phi^t \circ i_0$ and $F_t = F \circ i_t$.

For any r-form α on X we have
$$F_1^* \alpha - F_0^* \alpha = \int_0^1 \left(\frac{d}{dt} F_t^* \alpha \right) dt$$
$$= \int_0^1 \left[\frac{d}{dt} (F \circ \Phi^t \circ i_0)^* \alpha \right] dt$$
$$= \int_0^1 \left[i_0^* \frac{d}{dt} (\Phi^t)^* (F^* \alpha) \right] dt$$
$$= \int_0^1 \left[i_0^* (\Phi^t)^* \mathcal{L}_{\frac{\partial}{\partial t}} F^* \alpha \right] dt$$

Assume α is closed. Then by Cartan's magic formula
$$\mathcal{L}_{\frac{\partial}{\partial t}} F^* \alpha = d \iota_{\frac{\partial}{\partial t}} F^* \alpha + \iota_{\frac{\partial}{\partial t}} \underbrace{d F^* \alpha}_{\text{zero}}$$

So
$$(F_1^* - F_0^*) \alpha = \int_0^1 i_t^* d(\iota_{\frac{\partial}{\partial t}} F^* \alpha) \, dt$$
$$= \int_0^1 d(\underbrace{i_t^* \iota_{\frac{\partial}{\partial t}} F^* \alpha}_{(r-1)\text{-form on } X}) \, dt$$
$$= d \left(\int_0^1 i_t^* \iota_{\frac{\partial}{\partial t}} F^* \alpha \, dt \right)$$

which is exact. $\square$

§5 Submanifolds, foliations, and Frobenius integrability

§5.1 Immersions, submersions and local diffeomorphisms

Fix manifolds X, Y of dimensions n, m, and let $F: X \to Y$ be a smooth map.

<u>Definition 5.1</u>: F is an <u>immersion</u> / <u>submersion</u> / <u>local diffeomorphism</u> (at p) if DF is injective / surjective / an isomorphism (at p).

If D_pF is surjective we say p is a <u>regular point</u> of F.

If q is not a regular point then it's a <u>critical point</u>.

A point $q \in Y$ is a <u>regular value</u> if $\forall p \in F^{-1}(q)$, p is a regular point. Otherwise q is a <u>critical value</u>.

—— · ——

The name local diffeomorphism is justified by the following:

<u>Lemma 5.2</u>: If D_pF is an isomorphism then $\exists$ open ngbds $U \ni p$, $V \ni F(p)$ s.t. $F|_U : U \to V$ is a diffeomorphism.

<u>Proof</u>: Pick charts φ about p, ψ about $F(p)$.

Then $G = \psi \circ F \circ \varphi^{-1}$ is a map between open sets in $\mathbb{R}^n$ with invertible derivative at $\varphi(p)$.

By the inverse function theorem, $\exists$ open ngbds $U' \ni \varphi(p)$, $V' \ni \psi(F(p))$ s.t. $G|_{U'} : U' \to V'$ is a diffeomorphism.

Let $U = \varphi^{-1}(U')$, $V = \psi^{-1}(V')$; done. $\square$

<u>Example 5.3</u>: Consider the map

$$(0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

This is a local diffeomorphism. So if we restrict the domain to

$$(0,\infty) \times (\theta_0, \theta_0 + 2\pi)$$

then it gives a diffeomorphism onto $\mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \cdot (\cos\theta_0, \sin\theta_0)$

So (r,θ) give local coordinates on $\mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \cdot (\cos\theta_0, \sin\theta_0)$ without having to invert any trig functions. ⌜BUT AT WHAT COST⌟

<u>Notice</u>: if $F: X \to Y$ is a local diffeo at $p \in X$ and $y_1, \dots, y_n$ are local coords about $F(p)$, then $y_1 \circ F, \dots, y_n \circ F$ are local coords about p. In these coordinates, F is the "identity"

Similarly if $x_1, \dots, x_n$ are local coordinates about p, then $x_1 \circ (F|_U)^{-1}, \dots, x_n \circ (F|_U)^{-1}$ are local coords about $F(p)$ in which F is the identity.

<u>Proposition 5.4</u>: Suppose $F: X \to Y$ is an immersion at p, and $x_1, \dots, x_n$ are coords about p. Then there are coordinates $y_1, \dots, y_m$ about $F(p)$ such that

$$y \circ F = (x_1, \dots, x_n, 0, \dots, 0)$$

(i.e. in these coordinates F looks like

$$\mathbb{R}^n = \mathbb{R}^n \oplus 0 \hookrightarrow \mathbb{R}^n \oplus \mathbb{R}^{m-n} = \mathbb{R}^m \,)$$

Similarly if F is a submersion at p and $y_1, \dots, y_m$ are coords about $F(p)$ then $\exists$ coords $x_1, \dots, x_n$ about p s.t. F looks like projection onto the first m factors.

<u>Proof</u>: half is on Ex Sheet 3, other half similar $\square$

§5.2 Submanifolds

Fix an n-manifold X.

Definition 5.5: A codimension-k submanifold of X is a subset Z s.t. $\forall p \in Z$ there exist local coordinates $x_1, \dots, x_n$ about p in which Z is given by $x_1 = \dots = x_k = 0$

Warning!: This holds $\forall p \in Z$ not $\forall p \in X$

e.g. $Z = (\mathbb{R}^2 \setminus \{0\}) \times \{0\} \subset X = \mathbb{R}^3$

is a submanifold but near the origin it's not defined by the vanishing of coordinates.

Note that:
- Z inherits a topology from X which is automatically Hausdorff & 2nd countable
- about each $p \in Z$, have nice coordinates $x_1, \dots, x_n$ on X then $x_{k+1}, \dots, x_n$ give nice coordinates on Z
- the transition functions for these coords on Z are smooth

Equivalent atlases on X give equivalent atlases on Z.

Upshot: **Proposition 5.6**: If $Z \subset X$ is a codimension-k submanifold, then it's naturally an $(n-k)$-manifold.

Moreover, the inclusion $\iota : Z \hookrightarrow X$ is a smooth immersion that's a homeomorphism onto its image. And composition with ι induces a bijection

$$\{\text{smooth maps } Y \to Z\} \xrightarrow{\iota \circ -} \left\{\begin{array}{c}\text{smooth maps } Y \to X \\ \text{with image} \subset Z\end{array}\right\}$$

Definition 5.7: A smooth immersion that's a homeomorphism onto its image is called an embedding

Lemma 5.8: If $F : Y \to X$ is an embedding with image Z then Z is a submanifold of X and F induces a diffeomorphism $Y \to Z$.

<u>Example 5.9</u>: The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding.
Hence S^n is a submanifold of $\mathbb{R}^{n+1}$ and the smooth structure we defined on it coincides with the submanifold smooth structure.
Finding nice coordinates is hard, but there's an easier way to check if a subset of X is a submanifold.

<u>Proposition 5.10</u>: If $F: X \to Y$ is a smooth map and $q \in Y$ is a regular value for F, then $F^{-1}(q)$ is a submanifold of X of codimension $\dim(Y)$.

⌜If $\dim Y = \dim X + a$, $a > 0$; then $F^{-1}(q) = \emptyset$⌟

<u>Proof</u>: Take $p \in F^{-1}(q)$. Pick local coordinates $y_1, \ldots, y_m$ about q with $y(q) = 0$.
Since q is a regular value, F is a submersion at p, so $\exists$ coords $x_1, \ldots, x_n$ about p in which F is projection onto the first m coords $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$
So locally near p, $F^{-1}(q)$ is $\{x_1 = \cdots = x_m = 0\}$. $\square$

<u>Example 5.11</u>: Consider $F: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $x \mapsto \|x\|^2$.
Then $DF = 2 \sum_i x^i dx^i$
so D_pF is surjective for all $p \neq 0$.
Hence $\forall r \in \mathbb{R} \setminus \{0\}$ the set $F^{-1}(r)$ is a codimension-1 submanifold of $\mathbb{R}^{n+1}$, e.g. $F^{-1}(1)$ is S^n, a submanifold.
Most points $q \in Y$ are regular values:

<u>Theorem 5.12</u> (Sard's theorem): For any smooth map $F: X \to Y$, the set of critical values has measure zero in Y. More precisely, if $\varphi: U \to V$ is a chart on Y, then $\varphi(\{\text{critical values in } U\}) \subset V$ has measure zero with respect to the usual Lebesgue measure on $\mathbb{R}^{\dim Y}$.

<u>Pf</u>: Theorem 6.10 in Lee (2^{nd} edition)
Theorem 2.1.18 in Nicolaescu (Sept 2018 ver) $\square$

We'll only use the following weaker version:

Corollary 5.13: Regular values are dense in Y.

In particular regular values exist.

Warning!: Sard's Thm is about regular values, says nothing about regular points.

Example $5.13\frac{1}{2}$: If $\dim X < \dim Y$ then there are no regular points. So $\{\text{regular values}\} = Y \setminus F(X)$.

Definition 5.14: Submanifolds $Y, Z \subset X$ are transverse if $\forall p \in Y \cap Z$ we have $T_pY + T_pZ = T_pX$.

Write $Y \pitchfork Z$.

Proposition 5.15: If $Y, Z \subset X$ are codimension k, l transverse submanifolds, then $Y \cap Z$ is a submanifold of codimension $k+l$

Proof: For $p \in Y \cap Z$. There exist coordinates

$y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ about p

s.t. $Y = \{y_1 = \cdots = y_k = 0\}$,

$Z = \{z_1 = \cdots = z_l = 0\}$.

So can define a smooth map

$F : U \to \mathbb{R}^{k+l}$, for $U \ni p$ a small open

by taking $F = (y_1, \ldots, y_k, z_1, \ldots, z_l)$.

If $Y, Z \subset X$ intersect transversely then $Y \cap Z$ is a submanifold.

Take $p \in Y \cap Z$, local coords $y_1, \dots, y_n$,

$z_1, \dots, z_n$,

about p s.t. $Y = \{y_1 = \cdots = y_k = 0\}$,

$Z = \{z_1 = \cdots = z_l = 0\}$.

Consider $F : U \to \mathbb{R}^{k+l}$ given by

$$q \longmapsto (y_1(q), \dots, y_k(q), z_1(q), \dots, z_l(q))$$

By transversality, $T_pX \to T_pX/T_pY \oplus T_pX/T_pZ$ is surjective.

So F is a submersion at p.

So $\exists$ coords $x_1, \dots, x_n$ about p s.t.

$x_1 = y_1, \dots, x_k = y_k$, $x_{k+1} = z_1, \dots, x_{k+l} = z_l$

So near p, $Y \cap Z = \{x_1 = \cdots = x_{k+l} = 0\}$. $\square$

§5.3 Frobenius integrability

Fix an n-manifold X.

<u>Def^n 5.16</u> A <u>k-plane distribution</u> D of X is a rank k subbundle of TX.

<u>Example 5.17</u> In $\mathbb{R}^3$, $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ 2-plane distribution or $\left\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right\rangle$.

The first one can be described as $\ker dz$.

The second, as $\ker(dz - y\,dx)$.

In general, a k-plane distribution is given by the vanishing of <u>$n-k$</u> fiberwise linearly indep 1-forms.

Given a k-plane distribution, and an immersed curve γ in X you can ask whether $\dot\gamma$ lies in D everywhere.

This is a system of $n-k$ ODEs: if D is locally

$\ker(\alpha_1, \dots, \alpha_{n-k})$

then the ODEs are $\alpha_i(\dot{\gamma}) = 0$

These are invariant under reparametrisation, of γ.

If $k=1$, then $\exists!$ local solution curve through each point, modulo reparametrisation.

Can pick a small $(n-1)$-dim disk in X transverse to D.

Then get local coordinates on X

$$x^1, \underbrace{y^1, \dots, y^{n-1}}_{\text{coords on disc}}$$

coord along solution curves

Then the y^i are "conserved quantities" locally along solution curves.

Conversely, any curve contained (locally) within level sets of the y^i is a solution curve.

If $k>1$ then the system of ODEs is <u>under-determined</u>.

The nicest possible situation is that there exist $n-k$ local conserved quantities along solution curves, and a curve solves the system of ODEs if it lies locally in these level sets.

<u>Def^n 5.18</u> Such a system of ODEs is called <u>integrable</u>

We formalise the notion of local level sets as follows:

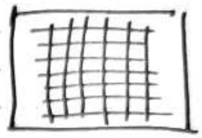
<u>Def^n 5.19</u> A smooth atlas on X is <u>k-foliated</u> if transition functions have the form

$$(x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}) \mapsto (\zeta(x,y), \eta(y))$$

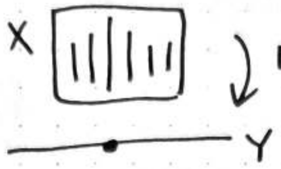
This respects the decomposition of $\mathbb{R}^n$ into slices $\mathbb{R}^k \times \{pt\}$.

A <u>k-foliation</u> on X is an equivalence class of k-foliated atlas under the natural notion of equivalence.

<u>Example 5.20</u> (i) If $X = Y \times Z$ then X is $(\dim Y)$-foliated by slices $Y \times \{pt\}$ (use product charts)
It is also $(\dim Z)$-foliated by slices $\{pt\} \times Z$.



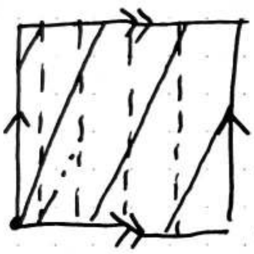
(ii) If $F : X \to Y$ is a submersion, then X is foliated by fibres



(iii) Consider the map $\mathbb{R}^2 \longrightarrow T^2 = S^1 \times S^1$

$$(x,y) \longmapsto (e^{ix}, e^{i(\alpha x + y)}) \quad \text{for } \alpha \in \mathbb{R}$$

This induces local coords on T^2 definining a foliation



slope α in $\frac{\partial}{\partial x}$ dir^n
vertical in $\frac{\partial}{\partial y}$ dir^n

Can foliate T^2 by the yellow slices ; if α is irrational then each leaf is dense in T^2.

Given a k-foliation of X, there's an induced k-plane distribution $D = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \rangle$

coordinates from foliated atlas

These are the tangent spaces to the local slices.

Conversely, given a k-plane distribution D, it arises from a k-foliation in this way iff the ODE system is integrable.
(The foliation coords $y^1, \dots, y^{n-k}$ correspond to the local conserved quantities)

<u>Theorem 5.2'</u> (Frobenius Integrability) A k-plane distribution D ariese from a foliation iff D is closed under the Lie bracket, i.e. if v, w vector fields in X lying in D,

then their Lie bracket $[v,w]$ also lies in D.
Such a distribution is called <u>integrable</u>.

<u>Example 5.23</u> Recall our 2-plane distributions on $\mathbb{R}^3$

(i) $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ arises from the 2-foliation induced by the standard chart on $\mathbb{R}^3$.

Can check it's closed under the Lie bracket.

(ii) $\langle \partial_x + y\partial_z , \partial_y \rangle$

Have $[\partial_y, \partial_x + y\partial_z] = \partial_z$.

So not closed under the Lie bracket.

Suppose F is a conserved quantity for the corresponding ODE system. Then

$$\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial y} = 0$$

So $f(x,y,z) = f(0,y,z-xy) = f(0,0,z-xy)$

So f is a function of $z-xy$ only.

But it's independent of y! So it's constant.

— · —

1-form $\alpha \rightsquigarrow \ker\alpha$

$\alpha\wedge d\alpha \overset{?}{=} 0$

— · —

Proof of Frobenius integrability

Both conditions are local, so it suffices to work in on arbitrary neighbourhood of a small point $p \in X$.

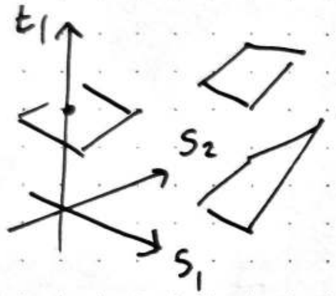
Suppose D arises from a foliation. Then locally have coords

$$x^1, \dots, x^k, y^1, \dots, y^{n-k} \quad \text{s.t.} \quad D = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$$

From our formula for $[\cdot,\cdot]$, this D is easily seen to be closed.

Conversely, suppose D is closed under Lie bracket and pick arbitrary coords $s_1, \dots, s_k, t_1, \dots, t_{n-k}$ about p.

WLOG $\langle \partial_{t_1}, \dots, \partial_{t_{n-k}} \rangle$ is transverse to D.



(can ensure this holds at p, hence on our whole coordinate patch, after shrinking if necessary)

Then for each i there $\exists!$ smooth functions a_{ij} s.t. $v_i := \partial_{s^i} + \sum_{j=1}^{n-k} a_{ij} \partial_{t^j}$ lies in D

Let Φ_i be a local flow of v_i.

Also WLOG $p = 0$ in the s, t coords

Define $F : U \longrightarrow X$ by $F(x^1, \dots, x^k, y^1, \dots, y^{n-k})$

$$= \Phi_1^{x^1} \circ \dots \circ \Phi_k^{x^k}(s = 0, t = y)$$

U: small nbhd of 0 in $\mathbb{R}^n$

Have $D_0 F(\partial_{x^i}) = v_i(p)$, $D_0 F(\partial_{y^i}) = \partial_{t_i}(p)$

So $D_0 F$ is invertible and F defines a parametrisation near p.

Left to show ∂_{x^i} is tangent to D everywhere.

Suppose the flows Φ_i commute. Then

$$\partial_{x^i} = \frac{d}{dt}\Big|_{t=0} \Phi_1^{x^1} \circ \dots \circ \Phi_i^{x^i + t} \circ \dots \circ \Phi_k^{x^k}(s = 0, t = y)$$

$$= \frac{d}{dt}\Big|_{t=0} \Phi_i^{x^i + t} \circ \Phi_1^{x^1} \circ \dots \circ \widehat{\Phi_i^{x^i}} \circ \dots \circ \Phi_k^{x^k}(s = 0, t = y)$$

$$= v_i(\text{blah}) \in D$$

So STP the $\Phi_i^{x^i}$ commute. By Example Sheet 3, this reduces to $[v_i, v_j] = 0 \quad \forall i, j$

We have

$$[v_i, v_j] = \sum_l \frac{\partial a_{jl}}{\partial s^i}\partial_{t^l}$$
$$+ \sum_{m,l}\left(\frac{\partial a_{jl}}{\partial t^m}\overset{a_{im}}{\swarrow}\partial_{t^l} - \frac{\partial a_{im}}{\partial t^l}\overset{a_{jl}}{\swarrow}\partial_{t^m}\right)$$
$$- \sum_m \frac{\partial a_{im}}{\partial s^j}\partial_{t^m}$$

We're assuming $[v_i, v_j] \in D$.

But $[v_i, v_j] \in \operatorname{span}\{\partial_{t^i}\}$, transverse to D.

So $[v_i, v_j] = 0$ as desired. $\square$

<u>Theorem 5.24</u> (Frobenius integrability – alternative version)

A distribution D arises from a foliation iff the <u>annihilator</u>

$I(D) := \{\alpha \in \Omega^\bullet(X) : \alpha(v_1, \dots, v_r) = 0 \text{ for any } v_1, \dots, v_r \in D\}$

is closed under d.

<u>e.g.</u> $D = \langle \partial_x, \partial_y \rangle$ has $I(D) = \Omega^\bullet(\mathbb{R}^3) \wedge dz$

So if $\alpha \in I(D)$ then $\alpha = \beta \wedge dz$ for some β.

Thus $d\alpha = d\beta \wedge dz \in I(D)$ ✓

<u>But</u> $D = \langle \partial_x + y\partial_z, \partial_y \rangle$ has $I(D) = \Omega^\bullet(\mathbb{R}^3) \wedge (dz - y\,dx)$

<u>Claim</u> $d(dz - y\,dx) \notin I(D)$

<u>Pf</u> This holds since

$$d(dz - y\,dx) \wedge (dz - y\,dx)$$
$$= -dy \wedge dx \wedge (dz - y\,dx)$$
$$= dx \wedge dy \wedge dz \neq 0. \quad \square$$

<u>Pf of Theorem 5.24</u> Both conditions are local, so we can work in a small neighbd of some p.

Then $\exists$ vector fields $v_1, \dots, v_k$ near p s.t. $D = \langle v_1, \dots, v_k \rangle$

and $\exists$ 1-forms $\alpha_1, \dots, \alpha_{n-k}$ s.t. $D = \ker\alpha_1 \cap \dots \cap \ker\alpha_{n-k}$.

Then $I(D) = \Omega^\bullet(X) \wedge \alpha_1 + \dots + \Omega^\bullet(X) \wedge \alpha_{n-k}$.

So $I(D)$ is closed under ~~$\not d$~~ d iff
$\forall i,\ d\alpha_i \in I(D)$.

This holds iff $\forall i, l, m$
$$d\alpha_i(v_l, v_m) = 0$$

<u>Claim</u> For any 1-form α and v. fields S, T,
$$d\alpha(S,T) = \iota_S d \iota_T \alpha - \iota_T d \iota_S \alpha - \iota_{[S,T]}\alpha$$

Apply this to $d\alpha_i(v_l, v_m)$ get
$$d\alpha_i(v_l, v_m) = -\alpha_i([v_l, v_m])$$

$I(D)$ is closed under d iff LHS $= 0\ \forall i, l, m$
iff $[v_l, v_m] \in D\ \ \forall l, m$
iff D is closed under $[\cdot,\cdot]$.

So done by first version of Frobenius. $\square$

<u>Pf of claim</u>

We have $\iota_S d \iota_T \alpha = \mathcal{L}_S(\iota_T \alpha)$
$$= \iota_{[S,T]}\alpha + \iota_T \mathcal{L}_S \alpha \quad \text{by Leibniz}$$

And $\mathcal{L}_S \alpha = \iota_S d\alpha + d\iota_S\alpha$.

So $\iota_S d \iota_T \alpha = \iota_{[S,T]}\alpha + \underbrace{\iota_T \iota_S d\alpha}_{d\alpha(S,T)} + \iota_T d \iota_S \alpha$ $\square$

§6 Lie groups & Lie algebras

§6.1 Lie groups

Def^n 6.1 A Lie group is a manifold G, equipped with a group structure, s.t.

multiplication $m : G \times G \longrightarrow G$

inversion $i : G \longrightarrow G$

are both smooth.

Example 6.2 $GL(n, \mathbb{R})$ with its manifold structure as an open subset of $\text{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.

<u>Def^n 6.3</u>: An <u>embedded Lie subgroup</u> of a Lie group G is a subgroup H that is also a submanifold.

The restrictions of the group operations from G to H are smooth, so H inherits a Lie group structure.

<u>Example 6.4</u>: $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$ are embedded Lie subgroups of $GL(n,\mathbb{R})$.

$GL(n,\mathbb{C})$, $U(n)$, $SU(n)$ are embedded Lie subgroups of $GL(2n,\mathbb{R})$

<u>Def^n 6.5</u>: For $g \in G$ an element of the Lie group G, have maps

$$L_g : G \to G \qquad R_g : G \to G \qquad C_g : G \to G$$
$$h \mapsto gh \qquad h \mapsto hg \qquad h \mapsto ghg^{-1}$$

These are <u>left-translation</u>, <u>right-translation</u> and <u>conjugation</u> by g.

These are diffeo^{m}s, with inverses

$$L_{g^{-1}}, \quad R_{g^{-1}}, \quad C_{g^{-1}}$$

<u>Def^n 6.6</u>: A tensor T on G is <u>left-invariant</u> if

$$\forall g \in G, \quad L_g^* T = T$$

Similarly define <u>right-invariant</u>, <u>conjugation-invariant</u>

T is <u>bi-invariant</u> if it's both left-invariant and right-invariant

<u>Lemma 6.7</u>: For any $h \in G$, the map

$$\left\{\begin{matrix}\text{Left-invariant} \\ (p,q)\text{-Tensors on } G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{tensors } (p,q) \\ \text{at } h\end{matrix}\right\},$$

given by evaluation at h, is a bijection.

Similarly for right-invariant.

<u>Pf</u> If T is left-invariant then $\forall g \in G$, we have

$$T_g = (L_{gh^{-1}})_* T_h = (L_{hg^{-1}})^* T_h \qquad (*)$$

So the map is injective.

[smoothness]

Conversely, given T_h at h, the formula (*) defines a left-invariant extension of T_h to G. $\square$

<u>Corollary 6.8</u> Any Lie group G is <u>parallelizable</u> (i.e. has trivial tangent bundle)

<u>Pf</u> Pick a basis for $T_e G$. The extensions of each to a left-invariant vector field form a fiberwise basis for TG, trivialising it. $\square$

<u>Example 6.9</u> For even $n \geq 2$, S^n does not admit a Lie group structure. On the other hand, S^3 is parallelizable, as it is diffeomorphic to $SU(2)$:

$$SU(2) = \left\{ \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} : |u|^2 + |v|^2 = 1 \right\} = S^3 \subseteq \mathbb{C}^2.$$

<u>§6.2 Lie algebras</u>

Fix a Lie group G.

<u>Defn</u>: The <u>Lie algebra</u> of G, denoted $\mathfrak{g}$, is $T_e G$.

<u>Example 6.11</u>: For $G = GL(n, \mathbb{R})$, we have

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) := Mat_{n \times n}(\mathbb{R})$$

Recall a Lie algebra is a vector space equipped with a bilinear alternating bracket $[\cdot, \cdot]$ satisfying the Jacobi identity.

<u>Propn 6.12</u>: $\mathfrak{g}$ carries a natural bracket operation making it into a Lie algebra

<u>Pf</u>: To each element $\xi \in \mathfrak{g}$ of the Lie algebra there is an associated left-invariant vector field, l_ξ.

We claim the bracket of two left-invariant v. fields is left-invt, so we can define $[\xi, \eta]$ by

$$l_{[\xi,\eta]} = [l_\xi, l_\eta].$$

This inherits the Lie algebra properties from $\mathfrak{X}(G)$.

It remains to prove our claim.

If $\xi, \eta \in \mathfrak{g}$ and $g \in G$, we have

$$L_g^*[l_\xi, l_\eta] = [L_g^* l_\xi, L_g^* l_\eta] \quad \text{by diffeo-invariance of } [\cdot,\cdot]$$
$$= [l_\xi, l_\eta] \quad \text{by left-invariance}$$

$\square$

Proposition 6.13: For all $\xi \in \mathfrak{g}$, the vector field l_ξ is complete

Pf Consider the ODE

$$\begin{cases} \gamma(0) = e, \\ \dot{\gamma} = l_\xi(\gamma). \end{cases}$$

This has a solution on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

This curve satisfies

$$\gamma(s+t) = \underbrace{\gamma(s) \cdot \gamma(t)}_{\text{group operation}} \quad \text{for small } s, t.$$

(Both sides solve $\delta(0) = \gamma(s)$, $\dot{\delta} = l_\xi(\delta)$
Recall left-invariance)

Now extend γ to $\mathbb{R}$ by defining

$$\gamma(t) = \gamma\left(\tfrac{t}{N}\right)^N \text{ for } N \gg 0.$$

Now define a global flow Φ for l_ξ by

$$\Phi^t(g) = g \cdot \gamma(t). \qquad \square$$

We'll write Φ_ξ for the flow of l_ξ.

Definition 6.14 The exponential map

$$\exp : \mathfrak{g} \to G$$

is given by

$$\exp(\xi) = \Phi_\xi^1(e).$$

Lemma 6.15: We could have used right-invariant vector fields instead, and we'd get the same exp.

Pf Let γ_ξ be the integral curve of l_ξ starting at e.
So $\exp(\xi) = \gamma_\xi(1)$.

STP that γ_ξ is an integral curve of the right-invariant r_ξ.
This holds since

$$\forall t \text{ we have } \quad \dot{\gamma}_\xi(t) = \frac{d}{ds}\Big|_{s=0} \gamma_\xi(t+s)$$

$$= \frac{d}{ds}\Big|_{s=0} \gamma_\xi(s)\gamma_\xi(t)$$

$$\overset{!}{=} r_\xi(\gamma_\xi(t)). \qquad \square$$

Lemma 6.16: exp is smooth

Pf Consider the vector field v on $\mathfrak{g} \times G$ given by

$$v(\xi, g) = (0, l_\xi(g))$$

⌜this is smooth!⌟

This has a local flow Φ which preserves slices $\{\xi\} \times G$.
On $\{\xi\} \times G$ it's exactly the flow of l_ξ.
So $\exp(\xi) = pr_2(\Phi^1(\xi, e))$.
Which is smooth. $\square$

Example 6.17: For $A \in \mathfrak{gl}(n, \mathbb{R})$, define e^A by

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

This converges locally absolutely uniformly.
Consider $\gamma(t) := e^{tA}$.
Then

$$\dot{\gamma}(t) = A + tA^2 + \frac{t^2A^3}{2!} + \cdots$$

$$= \underset{r_A(\gamma(t))}{\underset{\shortparallel}{Ae^{tA}}} = \underset{l_A(\gamma(t))}{\underset{\shortparallel}{e^{tA}A}}$$

So γ is the integral curve γ_A.
Hence $\exp(A) = \gamma(1) = e^A$.

Warning! At $e \in G$, the derivative $D_0 \exp : \mathfrak{g} \to \mathfrak{g}$ is $\mathrm{id}_{\mathfrak{g}}$,

so exp is a local diffeo near 0. ⌜$\exp : \mathfrak{g} \to G$⌟

But exp need not be globally injective or surjective.

E.g. for $SL(2,\mathbb{R})$ it's neither.

Lemma 6.18: For $\xi, \eta \in \mathfrak{g}$ we have

$$[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} (C_{\exp(t\xi)})_* \eta$$

Proof: We have ⌜$\Phi_\xi^t(g) = g \cdot \exp(t\xi)$⌟

$$[\xi,\eta] = \frac{d}{dt}\Big|_{t=0} \frac{d}{du}\Big|_{u=0} \Phi_\xi^{-t} \circ \Phi_\eta^{u} \circ \Phi_\xi^{t}(e)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{du}\Big|_{u=0} \exp(t\xi)\exp(u\eta)\exp(-t\xi)$$

$$= \frac{d}{dt}\Big|_{t=0} (C_{\exp(t\xi)})_* \eta \quad . \quad \square$$

Corollary 6.19: For $A, B \in \mathfrak{gl}(n,\mathbb{R})$, we have $[A,B] = AB - BA$.

Proof: By previous lemma,

$$[A,B] = \frac{d}{dt}\Big|_{t=0} e^{tA} B e^{-tA} = AB - BA . \quad \square$$

Corollary 6.20: If $\xi, \eta \in \mathfrak{g}$ satisfy $[\xi,\eta] = 0$ then

$$\exp(\xi + \eta) = \exp(\xi)\exp(\eta).$$

In particular $\exp(\xi)$ and $\exp(\eta)$ commute.

Proof Define $\gamma(t) = \exp(t\xi)\exp(t\eta)$.

We have $\dot\gamma(t) = \exp(t\xi)\,\xi\,\exp(t\eta)$ ← ⌜$(L_{\exp(t\xi)})_*(R_{\exp(t\xi)})_* \xi$ etc.⌟

$$+ \exp(t\xi)\exp(t\eta)\,\eta$$

$$= \exp(t\xi)\exp(t\eta)(\xi' + \eta)$$

where $\xi' = (C_{\exp(-t\eta)})_* \xi$.

We claim $\xi' = \xi \ \forall t$, so $\dot\gamma(t) = \ell_{\xi+\eta}(\gamma(t))$.

Then γ solves the ODE defining $\exp(t(\xi+\eta))$.

So we're done.

At $t=0$, $\xi' = \xi$.

And $\frac{d}{dt}\xi' = \frac{d}{dh}\Big|_{h=0} (C_{\exp(-(t+h)\eta)})_* \xi$

$$= -(C_{\exp(-t\eta)})_* [\eta, \xi]$$

$$= 0 \text{ by our assumption that } [\eta, \xi] = 0. \qquad \square$$

Warning! For general ξ, η it's not true that

$$\exp(\xi + \eta) = \exp(\xi)\exp(\eta).$$

§6.3 Lie group actions

Fix a Lie group G and a manifold X.

Definition 6.21: An action

$$\sigma : G \times X \to X \text{ of } G \text{ on } X$$

is smooth if the map σ is smooth.

Examples 6.22: (i) Action of G (or embedded Lie subgroups of) on G by left/right translation or by conjugation.

(ii) $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$ or on $\mathbb{RP}^{n-1}$.

(iii) $O(n)$ or a subgroup of it, acting on S^{n-1}.

Def^n 6.23: A smooth action of G on a vector space by linear maps is a smooth representation of G. This is the same as a Lie group homomorphism $\rho : G \to GL(V)$.

Example 6.24: The adjoint representation is the action of G on $\mathfrak{g}$ by conjugation:

$$\mathrm{Ad}_g(\xi) := (C_g)_* \xi$$

The dual representation is the coadjoint.

All actions and representations are smooth from now on.

Def^n 6.25: The infinitesimal action of $\xi \in \mathfrak{g}$ on $x \in X$ is $\xi \cdot x := D_{(e,x)}\sigma(\xi, 0) = (\exp(t\xi)x)^{\cdot}(0) \in T_x X$

Example 6.26: The infinitesimal adjoint action ~~X~~ of ξ on η is $(\mathrm{Ad}_{\exp(t\xi)}\eta)^{\cdot}(0) = [\xi, \eta]$.

§ 6.4 Quotients & homogeneous spaces

If a Lie group G acts on a manifold X, then have a quotient space X/G and a continuous projection $\pi: X \to X/G$.

Sometimes this quotient is nice

$$(\mathbb{R}^n \setminus \{0\}) / \mathbb{R}^* \cong \mathbb{RP}^{n-1}$$

But sometimes it's horrible!

$$\mathbb{R}^n / GL(n,\mathbb{R}) = \text{two points with a non-Hausdorff topology}$$
"connected doubleton"

Theorem 6.27 (Lee Theorem 21.10)

If the G-action is free and proper then X/G is a topological manifold of dimension $\dim X - \dim G$, and it has a unique smooth structure making $\pi: X \to X/G$ a submersion.

Def^n 6.28 : The action is <u>proper</u> if the following map

$$G \times X \longrightarrow X \times X$$
$$(g,x) \longmapsto (x, gx)$$

is proper (preimages of compact sets are compact). This is equivalent (Lee Prop^n 21.5) to the following: if (g_i) and (x_i) are sequences in G and X such that (x_i) and $(g_i x_i)$ converge, then (g_i) has a convergent subsequence.

Def^n 6.29 A <u>homogeneous space</u> for G is a manifold X carrying a transitive G-action.

A <u>principal homogeneous space</u> is a manifold with a transitive and free G-action; sometimes also called a <u>G-torsor</u>.

—— ∘ ——

If X is a G-torsor, then for any $x \in X$, the orbit map

$$G \to X, \quad g \mapsto gx$$

is a diffeomorphism. So X looks like a copy of G but with no distinguished identity element.

⌜local diffeo?⌟ ✓ but? defo not immediate

NEEDS SARD for outruling $\dim X > \dim G$

<u>Example 6.30</u> : (i) S^{n-1} is a homogeneous space for $SO(n)$

In fact it's $SO(n) / SO(n-1)$.

(ii) If H is an embedded Lie subgroup of G then the right translation action of H on G is proper (Example Sheet 4), so G/H is naturally a smooth manifold. The left-translation action of G descends to G/H, making G/H into a homogeneous space.

(In fact, <u>all</u> homogeneous spaces arise in this way.)

(iii) The space $F(V)$ of ordered bases in V carries a left action of $GL(V)$ making $F(V)$ into a $GL(V)$-torsor. There's also a right action of $GL(n, \mathbb{R})$, where $n = \dim V$, given by:

if $e_1, \dots, e_n$ is a basis for V, and $A \in GL(n, \mathbb{R})$

then $(e_1 \cdots e_n) A := (f_1 \cdots f_n)$

defines a new basis $f_1, \dots, f_n$.

This action is also free and transitive! So $F(V)$ is a torsor for $GL(n, \mathbb{R})$ acting on the right.

Correction to def^n 5.19 (foliated atlas)

The transition functions look like

$$(x,y) \mapsto (\zeta(x,y), \eta(y)) \text{ locally} \qquad \text{i.e. } \frac{\partial \eta}{\partial x^i} = 0 \ \forall i$$

§7. Principal bundles and connections

§7.1 Connections by hand

Fix a vector bundle $E \xrightarrow{\pi} B$ covered by trivialisations Φ_α in the usual way.

Given a section s, under Φ_α it becomes an $\mathbb{R}^k$-valued function, $v_\alpha : U_\alpha \to \mathbb{R}^k$. The naïve derivative is dv_α, an $\mathbb{R}^k$-valued 1-form. Under a different trivialisation Φ_β, v_α becomes $v_\beta = g_{\beta\alpha} v_\alpha$.

Let's take the naïve derivative and then pass the result back to the Φ_α trivialisation:

$$\begin{aligned} g_{\beta\alpha}^{-1}\, dv_\beta &= g_{\beta\alpha}^{-1}\, d(g_{\beta\alpha} v_\alpha) \\ &= dv_\alpha + g_{\beta\alpha}^{-1}(d g_{\beta\alpha})\, v_\alpha \end{aligned}$$

So the result is trivialisation-dependent via the action of the $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form $g_{\beta\alpha}^{-1} dg_{\beta\alpha}$ on v_α.

Definition 7.1 (Preliminary) A connection on E is a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form A_α on each trivialisation patch $U_\alpha \subset B$ s.t. on overlaps

$$A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta\, g_{\beta\alpha}$$

The covariant derivative of a section s wrt this connection $\mathcal{A}$ is the E-valued 1-form $d^{\mathcal{A}} s$ defined locally under the Φ_α by

$$dv_\alpha + A_\alpha v_\alpha .$$

This is consistent on overlaps:

$$\begin{aligned} g_{\beta\alpha}^{-1}(dv_\beta + A_\beta v_\beta) &= dv_\alpha + g_{\beta\alpha}^{-1}(dg_{\beta\alpha})\, v_\alpha \\ &\quad + g_{\beta\alpha}^{-1} A_\beta\, g_{\beta\alpha} v_\alpha \\ &= dv_\alpha + A_\alpha v_\alpha . \end{aligned}$$

⌜$\mathcal{A}$ vs $\mathcal{A}$⌟

Say s is <u>horizontal</u> or <u>covariantly constant</u> if $d^{\mathcal{A}} s = 0$. L20.2

<u>Example 7.2</u> Suppose E splits as $F \oplus F'$ for some rank l sub-bundle F.

We can cover E by trivialisations Φ_α in which the splitting becomes $\mathbb{R}^k = \mathbb{R}^l \times \mathbb{R}^{k-l}$.

Given a connection $\mathcal{A}$ on E, we can define a connection on F by taking the top-left $l \times l$ submatrix of each A_α. The covariant derivative of a section s of F is given by taking $d^{\mathcal{A}} s$ in E and projecting onto F along F'.

In particular if $\iota : X \to \mathbb{R}^N$ is an embedding, then $E = \iota^* T\mathbb{R}^N$ has a canonical trivialisation Φ_α and hence a canonical connection with $A_\alpha = 0$.

⌈inner product dependent⌋

The splitting $E = TX \oplus (TX)^\perp$ then induces a connection on TX.

<u>Definition 7.3</u> The <u>frame bundle</u> $F(E)$ of E is the space of ordered bases in each fibre, i.e.

$$F(E) = \left(\coprod_\alpha U_\alpha \times F(\mathbb{R}^k) \right) \Big/ \left(\begin{matrix} b \in U_\alpha, (v_1, \dots, v_k) \\ \sim \\ b \in U_\beta, (g_{\beta\alpha} v_1, \dots, g_{\beta\alpha} v_k) \end{matrix} \right)$$

This has a projection $\pi_F : F(E) \to B$.

It carries a right $GL(k, \mathbb{R})$-action, making each fiber $\pi_F^{-1}(b)$ into a principal homogeneous space.

A <u>section</u> of $F(E)$ over U is a map $f : U \to F(E)$ s.t. $\pi_F \circ f = \mathrm{id}_U$.

<u>Note</u>: sections of $F(E)$ over U correspond to trivialisations of E over U. Let f_α be the section of $F(E)$ corresponding to the trivialisation Φ_α of E.

We get for each α a diffeo

$$\begin{aligned} \Phi_\alpha^F : \pi_F^{-1}(U_\alpha) &\longrightarrow U_\alpha \times GL(k, \mathbb{R}) \\ f_\alpha(b) g &\longmapsto (b, g) \end{aligned}$$

Take a connection $\mathcal{A}$ on E. For each α, we can build a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $U_\alpha \times GL(k,\mathbb{R})$ as follows:

$$(v \in T_b U_\alpha,\ g\cdot\xi \in T_g GL(k,\mathbb{R}))$$
$$\mapsto \mathrm{Ad}_{g^{-1}} A_\alpha(v) + \xi$$

Pulling back by Φ_α^F gives a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $\pi_F^{-1}(U_\alpha)$.

Proposition 7.4: These local constructions agree on overlaps, and hence define a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form $\mathcal{A}$ on $F(E)$ satisfying

- $\mathcal{A}_p(p\cdot\xi) = \xi$ for all $p \in F(E)$, $\xi \in \mathfrak{gl}(k,\mathbb{R})$,
- $R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$ for all $g \in GL(k,\mathbb{R})$.

Conversely, any $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $F(E)$ satisfying these conditions defines a connection according to def^n 7.1, via taking $A_\alpha = f_\alpha^* \mathcal{A}$.

Proof Example Sheet 4. $\square$

Def^n 7.5: A connection on E is a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $F(E)$ satisfying the two conditions above.

§7.2 Principal bundles

Fix a Lie group G.

Def^n 7.6: A (principal) G-bundle over a manifold B is a manifold P equipped with

- a smooth surjection $\pi : P \to B$
- a collection of open sets U_α covering B and for each α a diffeo $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that
  - $pr_1 \circ \Phi_\alpha = \pi$
  - $\Phi_\beta \circ \Phi_\alpha^{-1}(b, g) = (b, g_{\beta\alpha}(b)\, g)$

for some smooth maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$.

P is the <u>total space</u>.

B is the <u>base</u>.

The Φ_α are <u>trivialisations</u>.

The $g_{\beta\alpha}$ are <u>transition functions</u>.

Lots of concepts transfer over from vector bundles.

E.g. pullbacks, sections, construction by gluing [that thing with cocycle]

Each trivialisation gives a section

$$b \mapsto \Phi_\alpha^{-1}(b, e)$$

over U_α. Conversely, a section s over U defines a trivialisation via

$$\Phi^{-1}(b, g) = s(b) \cdot g$$

Here we are using the <u>right</u> G-action on P, which one defines on trivialisations.

$F(E)$ $\boxed{\text{ordered bases of } E_b \xrightarrow{f_\alpha}}$ $\leftarrow \mathcal{A}$ $\qquad \mathcal{A}(p \cdot \xi) = \xi$

$$A_\alpha = f_\alpha^* \mathcal{A}$$

- B — b, U_α, A_α

If $P \to B$ is a principal G-bundle, then P has a right G-action, defined in trivialisations, i.e. if $\Phi_\alpha(p) = (b, g)$ then $p \cdot h = \Phi_\alpha^{-1}(b, gh)$

This gives a correspondence between sections of P and trivialisations $\Phi \mapsto s$ defined by

$$s(b) = \Phi^{-1}(b, e)$$

- $\Phi \longleftrightarrow s$ defined by $\Phi^{-1}(b,g) = s(b)g$ [this is smooth via other trivialisat^{n}s]

<u>Example 7.7</u>: (i) If E is a rank k vector bundle over B, then $F(E)$ is a principal $GL(k,\mathbb{R})$-bundle.

(ii) $B \times G \longrightarrow B$ is the <u>trivial</u> G-bundle

(iii) A G-bundle over a point is a G-torsor

<u>Warning!</u> A rank k vector bundle is <u>not</u> the same as a principal <u>$\mathbb{R}^k$-bundle</u>.

- The right G-action on a G-bundle is free and proper, and P/G is B. Conversely if P is a manifold carrying a right G-action that's free and proper, then the quotient map $\pi: P \to P/G$ is a principal G-bundle.

(π is a submersion so has local sections, and these induce trivialisations via the right G-action)

<u>Example 7.8</u>: Recall the Hopf map

$$H: S^{2n+1} \longrightarrow \mathbb{CP}^n$$

- $\underset{\cap}{}\ \mathbb{C}^{n+1}$ — The sphere in $\mathbb{C}^{n+1}$ carries a free, proper $U(1)$-action, by scalar multiplication, and the quotient map is H.

So H is a principal $U(1)$ bundle.

<u>Definition 7.9</u>: If $P \to B$ is a G-bundle and $\rho: G \to GL(V)$ is a representation, then the <u>associated vector bundle</u> is

$$P \times_G V = \{(p,v) \in P \times V\} / (p \cdot g, v) \sim (p, \rho(g) v).$$

⌜this is a quotient of $P \times V$ by G⌟

If P is trivialised over U_α with transition functions $g_{\beta\alpha}$, then $P \times_G V$ is trivialised over U_α with transition functions $\rho(g_{\beta\alpha})$. ⌜it is!⌟

<u>Example 7.10</u>: (i) If $P = F(E)$, and $\rho: GL(k, \mathbb{R}) \to GL(k, \mathbb{R})$ is the identity, then the associated vector bundle is E itself.

(ii) If $P = F(E)$, ρ is the dual rep, then the associated v.b is $E^\vee$.

Similarly can get tensor powers of $E, E^\vee$.

(iii) If $\rho: G \to GL(\mathfrak{g})$ is the adjoint representation ($\rho(g) \cdot \xi = g \xi g^{-1}$) then the associated v.b. is the <u>adjoint bundle</u> $\operatorname{ad} P$.

If $P = F(E)$ then $\operatorname{ad} P = \operatorname{End}(E) = E^\vee \otimes E$

§7.3 Connections

Let $\pi: P \to B$ be a G-bundle.

<u>Definition 7.11</u>: A <u>connection</u> on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P satisfying:

- $\mathcal{A}_p(\underbrace{p \cdot \xi}_{\in T_pP}) = \xi$
- $R_g^* \mathcal{A} = \operatorname{Ad}_{g^{-1}} \mathcal{A}$

If Φ_α is a trivialisation of P, corresponding to a section s_α, then $A_\alpha := s_\alpha^* \mathcal{A}$ is the <u>local connection 1-form</u>.

Lemma 7.12: On overlaps,

$$A_\alpha = g_{\beta\alpha}^{-1}\, dg_{\beta\alpha} + \mathrm{Ad}_{g_{\beta\alpha}} A_\beta \quad . \quad \square$$

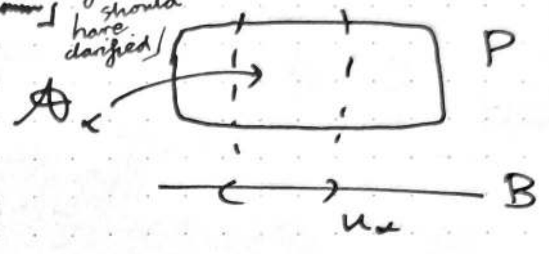
[this is very non-obvious]

Propositon 7.13: Every principal bundle (and hence every vector bundle) admits a connection.

Proof We can cover P by trivialisations Φ_α over U_α, and define a connection on $\pi^{-1}(U_\alpha)$, by taking $\mathcal{A}_\alpha$ to correspond to $A_\alpha = 0$.

[hmm] [yeah should have clarified]

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to this cover.



Then define

$$\mathcal{A} := \sum_\alpha (\rho_\alpha \circ \pi)\mathcal{A}_\alpha \quad \text{to get a connection on } P.$$

For $p \in P$, $\xi \in \mathfrak{g}$,

$$\mathcal{A}_p(p \cdot \xi) = \sum_\alpha (\rho_\alpha \circ \pi)(\mathcal{A}_\alpha)_p (p \cdot \xi) = \sum_\alpha (\rho_\alpha \circ \pi)\, \xi = \xi$$

Similarly $R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$. $\square$

Proposition 7.14: The space of all connections on P is a torsor for the space of ~~Ad~~ $\mathrm{ad}P$-valued 1-forms on the base.

Proof: Fix a reference connection $\mathcal{A}^0$ on P.

Now let $\mathcal{A}$ be any other connection.

Consider the $\mathfrak{g}$-valued 1-forms

$$A_\alpha - A^0_\alpha \quad \text{on } U_\alpha \subset B.$$

On overlaps, we have

$$A_\alpha - A^0_\alpha = \mathrm{Ad}_{g_{\beta\alpha}}(A_\beta - A^0_\beta).$$

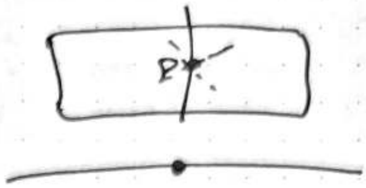
So they glue to give an $\mathrm{ad}P$-valued 1-form.

Conversely, if D is an $\mathrm{ad}P$-valued 1-form, then the Lie algebra-valued 1-forms $A^0_\alpha + D_\alpha$ define a connection $\mathcal{A}$.

These two constructions are inverses. $\square$

Def^n 7.15: For $p \in P$, the <u>vertical subspace</u> at p is

$$
\begin{aligned}
T_p{}^v P &:= \ker D_p \pi \\
&= T_p\, \pi^{-1}(\pi(p)) = T_p\, P_{\pi(p)} \\
&= p \cdot \mathfrak{g}
\end{aligned}
$$



A <u>horizontal subspace</u> is any complementary subspace.

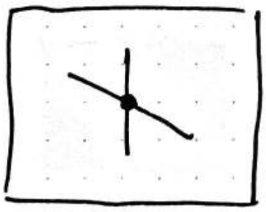
A <u>horizontal distribution</u> is a distribution H on P which is a horizontal subspace at every point.

Given a connection $\mathscr{A}$ on P, $H := \ker \mathscr{A}$ is a horizontal distribution (by rank-nullity, $\dim \ker \mathscr{A} = \dim P - \dim \mathfrak{g}$
$= \dim P - \dim T^v P$)

and $\ker \mathscr{A} \cap T^v P = 0$

Because $\mathscr{A}$ is right-equivariant, H is right-invariant, i.e.
$(R_g)_* H = H$.

Conversely, given a right-invariant horizontal distribution, $\exists!$ connection $\mathscr{A}$ whose kernel is this.



$\mathscr{A}$ $\mathscr{A}$ $\mathscr{A}$ $\mathscr{A}$

$\mathscr{A}, \mathscr{A}, \mathscr{A}$

<u>Last time</u>: connection $\mathcal{A}$ on $P \to B$

$\rightsquigarrow$ horizontal distribution $\ker \mathcal{A}$

AND horizontal distribution $\rightsquigarrow \mathcal{A}$

↳ invariant under right G-action on P

Any vector $v \in T_pP$ can be decomposed uniquely as

$$\begin{array}{ccc} p\cdot\xi & + & h \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \mathfrak{g} & & H \end{array} \qquad \text{Then define } \mathcal{A}(v) = \xi.$$

A section s of P is <u>horizontal</u> iff it's tangent to the horizontal distribution, i.e. $s^*\mathcal{A} = 0$.

<u>Example 7.16</u>: (i) Consider $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,z) \longmapsto (x,y)$$

viewed as a (trivial) principal $\mathbb{R}$-bundle.

The distributions $\langle \partial_x, \partial_y \rangle$ and $\langle \partial_x + y\partial_z, \partial_y \rangle$ are horizontal and are right-invariant. So they give a connection each on the bundle.

These have $\mathcal{A} = dz$ and $\mathcal{A} = dz - y\,dx$ respectively.

(Or $A = 0$, and $A = -y\,dx$)

(ii) Recall the Hopf bundle

$$H : \underset{\substack{\cap \\ \mathbb{C}^{n+1}}}{S^{2n+1}} \longrightarrow \mathbb{CP}^n \qquad \ulcorner U(1)\text{-bundle}\lrcorner$$

View T_pS^{2n+1} as a subspace of $\mathbb{C}^{n+1}$. Consider $T_pS^{2n+1} \cap i\,T_pS^{2n+1}$. This defines a $U(1)$-invariant horizontal distribution and hence a connection.

<u>Recall</u>: a section of $E \to B$ is horizontal iff it's covariantly constant. Can check that a connection on E induces a horizontal distribution on E s.t. a section is horizontal in the 'old' sense iff it is tangent to this distribution. Recall that a section f of $F(E)$ is a k-tuple of sections $s_1, \dots, s_k$ of E. Then f is

horizontal iff the s_i are horizontal.

Using the horizontal distribution, we can define parallel transport on $P \to B$ or $E \to B$ as in Example Sheet 3, Q7.

§7.4 Curvature

Fix a G-bundle P with a connection $\mathcal{A}$.

Definition 7.17: $\mathcal{A}$ is flat iff the horizontal distribution is integrable.

Proposition 7.18: TFAE:

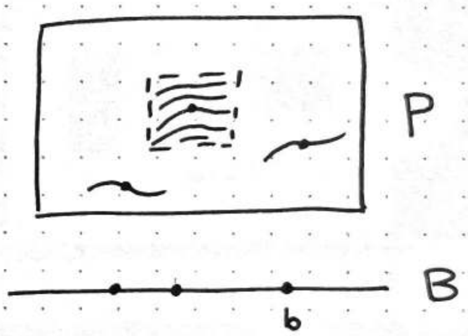
(i) $\mathcal{A}$ is flat,

(ii) P is foliated by local horizontal sections,

(iii) P has a horizontal section locally over each point in B,

(iv) P can be covered by trivialisations Φ_α s.t. all A_α are 0.

Proof: (i) $\Leftrightarrow$ (ii): because (ii) just spells out what it means for the horizontal distribution to arise from a foliation.



(ii) $\Rightarrow$ (iii): immediate

(iii) $\Rightarrow$ (ii): given $p \in P$, find a horizontal section s over $\pi(p)$. Then the right translates of s foliate P over a nghbd of $\pi(p)$, in particular near p.

(iii) $\Leftrightarrow$ (iv): given a trivialisation Φ_α, the corresponding section s_α is horizontal iff $s_\alpha^* \mathcal{A} = A_\alpha = 0$. $\square$

Slogan: Curvature is the obstruction to flatness.

Definition 7.19: The curvature $\mathcal{F}$ of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form $d\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$

Notation: For $\mathfrak{g}$-valued p,q forms $\sigma = \sum_i \xi_i \otimes \sigma_i$, $\tau = \sum_j \eta_j \otimes \tau_j$ we write $[\sigma \wedge \tau]$ for

$$\sum_{i,j} [\xi_i \wedge \eta_j] \otimes (\sigma_i \wedge \tau_j)$$

Warning! $[\sigma \wedge \tau] = (-1)^{pq+1} [\tau \wedge \sigma]$

Theorem 7.20: $\mathcal{A}$ is flat iff $\mathcal{F} = 0$

Proof Claim that $\mathcal{F}(v,w) = 0$ if (wlog) v is vertical.
Then by Frobenius, $\mathcal{A}$ is flat iff $d\mathcal{A} \in I(\ker \mathcal{A})$,
iff $d\mathcal{A}(v,w) = 0$ for all horizontal v, w,
iff $\mathcal{F}(v,w) = 0$ " (since $[\mathcal{A} \wedge \mathcal{A}]$ vanishes on horizontal sections)
iff (by the claim) $\mathcal{F} = 0$.

$[[\mathcal{A} \wedge \mathcal{A}](v,w) = [\mathcal{A}(v), \mathcal{A}(w)] - \langle \rangle]$

It remains to prove the claim, so take v to be the vertical vector field $v(p) = p \cdot \xi$ ($\xi \in \mathfrak{g}$ fixed).

Want to prove $\iota_v \mathcal{F} = 0$.

We have
$$\iota_v \mathcal{F} = \iota_v d\mathcal{A} + [\mathcal{A}(v), \mathcal{A} \cdot]$$
$$= \iota_v d\mathcal{A} + [\xi, \mathcal{A} \cdot]$$

So it's left to show $[\xi, \mathcal{A} \cdot] = -\iota_v d\mathcal{A}$.

We have
$$[\xi, \mathcal{A} \cdot] = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(t\xi)} \mathcal{A} \cdot$$
$$= \frac{d}{dt}\Big|_{t=0} (R_{\exp(-t\xi)})^* \mathcal{A}$$
$$= -\mathcal{L}_v \mathcal{A}$$
$$= -\iota_v d\mathcal{A} - \underbrace{d \underbrace{\iota_v \mathcal{A}}_{\xi \text{ (const.)}}}_{=0} \qquad \square$$

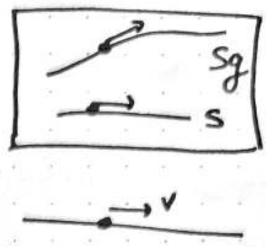
Given a section s_α corresponding to a trivialisation Φ_α we have $F_\alpha = s_\alpha^* \mathcal{F}$ a $\mathfrak{g}$-valued 2-form on U_α.

Proposition 7.21: These local expressions glue together to give an $\mathrm{ad}\, P$-valued 2-form on B.

Proof: On overlaps, we have $s_\beta = s_\alpha g_{\beta\alpha}^{-1}$ and want to show $F_\beta = \mathrm{Ad}_{g_{\beta\alpha}} F_\alpha$. Write $s = s_\alpha$, $g = g_{\beta\alpha}^{-1}$.

For any vector $v \in T_b(U_\alpha \cap U_\beta)$

$$(sg)_* v - (R_g)_*(s_* v) \text{ is vertical}$$



Since $\mathcal{F}$ annihilates vertical vectors, get

$$\begin{aligned} F_\beta = (sg)^* \mathcal{F} &= s^* R_g^* \mathcal{F} \\ &= s^* \mathrm{Ad}_{g^{-1}} \mathcal{F} \qquad \text{from } R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A} \\ &= \mathrm{Ad}_{g^{-1}} F_\alpha \quad . \qquad \square \end{aligned}$$

<u>Example 7.22</u>: For our two connections on $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ we have

$F = 0$ for $\mathcal{A} = dz$

$F = dx \wedge dy$ for $\mathcal{A} = dz - y\,dx$.

Last time: The local $\mathfrak{g}$-valued 2-forms

$$F_\alpha = dA_\alpha + \tfrac{1}{2}[A_\alpha \wedge A_\alpha]$$

- define an $\mathrm{ad}P$-valued 2-form on B, i.e. a section of

$$(\mathrm{ad}P) \otimes \Lambda^2 T^*B.$$

Proof used

$$R_g^* \mathcal{F} = \mathrm{Ad}_{g^{-1}} \mathcal{F}$$

true!

Consequence of (?)

$$R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}.$$

not true in general if g varies

Easier to work at fixed $b \in U_\alpha \cap U_\beta$ and say

$$R^*_{g(b)} \mathcal{F} = \mathrm{Ad}_{g(b)^{-1}} \mathcal{F}.$$

- Notation:

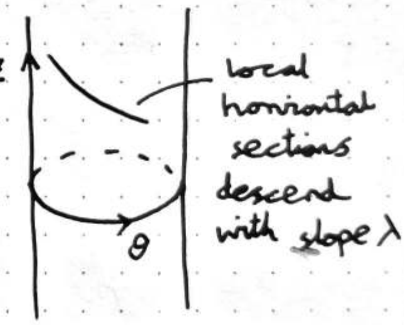
$$\tfrac{1}{2}[A_\alpha \wedge A_\alpha](v,w) = \tfrac{1}{2}[A_\alpha(v), A_\alpha(w)] - \tfrac{1}{2}[A_\alpha(w), A_\alpha(v)]$$

$$\text{"}[A_\alpha, A_\alpha]\text{"} = [A_\alpha(v), A_\alpha(w)]$$

$$\text{"}A_\alpha \wedge A_\alpha\text{"} = A_\alpha(v)A_\alpha(w) - A_\alpha(w)A_\alpha(v).$$

Warning: even if $\mathcal{F} = 0$, global horizontal sections need not exist, e.g. trivial principal $\mathbb{R}$-bundle over S^1 with $A = \lambda d\theta$ and fibre coordinate z. (so $\mathcal{A} = dz + \lambda d\theta$)

- So if $\lambda \neq 0$ then $\nexists$ global horizonal section.



§7.5 Algebraic structures

Given a connection $\mathcal{A}$ on G-bundle $P \to B$, and a representation $\rho: G \to GL(V)$, there's an induced connection on the associated vector bundle $E = P \times_G V$.

It's defined by local connection 1-forms

$$D_e\rho(A_\alpha).$$

[this is a connection, can proof]

- Example: If P is the frame bundle of a v.b. F then a connection on P induces connections on $F^\vee$, $F \otimes F^\vee$, etc.

Can also extend the covariant derivative $d^{\mathcal{A}}$ to an <u>exterior covariant derivative</u> using the Leibniz rule: an E-valued p-form σ can locally be written as a sum of expressions $s \otimes \alpha$ where s is a section of E, α is a p-form.

Then define $d^{\mathcal{A}}(s \otimes \alpha)$ to be

$$(d^{\mathcal{A}} s) \wedge \alpha + s \otimes d\alpha$$

⌜this is well defined!!!⌟

<u>Proposition 7.24</u> ((Second) Bianchi identity)

$$d^{\mathcal{A}} F = 0$$

(Here F is an $\mathrm{ad}P$-valued 2-form on B, and $d^{\mathcal{A}}$ is the exterior covariant derivative)

<u>Proof</u> Locally in a trivialisation we write F as F_α, a $\mathfrak{g}$-valued 2-form. Then locally

$$d^{\mathcal{A}} F \overset{!}{=} dF_\alpha + (\underset{\text{Def}}{\mathrm{ad} A_\alpha}) \wedge F_\alpha = dF_\alpha + [A_\alpha \wedge F_\alpha]$$

$$= d^2 A_\alpha + \tfrac{1}{2} d[A_\alpha \wedge A_\alpha] + [A_\alpha \wedge dA_\alpha] + [A_\alpha \wedge \tfrac{1}{2}[A_\alpha \wedge A_\alpha]]$$

First term is zero since $d^2 = 0$.

Last term too, by Jacobi identity.

Second & third terms cancel by Leibniz:

$$\tfrac{1}{2} d[A_\alpha \wedge A_\alpha] = \tfrac{1}{2}[(dA_\alpha) \wedge A_\alpha] - \tfrac{1}{2}[A_\alpha \wedge (dA_\alpha)] = -[A_\alpha \wedge dA_\alpha]. \qquad \square$$

<u>Warning</u>! $(d^{\mathcal{A}})^2 \neq 0$ in general

In fact $\underset{\substack{\uparrow \\ E\text{-valued} \\ p\text{-form}}}{(d^{\mathcal{A}})^2 \sigma} = \underbrace{\mathrm{Def}(F)}_{\substack{\text{End}(E)\text{-valued} \\ 2\text{-form}}} \wedge \sigma$

⌜$E = P \times_G V$⌟

§8 Riemannian Geometry

§8.1 Metrics

Given a vector bundle $E \to B$, sections of $(E^\vee)^{\otimes 2}$ correspond to fiberwise bilinear forms on E.

Def^n 8.1: An inner product on E is a section of $(E^\vee)^{\otimes 2}$ which is fiberwise symmetric & positive definite.

A Riemannian metric on X is an inner product on TX.

Lemma 8.2: Every vector bundle $E \to B$ admits an inner product. Hence every manifold admits a Riemannian metric.

Proof Cover E with trivialisations

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^k \times U_\alpha$$

On each $\pi^{-1}(U_\alpha)$, there's an inner product g_α corresponding to the standard one in $\mathbb{R}^k$.

Take a partition of unity $\{\rho_\alpha\}$ and set $g = \sum \rho_\alpha g_\alpha$. □

Def^n 8.3 A Riemannian manifold (X, g) is a manifold equipped with a Riemannian metric.

Write $g = g_{ab}$.

Let g^{ab} be the dual metric, defined by

$$g^{ab} = g^{ba}, \quad g^{ab} g_{bc} = \delta^a{}_c$$

i.e. if $\{v_i\}$ o.n. then dual basis is o.n

Write contractions with g_{ab}, g^{ab} by raising, lowering indices

e.g. $g^{bd} T^a{}_{bc} = T^{ad}{}_c$

Notation: $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$

Def^n 8.4: A connection $\mathcal{A}$ on E is compatible with an inner product g if g is covariantly constant with respect to the induced connection on $(E^\vee)^{\otimes 2}$.

§8.2 Connections on TX

Fix a manifold X.

<u>Def^n 8.5</u>: A <u>connection</u> on X is a connection on TX.

We'll think of this as a connection on E, where E is identified with TX via an E-valued 1-form θ.

(For $x \in X$, $\theta_x \in E_x \otimes T^*_x X = \mathrm{Hom}(T_x X, E_x)$)

Usually the covariant derivative is written ∇, its contraction with a vector v is written as ∇_v.

<u>Def^n 8.6</u> The <u>torsion</u> of a connection $\mathcal{A}$ on $E \cong TX$ is $T = d^{\mathcal{A}}\theta$, an E-valued 2-form.

(Sheet 4: $\nabla_v w - \nabla_w v = [v, w] + T(v, w)$)

The connection is <u>torsion-free</u> if $T = 0$.

<u>Proposition 8.7</u> (First Bianchi identity)

$$d^{\mathcal{A}} T = F \wedge \theta$$

<u>Proof</u> Both sides are $(d^{\mathcal{A}})^2 \theta$. $\square$

<u>Theorem 8.8</u> (Fundamental Theorem of Riemannian Geometry)

Given a Riemannian manifold (X, g) there's a unique torsion-free connection on X compatible with g.

This is the <u>Levi-Civita connection</u>. ⌜one guy⌟

<u>Proof</u> We'll show that the map

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{all } g\text{-compatible} \\ \text{connections} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} E\text{-valued} \\ 2\text{-forms} \end{array} \right\} \\ \mathcal{A} & \longmapsto & T_{\mathcal{A}} \end{array}$$

is a bijection.

Recall Connection on E

$$\updownarrow$$

Connection on $F(E)$

Connection on X = connection on $E \cong TX$
identified with TX via θ

Torsion $= d^{\mathcal{A}}\theta$

$$\{g\text{-compatible connections on } X\} \longleftrightarrow \{E\text{-valued 2-forms on } X\}$$
$$\mathcal{A} \longmapsto \underbrace{T_{\mathcal{A}}}_{\text{torsion}}$$

is a bijection.

Let $F_o(E)$ be the orthogonal frame bundle of E – a principal $O(n)$-bundle. Note that E is an associated bundle of $F_o(E)$,

$$E = F_o(E) \underset{O(n)}{\times} \mathbb{R}^n$$

via the standard representation of $O(n)$. So connections on $F_o(E)$ induce connections on E, and a connection on E is compatible with g iff it arises in this way (Ex Sheet 4)

Fix a connection $\mathcal{A}_o$ on $F_o(E)$.

We get a bijection

$$\left\{\begin{matrix} g\text{-compatible} \\ \text{connections} \\ \text{on } X \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \operatorname{ad} F_o(E)\text{-valued} \\ 1\text{-forms on } X \end{matrix}\right\}$$
$$\mathcal{A} \longmapsto \Delta := \mathcal{A} - \mathcal{A}_o$$

We also have

$$\operatorname{ad} F_o(E) \cong \mathfrak{o}(E) = \{\text{skew-adjoint endomorphisms of } E\}$$

So it's left to show that

$$\left\{\begin{matrix} \mathfrak{o}(E)\text{-valued} \\ 1\text{-forms on } X \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} E\text{-valued} \\ 2\text{-forms} \end{matrix}\right\}$$
$$\Delta \longmapsto T_{\mathcal{A}_o+\Delta} - T_{\mathcal{A}_o} \quad \text{is a bijection.}$$

We can view both bundles as sub-bundles of
$TX \otimes T^*X \otimes T^*X$, of rank $\frac{1}{2}n^2(n-1)$.

$\{\mathfrak{o}(E)\text{-valued 1-forms}\}$
$= \{$sections $\Delta^a{}_{bc}$ of $TX \otimes T^*X \otimes T^*X$ s.t.
$g_{ad}\Delta^d{}_{bc} + g_{db}\Delta^d{}_{ac} = 0 \}$ ⌜i.e. $\Delta_{abc} = -\Delta_{bac}$⌟

$\{E\text{-valued 2-forms}\}$
$= \{ \Delta^a{}_{bc} : \Delta^a{}_{bc} = -\Delta^a{}_{cb} \}$

And the map $\Delta \longmapsto T_{\nabla_0 + \Delta} - T_{\nabla_0}$
is $\Delta \longmapsto (\Delta \wedge \theta)^a{}_{bc} = \Delta^a{}_{cb} - \Delta^a{}_{bc}$
which is fiberwise linear, so STP it's an isomorphism fiberwise.

Since both have the same rank, it's STP the map is fiberwise injective.

So suppose Δ satisfies $\Delta_{abc} = -\Delta_{bac}$
and it's in the kernel, i.e. $\Delta^a{}_{cb} = \Delta^a{}_{bc}$.
Want to show $\Delta = 0$.

We have

$$\Delta_{abc} = -\Delta_{bac} = -\Delta_{bca} = \Delta_{cba}$$
$$= \Delta_{cab} = -\Delta_{acb} = -\Delta_{abc}. \qquad \square$$

Given local coordinates on X, get a trivialisation of $E \cong TX$.
The components of the associated local connection 1-forms are the <u>Christoffel symbols</u> $\Gamma^i{}_{jk}$.

<u>Definition 8.9</u> The curvature of the Levi-Civita connection is the <u>Riemann tensor</u>, $R = R^a{}_{bcd}$.

This is an $\mathfrak{o}(E)$-valued 2-form on X, so we can view it as a tensor of type $(1,3)$.

§8.4 Hodge theory

Let (X, g) be an oriented Riemannian manifold.

The metric g induces inner products on each $\Lambda^p T^* X$.

(If $\alpha^1, \dots, \alpha^n$ are orthonormal 1-forms, then α^I give a fibrewise orthonormal basis for $\Lambda^p T^* X$)

We get a distinguished volume form ω, defined by being positively oriented and of unit length.

Given a p-form β, there's a unique $(n-p)$-form $*\beta$ s.t. $\forall$ p-forms α,

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega .$$

<u>Definition 8.10</u> The map

$$* : \Omega^p(X) \to \Omega^{n-p}(X)$$

is the <u>Hodge star operator</u>.

It's a fiberwise linear isometry $\Lambda^p T^* X \to \Lambda^{n-p} T^* X$, and squares to $(-1)^{p(n-p)}$.

<u>Example 8.11</u> Take $\mathbb{R}^3$ with the standard orientation and metric. So $\omega = dx^1 \wedge dx^2 \wedge dx^3$.

And $*dx^1 = dx^2 \wedge dx^3$

$*dx^1 \wedge dx^2 = dx^3$.

Now assume X is compact. Then we can define an inner product on $\Omega^p(X)$ via

⌜X closed⌟

$$\langle \alpha, \beta \rangle_X = \int_X \langle \alpha, \beta \rangle \omega = \int_X \alpha \wedge *\beta$$

Given $(p-1)$-form α, p-form β, we have

$$\langle d\alpha, \beta \rangle_X = \int_X (d\alpha) \wedge *\beta$$

Leibniz

$$= \int_X d(\alpha \wedge *\beta) - (-1)^{p-1} \alpha \wedge d*\beta$$

Stokes

$$= (-1)^p \int_X \alpha \wedge d*\beta$$

$$\vdots$$
$$= \langle \alpha, (-1)^p *^{-1} d * \beta \rangle_X$$

So the operator
$$\delta := (-1)^p *^{-1} d * : \Omega^p(X) \to \Omega^{p-1}(X)$$
is the adjoint to d.

Definition 8.12 δ is called the codifferential, if $\delta\beta = 0$ then β is co-closed, if $\beta = \delta\gamma$ then β is co-exact.
(Can check $\delta^2 = 0$)

Definition 8.13 The Laplace-Beltrami operator is
$$\Delta := d\delta + \delta d = (d+\delta)^2 : \Omega^p(X) \to \Omega^p(X).$$

If $\Delta\alpha = 0$, say α is harmonic.
Write $\mathcal{H}^p$ for the space of harmonic p-forms.
Example sheet 4: α is harmonic iff closed and coclosed

Theorem 8.14 The map
$$\mathcal{H}^p(X) \longrightarrow H^p_{dR}(X) \qquad \text{(Hodge)}$$
$$\alpha \longmapsto [\alpha]$$
is an isomorphism (!)

Idea: "$H^p_{dR} = \ker d / \operatorname{im} d$
$$= \ker d \cap (\operatorname{im} d)^\perp$$
$$= \ker d \cap (\ker \delta)$$
$$= \mathcal{H}^p \text{"}$$

To make this precise:

Theorem 8.15 (Hodge decomposition) For all p, $\mathcal{H}^p$ is f. dim and we get orthogonal decompositions
$$\Omega^p(X) = \mathcal{H}^p(X) \oplus \Delta\Omega^p(X) = \mathcal{H}^p \oplus d\Omega^{p-1}(X) \oplus \delta\Omega^{p+1}(X).$$

Pf of 8.14: using last line of Hodge, it's STP $\ker d = (\operatorname{im}\delta)^\perp$.
Have $\alpha \in \ker d$ iff $\forall \beta$, $\langle d\alpha, \beta \rangle = 0$ i.e. $\langle \alpha, \delta\beta \rangle = 0$. □

Proof: See §10.4.3 in Nicolaescu. □