Four-Manifolds (Ivan Smith, is200)

Examples: (of closed 4-manifolds)

1. $S^4 = \{ x \in \mathbb{R}^5 \mid \|x\| = 1 \}$
2. Products: $\Sigma_g \times \Sigma_h$, $S^1 \times Y^3$ ← any closed 3-manifold
3. Twisted products, i.e. non-trivial fiber bundles

   e.g. if $\phi, \psi \in \mathrm{Diff}(\Sigma_g)$, $[\phi, \psi] = 1$, then $\exists$ a fiber bundle

   $$\begin{tikzcd} \Sigma_g \arrow[r] & X \arrow[d] \\ & T^2 \end{tikzcd}$$

   with monodromies ϕ, ψ

4. Complex hypersurfaces

   e.g. $V_d = \{ x_0^d + x_1^d + \dots + x_3^d = 0 \} \subseteq \mathbb{CP}^3$

If $d = 1$, $V_1 = \mathbb{P}^2 \subseteq \mathbb{CP}^3$

$d = 2$, $V_2 = \mathbb{P}^1 \times \mathbb{P}^1 \; (= S^2 \times S^2) \subseteq \mathbb{CP}^3$ (not obvious) ← Segre!

5. Connect sums; given 4-manifolds X_1, X_2 I can form $X_1 \# X_2 = (X_1 \setminus B^4) \cup_{S^3} (X^2 \setminus B^4)$

   (can be made smooth if the X^i are)

Classification is hopeless:

(a) [Markov] There is no algorithm to recognise when a finitely presented group is trivial.

(b) If $G = \langle g_1, \dots, g_k \mid r_1, \dots, r_s \rangle$ then $\exists$ a closed 4-mfd X_G with $\pi_1(X_G) = G$.

Proof of (b) Seifert-van Kampen:

$$\pi_1(X \# Y) = \pi_1 X * \pi_1 Y$$

$N = \#_{i=1}^k (S^1 \times S^3)$; $\pi_1(N) = \mathrm{Free}_k = \langle g_1, \dots, g_k \rangle$

Each relation r_i for my presentation is a word in the $\{g_i, g_i^{-1}\}$, which gives a loop $\gamma_i : S^1 \to N$ & wlog γ_i is a smooth embedding.

Surgery: N & the γ_i are orientable, so

$$\nu_{\gamma_i/N} = S^1 \times B^3$$

We cut out this $S^1 \times B^3$ & glue back $B^2 \times S^2$ along the boundary $S^1 \times S^2$.

Observe:

$$N = N \setminus (S^1 \times B^3) \bigcup_{S^1 \times S^2} (S^1 \times B^3)$$

$$\Downarrow \text{ SvK}$$

$$\pi_1(N) = \pi_1(N \setminus S^1 \times B^3) \underset{\langle \gamma_i \rangle}{*} \langle \gamma_i \rangle$$

$$= \pi_1(N \setminus S^1 \times B^3)$$

$$\tilde{N} = N \setminus (S^1 \times B^3) \bigcup_{S^1 \times S^2} (B^2 \times S^2)$$

$$\Downarrow \text{ SvK}$$

$$\pi_1(\tilde{N}) = \pi_1(N \setminus S^1 \times B^3) \underset{\langle \gamma \rangle}{*} \{e\}$$

$$= \pi_1(N) / \langle\langle \gamma \rangle\rangle$$

Now do surgery simultaneously on all the $\{\gamma_i\}$ & I obtain a 4-mfd X_G with

$$\pi_1 \cong \underbrace{\text{Free}_k / \langle\langle r_i \mid i = 1, \dots, s \rangle\rangle}_{G}. \qquad \square$$

Upshot: For most of the course, we will focus attention on simply-connected (compact) 4-mfds.

(c.f. our $V_d \subseteq \mathbb{CP}^3$ are examples of these, Lefschetz hyperplane theorem)

Homology & Cohomology

Suppose $\pi_1(X) = 0$, so $H_1(X;\mathbb{Z}) = 0$.

X is orientable : $H^*(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0,4 \\ \mathbb{Z}^b, & * = 2 \end{cases}$

$b = b_2(X) = 2^{nd}$ Betti number of X (cohomology is torsion-free)

P.Duality says $H^2(X;\mathbb{Z}) \xrightarrow{\sim} H_2(X;\mathbb{Z})$

Lemma : If X is an oriented 4-manifold, then every $A \in H_2(X;\mathbb{Z})$ is represented by an embedded oriented surface $\Sigma_A \subseteq X$.

Proof Let $A \in H_2(X;\mathbb{Z})$ & $\alpha \in H^2(X;\mathbb{Z})$ its Poincaré dual.

There's a complex line bundle $\mathcal{L} \to X$ with Euler class α. (Equivalently $\exists\ X \xrightarrow[u]{} \mathbb{CP}^\infty$

s.t. $u^*(\mathcal{L}_{taut}) = \mathcal{L}$

& $u^*(x) = \alpha$, where

$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x]$, $|x| = 2$)

Let $s : X \to \mathcal{L}$ be a section of $\mathcal{L}$ which intersects transversely.

Then $s^{-1}(0) = \Sigma \subseteq X$ is a surface which represents $A = PD(\alpha)$. $\square$

Important remark

If $\pi_1(X^4) = \{1\}$, the Hurewicz theorem says

$$\pi_2(X) \xrightarrow{\sim} H_2(X;\mathbb{Z})$$

is an isomorphism.

So given $A \in H_2(X;\mathbb{Z})$, $\exists$ a map $f : S^2 \to X$ s.t.

$$f_*[S^2] = A \in H_2(X;\mathbb{Z})$$

BUT transversality says I can assume f is an immersion but not an embedding.

Minimal genus problem

Given X^4, a compact smooth 4-mfd, & $A \in H_2(X;\mathbb{Z})$

Say X is oriented. Find minimal genus of an embedded oriented (connected) surface representing A.

This is (one of) the most fruitful questions in the subject.

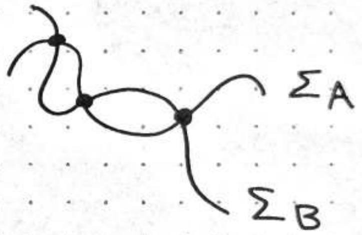
Recall Have cup product

$$\begin{aligned} H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \alpha \otimes \beta &\longmapsto \langle \alpha \cup \beta, [X] \rangle \end{aligned}$$

fundamental class

dual to the intersection product

$$\begin{aligned} H_2(X) \otimes H_2(X) &\longrightarrow \mathbb{Z} \\ A \otimes B &\longmapsto \cdot \end{aligned}$$



Count isolated set of intersections of Σ_A, Σ_B with signs.

Pick a $\mathbb{Z}$-basis for $H^2(X;\mathbb{Z})$

Cup-product is represented by a symmetric $\mathbb{Z}$-matrix Q_X (called the intersection form)

Poincaré Duality says this is unimodular, of det ± 1

If I change the $\mathbb{Z}$-basis for $H^2(X;\mathbb{Z})$, I change Q by similarity i.e. $Q \sim PQP^t$, $P \in GL(b;\mathbb{Z})$

Theorem If X_1, X_2 are closed, oriented, simply connected 4-manifolds, then X_1, X_2 are oriented homotopy-equivalent $\Leftrightarrow$ Q_{X_1}, Q_{X_2} are similar over the integers.

<u>Examples</u>

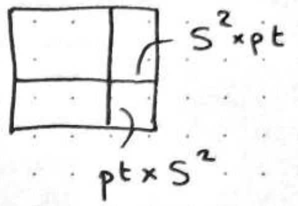
1. $Q_{S^4} = (\ )$

2. $Q_{\mathbb{CP}^2} = (1)$ BUT $Q_{\overline{\mathbb{CP}^2}} = (-1)$

   (so $\nexists\ f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ which reverses orientation)

3. $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ hyperbolic form U

4. $Q_{\mathbb{P}^2 \# \overline{\mathbb{P}^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,



in general

$$Q_{X\#Y} = \begin{pmatrix} Q_X & 0 \\ 0 & Q_Y \end{pmatrix}$$ by Mayer-Vietoris thm

A simpler invariant than $(H^2(X;\mathbb{Z}), Q_X)$ is $(H^2(X;\mathbb{R}), Q_X \otimes \mathbb{R})$.

A real non-degenerate symmetric bilinear form is determined by <u>rank</u> & <u>signature</u>.

$$\text{rank} = b = \dim_{\mathbb{R}} H^2(X;\mathbb{R})$$

Over $\mathbb{R}$, $Q_X \otimes \mathbb{R} \sim \text{diag}(\underbrace{1, \dots, 1}_{b_+}, \underbrace{-1, \dots, -1}_{b_-})$

$$b_+ + b_- = b_2$$

Def

<u>Signature</u> of X is $b_+ - b_- = \sigma(X)$

<u>Remark</u>: The forms $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are real similar, both of rank 2, signature 0, but they're <u>not</u> similar over $\mathbb{Z}$.

For U, if $\alpha \in \mathbb{Z}^2$ then $\alpha \cdot \alpha = Q(\alpha, \alpha) \in 2\mathbb{Z}$

But for diagonal form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ this is <u>not</u> true.

Shows $(H^2(X;\mathbb{Z}), Q_X)$ has more information than its realification.

Signature

If X_1, X_2 are simply-connected (oriented) 4-manifolds, then $X_1 \simeq X_2 \iff Q_{X_1}, Q_{X_2}$ are similar over $\mathbb{Z}$

Using Mayer-Vietoris / Seifert-van Kampen:

$$\pi_1(X \setminus B^4) = 0$$

$$\begin{array}{ccc} H_2(X\setminus B^4) & \underset{\text{Hurewicz}}{\cong} & \pi_2(X\setminus B^4) \\ \text{\rotatebox{90}{$\cong$}} & & \\ H_2(X) & & \end{array}$$

I can pick maps $f_i : S^2 \to X \setminus B^4$, $1 \leq i \leq b = b_2(X)$ s.t.

$$\bigvee_{i=1}^{b} S^2 \longrightarrow X \setminus B^4 \quad \text{is an iso on } \pi_i \text{ and } H_*$$

So this is a homotopy equivalence, & $X = \bigvee_{i=1}^{b} S^2 \cup e^4$ glue along $\partial e^4 = S^3 \longrightarrow \bigvee_{i=1}^{b} S^2$.

<u>AIM</u>: Show the homotopy class of $S^3 \longrightarrow \bigvee_{i=1}^{b} S^2$ determines Q_X. (moreso converse)

Consider $S^2 = \mathbb{P}^1 \subset \mathbb{P}^\infty_{\mathbb{C}}$, $\vee S^2 \subseteq \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$

$K(\mathbb{Z}, 2)$, $\pi_2(\mathbb{P}^\infty) = \mathbb{Z}$

$\pi_i(\mathbb{P}^\infty) = 0$, $i > 0$, $i \neq 2$

$$\pi_4((\mathbb{P}^\infty)^b, \vee S^2) \xrightarrow{\sim} \pi_3(\vee S^2)$$

LES of htpy groups

$$\tilde{\varphi}\left(D^4 \to (\mathbb{P}^\infty)^b,\ \tilde{\varphi}|_{\partial D^4} = \varphi\right) \longleftarrow\!\shortmid\ \varphi(S^3 \to \vee S^2)$$

Htpy exact sequence <u>also</u> says $\pi_i((\mathbb{P}^\infty)^b, \vee S^2) = 0$ for $i = 2, 3$.

So by Hurewicz

$$\pi_4((\mathbb{P}^\infty)^b, \vee S^2) \cong H_4((\mathbb{P}^\infty)^b, \vee S^2)$$
$$\underset{MV}{\cong} H_4((\mathbb{P}^\infty)^b)$$

(moreso LES)

So
$$\begin{tikzcd} D^4 \arrow[r, "\tilde{\varphi}"] & (\mathbb{P}^\infty)^b \\ \partial D^4 \arrow[u, phantom, "\cup"] \arrow[r] & \vee S^4 \arrow[u, phantom, "\cup"] \end{tikzcd}$$

is completely determined by evaluating $\langle \alpha, \tilde{\varphi}_*[D^4] \rangle$ $\rightsquigarrow$ for $\alpha \in H^4((\mathbb{P}^\infty)^b)$.

If $\omega_i \in H^2((\mathbb{P}^\infty)^b)$ is the dual to the i^{th} 2-sphere $\mathbb{P}^1_i \subseteq (\mathbb{P}^\infty)^b$

then $\{\omega_i \cdot \omega_j\}$ form a basis for $H^4((\mathbb{P}^\infty)^b)$.

We had $\vee S^2 \longrightarrow (\mathbb{P}^\infty)^b$ & $\tilde{\varphi} : D^4 \longrightarrow (\mathbb{P}^\infty)^b$

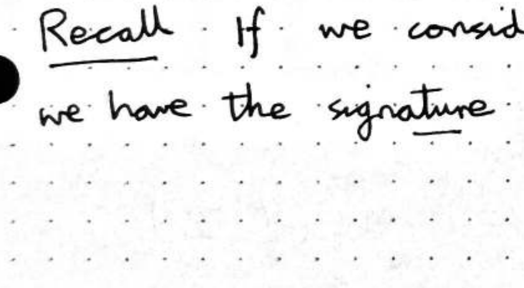
$\rightsquigarrow X \longrightarrow (\mathbb{P}^\infty)^b$ defined from $\vee f_i \cup \underbrace{\tilde{\varphi}}_{\substack{\text{extends my map} \\ \text{over 4-cell in } X}}$

$$\langle \omega_i \cdot \omega_j , \tilde{\varphi}_*[D^4] \rangle_{(\mathbb{P}^\infty)^b} = \langle \iota^*\omega_i \cdot \iota^*\omega_j , [X] \rangle_X$$
$$= Q_X(\underbrace{f_i, f_j}_{\text{in our basis for } H_2(X)}) \quad \square$$

Remark $\pi_3(\bigvee_{i=1}^{b} S^2) \cong \mathbb{Z}^{\frac{b(b+1)}{2}} = \{\text{symmetric matrices } b\times b \text{ over } \mathbb{Z}\}$

Recall If we consider $(H^2(X;\mathbb{R}), Q_X \otimes \mathbb{R})$ we have the signature $\sigma(X) = \underbrace{b_+}_{\substack{+\text{ve} \\ \text{eigenvalues of} \\ Q_X \otimes \mathbb{R}}} - b_-$

Proposition If oriented $X^4 = \partial W^5$ is the boundary of an oriented compact 5-manifold, then $\sigma(X) = 0$.



Proof P. Duality (for manifolds w/ boundary) says
$$H^*(W, \partial W) \xrightarrow{\sim} H_{n-*}(W)$$
$$H^*(W) \xrightarrow[\cdot \frown [W, \partial W]]{\sim} H_{n-*}(W, \partial W)$$

Here $[W, \partial W] \in H_n(W, \partial W) \cong \mathbb{Z}$
$$\begin{tikzcd} {[W,\partial W]} \arrow[d, mapsto] & H_n(W,\partial W) \arrow[d] \\ {[\partial W]} & H_{n-1}(\partial W) \end{tikzcd}$$
$[\partial W] \in H_{n-1}(\partial W) \cong \mathbb{Z}$

Write $i : \partial W \longrightarrow W$

If $\alpha, \beta \in H^2(W)$, $Q_{\partial W}(i^*\alpha, i^*\beta) = \langle i^*\alpha \cup i^*\beta, [\partial W] \rangle$
$= \langle \alpha \cup \beta, [i_* \partial W] \rangle = 0$

$$H_n(W, \partial W) \longrightarrow H_{n-1}(\partial W) \xrightarrow{i_*} H_{n-1}(W)$$

($i_*[\partial W] = 0$ as $[\partial W]$ comes from LHS in exact seq)

Now consider the diagram

$$\begin{tikzcd}
H^2(W) \arrow[r, "i^*"] \arrow[d, phantom, "\cong"] & H^2(\partial W) \arrow[r, "\delta^*"'] \arrow[d, phantom, "\cong"] & H^3(W, \partial W) \arrow[d, phantom, "\cong"] \\
H_3(W, \partial W) \arrow[r, "\delta"'] & H_2(\partial W) \arrow[r, "i_*"'] & H_2(W)
\end{tikzcd}$$

The maps δ^* & δ are adjoint

Rank (cok δ) = rank (ker δ^*)
= rank (im i^*)
= rank (im δ)

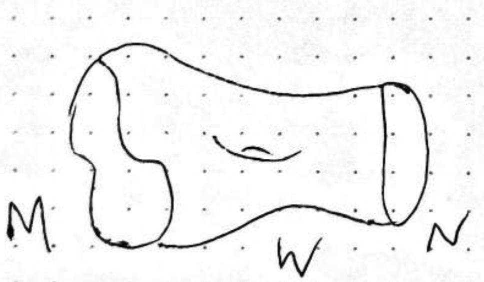
So im(i^*) has half the rank of $H^2(\partial W)$

Now $H^2(\partial W; \mathbb{R})$ has a half-dim isotropic subspace

$\Longrightarrow$ signature of ∂W must vanish (linear algebra) $\square$

Recall manifolds M^n, N^n (oriented) are cobordant if $\exists$ oriented W^{n+1} s.t. $\partial W^{n+1} = M \amalg \overline{N}$

└ opposite orientation



Let Ω^4 = 4-dim oriented manifolds up to oriented cobordism, a group under $\amalg$.

The previous result $\wedge$ says that $\sigma : \Omega^4 \longrightarrow \mathbb{Z}$.
$\ulcorner$and $\sigma(M_1 \amalg M_2) = \sigma(M_1) + \sigma(M_2)\lrcorner$

Deeper fact (which we won't prove)

$$\sigma : \Omega^4 \xrightarrow{\sim} \mathbb{Z} \qquad (!!!)$$

i.e. if $\sigma(X) = 0$ then $\exists$ oriented W^5 s.t. $X = \partial W^5$

Corollary $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$, ... are not oriented boundaries

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Crash Course on Characteristic Classes

Let E be a complex v. bundle over a space X (X compact Hausdorff)

Suppose E has rank k (fibres $\mathbb{C}^k$)

Then $\exists$ a map $X \xrightarrow[\phi]{} Gr(k; \mathbb{C}^\infty)$ s.t. $E = \phi^* E_{taut}$

tautological bundle

The classifying map φ is well-defined up to homotopy.

Fact $H^*(Gr(k, \mathbb{C}^\infty); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k]$, $c_i \in H^{2i}$

So we get elements $\phi^* c_i =: c_i(E) \in H^{2i}(X; \mathbb{Z})$,

called Chern classes of E.

Notation: $c_0(E) = 1 \in H^0(X; \mathbb{Z})$

total Chern class

$$c(E) = c_0(E) + c_1(E) + \dots + c_k(E) \in H^*(X; \mathbb{Z})$$

Characterising properties:

(a) (Naturality) If $f : X \to Y$ is a map & $E \to Y$,
then $c_i(f^*E) = f^* c_i(E)$

(b) (Dual) $c_i(E^*) = (-1)^i c_i(E)$

(c) (Sum) $c(E \oplus F) = c(E) \cdot c(F)$

(d) (Normalisation) $c(T\mathbb{CP}^n) = (1+h)^{n+1}$
where $h = PD[\mathbb{P}^{n-1}] \in H^2(\mathbb{P}^n; \mathbb{Z})$

Definition

If $E \to X$ is a real v. bundle, then

$$\underset{\text{real rank } k}{p_i(E)} = (-1)^i c_{2i}(\underset{\text{complex rank } k}{E \otimes \mathbb{C}}) \in H^{4i}(X;\mathbb{Z})$$

is called the i^{th} Pontrjagin class of E.

Remark If a complex bundle is the complexification of a real bundle, then its odd Chern classes vanish

So on a smooth 4-manifold we have

$$p_1(TX) \in H^4(X;\mathbb{Z})$$

Theorem $p_1(TX) = p_1(X) = 3\sigma$ (Hirzebruch's signature theorem)

Lattices

Recall: for a cx v. bundle $E \to X$, $c_i(E) \in H^{2i}(X;\mathbb{Z})$

real v. bundle $E \to X$;

$$p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X;\mathbb{Z})$$

Theorem If X is a smooth closed oriented 4-mfd,

$$\begin{array}{l} p_1(TX) = 3\sigma(X) \\ \rotatebox[origin=c]{90}{$\in$} \\ H^4(X;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \end{array}$$

Sketch proof Since $H^4(X;\mathbb{Z})$ is torsion-free, it suffices to understand $p_1(TX) \in H^4(X;\mathbb{R})$.

Chern-Weil theory gives differential forms representing real characteristic classes in terms of the curvature of a connection on E:

Here we pick a connection A on TX,

with curvature $F_A \in \Omega^2(\operatorname{End} TX)$

locally a matrix of 2-forms

We consider $F_A \wedge F_A$ a matrix of 4-forms &

$$p_1(TX) = c \int_X \operatorname{tr}(F_A \wedge F_A), \quad c \in \mathbb{R} \ \left(-\frac{1}{8\pi^2}\right)$$

If $X = \partial W^5$, then by Stokes

$$\int_X \alpha = \int_W \cancel{X}\, d\alpha \quad \text{for any } \alpha \in \Omega^4(X)$$

exterior derivative $D_{A^* \otimes A}$ on End TX

But $d(\operatorname{tr}(F_A \wedge F_A)) = \operatorname{tr}(DF_A \wedge F_A + F_A \wedge DF_A)$

So if $X = \partial W$, then $p_1(TX) = 0$, so

$p_1 : \Omega^4 \to \mathbb{Z}$ is a hom

$\uparrow$ 4-dim mfds up to cobordism

But $\sigma : \Omega^4 \xrightarrow{\sim} \mathbb{Z}$ (from before)

so $p_1 = \lambda \sigma$ for some universal $\lambda \in \mathbb{Q}$

Example

Take $\mathbb{CP}^2$, then $\sigma(\mathbb{CP}^2) = 1$ $\left(Q_{\mathbb{CP}^2} = (1) \right)$

If X is a complex surface, $TX \otimes \mathbb{C} \cong T^{hol}X \oplus \underbrace{\overline{T^{hol}(X)}}_{\cong\, T^{hol}(X)^*}$

by choosing Hermitian metric

So if $c(T^{hol}X) = 1 + c_1(X) + c_2(X)$

then
$$c(TX \otimes \mathbb{C}) = (1 + c_1(X) + c_2(X)) \cdot (1 - c_1(X) + c_2(X))$$
$$\Rightarrow p_1(X) = -2c_2(X) + c_1^2(X)$$

So $p_1(X) = -2c_2(X) + c_1(X)^2 \in H^4(X;\mathbb{Z})$

$c(T^{hol}\mathbb{P}^2) = (1+h)^3 = 1 + 3h + 3h^2 \in H^*(\mathbb{P}^2)$

so $c_2 = 3$, $c_1(\mathbb{P}^2)^2 = (3h)^2 = 9$

$\Rightarrow p_1 = 3 = 3\sigma$ so our universal constant was 3. □

<u>Digression</u> S^4 admits no (almost) complex structure

o/w $p_1(TS^4) = 3\sigma = 0$

$p_1 = -2c_2 + c_1(\cancel{S^4})^2 = -2\chi(X)$ ⌜for any cx surface $c_2(X) = \chi(X)$ ⌟

$\in H^2(S^4)$

$\divideontimes$ to $\chi(S^4) = 2$

Next: discuss invariants of intersection forms

We have rank & signature, invariants over $\mathbb{R}$

<u>Definition</u> An integral quadratic form (symmetric matrix over $\mathbb{Z}$ with non-zero determinant) is <u>indefinite</u> if the mx has +ve, and -ve, eigenvalues & <u>definite</u> o/w

An integral q.f. is called <u>even</u> if $Q(\alpha,\alpha) \in 2\mathbb{Z}$ $\forall \alpha \in \Lambda = \mathbb{Z}^2$ & <u>odd</u> o/w

Even/odd is called the <u>parity</u> of Q.

<u>Remark</u> For us, P. Duality says Q_X has determinant ± 1, so we're interested in unimodular forms.

Examples

1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the hyperbolic form U, realised by Q_X for $X = S^2 \times S^2$

2. Diagonal forms $b^+(1) \oplus b^-(-1)$

$$\underbrace{(1) \oplus (1) \oplus \cdots \oplus (1)}_{b^+ \text{ copies}} \qquad \underbrace{(-1) \oplus (-1) \oplus \cdots \oplus (-1)}_{b^- \text{ copies}}$$

realised as Q_X for $X = \mathop{\#}\limits_{j=1}^{b^+} \mathbb{P}^2 \# \mathop{\#}\limits_{j=1}^{b^-} \overline{\mathbb{P}^2}$

3. Define the E_8-lattice to be

$$\left\{ a \in \mathbb{R}^8 \;\middle|\; \begin{matrix} a_i \in \mathbb{Z} \ \forall i \\ \text{or} \\ a_i \in \mathbb{Z} + \tfrac{1}{2} \ \forall i \end{matrix} \quad \underline{\text{and}} \quad \sum_{i=1}^{8} a_i \in 2\mathbb{Z} \right\}$$

& E_8 quadratic form is just the restriction of Euclidean inner product to this lattice.

In the basis

$$e_1 + e_2,\ e_2 + e_3, \ldots, e_7 + e_8,\ \tfrac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8)$$

the E_8 q.f. has matrix

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & 0 & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & 0 & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\triangle$ 2

is the Cartan mx of E_8

Diagonal entries even $\Rightarrow$ E_8 even

Being a Cartan mx shows E_8 is definite

Remarks (i) If I look at $\underset{e_1, \ldots, e_8}{8(1)} \oplus \underset{e_0}{(-1)} \ni e_1 + \cdots + e_8 + 3e_0$

then $\underset{Q(\kappa,\kappa)}{\kappa \cdot \kappa} = -1$

Then $\underset{\underset{E_8}{\cong}}{\langle \kappa \rangle^{\perp}} \subseteq 8(1) \oplus (-1)$

(2) $\forall n \in 2\mathbb{Z}$, $\exists$ lattice $E_n \subseteq \mathbb{R}^n$, spanned by

$$\{ x \in \mathbb{Z}^n \mid \Sigma x_i \in 2\mathbb{Z} \} \cup \{ (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \}$$

& $\langle \, , \rangle_{\text{Eucl}} \big|_{\text{lattice}}$ is always unimodular, integral $4 \mid n$

even $8 \mid n$

<u>Definition</u> Let (Λ, Q) be a lattice. We say $w \in \Lambda$ is <u>characteristic</u> if $w \cdot \alpha = \alpha \cdot \alpha \mod 2 \quad \forall \alpha \in \Lambda$

<u>Example</u> (Λ, Q) is even $\Leftrightarrow$ 0 is a characteristic vector

The map $\alpha \mapsto \alpha \cdot \alpha \pmod 2$ is linear for any integral q.f. So characteristic vectors exist and are "unique mod 2"

i.e. w, w' both characteristic $\Rightarrow$ $w' = w + 2\alpha$ some $\alpha \in \Lambda$

$$\begin{aligned} w' \cdot w' &= w \cdot w + 4 w \cdot \alpha + 4 \alpha \cdot \alpha \\ &= w \cdot w + 8 \alpha \cdot \alpha \mod 8 \\ &= w \cdot w \mod 8 \end{aligned}$$

<u>Lemma</u> If w is characteristic, then $w \cdot w = \sigma(Q) \mod 8$

So if Q is even, then $8 \mid \sigma(Q)$.

We will deduce this from ~~[illegible]~~

<u>Lemma</u> If Q, Q' are integral unimodular q.f. which are indefinite with the same rank, signature & parity, then Q, Q' are equivalent (similar over $\mathbb{Z}$)

<u>Corollary</u> The indefinite unimodular forms are

$Q \sim b^+(1) \oplus b^-(-1)$ if odd $(b_1, b_2 \neq 0)$

or $Q \sim \frac{\sigma}{8} E_8 \oplus \left(\frac{rk - |\sigma|}{2} \right) U$ if even

<u>Pf of Cor</u> These realise all possible rank, signature & parity, subject to $8 \mid \sigma(Q)$ if Q is even, $|\sigma| < r$, & $r \equiv \sigma \mod 2$ $\square$

Proof of Lemma 2 uses the following 3 facts

(i) If $0 \neq Q$ is indefinite, then it has an isotropic vector
$\exists\, 0 \neq \alpha$ s.t. $\alpha \cdot \alpha = 0$

(ii) If $Q = n(1) \oplus m(-1)$ diagonal indefinite $(n, m \neq 0)$
& x, y are primitive with $x \cdot x = y \cdot y$, then
$\exists$ an auto^{m} of (Λ, Q) taking x to y provided
both characteristic or both non-characteristic

(iii) If $U \oplus (-1) \simeq (1) \oplus 2(-1)$ $\ulcorner \mathbb{P}^2 \# 2\overline{\mathbb{P}^2} \cong (\mathbb{P}^1 \times \mathbb{P}^1) \# \overline{\mathbb{P}^2} \lrcorner$

Remark (iii) easy, (ii) is direct argument if $n = m = 1$ in $GL(2, \mathbb{Z})$ & induction on rank

(i) uses Hasse-Minkowski results in number theory

$\ulcorner \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \lrcorner$

Erratum : The basis for E_8 was

$$e_1+e_2,\ e_2+e_3,\ \dots,\ e_6+e_7,\ e_7-e_8,$$
$$\tfrac{1}{2}(e_1-e_2+e_3-e_4+e_5+e_6-e_7-e_8)$$

(De minimis non curat lex)

L4) More lattices

Recall Lemma If Q, Q' are indefinite (unimodular q.f. $/\mathbb{Z}$) with same rank, signature & parity then $Q \sim Q'$.

Proof : Case (i) $\sigma(Q) = 0$

Pick primitive isotropic vector x for Q, & (unimodularity) pick y s.t. $x \cdot y = 1$

$\{x, y\}$ span a copy of $\overset{②}{U}$, so our lattice $\Lambda \simeq U \oplus U^{\perp}$

By induction on rank, I can assume result holds for $U^{\perp}$

Finally, using $U \oplus (-1) \simeq 2(1) \oplus (-1)$,

I see $Q \sim kU$ (even) or $Q \sim l(1) \oplus l(-1)$ if odd

Case (ii) $\sigma(Q) \neq 0$ Swapping Q for $-Q$ if necessary, just consider the case $\sigma(Q) > 0$

Induct on σ, & assume that

$$Q \oplus \underset{\langle x\rangle}{\overset{\shortparallel}{(-1)}} \simeq Q' \oplus \underset{\langle y\rangle}{\overset{\shortparallel}{(-1)}} \underset{\text{induct on }\sigma}{\simeq} \overbrace{p(1) \oplus q(-1)}^{\tilde{\Lambda}} \qquad p \geq q$$

$Q = \langle x\rangle^{\perp} \subseteq \tilde{\Lambda}$
$Q' = \langle y\rangle^{\perp} \subseteq \tilde{\Lambda}$ $\quad$ so if $\exists\, \phi : \tilde{\Lambda} \to \tilde{\Lambda}$ auto^{m} taking x to y, done

$x \cdot x = -1 = y \cdot y \implies x, y$ primitive

But x is characteristic $\iff Q$ even & y char $\iff Q'$ even

$x \cdot \alpha = \alpha \cdot \alpha \pmod 2\ \forall \alpha$

So "Fact ii" last time $\implies \phi$ exists. $\square$

sneaky!

Lemma 2 If w is char. for Q, $w \cdot w = \sigma(Q) \bmod 8$

Proof If w is char for Q, & $\tilde{Q} = Q \oplus \underset{\langle e_+ \rangle}{(1)} \oplus \underset{\langle e_- \rangle}{(-1)}$

then ~~W=~~ $w + e_+ + e_-$ is char for $\tilde{Q}$.

Then by classification

$$\tilde{Q} \simeq (b_Q^+ + 1)(1) \oplus (b_Q^- + 1)(-1) \quad \text{(since odd)}$$

In "diagonal" basis for $\tilde{Q}$, any char vector has odd entries

Now $Q(w,w) = \tilde{Q}(w + e_+ + e_-, w + e_+ + e_-)$

$= (b^+ + 1) - (b^- + 1) \bmod 8$

$= \sigma(Q) \bmod 8$ & we're done. □

Cor Indefinite intersection forms are

$$\begin{cases} a(1) \oplus b(-1) & \text{(odd)} \\ aE_8 \oplus b\mathcal{U} & \text{(even)} \end{cases}$$ □

Definite forms

Let (Λ, Q) be a definite quadratic form $/\mathbb{Z}$.

Wlog $\sigma(Q) > 0$ $(= \text{rank } Q)$

So over $\mathbb{R}$, $Q \sim_{\mathbb{R}}$ (usual Euclidean inner product)

Definition The theta function of (Λ, Q) is

$$\Theta_\Lambda(q) = \sum_{x \in \Lambda} q^{\frac{x \cdot x}{2}}$$

This is a power series in q with non-negative half-integer exponents.

It records how many vectors of a given squared length there are.

If $q = e^{2\pi i \tau}$ then

$$\Theta_\Lambda(\tau) = \sum_x e^{i\pi\tau(x \cdot x)}$$

& using ~~Θ~~ $\Theta_\Lambda(q) \sim (\deg \frac{n}{2} \text{ poly})$, $\Theta_\Lambda(\tau)$ converges uniformly on compact subsets of $\mathfrak{h} = \{\operatorname{Im} \tau > 0\}$ so defines a holomorphic function.

Example $\theta_\Lambda(\tau) = 1 + 2(e^{i\pi\tau} + e^{4i\pi\tau} + e^{\underset{9}{\cancel{\ }}\pi i\tau} + \cdots)$
for $\Lambda = \mathbb{Z}$

Lemma (i) $\theta_\Lambda(\tau) = \theta_\Lambda(\tau+2)$ — Q is $\mathbb{Z}$-valued

(ii) $\theta_\Lambda(-\frac{1}{\tau}) = (\frac{\tau}{i})^{n/2}\,\theta_\Lambda(\tau)$ — more delicate

Corollary If Q even, $\theta_\Lambda(1+\tau) = \theta_\Lambda(\tau)$

$$\theta_\Lambda(-\tfrac{1}{\tau}) = \tau^{n/2}\,\theta_\Lambda(\tau)$$

⌜$8|\sigma$ iii⌟

i.e. θ_Λ is a modular form of weight $\frac{n}{2}$

Note $\theta_\Lambda(q) = 1 + \cdots$

so if Λ, Λ' have same rank, $\theta_\Lambda - \theta_{\Lambda'}$ is a "cusp form"
(modular form vanishing at ∞)

There are no cusp forms of weight < 12

Corollary For $E_8 \oplus E_8$, E_{16}

must have $\theta_{E_8 \oplus E_8} = \theta_{E_{16}}$

$\Rightarrow \mathbb{R}^{16}/_{E_8 \oplus E_8}$ & $\mathbb{R}^{16}/_{E_{16}}$ are "isospectral but not isometric" — Lattices not iso^m

same length spectrum for closed geodesics

(c.f. can you hear the shape of a drum)

Siegel mass formula

For each $8|n$, $\exists$ only finitely many even definite unimodular lattices of rank n; &

$$\sum_{[\Lambda]\,\text{iso}^n\text{classes}} \frac{1}{|\mathrm{Aut}(\Lambda)|}\,\theta_\Lambda = c_{n/2}\,(\mathrm{Eis})_{n/2}$$

$c_{n/2} \in \mathbb{Q}$ (known constants)

$$(\mathrm{Eis})_{n/2} = \sum_{(a,b)\in\mathbb{Z}^2\setminus 0} \left(\frac{1}{a+b\tau}\right)^{n/2}$$

Studying growth of $c_{n/2}$ & bounds on $\mathrm{Aut}(\Lambda)$ shows

Dimension	8	16	24	32	40
# Lattices / iso	1 (E_8)	2 ($E_8 \oplus E_8$, E_{16})	24 (Niemeier tallices)	~80 million	$> 10^{51}$

Back to 4-manifolds

The theory of smooth 4-manifolds was really launched by the following theorems.

Theorem (Rokhlin, 1952)

Let X be a compact 1-connected 4-manifold, smooth. If Q_X is even then $16 \mid \sigma(X)$.

Theorem (Donaldson, 1983) (*)

Let X be a smooth compact 4-manifold with definite intersection form. Then Q_X is diagonalisable so

$$Q_X \sim a(1) \quad \text{or} \quad a(-1)$$

Theorem (Freedman, 1982)

For every unimodular q.f. $/\mathbb{Z}$ there's a topological 1-connected 4-manifold (compact) realising that int^n form. If Q is even then X is unique up to homeo. If Q is odd $\exists$ 2 homeo types, of which at most one is smoothable.

Corollary of Freedman

If X is smooth 1-connected compact 4-mfd then the homeo type of X is determined by Q_X.

Examples There is a topological mfd with $Q = E_8$, not smooth by Rokhlin. $\exists$ with $Q = E_8 \oplus E_8$, not smooth by Donaldson.

Proof of Lemma **Claim** For a unimodular lattice,

$$\vartheta_\Lambda\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{n/2} \vartheta_\Lambda(\tau)$$

$$\forall \tau \in \mathfrak{h},\ \Lambda \text{ definite}$$

Poisson summation formula (Fourier theory)

If $f \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing, & $\hat{f}$ is its Fourier transform, then

$$\det \Lambda \left(\sum_{v \in \Lambda} f(v) \right) = \sum_{\hat{v} \in \Lambda^*} \hat{f}(\hat{v}) \qquad (+)$$

If $f = e^{-\pi |x|^2}$, $\hat{f} = e^{-\pi |y|^2}$

Our Λ unimodular so $\det \Lambda = 1$

Rescaled lattice $t^{1/2}\Lambda$ has dual $t^{-1/2}\Lambda^* = t^{-1/2}\Lambda$

The identity $(+)$ becomes

$$t^{\frac{n}{2}} \underbrace{\theta_\Lambda(it)}_{} = \theta_\Lambda\left(-\frac{1}{it}\right)$$

det of rescaled Lattice

$$\sum_{v \in \Lambda} e^{i\pi i t\, v \cdot v} = \sum_{v \in \Lambda} e^{-\pi t |v|^2} = \sum_{v \in t^{1/2}\Lambda} e^{-\pi |v|^2}$$

If $\tau = it$, this proves claim holds on $i\mathbb{R}_+ \subset h$

But both sides are holo^c, so holds everywhere. □

We deduce $(*)$ from

Prop^n (Elkies) Let Λ be a unimodular integral lattice in $\mathbb{R}^n$ with no characteristic element w s.t. $w \cdot w < n$.
Then $\Lambda \cong (\mathbb{Z}^n, \langle \cdot \rangle_{Eucl})$

Recall: Elkies' Theorem:

Let Λ be a unimodular lattice in $\mathbb{R}^n$ with no characteristic vector w s.t. $w \cdot w < n$.

Then $\Lambda \; (\cong \mathbb{Z}^n, \langle , \rangle_{eucl})$

Proof (sketch)

Aim: Prove that $\theta_\Lambda = \theta_{\mathbb{Z}^n} \Rightarrow \forall L$, Λ & $\mathbb{Z}^n$ have same no. of vectors of square length L.

Taking $L=1$, $\exists$ n pairs $\pm e_j$, $1 \leq j \leq n$ of unit vectors for Λ

Note $e_i \cdot e_j < 1$ $(|e_i||e_j| \cos\theta)$ & Λ is integral, so e_i, e_j orthogonal when $i \neq j$

$\Rightarrow \mathbb{Z}^n \hookrightarrow \Lambda$ & both unimodular, so an iso^m

For any Λ_1, Λ_2, $\theta_{\Lambda_1 \oplus \Lambda_2} = \theta_{\Lambda_1} \cdot \theta_{\Lambda_2}$

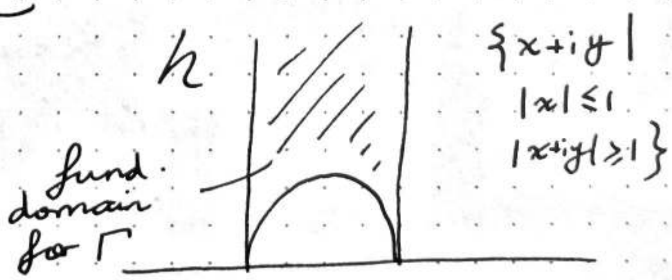
Recall $\theta_{\mathbb{Z}}(\tau) = 1 + 2(e^{i\pi\tau} + e^{4i\pi\tau} + e^{9i\pi\tau} + \cdots)$

Fact $\theta_{\mathbb{Z}} \to 1$ as $\tau \to i\infty$ & $\theta_{\mathbb{Z}}$ never vanishes in h (Jacobi triple product identity)

Let $R(\tau) = \dfrac{\theta_\Lambda(\tau)}{\theta_{\mathbb{Z}^n}(\tau)}$

If $S(\tau) = -\frac{1}{\tau}$, $T(\tau) = \tau + 1$, then θ_Λ & $\theta_{\mathbb{Z}^n}$ transform under same way for $\underbrace{\langle S, T^2 \rangle}_{\Gamma} \leq \langle S, T \rangle = SL(2, \mathbb{Z})$

Note R is invariant under Γ



For any lattice Λ,

$$\hat{\theta}_\Lambda(\tau) = \sum_{x \in \Lambda + \frac{w}{2}} e^{i\pi(x \cdot x)\tau}$$ where $w \in \Lambda$ is a fixed char vector

This is a generating function encoding the lengths of all the char. vectors in Λ.

<u>Example</u>: $\hat{\theta}_{\mathbb{Z}}(\tau) = 2\sum_{m\geq 0} e^{i\pi(m+\frac{1}{2})^2\tau}$

char vector are odd $\quad = 2e^{i\pi\tau/4}\left(1+e^{2\pi i\tau}+e^{6\pi i\tau}+e^{12\pi i t}+\cdots\right)$

$$\sim 2e^{i\pi\tau/4} \longrightarrow 0 \quad \text{as } \tau \to i\infty$$

Hypothesis on Λ $\Rightarrow$ $\hat{\theta}_\Lambda(\tau) \ll e^{i\pi n\tau/4}$ as $\tau \to i\infty$

$\Rightarrow \dfrac{\hat{\theta}_\Lambda}{\hat{\theta}_{\mathbb{Z}^n}}$ bounded as $\tau \to i\infty$ (shortest char vector has length$^2 < n$)

Poisson summation:

$$\theta_\Lambda\left(-\frac{1}{\tau}+1\right) = \left(\frac{\tau}{i}\right)^{n/2}\hat{\theta}_\Lambda(\tau)$$

True for any Λ

c.f. consider $t^{1/2}\Lambda$

$\Rightarrow R(\tau)$ bounded as $\tau \to 1$

$\Rightarrow R$ constant by Liouville

In fact equal by evaluating at $i\infty$. $\square$

<u>Example</u> $\nexists\ u : S^2 \hookrightarrow \mathbb{CP}^2$ smooth embedding with $u_*[S^2] = 3H \in H_2(\mathbb{CP}^2;\mathbb{Z})$

[A cubic curve has genus 1, which is therefore the <u>minimal genus</u> for this H_2-class]

<u>Proof of Example</u>

Recall the blowup: replace a point $p \in X^4$ by a $\mathbb{CP}^1$ with normal bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$; <u>note</u>

$$\partial(U(p)) = \partial(B^4) = \partial(\mathcal{O}_{\mathbb{P}^1}(-1)) = S^3$$

Topologically,

$$Bl_p(X) = X \# \overline{\mathbb{CP}^2}$$

Blow up $\mathbb{P}^2$ eight times $\rightsquigarrow X$ with intersection form

$$\underbrace{(1)\oplus 8(-1)} \cong -E_8 \oplus (1)$$

If $\exists\, u : S^2 \to \mathbb{P}^2$ in class $3H$, then $\exists\, S^2$ in $\mathbb{P}^2 \# 8\overline{\mathbb{P}^2}$ representing class $3H + \underbrace{E_1 + \cdots + E_8}_{\text{exceptional divisors of blowup, i.e. the } \mathbb{P}^1\text{'s}}$

$w \cdot w = 1$

$\Rightarrow$ locally X looks like $\mathcal{O}_{\mathbb{P}^1}(+1)$ near the S^2 representing w

$\Rightarrow X = X' \# \mathbb{P}^2$ for some X' (surgered out the $\mathcal{O}_{\mathbb{P}^1}(1)$)

(has intersection form $(-E_8)$

※ $16 \mid \sigma(X')$ by Rokhlin (noting $\pi_1(X') = \{1\}$)

Example #2

Consider $\mathbb{CP}^2$ blown up nine times $\mathbb{CP}^2 \# 9\overline{\mathbb{P}^2} =: E(1)$ "first elliptic surface"

with intersection form $(1) \oplus 9(-1) = -E_8 \oplus \underset{\alpha}{\underset{\cup}{(1)}} \oplus \underset{\gamma}{\underset{\cup}{(-1)}}$

Now $\nexists$ smooth X' s.t.

$E(1) \cong X' \# \mathbb{CP}^2$, with $H_2(\mathbb{P}^2) = \alpha$

If there was, $Q_{X'} = -E_8 \oplus (-1)$

odd, so Rokhlin says nothing

but it's excluded by Donaldson as not diagonal

But Freedman's corpus includes a "sphere embedding theorem" which says: if $\pi_1 X = 1$, $\alpha \in H_2(X;\mathbb{Z})$ & $\exists\, \beta \in H_2(X,\mathbb{Z})$

s.t. $\alpha \cdot \beta = 1$ & $\beta \cdot \beta \in 2\mathbb{Z}$,

then $\exists$ a "locally flat" embedded (topological) S^2 in α

So we take our class α, & note that $\beta := \alpha + \gamma$ is a suitable class for Freedman.

So $\exists\, S^2 \xhookrightarrow[\text{loc. flat}]{} E(1)$ representing α.

This has an open nghbd $\mathcal{U}$ homeo^c to the $+1$-disc bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ over S^2.

So $\exists$ a topological embedding

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook, "i", "C^0"'] & \mathbb{CP}^2 \\ S^2 \arrow[u, hook, "j"] & \end{tikzcd}$$

Consider $\mathbb{CP}^2 \setminus \underbrace{S^2}_{\text{image}(i \circ j)}$

This is homeo to $\mathbb{R}^4$; it's contractible & then Freedman shows it really is homeo^c to $\mathbb{R}^4$.

But not diffeo^c to $\mathbb{R}^4$. If it was, any compact subset of $\mathbb{CP}^2 \setminus S^2$ would be contained in a smooth S^3, e.g. $\mathbb{CP}^2 \setminus \mathcal{U}$.

This would transport to $E(1)$ & violate our previous result. □

Remark In $\mathbb{P}^2 \# 8\overline{\mathbb{P}^2}$ with $(1) \oplus 8(-1) = -E_8 \oplus \underset{\alpha}{\underset{\cup}{(1)}}$

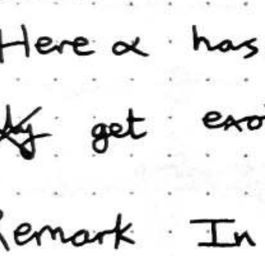
Here α has no dual class & one can't ~~only~~ get exotic $\mathbb{R}^4$ out of Rokhlin.

Remark In $\mathbb{CP}^2$, $3H$ can't be represented by an embedded S^2, BUT in $\mathbb{CP}^2 \# \mathbb{CP}^2$, the homology class $(3H, 0)$ is represented by an embedded S^2.

We have $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{CP}^2$

We plumb 3 fibres of the disc bundle

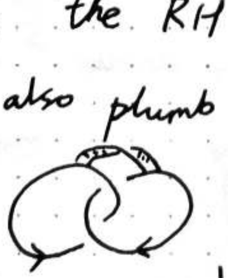
Picture in $S^3 = \partial \mathcal{U}$



After plumbing, this boundary is the RH trefoil knot

$1+1+1=3H$

I can also plumb 2 fibre discs to get a LH trefoil



has boundary a LH trefoil

$1-1=0$

Now I construct my copy of $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ by gluing two copies of $\mathcal{U}$ so as to identify the LH & RH trefoils in boundary S^3.

The discs with boundary the knots glue into an S^2 with class $(3H, 0)$

Related In $\mathbb{P}^2 \# \mathbb{P}^2$, the sphere I just built in $(3H, 0)$ & the usual line in class $(0, H)$ have intersection no. 0 but can't be disjointed by smooth isotopy

o/w could represent $(3H, H)$ by an embedded S^2

But this has square $10 \not\equiv \sigma(\mathbb{P}^2 \# \mathbb{P}^2)$ mod 16, & now could run an analogous argument & contradict Rokhlin.

K3 surfaces

Recall: a complex v. bundle $E \to X$ of rank k is classified by $X \longrightarrow Gr(k, \mathbb{C}^\infty)$ $\rightsquigarrow$ $c_i(E) \in H^{2i}(X;\mathbb{Z})$
Chern classes

A real v. bundle $E \to X$ of $\text{rank}_{\mathbb{R}}\, k$ is classified by
$X \to Gr(k, \mathbb{R}^\infty)$, $H^*(Gr(k,\mathbb{R}^\infty);\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\dots,w_k]$
$\rightsquigarrow$ $w_i(E) \in H^i(X;\mathbb{Z}/2)$ the Stiefel-Whitney classes of E

Properties (i) Let $w(E) = 1 + w_1(E) + \cdots + w_k(E)$
Then $w(E \oplus F) = w(E)\cdot w(F)$

(ii) If $E \to X$ is a cx v. bundle of rank k (so real rank $2k$)
then $c_i(E) \bmod 2 = w_{2i}(E) \in H^{2i}(X;\mathbb{Z}/2)$

(iii) $w_k(E) = e(E) \bmod 2$ if $k = \text{rank}_{\mathbb{R}} E$

(iv) $w_1(TX) = 0 \iff X$ orientable (Usually write $w_i(X)$ for $w_i(TX)$ if X smooth mfd)

Lemma If X is orientable closed 4-mfd,
then $w \in H^2(X;\mathbb{Z})$ is characteristic for Q_X
$\iff$ w reduces mod 2 to $w_2(TX)$

Pf Recall if $\alpha \in H^2(X;\mathbb{Z})$, $\exists$ embedded oriented surface $\Sigma \subseteq X$ representing $PD(\alpha) = A \in H_2(X;\mathbb{Z})$

$$TX|_\Sigma = T\Sigma \oplus \nu_{\Sigma/X} \Rightarrow w(TX|_\Sigma) = w(T\Sigma)\cdot w(\nu_{\Sigma/X})$$
$$= (1+w_2(T\Sigma))(1+w_2(\nu_{\Sigma/X}))$$

$$\Rightarrow w_2(TX)\cdot\alpha = w_2(T\Sigma)[\Sigma] + \underbrace{w_2(\nu_{\Sigma/X})[\Sigma]}_{\substack{e(\nu_{\Sigma/X}) = [\Sigma]\cdot[\Sigma] \pmod 2 \\ = \alpha\cdot\alpha \pmod 2}}$$

$$\underbrace{\overset{''}{A}}_{e(\Sigma) \equiv 0 \bmod 2}$$

$\square$

Now let $Z_d \subseteq \mathbb{CP}^3$ be the smooth 4-mfd which is the zero set of a homogeneous degree d polynomial vanishing transversely.
If $h = PD(\mathbb{P}^2) \in H^2(\mathbb{P}^3;\mathbb{Z})$, then $c_1(\nu_{Z_d/\mathbb{P}^3}) = dh$
$c(T\mathbb{P}^3) = (1+h)^4$ (normalisation axiom)

$c(T\mathbb{P}^3|_{Z_d}) = c(TZ_d)\, c(\nu_{Z_d/\mathbb{P}^3})$

$(1+h^*)^4|_{Z_d} = (1+c_1(Z_d)+c_2(Z_d))(1+dh)$

$\Rightarrow c_1(Z_d) = (4-d)h$

$c_2(Z_d) = (d^2-4d+6)h^2$

So $\chi(Z_d) = \langle c_2(Z_d), [Z_d]\rangle = d^3-4d^2+6d$

& $p_1(TZ_d) = -c_2(TZ_d \otimes \mathbb{C})$

$= -c_2(TZ_d \oplus (TZ_d)^*)$

$= c_1^2(TZ_d) - 2c_2(TZ_d)$

$\Rightarrow \sigma(Z_d) \underset{\text{(signature thm)}}{=} p_1/3 = \frac{d(4-d^2)}{3}$

Finally, Lefschetz hyperplane theorem $\Rightarrow \pi_1(Z_d) = \{1\}$

<u>Note</u> $c_1(Z_d) = (4-d)h$ vanishes mod 2 $\Leftrightarrow$ d even so $w_2(TZ_d) = 0 \Leftrightarrow d$ even so Q_{Z_d} is even iff d even

<u>$d=4$</u> $c_1 = 0$ ("Calabi-Yau")

$\chi(Z_4) = 24$, $\sigma(Z_4) = -16$, Q_{Z_4} is even & indefinite
$(b_2 = 22)$

$\underset{\text{(classification)}}{\Longrightarrow} Q_{Z_4} = 2(-E_8) \oplus 3U$

<u>Remark</u> Donaldson says definite Q_X are diagonalisable

The <u>$11/8$ conjecture</u> says that for an indefinite 4-manifold

$$\frac{\text{rank}(Q_X)}{|\text{signature}|} \geq \frac{11}{8}$$

If $2aE_8 \oplus bU$ is Q_X, conjecturally $\frac{16|a|+2b}{16|a|} \geq \frac{11}{8}$

i.e. $b \geq 3|a|$

For $K3$ we see the <u>simplest</u> Q_X compatible with $11/8$, remembering (for $\pi_1 X = 0$), $16 \mid \sigma(X)$ so the coeff of E_8 was even.

So Z_4 is the "simplest" 1-connected X^4 not a # of $S^2 \times S^2$, $\mathbb{C}P^2$ & $\overline{\mathbb{C}P^2}$.

Fact: If X is a smooth algebraic surface, & $b^1(X) = 0$, $c_1(X) = 0 \in H^2(X;\mathbb{Z})$ then $X \underset{C^\infty}{\cong} Z_4$. All such surfaces are called K3 surfaces.

Example Kummer K3's

Take $T^4/\pm I$. This has 16 singular points, locally modelled on $\mathbb{C}^2/\pm I$ which are cones on $\mathbb{R}P^3$.

There's a resolution of this space, which replaces ngbds of singular points by $\mathcal{O}_{\mathbb{P}^1}(-2)$ (or a disc bundle therein, also with $\partial = \mathbb{R}P^3$) $\ulcorner SO(3) \lrcorner$

(Blow up the singular points)

Easy check: $c_1(\mathcal{O}_{\mathbb{P}^1}(-2))$ & result of the blowup has $c_1 = 0$

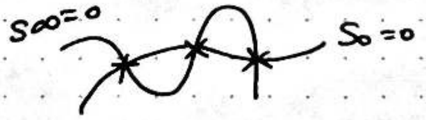
Also $H^1(T^4/\pm I) = 0$, so result of desingularisation is K3.

Note this K3 contains 16 $\mathbb{P}^1$'s with normal bundle $\mathcal{O}(-2)$

so $16(-2) \hookrightarrow 2(-E_8) \oplus 3\mathcal{U}$

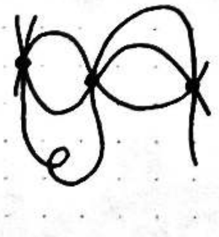
Example (Elliptic Surface)

Start in $\mathbb{C}P^2$ with a pencil of ~~conics~~ cubics i.e. sections $s_0, s_\infty \in \Gamma(\mathcal{O}(3))$

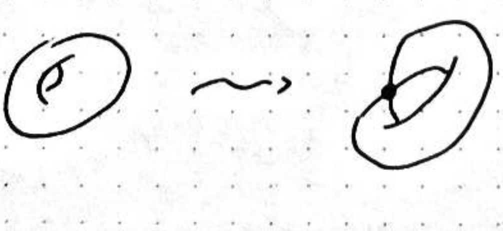


2 cubics intersecting in 9 points

Consider $\{\lambda s_0 + \mu s_\infty = 0 \mid [\lambda, \mu] \in \mathbb{P}^1\}$



Generically, $\exists$ 12 elements in this 1-parameter family which define cubics with a single node



(Lefschetz pencils)

We blow up the 9 <u>base-points</u> of this pencil ($s_0 = s_\infty = 0$)

This replaces B^4 by $\underbrace{\mathrm{Disc}(\mathcal{O}_{\mathbb{P}^1}(-1))}_{\partial = S^3}$

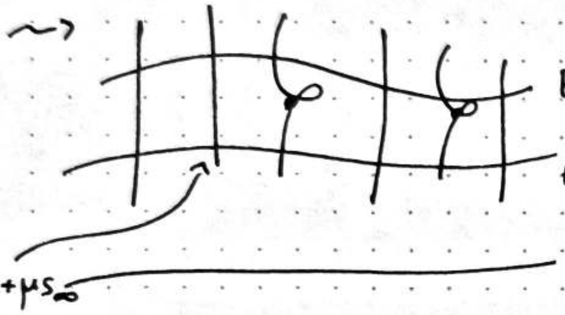
Locally, consider

$$\begin{tikzcd} & \mathbb{C}^2 \times \mathbb{P}^1 \supseteq \left\{ ((z,w),[\lambda,\mu]) \;\middle|\; \begin{array}{l}(z,w) \text{ lies on} \\ \text{line param}^d \text{ by} \\ [\lambda,\mu]\end{array} \right\} \arrow[dl, "\text{fibres points, } \exists\text{xcept over zero}"'] \arrow[dr, "\text{complex line param}^d \text{ by point}"] & \\ \mathbb{C}^2 & & \mathbb{P}^1 \end{tikzcd}$$

$\Rightarrow$ we have $\mathcal{O}_{\mathbb{P}^1}(-1)$

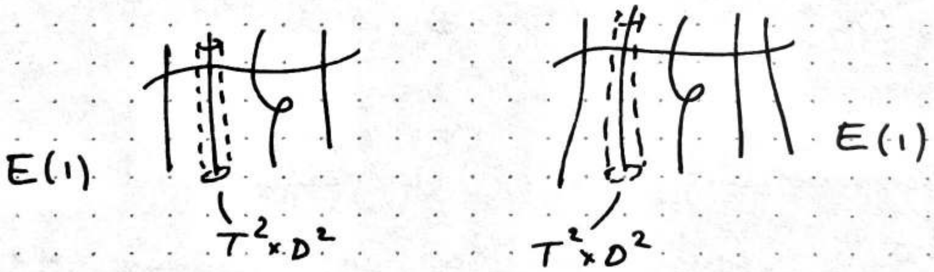
This separates all the cubics at each basepoint

$\rightsquigarrow$



This $E(1) \underset{C^\infty}{\cong} \mathbb{P}^2 \# 9\overline{\mathbb{P}^2}$

We now form $E(2) = E(1) \underset{\text{fibre}}{\#} E(1)$



$$\rightsquigarrow \underset{\substack{| \\ \text{24 singular nodal fibers}}}{E(2)} \longrightarrow \mathbb{P}^1$$

<u>Exercise</u> This fibre sum is also the 2 : 1 branched cover of $E(1)$, branched along 2[Torus fibre]

The $c_1(E(1)) = \underset{\substack{/ \\ c_1(\mathbb{P}^2)}}{3h} \underbrace{- E_1 \cdots - E_9}_{\text{exce curves}} = [\text{Torus fibre}]$

If $X \xrightarrow[d:1]{} Y \supseteq \Sigma$ is a branched cover of 4-mfds, branched over a smooth $\Sigma \subseteq Y$,

$$c_1(X) = d\left(f^* c_1(Y) - [\text{Ramification}]\right) \qquad \ulcorner d=2 \text{ at least} \lrcorner$$

$$\Rightarrow c_1(E(2)) = 0 \text{ so } E(2) \underset{C^\infty}{\cong} K3$$

The K3 surface is *hyperkähler*: it has 3 complex structures I, J, K satisfying the relations of the quaternions.

$(\text{c.f. } T^4/\pm I = (\mathbb{H}/\mathbb{Z}^4)/\pm I)$

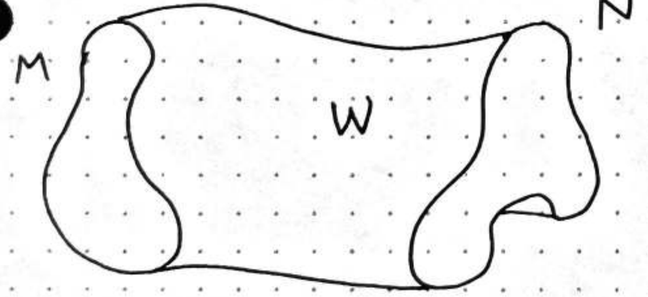
Definition A framing of a manifold is a trivialisation of its tangent bundle; a *stable framing* is a triv^n of $TM \oplus \mathbb{R}^k$ (view these as up to homotopy)

Example A Lie group G has a canonical framing

$$TG = G \times \mathfrak{g}$$

So $T(SU_2) = T(S^3)$ is canonically framed.

We say 2 (stably) framed M, N are *framed cobordant* if $\exists$ a cobordism W with TW trivial & trivialisation at ends $TW|_M = TM \oplus \mathbb{R}$ agrees w/ the given one.



Rokhlin

Recall : stably framed manifold M, trivialisation of $TM \oplus \mathbb{R}^k$

stably framed cobordism : $\exists$ a group Ω^{fr}_k of framed cobordism classes of k-dim^l mfds

Suppose $S^{n+k} \longrightarrow S^n$ & $y \in S^n$ is regular.

Then $f^{-1}(y)$ is a k-mfd with trivialised normal bundle

as $\nu_{f^{-1}(y)/S^{n+k}} = f^* \nu_{y/S^n}$

Viewing $f^{-1}(y) \subseteq \mathbb{R}^{n+k} \subset S^{n+k}$, then

$\nu_{f^{-1}(y)} \oplus Tf^{-1}(y) \cong T\mathbb{R}^{n+k}$ trivial, so we have stably framed the tangent bundle.

So $f : S^{n+k} \longrightarrow S^n$ gives an elt of Ω^{fr}_k (different regular values y_i lead to framed cobordant $f^{-1}(y_i)$) "yes! see Milnor"

But Freudenthal Suspension says that

$$\pi_{n+k}(S^n) \longrightarrow \pi_{n+k+1}(S^{n+1})$$

⌜ $\pi_k X \to \pi_{k+1}(\Sigma X)$ ⌟ limit called $\pi_k^{st} :=$ k^{th} stable hmpty

So $\exists$ maps $\pi_7(S^4) \longrightarrow \underbrace{\pi_8(S^5) \longrightarrow \cdots}_{\text{stable}}$

$$\pi_3^{st} \longrightarrow \Omega_3^{fr}$$

The quaternionic Hopf map

$$S^3 \longrightarrow S^7(=S(\mathbb{H}^2)) \longrightarrow S^4(=\mathbb{HP}^1)$$

c.f. $S^1 \to S^3 \to S^2$

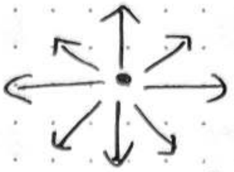
LES of htpy groups for this fibration shows

$$\begin{tikzcd}
\text{torsion} \arrow[r, "0"'] & \pi_7(S^7) \arrow[r] \arrow[d, equal] & \pi_7(S^4) \arrow[r] \arrow[d] & \pi_6(S^3) \arrow[r] & 0 \\
 & \mathbb{Z} & \pi_8(S^5) = \pi_3^{st} & &
\end{tikzcd}$$

Fact : $\pi_3^{st} = \mathbb{Z}/24$ generated by S^3 with the Lie group framing

Recall K3 surface has $\chi(K3) = 24$

So I can find a v.field Z on K3 with 24 simple zeroes at each of which Z looks like



(c.f. elliptic fib^n with 24 singular pts)

And K3 is Hyperkähler: take $\{Z, IZ, JZ, KZ\}$ frame $TK3$ on complement of 24 points

View $K3 \setminus 24$ open 4-balls as a null-cobordism of $\coprod_{i=1}^{24} S^3$

Then $\langle IZ, JZ, KZ \rangle$ frame each S^3 with the Lie group framing, so $24\,[S^3_{Lie}] = 0 \in \Omega_3^{fr}$

Background (a) If $O(n)$ is the orthogonal group,

$$\exists\ O(n) \to H(n) = \text{self-htpy equivalences of } S^n = \mathbb{R}^n \cup \{\infty\}$$

$$\to \Omega^n S^n$$

So $\exists\ \pi_k O(n) \longrightarrow \pi_k(\Omega^n S^n) = \pi_{k+n}(S^n)$

$\exists$

$$\begin{tikzcd} O(n) \arrow[r] \arrow[d] & H(n) \arrow[d] \\ O(n+1) \arrow[r] & H(n+1) \end{tikzcd}$$

$\leadsto \exists$ a map (J-homo^m)

$$\pi_i O \longrightarrow \lim_{n\to\infty} \pi_{i+n}(S^n) = \pi_i^{st}$$

$$O = \lim_{n\to\infty} O(n) \qquad\qquad = \Omega_i^{fr}$$

Classical $\pi_3 O = \mathbb{Z}$ & in fact $\mathbb{Z} \to \mathbb{Z}/24$ is onto

Background If M is a monoid (abelian), it has a Grothendieck group i.e. the abelian group "generated" by M

— universally: given a monoid hom $M \to A$ (abelian gp) $\exists$ homo^m of abelian groups $\text{Groth}(M) \to A$

$$\begin{tikzcd} \text{Groth}(M) \arrow[r] & A \\ M \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

— construction: take $M \times M$ & equivalence relation

$(m_1, m_2) \sim (n_1, n_2)$

iff $m_1 + n_2 + k = m_2 + n_1 + k$, some $k \in M$

Think of (m_1, m_2) as "$m_1 - m_2$"

If X is a space, $\exists$ monoid $\mathrm{Vect}(X)$ of $\underset{\text{real}}{\uparrow}$ v. bundles over X with $\oplus$ as addition.

$\mathrm{Groth}(\mathrm{Vect}(X), \oplus) =: KO(X)$ real K-theory

$\widetilde{KO}(X) = \ker(KO(X) \xrightarrow[\text{restr}]{} KO(\underset{\text{basepoint}}{\underset{\uparrow}{x_0}}))$, $KO(x_0) \cong \mathbb{Z}$

(reduced K-theory)

<u>Fact</u> (Bott periodicity) $\widetilde{KO}(S^4) = \mathbb{Z}$

A bundle on S^4 is trivial on both hemispheres & so is clutched by a map $S^3 \to GL(n, \mathbb{R})$ if rank $= n$

$\widetilde{KO}$ classifies bundles up to stable iso^m & so really

$\widetilde{KO}(S^4) = \pi_3(O)$

Back to Rokhlin's Theorem! [sic]

We stated this as: if X is 1-connected, & Q_X even then $16 \mid \sigma(X)$.

A better formulation involves <u>Spin</u> structures

<u>Recall</u> $SO(n)$ has $\pi_1 = \mathbb{Z}/2$ & a 2:1 cover $\mathrm{Spin}(n)$ $(n>2)$

A <u>spin structure</u> on a manifold is a lift of the $SO(n)$-frame bundle associated to TM to a spin bundle

<u>Lemma</u> An oriented manifold is spin $\iff$ $w_2(TX) = 0$

(a) If X^4 is a spin 4-mfd, then X is "almost framed", meaning $TX|_{X \setminus B^4}$ is trivialisable.

<u>Sketch</u> (·) Take a trivialising cover for the principal SO_n-frame bundle; has transition maps $t_{ij} : U_i \cap U_j \to SO(n)$

Lift these $\underset{\sim}{\text{arbitrarily}}$ to $\tilde{t}_{ij} : U_i \cap U_j \longrightarrow \mathrm{Spin}(n)$ [good cover]

$\tilde{t}_{ijk} := \tilde{t}_{ij}\, \tilde{t}_{jk}\, \tilde{t}_{ki} \in \check{C}^2(\mathcal{U}, \mathbb{Z}/2)$

Claim these satisfy the cocycle condition, so define a cohomology class, & this is $w_2(E)$

(a) TX is classified by $X \longrightarrow \mathrm{Gr}_{\substack{\cancel{E}\\4}}(\mathbb{R}^\infty)$

Oriented: this is trivial on 1-skeleton

Spin: " 2-skeleton

& $X \setminus B^4 \simeq$ (3d cell complex)

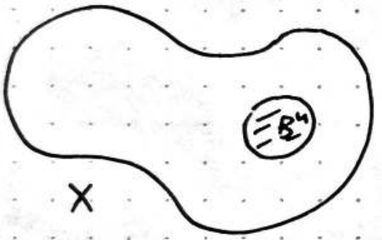
But $\pi_3(\mathrm{Gr}_4 \mathbb{R}^\infty) = \pi_2(O_4) = 0$. $\square$

<u>Theorem</u> (Rokhlin)

If X is an oriented Spin 4-mfd, then $16 \mid \sigma(X)$

(<u>Recall</u>: $w_2(TX)$ is characteristic for Q_X
so $w_2(TX) = 0$ says Q_X even)

<u>Proof</u> Take a ball $B^4 \subseteq X$ & collapse $X \setminus B^4$ to get a degree 1 map to S^4.



$\xrightarrow[p]{} S^4$

Since $TX|_{X \setminus B^4}$ is trivial, $\exists$ bundle $E \longrightarrow S^4$ s.t.

$$TX \cong p^* E$$

Now $p^*: H^4(S^4) \xrightarrow[\cong]{} H^4(X)$ so

$$\langle p_1(TX), [X] \rangle = \langle p_1(E), [S^4] \rangle$$

(Recall signature theorem says $p_1(TX) = 3\sigma(X)$)

But total Pontrjagin class

$$p(\tilde{E}) = 1 + p_1(\tilde{E}) + \cdots \in H^{4*}(Z) \quad \text{for } \tilde{E} \to Z$$

is multiplicative under $\oplus$, so

$$p_i(F \oplus F') = p_i(F) + p_i(F') + \underbrace{\text{l.o.t.}}_{\text{products of lower degree classes}}$$

$\Rightarrow p_1 : \widetilde{KO}(S^4) \longrightarrow \mathbb{Z}$ homo$^{\text{m}}$

$p_1(F) = \pm c_2(F_{\mathbb{C}})$ & recall $c_i(L) = w_{2i}(L_{\mathbb{R}})$ (mod 2)

So $p_1(F) = \pm W_4(F \oplus F)$ mod 2

$\underset{\substack{\text{product}\\ \text{formula}\\ \text{for}\\ \text{Stiefel}\\ \text{-Whitney}}}{=} \quad = 2\underbrace{\cancel{W_4(F)}}_{0} - \cancel{W_2(F)^2} \to 0$ for $H^2(S^4) = 0$

$\underline{\text{so}}$ $p_1 : \widetilde{KO}(S^4) \longrightarrow \mathbb{Z}$

$n \longmapsto \lambda n$ for λ even

$$\begin{tikzcd}
\widetilde{KO}(S^4) = \mathbb{Z} \arrow[r] \arrow[d, "p_1"] & \pi_3^{st} = \mathbb{Z}/24 = \langle S^3, \text{Lie group} \rangle \\
\mathbb{Z} &
\end{tikzcd}$$

mult^n by even $\to p_1$

$\underline{\text{But}}$: we have $E \to S^4$ s.t. $p^*E = TX$ which is $\underline{\text{trivialised}}$ on $X \setminus B^4$.

So E on $S^3 = \partial B^4$ $\cancel{X}$ does framed bound

So $24 \mid [E]$ & so $48 \mid p_1(E)$

& $p_1(E) = 3\sigma$

So $16 \mid \sigma(X)$. $\square$

Differential Operators

Having proved Rokhlin's theorem, our next goal is to prove Rokhlin's theorem.

Let M be a manifold, $E \to M$, $F \to M$ real / cx v. bundles;

we say $P : \underset{\text{global sections}}{\Gamma(E)} \to \Gamma(F)$ is a linear differential operator of order k

if in local co-ords on M,

$$Pu = \sum_{|I| \leq k} P^I(x) \partial_I u , \quad \partial_I = \partial_{i_1} \cdots \partial_{i_k} \quad I = \{ i_1, \dots, i_k \}$$

Then $P^I(x) : E_x \to F_x$ linear & symmetric in $\{ i_1, \dots, i_k \}$

The local expression is not really well-defined (if we change local coordinates) but the "top order" piece is well-defined.

Definition The symbol $\sigma(P)$ of P is the expression

$$\sigma(P)(x, \xi) = \sum_{|I| = k} P^I(x) \xi^I , \quad \xi^I = \xi_{i_1} \cdots \xi_{i_k}$$

Especially important:

Fix $\xi \in T_x^* M$ & evaluate $\sigma(P)(x, \underbrace{\xi, \dots, \xi}_{\text{all equal}})$

Another viewpoint:

Fix a locally defined function $f : M \to \mathbb{R}$, $df|_x = \xi \in T_x^* M$

If I consider $P \circ e^{itf}$ & use Leibniz rule,

I'll get e^{itf} (lots of terms involving $(it)^L$, $L \leq k$)

So $$e^{-itf} \circ P \circ e^{itf} \underset{\text{Leibniz}}{=} (it)^k \sum_{|I| = k} P^I(x) \cancel{x} (\partial_1^{i_1} f \cdots \partial_n^{i_n} f) + \text{l.o.t}$$

So $$\lim_{t \to \infty} (it)^{-k} (e^{-itf} \circ P \circ e^{itf}) \Big|_x = \sigma(P)(x, \xi)$$

If I write

$$e^{itf} = 1 + itf + \frac{(it)^2 f^2}{2} + \cdots \quad \text{I find}$$

$$e^{-itf} \circ P \circ e^{itf} = P + it[P,f] - \frac{t^2}{2}[[P,f],f] + \cdots$$

So if P is 1st order, $\sigma(P)(x,\xi) = [P,f]$ for f s.t. $df|_x = \xi$

Example Exterior differentiation $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

Then
$$[d,f]\omega = d(f\omega) - f\,d\omega$$
$$= df \wedge \omega$$

So $\sigma(d)(x,\xi) : \Lambda^k T^*_x \longrightarrow \Lambda^{k+1} T^*_x$ which has quite a big kernel
$\xi \wedge \cdot$

Similarly: if E is a bundle on M & A a connexion in E with covariant derivative d_A, then

$$\sigma(d_A)(x,\xi) : \Gamma(\Lambda^k T^*_x \otimes E_x) \rightarrow \Gamma(\Lambda^{k+1} T^*_x \otimes E_x)$$

is again $\xi \wedge \cdot$

Hodge theory in 10 minutes

Let (M,g) be oriented Riemannian, metric g, $\mathrm{vol}_g \in \Omega^n(T^*M)$

Define Hodge star $* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ by

$$\beta \wedge * \alpha = \underbrace{\langle \alpha, \beta \rangle}_{\text{inner product on forms induced by } g} \mathrm{vol}_g \qquad \forall\, \alpha, \beta \in \Omega^k$$

Check $(*)^2 = (-1)^{k(n-k)}$

Lemma Formal adjoint to $d : \Omega^k \rightarrow \Omega^{k+1}$ is

$$d^* = (-1)^{n(k+1)+1} * \circ\, d \circ *$$

Proof Want
$$\langle \alpha, d\beta \rangle_{L^2} = \int_M \langle \alpha, d\beta \rangle \mathrm{vol}_g$$
$$\parallel$$
$$\langle d^*\alpha, \beta \rangle_{L^2} \quad \text{for } \alpha \in \Omega^k, \beta \in \Omega^{k-1}$$

$\langle \alpha, d\beta \rangle_{L^2} = \int_M d\beta \wedge * \alpha$

& $\int_M d(\beta \wedge * \alpha) = 0$ by Stokes (assume M closed)

$\therefore \langle \alpha, d\beta \rangle_{L^2} = (-1)^k \int_M \beta \wedge d(*\alpha)$

$= (-1)^k (-1)^{(n-k)k^{k \text{ sws!}}} \int_M \beta \wedge * * d(*\alpha)$

$= \langle d^* \alpha, \beta \rangle_{L^2}$ & result will now follow $\square$

<u>Definition</u> The Laplacian $\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$

linear differential operator of order 2

<u>Definition</u> $\alpha \in \Omega^k(M)$ is <u>harmonic</u> if $\Delta \alpha = 0$

Write $\mathcal{H}^k(M)$ for the space of harmonic k-forms

<u>Hodge Theorem</u>: Harmonic forms are <u>closed</u>, &

$\mathcal{H}^k(M) \hookrightarrow \Omega^k(M)$ induces $\mathcal{H}^k(M) \xrightarrow{\cong} H^k(M; \mathbb{R})$

<u>Non-proof</u> Let $\cdots \to E^{i-1} \xrightarrow{d} E^i \xrightarrow{d} E^{i+1} \to \cdots$

be a chain cx (bdd) of finite dim Hilbert spaces

Write d^* = adjoint of d,

$\Delta = dd^* + d^*d$.

(i) $\langle \Delta\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle d^*\alpha, d^*\alpha \rangle$

$= \| d\alpha \|^2 + \| d^*\alpha \|^2$

$\Rightarrow \boxed{\ker \Delta = \ker(d) \cap \ker(d^*)}$

(ii) Δ is self-adjoint so E^k splits into orthogonal eigenspaces

i.e. $\ker \Delta$ & (rest)

$\Rightarrow \boxed{E^k = \ker(\Delta) \oplus \mathrm{im}(\Delta)}$

(iii) $\Delta \alpha = 0 \Rightarrow d\alpha = 0$

so $\langle \alpha, d^*\beta \rangle = \langle d\alpha, \beta \rangle = 0$

so $\ker \Delta \perp \mathrm{im}(d^*)$ & similarly $\perp \mathrm{im}(d)$

$\Rightarrow \boxed{E^k = d(E^{k-1}) \oplus d^*(E^{k+1}) \oplus \ker(\Delta)}$ orthogonal

(iv) $\mathrm{im}\,\Delta \subseteq \underbrace{d(E^{k-1}) + d^*(E^{k+1})}_{\in (\ker\Delta)^\perp} \subseteq \mathrm{im}\,\Delta$ by (ii), (iii)

$\Rightarrow \ker\Delta = \mathcal{H}^k \hookrightarrow E^k$ induces $\boxed{\mathcal{H}^k \xrightarrow{\cong} \left(\frac{\ker d|_{E^k}}{dE^{k-1}} \right)}$

i.e. Hodge theorem would be trivial if the $\Omega^i(M)$ were finite dim^L

The fact that they behave as if they were is because Δ is elliptic.

Definition Let P be a linear diff operator

Then P is elliptic if $\sigma(P)(x, \xi) : E_x \longrightarrow F_x$ is an isomorphism $\forall x, \forall \xi \neq 0$

Example (i) d isn't

(ii) Δ is ; note that on a Riemannian (M, g), M closed

$dd^* + d^*d$ is intrinsically defined

So its symbol $\in S^2(T^*_x)$ is invariant under the natural $O(n)$ - action. If V is the standard rep^n of $O(n)$, then $\mathrm{Sym}^2(V)$ has a ! invt vector, the metric

$\Rightarrow \sigma(\Delta) = g$ up to scalar

Explicitly $\sigma(\Delta)(\xi) = \cdot\, |\xi|_g^2$

If P is an <u>elliptic</u> operator $P : \Gamma(E) \to \Gamma(F)$

(i) P extends to suitable Sobolev completions

$$\underset{\substack{k \text{ derivatives} \\ \text{in } L^2}}{L^{2,k}}(E) \longrightarrow L^{2,k-l}(F) \quad , \quad l = \text{order}(P)$$

(ii) which satisfies the elliptic estimate

$$\|u\|_{L^{2,k}} \leq C\left(\|Pu\|_{L^{2,k-l}} + \|u\|_{L^{2,k-1}} \right)$$

Rellich lemma:

$L^{2,k} \hookrightarrow L^{2,k-1}$ is a "compact operator",

takes bounded sets to relatively compact sets

Suppose $(u_n) \subseteq L^{2,k}(E)$ all have norm 1, & $Pu_n = 0$

Then $\|u_n - u_m\|_{L^{2,k}} \leq C\left(\|u_n - u_m\|_{L^{2,k-1}} \right)$

& in $L^{2,k-1}$, $\{u_n\}$ lie in a rel cpct set, so

RHS has a convergent subsequence

=> original seq has a <u>Cauchy</u> subsequence

So unit ball in ker(P) is compact so ker P is <u>finite dimensional</u>.

In fact, if P is elliptic then formal adjoint P^* has $\sigma(P^*) = \pm\, \sigma(P)$ & so P^* is elliptic & P has f.d. kernel & $\underset{\substack{\uparrow \\ \ker(P^*)}}{\underline{\text{cokernel}}}$

<u>Definition</u> The <u>index</u> $\text{ind}(P) = \dim(\ker P) - \dim(\text{coker } P)$
(the "analytical index" a-ind(P))

The index theorem

Def^n If V_1, V_2 are Banach spaces & $P : V_1 \to V_2$ is a bounded linear operator, then P is Fredholm if it has finite-dim kernel & cokernel

The analytic index $\text{a-ind}(P) := \dim(\ker P) - \dim(\operatorname{coker} P)$

Lemma $\pi_0(\text{Fred}(V_1, V_2)) \xrightarrow{\text{a-ind}} \mathbb{Z}$ (in fact an iso^m)

Examples (i) Topelitz operators :

$$H(S^1) = \{ f \in L^2(S^1) \mid f = \sum_{n \geq 0} c_n z^n \text{ in } L^2 \}$$

If $g : S^1 \to \mathbb{C}$, there's an operator

$$T_g : f \mapsto \text{Proj}_H \circ \text{Mult}_g(f) = P_H(gf)$$

If $g : S^1 \to \mathbb{C}^*$, then T_g is Fredholm of index $-\deg(g)$.

Well $[S^1, \mathbb{C}^*] = \mathbb{Z}$ & index is htpy invariant, so take $g(z) = z^m$.

If $e_0, e_1, \ldots$ is the basis for H of monomials $e_i = z^i$, then T_g is a shift e.g. T_z has kernel 0

& cokernel $\langle 1 \rangle_{\mathbb{C}}$

(ii) Elliptic operators are Fredholm

The Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$

(or acting on some Sobolev completion)

is self-adjoint so $\operatorname{coker}(\Delta) = \ker(\Delta^*) = \ker(\Delta)$ so index 0.

But $\Omega^{ev}(M) \xrightarrow[d+d^*]{} \Omega^{odd}(M)$

$\sigma(d)(x, \xi) = \xi \wedge -$, $\xi \in T_x^*$

$\sigma(d^*)(x, \xi) = \iota_\xi(-)$, $\xi \in T_x$ metric dual

Then $\sigma(d + d^*)$ is invertible $\forall \xi \neq 0$ so $d + d^*$ is elliptic

& $\ker(d + d^*) = \ker d \cap \ker d^* = \ker \Delta$

Now $\text{a-ind}(d + d^*) = \chi(M)$ ☺

● How do we compute the index?

Natural setting is again K-theory

If X is compact, $K(X)$ = Grothendieck gp of monoid of complex vector bundles on X

If I have a bounded cx of vector bundles

$$E^0 \xrightarrow[d]{} E^1 \xrightarrow[d]{} E^2 \longrightarrow \cdots \longrightarrow E^k$$

I get an ellement $\sum (-1)^i [E^i] \in K(X)$, & chain homotopic complexes give the same element of $K(X)$.

●

If Y is locally compact (e.g. vector bundle over compact),

$$\underbrace{K(Y) := \widetilde{K}}_{\text{reduced K-theory}}(\underbrace{Y_+}_{\text{1-point compactification}})$$

If $E^\bullet \to Y$ is a bounded cx of v. bundles on Y, & exact away from a compact set, then $E^\bullet \in K(Y)$.

● Also note: If $\underset{\text{open}}{U \subseteq Y}$, $Y_+ \longrightarrow Y_+/_{Y_+ \setminus U} = U_+$

so pullback gives $K(U) \longrightarrow K(Y)$ ("extension by zero")

(ii) There's a K-theory Thom isoᵐ:

If $V \to M$ is a v. bundle "K-theory oriented" (incl complex) then $K(M) \xrightarrow{\cong} K(V)$

Suppose $P : \Gamma(E) \longrightarrow \Gamma(F)$ is an elliptic diff op.

● $E \to M$
$F \to M$ v. bundles over cpct mfd M
& supress Sobolev spinachi

If $\pi : T^*M \to M$ is projection,

$$\pi^* E \xrightarrow[\sigma(P)]{} \pi^* F$$

Recall $\sigma(P)(x,\xi) : E_x \to F_x$
linear

Ellipticity $\Rightarrow$ this complex is exact away from $M \subseteq T^*M$
0-section

Def^n The topological index of P is

$$t\text{-ind}(P) := [\pi^*E \xrightarrow[\sigma]{} \pi^*F] \in K(T^*M)$$

Theorem (Atiyah-Singer)

$$a\text{-ind}(P) = t\text{-ind}(P)$$ (for P elliptic, M compact)

Rmk We obtain an integer from t-ind(P) via:

Embed $M \hookrightarrow \mathbb{R}^N$, $TM \hookrightarrow T\mathbb{R}^N$

& $\nu_{TM/T\mathbb{R}^N} = \nu_{M/\mathbb{R}^N} \otimes \mathbb{C}$ is complex

Now
$$\begin{tikzcd} K(T^*M) \arrow[r, equal, "" ', "\text{metric}" '] & K(TM) \arrow[r, "\text{Thom}"] & K(\nu_{TM/T\mathbb{R}^N}) \arrow[d, equal] \\ & & K\left(\substack{\text{open suset} \\ \text{of } \mathbb{R}^{2N}}\right) \arrow[d] \\ & & K(\mathbb{R}^{2N}) \arrow[d, equal] \\ & & \widetilde{K}(S^{2N}) \cong \mathbb{Z} \end{tikzcd}$$

Def^n The K-theory index

K-ind (P) is the above element of $K(T^*M)$.

$t\text{-ind}(P) \in \mathbb{Z}$

Sketch The t-ind satisfies the following

(i) $K(T^*X) \cong K(TX) \to \mathbb{Z}$ defined for all cpct mfds X

(ii) $K(pt) \longrightarrow \mathbb{Z}$ is naturally the identity

(iii) If $X \hookrightarrow Y$ is the inclusion of a compact submanifold,

we have
$$\begin{tikzcd} K(TX) \arrow[r, "i_!"] \arrow[d, dashed] & K(TY) \\ K(\nu_{TX/TY}) \arrow[r] & K\left(\substack{\text{open subset} \\ \text{of } TY}\right) \arrow[u, dashed] \end{tikzcd}$$

and t-ind commutes with the $i_!$ maps.

Observe: there is a unique collection of hom's
$K(TX) \to \mathbb{Z}$ defined ∀ cpct X which satisfies these properties & diffeo^n invariance.

Since $X \xhookrightarrow[i]{} \mathbb{R}^N \hookleftarrow pt$

$$\begin{tikzcd} K(TX) \arrow[r, "i_!"'] \arrow[dr] & K(T\mathbb{R}^N) \arrow[d] & K(pt) \arrow[l, "\cong", "\text{Bott}"'] \arrow[dl, "\text{given}"] \\ & \mathbb{Z} & \end{tikzcd}$$

If a-ind satisfies these properties we're done.

Want (i) Given $\alpha \in K(TX) = K(T^*X)$, find P s.t.
$\sigma(P) = \alpha$,

(ii) Show the association $\alpha \longmapsto$ a-ind satisfies being a well-defined hom $K(TX) \to \mathbb{Z}$, & (ii), (iii) above

(i) Fails, but works if you allow "pseudo-differential operators"
The fact that (iii) holds is "locality" of diff^l operators (excision). □

Cohomological form

Recall total Chern class of cx vector bundles satisfied
$c(E \oplus F) = c(E) \cdot c(F)$
So defines hom $(K(X), +) \longrightarrow (H^{ev}(X), \cdot)$
In fact, ∃ 'ring' hom $K(X)_{\mathbb{Q}} \xrightarrow[ch]{\cong} H^{ev}(X, \mathbb{Q})$ compact X
Chern character

Since $L \mapsto c_1(L)$ is a 'ring' hom
$\{\text{line bundles}, \otimes\} \longrightarrow (H^2(X, \mathbb{Z}), +)$
must be that $ch(L) = e^{c_1(L)}$
$\Rightarrow ch(L_1 \oplus L_2) = \Sigma e^{c_1(L_i)}$
If $E = \oplus L_i$, can write $c_k(E)$ in terms of $\{c_1(L_i)\}$

"Splitting principle"

(E is pullback of $\oplus L_i$ for all E) ⌈spinach, look up Splitting Principle ~⌋

says char class identities hold $\forall E$ if for $E = \oplus L_i$

If $E = \oplus L_i$, $ch(E) = \sum e^{c_1(L_i)}$

$$= rk(E) + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots$$

We now have:

$$\begin{tikzcd}
K(T^*M) \arrow[r, "ch"] \arrow[dr, bend right, "\phi"'] & H^*_{cpct}(T^*M)_{\mathbb{Q}} \arrow[d, "Thom"] \\
& H^*(M, \mathbb{Q})
\end{tikzcd}$$

<u>Atiyah-Singer #2</u>

$$a\text{-}ind(P) = \langle \phi(\sigma(P)) \cdot Td(TM_{\mathbb{C}}), [M] \rangle$$

$Td(-)$ = Todd class, a certain characteristic class

<u>Remark</u> If $V \to M$ is a cx vector bundle,

$$\begin{tikzcd}
K(M) \arrow[r, "\cong"] \arrow[d, "ch"'] & K(V) \arrow[d, "ch"] \\
H^*(M) \arrow[r, "\cong"', "Thom"'] & H^*_c(V)
\end{tikzcd}$$

⌈over $\mathbb{Q}$⌋

does <u>not</u> commute.

The Todd class measures this failure of commutativity.

Explicitly: If $V \to M$ is complex, and A is a unitary connection in V, then $F_A \in \Omega^2(End V)$ lies in $\Omega^2(u(V))$ is locally a skew-hermitian mx of 2-forms

Chern-Weil theory says char. classes are poly's in the eigenvalues $\{x_i\}$ of this matrix.

e.g. $c_k(E) = (-1)^k \sum_{|I|=k} x_{i_1} \cdots x_{i_k}$

$$Td(V) = \prod \frac{x_j}{1-e^{-x_j}} = 1 + c_1(V) + \frac{c_1^2(V) + c_2(V)}{12} + \cdots$$

The Dirac Operator

Definition A Clifford algebra is a real algebra generated by elements $\{a_i\}$ s.t. $a_i^2 = -1$, $a_{ij} + a_{ji} = 0$, $i \neq j$

If $(V, \langle , \rangle)$ is a real inner product space, there's an associated Clifford algebra $\bigoplus_{n \geq 0} V^{\otimes n} / v \otimes v = -\|v\|^2 \cdot 1$

If $e_1, \dots, e_n$ is an o.n. basis for V, then an additive basis for $Cl(V)$ is given by

$$\{ e_I = e_{i_1} \cdots e_{i_r} \mid i_1 < i_2 < \cdots < i_r , \ 1 \leq r \leq n \}$$

So $Cl(V) \underset{\oplus}{\cong} \Lambda^* V$, but mult^n is deformed so $e_i \cdot e_i = -1$

Examples (i) $Cl(\mathbb{R}) = \mathbb{R}[x]/_{x^2 = -1} \cong \mathbb{C}$

(ii) $Cl(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_1e_2$

$\quad \& \quad e_1^2 = -1 = e_2^2$

$\qquad e_1 e_2 = -e_2 e_1 \quad$ so $\quad Cl(\mathbb{R}^2) \cong \mathbb{H}$

(iii) Since $v \otimes v = -\|v\|^2 \cdot 1$ is mod 2 homogeneous, we always get $Cl(V) = Cl_0(V) \oplus Cl_1(V)$

$\& \quad Cl(V) \xrightarrow{\sim} Cl_0(V \oplus \mathbb{R})$

$$v_0 + v_1 \longmapsto v_0 + v_1 \cdot e \quad , \ e \in \mathbb{R}, \|e\| = 1$$

Thus $Cl_0(\mathbb{R}^3) \cong Cl(\mathbb{R}^2) \cong \mathbb{H}$

$\qquad Cl_0(\mathbb{R}^4) \cong Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}$

In fact: if $\underset{e_0}{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}, \underset{e_1}{\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$

then $e_i^2 = -1$, $e_i e_j + e_j e_i = 0 \quad (i \neq j)$

& then define hom $Cl(\mathbb{R}^4) \underset{\substack{\uparrow \\ \text{count} \\ \text{dim}}}{\xrightarrow{\sim}} Mat(2; \mathbb{H})$

Definition If $(V, \langle ; \rangle)$ is a real inner product space, then a Clifford module for V is a skew-Hermitian rep^n of $Cl(V)$, i.e.

we have a Hermitian cx v. space S, & Clifford mult^n $\gamma : V \longrightarrow \mathrm{End}(S)$ & s.t.

- if $\|v\| = 1$, $\gamma(v)^2 = -1$
- $\langle v_1, v_2 \rangle = 0$, $\gamma(v_1)\gamma(v_2) + \gamma(v_2)\gamma(v_1) = 0$
- $\gamma(v)^* = -\gamma(v)$

Artin - Wedderburn Theorem

If $\dim V = 2k$ is even, then $\exists!$ irred Clifford module S, & $\dim_{\mathbb{C}} S = 2^k$. (f. dim)

If $\dim V = 2k+1$ is odd, $\exists$ 2 f.d. irred Clifford modules both of $\dim_{\mathbb{C}} = 2^k$ (differ by $\gamma \mapsto -\gamma$)

Examples If $\dim V = 3$, say e_1, e_2, e_3 basis, then $S \cong \mathbb{C}^2$ & the module structure $\gamma(e_i) = B_i$ has

$$B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If $\dim V = 4$, $V = \langle e_0, \dots, e_3 \rangle$ then $S \cong \mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ and the Clifford module has

maybe + for i>0

~~$\gamma(e_0) = i\,Id$~~, $\quad \gamma(e_i) = \begin{pmatrix} 0 & -B_i \\ B_i & 0 \end{pmatrix}$ $\quad 0\,\cancel{1} \leq i \leq 3$

$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Remark Let $\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdot e_2 \cdots e_n \in CL(V) \otimes \mathbb{C}$ where $V = \langle e_1, \dots, e_n \rangle$

Check $\omega_{\mathbb{C}}^2 = \pm 1$, so its eigenspaces decompose $S = S^+ \oplus S^-$

Moreover, $\omega_{\mathbb{C}}$ commutes with $Cl_0(V) \otimes \mathbb{C}$

skew-commutes with $Cl_1(V) \otimes \mathbb{C}$ (·‿·?)

If n even, the irred module $S = S^+ \oplus S^-$ splits as a Cl_0 - rep^n, & Cl_1 swaps $S^\pm$

Let (M, g) be a Riemannian manifold.
Then $(T_x M, g_x)$ is an i.p. space.
So $\exists$ Clifford bundle $Cl(TM) \longrightarrow M$
with fibre $\underset{\bullet}{\cong} \Lambda^* T_x M$

Definition A spin^c-structure on (M, g) is a Hermitian cx v. bundle $S \to M$ & a map $\rho : TM \to \mathrm{End}_{\mathbb{C}}(S)$ s.t. $\forall x \in M$, $T_x M \xrightarrow[\rho]{} \mathrm{End}(S_x)$ is an irreducible Clifford module.

Example If $\dim M = 4$, we have $S^{\pm} \longrightarrow M$ rank 2 Hermitian bundles & $\rho : TM \to \mathrm{End}(S^+ \oplus S^-)$
(induces a map $\gamma : TM \longrightarrow \mathrm{Hom}(S^+, S^-)$)

Definition If (M, g) is Riemannian, and (S, ρ) is a Spin^c-structure, then a Spin^c-connection is a unitary connection A on S s.t. the associated covariant derivative satisfies

$$d^A(v \cdot s) = v \cdot d^A(s) + \overset{\text{Levi-Civita}}{d^{LC}}(v) \cdot s \qquad \left(\cdot = \text{Clifford mult}^{\text{n}} \right)$$

$v \in \Gamma(TM)$, $s \in \Gamma(S)$ & $d^A : \Gamma(S) \to \Gamma(T^*M \otimes S)$

Remarks (i) These exist, by a partition-of-unity argument
(ii) If A, B are 2 Spin^c connections, then if $\theta = d_A - d_B$,

$$\theta(v \cdot s) = v \cdot \theta(s)$$

so this is naturally

$$\theta \in \Omega^1(M, \underbrace{\mathrm{End}(S, \rho)}_{\substack{\text{Clifford} \\ \text{connection}}} \cap \underbrace{\mathfrak{u}(S)}_{\text{unitary}}) = \Omega^1(M; \mathbb{R})$$

(since (S, ρ) irred $\Rightarrow$ only auto$^{\text{m}}$ are scalars)

Definition The Dirac operator associated to (S,ρ) & the Spin^c-connection A is

$$D_A : \Gamma(S) \xrightarrow[d_A]{} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow[\text{cliff}]{\cdot} \Gamma(S)$$

(a 1st order linear differential operator on $\Gamma(S)$)

Example

Let $(M,g) = (\mathbb{R}^4, g_{\text{Eucl}})$, $S = S^+ \oplus S^- = \mathbb{C}^2 \oplus \mathbb{C}^2$

The usual exterior derivative d is a Spin^c-connection & the Dirac operator $D_{A=d}$ is

$$s \xmapsto[d_A]{} ds \xmapsto[g]{} \sum_{i=0}^{3} e_i \frac{\partial s}{\partial x_i} \longmapsto \sum (A_i \, {\partial}/{\partial x_i}) \, s$$

where $\rho(e_i) = \begin{pmatrix} 0 & -B_i \\ B_i & 0 \end{pmatrix} = A_i$

Then $D_A^2(s) = \Delta s = -\sum \frac{\partial^2}{\partial x_i^2}(s)$ using $A_i^2 = -1$

$A_{ij} + A_{ji} = 0$

Lemma D_A is elliptic

Proof Recall for any connection A on E,

$$\sigma(d_A)(x,\xi) = \xi \wedge \cdot$$

So $\sigma(D_A)(x,\xi) = \cdot \, \xi^\flat$, $\xi^\flat = (\text{musical iso}^{\text{m}})\, \xi$

But $(\cdot \xi^\flat)(\cdot \xi^\flat) = -|\xi^\flat|_g^2$

$\left[T^*M \xrightarrow{\flat} TM \right.$

$\left. TM \xrightarrow{\#} T^*M \right]$

So $\sigma(D_A)(x,\xi)$ is invertible $\forall \xi \neq 0$ so D_A is elliptic. $\square$

Atiyah-Singer theorem now says:

$$\text{a-ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D)$$

$$= (-1)^{\frac{n(n+1)}{2}} \left\langle \frac{\operatorname{ch} \mathcal{E}_0}{\operatorname{eul}(M)} \operatorname{Td}(TM \otimes \mathbb{C}), [M] \right\rangle$$

Here $D_A : \Gamma(S^+) \to \Gamma(S^-)$ with $\dim(M)$ even

$$\operatorname{ch}(\mathcal{E}_0 \to \mathcal{E}_1) = \operatorname{ch}(\mathcal{E}_0) - \operatorname{ch}(\mathcal{E}_1)$$

& $\frac{1}{\mathrm{eul}(M)}$ comes from inverting the Thom iso^n

Recall : we had this calculus for characteristic classes in terms of "Chern roots" $\{x_i\}$ eigenvalues of a unitary connection in a bundle.

$$Td(E) := \prod \left(\frac{x_j}{1-e^{-x_j}} \right) \quad \& \quad \mathrm{eul}(M) = (-1)^n x_1 \cdots x_{n/2}$$

(c.f. euler class = top Chern classes)

Lemma $ch(S^+) - ch(S^-) = \prod_{i=1}^{n/2} (e^{x_j/2} - e^{-x_j/2})$

in the case when $\det(S^+) = \det(S^-) = \underline{\mathbb{C}}$

Upshot : $\mathrm{index}(D_A : \Gamma(S^+) \longrightarrow \Gamma(S^-))$

$n = \dim(M)$ even

& Spin^c-str has $\det(S^+)$ trivial

$$\overset{\shortparallel}{} \; (-1)^{n/2} \int_M \underbrace{\prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)}}_{\hat{A}\text{-genus}}$$

$$\shortparallel$$

$$1 - \frac{1}{24} p_1(M) + \frac{1}{5760}(7p_1^2 - p_2) + \cdots$$

On a 4-manifold, we get

$\mathrm{index}(D_A) = -\frac{\sigma}{8}$ for a Spin^c-str with $\det(S^+)$ trivial

Spin^c - structures

Recall: if V is an inner product space,

$$Cl(V) \underset{\oplus}{\cong} \Lambda^* V, \quad e_i e_j + e_j e_i = 0, \quad e_i^2 = -1$$

for $\{e_1, \dots, e_n\}$ o.n.b. of V

We said a Spin^c structure on (M, g) was a Hermitian v.b. $S \to M$ & a Clifford mult^n $\rho : TM \to \text{End}_{\mathbb{C}}(S)$ s.t. fibrewise irred $Cl(T_xM, g_x)$-module

Definition: $\text{Pin}(V) = \{$ units in $Cl(V)$ generated by norm 1 elements $\}$

$$= \{ v_1 \cdots v_k \in Cl(V), \ \|v_i\| = 1 \}$$

$\text{Spin}(V) = \text{Pin}(V) \cap Cl_0(V) = \{$ all even length expressions $\}$

Lemma $\exists$ natural surjection

$\text{Spin}(V) \longrightarrow SO(V)$ with kernel of size 2

So $\text{Spin}(V) \cong \text{Spin}(n)$.

(The double-cover is non-trivial)

Proof If $u \in V$ has $\|u\| = 1$, $u^{-1} = -u$

Conside conjugation action of u:

$$\begin{aligned} x \longmapsto u x u^{-1} &= -uxu \\ &= -u(-ux - 2\langle x, u \rangle) \\ &= -x + 2u\langle x, u \rangle \\ &= -\text{Refl}_{u^\perp}(x) \end{aligned}$$

So $\exists$ hom $\text{Pin}(V) \longrightarrow O(V)$

$$u \longmapsto -\text{Refl}_{u^\perp} \text{ for } \|u\| = 1.$$

This induces $\text{Spin}(V) \twoheadrightarrow SO(V)$.

If $u \in \ker(\text{Spin}(V) \longrightarrow SO(V))$, u acts trivially on $V \subseteq Cl(V)$, so u acts on $Cl(V)$ centralising the action of V & hence acts centrally.

n odd : $Z(Cl(V)) = \mathbb{R}\langle 1, e_1 \cdots e_n \rangle$

n even : $Z(Cl(V)) = \mathbb{R}\langle 1 \rangle$

So $Z(Cl(V)) \cap Cl_0(V) \cong \mathbb{R}$.

So kernel of $Spin(V) \longrightarrow SO(V)$ is ± 1. $\square$

Definition $Spin^c(n) = \left(Spin(n) \times U(1)\right)/_{\pm 1}$

So $\exists$ exact sequence

$$1 \to U(1) \to Spin^c(n) \xrightarrow{\alpha} SO(n) \longrightarrow 1$$

& $\exists$ hom $Spin^c(n) \underset{\lambda}{\longrightarrow} U(1)$

$$[g, z] \longmapsto z^2$$

$Spin^c(V)$ is the subgroup of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $Spin(V)$ & $U(1) \subseteq \mathbb{C}$

Remark

$$\begin{tikzcd}
Spin^c(n) \arrow[r, "(\lambda{,}\alpha)"'] \arrow[d] \arrow[dr, phantom, "\square"] & SO(n) \times SO(2) \arrow[d] \\
Spin(n+2) \arrow[r] & SO(n+2)
\end{tikzcd}$$

Can also define $Spin^c(n)$ to make this a pull-back diagram.

Lemma If X is an oriented closed 4-mfd w/ metric g, then X admits a $Spin^c$-structure & the set of such is affine over $H^2(X; \mathbb{Z})$.

(Proof 1) $b^1(X) = 0$, for simplicity

If (S, ρ) is a Clifford module, recall $Aut(S, \rho) = S^1$

Take a trivialising cover $\{U_\alpha\}$ for TX, & a standard Clifford module S_α over U_α.

This has transition maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to Aut(S_{std}, \rho) = S^1$

Want $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} : \underset{\text{triple int}}{U_{\alpha\beta\gamma}} \longrightarrow S^1$ is identically 1

Let $C^\infty_X(S^1)$ be the sheaf on X of smooth S^1-valued functions

$\{\varphi_{\alpha\beta}\} \in H^2(X, C^\infty_X(S^1))$

Exponential exact sequence

$$0 \to \mathbb{Z}_X \to C^\infty_X(\mathbb{R}) \longrightarrow C^\infty_X(S^1) \to 0$$

$\rightsquigarrow$

$$\begin{tikzcd} H^2(X, \underbrace{C^\infty_X(\mathbb{R})}_{\text{flasque (?)}}) \arrow[r] \arrow[d] & \underbrace{H^2(X, C^\infty_X(S^1))}_{\text{vanishes}} \arrow[r] & H^3(X; \mathbb{Z}) \arrow[d] \\ 0 & & 0 \text{ as } b^1(X) = 0 \end{tikzcd}$$

⌈more like soft⌋

So we can change $\{\varphi_{\alpha\beta}\}$ by a coboundary to make it vanish & hence get a Spin^c-str.

If (S, ρ), $(\tilde{S}, \tilde{\rho})$ are two such,

$(S, \rho)|_{U_\alpha} \cong (\tilde{S}, \tilde{\rho})|_{U_\alpha} \quad \forall \alpha$ (!ness of irred Clifford mod.)

Same way: $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$ define a class in $H^1(X, C^\infty_X(S^1))$

The same exponential sequence identifies this with $H^2(X; \mathbb{Z})$. $\square_1$

(Proof 2) From the pullback diagram

$$\begin{tikzcd} \mathrm{Spin}^c(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & SO(n) \times SO(2) \arrow[d] \\ \mathrm{Spin}(n+2) \arrow[r] & SO(n+2) \end{tikzcd}$$

note (i) a Spin^c-str on X defines a complex line bundle $\mathcal{L} \to X$, via hom $\alpha : \mathrm{Spin}^c(n) \to U(1)$

(this would be $\det S^+$ or $\det S^-$)

⊗ (ii) given an oriented real rank 2 bundle $L \to X$, $\mathrm{Spin}^c(n)$-lifts of $Fr(TX)$ correspond to Spin structures on $TX \oplus L$

Note Giving a Spin^c-str on (X,g) as an irred Clifford module is equivalent to giving a $\mathrm{Spin}^c(n)$-lift of $Fr(TX)$

Question becomes: when does $TX \oplus L$ have a spin structure?

From long ago, we saw that $w_2(TX)$ always admits lifts to $H^2(X;\mathbb{Z})$ & these are the char. elts of int form Q_X.

Given such a char elt, can take $L \to X$ the cx line bundle with $c_1(L) =$ that char elt

$$(\Rightarrow w_2(L) = c_1(L) \bmod 2)$$

& then $TX \oplus L$ is spin. $\square$

Corollary The map

$$\mathrm{Spin}^c\ \mathrm{Str}(X) \longrightarrow H^2(X,\mathbb{Z})$$
$$(S,\rho) \longmapsto c_1(\mathcal{L}_{(S,\rho)})$$

has image $\mathrm{Char}(Q_X) = \{\text{char elts for } Q_X\}$.

Recall If $\dim X$ is even, & $D(\Gamma(S^+) \to \Gamma(S^-))$ is the Dirac operator, we needed for computing its index that

$$ch(S^+) - ch(S^-) = \prod_{j=1}^{n/2} (e^{x_j/2} - e^{-x_j/2})$$

$\{x_j\}$ eigenvalues of $\frac{i\Omega}{2\pi}$ curv of unitary connection in S

Sketch **If** metric had holonomy in $SO(2) \times \cdots \times SO(2) \leq \underset{\substack{\| \\ SO(\underset{\substack{| \\ \dim X}}{n})}}{SO(2m)}$

a spin str on a line bundle L, just a square root

i.e. $\hat{L}$ s.t. $\hat{L} \otimes \hat{L} \cong L$.

If $(TX) \otimes \mathbb{C} = \bigoplus_{i=1}^{n} (L_i \oplus \bar{L}_i)$, then the spin structure on X gives $(\hat{L}_1)_{\mathbb{C}} \oplus \cdots \oplus (\hat{L}_n)_{\mathbb{C}}$

& the spinors $S^\pm$ are the (± 1)-$\omega_{\mathbb{C}}$ eigenspaces acting on the Clifford module.

$\omega_{\mathbb{C}}$ acts on $(\hat{L}_i)_{\mathbb{C}} \cong \hat{L}_i \oplus (\hat{L}_i)^{-1}$ via $(1, -1)$

<u>Find</u>: $S^{\pm} = \bigoplus_{\substack{a_i \in \{\pm 1\} \\ \prod a_i = \pm 1}} \hat{L}_1^{a_1} \otimes \cdots \otimes \hat{L}_n^{a_n}$

& since $c_1(\hat{L}_i) = \frac{c_1(L_i)}{2} = \frac{x_i}{2}$

$\Rightarrow ch(S^{\pm})$ = terms in expansion of $\prod (e^{x_i/2} + e^{-x_i/2})$

with an $\underset{S^+}{\text{even}}$ resp $\underset{S^-}{\text{odd}}$ number of "$x_i/2$" factors □

Given that, we saw that

if (S,ρ) is a Spin^c-structure s.t. $\mathcal{L}_{(S,\rho)} \cong \underline{\mathbb{C}}$ is trivial

e.g. if (S,ρ) comes from a Spin-structure via the hom^m

$\mathrm{Spin}(n) \times U(1) \longrightarrow \mathrm{Spin}^c(n)$

then the a-ind$(D : \Gamma(S^+) \to \Gamma(S^-))$

$= \hat{A}$-genus(X)

$= 1 - \frac{1}{24} p_1 + \frac{1}{5760}(7p_1^2 - p_2) + \cdots$

& if $\dim_{\mathbb{R}} X = 4$, a-ind$(D) = -\frac{\sigma}{8}$ $(p_1 = 3\sigma)$

<u>Lemma</u> index$(D : \Gamma(S^+) \to \Gamma(S^-))$ for a spin 4-manifold is even (So $16 \mid \sigma(X)$, recovering Rokhlin's theorem)

<u>Proof</u> Claim: $\exists$ quaternionic structure structure on ker / coker of Dirac (c.f. $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$)

<u>Fact</u> (a) $\exists$ real structure on $\mathbb{C}^2$ i.e. $c : \mathbb{C}^2 \to \mathbb{C}^2$ complex anti-linear, s.t. $c^2 = +1$, which commutes with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ & $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

(b) $\exists$ quaternionic str on $\mathbb{C}^2$, $j : \mathbb{C}^2 \to \mathbb{C}^2$ complex anti-linear with $j^2 = -1$ s.t. j anticommutes with same Pauli matrices

$c \otimes j : \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{S} \to \mathbb{S}$ anticommutes w/ Clifford mult^n by $\gamma_0 \cdots \gamma_5$ but commutes with $\omega_{\mathbb{C}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

recall $\mathbb{S}$ for $CL(\mathbb{R}^4)$ was $\mathbb{H} \oplus \mathbb{H}$

Now $J = c \otimes j$ satisfies $DJ = -JD : \Gamma(S^+) \to \Gamma(S^-)$

Hence ind (D) is nec. even. $\square$

Remark (i) A spin str on X determines a canonical spin^c-structure

(ii) An (almost) cx structure on X^4 determines a canonical Spin^c - structure

$$\mathrm{Spin}^c(4) = \{(\alpha, \beta) \in U(2) \times U(2) \mid \det \alpha = \det \beta\}$$

There's a canonical hom $U(2) \to \mathrm{Spin}^c(4)$

If (X, J) is a complex surface, the

$$\underset{\text{canonical}}{\mathcal{L}_{(S,\rho)}} \cong \underset{\text{canonical bundle}}{K_S}$$

Seiberg - Witten equations

Recall If (X, g) oriented Riemannian 4-mfd, then it admits Spin^c structures, & in fact

$$\mathrm{Spin}^c(X)/_{\sim} \cong H^2(X, \mathbb{Z})$$

$$\& \ \exists\ c_1 : \mathrm{Spin}^c(X) \longrightarrow H^2(X; \mathbb{Z})$$

$$S \longmapsto c_1(\det S^{\pm})$$

has image the set $\mathrm{Char}(X)$ of char elements

Recall: the Hodge star $* : \Omega^2(X) \to \Omega^2(X)$ satisfies $*^2 = 1$ & so $\underbrace{\Omega^2(X) \cong \Omega^{2+}(X)}_{\text{self-dual}} \oplus \underbrace{\Omega^{2-}(X)}_{\text{anti-self-dual}}$

eigenspace decompⁿ for $*$.

If $\alpha \in \Omega^2(X)$, I can form
$$\alpha^+ = \tfrac{1}{2}(1 + *)\alpha$$
$$\alpha^- = \tfrac{1}{2}(1 - *)\alpha$$
its (anti-) self-dual parts, & then $\alpha = \alpha^+ + \alpha^-$.

Lemma Let S be a Spin^c-structure on X with spinor bundle S. Then Clifford multⁿ,

$$\Lambda^2(TX) \xrightarrow[\rho]{} \mathrm{End}(S)$$

induces an isoᵐ $\Lambda^2_+ \xrightarrow[\sim]{} \mathfrak{su}(S^+)$

$$\llcorner \{\alpha \mid \alpha^* = -\alpha,\ \mathrm{tr}(\alpha) = 0\}$$

Proof These are both rank 3 bundles, & we just check pointwise.

Recall $\rho : TX \to \mathrm{End}(S)$ had, in a local o.n. basis for $T_x X$, that

$$e_1 \longmapsto \begin{pmatrix} 0 & \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \\ \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} & 0 \end{pmatrix}, \quad e_2 \longmapsto \begin{pmatrix} 0 & \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} & 0 \end{pmatrix}$$

$$e_3 \longmapsto \begin{pmatrix} 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \quad e_4 \longmapsto \begin{pmatrix} 0 & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \end{pmatrix}$$

Λ^2_+ is generated by $e_1e_2 + e_3e_4$, $e_1e_3 + e_2e_4$, $e_1e_4 + e_2e_3$.
(from defⁿ of Hodge star; $\mathrm{vol} = e_1e_2e_3e_4$)

Check these are

- $$\begin{pmatrix} \begin{matrix} -2i & 0 \\ 0 & 2i \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \begin{matrix} 0 & 2 \\ -2 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \begin{matrix} 0 & -2i \\ -2i & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

(so trivial on S^-) $\square$

<u>Recall</u>: If S is the spinor bundle, we have Spin^c-connection & $\{\mathrm{Spin}^c(S)\} \underset{\text{affine}}{\cong} \Omega^1(X, i\mathbb{R})$

(so if I fix one A_0, other Spin^c-connections are $a \in \Omega^1(X; i\mathbb{R})$)

- If A is a Spin^c-connection, then $F_A = d_A d_A \in \Omega^2(X; i\mathbb{R})$

If $\Phi \in \Gamma(S)$, $\Phi\Phi^* \in \Gamma(\mathrm{End} S) = \Gamma(S \otimes S^*)$

& $\underbrace{i\Phi\Phi^*}_{\hat{\Phi}} \in$ Skew-adjoint $\mathrm{End}(S)$

then $\hat{\Phi} - \frac{1}{2}(\mathrm{tr}\,\hat{\Phi})\,\mathrm{Id}$ = trace-free part is an elt of $su(S)$

So if $\Phi \in \Gamma(S^+)$, then the preimage of $\hat{\Phi} - \frac{1}{2}\mathrm{tr}(\hat{\Phi})\,\mathrm{Id}$ under $\Lambda^2_+ \longrightarrow su(S^+)$ is called $q(\Phi)$, q for quadratic, & is naturally an elt of $\Omega^2(X; i\mathbb{R})$

- <u>Seiberg - Witten equations</u>

are for a pair (A, Φ), $A \in$ Spin^c-connection
$\Phi \in \Gamma(S^+)$ positive spinor

$$\begin{cases} F_A^+ = q(\Phi) \in \Omega^{2+}(X, i\mathbb{R}) \\ D_A^+ \Phi = 0 \in \Gamma(S^-) \end{cases}$$

(recall $D_A = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, $\Gamma(S^+) \xrightarrow{D_A^+} \Gamma(S^-)$ in even dim^n)
$\Gamma(S^-) \xrightarrow{D_A^-} \Gamma(S^+)$

- <u>Remark</u> We will often want to fix a perturbation $\eta \in \Omega^2(X; i\mathbb{R})$ & consider $F_A^+ - q(\Phi) = \eta$ rather than $F_A^+ - q(\Phi) = 0$

First key point: there's a large symmetry group

Let $\mathcal{G} = \mathrm{Aut}(X, \mathfrak{s})$ gauge group $= C^\infty(X, S^1)$

If $u \in \mathcal{G}$, u acts on $\Gamma(S^\pm)$ by mult^n, $\psi \mapsto u\psi$

& it acts on Spin^c-connections via

$$A \longmapsto u\cdot A \quad \text{whoe} \quad d_{u\cdot A} = d_A + u\cdot du^{-1}$$

(If $\pi_1 X = 1$, we can write $u = e^{if}$ for some $f: X \to \mathbb{R}$,
& $u \cdot du^{-1} = i\,df$. In general, it's $i\,d(\log u^{-1})$)

If fix a reference connection A_0 on S, then another $A = A_0 + ia$ for $a \in \Omega^1$.

And $u\cdot A = A_0 + ia + udu^{-1}$

<u>Lemma</u>: $\mathcal{G}$ preserves $\{$solutions to $\mathcal{M}(SW_\eta)\}$
for any $\eta \in \Omega^2(X; i\mathbb{R})$

<u>Proof</u>: F_A, F_A^+ are unchanged by $\mathcal{G}$-action, since $U(1)$ abelian

(locally $F_{A+a} = F_A + da + a\wedge a$, if $a \in \Omega^1$ the last term zero)

Also, $q(e^{if}\cdot\Phi) = e^{if}\Phi \cdot e^{-if}\Phi^* - |\Phi|^2\,\mathrm{id} = q(\Phi)$

& general case is similar.

Finally, if $D_A^+(\Phi) = 0$, we want

$$D^+_{u\cdot A}(u\Phi) = 0$$

But $\underset{d_A}{u\cdot\cancel{\nabla_A}} = u \circ d_A \circ u^{-1}$ from our local expression

& $D_A = \rho \circ d_A$ & then $D_{u\cdot A}(u\Phi) = \rho(u d_A u^{-1}(u\Phi))$

& Clifford mult^n commutes with scalar mult^n by u,

so $D_{u\cdot A}(u\Phi) = u\rho\, d_A(\Phi) = u\, D_A\Phi$

So 2^{nd} eq^n is preserved. $\square$

Def^n The based gauge group $\mathcal{G}_0$ is

$\{u \in \mathcal{G} \mid u(x_0) = 1\}$ where $x_0 \in X$ is a basepoint

(Alternative: if $\pi_1 X = \{1\}$, can set $\mathcal{G}_0 = \{u = e^{if} \in \mathcal{G} \mid \int_X f = 0\}$)

Then there's a SES

$$1 \to \mathcal{G}_0 \to \mathcal{G} \to S^1 \to 0$$

Lemma $\mathcal{G}_0$ acts freely on $\mathcal{M}(SW_\eta)$

In fact, each orbit contains a unique representative (d_A, Φ) s.t. $d_A = d_{A_0} - ia$ & $d^* a = 0$ when $b_1(X) = 0$.

(where A_0 is a fixed Spin^c-connection)

Def^n Having fixed $A_0 \in \text{Spin}^c(S)$, we say A is in Coulomb gauge if $A = A_0 - ia$, with $d^* a = 0$.

Thus $\mathcal{M}(SW_\eta)/\mathcal{G} = \mathcal{M}(SW_\eta)^{\text{Coulomb}}/S^1$ $\left[\left[b'(X) = 0 \right]\right]$

Proof of Lemma If $u \in \mathcal{G}_0$ acts trivially (on $-$), then $u \cdot du^{-1} = 0$ so u^{-1} & u are constant functions.

So in fact $u \equiv 1$. So $\mathcal{G}_0$ acts freely

The Hodge theorem says

$$\Omega^1(X) = \ker d^* \oplus \operatorname{im} d$$

So given $ia \in i\Omega^1$, $\exists!\, df$ s.t. $d^*(-ia + df) = 0$

So $\mathcal{G}_0$-orbits on $\text{Spin}^c(S)$ have ! Coulomb gauge representatives if $b'(X) = 0$. $\square$

Remark In general, $H^1(X, \mathbb{Z}) = [X, U(1)]$

Then $\mathcal{M}(SW_\eta)/\mathcal{G} = \mathcal{M}(SW_\eta)^{\text{Coulomb}}/H^1(X, \mathbb{Z}) \times S^1$

The next key feature of the SW equations is their compactness properties.

Recall On $\mathbb{R}^n$, $\Delta = -\sum (\frac{\partial}{\partial x_i})^2$

& d was the "trivial" Spin^c-connection, we saw $D^2 = \Delta$

In general, something close to this happens

Theorem (Weitzenbock formula)

Let (X^4, g) be an oriented Riemannian 4-mfd (closed)

We have a Spin^c structure $\mathfrak{s}$ & Spin^c connection A,

& let $d_A : \Gamma(S) \longrightarrow \Gamma(T^*X \otimes S)$

$$D_A : \Gamma(S) \longrightarrow \Gamma(S)$$

Then $D_A^2 \Phi = d_A^* d_A \Phi + \frac{s}{4} \Phi + \frac{1}{2} \rho(F_A) \cdot \Phi$

where $s = \text{scal}(X)$ is the scalar curvature of (X, g)

& $F_A \in \Omega^2(X; i\mathbb{R})$ & $\rho(F_A)$ acts via Clifford mult^n.

Remark In fact this holds on any even-dim^L Spin^c-mfd.

On an even-dim Spin manifold, we have a slightly simpler formula: $D_A^2 = d_A^* d_A + \frac{s}{4}$

Here A is the spin connection of the Spin structure

Consequence: if $\langle D_A^2 \phi, \phi \rangle = \langle d_A^* d_A \phi, \phi \rangle + \langle {}^s/_4 \, \phi, \phi \rangle$

$\phi \in \Gamma(S)$ $\qquad = \| d_A \phi \|^2 + \frac{s}{4} \| \phi \|^2$

& so if $\text{scal}(X) > 0$, then $(X, \mathfrak{s})$ has no harmonic spinors, i.e. elts s.t. $D\phi = 0$

Similarly, $D_A^+ \phi = 0$ has no solution

So on a spin mfd with scal > 0, $\hat{A}(X) = 0$.

Compactness

- If (M, g) is Riemannian, it has a distinguished Levi-Civita connection ∇_{LC} with curvature $F_{\nabla_{LC}} \in \Omega^2(\operatorname{End} TM)$

$$\overset{\shortparallel}{\Gamma(\Lambda^2 T^* \otimes (T^* \otimes T))}$$

with components $R_{ijkl} = \langle R(e_i, e_j) e_k, e_l \rangle$ $\quad$ $\{e_i\}$ o.n.b. of TM

where $R(e_i, e_j) : TM \to TM$

$[\, R(X,Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, where ∇_X is the endo^m of covariant differentiation defined by ∇_{LC} $]$

- The Ricci curvature $Ric \in \Gamma(T^* \otimes T^*)$

$$Ric(e_i, e_j) = -\sum_k R_{kikj}$$

The scalar curvature

$$s = \operatorname{trace}(R_{ii}) = -\sum_i Ric(e_i, e_i)$$

$$= -\sum_{i,k} R_{kiki} = \sum_{i,k} R_{ikki}$$

(R_{ijkl} transforms by sign : $S_4 \to \{\pm 1\}$ under permuting indices)

↖ ? ⌜really⌟

Bianchi identity

- $$\sum_{\substack{\text{cyclic} \\ (i,j,k)}} R_{ijkl} = R_{[ijk]l} = 0$$

We now have even-dim Spin mfd, ∇_{LC} & the Spin lift ∇ of the LC connection : the Dirac operator

$$D : \Gamma(S) \to \Gamma(S) \quad , \quad \psi \mapsto \sum_i e_i \cdot \nabla_{e_i} \psi$$

where $\{e_i\}$ are local o.n.b of TM

Weitzenbock / Lichnerowicz :

- $D^2 = \nabla^* \nabla + \frac{s}{4}$, $\quad s =$ scalar curvature

(In Spin^c-case, we had $D^2 = \nabla^* \nabla + \frac{s}{4} + \rho(F_A) \cdot$, where A Spin^c-connection)

Remark $D^2 - \nabla^*\nabla$ is necessarily a 1st order operator

relatively little work to show 0th order, so a function of curvature, & we really need $+\frac{3}{4}$

Proof (with even no. of sign errors)

Everything in sight is intrinsically defined, so choose to work in coordinates s.t. if $\{e_i\}$ o.n.b of local v. fields,

& $\nabla^{LC}_{e_i}(e_j)\big|_p = 0$

∇^{LC} defined so $\nabla^{LC}_Y X - \nabla^{LC}_X Y = [Y, X]$, so $[e_i, e_j]\big|_p = 0$

If $\phi \in \Gamma(\mathbb{S})$ near p,

$$D^2\phi\big|_p = \sum e_i \cdot \nabla_{e_i}\left(\sum e_j \cdot \nabla_{e_j}\right)(\phi)\big|_p$$

So $D^2\phi\big|_p = \sum_{i,j} e_i e_j \nabla_{e_i}\nabla_{e_j}(\phi)\big|_p$ using that ∇_{e_i} is a derivation for Clifford mult^n

$$= -\sum_i \nabla^2_{e_i}\phi\big|_p + \sum_{i<j} e_i e_j \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right)\phi\big|_p$$

↖ Clifford relations

$$= \nabla^*\nabla\phi\big|_p + \sum_{i<j} e_i e_j F_{\nabla^{spin}}(e_i, e_j)\phi\big|_p$$

↖ curvature of spin connection in $\Omega^2(\mathrm{End}\, S)$

∇^{LC} locally defined by an $so(n)$-valued 1-form

∇^{spin} on S locally defined by an $so(2^{n/2})$-valued 1-form

$\mathrm{Spin}(V) = \{$ even length products of length 1 vectors in $V \}$

& $\mathrm{Lie\,Spin}(V) = \mathrm{spin}(V) = \mathbb{R}\langle v_1 v_2 \mid v_1 \perp v_2 \in V \rangle \subseteq \mathrm{Cliff}(V)$

Explicitly, if $M_{k,l}$ is mx with 1 in (k,l), -1 in (l,k) $\quad so(n)$

then I associate to this Clifford mult^n by $e_k e_l$ as an endo^m of S.

So if $F_{\nabla^{LC}} = \sum_{k,l} \omega_{kl} M_{kl}$ locally written as sum of 2-forms ω_{kl}, then

$$F_{\nabla^{spin}} = \sum_{k,l} \omega_{kl} \cdot (e_k e_l)$$

in spin via Clifford mult^n on $\mathbb{S}$

$$\text{So } F_{\nabla^{spin}}(e_i, e_j) \underset{\substack{\uparrow \\ \sum_{k<l}}}{=} \underbrace{\langle R(e_i,e_j) e_k, e_l \rangle}_{R_{ijkl}} \cdot e_k e_l$$

<u>Upshot</u>:
$$D^2\phi|_p = \nabla^*\nabla\phi|_p + \underbrace{\frac{1}{2} \sum_{\substack{i<j \\ k<l}} R_{ijkl}\, e_i e_j e_k e_l \, \phi|_p}_{\frac{1}{8} \sum_{ijkl} R_{ijkl}\, e_i e_j e_k e_l \, \phi|_p}$$

using $R_{ijkl} = -R_{jikl} = R_{jilk}$

Use Bianchi: If we cycle 3 $\underset{\text{distinct}}{\uparrow}$ indices i, j, k, then $e_i e_j e_k$ unchanged,

but $R_{[ijk]l} = 0$.

End up with

$$\sum_{ijl} R_{ijjl}\, e_i e_j e_j e_l + \sum_{ijl} R_{ijil}\, e_i e_j e_i e_l$$

$$\sum_{ijl} R_{ijjl}\, e_i e_j e_j e_l = -\sum_{ijl} R_{ijjl}\, e_i e_l$$

& terms with $i \neq l$ cancel in pairs, so get

$$= \sum_{ij} R_{ijji} = \mathrm{scal}(M)|_p .$$ Other term analogous. $\square$

<u>Remark</u> $\mathrm{spin}^c = \mathrm{spin} + \mathrm{Lie}(S^1)$

$$F_{\nabla^A} = \underset{F_{\nabla^{LC}}}{F_{\nabla^{spin}}} + \underset{\Omega^2(i\mathbb{R})}{\underset{\cap}{F_A}} \qquad \left[\sum \underbrace{F_{\nabla^A}(e_i, e_j)}_{(\rho(F_A) + \sum R_{ijkl} \cdots)} e_i e_j \right]$$

Back to S-W

Recall : we have this quadratic map

$$q : S^+ \to i\,\mathfrak{su}(S^+) \subseteq \mathrm{End}(S^+)$$
$$\psi \longmapsto \psi\psi^* - \frac{|\psi|^2}{2} \cancel{X}\, \mathrm{Id}$$

As an endo^{m} this sends $\alpha \mapsto \langle \alpha, \psi \rangle \psi - \frac{1}{2}|\psi|^2 \alpha$

<u>Exercise</u> $\langle q(\psi)\underset{\text{extra}\,\psi}{,} , \psi \rangle = \frac{1}{2}|\psi|^4$ pointwise

<u>Prop^{n}</u> If (A, Φ) solves SW_η equations , then

$$|\Phi|^2 \leq S_{max} = \max_{x \in X} \{ \max(0, -\mathrm{scal}(x)) \}$$

<u>Proof</u> (When $\eta = 0$, general case similar)

$$D_A^2 \Phi = \nabla^* \nabla \Phi + \frac{s}{4}\Phi + \frac{1}{2}\rho(F_A)\cdot\Phi$$

& $\Phi \in \Gamma(S^+) \subseteq \Gamma(S)$, so $\rho(F_A)\cdot\Phi = \gamma(F_A^+)\cdot\Phi$

$$\left.\begin{aligned} D_A \Phi &= 0 \\ F_A^+ &= q(\Phi) \end{aligned}\right\} \text{SW eq}^{n}\text{'s for } \eta = 0$$

$$0 = \langle D_A^2 \Phi, \Phi \rangle = \langle \nabla^*\nabla\Phi, \Phi \rangle + \frac{s}{4}\langle \Phi, \Phi \rangle + \frac{|\Phi|^4}{4}$$

We also have that :

<u>Sublemma</u> If A is a metric connection in a bundle E, with cov. derivative ∇_A & ∇_A^* its formal adjoint,

$$\Delta_g |\phi|^2 = 2\langle \nabla^*\nabla\phi, \phi \rangle - |\nabla\phi|^2 , \quad \phi \in \Gamma(E)$$

Given that , pick $x_0 \in X^4$ s.t. $|\Phi(x_0)|$ is maximal

Then $\Delta|\Phi(x)|^2 \geqslant 0$ at $x = x_0$

But $\Delta|\Phi(x)|^2 + |\nabla\Phi|^2 = 2\langle \nabla^*\nabla\Phi, \Phi \rangle$

$$= 2\left(-|\Phi|^2 \cdot \frac{s}{4} - \frac{1}{4}|\Phi|^4\right)$$

$$\Rightarrow \frac{s}{2}|\Phi(x_0)|^2 + \frac{|\Phi(x_0)|^4}{2\cancel{4}} \leq 0$$

So $\Phi(x_0) = 0$, & then $\Phi \equiv 0$ <u>or</u>

$$|\Phi(x_0)|^2 \le - s(x_0) \qquad \square$$

If (A, Φ) solves SW_η,

$$F_A^+ = q(\Phi) + \eta,$$

so $|F_A^+| < C\,|\Phi(x)|^2 + \text{Const}_\eta$

$\underline{\text{so}}$ $|F_A^+|$ is pointwise bounded

Fix one Spin^c connection A_0, so $A = A_0 + ia$

$$F_A^+ = F_{A_0}^+ + i d^+ a, \qquad a \in \Omega^1(X; i\mathbb{R})$$

Know $d^+ a$ is pointwise bounded.

Using Coulomb gauge, we can suppose $d^* a = 0$.

Consider $d^+ + d^* : \Omega^1(X) \longrightarrow \Omega^{2+}(X) \oplus \Omega^0/_{\mathbb{R}}$

The operator $d^+ + d^*$ is Fredholm with kernel $H^1(X, \mathbb{R})$

(Remark; if $\alpha \in \Omega^1(X)$, $d\alpha = d^+\alpha + d^-\alpha \in \Omega^{2+} \oplus \Omega^{2-}$

$$0 = \int_X d(\alpha \wedge d\alpha) = \int_X d\alpha \wedge d\alpha = \int_X \langle d\alpha, * d\alpha \rangle \, \text{vol}$$

$$= \|d^+\alpha\|^2 - \|d^-\alpha\|^2$$

$$\Rightarrow \ker d = \ker d^+ = \ker d^- \quad)$$

Reducibles

Let (X^4, g) be an oriented Riemannian 4-mfd.

● The self-duality complex

$$\mathcal{E}^\bullet : 0 \to \Omega^0(X;\mathbb{R}) \to \Omega^1(X;\mathbb{R}) \to \underset{\llcorner \text{depends on } g \text{ via } *}{\Omega^{2,+}_g(X;\mathbb{R})} \to 0$$

Lemma : $H^0(\mathcal{E}^\bullet) = H^0_{dR}(X)$

$H^1(\mathcal{E}^\bullet) = H^1_{dR}(X)$

$H^2(\mathcal{E}^\bullet) = \mathcal{H}^+_g(X) \cong H^{2+}(X;\mathbb{R})$

Proof : H^0 is obvious

● If $\alpha \in \Omega^1(X)$ & $d\alpha = d^+\alpha + d^-\alpha$, $d^\pm\alpha = \frac{1}{2}(1 \pm *)d\alpha$

then $0 = \int_X d(\alpha \wedge d\alpha) = \int_X d\alpha \wedge d\alpha = \|d^+\alpha\|^2_{L^2} - \|d^-\alpha\|^2_{L^2}$

& $\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$

So $\ker d^+ = \ker d^- = \ker d$, so $H^1(\mathcal{E}^\bullet) = H^1_{dR}(X)$.

For $H^2(\mathcal{E}^\bullet)$, fact (check!) $\mathcal{H}^+$ is orthogonal to $\mathrm{im}(d^+)$

So

$$\begin{tikzcd} \mathcal{H}^+_g \arrow[r, hook] \arrow[rr, dashed, bend right, "\text{injective}"'] & \Omega^{2+}_g \arrow[r] & \Omega^{2+}/\mathrm{im}(d^+) \end{tikzcd}$$

& we want it to be onto.

●

If $\omega \in \Omega^{2+}(X)$, $\omega = \omega_{\text{Harmonic}} + d\alpha + *d\beta$

by Hodge theory $\in \mathcal{H}^2_g + \mathrm{im}\, d + \mathrm{im}(d^*)$

& decomposition is unique.

If $*\omega = \omega$, you see $*\omega_{\text{Harmonic}} = \omega_{\text{Harmonic}}$ & $d\alpha = d\beta$

So $\omega = \omega_{\text{Harmonic}} + 2d^+\alpha \in \mathcal{H}^+_g \oplus \mathrm{im}(d^+)$. $\square$

Back to compactness

● We showed that if (A, Φ) solves S-W , then

$$|\Phi(x)| \le \max_{x \in X} \{ \max\{-s(x), 0\} \}$$

& $F_A^+ = q(\Phi) \Rightarrow |F_A^+(x)| \leq \frac{|\Phi(x)|^2}{2} < \text{const}$.

● Fix Spin^c - connection A_0 & write $A = A_0 + ia$,

$F_A^+ = F_{A_0}^+ + i d^+ a$

We work in Coulomb gauge so $d^* a = 0$.

Now $d^+ + d^* = P : \Omega^1(X) \longrightarrow \Omega^{2+} \oplus \Omega^0$ is linear Fredholm with kernel $\mathcal{H}^1_g \cong H^1_{dR}(X)$

The elliptic estimate : $\|a\|_{L^{k,p}} \leq C \left(\underbrace{\|Pa\|_{L^{k-1,p}}}_{\text{pointwise bounded}} + \|a\|_{C^0} \right)$ ⌈take $p=2$⌋

● But also, we can write

$a = a_{harm} + a'$, $a' \in (\ker P)^\perp$ L^2-orthogonal

Strengthened elliptic estimate:

$\|a'\|_{L^{2,k}} \leq C \, \|Pa\|_{L^{2,k-1}}$

So a' is ptwise bded.

Recall gauge transformations act on $M(SW_\eta)$ & recall $C^\infty(X, S^1)$ has components $H^1(X;\mathbb{Z})$ so via $\mathcal{G}$-action,

● I can move a_{harm} into a fund. domain for action of $H^1(X;\mathbb{Z})$ on $H^1(X;\mathbb{R})$.

So now I get pointwise bounds on (A, Φ) & "elliptic bootstrapping" gives C^∞-bounds. □

Corollary The set $\{ s \in \mathrm{Spin}^c(X) \mid \mathcal{M}(SW, s) \neq \phi \}$ is finite.
↑ unperturbed or fixed perturbation

● Definition We say a solution to (S-W) quations, say (A, Φ), is reducible if $\Phi \equiv 0$, irreducible o/w

Recall The space of sol^{n}s has an action of S^1 constant gauge transformations & the reducibles are the fixed points. When $b'(X) = 0$, $M(SW)^{\text{Coulomb}}$ was compact, & S^1 acts freely on irreducibles & fixes reducibles.

Lemma If $b'(X) = 0$, there are either 0 or 1 reducible solutions, in $M(SW)^{\text{Coulomb}}$

Proof $D_A^+ \Phi = 0$

$F_A^+ = q(\Phi) + \eta$ so if $\Phi = 0$, then $F_A^+ = \eta$

& recall $F_A^+ = F_{A_0}^+ + i\, d^+ a$,

$d^* a = 0$, $a \in \Omega^1(X; \mathbb{R})$

So $\{ a \in \Omega^1(X; \mathbb{R}) \mid d^* a = 0,\ d^+ a = \eta - F_{A_0}^+ \}$ for A_0 fixed Spin^c-connection.

But $d^+ + d^*$ has kernel $\mathcal{H}^1_g(X) = \{0\}$.

So $d^+ + d^*$ is injective

So $\exists$ at most one sol^n a & hence at most 1 reducible

$\square$

Upshot SW_η has a reducible sol^n exactly if $\eta - F_{A_0}^+ \in \operatorname{im}(d^+)$, which is a "codimension b^+" condition.

Expect: $b^+ = 0 \Rightarrow \exists$ reducible

$b^+ = 1$, generic η, there is not

$b^+ \geq 2$, generic paths of η's, you never encounter a reducible

Recap: $SW : \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \longrightarrow \Omega^{2+}(X; i\mathbb{R}) \oplus \Gamma(S^-) \oplus \Omega^0(X)$

$(a, \Phi) \longmapsto (F_A^+ - q(\Phi), D_A^+ \Phi, d^* a)$

$A = A_0 + i a$

The η-perturbed sol^n space is the set $SW^{-1}(\eta, 0, 0)$

The linearisation of the SW-operator $\mathcal{D}(SW)|_{(A, \Phi)}$ has form $(d^+ + L(\Phi), D_A^+, d^*)$ for some 0^{th} order L.

This has index

$$\frac{c_1(S)^2 - \sigma(X)}{4} - b^+(X) + b^1(X)$$

Remark $\mathrm{Index}_{\mathbb{R}}(D^+)$
$= 2\,\mathrm{Ind}_{\mathbb{C}}(D^+)$
$(= -\sigma/8$ in Rokhlin $)$

If $f: M \to N$ is a map of f.d mfds

(i) if $p \in N$ is regular, $Df|_x$ onto $\forall x \in f^{-1}(p)$,
then $f^{-1}(p) \subseteq M$ smooth submfd

(ii) almost all $p \in N$ are regular (Sard)

The Sard-Smale theorem extends this to smooth Fredholm maps of Banach mfds.

So

$$SW : L^{2,k}(;T^*X) \times L^{2,k}(S^+) \\ \downarrow \\ L^{2,k}(i\,su(S^+)) \times L^{2,k-1}(S^-) \times L^{2,k-1}(X)$$

this a Fredholm extension, & for generic $\eta \in \Omega^{2+}(X; i\mathbb{R})$, $SW^{-1}(\eta, 0, 0)$ smooth of dimension $d(s)$ given before

Back to Donaldson: Let X be a smooth oriented 4-mfd with Q_X definite.

Claim: Q_X is diagonalisable

First reductions:

(i) by surgery it suffices to consider $b^1(X) = 0$

(ii) reversing orientation if necessary, consider Q_X negative definite

So $b^+(X) = 0$, $b^-(X) = b^2(X) = -\sigma(X) = r$

Pick metric g & spin^c-structure s s.t. $c_1(s) = \kappa$ in $H^2(X; \mathbb{Z})$ is a characteristic element.

Goal $\kappa^2 \leq -b_2(X)$

● In that case,

$-Q_X : \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$ +ve definite form on $\mathbb{Z}^r$

has $v \cdot v \geqslant r$ ∀ char vectors v

which (Elkies theorem) characterises the diagonal form

We have $\mathcal{M}(SW_\eta) = \{ (A, \Phi) \mid F_A^+ = q(\Phi) + \eta,\ D_A^+ \Phi = 0,\ d^*(A - A_0) = 0 \}$

S^1 acts by constant gauge transformations

& $M_\eta = \mathcal{M}(SW_\eta)/S^1$

● Since $H^1 = 0$, $H^{2+} = 0$, the self-duality complex shows $\{ a \mid d^+a = \eta - F_{A_0}^+ \}$ has a sol^n.

So we have a reducible sol^n.

Away from reducible, $\mathcal{M}_\eta = \mathcal{M}(SW_\eta)/S^1$ is smooth of dimension $d(s) - 1$

Note $d(s) = \frac{c_1^2(s) - \sigma}{4} - \underset{0}{\cancel{b^+}} + \underset{0}{\cancel{b^1}} = \frac{\kappa^2 + b_2(X)}{4}$

● Key Claim: Near the reducible sol^n (A_{red}, Φ), there is a local model

$\mathcal{M}_\eta \underset{\text{locally}}{\sim}$ the quotient of S^1 acting on an even dim^L v. space

$\underset{\text{locally}}{\sim}$ cone over $\mathbb{CP}^{L-1}$, $L = \frac{d(s)+1}{2}$ $\ddot{\sim}$

(Recall, κ char. says $\kappa^2 \equiv \sigma(X)$ mod 8

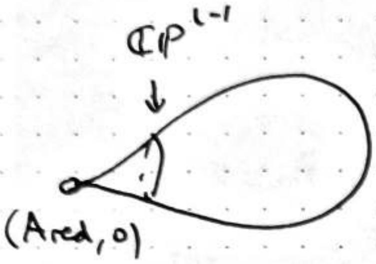
so $d(s) + 1$ is even)

● (Content of claim: reducible sol^n is regular, & hence locally $\mathcal{M}(SW_\eta)$ is modelled on $\ker(D_{A_{red}}^+)$)

Consider

$$\mathcal{M}(SW_\eta) \setminus \{\text{reducible}\} \to \mathcal{M}_\eta \setminus \{\text{reducible}\}$$

S^1 fibration, fibre at (A, Φ) is $\{(A, z\Phi)\}_{z \in S^1}$



See (i) S^1-fibration over $\mathbb{CP}^{L-1}$ is tautological circle bundle, & so $\langle c_1(\mathcal{L})^{L-1}, [\mathbb{CP}^{L-1}] \rangle \neq 0$

But (ii) $\mathbb{CP}^{L-1}$ <u>bounds</u> in smoooth $\mathcal{M}_\eta \setminus \{\text{reducible}\}$ ⨳

Kähler Surfaces

SW : $\Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \to \Omega^{2+}(X; i\mathbb{R}) \oplus \Gamma(S^-)$

$\oplus\ \Omega^0(X)/\mathbb{R}$

$SW(a, \Phi) \mapsto (F_A^+ - q(\Phi), D_A^+ \Phi, d^* a)$

↑ or functions of mean zero

$A = A_0 + ia$

$\text{Index}(D^+) = \dfrac{c_1^2(S) - \sigma(X)}{4}$

& $d^+ + d^* : \Omega^1 \to \Omega^{2+} \oplus \Omega^0$

kernel H^1

cokernel $H^0 \oplus H^{2+}$

has index $b_1 - (1 + b^+)$

SW above has index $\dfrac{c_1^2(S) - \sigma}{4} + b_1 - b^+$

(before dividing out by S^1 gauge transformations)

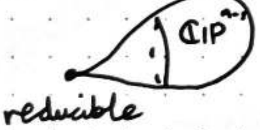
In Donaldson's theorem,

$b_1 = 0$ & Q_X negative definite so $b^+ = 0$

& $\mathcal{M}(SW)$ has dim $\dfrac{c_1^2(S) + b_2}{4}$ as $\sigma = -b_2$

↑ even, as $c_1(S) = \kappa$ is characteristic

So $\mathcal{M}(SW)/S^1$ is odd-dim



Contradiction since $\mathcal{M}(SW) \to \mathcal{M}(SW)/S^1$ is a copy of taut. bundle over $\mathbb{CP}^{L-1}$

so $\langle c_1(\mathcal{L}_{taut})^{L-1}, [\mathbb{CP}^{L-1}] \rangle \neq 0$

<u>BUT</u> $\mathbb{CP}^{L-1}$ bounds in smooth locus $\mathcal{M}(SW)/S^1$

Resolution to contradiction:

there are $\underline{\underline{\text{no}}}$ irreducibles

which forces $\text{vdim}(\mathcal{M}(SW)) < 0$

Lemma The reducible sol^n is generically regular

Proof Consider the map

$$D^{par} : \Omega^1(X; i\mathbb{R}) \times \Gamma(S^+) \longrightarrow \Gamma(S^-)$$
$$(a, \phi) \longmapsto D^+_{A_0+a}(\phi)$$

which we view as a parametrised Dirac operator, param^d by a.

Restrict domain to $\Omega^1 \times \Gamma(S^+) \setminus \{0\}$

Key claim $0 \in \Gamma(S^-)$ is a regular value

In that case, if we consider projection

$$(D^{par})^{-1}(0) \xrightarrow[proj]{} \Omega^1(X; i\mathbb{R}) \quad \text{is Fredholm}$$

since $D^+_{A_0+a}$ is,

& $(proj)^{-1}(a) = \ker(D^+_{A_0+a}) \setminus \{0\}$

Regular values of proj exactly correspond to a s.t.

$$\text{Coker}(D^+_{A_0+a}) = \{0\}$$

$$D(D^{par})\big|_{(a,\phi)} : (b, \psi) \mapsto D^+_{A_0 \underset{+a}{\uparrow}}\psi + \rho(a)\cdot\psi + \rho(b)\cdot\phi$$

We want to show $\forall (a, \phi) \in (D^{par})^{-1}(0)$, this is onto.

If $\alpha \perp_{L^2}$ (image of $D(D^{par})$) & $b = 0$,

$\alpha \perp_{L^2} D^+_{A_0+a}\psi \quad \forall \psi$

So $\alpha \in \ker(D^-_{A_0+a})$ the adjoint.

By hypothesis, $\phi(x)$ is non-zero in some open set, $\mathcal{U} \subseteq X$.

So if I take $\psi = 0$, I see $\alpha \perp_{L^2} \rho(b)\cdot\phi \quad \forall b$

so $\alpha \equiv 0$ on $\mathcal{U} \underset{open}{\subseteq} X$

But elements in the kernel of a diff operator like $D^-_{A_0+a}$ satisfy unique continuation, so $\alpha = 0$.

So param^d Dirac operator has smooth 0-locus (Sard-Smale theorem) □

Recall: an almost cx str J on X defines a canonical spin^c structure $(U(n) \to Spin^c(n))$

& $Spin^c(X) \xleftarrow{\sim} H^2(X;\mathbb{Z})$

$S_{can} \otimes L \longleftarrow\!\!| \quad L$

The canonical str. has $\det(S^{\pm}) = K_X^{-1}$

Lemma For the spin^c-str $S_{can} \otimes \underset{\overset{\uparrow}{\text{trivial!}}}{L}$ on a Kähler surface (X, ω, J)

$$\mathbb{S}^+ \cong \Omega^0(X;\mathbb{C}) \oplus \Omega^{0,2}(X;\mathbb{C})$$

$$\mathbb{S}^- \cong \Omega^{0,1}(X)$$

& $D^+ : \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ becomes Dolbeault operator $\bar\partial_A + \bar\partial_A^*$ $\leftarrow$ A connection in L_{triv}

Sketch V real v. space with cx str J

$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ $\pm i$ eigenspaces for J

Then V acts on $\Lambda^* V^{1,0}$ by $v \cdot (v_1 \wedge \cdots \wedge v_k)$
$= v^{1,0} \wedge v_1 \wedge \cdots \wedge v_k$
$- \iota_{v^{1,0}}(v_1 \wedge \cdots \wedge v_k)$

Check: $v \cdot v (-) = -|v|^2 (-)$ so this extends to make $\Lambda^*(V^{1,0})$ a $Cl(V)$-module

$\cong$ $\mathbb{C}$-linear

$$\Lambda^*(V^{0,1})^* = \bigoplus_k \Omega^{0,k}(X)$$

D_A^+ & $\bar\partial_A + \bar\partial_A^*$ have same symbol

So their difference is order 0 diff^l operator

So the difference is linear over functions. If $\phi_{can} \equiv 1 \in \Gamma(\mathbb{S}^+)$ then D_A^+, $\bar\partial_A + \bar\partial_A^*$ both kill ϕ_{can}, so agree on $\Omega^0(X;\mathbb{C})$ i.e. $D_A^+|_{\Omega^0} = \bar\partial_A$

Also: $D^+|_{\Omega^{0,2}} = \bar\partial_A^*$, by checking they behave same way on $\rho(w)\cdot\phi_{can}$ $\square$

~~We can write $\Phi = (\alpha,\beta) \in \Omega^0$~~

More generally, for $S_{can}\otimes L$, $S^+ = \Omega^0(L)\oplus\Omega^{0,2}(L)$

$S^- = \Omega^{0,1}(L)$

& if $\Phi\in\Gamma(S^+)$, I can write $\Phi = (\alpha,\beta)$ wrt splitting & Dirac eqⁿ is $\boxed{\bar\partial_A\alpha + \bar\partial_A^*\beta = 0}$

Now $\Omega^{2+}(X)\otimes\mathbb{C} = \Omega^0(X,\mathbb{C})\cdot\omega \oplus (\Omega^{2,0}\oplus\Omega^{0,2})$

& $\underset{\overset{\cap}{\Omega^{2,+}(i\mathbb{R})}}{F_A^+} = if\cdot\omega + (\mu-\bar\mu)$, $\mu\in\Omega^{0,2}(X,\mathbb{C})$

$f\in C^\infty(X,\mathbb{R})$

$F_A^+ = (\Phi\Phi^*)_{\text{trace-free}}$ becomes
$$\boxed{\begin{aligned}(F_A^+)^{1,1} &= \tfrac{1}{4}(|\alpha|^2-|\beta|^2)\,\omega\\ (F_A^{0,2}) &= \frac{\bar\alpha\beta}{2}\end{aligned}}$$

<u>Useful perturbation</u>: $(F_A^+)^{1,1} = \frac{1}{4}(|\alpha|^2-|\beta|^2-t^2)\cdot\omega$, $t\in\mathbb{R}$

<u>Key Lemma</u> $\bar\alpha\beta = 0$ $(t=0)$

Proof $\bar\partial_A\alpha + \bar\partial_A^*\beta = 0 \Rightarrow \underbrace{\bar\partial_A\bar\partial_A}_{F_A^{0,2}=\frac{\bar\alpha\beta}{2}}\alpha + \bar\partial_A\bar\partial_A^*\beta = 0$

$\therefore \frac{1}{2}|\alpha|^2\beta + \bar\partial_A\bar\partial_A^*\beta = 0$

Take inner product with β, & $\int_X$

$\therefore \int_X|\alpha|^2|\beta|^2\,\mathrm{vol} + \|\bar\partial_A^*\beta\|^2_{L^2} = 0$

So $\bar\alpha\beta = \alpha\bar\beta = 0$. $\square$

Deduce: $F_A^{0,2} = 0$

So A defines a holomorphic structure on line b L,
& $\alpha \in H^0(L)$ is a global holo. section

Moreover, $\alpha\bar{\beta} = 0 \Rightarrow$ one of α, β vanish identically
(by unique continuation ideas)

(Depending on $\deg(L)$)

Theorem Suppose X is a Kähler surface of general type $b^+ > 1$, $K_X = [\omega]$. Then for S_{can} & perturbation $t > 0$, then $\mathcal{M}(SW_t)/S^1 = \{pt\}$ (In fact, regular)

Remark We said that expected $\dim(\mathcal{M}(SW)) = \frac{c_1^2(S) - \sigma}{4} + b_1 - b^+$

mean zero f^ns in codomain

So $M(SW)/S^1$ has dim $\frac{c_1^2(S)-\sigma}{4} - \underbrace{(1 - b_1 + b^+)}_{\frac{e+\sigma}{2\cancel{4}}}$

$$\frac{c_1^2(s)}{4} - \frac{(2e + 3\sigma)}{4}$$

$e = 2 - 2b_1 + b^+ + b^-$
$\sigma = b^+ - b^-$

But on a complex surface, $c_1(TX, J)^2 = \frac{2e + 3\sigma}{4} = K_X^2$

If $L \in H^2(X, \mathbb{Z})$,
$S_{can} \otimes L$ has $\det(S^+) = L^2 \otimes K^{-1} = 2L - K$

So $\text{vdim}(\mathcal{M}(SW)/S^1) = (2L - K)^2/4 - K^2 = L(L-K)$

Recall also reducible sol^{n}s are a codim (b^+) phenomenon, so if $b^+ \geq 2$, then $\mathcal{M}(SW)/S^1$ is generically a mfd, well-defined up to cobordism.

If ~~D~~ $L(L-K) = 0$, then I expect $\mathcal{M}(SW)/S^1$ to gen. be a finite set.

Then $SW_X : H^2(X; \mathbb{Z}) \to \mathbb{Z}/2$ is the parity of this set, defined on $\{L \in H^2(X; \mathbb{Z}) \mid L(L-K) = 0\}$.

(where X Kähler, $c_1(TX, J) = -K$)

Theorem 2 In same setting, if $SW_X(E) \neq 0$, then $E \in \{0, K_X\}$

Corollary In the same setting, if $f : X \longrightarrow X$ is any diffeo, then f preserves K_X i.e. $f^*(K_X) = \pm K_X$

Finite-dimensional approximation

Theorem Let (X, ω) be a minimal Kähler surface,
$K_X = [\omega]$, $b^+ > 1$, $b^1(X) = 0$. Then
$SW : \mathrm{Spin}^c(X) \to \mathbb{Z}$ was $s \mapsto \begin{cases} \pm 1 & \text{canonical str or its dual} \\ 0 & \text{o/w} \end{cases}$

Proof We saw SW eqns become

$$\begin{cases} (F_A^+)^{1,1} = \frac{1}{4}\left(|\alpha|^2 - |\beta|^2 - t^2\right)\omega, & t \in \mathbb{R} \text{ perturbation} \\ F_A^{0,2} = \frac{\bar{\alpha}\beta}{2} & \text{where } \Phi = (\alpha, \beta) \\ \bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0 \end{cases}$$

& we showed $\bar{\alpha}\beta \equiv 0$ so A defines a holo^c str on the line bundle L, & $\alpha \in H^0(L)$, $\bar{\beta} \in H^0(K - L)$

Recall in general L was a square root of $\det(s) \otimes K_X$ so for s_{can}, $\alpha \in H^0(\underset{\substack{\uparrow \\ \text{trivial} \\ \text{line b.}}}{\mathbb{C}})$ so α is constant,

& $\beta \equiv 0$ so then $(F_A^+)^{1,1} = (|\alpha|^2 - t^2)\,\omega$ shows RHS $\equiv 0$ since $A \in \mathrm{Conn}(L)$ & L is now trivial.

So $|\alpha| = t$, so up to S^1-action of constant gauge transformations α is completely determined.

Now since L is trivial, A is the trivial connection, F_A cannot have interesting $(1,1)$-part.

Shows for $s = s_{can}$, $\mathcal{M}(SW_t)/_{S^1} = \{pt\}$ & we quote this solution is regular.

To show $SW(s) = 0$ if $s \notin \{s_{can}, s_{can}^*\}$, we use the Hodge index theorem which says that the signature of $Q_X|_{\underset{H^{1,1}(X,\mathbb{R})}{P_{\mathbb{R}}(X)_{\mathbb{R}}}}$ is $(1, \rho - 1)$ for $\rho = \mathrm{rank}(H^{1,1})$

Suppose $L \in H^2(X, \mathbb{Z})$ s.t. $SW(s_L) \neq 0$

Certainly $\mathrm{vdim}(\mathcal{M}(SW(s_L))) \geq 0$ & so $\frac{1}{4}\left(c_1(s)^2 - (2e + 3\sigma)\right) \geq 0$

& $c_1(s) = 2L - K$ $\Rightarrow$ $L(L-K) \geqslant 0$ so $L^2 \geqslant KL$

Also $\exists$ holo^c str on L s.t. $\alpha \in H^0(L)$, $\bar{\beta} \in H^0(K-L)$, &

$[\omega] = K_X$ so $K.L \geqslant 0$ (>0 if L is not torsion)

$K\cdot(K-L) \geqslant 0$ (>0 if $K.L$ is not torsion)

So if $H^2(X,\mathbb{Z})$ is torsion-free e.g. $\pi_1 X = 0$ then

$\begin{pmatrix} K^2 & K.L \\ K.L & L^2 \end{pmatrix}$ has +ve trace, & +ve determinant so is +ve definite $\#$ (Hodge index)

Step back:

$\mathcal{W}^+ = \Gamma(S^+) \oplus \Omega^1(X; i\mathbb{R})$

$\mathcal{W}^- = \Gamma(S^-) \oplus \Omega^{2+}(X;\mathbb{R}) \oplus \Omega^0(X)/\mathbb{R}$

& $SW : \mathcal{W}^+ \to \mathcal{W}^-$ has form $SW = \ell + c$

where $\ell = D \oplus (d^+, d^*)$ linear Fredholm

$c = (\gamma(ia)(\phi), q(\phi), 0)$ non-linear compact operator

(i.e. c maps bounded sets to rel. cpct sets)

Weitzenbock theorem (ptwise bounds on Φ if $SW(A,\Phi) = 0$) shows SW^{-1}(bounded) is bounded

Lemma If $f = \ell + c : H' \to H$ is such a map of Hilbert spaces, & if $\mathbb{W}^- \subseteq H$ is a suff large f.d. subspace, & if $\mathbb{W}^+ = \ell^{-1}(\mathbb{W}^-)$, then

$$\text{image}(f|_{\mathbb{W}^+}) \cap \text{Sphere}(\mathbb{W}^-)^\perp = \phi$$

Consequence: Let $p : \mathcal{W}^- \setminus S(\mathbb{W}^-)^\perp \to \mathbb{W}^-$ projection

Consider the map $p \circ SW|_{\mathbb{W}^+} : \mathbb{W}^+ \to \mathbb{W}^-$

Now $\mathbb{W}^\pm$ are f.dim & our map sends bounded sets to bounded sets, so it extends to one-point compactifications, & get $sw \in [\mathbb{W}^+_\infty, \mathbb{W}^-_\infty]$ & this has a htpy class

& this has a htpy class well-defined stably

$\in \pi_d^{st}$, $d = \frac{c_1(s)^2 - (2e+3\sigma)}{4} + 1$ ($b^1=0$ so only S^1-gauge transform^n)

Example If X min Kähler gen type, $b^+>1$, $b^1=0$, s_{can},
$SW_t^{-1}(0) \cong S^1$ of gauge transf. And the circle with its Lie group framing is unique non-trivial elt $\eta \in \pi_1^{st} = \mathbb{Z}/2$

Remark: The original SW map $\mathcal{W}^+ \to \mathcal{W}^-$ also extends to 1-pt compactifications $\underset{S^\infty =}{(\mathcal{W}^+)_\infty} \to \underset{= S^\infty}{(\mathcal{W}^-)_\infty}$

& $S^\infty \simeq pt$ is contractible, so there is no homotopy theoretic content

Remark Bauer's gluing theorem says $SW_{X\#X} = SW_X \underset{\substack{\uparrow \\ \text{smash}}}{\wedge} SW_X$
when $b_1(X) = 0$, $b^+(X) > 1$
And $\eta \wedge \eta \neq 0$ in π_2^{st}. But the "ordinary" SW-invts vanish for any $X \# X$.

Proof of Lemma If $f = l + c : H' \to H$, let $D \subseteq H$ be closed unit ball.
Then $f^{-1}(D) \subseteq H'$ is bounded, so $C = \text{closure}\,(c(f^{-1}(D)))$ is compact in H.
Cover C by finite set of radius $\varepsilon < \frac{1}{4}$ balls, centres v_i
Let $\mathbb{W}^- \supseteq \mathrm{Span}\langle v_i \rangle + (\mathrm{im}\, l)^\perp$
⌞ l Fredholm so coker is f.d.
Suppose $w \in S(\mathbb{W}^-)^\perp$ & image $f|_{\mathbb{W}^+}$ ($\mathbb{W}^+ := l^{-1}(\mathbb{W}^-)$)
Then ~~$(l+c)$~~ consider $\underbrace{f^{-1}(w) \cap \mathbb{W}^+}_{(*)} \subseteq f^{-1}(D)$ (w has unit length)
So $(*)$ is mapped by $(l+c)|_{\mathbb{W}^+}$ into $\mathbb{W}^- + C$

& these sets are distance $1-\varepsilon$ apart.

Intersection forms of 1-connected 4-mfds were governed by:

Rokhlin	X spin, $16 \mid \sigma(X)$
Donaldson	Q_X definite $\Rightarrow$ Q_X diagonalisable
Furuta	X spin & Q_X indefinite $b_2(X) \geqslant \frac{10}{8}\lvert\sigma(X)\rvert + 2$
	($b_2(X) \geqslant \frac{11}{8}\lvert\sigma(X)\rvert$ conjectured but open)

Furuta's theorem uses f.d. reduction

We have $SW : \mathbb{W}^+ \longrightarrow \mathbb{W}^-$ & we write $\mathbb{W}^+ = \mathbb{H}^a \oplus \mathbb{R}^b$

$\mathbb{W}^- = \mathbb{H}^\alpha \oplus \mathbb{R}^\beta$, recalling X spin, $\Gamma(S^\pm)$ naturally quaternionic.

(so $\mathbb{W}^- = \mathbb{H}^\alpha \oplus \mathbb{R}^\beta \subseteq \Gamma(S^-) \oplus (\Gamma(\Omega^{2+} \oplus \Omega^0/_{\mathbb{R}})))$

Atiyah - Singer theorem says

$$\alpha = a - \frac{\sigma}{16} \qquad (\text{on Spin mfd, so } c_1(s) = 0)$$
$$\beta = b - b_2^+$$

Definition The group

$$\begin{array}{c} \mathrm{Pin}(2) \underset{C^\infty}{\cong} S^1 \amalg S^1 \\ \| \\ S^1 \cup_j S^1 \subseteq \mathrm{Unit}(\mathbb{H}) \end{array}$$

If $\mathbb{H} = \langle 1, i, j, k \rangle$ then $\mathrm{Pin}(2) = \begin{array}{c} \{\cos t + i \sin t\} \\ \amalg \\ \{j \cos t - k \sin t\} \end{array}$

$\mathrm{Pin}(2)$ acts on $\Gamma(S^\pm)$ as these are quaternionic

$\mathrm{Pin}(2) \to \mathbb{Z}/2$ & $\mathbb{Z}/2$ acts on $\Omega^*(X)$ by mult^n by -1
$j \mapsto 1$

Lemma SW is $\mathrm{Pin}(2)$-equivariant

Upshot: we can extract a $\mathrm{Pin}(2)$-equiv^t map of f.d. spheres $(\mathbb{W}^+)_\infty \to (\mathbb{W}^-)_\infty$ & hence an elt of $\mathrm{Pin}(2)$-π^{st}

Classical Borsuk-Ulam thm

$S^n \to S^m$ $\underset{\text{antipodal}}{\mathbb{Z}/2}$-equivt then $m \geqslant n$

<u>Fact</u> there is no Pin(2)-equivt map from $\mathbb{H}^a \oplus \mathbb{R}^b \to \mathbb{H}^\alpha \oplus \mathbb{R}^\beta$ unless Furuta's constraint holds.