Knots

Jake Rasmussen
jar60@cam.ac.uk
Office E2.02

0) Foundations

① Isotopies

Spose $f: M \to N$ is a map of smooth manifolds

Def: f is an embedding ($f: M \hookrightarrow N$) if

$$df: TM \to TN \text{ is injective}$$

Inverse Function Theorem: If $df|_x$ is injective, then $\exists\, U$ ngbd of x w/ $f|_U$ an embedding.

Def: If $f_0, f_1: M \hookrightarrow N$ are embeddings, say f_0 is isotopic to f_1 ($f_0 \sim_i f_1$), if there is a smooth map $F: M \times I \to N$ with

$$\left.\begin{aligned} f_0(x) &= F(x, 0) \\ f_1(x) &= F(x, 1) \end{aligned}\right\} \text{ all } x$$

and f_t is an embedding for all $t \in I$, where $f_t(x) = F(x, t)$.

Isotopy $\Rightarrow$ homotopy.

F is an isotopy.

Lemma If $f_0 \sim_i f_1$ via F, then $\exists\, \hat{F}$ s.t. $f_0 \sim_i f_1$ via $\hat{F}$ with

$$\hat{F}(x, t) = f_0(x) \quad \forall t \leq \tfrac{1}{4}$$
$$\hat{F}(x, t) = f_1(x) \quad \forall t \geq \tfrac{3}{4}$$

Proof: Choose $g: I \to I$ smooth s.t.

$$g(t) = 0 \text{ for } t \leq \tfrac{1}{4}$$
$$g(t) = 1 \text{ for } t \geq \tfrac{3}{4}$$

and set $\hat{F}(x, t) = F(x, g(t))$. $\square$

Corollary Isotopy is an equivalence relation.

Ex 1 If $\vec{v}'(t)$ is a smooth compactly supported, time dependent vector field on M, then there's an isotopy (the flow of $\vec{v}$) $\Phi : M \times [0,1] \longrightarrow M$

with $\Phi_0 = id_M$

$$\left.\frac{d\Phi}{dt}\right|_{(x,t)} = v(\underset{\underset{\Phi(\cdot)}{\uparrow}}{x}, t)$$

Ex 2 If $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $df|_0$ injective then $\exists\, U$ nbhd of $0 \in \mathbb{R}^m$ s.t. $f|_U \sim_i df|_0|_U$

Proof $F(x,t) = t f(x) + (1-t)\, df|_0(x)$

Then $df_t|_0 = t\, df|_0 + (1-t)\, df|_0 = df|_0$

$\Rightarrow \exists\, U_t$ s.t. $f_t|_{U_t}$ is an embedding

To get a uniform U, consider

$dF|_{(0,t)} = df|_0 \oplus id$ injective

⌈map into $\mathbb{R}^n \times I$?⌋

$\Rightarrow$ can find ε s.t. $F|_{B_\varepsilon(0) \times [t-\varepsilon, t+\varepsilon]}$ is an embedding, use compactness. ?

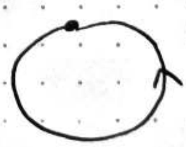
② ~~Isotopies~~ Knots

Def An oriented knot in $\mathbb{R}^3$ is an isotopy class of ~~an~~ embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$

Ex The unknot is the class of

$U : S^1 \longrightarrow \mathbb{R}^3$

$(x,y) \mapsto (x,y,0)$



Exercise: If $\varphi : S^1 \to S^1$ is an orientation-preserving diffeo, then $\varphi \sim_i \mathrm{id}_{S^1}$

$\Rightarrow K \circ \varphi \sim_i K \circ \mathrm{id}_{S^1} = K$

(reparametrise w/o changing isotopy class)

Def: The reverse of K is $r(K) = K \circ r$

where $r : S^1 \to S^1$

$(x, y) \mapsto (x, -y)$

Ex: $U \sim_i r(U)$

Def: $\{\text{Knots in } \mathbb{R}^3\} = \{\text{oriented knots in } \mathbb{R}^3\} / \sim$

where $K \sim r(K)$

Diagrams:

Def: A knot diagram is a) a smooth map $\gamma : S^1 \to \mathbb{R}^2$ s.t.

① $\gamma'(t) \neq 0$ for all $p \in S^1$

No!

② If $\gamma(p) = \gamma(p')$, $p \neq p'$, then $\gamma'(p), \gamma'(p')$ are linearly indep. (transverse double pts)

yes

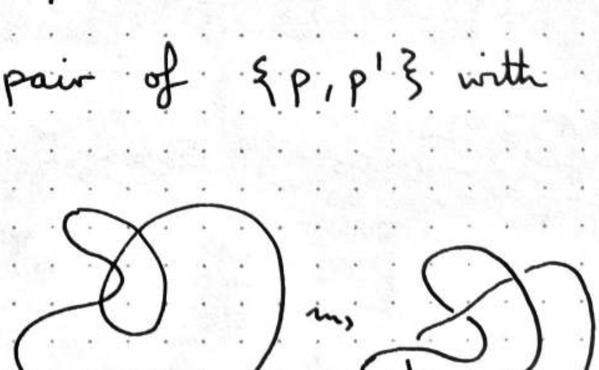
No!

③ $\nexists$ distinct p, q, r s.t. $\gamma(p) = \gamma(q) = \gamma(r)$ (no triple intersections)

No!

b) an ordering $p > p'$ on each pair of $\{p, p'\}$ with $\gamma(p) = \gamma(p')$

"on top" = >



Choose $z : S^1 \longrightarrow \mathbb{R}$ with $z(p) > z(p')$ whenever $\gamma(p) = \gamma(p')$ and $p > p'$

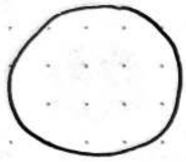
Define $K : S^1 \longrightarrow \mathbb{R}^3$

$$p \longmapsto (\gamma(p), z(p))$$

Choice of z doesn't matter; if $\hat{z}$ is another choice then $K \sim \hat{K}$ via

$$F(p,t) = (\gamma(p),\ t z(p) + (1-t)\hat{z}(p))$$

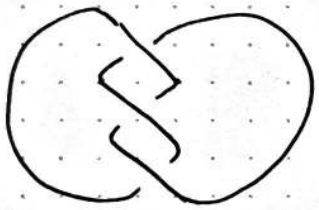
Ex:



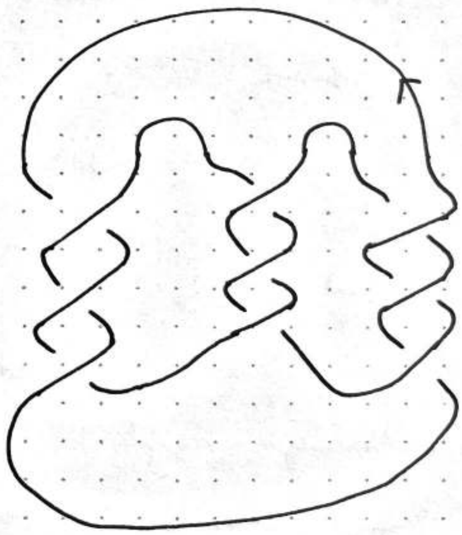
unknot

unknot

unknot



negative trefoil



$K \neq r(K)$

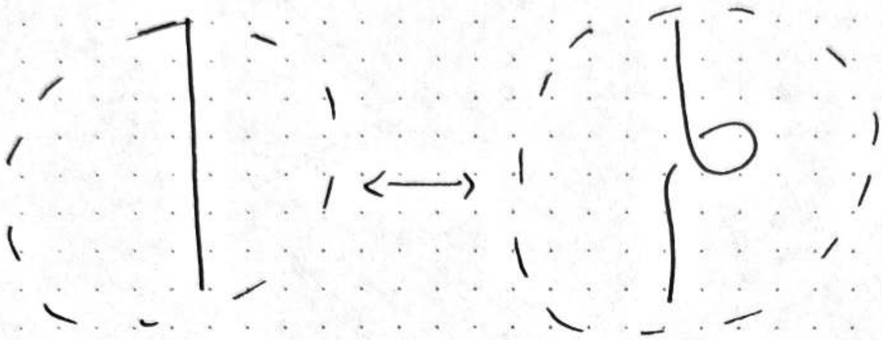
If $\vec{v} \in S^2$, $\pi_v : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is orthogonal projection.

Theorem: Given knot $K : S^1 \hookrightarrow \mathbb{R}^3$. There's an open dense subset $U \subset S^2$ s.t. $\forall v \in U$, $\pi_v \circ K$ gives a knot diagram; $p > p'$ if $v \cdot K(p) > v(K(p'))$

Reidemeister moves

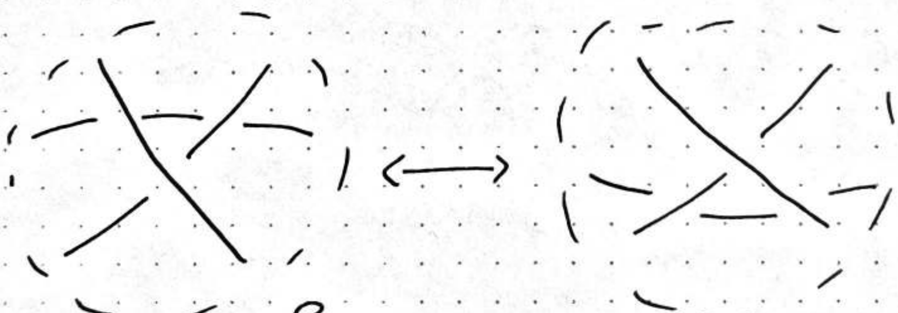
- Given D, D' diagrams representing the same K, how are D, D' related?

Reidemeister moves:



R_I $\quad$ R_I'

R_{II} $\quad$ R_{II}'



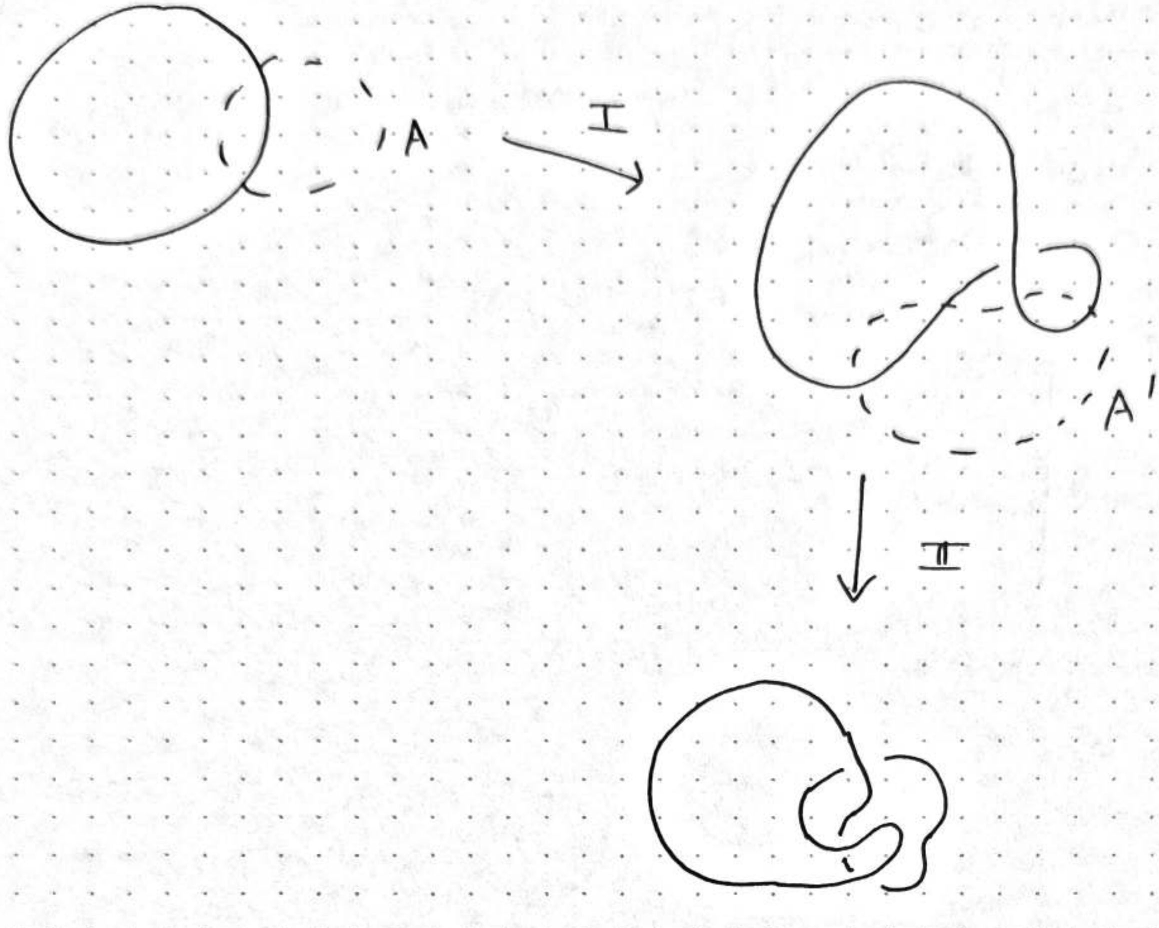
R_{III} $\quad$ R_{III}'

<u>Def</u>: diagrams D, D' are locally equivalent if there is $A \subset \mathbb{R}^2$, $A \cong D^2$ s.t. $D \cap (\mathbb{R}^2 \setminus A) = D' \cap (\mathbb{R}^2 \setminus A)$

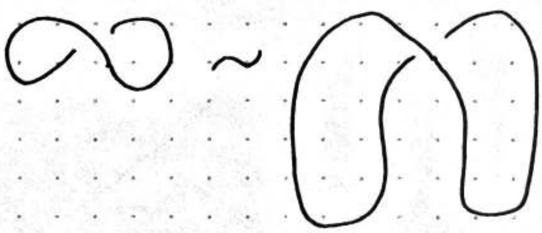
and there are homeomorphisms

$$\varphi : (A, D \cap A) \xrightarrow{\sim} (D^2, R_i)$$
$$\varphi' : (A, D' \cap A) \xrightarrow{\sim} (D^2, R_i')$$

[I think we need φ, φ' to be identity ~ equal on boundary]



<u>Thm</u>: (Reidemeister) Let $\sim$ be the equivalence relation on diagram regenerated by local moves and ~~homeomorphisms~~ diffeomorphisms of $\mathbb{R}^2$. If D, D' represent isotopic knots K, K' then $D \sim D'$.



0) Continued

$\vec{v} \in S^2$, $\pi_v : \mathbb{R}^3 \to \mathbb{R}^2$

$K : S^1 \hookrightarrow \mathbb{R}^3$ is a knot

$\gamma_v : S^1 \to \mathbb{R}^2$, $\gamma_v = \pi_v \circ K$

γ_v gives a knot diagram if

1) $\gamma_v'(p) \neq 0$ for all p (γ_v is immersed)
2) double points are transverse
3) no triple points

Thm: There is a dense open set $\mathcal{U} \subset S^2$ s.t. if $v \in \mathcal{U}$, then γ_v is a knot diagram

Spose $f : M \to N$ is a smooth map.

Say $\vec{x} \in M$ is a critical point for f if $df|_{\vec{x}}$ is not surjective and say $\vec{y} \in N$ is a critical value if $f^{-1}(\vec{y})$ contains a critical point.

Sard's Thm: The set of critical values of f has measure zero in N

(!)

So if M is compact, the set of regular values of f is open & dense in N.

needs tweaking: $p>q$ vs $p<q$

Consider $\varphi : S^1 \times S^1 \to S^2$

$$(p,q) \longmapsto \rho(K(p) - K(q))$$
$$(p,p) \longmapsto \rho(K'(p))$$

$\rho : \mathbb{R}^3 \setminus 0 \to S^2$

$\vec{x} \mapsto \dfrac{\vec{x}}{\|\vec{x}\|}$

① $\gamma_v'(p) \neq 0 \iff \pi_v(K'(p)) \neq 0$

$\iff \rho(K'(p)) \neq \pm v$

so 1) holds $\forall p \iff \pm v \notin \varphi(\Delta)$, $\Delta = \{(p,p)\} \subset S^1 \times S^1$

② (p,q) is a double point of γ_v

$\iff K(p) - K(q) = \lambda \vec{v}$

$\iff \varphi(p,q) = \pm \vec{v}$

If the above holds,

$$d\varphi|_{(p,q)}(\alpha,\beta) = d\rho_{\lambda v}(\alpha K'(p) - \beta K'(q))$$

Now $d\rho_{\lambda v}(w) = \frac{1}{\lambda}\pi_v(w)$

So
$$d\varphi|_{(p,q)}(\alpha,\beta) = \frac{1}{\lambda}\pi_v(\alpha K'(p) - \beta K'(q))$$
$$= \frac{1}{\lambda}(\alpha \gamma_v'(p) - \beta \gamma_v'(q))$$

So $d\varphi|_{(p,q)}$ is surjective $\iff$ $\gamma_v'(p), \gamma_v'(q)$ are l.i.

By Sard's thm, $\exists$ open dense set $U \subset S^2$ s.t.

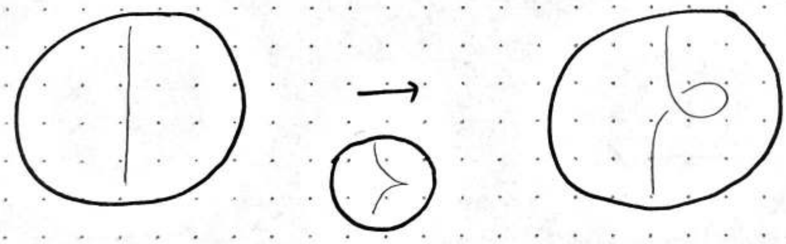
1) $U \cap \pm\varphi(\Delta) = \emptyset$

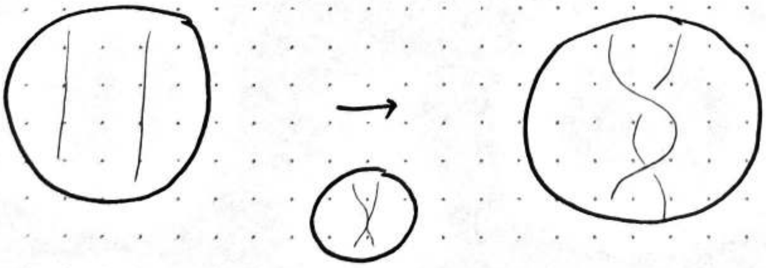
2) $v \in U \Rightarrow v$ is a regular value of φ

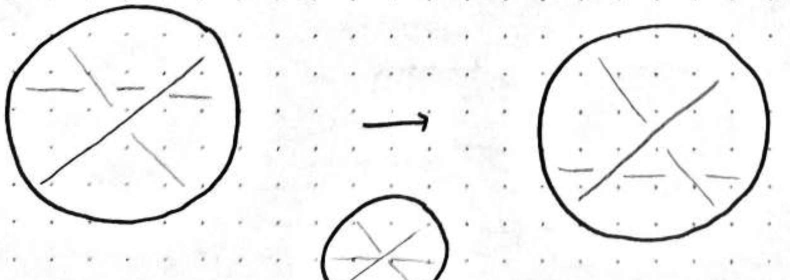
$\Rightarrow$ ① & ② hold

Condition 3) is similar: show that if 1) & 2) hold then there is a nearby v' for which 3) holds.

Reidemeister Moves:







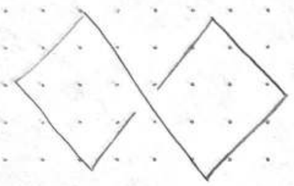
PL Knots:

If $v = (v_0, \ldots, v_n) \in (\mathbb{R}^3)^{n+1}$ with $v_0 = v_n$,
then let $K(v) = \bigcup_{i=1}^{n} \overline{v_{i-1} v_i}$

line segment

Def: $K(v)$ is a PL knot if

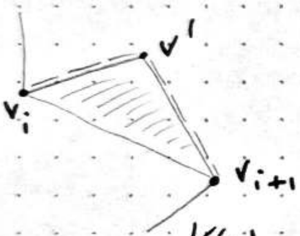
$$\overline{v_{i-1} v_i} \cap \overline{v_{j-1} v_j} = \emptyset \quad \text{for } i \notin \{j-1, j, j+1\}$$

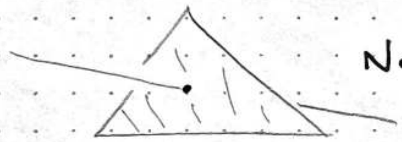


Def Spose $K(v)$ is a PL knot
Choose $v' \in \mathbb{R}^3$ with

$$(\Delta v_i v' v_{i+1}) \cap K(v) = \overline{v_i v_{i+1}}$$

We say $K(v)$ is locally equivalent to
$K(v_0, \ldots, v_i, v', v_{i+1}, \ldots, v_n)$.





Not allowed

Def PL equivalence is the equivalence relation on PL knots generated by local equivalence

Thm: There's a bijection

$$\left\{ \begin{matrix} \text{smooth} \\ \text{knots} \end{matrix} \right\} \longleftrightarrow \left\{ \text{PL Knots} \right\} / \text{PL isotopy}$$

$$K \longmapsto L(K)$$

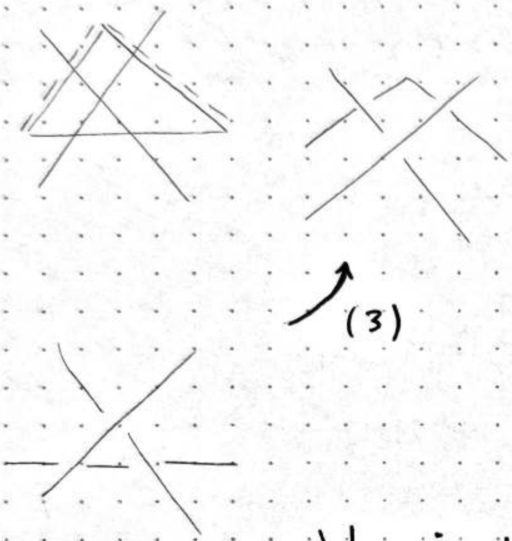
if K is piecewise smooth isotopic to $L(K)$

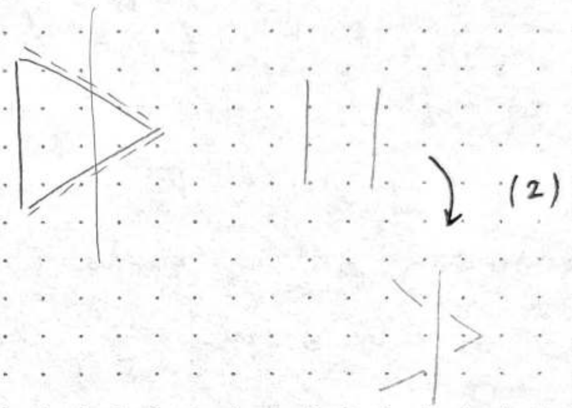
Spose K, K' are locally equivalent.
After subdividing triangles, can assume that

$$\pi(\Delta v_i v' v_{i+1}) \text{ intersects } \pi(K \setminus \overline{v_i v_{i+1}})$$

in either ① a line segment

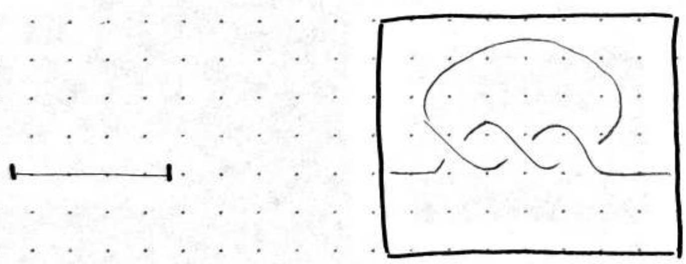
② two line segments w/ a single crossing





<u>Warning</u>: Continuous maps are not your friends

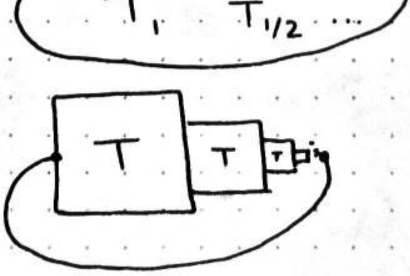
<u>Ex</u>: $T : [-1,1] \longrightarrow [-1,1]^3$



$T_s(t) = s\,T(t/s)$

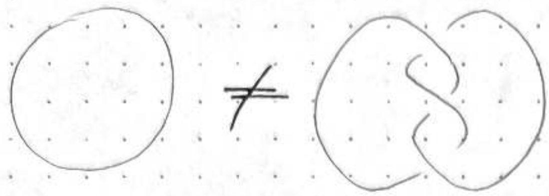
$T_s : [-s,s] \to [-s,s]^3$

$f : I \longrightarrow \mathbb{R}^3$



I) Jones Polynomial

- Want to show



Idea: $I : \{\text{diagrams}\} \longrightarrow S$

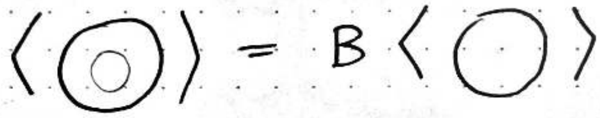
If I doesn't change under R. moves, it descends to a map $I : \{\text{knots}\} \longrightarrow S$.

3.1) Kauffman Bracket

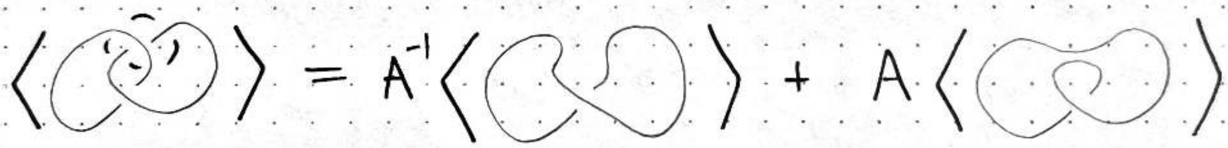
- Prop^n There is a unique map $\langle \ \rangle : \{\text{Knot diagrams}\} \to \mathbb{Z}[A^{\pm}, B]$ satisfying 0) $\langle \emptyset \rangle = 1$

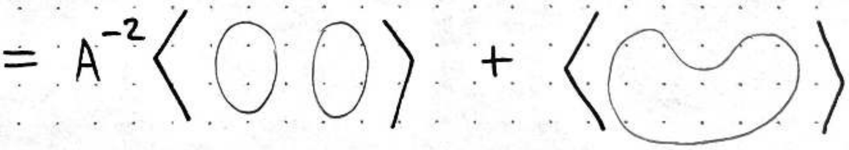
and the local rules

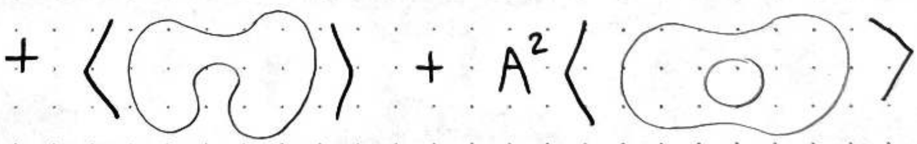
1) 

2) 

Ex:



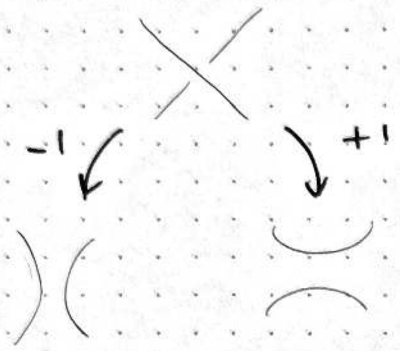




$$= A^{-2}B^2 + 2B + A^2B^2$$

Proof

If D has n crossings, can apply rule 1) to every one. Set of possible resolutions is in bijection with $\{\pm 1\}^n$



So given $v \in \{\pm 1\}^n$, get a diagram D_v with no crossings.

Then $\langle D_v \rangle := B^{|D_v|}$

for $|D_v| = \#$ components of D_v.

So if $\langle \ \rangle$ exists, it's given by

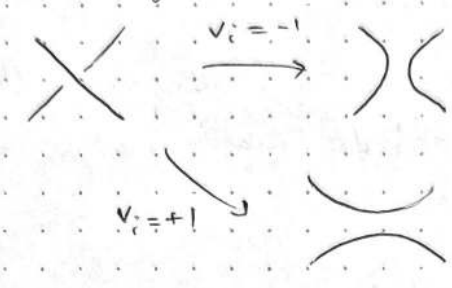
$$\langle D \rangle := \sum_{v \in \{-1\}^n} A^{\Sigma v_i} B^{|D_v|}$$

1.1) Kauffman Bracket (ct'd)

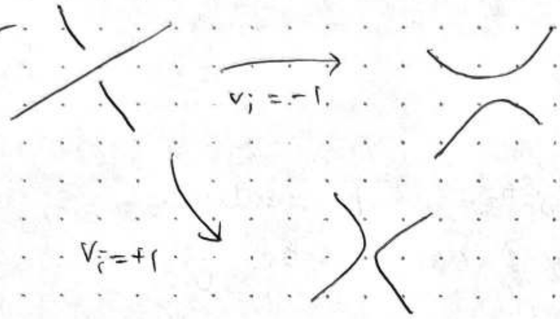
- Planar diagram D with n-crossings

  Order crossings.

  Given $v \in \{\pm 1\}^n$, assign D_v to it by resolving the i^{th} crossing according to v_i:



- So for



Define

$$\langle D \rangle = \sum_v A^{\Sigma v_i} B^{|D_v|}$$

where $|D_v| = \#$ cpts of D_v

Prop: $\langle\ \rangle$ satisfies

① $\langle \emptyset \rangle = 1$

② $\langle \times \rangle = A^{-1} \langle)(\rangle + A \langle \asymp \rangle$

③ $\langle \bigcirc \rangle = B \langle\ \rangle$

Proof: 1) obvious

- 2) $\displaystyle \langle \times \rangle = \sum_{v_j = -1} A^{\Sigma v_i} B^{|D_v|} + \sum_{v_j = +1} A^{\Sigma v_i} B^{|D_v|}$

  j^{th} crossing

  $= A^{-1} \langle)(\rangle + A \langle \asymp \rangle$

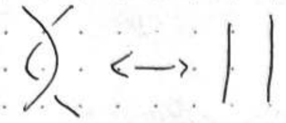
3) If $D = \bigcirc$, $D' =$, then $|D_v| = |D'_v| + 1 \quad \forall v$ $\square$

Effect of R. Moves on $\langle\ \rangle$

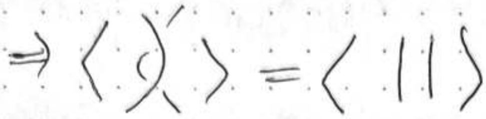
$R\mathrm{II}$) $\langle\ \rangle = A^{-1} \langle\ \rangle + A \langle\ \rangle$

- $= A^{-2} \langle\ \rangle + \langle\ \rangle + \cancel{A} \langle\ \rangle + A^2 \langle\ \rangle$

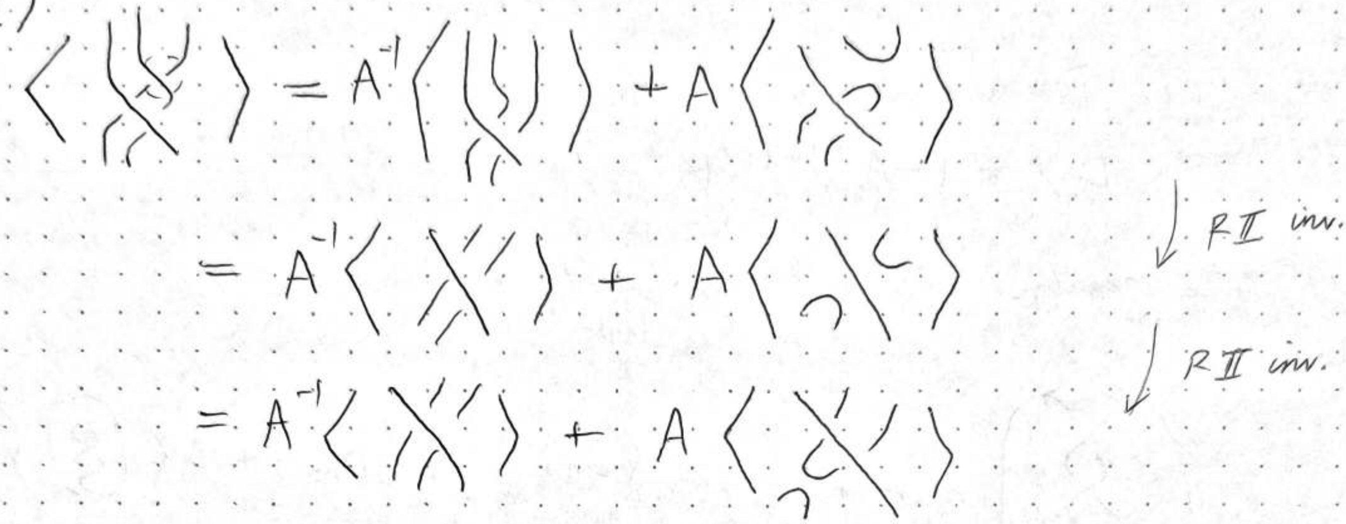
$= \langle)(\rangle + (A^{-2} + B + A^2) \langle\ \rangle$



So from now on take $\boxed{B = -A^2 - A^{-2}}$

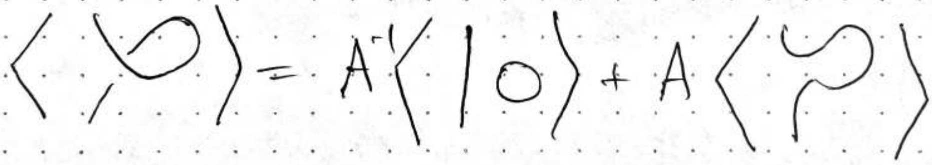


RII)



So $\langle\ \rangle$ is invariant under RIII.

RI)

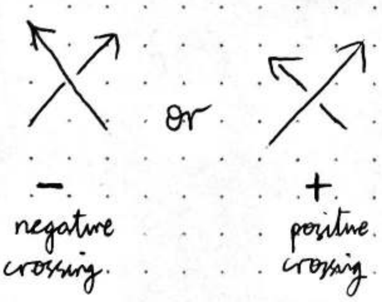


$$= \left(A^{-1}(-A^2 - A^{-2}) + A\right)\langle\, | \,\rangle$$

$$= -A^{-3}\langle\, | \,\rangle$$

Not invariant under RI!

<u>The Fix</u>: If D is oriented, then every crossing looks like



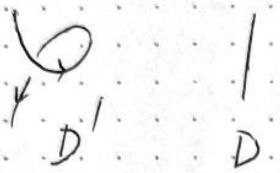
$n_\pm(D) = \#$ of $\pm$ crossings

<u>Def</u> The <u>writhe</u> of D is

$$w(D) = n_+(D) - n_-(D)$$

Lemma If D_i and D_i' are related by the i^{th} Reidemeister move, then
1) $w(D_1') = w(D_1) - 1$
2) $w(D_2') = w(D_2)$
3) $w(D_3') = w(D_3)$

Proof 1)

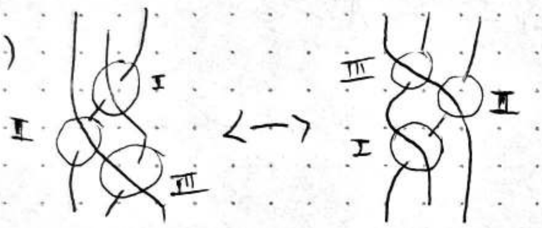


D' has one more crossing than D, and it's negative

2)

D' has 2 more crossings w/ opposite signs

3)



No matter what orientations are, sign of c_i = sign of c_i' $\square$

Thm If D is a link diagram,

$$\overline{V}(D) := (-A^3)^{-w(D)} \langle D \rangle$$ does not change under R-moves

Proof (1) $\overline{V}(D_1') = (-A^3)^{-w(D_1)+1}(-A^{-3}) \langle D_1 \rangle$

$$= \overline{V}(D_1)$$

(2), (3) Both w, $\langle\ \rangle$ are invt under R2, R3 $\square$

Def An oriented n-cpt link in $\mathbb{R}^3$ is an isotopy class of embeddings

$$i : \coprod_{j=1}^{n} S^1 \hookrightarrow \mathbb{R}^3$$

Same proof as for knots shows links have diagrams, and two diagrams of L are related by R moves.

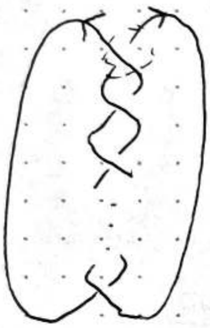
<u>Def</u> If L is an oriented link, $\overline{V}(L) = (-A^3)^{-w(D)} \langle D \rangle$, where D is any diagram of L, is the <u>unnormalised Jones polynomial</u> of L.

<u>Ex</u>: $\overline{V}(\bigcirc) = B = -A^{-2} - A^{2}$

<u>Cor</u>: If D is a diagram of the unknot, then

$$\langle D \rangle = (-A^3)^{w(D)} B$$

<u>Ex</u>: The negative $(2,n)$ torus ~~knot~~ link is represented by D_n



n crossings

$\langle D_n \rangle = A^{-1} \langle D_{n-1} \rangle + A \langle$  $\rangle$

$= A^{-1} \langle D_{n-1} \rangle + A(-A^3)^{n-1} B$

So $\langle D_1 \rangle = (-A^{-3}) B$

$\langle D_2 \rangle = (-A^{-4} - A^{4}) B$

$\langle D_3 \rangle = (-A^{-5} - A^{3} + A^{7}) B$

$\vdots$

$$\langle D_n \rangle = -A^{-n-2} (1 + A^{8} - A^{12} + A^{16} - \cdots \pm A^{4n}) B$$

$$w(D_n) = -n$$

So $\overline{V}(T(2,-n)) = (-A^3)^n (-A^{-n-2})(1 + A^8 - A^{12} + \cdots \pm A^{4n}) B$

<u>Cor</u> $T(2,-n) = T(2,-m) \Rightarrow n = m$

Infinitely many different knots

Better normalisation

Def: The normalized Jones polynomial of V is

$$V_L(q) = \frac{\overline{V}(L)}{\overline{V}(O)} = \frac{\overline{V}(L)}{B}\Bigg|_{q=-A^{-2}}$$

Ex: $V(O) = 1$

$$V(T(2,-n)) = q^{1-n}(1+q^{-4}-q^{-6}+\dots\pm q^{-2n})$$

Depending on whether I want to emphasize q or L, I'll write $V_L(q)$ or $V(L)$ interchangeably.

Exercise: $V_L(q) \in \mathbb{Z}[q;q^{-1}]$ ← ok!

all powers of q are odd if $|L|$ is ~~odd~~ even ← how?

# of components

even $|L|$ is ~~even~~ odd

E.g. if K is a knot $V_K(q) \in \mathbb{Z}[q^{\pm 2}]$

$$V(T(2,-3)) = q^{-2} + q^{-6} - q^{-8}$$

Recall: If D is a planar diagram of L

- $\overline{V}(L) = (-A^3)^{-w(D)} \langle D \rangle$

  $V(L) = \left. \dfrac{\overline{V}(L)}{\overline{V}(O)} \right|_{q=-A^{-2}}$

Operations on Knots / links

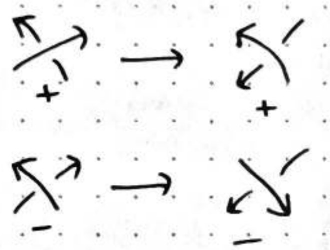
Orientation Reversal:

$r : \coprod_{i=1}^{n} S^1 \longrightarrow \coprod_{i=1}^{n} S^1$ reverses orientation

on every component

- $r(L) = L \circ r, \quad L : \coprod_{i=1}^{n} S^1 \hookrightarrow \mathbb{R}^3$

Effect on signs:



so $w(r(D)) = w(D)$.

$\langle r(D) \rangle = \langle D \rangle \implies \overline{V}(r(L)) = \overline{V}(L)$,

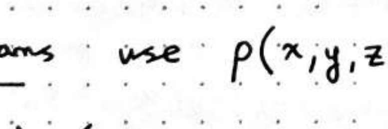
$V(r(L)) = V(L)$

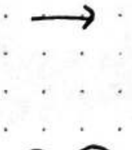
Mirror:

- the mirror of L is $\rho \circ L$ where $\rho : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a

reflection. $\quad$ " $\overline{L}$

Diagrams use $\rho(x,y,z) = \rho(x,y,-z)$

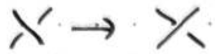


$K =$  $\quad \overline{K} =$  positive trefoil

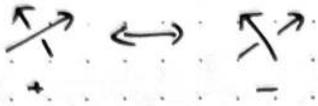
- Since $\langle O \rangle = (-A^2 - A^{-2}) \langle \ \rangle$

  $\langle$  $\rangle = A^{-1} \langle \ )(\ \rangle + A \langle \asymp \rangle$

  $\langle$  $\rangle = A^{-1} \langle \asymp \rangle + A \langle \ )(\ \rangle$

$\langle\ \rangle$ is invariant under the operation of simultaneously sending  and $A \to A^{-1}$

i.e. $\langle \bar{D} \rangle = \langle D \rangle |_{A \to A^{-1}}$

signs : 

so $w(\bar{D}) = -w(D)$

$\Rightarrow \bar{V}(\bar{L}) = \bar{V}(L)|_{A \to A^{-1}}$

$V(\bar{L}) = V(L)|_{q \to q^{-1}}$

Ex: $T(2,n) = \overline{T(2,-n)}$  $T(2,4)$

so $V(T(2,n))$

$= q^{n-1}(1 + q^4 - q^6 + \cdots \pm q^{2n})$

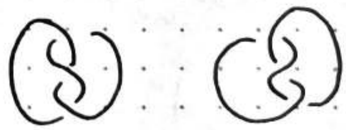
positive torus knots have positive powers of q

$\Rightarrow T(2,n) \neq T(2,-m)$ for all n, m as long as $n > 1$

$U = T(2,1) = T(2,-1)$

Disjoint union :

Diagrams: $D_1 \sqcup D_2$ $\boxed{D_1}$ $\boxed{D_2}$

$T(2,-3) \sqcup T(2,3)$ 

If $L_1 : \coprod^{n} S^1 \hookrightarrow \mathbb{R}^3 \subset S^3_1$

$L_2 : \coprod^{m} S^1 \hookrightarrow \mathbb{R}^3 \subset S^3_2$

$L_1 \sqcup L_2 : \coprod^{n} S^1 \sqcup \coprod^{m} S^1 \hookrightarrow S^3_1 \# S^3_2 = S^3$

$\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$

Proof by induction on # crossings in D_2

$w(D_1 \sqcup D_2) = w(D_1) + w(D_2)$

$\Rightarrow \bar{V}(L_1 \sqcup L_2) = \bar{V}(L_1)\,\bar{V}(L_2)$

$V(L_1 \sqcup L_2) = V(L_1)\,V(L_2)\,\bar{V}(0)$ $\left[\bar{V}(0) = -A^{-2} - A^2 = q + q^{-1}\right]$

Connected sum:

If $K_1 : S^1_1 \hookrightarrow \mathbb{R}^3 \subset S^3_1$

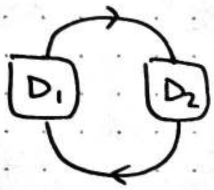
$K_2 : S^1_2 \hookrightarrow \mathbb{R}^3 \subset S^3_2$

are knots, I get

$$K_1 \# K_2 : \underset{\overset{\shortparallel}{S^1}}{S^1_1 \# S^1_2} \hookrightarrow \underset{\overset{\shortparallel}{S^3}}{S^3_1 \# S^3_2}$$

Diagrams:

$D_1 =$   $= D_2$

$D_1 \# D_2 =$ 

This doesn't depend on # point

Ex. $V(K_1 \# K_2) = V(K_1)V(K_2)$

1.2) Crossing Number

$$\langle D \rangle = \sum_{v \in \{\pm 1\}^n} A^{\Sigma v_i} B^{|D_v|}$$

$$=: \sum_v \langle D \rangle_v$$

⌜Knot Info or Knot Atlas⌟

Let $M(D) =$ maximum power of A in $\langle D \rangle$

$m(D) =$ minimum "

Ex: ∵ $M_v(D) =$ maximum " $\langle D \rangle_v = \Sigma v_i + 2|D_v|$

$m_v(D) =$ minimum " $= \Sigma v_i - 2|D_v|$

If $v, v' \in \{\pm 1\}^n$, say $v \leq v'$ if $v_i \leq v_i'$ all i;

$v_+ = (+1, \ldots, +1)$ then $v_- \leq v \leq v_+$ all v

$v_- = (-1, \ldots, -1)$

Say $v <_j v'$ if $v_j = -1$, $v_j' = +1$

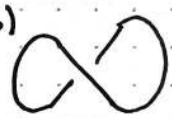
$v_i = v_i'$ all $i \neq j$

<u>Lemma</u>: If $v <_j v'$ then

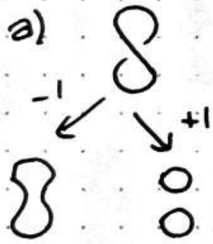
$$|D_{v'}| = |D_v| \pm 1$$

<u>Proof</u>: Let $D_{\hat{v}_j}$ be the diagram obtained by resolving all crossings according to $v_i = v_i'$ except the j^{th} one.

$D_{\hat{v}_j}$ has 1 crossing, must look like

a)  or b)  but <u>not</u> 

plus a bunch of circles.

a) 

$\Rightarrow |D_{v'}| = |D_v| + 1$

Similarly in b)

$|D_{v'}| = |D_v| - 1$. $\square$

<u>Prop</u> For all $v \in \{\pm 1\}^n$, $M_v(D) \leq M_{v_+}(D)$

$m_v(D) \geq m_{v_-}(D)$

<u>Proof</u> If $v <_j v'$, then

$m_v(D) = \Sigma v_i - 2|D_v|$

etc.

$M_v(D) = \Sigma v_i + 2|D_v|$

$M_{v'}(D) = \Sigma v_i' + 2|D_{v'}|$

$\geq \Sigma v_i + 2 + 2(|D_v| - 1)$

$= M_v(D)$

For any v we can find a chain $v <_{i_1} v_1 < \cdots <_{i_k} v_+$

$$\Rightarrow M_v(D) \leq M_{v_1}(D) \leq \cdots \leq M_{v_+}(D)$$

Similarly for the second statement. $\square$

So <u>Cor</u> $M(D) \leq M_{v_+}(D)$

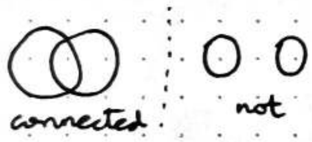
$m(D) \geq m_{v_-}(D)$

<u>Pf</u> $\langle D \rangle = \sum_v \langle D \rangle_v$ $\square$

So
$$\begin{aligned} M(D) - m(D) &\leq M_{v_+}(D) - m_{v_-}(D) \\ &= 2n + 2|D_{v_+}| - (-2n) + 2|D_{v_-}| \\ &= 4n + 2(|D_{v_+}| + |D_{v_-}|) \end{aligned}$$

$n = \#$ of crossings in D

~~Lemma~~ Say D is <u>connected</u> if space of underlying plane curves (forgetting over/under crossings) is connected.



<u>Lemma</u> If D is a connected planar diagram, w/ n crossings

$$|D_{v_+}| + |D_{v_-}| \leq n+2$$

<u>Proof</u> By induction on n.

<u>$n=0$</u> $D = \bigcirc$ $|D_{v_+}| = |D_{v_-}| = 1$

In general, choose a crossing of D; let D^-, D^+ be the diagrams obtained by resolving that crossing.

At least one of D_-, D_+ is connected, since D is.

Suppose it's D^-. Then

$$|(D^-)_-| + |(D^-)_+| \leq (n-1)+2 \quad \text{by induction}$$

$$(D^-)_+ <_j D_+ \Rightarrow |D_+| \leq |(D^-)_+| + 1 \quad \text{by Lemma}$$

$$\Rightarrow |D_-| + |D_+| \leq (n-1) + 2 + 1 = n+2$$

<u>Cor</u> If D is a connected diagram with n crossings, then $M(D) - m(D) \leq 6n + 4$

I

If D is a planar diagram, let $c(D)$ be the # of crossings in D.

Then <u>Thm</u>: If D is a connected planar diagram then

$$M(D) - m(D) \leq M(D_{v_+}) - M(D_{v_-})$$

$\langle D \rangle_v = A^{\Sigma v_i}(-A^2 - A^{-2})^{|D_v|}$

$$\begin{aligned} &= (c(D) + 2|D_{v_+}|) \\ &\quad - (-c(D) - 2|D_{v_-}|) \\ &= 2c(D) + 2(|D_{v_+}| + |D_{v_-}|) \\ &\leq 2c(D) + 2(c(D) + 2) \\ &= 4c(D) + 4 . \end{aligned}$$

<u>Def</u> Say L is <u>non-split</u> if every diagram D representing L is connected. $\Leftrightarrow$ $L \neq L_1 \amalg L_2$ for $L_1, L_2 \neq \phi$

<u>Def</u> If L is a link, its <u>crossing number</u> is

$$\min\{c(D) \mid D \text{ is a diagram of } L\} = c(L)$$

[Write $M_q(V(L))$ = max power of q in $V(L)$, similarly for $m_q(V(L))$]

<u>Thm</u> (Kauffman) If L is a <u>non-split</u> link then $c(L) \geq \frac{1}{2}(M_q(V(L)) - m_q(V(L)))$

<u>Proof</u> If D is a diagram of L, D is connected, and $V(L) = \left.\dfrac{\bar{V}(L)}{-A^2 - A^{-2}}\right|_{q = -A^{-2}}$

$$\begin{aligned} M_q(V(L)) - m_q(V(L)) &= \tfrac{1}{2}(M_A(\bar{V}(L)) - m_A(\bar{V}(L)) - 4) \\ &\leq \tfrac{1}{2}(4c(D) + 4 - 4) \\ &= 2c(D). \quad \square \end{aligned}$$

<u>Ex</u>: $L = T(2,n)$

$V(L) = q^{n-1}(1 + q^4 - q^6 + \cdots \pm q^{2n})$

$M_q(V(L)) - m_q(V(L)) = 2n$

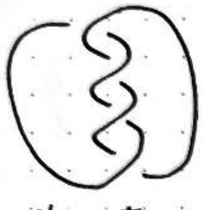
$\Rightarrow c(T(2,n)) \geq n$

But $T(2,n) =$  n crossings, so $c(T(2,n)) = n$

N.B. $T(2,n)$ is non-split since $(q+q^{-1}) \nmid V(L)$

1.3) Alternating Knots

Def^n A diagram D is <u>alternating</u> if as we traverse D, crossings alternate between over + under

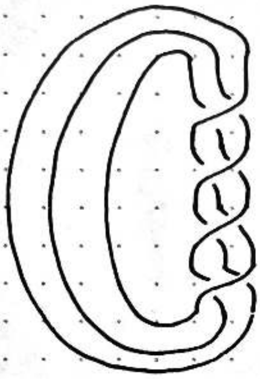


alternating

"figure 8 knot"

alternating

not alternating but boring

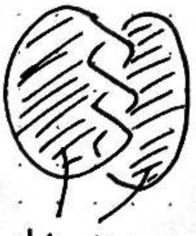


$T(3,4)$

L is <u>alternating</u> if it has an alternating diagram.
L is non-alternating o/w

Def If D is a planar diagram, a <u>checkerboard colouring</u> of D is an assignment of colours (black or white) to each region of the complement of D_{graph} s.t. the colours on either side of every edge differ.

D_{graph} = underlying planar 4-valent graph of D
(vertex at each crossing)



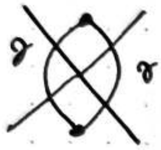
black
(white on board)

Lemma Every planar diagram has exactly two checkerboard colourings (related by swapping colours)

<u>Proof</u> Fix a region R_0. For any other region R, pick a path γ from R_0 to R which misses the vertices of Dgraph.

Then mod 2 intersection # of γ with Dgraph determines the relative colours of R_0, R.

This does not depend on the choice of γ since every vertex of Dgraph has even valency (4). $\square$



Given a checkerboard colouring of D, can form two new planar graphs $B(D)$, $W(D)$.

Vertices of $B(D)$ are black regions, edges $\to$ crossings in D

Similarly for $W(D)$.

<u>Ex</u>:

$B(D)$ $W(D)$

$B(D)$

$W(D)$

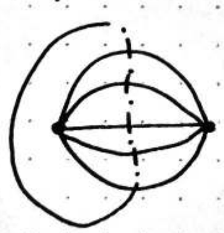
Observe that $B(D)$ and $W(D)$ are dual planar graphs,

i.e. vertices of $W(D)$ $\longleftrightarrow$ complementary regions of $B(D)$

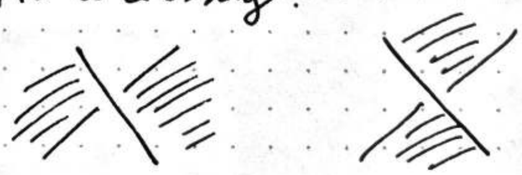
edges of $W(D)$ $\longleftrightarrow$ edges of $B(D)$

Ex: $T(2,n)$

$B(D)$



At a crossing:



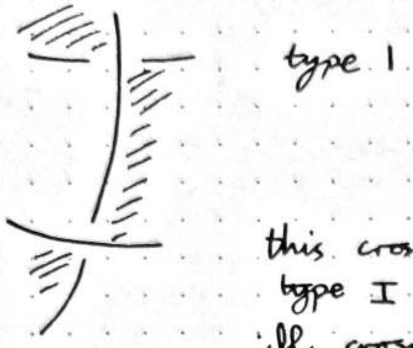
2 possibilities

I II

Say that colouring is <u>consistent</u> if all crossings are type I, or all crossings are type 2.

<u>Lemma</u> If D is a <u>connected</u> planar diagram, then D is consistent $\iff$ D is alternating.

<u>Proof</u>



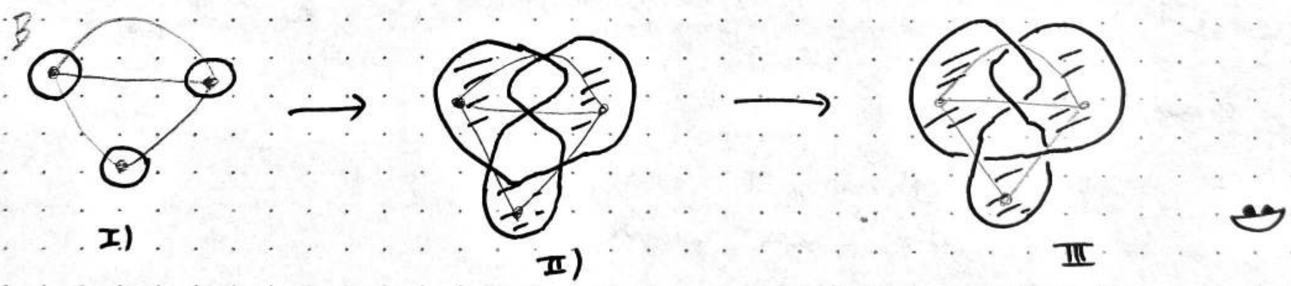
this crossing has type I iff crossings alternate as I go from c_1 to c_2

D is connected $\Rightarrow$ I get all crossings using this method
So D is alternating $\iff$ D is consistent. $\square$

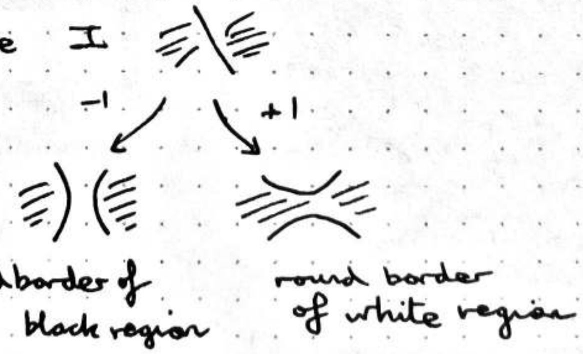
Conversely, given a connected planar graph B, there's a unique alternating diagram D with $B(D) = B_0$.

To construct D,

1) Start with a disk around each vertex of B
2) Add a crossing along each edge (this determines D_{graph})
3) Choose all crossings to be consistent with colouring.

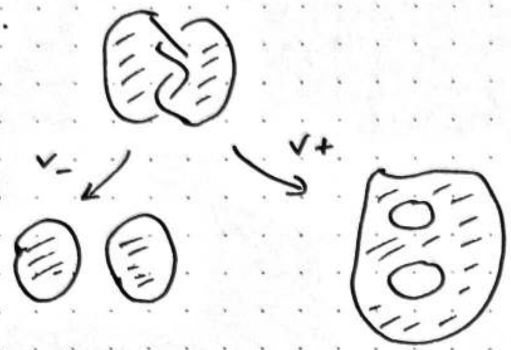


If every crossing is type I



So components of D_{v_-} are boundaries of black regions
" D_{v_+} " white "

E.g.



Lemma If D is a connected alternating diagram, then
$$|D_{v_-}| + |D_{v_+}| = c(D) + 2$$
[we have $\leq$ in general]

Proof $|D_{v_-}| = \#$ vertices in $B(D)$

$|D_{v_+}| = \#$ " in $W(D)$

$= \#$ faces for $B(D)$

$B(D)$ lies on a sphere, so

$$\underset{|D_{v_-}| - c(D) + |D_{v_+}|}{\underset{\shortparallel}{V - E + F}} = 2 \quad ☺$$

$|D_{v_-}| - c(D) + |D_{v_+}|$. $\square$

Ex sheet #5 :

$V(L)$ has even powers of q $\iff$ $|L|$ is odd,

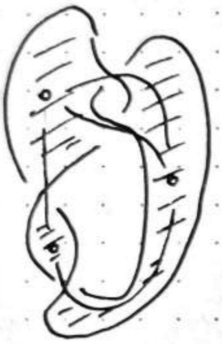
Last Lecture :

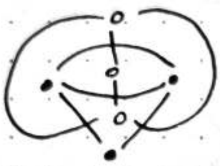
Prop : There's a bijection

$$\left\{\begin{matrix}\text{connected}\\ \text{alternating}\\ \text{diagrams}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{connected}\\ \text{planar}\\ \text{graphs}\end{matrix}\right\}$$

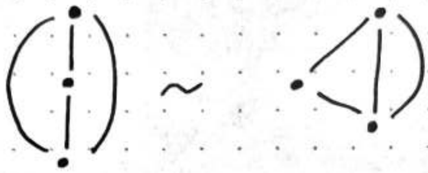
$$D \longrightarrow B(D)$$ where we use type I colouring

Ex :

$D =$ 



black graph of $\bar{D}$



$\Rightarrow D = \bar{D}$ (i.e. K is amphichiral $K = \bar{K}$)

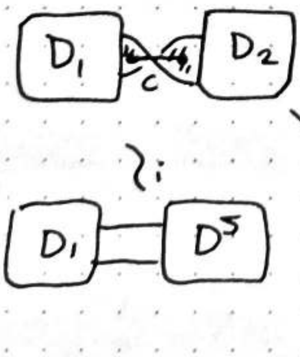
D_{v_-} = boundary of black regions

D_{v_+} = " white "

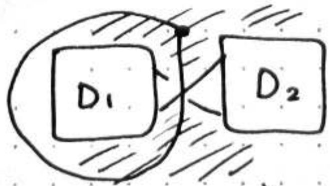
$$\underset{\text{Euler}}{\Longrightarrow} \quad |D_{v_-}| + |D_{v_+}| = c(D) + 2$$

Ex:  alternating diagram doesn't have minimal crossing number

Def : A crossing c of D is nugatory if D looks like



or



B(D) has a bridge, removal disconnects

$\Updownarrow$

W(D) loop

B(D) has a loop, edge with both ends at same vertex

$\Updownarrow$

W(D) bridge

Say D is <u>reduced</u> if it has no nugatory crossing

$\iff$ $B(D)$ has no loops, no bridges

$\iff$ $B(D), W(D)$ have no loops

<u>Lemma</u> If D is a reduced alternating diagram, then

$$m(D) = m(\langle D\rangle_{v_-}) = -n - 2|D_{v_-}|$$

$$M(D) = M(\langle D\rangle_{v_+}) = n + 2|D_{v_+}|$$

<u>Proof</u> Suppose $v_- <_i v'$

The diagram obtained by resolving all but the i^{th} crossing

A)  looks like A) instead of B)

A') 

since D is reduced, no edge of $B(D)$ is a loop.

B) 

$A \Rightarrow |D_{v_-}| = |D_{v'}| + 1$

$\Rightarrow m(\langle D_{v_-}\rangle) < m(\langle D\rangle_{v'})$

$\Rightarrow m(\langle D_{v_-}\rangle) < m(\langle D\rangle_v)$ for all $v \neq v_-$

Since $\langle D\rangle = \Sigma \langle D\rangle_v$,

$m(D) = m(\langle D\rangle_{v_-})$ & similarly for $M(D)$ and D_{v_+}. $\square$

<u>Cor</u>: If D is a reduced alternating diagram,

$$\begin{aligned} M(D) - m(D) &= M(\langle D\rangle_{v_+}) - m(\langle D_{v_-}\rangle) \\ &= n + 2|D_{v_+}| - (-n - 2|D_{v_-}|) \\ &= 2n + 2(|D_{v_+}| + |D_{v_-}|) \\ &= 2n + 2(n+2) \\ &= 4n + 4 \quad \text{where } n = c(D), \end{aligned}$$

<u>Thm</u> If L is a non-split link, and D is a reduced alternating diagram of L, then $c(L) = c(D)$.

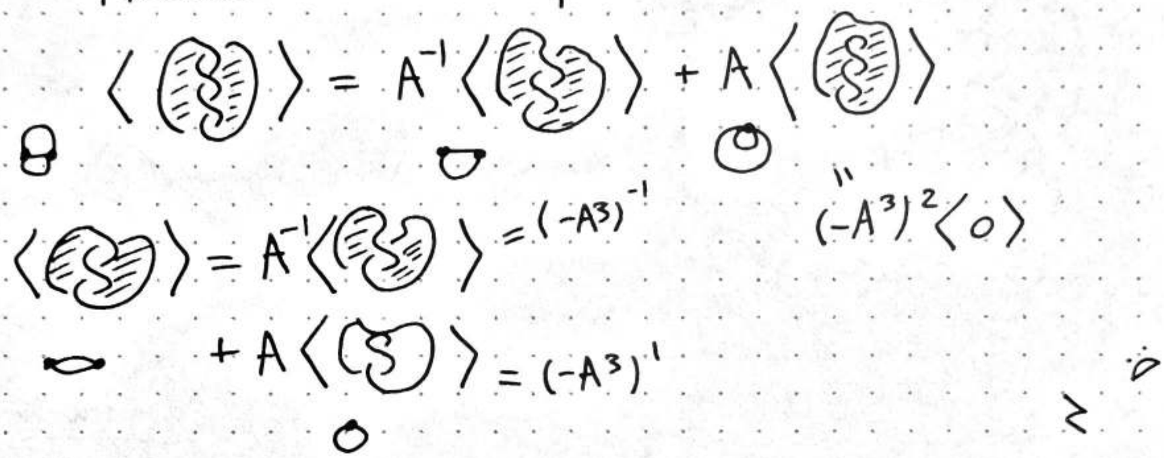
<u>Pf</u> If D' is a diagram,

$$c(D') \geq \tfrac{1}{2}\left(M_q(V(L)) - m_q(V(L))\right)$$
$$\geq \tfrac{1}{2}\left(\tfrac{1}{2}(M(D) - m(D) - 4)\right) \geq c(D) \quad \text{by Cor.} \qquad \square$$

↑
(First Tait Conjecture)

1.5) Maximal Trees

Recall how we compute

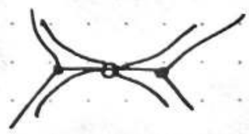


<u>Operations on planar graphs</u>

If e is an edge of G (connected planar graph) which is neither a loop nor a bridge, then $G \setminus e$ remove e

G/e collapse e to a point

are also connected planar graph.



If D is a connected planar diagram (w/ checkerboard colouring) and c is a crossing of D

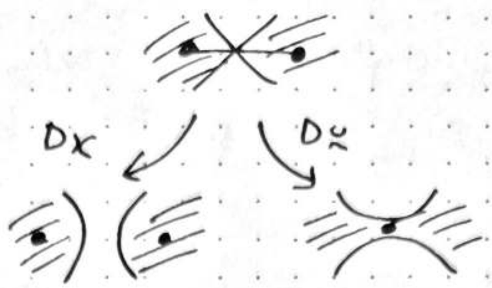
diagram

black graph

Assume crossing is non-nugatory $\Leftrightarrow$ corresponding edge is neither a loop nor a bridge

$B(D_X) = B(D) \| e$

$B(D_{\simeq}) = B(D)/e$



<u>Def</u> If G is a graph, a maximal tree of G is a subgraph which is a tree and contains every vertex of G.

<u>Ex</u>:  has 3 maximal trees 

<u>Def</u>: A connected planar graph G is <u>small</u> if every edge is either a loop or a bridge.

<u>Prop</u>: If G is small then

a) G has a unique maximal tree

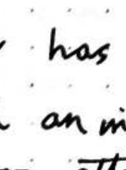
b) If D is any planar diagram with $B(D) = G$, then D can be unknotted using RI moves.

<u>Proof</u> a) easy. no loop can be an edge in a maximal tree

Let G' be the result of deleting all loops from G.

Every edge of G' is a bridge $\Rightarrow$ G' is a tree

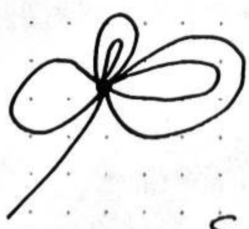
b) G' is a tree. Choose v which is a leaf of G'

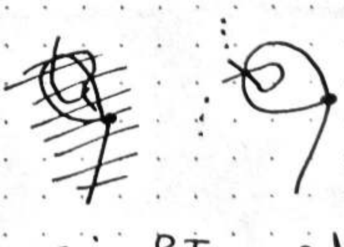


If v has no loops in G, then diagram D is as shown, and can be simplified by RI move

Reduces # edges.

If v has loops, find an innermost loop attached to v.





Simplified using RI, reduce # edges too.

After simplification, graph still has only bridges + loops.
So by induction corresponding diagram can be unknotted
via RI moves.

G is a connected planar graph

A subgraph $G' \subset G$ is a maximal tree $\iff$

a) G' is connected

b) $V_{G'} = V_G$ (*)

c) $\chi(G') = 1$

Say an edge e is <u>interesting</u> if it is neither a loop nor a bridge.

[<u>Def</u> $\mu(G) = \{ T \subset G \mid T \text{ a max tree} \}$]

If e is interesting, have two operations

$G \setminus e$ (remove e)

G/e (collapse e)

still connected planar graphs

<u>Lemma</u> If e is an interesting edge of G, then there's a bijection

$$\mu(G) \longleftrightarrow \mu(G\setminus e) \amalg \mu(G/e)$$

$$T \longmapsto T \subset G\setminus e \text{ if } e \notin T$$

$$\longmapsto T/e \subset G/e \text{ if } e \in T$$

<u>Proof</u> Using (*), these ↑ are maximal trees.

The inverse is given by

$$T \in \mu(G\setminus e) \mapsto T \subset G$$

$$T \in \mu(G/e) \mapsto T \cup e. \quad \square$$

<u>Standing assumptions</u>

D is a connected diagram of L

$$B(D) = G, \quad \langle D \rangle' = \langle D \rangle / \langle o \rangle = \frac{\langle D \rangle}{-A^2 - A^{-2}} = (-A^3)^{W(D)} V(L)\Big|_{q=-A^{-2}}$$

<u>Prop</u> $\langle D \rangle' = \sum_{T \in \mu(G)} A^{f(T)} \langle D_T \rangle'$

where $G_T = B(D_T)$ is small (no interesting edges)

<u>Proof</u> By induction on # of crossings of D

- $D = \bigcirc$ is obvious

Given a general D, if $G = B(D)$ is small, I'm done.
Otherwise choose an interesting edge of D and resolve the corresponding crossing:

$$\langle D\rangle' = A^{-1}\langle D_{)(}\rangle' + A\langle D_{\asymp}\rangle'$$

$$\{B(D_{)(}), B(D_{\asymp})\} = \{G/e, G\backslash e\}$$

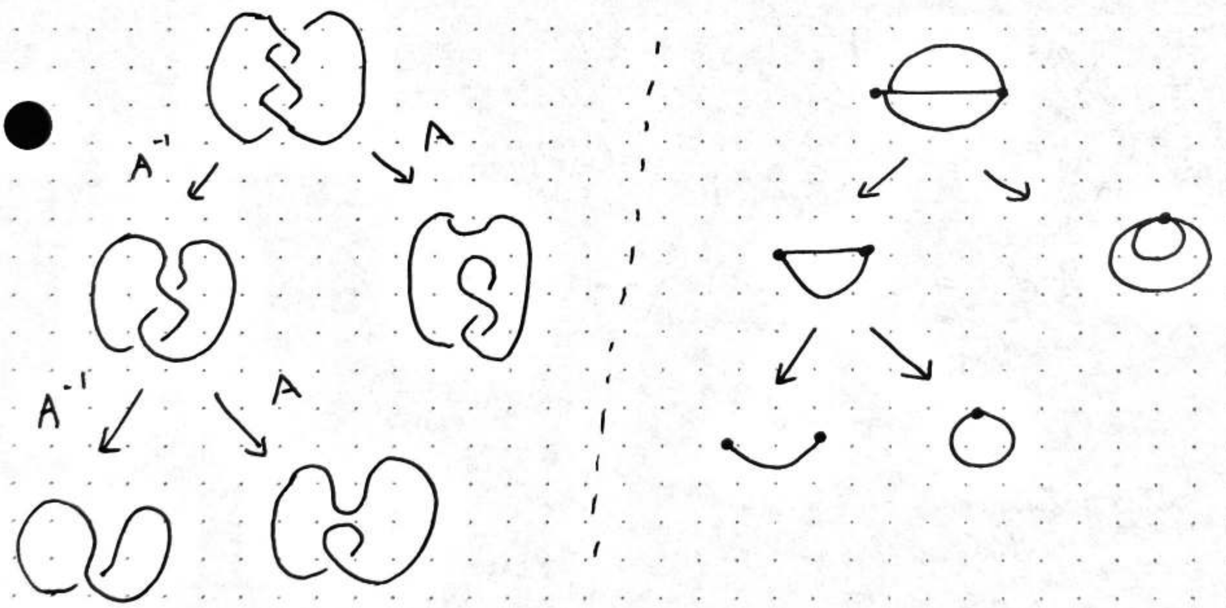
So by induction, applied to $D_{)(}$, $D_{\asymp}$, get

$$\langle D\rangle' = A^{\pm 1}\sum_{T\in\mu(G\backslash e)} A^{f_1(T)}\langle D_T\rangle' + A^{\mp 1}\sum_{T\in\mu(G/e)} A^{f_2(T)}\langle D_T\rangle'$$

By lemma, write as

$$\sum_{T\in\mu(G)} A^{f(T)}\langle D_T\rangle'. \qquad \square$$

Best way to think about this is via binary trees.



Cor If $V(L) = q^k(\sum a_i q^{2i})$
then $\sum |a_i| \leq \# \mu(G)$

Proof: D_T is small $\Rightarrow \langle D_T \rangle' = (-A^3)^{w(D_T)}$
$\Rightarrow \langle D \rangle'$ is a sum of $\mu(G)$ terms $\pm A^{g(T)}$. $\square$

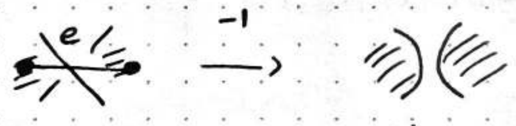
Def: polynomial $p(q) = q^k(\sum a_i q^{2i})$ is alternating
if $a_i a_{i+1} \leq 0$ for all i

Ex: $V(T(2,n)) = q^{n-1}(1 + q^4 - q^6 + q^8 - q^{10} \cdots \pm q^{2n})$

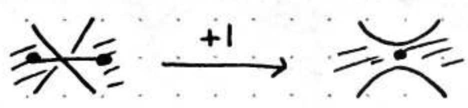
Thm: If D is a connected alternating diagram then
a) $V(L)$ is alternating and
b) $\sum |a_i| = \left| V(L)|_{q^2=-1} \right| = \# \mu(G)$

Proof Suppose D has type I colouring, so every crossing looks like 

Resolutions $D_{)(}$, $D_{\asymp}$ are both alternating w/ type I colouring

$D_{)(}$ 

$G \longmapsto G \setminus e$

$D_{\asymp}$ 

$G \longmapsto G/e$

Consider $h(G) = \underbrace{V_G}_{\#\text{ vertices}} + \chi(G)$

A^{-1} $G \to G \setminus e$ $\quad V \to V$, $\chi \to \chi + 1$ $\quad h \to h+1$

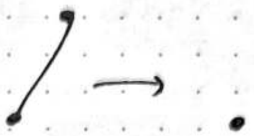
A^{+1} $G \to G/e$ $\quad V \to V-1$, $\chi \to \chi$ $\quad h \to h-1$

Net result:

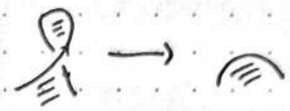
$$\langle D\rangle' = \sum A^{h(G)-h(G_T)} \langle D_T\rangle'$$

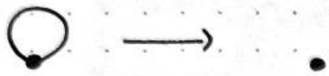
Simplify D_T using R I moves

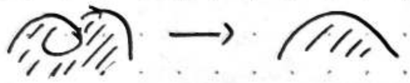
To simplify D_T, either remove a leaf



or remove a loop







$w \longrightarrow w+1$

$v \longrightarrow v-1$

$\chi \longrightarrow \chi$

$h \longrightarrow h-1$

$V \to V$

$\chi \to \chi+1$

$w \longrightarrow w-1$

$h \to h+1$

Change in W is opposite to change in h, so

$w=0$

$\chi=1 \Rightarrow w=2-h$ by induction

$v=1$

$$\text{So } \langle D\rangle' = \sum_{T\in\mu(G)} A^{h(G)-h(G_T)} \langle D_T\rangle'$$

$$= \sum_T A^{h(G)-h(G_T)} (-A^3)^{w(D_T)}$$

$$= \sum_T A^{h(G)-h(G(T)} (-A^3)^{2-h(G_T)}$$

$$= A^{h(G)+6} \sum_T (-1)^{h(G_T)} (A^4)^{-h(G_T)}$$

is alternating poly in A^4

$\Rightarrow V(L)$ is alternating

a) $\Rightarrow$ b) is easy (exercise). $\square$

<u>Def^n</u> If L is a link, it's <u>determinant</u> is

$$\det L = \left| V_L|_{q^2=-1} \right|$$

<u>Thm</u> If L is alternating, $\det L = \# \mu(G)$ where $G = B(D)$ for D any ~~reduced~~ ~~non-split~~ connected alternating diagrams of L.

If L is split, $(q+q^{-1}) \mid V(L) \Rightarrow \det L = 0$

<u>Cor</u> If D is a ~~non-split~~ connected alternating diagram of L then L is non-split.

<u>Proof</u> $\det L = \# \mu(G) > 0$ $\square$

<u>Open Question</u> Is $c(K_1 \# K_2) \overset{?}{=} c(K_1) + c(K_2)$?

True if K_1, K_2 are alternating.

Best general bound (Lackenby)

$$c(K_1 \# K_2) \geq \frac{c(K_1) + c(K_2)}{152} \qquad (!?)$$

II) Alexander polynomial

2.1) Knot exterior

Tubular Ngbd Thm: If $N \subset M$ is an embedded submanifold, then $\exists$ embedding

$$j : D(\nu_{N \subset M}) \hookrightarrow M$$

with $j \circ s_0 =$ inclusion of N into M.

<u>Idea of Proof</u> Use exponential map $\exp : TM|_x \to M$ which sends $v \in TM|_x$ to $\gamma_v(1)$ where $\gamma_v =$ unique geodesic s.t. $\gamma_v(0) = x$, $\gamma_v'(0) = v$. (pick a Riemannian metric)

Consider $\exp|_{\nu_{N \subset M}}$, with $\nu_{N \subset M} = TN^{\perp} \subset TM|_N$

Compute $d(\exp) = id$

$j(V) = \exp(V)$ and use inverse function theorem to see that j is an embedding.

2.1) The link exterior

- Links in S^3

Def: An oriented link in S^3 is an isotopy class of embeddings $L: \coprod^n S^1 \hookrightarrow S^3$.

$\mathbb{R}^3 \subset S^3$ (one point compactification)

So we get a map

$$\psi: \left\{ \begin{array}{c} \text{oriented links} \\ \text{in } \mathbb{R}^3 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{oriented links} \\ \text{in } S^3 \end{array} \right\}$$

Standard transversality arguments show that

- 1) ψ is surjective (any $L \hookrightarrow S^3$ is isotopic to L' which misses the point ∞)

2) ψ is injective (any $L \times I \to S^3$ generically misses ∞)

Links in $\mathbb{R}^3$ $\longleftrightarrow$ Links in S^3

Suppose $N \subset M$ is a smooth submanifold.

Let $\nu = \nu_{M/N}$ be the normal bundle

$s_0: N \to \nu$ zero-section

- Def^n $j: D(\nu) \hookrightarrow M$ is a closed tubular ngbd of N if

a) $j \circ s_0 = id_N$

b) $dj|_{s_0(x)}: T\nu|_{s_0(x)} \longrightarrow TM|_x$

$$\begin{array}{ccc} dj|_{s_0(x)} : T\nu|_{s_0(x)} & \longrightarrow & TM|_x \\ \| \qquad\qquad \wr\wr & & \wr\wr \\ id \qquad T_xN \oplus \nu_x & \longrightarrow & T_xN \oplus \nu_x \end{array}$$

[metric once & for all]

Tubular Ngbd Thm If $N \subset M$ is a smooth submanifold then

a) $\exists$ a tubular ngbd $j: D(\nu) \hookrightarrow M$

b) If $j, j': D(\nu) \hookrightarrow M$ are tubular ngbds then $j \sim_i j'$

- Idea a) define $j(v) = \exp_{\pi(v)}(\epsilon v)$ (small ϵ)

where $\exp_x(v) = \gamma(1)$, γ is the unique geodesic w/ $\gamma(0) = x$, $\gamma'(0) = v$

b). $\mu_\varepsilon : V \to \varepsilon V$ Then $j \sim_i j \circ \mu_\varepsilon$

$j' \sim_i j' \circ \mu_\varepsilon$

So enough to prove $j|_{D_\varepsilon(\nu)} \sim_i j'|_{D_\varepsilon(\nu)}$

Argue as in proof that $f|_{B_\varepsilon(x)} \sim_i df|_x$ (c.f. first lecture)

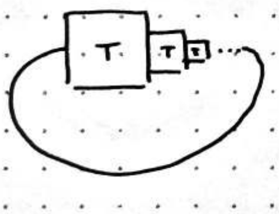
<u>Def</u> If $N \subset M$ is a smooth submanifold, M cpct;
and $j : D(\nu) \hookrightarrow M$ is a tubular ngbd,
the <u>exterior</u> of N is $M \setminus j(D^\circ(\nu))$
written E_N.

If $L : N \hookrightarrow M$ is an embedding, write $E_L = E_{\mathrm{im} L}$
E_L is a compact manifold with boundary $\partial E_L = S(\nu)$.
The <u>complement</u> of N is $M \setminus N$, which is a non-compact manifold.

We have $M \setminus N \cong E_N \cup_{\partial E_N} \partial E_N \times [0,\infty)$ $\left(D^n \setminus 0 \cong S^{n-1} \times [0,\infty) \right)$

In particular $E_N \sim M \setminus N$.

<u>Ex</u>: Let $W : S^1 \to S^3$ be the "wild embedding" from 2nd lecture



$\pi_1(S^3 \setminus \mathrm{im}(W))$ is not a finitely generated group

$\Rightarrow S^3 \setminus \mathrm{im}(W)$ is not $\sim$ to a compact 3-manifold w/ boundary

$\Rightarrow W$ has no tubular ngbd

<u>But</u>: $S^3 \setminus \mathrm{im}(W)$ is a smooth 3-manifold (wild end)

<u>Lemma</u> If $K : S^1 \hookrightarrow S^3$, then

$\nu_{S^3/K}$ is trivial

<u>Proof</u> $\nu_{S^3/K}$ is a 2-dim vector bundle over S^1
By the clutching construction, such vector bundles are in bijection with $\pi_0(O(2))$ has two elements.

Explicitly $T = I \times \mathbb{R}^2 / (0,v) \sim (1,v)$

et

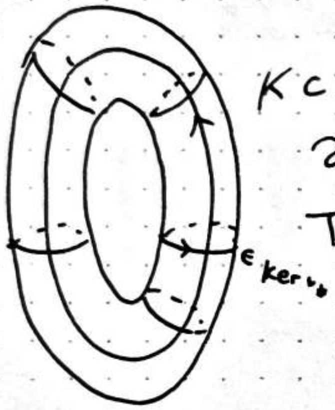
$$M = I \times \mathbb{R}^2 / (0,v) \sim (1, rv) \qquad r : \mathbb{R}^2 \to \mathbb{R}^2 \text{ reflection}$$

M is non-orientable as a vector bundle

$\Rightarrow D(M)$ is non-orientable (as a 3-manifold)

But S^3 is orientable, so $D(M) \not\hookrightarrow S^3$. $\square$

So $D(\nu_{S^3/K}) \simeq S^1 \times D^2$



$K \subset S^1 \times D^2$

$\partial \nu(K) \simeq T^2$

The map

$$\begin{array}{ccc} H_1(\partial\nu(K)) & \xrightarrow{\iota_*} & H_1(\nu(K)) \\ \| & & \| \\ H_1(S^1 \times S^1) & \longrightarrow & H^1(S^1 \times D^2) \\ \| & & \| \\ \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \end{array}$$

has kernel $\ker \iota_* \cong \mathbb{Z} = \langle [\partial D^2] \rangle$

An orientation on K determines a preferred generator m (meridian) for $\ker \iota_*$, according to the right-hand rule / intersection no.

$$K \cancel{M} \cdot [D^2] = 1 \quad \text{in } H_*(\nu(K))$$

$$\Leftrightarrow \quad K \cdot [\partial D^2] = 1 \quad \text{in } H_*(\partial\nu(K))$$

<u>Prop</u> Suppose $L : \coprod^n S^1 \hookrightarrow S^3$ is a link. Then

$$H_*(E_L) = \begin{cases} \mathbb{Z}^{n-1} & \text{if } * = 2 \\ \mathbb{Z}^n & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \end{cases}$$

and $H_1(E_L) = \langle m_1, \dots, m_n \rangle$ where m_i is the meridian of the i^{th} component of L.

<u>Proof</u> $S^3 = E_L \cup_{\partial\nu(L)} \nu(L)$ $\nu(L)$ = tub nghd

$\nu(L) \simeq \coprod^n S^1 \times D^2$

$\partial\nu(L) \simeq \coprod^n T^2$

<u>Mayer - Vietoris</u>

$$\begin{array}{ccccc} & & & & \langle [S^3] \rangle \\ & & & & \| \\ \langle [T_i^2] \rangle\, \mathbb{Z}^n & & H_3(E_L) \oplus H_3(\nu(L)) & \longrightarrow & H_3(S^3) \xrightarrow{\partial} \\ \| & & & & \\ H_2(\partial\nu(L)) & \longrightarrow & H_2(E_L) \oplus \cancel{H_2(\nu(L))}^{\,0} & \longrightarrow & \cancel{H_2(S^3)}^{\,0} \xrightarrow{\partial} \\ H_1(\partial\nu(L)) & \longrightarrow & H_1(E_L) \oplus H_1(\nu(L)) & \longrightarrow & \cancel{H_1(S^3)}^{\,0} \\ \| & & \| & & \\ \mathbb{Z}^{2n} & & \mathbb{Z}^n & & \end{array}$$

$$\partial[S^3] = \oplus [T_i^2]$$

← sure but like ... recall fundamental classes; Hatcher Lemma 3.27 ✓

$\therefore H_2(E_L) = \mathbb{Z}^n / \langle (1,1,\dots,1) \rangle \simeq \mathbb{Z}^{n-1}$

$$0 \to \mathbb{Z}^{2n} \xrightarrow{i_{1*} \oplus i_{2*}} H_1(E_L) \oplus \mathbb{Z}^n \longrightarrow 0$$

$\Rightarrow H_1(E_L) \cong \mathbb{Z}^n$

and $i_{2*}(m_i) = 0$

so $H_1(E_L) = \langle m_i \rangle$ $\square$

Disappointing since the answer doesn't depend on anything expect the # of cpts of L.

Let's also compute $H_*(E_K, \partial E_K)$ ¡¡ let's not rush this calculation

$H_*(E_K, \partial E_K) \overset{PD}{\simeq} H^*(E_K)$

Why? $H^*(E_K, \partial E_K) \simeq H^*_c(E_K) \simeq H_*(E_K)$

↑ I assume here E_K is open i.e. w/o ∂E_K

2.2) Seifert Surfaces

- Suppose $N \subset M$ is an n-dimensional closed, oriented, connected embedded submanifold, $i : N \hookrightarrow M$.

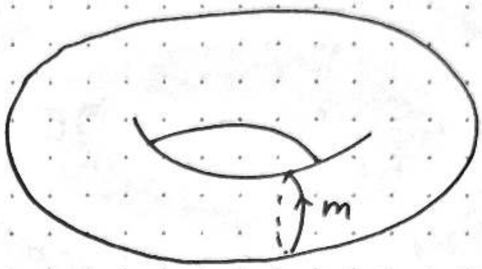
$$H_n(N) \simeq \mathbb{Z} = \langle [N] \rangle$$

Write $[N] = i_*([N]) \in H_n(M)$.

More generally, suppose

$$(N, \partial N) \hookrightarrow (M, \partial M)$$

where N is compact, connected, oriented manifold w/ ∂. (dim $= n$)



$[m] \in H_1(T^2)$

- LES of $(N, \partial N)$

$$0 \to H_n(N) \to H_n(N, \partial N) \to H_{n-1}(\partial N) \to H_{n-1}(N)$$

$$H_n(N) = 0, \quad H_n(N, \partial N) = \mathbb{Z}, \quad 1 \longmapsto [\partial N]$$

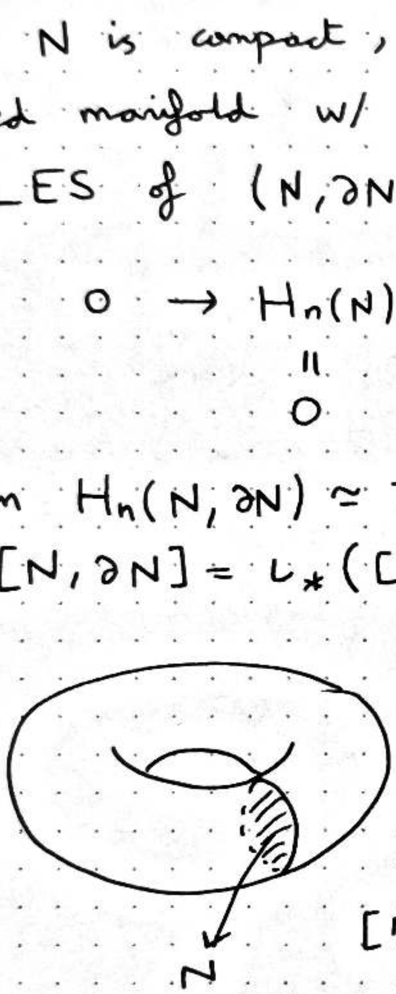
← yeslmao

Then $H_n(N, \partial N) \simeq \mathbb{Z} = \langle [N, \partial N] \rangle$, write

$$[N, \partial N] = \iota_*([N, \partial N]) \in H_n(M, \partial M).$$

Commuting map of LES of pairs

$$\begin{array}{ccccccc} H_n(N) & \to & H_n(N, \partial N) & \xrightarrow{\partial} & H_{n-1}(\partial N) & \to \\ \downarrow i_* & & \downarrow & & \downarrow & \\ H_n(M) & \to & H_n(M, \partial M) & \xrightarrow{\partial} & H_{n-1}(\partial M) & \to \end{array}$$

- 

$[N, \partial N] \in H_2(S^1 \times D^2, S^1 \times S^1)$

So $\partial [N, \partial N] = [\partial N] \in H_{n-1}(\partial M)$

$$[N, \partial N] \in H_n(M, \partial M)$$

$H_*(T^2)$

<u>Prop</u>: ① $H_1(T^2) = \mathbb{Z}^2$

- ② If $\alpha \in H_1(T^2)$, $\alpha \in [\gamma]$ where $\gamma : S^1 \hookrightarrow T^2$ is a simple closed curve if and only if α is <u>primitive</u> i.e. $\alpha \neq k\beta$ for some $k > 1$.

③ If $\gamma, \gamma' : S^1 \hookrightarrow T^2$ with $[\gamma] = [\gamma']$ $(\neq 0)$ then $\gamma \sim_i \gamma'$.

<u>Pf</u> Ask someone who went to MCG. $\square$

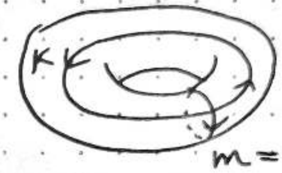
Spose $K \hookrightarrow S^3$ is an oriented knot.

$$S^3 = E_K \cup_{\partial \nu(K)} \nu(K)$$

where $\nu(K) \simeq S^1 \times D^2$, $\partial \nu(K) \simeq S^1 \times S^1 = T^2$

$$H_*(E_K) = \begin{cases} \mathbb{Z}, & * = 0,1 \\ 0, & \text{o/w} \end{cases}$$

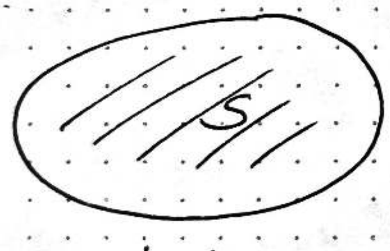
$H_1(E_K) = \langle m \rangle$.



$m = m(K)$ is <u>meridian</u> of K

<u>Def</u>

<u>Def^n</u> A <u>Seifert surface</u> of K is an embedded oriented $S \hookrightarrow S^3$ with $\partial S = K$ (as oriented manifolds)



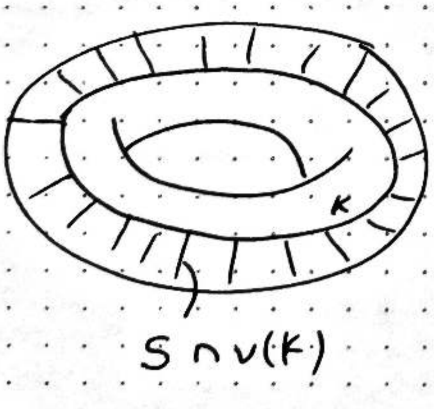
unknot

Tubular ngbd thm $\Rightarrow$ we can choose $\nu(K)$ so that $(\nu(K), S \cap \nu(K))$

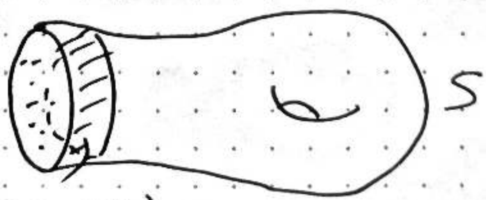
$$\cong$$

$$(D(\nu_{S^3/K}), TS \cap D(\nu_{S^3/K}))$$

$S \cap \nu(K)$ is a tubular ngbd of ∂S in S



$S \cap \nu(K)$



$S \cap \nu(K)$

$S' = S \cap E_K \simeq S$

Prop: ① $[\partial S'] \in H_1(\partial E_K)$ generates $\ker \iota_* : H_1(\partial E_K) \to H_1(E_K)$

② $\langle m, [\partial S'] \rangle$ is a basis for $H_1(\partial E_K)$

③ $H_*(E_K, \partial E_K) \simeq \begin{cases} \mathbb{Z}, & * = 2,3 \\ 0, & \text{o/w} \end{cases}$

and $H_2(E_K, \partial E_K) = \langle [S', \partial S'] \rangle$.

Proof LES of $(E_K, \partial E_K)$

$$0 \to \underset{0}{H_3(E_K)} \to H_3(E_K, \partial E_K) \to$$
$$\underset{\mathbb{Z}}{H_2(\partial E_K)} \to \underset{0}{H_2(E_K)} \to H_2(E_K, \partial E_K) \to$$
$$\underset{\mathbb{Z}^2}{H_1(\partial E_K)} \to \underset{\mathbb{Z}}{H_1(E_K)} \to H_1(E_K, \partial E_K) \to 0$$

[$m \in H_1(\partial E_K)$ $\Rightarrow i_*$ surjective]

$$\partial : \underset{\mathbb{Z}}{\underset{\shortparallel}{H_3(E_K, \partial E_K)}} \xrightarrow{\simeq} H_2(\partial E_K)$$

$$0 \to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_2(E_K, \partial E_K)}} \to \underset{\mathbb{Z}^2}{\underset{\shortparallel}{H_1(\partial E_K)}} \xrightarrow{i_*} \underset{\mathbb{Z}}{\underset{\shortparallel}{H_1(E_K)}} \to 0$$

Consider $[S', \partial S'] \in H_2(E_K, \partial E_K)$

Let $\ell = \partial[S', \partial S'] = [\partial S']$

By exactness, $\ell \in \ker i_*$

ℓ is primitive, since it's represented by the embedded curve $\partial S'$.

Consider $j_* : H_1(\partial E_K) \to H_1(\nu(K))$.

Then $j_*[\partial S'] = [K]$ generates $H_1(\nu(K))$.

So $\ell \neq 0$ in $H_1(\partial E_K)$.

① $\ell \in \ker \iota_* \simeq \mathbb{Z}$ is a non-zero primitive element, so it generates

② follows from ①, since

$$H_1(\partial E_K) = \langle m \rangle \oplus \ker \iota_*$$

③ Follows since

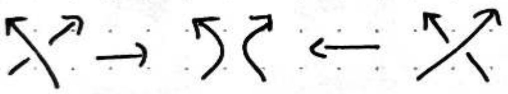
$$\ell = \partial[S', \partial S'] \text{ generates im } \partial = \ker \iota_* . \qquad \square$$

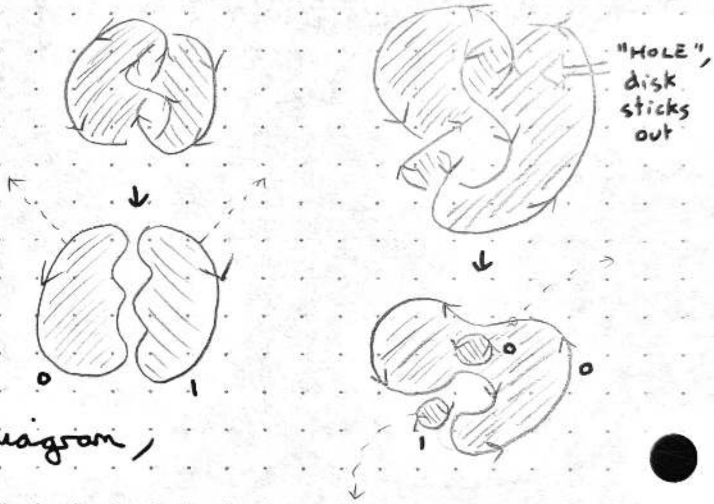
<u>Thm</u> Every $K \hookrightarrow S^3$ has a Seifert surface.

<u>2 proofs</u>: <u>Seifert's algorithm</u>:

Given a diagram D of K, construct a Seifert surface as follows.

① Give every crossing the oriented resolution





② resulting diagram has no crossings and a natural orientation

③ let C be a circle of resulting diagram, it bounds a disk $D_C \subset \mathbb{R}^2$

Let $n_C = \#$ of circles in resolved diagram that separate C from ∞ (mod 2)

Let $r_C = 0$ if C is oriented counter-clockwise

$= 1$ " clockwise

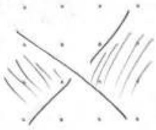
blue red

Orient D_C according to sum $n_C + r_C$

Colour D_C red if $n_C + r_C$ is odd

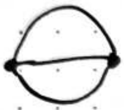
blue if $n_C + r_C$ is even

∴

④ Attach a band  of surface at each crossing

⑤ This surface retracts onto a graph with 1 vertex for each disk, 1 edge for each crossing.

$$\chi = \#\text{disks} - \#\text{crossings} = 1 - 2g(S)$$

 et "figure 8"

"trefoil"

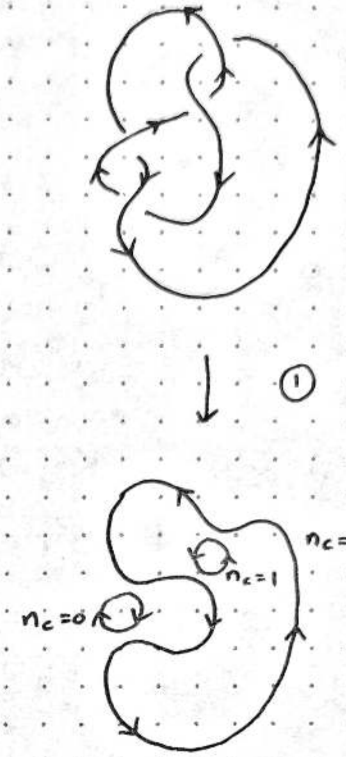
<u>Thm</u> Every $K \hookrightarrow S^3$ has a Seifert surface

● <u>Proof 1</u> (cont.) Seifert's algorithm

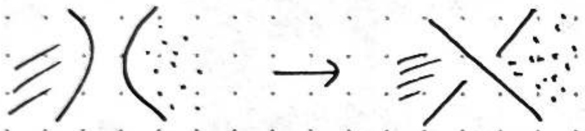
① give crossings oriented resolution

② each circle bounds a disc D_c
at height n_c above blackboard

③ orient D_c according to ~~the~~
~~sum~~ $\cancel{n_c} + r_c$ $\ast$ $\doteq$



Attach a twisted band to the D_c's
at each crossing

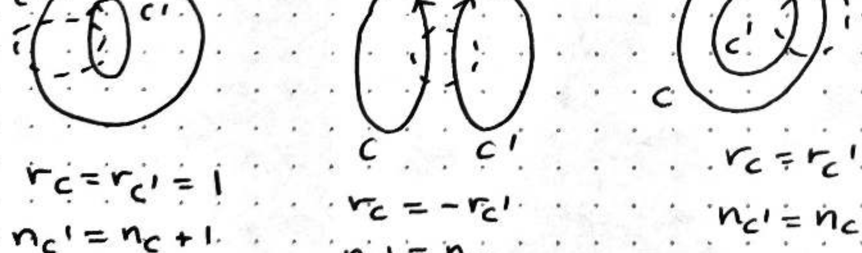
● 

Need to check that

① this is compatible w/ orientations (not really)

$\Leftrightarrow$ if c, c' are 2 circles at a crossing, then $(\ast)$
$n_c + r_c$ has opposite parity to $n_{c'} + r_{c'}$ mod 2

Resolve all crossings except this one, get 3 possible pics

● 

$r_c = r_{c'} = 1$
$n_{c'} = n_c + 1$

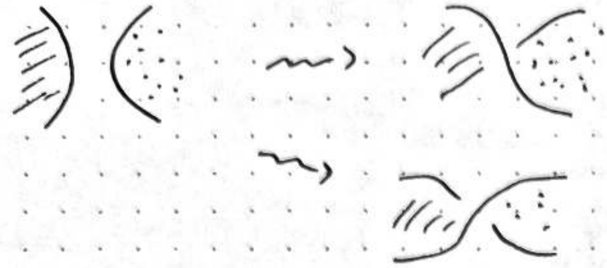
$r_c = -r_{c'}$
$n_{c'} = n_c$

$r_c = r_{c'} = 0$
$n_{c'} = n_c + 1$

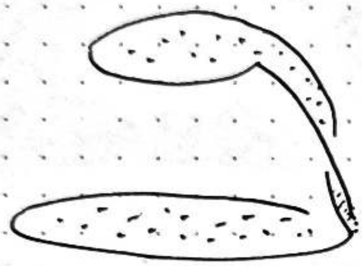
● $(\ast)$ holds in all 3 cases

To see that S is embedded, enough to look at a nbhd of a crossing.

$n_c = n_{c'}$ twist e 

- 

$n_c = n_{c'} + 1$

-  $\square$

Proof 2 (Sketch)

$$H^1(E_K; \mathbb{Z}) \subset H^1(E_K; \mathbb{R}) \simeq H^1_{dR}(E_K; \mathbb{R})$$

$H^1(E_K;\mathbb{Z}) \cong \mathbb{Z} = \langle a \rangle$, $H^1(E_K;\mathbb{R}) \cong \mathbb{R}$

Pick $\alpha \in \Omega^1(E_K)$ with $d\alpha = 0$ and $[\alpha] = a \in H^1(E_K; \mathbb{R})$

- Pick $p_0 \in E_K$, define $f_\alpha : E_K \to S^1 = \mathbb{R}/\mathbb{Z}$

$$f_\alpha(p) = \int_{\gamma_p} \alpha \quad \text{where } \gamma_p \text{ is a path from } p_0 \text{ to } p$$

If γ_p' is another such path, then

$$\int_{\gamma_p} \alpha - \int_{\gamma_p'} \alpha = \int_{\substack{\gamma_p - \gamma_p' \\ (\text{loop})}} \alpha = \langle a, [\gamma_p - \gamma_p'] \rangle \in \mathbb{Z}$$

since $a \in H^1(E_K; \mathbb{Z})$

- So $f_\alpha(p)$ is well-defined in $\mathbb{R}/\mathbb{Z}$.

f_α is a smooth map, so pick $x \in S^1$ a regular value (Sard)
Then $f_\alpha^{-1}(x)$ is a smooth submanifold S of E_K with $\partial S \subset \partial E_K$

We have $[\partial S] = PD(\iota^*(a)) \in H_1(\partial E_K)$ ⌜dual to meridian⌟
(exercise) ⌜∂S is level set of $\int \alpha$ in ∂E_K !⌟
is primitive in $H_1(\partial E_K)$

$\Rightarrow [S, \partial S]$ generates $H_2(E_K, \partial E_K) = \mathbb{Z}$

$\Rightarrow S$ is a Seifert surface. $\square$ ⌜after possible tweaking⌟

<u>Summary</u> ① Every $K \hookrightarrow S^3$ has a Seifert surface S, not unique
② The class $[\partial S] \in H_1(\partial E_K)$ generates

$$\ker(H_1(\partial E_K) \to H_1(E_K))$$

and satisfies $j_*[\partial S] = [K]$ for $j: \partial E_K \hookrightarrow \nu(K)$ inclusion
This does not depend on choice of S.

<u>Def^n</u> $\ell = [\partial S]$ is the <u>homological longitude</u>
a.k.a. <u>Seifert longitude</u> of K.

③ ∂E_K has a preferred basis $\langle m, l \rangle$ where m is the meridian for K.

<u>Links</u> If $L = \coprod^n L_i \hookrightarrow S^3$ is an oriented link, where L_i are the components of K, then

$$\partial E_L = \coprod^n \partial_i E_L, \quad \partial_i E_L = \partial(\nu(L_i))$$

$H_1(\partial_i E_L)$ has a preferred basis $\langle m_i, l_i \rangle$
where l_i is the Seifert longitude of L_i (<u>forget</u> other components of L)

Then $\langle m_1, \dots, m_n \rangle$ is a basis of $H_1(E_L)$

But usually $[L_i] \neq 0$ in $H_1(E_L)$.

Seifert surfaces : a Seifert surface of L is an embedded oriented $S \hookrightarrow S^3$ w/ $\partial S = \hat{\coprod} L_i$

These exist by Seifert's algorithm.

N.B. $\partial_i S \neq \ell_i$ usually

<u>Ex</u> : $L = T(2,4)$

$\chi = 2 - 4 = -2$ $\qquad\qquad$ $\chi = 4 - 4 = 0$

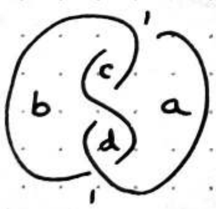
2.3) $\pi_1(E_L)$

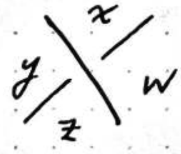
<u>2 presentations of $\pi_1(E_L)$</u> coming from a diagram D of L

<u>Dehn presentation</u> :

Generators are finite regions of $\mathbb{R}^2 \setminus D_{graph}$

Relations $\longleftrightarrow$ crossings





$\rightarrow \quad x y^{-1} z w^{-1} = 1$

infinite region $\rightarrow 1 \in \pi_1(E_L)$

generators a, b, c, d

relations $1 a^{-1} c b^{-1} = 1$

$c a^{-1} d b^{-1} = 1$

$d a^{-1} 1 b^{-1} = 1$

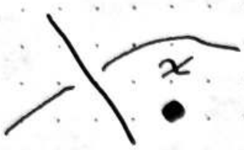
$\langle a, b, c, d \mid a^{-1} c b^{-1}, c a^{-1} d b^{-1}, d a^{-1} b^{-1} \rangle$

$c = ab \qquad d = ba$

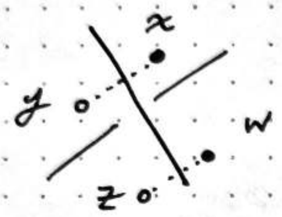
$\langle a, b \mid a b a^{-1} b a b^{-1} \rangle$

Basepoint $* = \infty \in S^3 = \mathbb{R}^3 \cup \infty$

- Generator associated to a region x is a vertical line (parallel to z axis passing through x)



$\bullet$ = line parallel to z-axis passing through $\bullet$



$$xy^{-1} = wz^{-1}$$

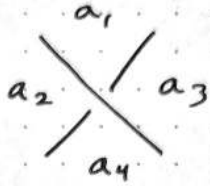
$\bullet$: going in

$\circ$: going out

Dehn presentation

- D a connected planar diagram with n crossings has $n+1$ finite complementary regions (exercise)

$$G_{Dehn} = \langle \underset{\substack{\uparrow \\ \text{regions}}}{a_1, \dots, a_{n+1}} \mid \underset{\substack{\uparrow \\ \text{crossings}}}{w_1, \dots, w_n} \rangle$$



$$w = a_1 a_2^{-1} a_4 a_3^{-1}$$

Wurtinger presentation

D is an oriented planar diagram

- An arc of D is part of D which I can draw without lifting up chalk

n crossings $\longrightarrow$ n arcs

three arcs

G_{Wirt} has generators $\gamma_1, \dots, \gamma_n$ $\updownarrow$ arcs

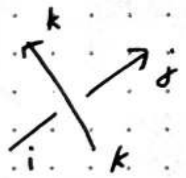
relations $w_1, \dots, w_n$ $\updownarrow$ crossings

$$\gamma_3 = \gamma_1 \gamma_2 \gamma_1^{-1}$$
$$\gamma_1 = \gamma_2 \gamma_3 \gamma_2^{-1}$$
$$\gamma_2 = \gamma_3 \gamma_1 \gamma_3^{-1}$$

eliminate γ_3

$$\gamma_2 = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_1 \gamma_1 \gamma_2^{-1} \gamma_1^{-1}$$
$$= \gamma_1 \gamma_2 \gamma_1 \gamma_2^{-1} \gamma_1^{-1}$$
$$\gamma_1 = \gamma_2 \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$$

$$G_{Wirt} = \langle \gamma_1 \gamma_2 : \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \rangle$$

- 

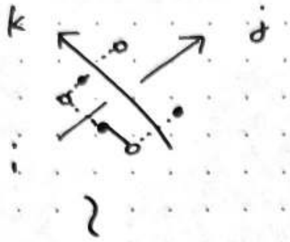
$$\gamma_j = \gamma_k \gamma_i \gamma_k^{-1}$$

$$\gamma_j = \gamma_k^{-1} \gamma_i \gamma_k$$

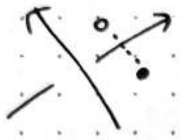
Geometry

γ_i is a loop starting from ∞ going around arc i compatible w/ right hand rule

Relation



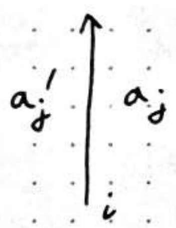
$\gamma_k \gamma_i \gamma_k^{-1}$



γ_j

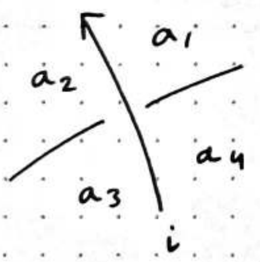
$G_{\text{wirt}} \cong G_{\text{Dehn}}$:

Define $\varphi : G_{\text{Wirt}} \longrightarrow G_{\text{Dehn}}$



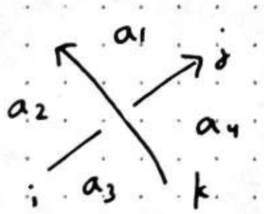
$\varphi(\gamma_i) = a_j (a_j')^{-1}$

This is well-defined :



$\varphi(\gamma_i) = a_1 a_2^{-1} = a_4 a_3^{-1}$

↑ Dehn relation

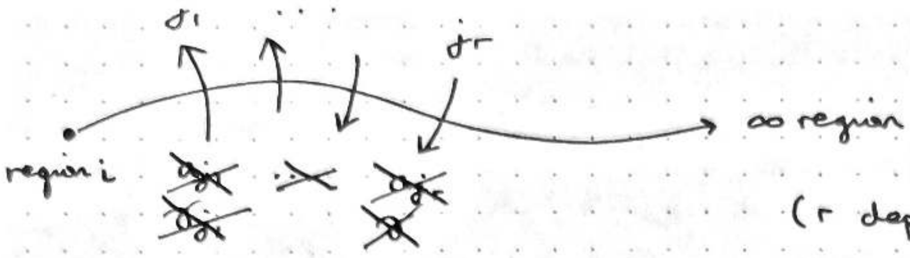


$$\begin{aligned}&\varphi(\gamma_k)\varphi(\gamma_i)\varphi(\gamma_k^{-1})\\ &= (a_4 a_3^{-1})(a_3 a_2^{-1})(a_2 a_1^{-1})\\ &= a_4 a_1^{-1} = \varphi(\gamma_j)\end{aligned}$$

so φ well-def group hom

Define $\psi : G_{\text{Dehn}} \longrightarrow G_{\text{Wirt}}$

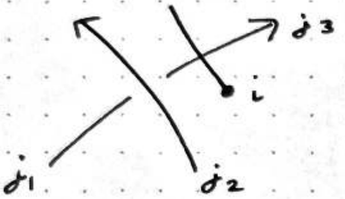
Pick a path c_i from region i to infinite region



(r depends on i & path)

$$\psi(a_i) = \gamma_{j_1}^{\pm 1} \cdots \gamma_{j_r}^{\pm 1}$$ where exponents determined by sign of intersection

Check that ~~$\varphi(a_i)$~~ $\psi(a_i)$ doesn't depend on choice of c_i

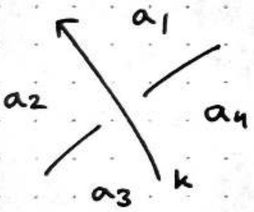


vs

γ_{j_3} $\quad$ $\gamma_{j_2}\gamma_{j_1}\gamma_{j_2}^{-1}$

same by Wirtinger relation

Check Dehn relation at a crossing

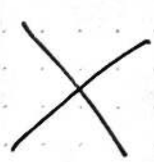


$$\psi(a_1 a_2^{-1}) = \gamma_k = \psi(a_4 a_3^{-1})$$

<u>Exercise</u> ψ, φ are inverse maps ✓

<u>Proof that $\pi_1(E_L, \infty) \simeq G_{Dehn}$</u> :

<u>Step 1</u> Let $\nu(D_{graph})$ be a union of balls around vertices and $D^2 \times e$ around edge e



D_{graph}

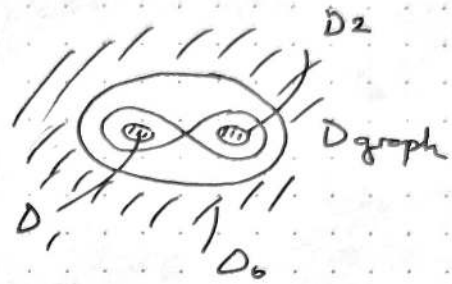
$\nu(D_{graph})$

Define $E_D = S^3 \setminus \operatorname{int}(\nu(D_{graph}))$

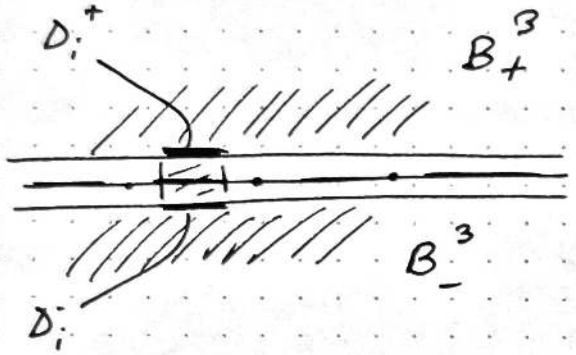
$E_D \cap S^2 = D_0 \cup \cdots \cup D_{n+1}$

plane of D

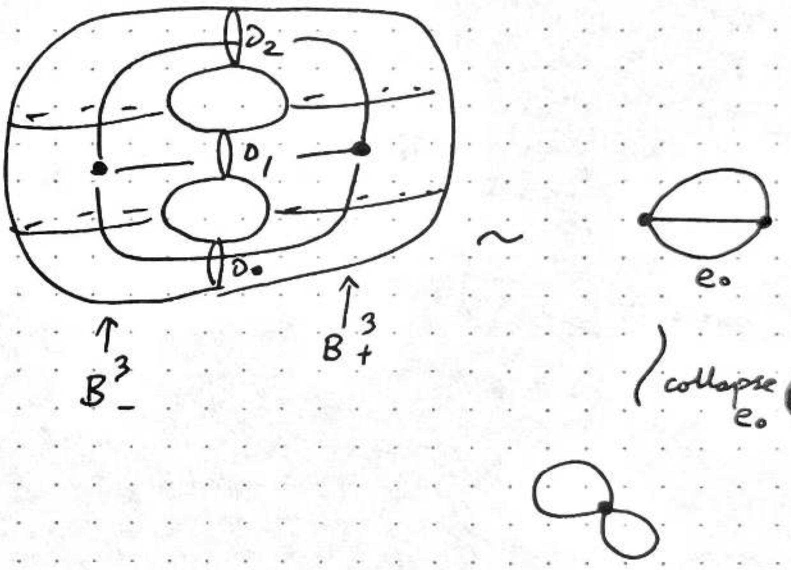
a union of discs one for each region, $D_0 \leftrightarrow$ infinite region



$E_D \cap (S^3 \setminus \nu(S^2)) = B^3_+ \cup B^3_-$



So $E_D \simeq B^3_+ \underset{D_i^+ \simeq D_i^{-1}}{\cup} B^3_-$ is a handlebody.

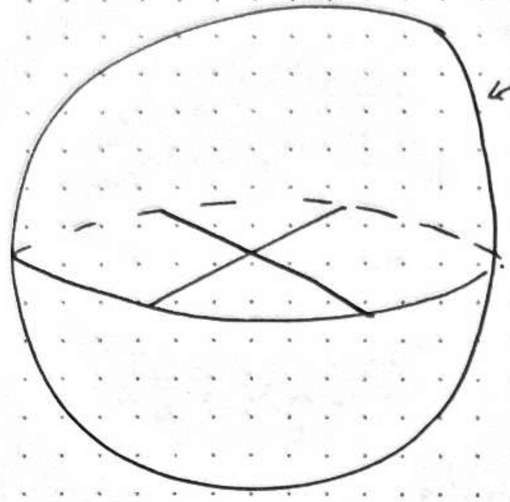


So $E_D \sim \bigvee_{i=1}^{n+1} S^1$

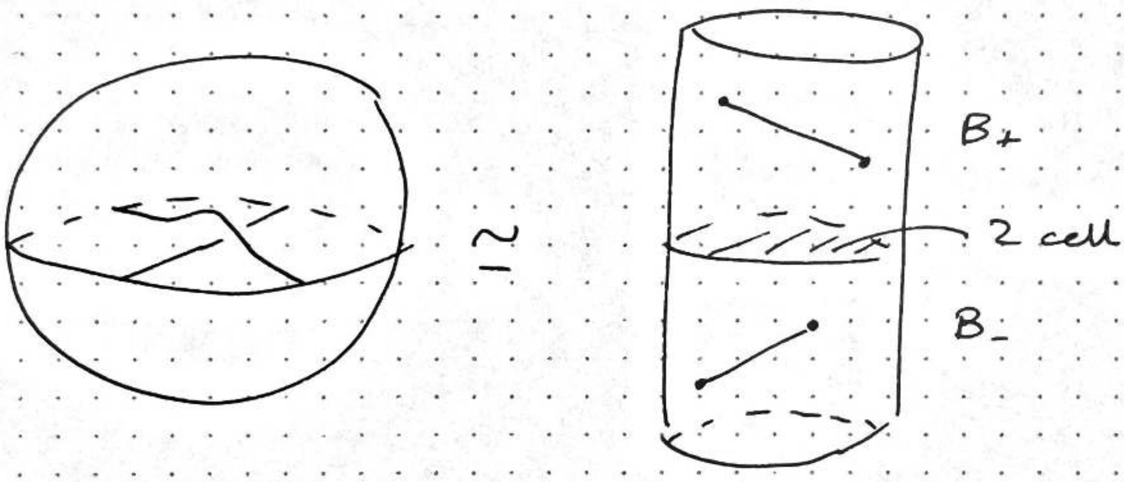
$\Rightarrow \pi_1(E_D) \simeq \langle a_1, \ldots, a_{n+1} \rangle$

is a free group with generators a_i which are Dehn generators

Step 2 Look at a crossing



$B^3 \setminus (B^3 \cap \text{Dgraph})$
strong def retrates to
$S^2 \setminus (S^2 \cap \text{Dgraph})$



$B_\pm \setminus (B_\pm \cap K)$ sdr to

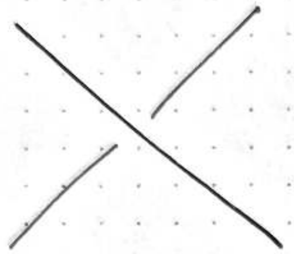
$\partial B_\pm \setminus (\partial B_\pm \cap K)$

$\therefore B^3 \setminus (B^3 \cap K)$ sdr to

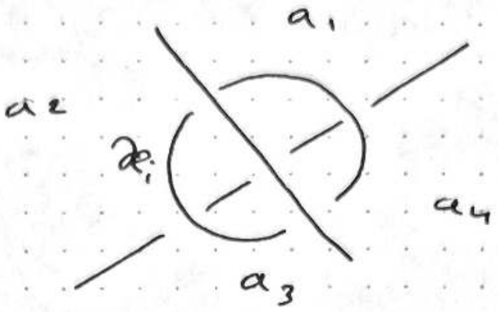
$(S^2 \setminus (S^2 \cap K)) \cup (2\text{-cell})$

So $E_L \sim E_D \cup \{n \text{ 2-cells}, 1 \text{ for each crossing}\}$

$\Rightarrow \pi_1(E_L) = \pi_1(E_D) / \langle \partial e_i \rangle_{i=1}^n$ whee e_i is the i^{th} 2-cell



top view



Dehn relator

$a_1 a_2^{-1} a_3 a_4^{-1}$ ✓

□

Ambient Isotopy

Def^n Suppose $N_0, N_1 \subset M$ are smooth submanifolds.
Say N_0, N_1 are <u>ambient isotopic</u> if there's a diffeo
$f : M \to M$ s.t. 1) $f(N_0) = N_1$
2) $f \sim_i \mathrm{id}_M$

$\Rightarrow f$ takes $M \setminus N_0 \xrightarrow{\cong} M \setminus N_1$

<u>Ambient isotopy $\Rightarrow$ isotopy</u>:
Let $i_0 : N_0 \hookrightarrow M$ be inclusion,
$i_1 : N_0 \hookrightarrow M$, $i_1 = f \circ i_0$
$i_0(N_0) = N_0$, $i_1(N_0) = N_1$
and $f \sim_i \mathrm{id}_M \Rightarrow f \circ i_0 \sim_i i_0$
$$\underset{\displaystyle i_1}{\overset{\displaystyle f\circ i_0}{\parallel}}$$

$\boxed{\text{Thm}}$ ~~If $N_0, N_1 \subset M$ are compact smooth submanifolds with $N_0 \sim_{ai} N$~~ If N is a compact manifold,
$i_1, i_0 : N \hookrightarrow M$ isotopic embeddings,
then $N_0 = i_0(N)$ is a.i. to $N_1 = i_1(N)$.

(Isotopy $\Rightarrow$ Ambient isotopy)

<u>Lemma</u> Suppose $N \subset M$ is a smooth submanifold, and that $v \in \Gamma(TN)$ is a vector field on N.
Then $\exists\, V \in \Gamma(TM)$ with $V|_N = v$

<u>Proof</u> Let $j : D(\nu_{M/N}) \hookrightarrow M$ be a tubular ngbd of N.
$D(\nu_{M/N}) = D$ $\qquad \pi : D \to N$ projection

Choose a splitting $TD = \pi^* TN \oplus \pi^* \nu_{M/N}$

Define $\hat{v} \in \Gamma(TD)$ by $\hat{v}|_w = \rho(\|\vec{w}\|)\, \pi^*(v|_{\pi(w)})$
$\in \pi^*(TN) \subset \pi^*(TD)$

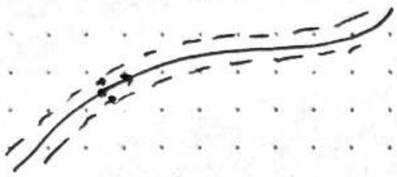
where $\rho : [0,1] \longrightarrow \mathbb{R}$

$\rho(x) = 0$ if $x \geq \frac{3}{4}$

$\rho(x) = 1$ if $x \leq \frac{1}{4}$

Now define $V|_x = \begin{cases} dj\,(\hat{v}|_w) & \text{if } x = j(w) \\ 0 & \text{else, i.e. } x \notin \text{im } j \end{cases}$

$\square$



Pf of Thm Spose $F : N \times I \longrightarrow M$ is isotopy. Then

$$\hat{F} : N \times I \longrightarrow M \times I$$
$$(x, t) \quad\quad (F(x,t), t)$$

is an embedding

Let $\hat{N} = \text{im } \hat{F}$, consider a vector field $\vec{v} \in \Gamma(T\hat{N})$

$$\vec{v} = d\hat{F}\left(\frac{\partial}{\partial t}\right) = \left(\frac{\partial F}{\partial t}, 1\right) \in TM \oplus TI = T(M \times I)$$

By Lemma, v extends to $V \in \Gamma(T(M \times I))$

$$V_{(p,t)} = (V_0(p,t), f(p,t)) \in TM \oplus TI$$

i.e. $V_0(p,t)$ is a time-dependent vector field on M

~~Ok~~ So let $\Phi : M \times I \to M$ be the flow of V_0,

$$\frac{d\Phi}{dt}\Big|_{(p,t)} = \cancel{V_0(p,t)}\; V_0(\Phi(p,t), t)$$

and $\dfrac{d\Phi}{dt}\Big|_{(F(x,t),t)} = \dfrac{dF}{dt}\Big|_{(F(x,t),t)}$

By uniqueness of solutions of ODEs, $\Phi|_{N \times I} = F$

so Φ is an ambient isotopy between $N_0 = F(N, 0)$

$N_1 = F(N, 1)$. $\square$

<u>Cor</u> If $\iota_0, \iota_1 : N \hookrightarrow M$ are isotopic

$N_j = \text{im}\, \iota_j$, $j_k : \cancel{\nu(D_{M/N_k})}$

$D(\nu_{M/N_k}) \hookrightarrow M$ a tub. ngbd

then $\text{im}\, j_0$ is ambient isotopic to $\text{im}\, j_1$.

<u>Proof</u> N_0, N_1 are a.i. via $f : M \xrightarrow{\sim} M$

$\Rightarrow j_0 \sim_i f \circ j_0$ is a tub ngbd of N_1

$\Rightarrow f \circ j_0 \sim_i j_1 \Rightarrow j_0 \sim j_1 \Rightarrow \text{im}\, j_0$ is a.i to $\text{im}\, j_0$

$\square$

<u>Cor</u> If $L_0, L_1 : \coprod^n S^1 \hookrightarrow S^3$ are isotopic,

j_0, j_1 are tubular ngbds of L_0, L_1, then

$S^3 \setminus j_0(L)^\circ \underset{\text{orientation preserving}}{\cong} S^3 \setminus j_1(L)^\circ$ $\square$

<u>Cor</u> Unknot $\not\sim_i T(2,3)$

<u>Pf</u> $E_U \simeq S^1 \times D^2$

$\pi_1(E_u) \cong \mathbb{Z}$

$\pi_1(E_{T(2,3)}) = \langle \gamma_1, \gamma_2 \mid \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle$

is non-abelian since it has a surjective map

$$\begin{aligned} \pi_1(E_{T(2,3)}) &\longrightarrow S_3 \\ \gamma_1 &\longmapsto (1\,2) \\ \gamma_2 &\longmapsto (2\,3) \end{aligned}$$

$\square$

<u>Rmk</u> $\pi_1(E_{T(2,3)}) = \langle x, y \mid x^2 = y^3 \rangle$

2.4) Alexander Polynomial

~~X~~ Let $K \subset S^3$ be a knot.

Consider abelianisation map

$$|\cdot| : \pi_1(E_K) \longrightarrow H_1(E_K) \cong \mathbb{Z}$$

$\ker |\cdot| \subset \pi_1(E_K)$

By the correspondence b/w covering spaces & subgroups of π_1, there's a connected covering space

$$p : \widetilde{E}_K \longrightarrow E_K \quad \text{with } \pi_1(\widetilde{E}_K) \cong \ker |\cdot|$$

$\ker|\cdot|$ is normal, so $\widetilde{E}_K$ is a normal covering with deck group $G_{Deck} = \pi_1(E_K)/\pi_1(\widetilde{E}_K) \cong \mathbb{Z}$

<u>Def^n</u> $\widetilde{E}_K$ is the <u>infinite cyclic cover</u> of E_K

<u>Fact</u>: $E_{T(2,3)} \sim X$ a cell complex with

1 0-cell p

2 1-cells a, b

1 2-cell attached along $w = aba b^{-1} a^{-1} b^{-1}$

$\ulcorner$ c.f. $\pi_1(X) = \langle a, b \mid w \rangle \cong \pi_1(E_{T(2,3)})$ $\lrcorner$

$$C_*^{cell}(X) = \underset{\langle w \rangle}{\mathbb{Z}} \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \underset{\langle a,b \rangle}{\mathbb{Z}^2} \xrightarrow{[0\ 0]} \underset{\langle p \rangle}{\mathbb{Z}}$$

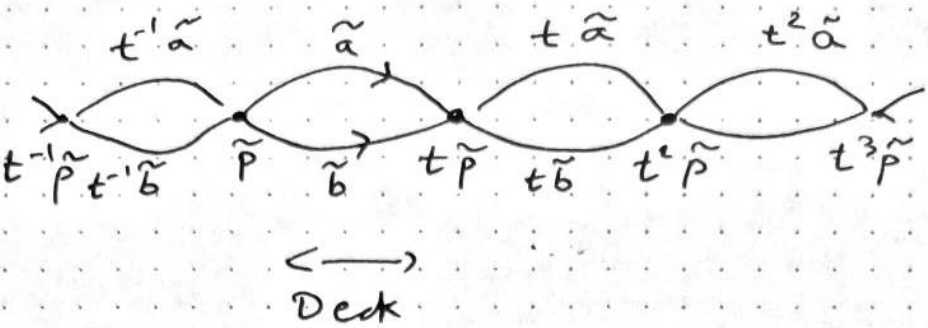
$da = db = 0$

$dw = a - b$

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ D^k \arrow[ur] \arrow[r] & X \end{tikzcd}$$

If $e : D^k \to X$ is a cell, $\pi_1(D^k) = 1$, so e lifts to $\widetilde{X} \sim \widetilde{E}_K$

~~Set~~ $G_{Deck} = \mathbb{Z} = \langle t \rangle$ acts freely & transitively on set of lifts.

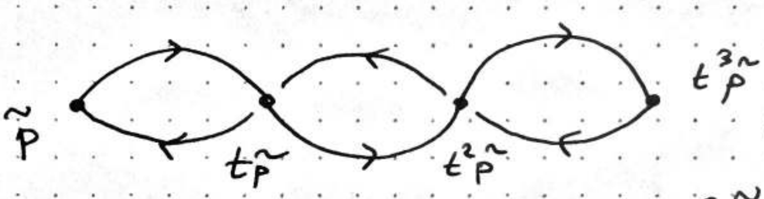


Let $\tilde{a}(0) = \tilde{p}$ for $\tilde{a}$ lift of a

Then $\tilde{a}(1) = t^{|a|} \cdot \tilde{p}$

Similarly $\tilde{b}(0) = \tilde{p}$, $\tilde{b}(1) = t^{|b|}\tilde{p} = tp$

Let $\tilde{w}$ be lift of w with $\tilde{w}(1) = \tilde{p}$



$aba\,b^{-1}a^{-1}b^{-1}$

$\partial\tilde{w}$

$C_*^{cell}(\tilde{X})$ is a module over $R = \mathbb{Z}[G_{Deck}] = \mathbb{Z}[\mathbb{Z}]$
$$= \mathbb{Z}[t^{\pm 1}]$$

$$C_*^{cell}(\tilde{X}) = R \xrightarrow{\begin{bmatrix} t^2-t+1 \\ -t^2+t-1 \end{bmatrix}} R \oplus R \xrightarrow{[t-1,\, t-1]} R$$
$$\langle \tilde{w} \rangle \quad \langle \tilde{a}, \tilde{b} \rangle \quad \langle \tilde{p} \rangle$$

$$d\tilde{a} = t\tilde{p} - \tilde{p} = (t-1)\tilde{p}$$
$$d\tilde{b} = t\tilde{p} - \tilde{p} = (t-1)\tilde{p}$$
$$d\tilde{w} = \tilde{a} + t\tilde{b} + t^2\tilde{a} - t^2\tilde{b} - t\tilde{a} - \tilde{b}$$
$$= (1 - t + t^2)(\tilde{a} - \tilde{b})$$

Recall $K \hookrightarrow S^3$ a knot

Infinite cyclic cover $p: \tilde{E}_K \to E_K$

Deck group $G_{Deck} \cong \mathbb{Z} \cong H_1(E_K) \cong \langle \varphi \rangle$

$\varphi: \tilde{E}_K \to \tilde{E}_K$

Def^n The <u>Alexander module</u> of K is

$$A(K) = H_1(\tilde{E}_K)$$

as a module over

$$R = \mathbb{Z}[H_1(E_K)] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$$

Where $t \cdot x = \varphi_*(x)$

<u>Ex</u> a) $K = U$, $E_K = S^1 \times D^2$

$$\tilde{E}_K = \mathbb{R} \times D^2$$

$$A(K) = H_1(\tilde{E}_K) = 0$$

b) $K = T(2,3)$

$$E_K \sim X \quad \text{with} \quad C_\cdot^{cell}(\tilde{X}) = R \xrightarrow[d_2]{\begin{bmatrix} t^2-t+1 \\ -t^2+t-1 \end{bmatrix}} R \oplus R \xrightarrow[d_1]{[t-1,\ t-1]} R$$

$\ker d_1 = \langle (1,-1) \rangle$

$\operatorname{im} d_2 = \langle (t^2-t+1, -t^2+t-1) \rangle$

$$\therefore A(K) = H_1(\tilde{E}_K) = R/(t^2-t+1) \neq 0$$

So $E_U \not\simeq E_{T(2,3)}$

$\therefore U \neq T(2,3)$

<u>Remark</u> If $\bar{K}$ is the mirror of K, then there's a o.r. diffeomorphism $(S^3, K) \to (S^3, \bar{K})$

$\Rightarrow E_{\bar{K}}$ is o.r. diffeo^c to E_K

$\pi_1(E_K)$, $A(K)$ are insensitive to orientation

i.e. $\pi_1(E_K) = \pi_1(E_{\bar{K}})$, $\quad A(K) = A(\bar{K})$

But we know $T(2,3) \neq \overline{T(2,3)} = T(-2,3)$

<u>Thm</u> (Gordon + Luecke) If E_K is O.P. homeo^c to $E_{K'}$, then $K \sim_i K'$.
(Knots are determined by their oriented complements)

$\Rightarrow E_{T(2,3)}$ does not have an orientation-reversing homeomorphism (unto itself) [P.S.A.]

Let $A(K;\mathbb{Q}) = H_1(\tilde{E}_K;\mathbb{Q})$
so $A(K;\mathbb{Q})$ is a module over $R_{\mathbb{Q}} = \mathbb{Q}[t^{\pm 1}]$.
$R_{\mathbb{Q}}$ is a PID
$E_K \sim X_K$ which is a cell cx with 1 0-cell
n 1-cells
n-1 2-cells
(from proof that $G_{Dehn} = \pi_1(E_K)$)
$\Rightarrow \tilde{E}_K \sim \tilde{X}_K$ which is a cell cx with cells
$t^i e$, $i \in \mathbb{Z}$, e a cell of X

$\Rightarrow C_*^{cell}(\tilde{X}): R^{n-1} \longrightarrow R^n \longrightarrow R$
is finitely generated over R
$\Rightarrow H_*(E_K;\mathbb{Q})$ is f.g. over $R_{\mathbb{Q}}$ [use that $R_{\mathbb{Q}}$ is Noetherian]

Structure Thm for f.g. modules over a PID
$\Rightarrow H_1(\tilde{E}_K;\mathbb{Q}) \simeq \underbrace{R_{\mathbb{Q}}^k}_{\text{free}} \oplus \underbrace{(R_{\mathbb{Q}}/P_1 \oplus \cdots \oplus R_{\mathbb{Q}}/P_r)}_{\text{torsion}}$

<u>Lemma</u> $H_1(\tilde{E}_K;\mathbb{Q})$ is a torsion module over $R_\mathbb{Q}$

<u>Proof</u> We can recover $C_*^{cell}(X_K)$ by setting $t=1$

i.e. $C_*^{cell}(X) \simeq C_*^{cell}(\tilde{X}_K) \underset{R_\mathbb{Q}}{\otimes} M_{t-1}$

for $M_{t-1} = \dfrac{R_\mathbb{Q}}{(t-1)}$

By UCT,

$$H_*(X_K)_\mathbb{Q} \simeq \overbrace{H_*(\tilde{X}_K) \otimes M_{t-1}}^{Ⓐ} \oplus \overbrace{\mathrm{Tor}^{R_\mathbb{Q}}(H_{*-1}(\tilde{X}_K), M_{t-1})}^{Ⓑ}$$

$H_0(\tilde{X}_K) \simeq \mathbb{Q}$ since $\tilde{X}_K$ is connected

$\simeq R_\mathbb{Q}/(t-1)$

$\simeq M_{t-1}$

$\Rightarrow$ Ⓑ $\simeq M_{t-1} \simeq \mathbb{Q}$

But $H_1(X_K;\mathbb{Q}) \simeq \mathbb{Q}$

So Ⓐ $= 0 \Rightarrow H_*(\tilde{X}_K)$ has no free part. □

So $A(K;\mathbb{Q}) \simeq \dfrac{R_\mathbb{Q}}{P_1} \oplus \cdots \oplus \dfrac{R_\mathbb{Q}}{P_r}$

Define the Alexander polynomial

$$\Delta_K(t) = \prod_{i=1}^{r} P_i \in R_\mathbb{Q}$$

to be the "order" of $A(K)$

This is well-defined up to multiplication by units in $R_\mathbb{Q}$

i.e. up to multiplication by ct^i, $c \in \mathbb{Q} \setminus \{0\}$

Ex: $\Delta_U(t) \sim 1$

$\Delta_{T(2,3)}(t) \sim (t^2 - t + 1)$

where $f \sim g$ means $f = ug$ where u is a unit

Fibred Knots

Def^{n} A smooth manifold M fibres over S^1 if there's a submersion $f: M \to S^1$

=> (Ehresman) M is a locally trivial fibre bundle over S^1 with fibre $F = f^{-1}(1)$ ⌜proper?⌟

Ex: M fibres over $S^1 \Rightarrow \chi(M) = 0$ ⌜nowhere-zero v. field⌟

so T^2 fibres over S^1

but Σ_g does not if $g > 1$

Given M as above, consider

$$\tilde{M} = \{(x,t) \in M \times \mathbb{R} \mid f(x) = p(t)\}$$

The map $\tilde{f}: \tilde{M} \to \mathbb{R}$

a) $(x,t) \mapsto t$

is a submersion,

the map $\tilde{p}: \tilde{M} \to M$

b) $(x,t) \mapsto x$

is a covering map with deck group $\mathbb{Z}$.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & S^1 \end{array}$$

$\tilde{M}$ is a fibre product of $\mathbb{R}$ and M

a) ⇒ there's a diffeo

$$\alpha: F \times \mathbb{R} \to \tilde{M}$$

since $\tilde{M}$ is trivial

b) => there's a deck transformation

$$\beta: \tilde{M} \to \tilde{M}$$
$$(x,t) \mapsto (x, t+1)$$

Composition

- $F \hookrightarrow F \times \{0\} \subset F \times \mathbb{R} \xrightarrow{\alpha} \tilde{M} \xrightarrow{\beta} \tilde{M} \xrightarrow{\alpha^{-1}} F \times \mathbb{R}$

$$F \times \{0\} \xrightarrow{\qquad\qquad \varphi \qquad\qquad} F \times \{1\}$$

defines $\varphi : F \to F$ which is <u>monodromy</u> of fibration

$$M = \tilde{M} / \cancel{\alpha}\, \beta$$
$$= F \times [0,1] \big/ (\varphi(x), 0) \sim (x, 1)$$
[maybe φ^{-1}]

<u>Def^n</u> $K \hookrightarrow S^3$ is *fibred* if E_K fibres over S^1

- If so, I can choose a fibration $f : E_K \to S^1$ s.t. $F = f^{-1}(1)$ is connected (exercise)

If K is fibred, have a covering map
$$\tilde{E}_K \longrightarrow E_K \quad \text{with deck group } \mathbb{Z}$$
by construction we just did.

This must be the infinite cyclic cover since the only surjective map $\pi_1(E_K) \longrightarrow \mathbb{Z}$ is abelianisation.

- So by (a), $\tilde{E}_K \simeq F \times \mathbb{R}$
$$H_1(\tilde{E}_K) = H_1(F)$$

How does t act?
$$t \cdot x = \beta_*(x)$$
i.e. $t : H_1(F) \longrightarrow H_1(F)$
is given by φ_*

So as a module over $R_{\mathbb{Q}}$, $H_1(\tilde{E}_K) = H_1(F) \otimes R_{\mathbb{Q}} \big/ t \cdot x = \varphi_*(x)$

- In other words,
$$H_1(\tilde{E}_K) = \operatorname{coker} \underbrace{\Phi_*}_{I \otimes t - \varphi_* \otimes 1} : H_1(F) \otimes R_{\mathbb{Q}} \longrightarrow H_1(F) \otimes R_{\mathbb{Q}}$$

$\Rightarrow \Delta_K(t) \sim \det(tI - \varphi_*)$

is the characteristic polynomial of
the monodromy action $\varphi_* : H_1(F) \to H_1(F)$

Torus Knots

$$S^3 = \{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \}$$

Identify S^3 w/ $\mathbb{R}^3 \cup \{\infty\}$ via stereographic projection from $(0,i)$. This identifies $S^1 \times 0$ with the unit circle in the x-y plane in $\mathbb{R}^3$.

Define $T(m,n)$ (the (m,n) torus knot) to be

$$T(m,n) = \{ (z,w) \in S^3 \mid z^n = w^m \}$$

If so, have

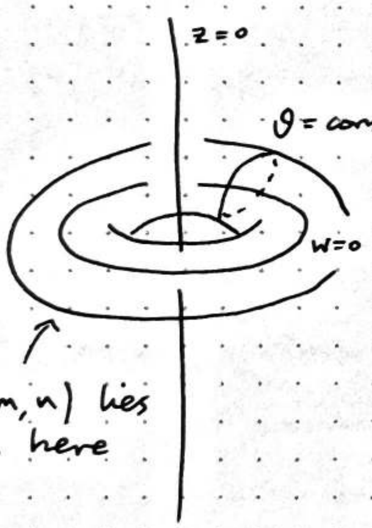
$$|z|^2 + |w|^2 = 1 \quad \text{and} \quad |z|^n = |w|^m \quad \text{so}$$

$$|z|^2 + |z|^{2n/m} = 1$$

As $f(r) = r^2 + r^{2n/m}$ is a monotonic function of r, $\exists!$ value of r s.t. $|z| = r$ satisfies the above

$T(m,n)$ lies on the torus $\{ (z,w) \in S^3 \mid |z| = r \}$

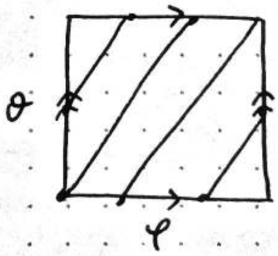
This torus looks like



$T(m,n)$ lies on here

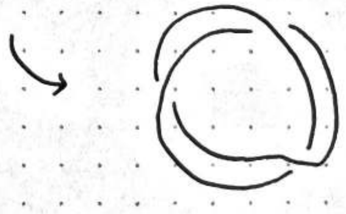
Write $z = re^{i\theta 2\pi}$, $w = r'e^{i\varphi 2\pi}$

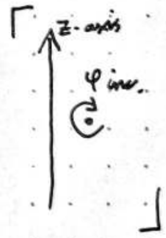
Then $n\theta - m\varphi \equiv 0 \mod 1$



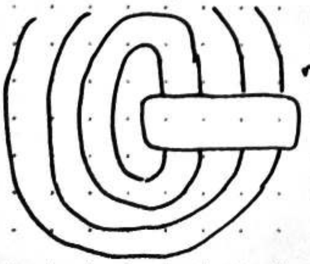
line with slope $\frac{n}{m}$

$n = 2,\ m = 3$





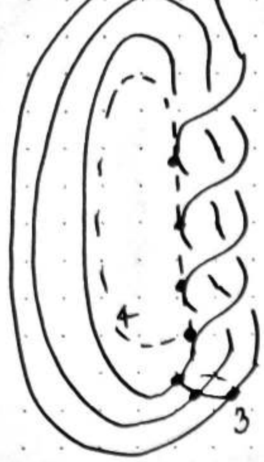
In general $T(m,n)$ has a diagram



m strands

$n\ \frac{1}{m}$ twists

$= \frac{1}{m}$ twist



sits on torus

$$\xi \longmapsto \xi/|\xi|$$

Consider $f: S^3 \setminus T(m,n) \to \mathbb{C}\setminus 0 \longrightarrow S^1$

$$(z,w) \longmapsto z^n - w^m$$

This map is a submersion and so $T(n,m)$ is fibred.

[it is a submersion!]

- Plenty of other knots are fibred e.g. figure 8.

Presentations:

Def: Say M is a module over a (commutative) ring R.
M is finitely presented if there's an exact sequence

$$R^n \xrightarrow{P} R^m \xrightarrow{\pi} M \longrightarrow 0$$

and this sequence is a presentation of M.

If $e_i = (0, \dots, 1, \dots, 0) \in R^m$, then $\pi(e_1), \dots, \pi(e_m)$ are generators of M; and $P(e_1), \dots, P(e_n)$ are relations between generators.

Write $P(e_j) = \sum P_{ij} e_i$. Say $m \times n$ matrix P_{ij} is a presentation matrix for M.

Fact: If P, P' are 2 presentations for M, they are related by a sequence of elementary moves and their inverses:

generators $a_i = \pi(e_i)$

relations $r_j = P(e_j)$

- Moves: 1) Add a new generator a_{m+1}, and relation: $a_{m+1} = 0$

$$P \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$$

2) Add a new relation $0=0$

$$P \longleftrightarrow (P\ 0)$$

3) replace a_i with $a_i + \alpha a_j$, $\alpha \in R$

$P \longleftrightarrow P'$ where j^{th} row of P'
$= j^{th}$ row of P
$- \alpha \cdot i^{th}$ row of P

4) replace r_i with $r_i + \beta r_j$, $\beta \in R$

$P \longleftrightarrow P''$ where i^{th} column of P''
$= i^{th}$ column of P
$+ j^{th}$ column of $P \cdot \beta$

5) Multiply rows/columns by units

Now suppose R is a UFD. Then if $\alpha_1, \dots, \alpha_k \in R$ there's a well-defined gcd $\gcd(\alpha_1, \dots, \alpha_k)$ up to units

<u>Def</u> If P is an $m \times n$ matrix over a UFD R, let
$e_0(P) = \gcd(\det(\tilde{P}))$ where $\tilde{P}$ ~~[illegible]~~ ranges over $m \times m$ submatrixes obtained by deleting columns of P (if $n \geq m$)

$= 0$ if $n < m$

<u>Lemma</u> If P and P' are related by an elementary move, then $e_0(P) \sim e_0(P')$, where $\sim$ means equal up to units.

<u>Proof</u> Just check for each move, using fact that det is linear on rows + columns and fact that

$$\gcd(x, y) \sim \gcd(x, y + \alpha x)$$

<u>Def</u> If M is a f.p. module over a UFD, let
$e_0(M) = e_0(P)$ where P is any presentation matrix of M.
Lemma $\Rightarrow$ this is well-defined

<u>Ex</u>: If R is a PID, then
$e_0(M) = \text{order } M$ if M is torsion
$\qquad\quad 0$ o/w

<u>Pf</u>: Choose a presentation matrix in Smith Normal Form

<u>Multivariable Alexander Polynomial</u>

Suppose L is a link, $L \hookrightarrow S^3$, with n components.

<u>Def</u>: The universal <u>abelian cover</u> $p: \tilde{E}_L \to E_L$ is the connected covering space given by the kernel of the abelianisation map $|\cdot| : \pi_1(E_L) \longrightarrow H_1(E_L) = \langle m_1, \dots, m_n \rangle \simeq \mathbb{Z}^n$

Then $\tilde{E}_L$ has $G_{Deck} \simeq H_1(E_L) = \mathbb{Z}^n$, so
$H_1(\tilde{E}_L)$ is a module over $\mathbb{Z}[H_1(E_L)] = \mathbb{Z}[\mathbb{Z}^n]$
$$= \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = R_L$$

$\mathbb{Z}[t_1, \dots, t_n]$ is a UFD $\Rightarrow$ R_L is a UFD (yeah)

<u>Def</u> The multivariable <u>Alexander polynomial</u>
$$\Delta(L) = e_0(H_1(\tilde{E}_L)) \in R_L$$
well-defined up to mult^n by a unit in R_L, $\pm t_1^{a_1} \dots t_n^{a_n}$

<u>Ex</u>: $H_1(E_{T(2,3)}) \simeq R_K / (t^2 - t + 1)$

$\Rightarrow \Delta(T(2,3)) \sim t^2 - t + 1$

2.5) Fox Calculus

- Suppose X is a cell cx with

  1 0-cell p

  m 1-cells $a_1, \ldots, a_m$

  n 2-cells attached along $w_1, \ldots, w_{\cancel{m}n}$
  words in the a_i's

$\pi_1(X) \simeq \langle a_1, \ldots, a_m \mid w_1, \ldots, w_n \rangle$

$H_1(X) \simeq \mathbb{Z}^k \oplus T$, T torsion

$\overline{H_1(X)} = H_1(X)/T \simeq \mathbb{Z}^k$

- $$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[rr, bend right, "\text{surjective}"'] & H_1(X) \arrow[r] & \overline{H_1(X)} \simeq \mathbb{Z}^k \end{tikzcd}$$

Let $p : \tilde{X} \to X$ be the covering map corresponding to the kernel of $|\cdot|$

Deck group is $G_{Deck} \simeq \mathbb{Z}^k$

$\tilde{X}$ will be a cell cx, cells are of the form $g\tilde{e}$ where $g \in G_{Deck}$, $\tilde{e}$ is a lift of a cell e based at

- $\tilde{p}$.

Since cells in $\tilde{X}$ are lifts of cells in X, boundary operator in $C_*^{cell}(X)$ commutes with the action of G_{Deck}

So $C_*^{cell}(\tilde{X})$ is a chain cx over $R_X = \mathbb{Z}[\overline{H_1(X)}]$

$$= \mathbb{Z}[\mathbb{Z}^k] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$$

$$C_*^{cell}(\tilde{X}) : \quad \underset{\langle \tilde{w}_j \rangle}{R_X^n} \xrightarrow{d_2} \underset{\langle \tilde{a}_i \rangle}{R_X^m} \xrightarrow{d_1} \underset{\langle \tilde{p} \rangle}{R_X}$$

- $\tilde{a}_i$ starts at $\tilde{p}$, ends at $|a_i|\tilde{p}$

$\Rightarrow d_1(\tilde{a}_i) = (|a_i| - 1)\tilde{p}$

$\therefore d_1$ has matrix $[\,|a_1|-1, \ldots, |a_m|-1\,]$ $1\times m$ matrix

- $d_2 : R_X^n \xrightarrow{A_X} R_X^m$

$A_X = [\, d_{a_i} w_j \,]$ where $d_{a_i} w$ is the Fox derivative

given by $d_{a_i}\left(\prod_{k=1}^{r} a_{i_k}^{\pm 1}\right) = \sum_{k=1}^{r} |a_{i_1}^{\pm 1} \ldots a_{i_{k-1}}^{\pm 1}|\, d_{a_i}(a_{i_k}^{\pm 1})$ $\in \mathbb{Z}[\overline{H_1(X)}]$

and $d_{a_i}(a_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$

$d_{a_i}(a_j^{-1}) = \begin{cases} -|a_i^{-1}| & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$

- <u>Proof</u> $d_{a_i}(\tilde{w}_j)$ counts segments of $\tilde{w}_j$ that run along

$g\tilde{a}_i$, $g \in G_{Deck}$

As we walk along $\tilde{w}_j$, segments we pass over correspond to the lifts of generators in w, we first pick out those that run over $g\tilde{a}_i \longleftrightarrow$ appearances of a_i in W

Lift of a_i corresponding to $\alpha a_i \beta$ is $|\alpha|\, \tilde{a}_i$

Lift corresponding to $\alpha a_i^{-1} \beta$ is $-|\alpha||a_i|^{-1}\tilde{a}_i$

- <u>Lemma</u> $d_{a_i}(ww') = d_{a_i}(w) + |w|\, d_{a_i}(w')$ "Leibniz"

<u>Proof</u> Almost follows from defⁿ, but should check

$$d_{a_i}(\alpha a_i)(a_i^{-1}\beta) = d_{a_i}(\alpha\beta)$$
$$\parallel \qquad\qquad \parallel ?$$
$$d_{a_i}\, \alpha(a_i a_i^{-1})\beta$$

follows from $d_{a_i}(a_i a_i^{-1}) = 1 + |a_i|(-|a_i|^{-1}) = 1 - 1 = 0$ $\square$

-

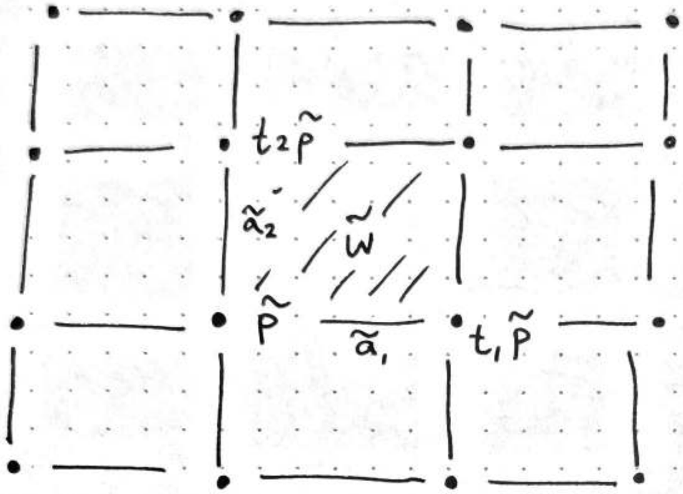
Ex1 $\pi_1(X) = \langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} \rangle$

$(X \sim E_{T(2,2)})$

Abelianize $\pi_1(X)$: $a_1 + a_2 - a_1 - a_2 = 0$

$\Rightarrow H_1(X) = \langle t_1, t_2 \rangle \cong \mathbb{Z}^2$, $|a_i| = t_i$

$\tilde{X}$



$$d_{a_1}(w) = d_{a_1}(a_1 a_2 a_1^{-1} a_2^{-1}) = 1 - |a_1 a_2||a_1^{-1}| = 1 - t_2$$

$$d_{a_2}(w) = d_{a_2}(a_1 a_2 a_1^{-1} a_2^{-1}) = |a_1| - |a_1 a_2 a_1^{-1} a_2^{-1}| = t_1 - 1$$

$$C_*^{cell} : R \xrightarrow{\begin{bmatrix} 1-t_2 \\ t_1 - 1 \end{bmatrix}} R^2 \xrightarrow{[t_1 - 1 \quad t_2 - 1]} R$$

$$d^2 = 0 \; ; \; [t_1 - 1 \quad t_2 - 1] \begin{bmatrix} 1 - t_2 \\ t_1 - 1 \end{bmatrix} = 0$$

Ex 2 : $\pi_1(X) = \langle a_1, a_2, a_3 \mid a_2^{-1} a_1 a_3 a_1^{-1},\ a_1^{-1} a_3 a_2 a_3^{-1} \rangle$

Abelianize: $-|a_2| + |a_1| + |a_3| - |a_1| = 0$, $|a_2| = |a_3|$

$-|a_1| + |a_3| + |a_2| - |a_3| = 0$, $|a_1| = |a_3|$

$H_1(X) = \mathbb{Z} = \langle t \rangle$, $|a_1| = |a_2| = |a_3| = t$

$$A_X = \begin{bmatrix} t^{-1}-1 & -t^{-1} \\ -t^{-1} & +1 \\ +1 & t^{-1}-1 \end{bmatrix}$$

Check : $d^2 \stackrel{?}{=} 0$

Group presentations

Suppose $G = \underbrace{\langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle}_{P}$ is a finitely presented group

Then to the presentation P we associate a 2-complex X_P as before

1-cells $\longmapsto a_i$

2-cells $\longmapsto w_j$

Let $A_P = A_{X_P}$ be the Alexander matrix

Tietze moves are elementary moves on presentations :

- add a generator a_{m+1}, and $w_{n+1} = a_{m+1}$

$P' = \langle a_1, \dots, a_m, a_{m+1} \mid w_1, \dots, w_n, a_{m+1} \rangle$

$$A_{P'} = \begin{bmatrix} A_P & 0 \\ 0 & 1 \end{bmatrix}$$

- add a trivial relation

$$P' = \langle a_1, \dots, a_m \mid w_1, \dots, w_n, \underset{\substack{\uparrow \\ \text{empty word}}}{1} \rangle$$

$$A_{P'} = (A_P \quad 0)$$

- multiply one relation by another

$$P' = \langle a_1, \dots, a_m \mid w_1, \dots, w_i w_j, \dots, w_n \rangle$$

$$w_i' = w_i w_j$$
$$w_j' = w_j$$
$$w_k' = w_k \quad , \; k \neq i, j$$

$$d_{a_k} w_i' = d_{a_k} w_i + |w_i| \, d_{a_k} w_j$$
$$= d_{a_k} w_i + d_{a_k} w_j$$

i.e. i^{th} ~~row~~ column of $A_{P'}$ = $i^{th} + j^{th}$ ~~rows~~ columns of A_P

- replace w_j by $w_j' = a_i w_j a_i^{-1}$

$$d_{a_k} w_j' = |a_i| \, d_{a_k} w_j \qquad \ulcorner\text{even if } k = i \; (!)\lrcorner$$

multiply j^{th} column of A_P by a unit $|a_i|$ to get $A_{P'}$

Thm (Tietze) If P, P' are presentations for isomorphic groups, then we can get from P to P' by a sequence of Tietze moves

Def If P is a group presentation w/ m gen's, n rels, let

$$\Delta(P) = e_1(A_P) = ~~\det~~ \gcd\left(\det \tilde{A} \;\middle|\; \tilde{A} \text{ an } (m-1)\times(m-1) \text{ submatrix of } A_P\right)$$

Thm If P, P' related by Tietze then $\Delta_P \sim \Delta_{P'}$

So if G is a f.p. group define

$$\Delta(G) = \Delta(P) \quad \text{where } P \text{ is a presentation}$$

multivariable Alexander poly

$$\Delta(L) = \Delta(\pi_1(EL))$$

Recall:

- $G = \underbrace{\langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle}_{P}$

$$\Delta(G) = \Delta(X_P) = e_1(A_P) = \gcd\left\{ \det \tilde{A} \,\middle|\, \begin{matrix}\tilde{A} \text{ is a } (m-1)\times(m-1) \\ \text{submatrix of } A_P\end{matrix} \right\}$$

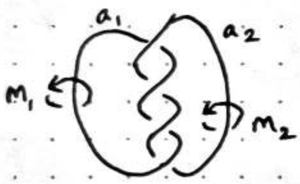
Ex: $G = \pi_1(E_{T(2,3)}) = \langle a_1, a_2, a_3 \mid a_2^{-1} a_1 a_3 a_1^{-1}, a_1^{-1} a_3 a_2 a_3^{-1} \rangle$

$$A_P = \begin{bmatrix} t^{-1}-1 & -t^{-1} \\ -t^{-1} & 1 \\ 1 & t^{-1}-1 \end{bmatrix}$$

all 3 determinants are $\sim t^2 - t + 1 \sim \Delta(T(2,3))$

- Ex 2: $L = T(2,4)$

$$\pi_1(E_L) = \langle a_1, a_2 \mid a_1 a_2 a_1 a_2 a_1^{-1} a_2^{-1} a_1^{-1} a_2^{-1} \rangle$$



Abelianize: $\left[|a_1|, |a_2| \,\middle|\, \begin{matrix} |a_1| + |a_2| + |a_1| + |a_2| \\ -|a_1| - |a_2| - |a_1| - |a_2| \end{matrix} \right]$

$$= \langle t_1, t_2 \rangle \cong \mathbb{Z}^2$$

$$H_1(E_L) \simeq \langle m_1, m_2 \rangle \quad \text{"meridians"}$$

$$A_P = \begin{bmatrix} d_{a_1} w \\ d_{a_2} w \end{bmatrix} = \begin{bmatrix} 1 + t_1 t_2 - t_1 t_2^2 - t_2 \\ t_1 + t_1^2 t_2 - t_1 t_2 - 1 \end{bmatrix}$$

- $$= \begin{bmatrix} (1 + t_1 t_2)(1 - t_2) \\ (1 + t_1 t_2)(t_1 - 1) \end{bmatrix}$$

$\therefore \Delta(L) \sim 1 + t_1 t_2$

Note $d^2 = 0 \Rightarrow$

$$[t_1 - 1 \quad t_2 - 1] \begin{bmatrix} d_{a_1} w \\ d_{a_2} w \end{bmatrix} = 0$$

Now suppose P has 1 more generator than relators, $n = m-1$

(e.g. $P = G_{Dehn}$ or $P = G_{Wirt}$, toss one relation)

- Then $\Delta(G) = \gcd(\det A_{P,\hat{i}})$ where $A_{P,\hat{i}}$ is A_P with i^{th} row deleted

<u>Prop</u> $(|a_j|-1)\det A_{p,\hat{i}} \sim (|a_i|-1)\det A_{p,\hat{j}}$

<u>Proof</u> $A_p = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$

$d^2 = 0$ in $C_{cell}(\tilde{X}_p)$

$\Rightarrow \sum_i (|a_i|-1)\, v_i = 0$

as $d_1 = [\,|a_1|-1 \;\cdots\; |a_m|-1\,]$

So $(|a_j|-1)\det A_{p,\hat{i}} = \det \begin{bmatrix} v_1 \\ \vdots \\ \cancel{v_i} \\ \vdots \\ (|a_j|-1)\,v_j \\ \vdots \\ v_m \end{bmatrix}$

$$= \det \begin{bmatrix} v_1 \\ \vdots \\ \cancel{v_i} \\ \vdots \\ -\sum_{j\neq k}(|a_k|-1)\,v_k \\ \vdots \\ v_m \end{bmatrix}$$

jth spot

$$= \det \begin{bmatrix} v_1 \\ \vdots \\ \cancel{v} \\ \vdots \\ -(|a_i|-1)\,v_i \\ \vdots \\ v_m \end{bmatrix}$$

$$\sim (|a_i|-1)\det A_{p,\hat{j}} \qquad \square$$

<u>Equivalently</u>: $\dfrac{\det A_{p,\hat{i}}}{|a_i|-1} \sim \dfrac{\det A_{p,\hat{j}}}{|a_j|-1}$

$\Rightarrow (|a_i|-1)\,\gcd(\det A_{p,\hat{k}}) = \alpha \det A_{p,i}$ [use UFD, like, aggresively]

where $\alpha = \gcd(|a_1|-1, \dots, |a_m|-1)$

Exercise $\alpha = \begin{cases} t-1 & \text{if } H_1(E_L) = \mathbb{Z} \\ 1 & \text{if } H_1(E_L) = \mathbb{Z}^k,\ k>1 \end{cases}$

E.g. if $(a_1, \dots, a_m)$ generate $\mathbb{Z}$,

$$\gcd(t^{a_1} - 1, \dots, t^{a_m} - 1) = t - 1$$

But $\gcd((t_1 - 1), (t_2 - 1)) = 1$

<u>Cor</u>: $(|a_i| - 1)\, \Delta(G) \sim \alpha \det A_{P, \hat{\imath}}$

<u>Prop</u>: Suppose K is a knot.
Then $\Delta(E_K) \sim e_0(H_1(\tilde{E}_K))$

<u>Proof</u>: Use $P = G_{\mathrm{Wirt}}$, so all a_i's are conjugate

$\Rightarrow |a_i| = |a_j| = t$

So $d_1 = [t-1, \dots, t-1]$

$\ker d_1 = \{ (x_1, \dots, x_m) \mid \sum x_i = 0 \}$

$\pi_{\hat{m}} : \ker d_1 \xrightarrow{\cong} R^{m-1}$ projection on first $m-1$ coords

So $H_1(\tilde{E}_K) = \dfrac{\ker d_1}{\operatorname{im} d_2} \xrightarrow{\pi} \dfrac{R^{m-1}}{\operatorname{im}(\pi_{\hat{m}} \circ A_P)} = \dfrac{R^{m-1}}{\operatorname{im}(A_{P, \hat{m}})}$

$\Rightarrow e_0(H_1(\tilde{E}_K)) = \det A_{P, \hat{m}} \sim \Delta(E_K)$ $\square$

2.6) <u>Seifert genus</u>

Recall that if $K \hookrightarrow S^3$, a <u>Seifert surface</u> of K is an embedded, compact, connected, oriented surface $S \hookrightarrow S^3$ with $\partial S = K$.

<u>Def^n</u> If K is a knot in S^3, its <u>Seifert genus</u> is

$$g(K) = \min\{ g(S) \mid S \text{ Seifert surface for } K \}$$

<u>Prop</u>: $g(K) = 0 \iff K = U$

<u>Proof</u>: $g(U) = 0$ is clear

If $g(K)=0$, let $\varphi : B^2 \hookrightarrow S^3$ be a genus 0 Seifert surface.

For $t \in (0,1]$, let $K_t = \varphi|_{\partial B_t}$

This is a knot with $K_t \sim_i K_1 = K$ via $\varphi|_{B^2 \setminus B_t^\circ}$

For small ε, $\varphi|_{B_\varepsilon} \sim_i d\varphi|_0 \big|_{B_\varepsilon}$

$\mathrm{Im}\ d\varphi|_0 \subset \underset{(\text{a plane})}{\mathbb{R}^2} \subset \mathbb{R}^3$

So $K_\varepsilon \sim_i K' \subset$ a plane $\Rightarrow K' = U$. $\square$

$\tilde{E}_K$ via Seifert surfaces ◡̈

Let S be a Seifert surface of K

Lemma $\nu_{S^3/S}$ is trivial

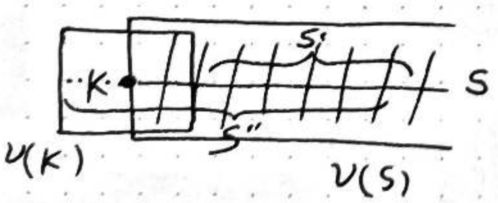
Proof A real line bundle $\begin{matrix} \mathbb{R} \to L \\ \downarrow \\ B \end{matrix}$ is trivial iff L is orientable.

S^3 is orientable, S is orientable

$\Rightarrow \nu_{S^3/S}$ is orientable. $\square$

Let $\nu(S)$ be a closed tubular nghbd of S.

$\nu(S) \simeq S \times [-1,1]$



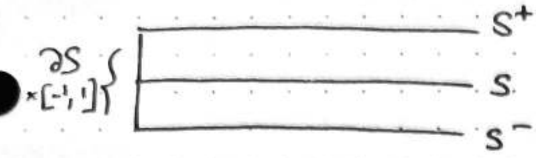
Then $S \sim_i S' \sim S''$, so

if $E_S = S^3 \setminus \mathrm{int}(\nu(S))$

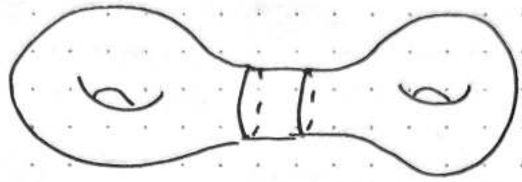
then $E_S \simeq E_{S'} \simeq E_{S''}$

$\Rightarrow E_S \simeq E_K \setminus E_{S'}$

$$\partial E_S = \partial \nu(S) = \underset{\substack{\uparrow \\ S_-}}{S \times \{-1\}} \underset{\partial S \times \{-1\}}{\cup} \partial S \times [-1,1] \underset{\partial S \times \{1\}}{\cup} \underset{\substack{\uparrow \\ S_+}}{S \times \{1\}}$$

- 

is double of S



so $g(\partial\nu(S)) = 2g(S)$

Lemma E_S is connected and $H_1(E_S) \simeq \mathbb{Z}^{2g(S)}$

Proof $S^3 = E_S \underset{\partial\nu(S)}{\cup} \nu(S)$

- Mayer–Vietoris says

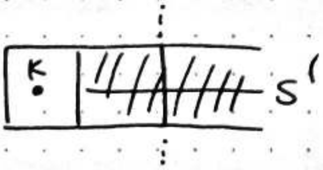
$$H_2(\cancel{S^3}) \xrightarrow{\text{zero}}$$

$$\underset{\mathbb{Z}^{4g(S)}}{H_1(\partial\nu(S))} \xrightarrow{\simeq} \underset{\mathbb{Z}^{2g(S)}}{H_1(\nu(S))} \oplus \underset{\therefore = \mathbb{Z}^{2g(S)}}{H_1(E_S)} \longrightarrow H_1(\cancel{S^3}) \xrightarrow{\text{zero}}$$

$$\underset{\mathbb{Z}}{\overset{\shortparallel}{H_0(\partial\nu(S))}} \longrightarrow \underset{\mathbb{Z}}{\overset{\shortparallel}{H_0(\nu(S))}} \oplus \underset{\therefore = \mathbb{Z}}{H_0(E_S)} \longrightarrow \underset{\mathbb{Z}}{\overset{\shortparallel}{H_0(S^3)}} \longrightarrow 0$$

Lemma $E_K = E_S/\sim$

- where $\iota_+(x) \sim \iota_-(x)$ where $\iota_\pm : S \to S_\pm \subset \partial\nu(S)$ are inclusions

Proof : 

$$E_S/\sim \; = E_{S'}/\sim \; \simeq E_K/\sim' \overset{(*)}{\simeq} E_K$$

where $\iota(x,t) \sim' \iota(x,t')$

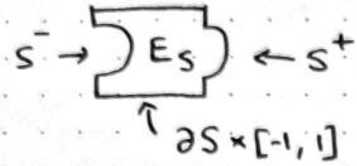
$\iota : S^1 \times [-1,1] \to S^3$ inclusion

- $(*)$ is since $[-1,1]/_{[-\frac{1}{2},\frac{1}{2}]} \sim [-1,1]$ $\square$

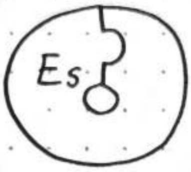
Suppose $K \hookrightarrow S^3$, S a Seifert surface

- $E_S = S^3 \setminus \nu(S)$, $\partial E_S = S_+ \cup \partial S \times [-1,1] \cup S_-$

  $\iota_\pm : S \xrightarrow{\sim} S_\pm$



$E_K \cong E_S / \sim$ where $\iota_+(x) = \iota_-(x)$



- $H_1(E_S) \cong \mathbb{Z}^{2g}$, $g = g(S)$

Consider $Y = E_S \times \mathbb{Z} / \sim$
where $(\iota_+(x), n) \sim (\iota_-(x), n+1)$

...  ...

$\mathbb{Z}$ acts freely on Y by $k \cdot (x, n) = (x, n+k)$

- $Y/\mathbb{Z} = E_S /_{\iota_+(x) \sim \iota_-(x),\ x \in S} \cong E_K$

So $p : Y \to Y/\mathbb{Z} \cong E_K$ is a covering map with deck group $\mathbb{Z}$. i.e. $Y \cong \tilde{E}_K$

<u>Lemma</u> As a module over $R = \mathbb{Z}[H_1(E_K)] = \mathbb{Z}[t^{\pm 1}]$,

$H_1(Y) \cong \operatorname{coker}(t\iota_{-*} - \iota_{+*})$

$\iota_{\pm *} : H_*(S) \otimes R \longrightarrow H_*(E_S) \otimes R$

<u>Proof</u> $E = \{e \in \mathbb{Z} \mid 2 \mid e\}$, $O = \{e \in \mathbb{Z} \mid 2 \nmid e\}$

- $\pi : E_S \times \mathbb{Z} \longrightarrow Y$

  $A = \pi(E_S \times E)$

  $B = \pi(E_S \times O)$

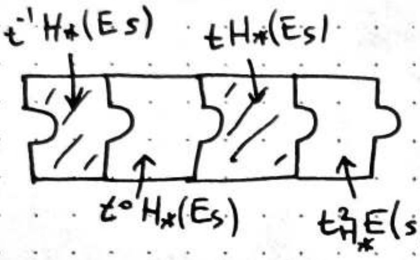
So $A \cup B = Y$, $A \cap B \simeq S \times \mathbb{Z}$.

- <u>Mayer-Vietoris</u>:

$$\to H_*(S \times \mathbb{Z}) \xrightarrow[\iota_{A*} - \iota_{B*}]{} H_*(A) \oplus H_*(B) \longrightarrow H_*(Y) \longrightarrow$$

$$H_*(S \times \mathbb{Z}) \simeq H_*(S) \otimes R$$

$$H_*(A) \oplus H_*(B) \simeq H_*(E_S) \otimes R$$



M-V becomes

$$H_1(S) \otimes R \xrightarrow[t\iota_{-*} - \iota_{+*}]{} H_1(E_S) \otimes R \longrightarrow H_1(Y)$$

$\therefore = 0$

$$H_0(S) \otimes R \xrightarrow[t\iota_{-*} - \iota_{+*}]{} H_0(E_S) \otimes R$$

$$\begin{array}{ccc} \wr & & \wr \\ R & \xrightarrow[\cdot(t-1)]{} & R \end{array}$$

injective!

$\therefore \; H_1(Y) = \operatorname{coker}(t\iota_{-*} - \iota_{+*})$. $\square$

<u>Thm</u> (Seifert): If K is a knot, then

$$\deg \Delta_K(t) \leq 2g(K)$$

difference between lowest, highest powers of t

- <u>Proof</u> If S is a Seifert surface for K, then

$$H_1(\tilde{E}_K) \simeq \operatorname{coker}(t\iota_{-*} - \iota_{+*})$$
$$\simeq \operatorname{coker}(tA_- - A_+)$$

where $A_\pm : H_1(S) \to H_1(E_S)$

are $2g(S) \times 2g(S)$ matrices with entries in $\mathbb{Z}$.

So $B = tA_- - A_+$ is a $2g \times 2g$ matrix whose entries are linear in t.

$\Rightarrow$ $\Delta_K(t) \sim \det B$ is a polynomial of degree $\leq 2g$

$\Rightarrow$ $\deg \Delta_K(t) \leq 2g(S)$

True for any S, so $\deg \Delta_K(t) \leq 2g(K)$. $\square$

<u>Remark</u>: $2g(K) = \deg \Delta_K(t)$ if K is alternating

if K has ≤ 10 crossings

But not always true, e.g. knots on the gates (to the CMS)

both have $\Delta(K) = 1 = \Delta(U)$ but $g(K_1) = 2$

$g(K_2) = 3$

<u>Fibred Knots</u>: Suppose E_K fibres over S^1, with connected fiber F

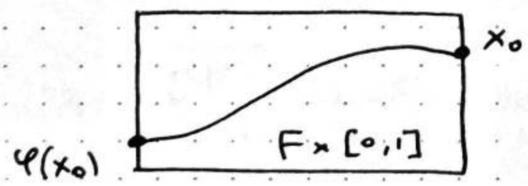
<u>Lemma</u>: F is a Seifert surface for K

<u>Proof</u>: $E_K \simeq F \times [0,1]/\sim$ where $(\varphi(x), 0) \sim (x, 1)$

$\varphi : F \xrightarrow{\sim} F$

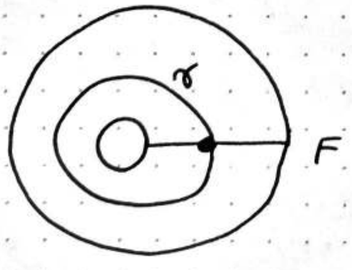
F is connected so $F \times [0,1]$ is connected

Fix $x_0 \in F$, choose a path γ from $(\varphi(x_0), 0)$ to $(x_0, 1)$



γ closes to give a loop in E_K

and $[\gamma] \cdot F = 1$.



$\Rightarrow$ F generates $H_2(E_K, \partial E_K) \simeq \mathbb{Z}$

$\Rightarrow$ F is a Seifert surface $\square$

Cor 1 : $g(K) = g(F)$

Proof : $\Delta_K(t) = \det(t\varphi_* - 1)$

where $\varphi_* : H_1(F) \xrightarrow{\sim} H_1(F)$ is an iso,

so deg $\det(\varphi_* - tI)$ has degree $2g(F)$

so $2g(F) \leq 2g(K)$

but F is a Seifert surface, so $2g(F) = 2g(K)$. $\square$

Cor 2 : If K is fibred, then $\Delta_K(t)$ is monic

This is an iff for alternating K

K with ≤ 10 crossings

3) Knots + 3 and 4-manifolds

3.1) Handlebodies

Def A n-dim^l k-handle is $\underset{\text{big}}{D^k} \times \underset{\text{small}}{D^{n-k}} = H_n{}^k$

$D^k \times 0$ is the core $\qquad$ $S^{k-1} \times 0$ is attaching sphere

$0 \times D^{n-k}$ is the cocore $\qquad$ $0 \times S^{n-k-1}$ is belt sphere

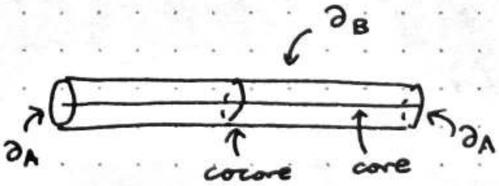
$$\partial H_n{}^k = \partial D^k \times D^{n-k} \cup_{S^{k-1} \times S^{n-k-1}} D^k \times \partial D^{n-k}$$

$$= \partial_A H_n{}^k \cup_{S^{k-1} \times S^{n-k-1}} \partial_B H_n{}^k$$

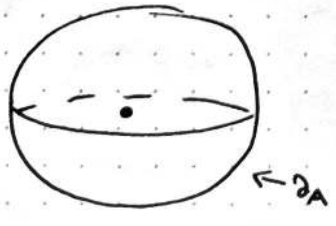
Pictures for $n = 3$:

$k=0$

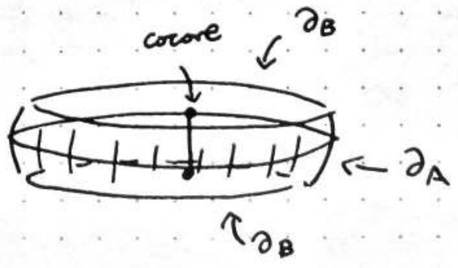
$k=1$



$k=3$

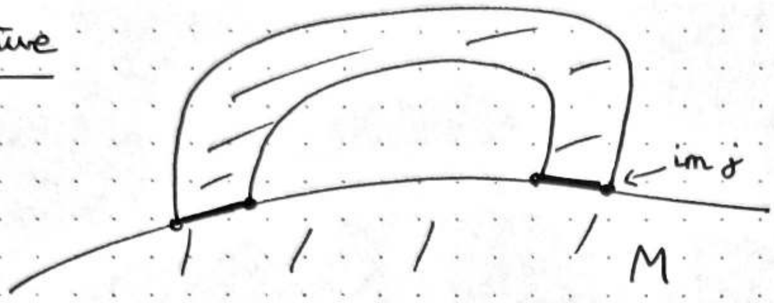


$k=2$



<u>Basic fact</u> If M is a smooth n-manifold with boundary and $j : \partial_A H_n^k \longrightarrow \partial M$ is an embedding, then $M \cup_j H_n^k$ is a smooth n-manifold w/ ∂

<u>Picture</u>



<u>Lemma</u>: $\partial_A H_n^k \cup (D^k \times 0)$ is a strong def retract of H_n^k

$\therefore M \cup_{j|_{S^{k-1} \times 0}} D^k$ is a sdr of $M \cup_j H_n^k$

Handlebodies

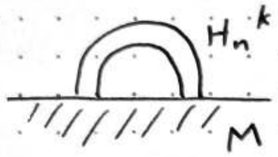
$H_n^k = D^k \times D^{n-k}$

$\partial_A H_n^k = \partial D^k \times D^{n-k}$

$\partial_B H_n^k = D^k \times \partial D^{n-k}$

If $j : \partial_A H_n^k \hookrightarrow \partial M$ is an embedding, $M = n$-manifold

$M(j) = M \cup_j H_n^k$ is an n-manifold



Cell Complexes

$f : S^{k-1} \longrightarrow X$

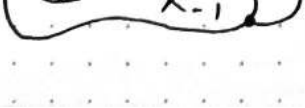
$X(f) = X \cup_f D^k$ add a k-cell

$f_0, f_1 : S^{k-1} \longrightarrow X$
with $f_0 \sim f_1$

$\Rightarrow X(f_0) \sim X(f_1)$

X is an n-dim finite cellcx rel $X_{-1} \subset X$ if there are subsets $X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ such that $X_k = X_{k-1}(F)$ for

$F : \coprod^{n_k} S^{k-1} \longrightarrow X_{k-1}$



Handlebodies

$j : \partial_A H_n^k \hookrightarrow \partial M$

$M(j) = M \cup_j H_n^k \sim M \cup_{j|_{S^{k-1}\times 0}} D^k$

<u>Lemma</u> If $j_0, j_1 : \partial_A H_n^k \hookrightarrow \partial M$ with $j_0 \sim_i j_1$, then
$M(j_0) \cong M(j_1)$

<u>Proof</u> Isotopy $\Rightarrow$ Ambient isotopy

So $\exists\ \varphi : M \xrightarrow{\cong} M$ s.t. $\varphi \circ j_0 = j_1$

Define $\Phi : M(j_0) \longrightarrow M(j_1)$

$x \in M \longmapsto \varphi(x)$

$y \in H_n^k \longmapsto y$ $\square$

<u>Def^n</u> An n-manifold M is a handlebody rel $M_{-1} \subset M$ if
(M_{-1}: closed n-dim submanifold)
there's a sequence $M_{-1} \subset M_0 \subset \cdots \subset M_n = M$, where $M_k = M_{k-1}(J)$ for $J : \coprod^{n_k} \partial_A H_n^k \hookrightarrow \partial M_{k-1}$

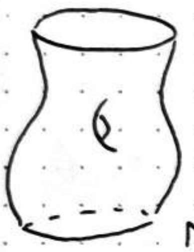
By induction, $M_K \sim X_K$ where X_K is a cell complex rel M_{-1}

k cells $\qquad$ k-handles

Slogan (Morse / Smale)

All smooth manifolds are divided into handles.

Def If N, N' are $n-1$ smooth manifolds w/o boundary, a cobordism $M : N \to N'$ is an smooth n-manifold with a diffeo $g : \partial M \longrightarrow -N \amalg N'$

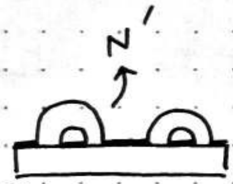


E.g. $N \times [0,1] : N \to N$

Thm (Morse / Smale)

If $M : N \to N'$ is a cobordism, then M is a handlebody rel $\nu(N) \simeq N \times [0, \varepsilon]$

All handles are attached on $N \times \varepsilon$ boundary



- Choose Morse function $f : M \to [0,1]$, $f|_N \equiv 0$, $f|_{N'} \equiv 1$

Indek critical pt $\longleftrightarrow$ k-handle

Cellular Chain C_X :

X is a cell cx rel X_{-1}, then $H_*(X, X_{-1}) \simeq H_*^{cell}(X, X_{-1})$ where $C_*^{cell}(X, X_{-1})$ is generated by $e_1^*, \dots, e_{n_*}^*$ $\leftarrow$ $*$-cells of X rel X_{-1}

$de_i^k = \sum n_i^j e_j^{k-1}$

$n_i^j = \deg f_i^j$

$$\begin{tikzcd}
S^{k-1} \arrow[r, "f_i"] \arrow[dr, "f_i^j"'] & X_{k-1} \arrow[r] \arrow[d, "\cancel{f_i^j}"] & X_{k-1}/X_{k-2} \cong \bigvee^{n_{k-1}} S^{k-1} \arrow[dl, "\pi_j"] \\
 & S^{k-1} &
\end{tikzcd}$$

Cell cx of a handlebody

Suppose M is a handlebody rel M_{-1}.

Then $M \sim X$, cell cx rel M_{-1}, so

$$H_*(M, M_{-1}) \cong H_*(X, M_{-1}) \cong H_*^{cell}(X, M_{-1})$$

What is $C_*^{cell}(X, M_{i-1})$?

$C_k^{cell}(X, M_{i-1})$ has generators $h_1^k, \dots, h_{n_k}^k$ corresponding to k-handles of M rel M_{-1} $H_{n,1}^k, \dots, H_{n,n_k}^k$

Let A_i^k = attaching sphere of $H_{n,i}^k$,
$\qquad = S^{k-1} \times 0 \subset \partial_A H_{n,i}^k \subset \partial M_{k-1}$, $\quad A_i^k \cong S^{k-1}$

B_j^{k-1} = belt sphere of $H_{n,j}^{k-1} = 0 \times S^{n-k} \subset \partial_B H_{n,j}^{k-1} \subset \partial M_{k-1}$, $\quad B_j^{k-1} \cong S^{n-k}$

$$\left.\begin{array}{l} A_i^k \simeq S^{k-1} \\ B_j^{k-1} \simeq S^{n-k} \end{array}\right\} \subset \partial M_{k-1} \text{ an } n-1 \text{ manifold}$$

Lemma: $d h_i^k = \sum n_i^j h_j^{k-1}$

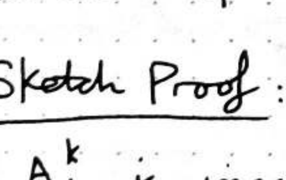
where $n_i^j = A_i^k \cdot B_j^{k-1}$ intersection number in ∂M_{k-1}

Sketch Proof: (no signs)

A_i^k is image of attaching map

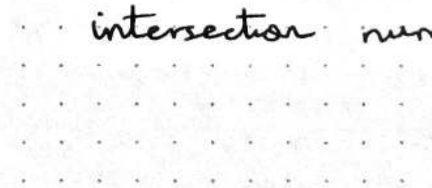
$f_i^k : S^{k-1} \to \partial M_{k-1}$

Picture: $(n=3,\ k=2)$



X

$\sim$



B_1' $\quad$ B_2' $\quad$ A_1^2

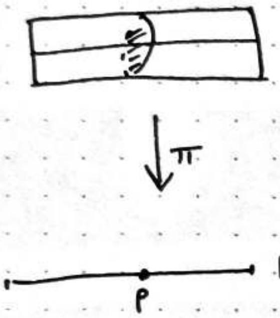
$n_1' = \#$ pts in $(f_1^2)^{-1}(p)$
$\quad = B_1' \cap A_1^2$ = cocore of H_1'

$$\begin{tikzcd}
S^{k-1} \arrow[r, "f_i"] \arrow[rrrd, bend right=10, "f_i^\gamma"'] & \partial M_{k-1} \arrow[r] & M_{k-1} \arrow[r, "\sim"] \arrow[d] & X_{k-1} \arrow[r] & X_{k-1}/X_{k-2} \arrow[dl, "\pi_\gamma"] \\
 & & \bullet \arrow[r, "\alpha"] & S^{k-1} &
\end{tikzcd}$$

$n_i^\gamma = \deg f_i^\gamma$

$\quad = \#(f_i^\gamma)^{-1}(p)$ 　　　　factors through

if f is transverse at p 　　$\bullet = M_{k-1}/(M_{k-1} \setminus H_\gamma^{k-1})$



$$\begin{tikzcd}
H_\gamma^{k-1} \arrow[d, "\pi"] \arrow[dr, "\alpha"] & \\
D^{k-1} \arrow[r] & S^{k-1}
\end{tikzcd}$$

$(f_{i\gamma})^{-1}(p) = f_i^{-1}(\underbrace{\pi^{-1}(p)}_{\text{cocore}}) = A_i^k \cap B_\gamma^{k-1}$

$\ulcorner p = \circ \lrcorner$

f_i^γ is transverse $\Leftrightarrow$ A_i^k intersects B_γ^{k-1} transversely. □

Cobordisms:

$M : N \to N'$ means $\partial M = \bar{N} \amalg N'$ where $\bar{N}$ is N with orientation reversed

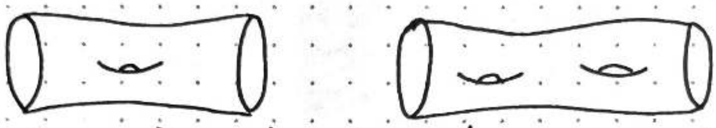
Then $\partial \bar{M} = N \amalg \bar{N}'$, so

$\bar{M} : N' \to N$ (orientation reversal)

If $M_0 : N \to N'$, $M_1 : N' \longrightarrow N''$, then I have

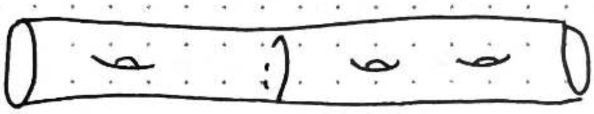
$M_1 \circ M_0 : N \longrightarrow N''$ where $M_1 \circ M_0 = M_0 \cup_{N'} M_1$

Picture:



$M_0 : N \longrightarrow N'$ $M_1 : N' \longrightarrow N''$

$M_1 \circ M_0$:



N N''

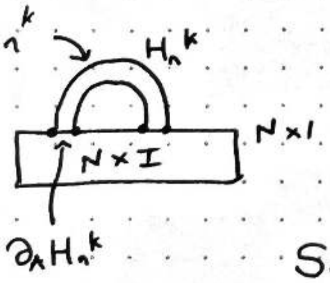
Exercise: $(M_2 \circ M_1) \circ M_0 = M_2 \circ (M_1 \circ M_0)$

Surgery:

Def: Suppose N is an $n-1$ manifold,

$j : \partial_A H_n^k \hookrightarrow N$.

Let $N[j] = N \times I (j \times 1) = N \times I \cup_{j \times 1} H_n^k$



$\partial N[j] = \bar{N} \times 0 \amalg \underbrace{(N \setminus int(im\, j)) \cup_{j(S^{k-1} \times S^{n-k-1})} \partial_B H_n^k}_{N'}$

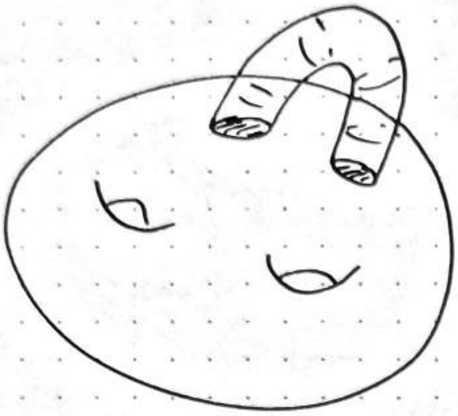
So $N[j] : N \to N'$

Say N' is the result of surgery on N along j "gaping wound"

$N[j]$ is the trace of the surgery

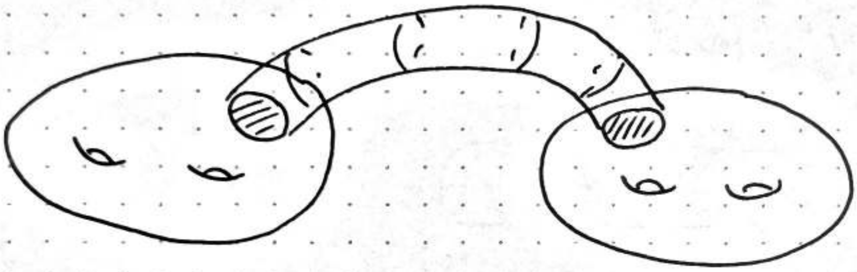
Ex : $n = 3$

Add 1-handle



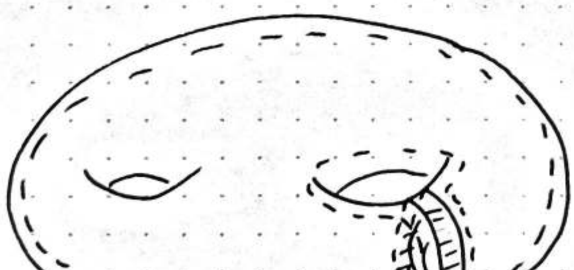
$N[j] : \Sigma_g \to \Sigma_{g+1}$

$N = \Sigma \amalg \Sigma'$



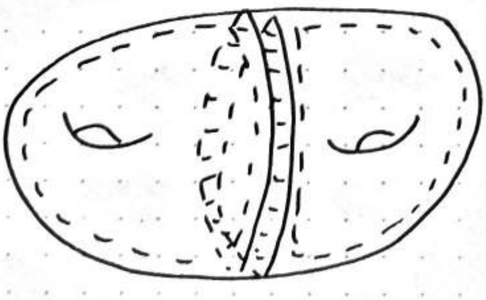
$N[j] : \Sigma \amalg \Sigma' \longrightarrow \Sigma \# \Sigma'$

2-handles :



N' ---

$N[j] : \Sigma_g \longrightarrow \Sigma_{g-1}$



$N[j] : \Sigma \# \Sigma' \longrightarrow \Sigma \amalg \Sigma'$

If $N' = N \setminus \operatorname{int}(\operatorname{im} j) \underset{j(S^{k-1} \times S^{n-k-1})}{\cup} \partial_B H_n^k$

then $j' : \partial_B H_n^k \longrightarrow N'$

$\qquad\qquad \partial_A H_n^{*n-k} \xrightarrow{\hat{j}} N'$, with $\partial_B H_n^k \cong \partial_A H_n^{*n-k}$

$$H_n^k = D^k \times D^{n-k} \simeq D^{n-k} \times D^k = H_n^{*n-k}$$

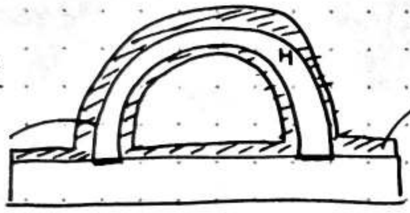
$$\partial_A H_n^k = \partial_B H_n^{*n-k}$$

$$\partial_B H_n^k = \partial_A H_n^{*n-k}$$

Lemma $N'[\hat{j}] \simeq -N[j] = \overline{N[j]}$

$N'[\hat{j}] : N' \to N$

Picture



Proof $\overline{N[j]} \simeq N[j] \underset{N'}{\cup} N' \times I$ (think about orientations)

$$\simeq N \times I \underset{j}{\cup} H_n^k \underset{N'}{\cup} N' \times I$$

$$\simeq N \times I \underset{N}{\cup} \left(H_n^{*n-k} \cup_{\hat{j}} N' \times I \right)$$

$$\simeq N \times I \underset{N}{\cup} N'[\hat{j}] \simeq N'[\hat{j}] \qquad \square$$

⌜1-handles can break orientation⌟

Cor: Suppose $M : N \to N'$ is a handlebody, so $M = M(r) \circ M(r-1) \circ \dots \circ M(1)$ is a composition of traces of surgeries, with k-handles $H^k_{n,1}, \dots, H^k_{n,n_k}$.

Then $\overline{M} : N' \to N$ is a handlebody, with $(n-k)$-handles $H^{*n-k}_{n,i}$ dual to $H^k_{n,i}$ $\quad$ ⌜$r = \sum_{i=0}^{n} n_i$⌟

Thm If $M : N \to N'$ is a handlebody, then
of dim n

$$H_k(M,N) \simeq H^{n-k}(M,N')$$
$$H^k(M,N) \simeq H_{n-k}(M,N')$$
(Poincaré – Lefschetz duality)

Proof (With $\mathbb{Z}/2$ coeffs, but works w/ $\mathbb{Z}$ coeffs if M is orientable)

Consider $H_*(M,N;\mathbb{Z}/2) \simeq H_*^{cell}(M,N;\mathbb{Z}/2)$, where

$$C_k^{cell}(M,N;\mathbb{Z}/2) = \langle h_1^k, \dots, h_{n_k}^k \rangle$$

$$dh_i^k = \sum n_{ij} h_j^{k-1} \quad \text{where } n_{ij} = A_i^k \cdot B_j^{k-1} \text{ in } \partial M_{k-1}$$

On the other hand, considering the dual handle decomposition $\overline{M} : N' \to N$, we see

$$H_*(M,N'; \mathbb{Z}/2) \simeq H_*^{cell}(M,N';\mathbb{Z}/2)$$

where $C_{k}^{cell}(M;N)$ is generated by

$$\langle h_1^{*n-k}, \dots, h_{a_k}^{*n-k} \rangle \quad \text{with}$$

$$dh_j^{*n-k} = \sum n'_{ij} h_i^{*n-k+1}$$

and $n'_{ij} = A_j^{*n-k} \cdot B_i^{*(n-k+1)}$

But $A(H^k_{n,i}) = B(H^{*n-k}_{n,i})$

and $B(H^k_{n,i}) = A(H^{*n-k}_{n,i})$.

So $n'_{ij} = \cancel{n_{ij}}\, n_{ji}$ i.e. $C_*^{cell}(M;N;\mathbb{Z}/2)$

is dual to $C_*^{cell}(M,N';\mathbb{Z}/2)$

hence $\simeq$ to $C^*_{cell}(M,N';\mathbb{Z}/2)$. $\square$

<u>Remark</u> We have proved "weak Poincaré duality"

$H_*(M,N) \cong H^{n-*}(M,N')$

with $\mathbb{Z}/2$-coeffs or with $\mathbb{Z}$-coeffs if M is orientable.

<u>Strong Poincaré duality</u>:

$$\smile : H^k(M,N) \otimes H^{n-k}(M,N') \to H^n(M, N \amalg N') = H^n(M,\partial M) \xrightarrow{\langle \cdot, [M] \rangle} \mathbb{F}$$

with coeffs in $\mathbb{F}$ a field, is a perfect (non-singular) pairing.

e.g. $\mathbb{F} = \mathbb{Z}/2$, $\mathbb{Q}$ (if M orientable)

$\cdot : H_k(M,N) \otimes H_{n-k}(M,N') \to \mathbb{F}$ dual pairing

3.2) Seifert matrix

If $M : N \to N'$ is an orientable cobordism of dim n, then $P.D. : H_k(M,N) \xrightarrow{\sim} H^{n-k}(M,N')$.

In particular $M : \emptyset \to \partial M$, so

$$P.D. : H_k(M,\partial M) \to H^{n-k}(M)$$
$$H_k(M) \longrightarrow H^{n-k}(M,\partial M)$$

Suppose $K \subset S^3$ a knot, $S \hookrightarrow S^3$ is a Seifert surface of K.

$\Delta_K(t) \sim \det(A^+ - tA^-)$ where $A^\pm$ matrices representing

$c_{\pm *} : H_1(S) \longrightarrow H_1(E_S) \simeq \mathbb{Z}^{2g(S)}$.

<u>Lemma</u> 1. $H_1(E_S) \simeq H^1(S)$

<u>Proof</u> $H_1(E_S) \overset{a)}{\simeq} H^2(E_S, \partial E_S) \overset{b)}{\simeq} H^2(S^3, \nu(S))$

$\underset{c)}{\simeq} H^1(\nu(S)) \underset{d)}{\simeq} H^1(S)$

a) is PD

b) is excision

c) follows from LES of $(S^3, \nu(S))$ since $H^1(S^3) = H^2(S^3) = 0$

d) $S \sim \nu(S)$ □

Consider $\alpha : H_1(E_S) \to H^1(S)$ as in Lemma

$\alpha = \delta^{-1} \circ PD$, where $\delta : H^1(S) \xrightarrow{\sim} H^2(E_S, \partial E_S)$ is

$$H^1(S) \xrightarrow[\pi^*]{} H^1(\nu(S)) \to H^2(S^3, \nu(S)) \longrightarrow H^2(E_S, \partial E_S)$$

everything is free over $\mathbb{Z}$, so δ is dual to

$$\partial : H_2(E_S, \partial E_S) \longrightarrow H_1(S)$$

given by

$$H_2(E_S, \partial E_S) \to H_2(S^3, \nu(S)) \to H_1(\nu(S)) \xrightarrow{\pi_*} H_1(S).$$

If $\Sigma \hookrightarrow S^3$, $\partial \Sigma \subset S$, then

$$\partial[\Sigma, \partial\Sigma] = [\partial\Sigma] \in H_1(S)$$

<u>Lemma 2</u> If $x \in H_1(E_S)$, $y \in H_1(S)$ are represented by embedded circles then $\langle \alpha(x), y \rangle = lk(x,y)$ in S^3

<u>Proof</u>
$$\begin{aligned}\langle \alpha(x), y \rangle &= \langle \delta^{-1} PD(x), y \rangle \\ &= \langle PD(x), \partial^{-1} y \rangle \\ &= \langle PD(x), [\Sigma, \partial\Sigma] \rangle \\ &= x \cdot \Sigma \\ &= lk(x,y) .\end{aligned}$$

Choose $\Sigma \hookrightarrow S^3$, $\partial\Sigma = y$ so $\partial[\Sigma, \partial\Sigma] = y$

$\square$

<u>Bases</u> Let $\{x_1, \dots, x_{2g}\}$ be a basis of loops for $H_1(S)$

$\{x^1, \dots, x^{2g}\}$ be the dual basis of $H^1(S)$

i.e. $\langle x^i, x_j \rangle = \delta^i_j$

So $\{y_1, \dots, y_{2g}\}$, $y_i = \alpha^{-1}(x^i)$ is a basis of $H_1(E_S)$

<u>Lemma 3</u> If $z \in H_1(E_S)$, then $z = \sum_{i=1}^{2g} lk(z, x_i) y_i$

<u>Proof</u>
$$\begin{aligned}\alpha(z) &= \sum \langle \alpha(z), x_i \rangle x^i \\ &= \sum lk(z, x_i) x^i \quad \text{by Lemma 2}\end{aligned}$$

Apply α^{-1} to both sides. $\square$

Let $A^{\pm} = [a^{\pm}_{ij}]$ be the matrix of $\iota_{\pm *} : H_1(S) \to H_1(E_S)$ wrt the bases $\{x_1, \dots, x_{2g}\}$ and $\{y_1, \dots, y_{2g}\}$,

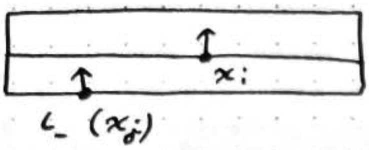
so $\iota_{\pm *}(x_j) = \sum a^{\pm}_{ij} y_i$

<u>Cor</u> $\boxed{a^{\pm}_{ij} = lk(\iota_{\pm}(x_j), x_i)}$

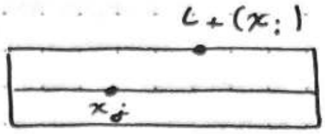
<u>Proof</u> Follows by putting $z = \iota_{\pm *}(x_j)$ in Lemma 3. $\square$

<u>Cor</u> $a^-_{ij} = a^+_{ji}$ i.e. $A^- = (A^+)^T$

<u>Proof</u>
$$\begin{aligned}a^-_{ij} &= lk(\iota_-(x_j), x_i) \\ a^+_{ij} &= lk(\iota_+(x_j), x_i)\end{aligned}$$



$S \times [-1,1]$



$S \times [-1,1]$

The link $\iota_-(x_j) \cup x_i \subset \nu(S) \subset S^3$

is isotopic to $x_j \cup \iota_+(x_i)$.

So $\operatorname{lk}(\iota_-(x_j), x_i) = \operatorname{lk}(\iota_+(x_i), x_j)$

$$\underset{a_{ij}^-}{\shortparallel} \qquad\qquad \underset{a_{ji}^+}{\shortparallel}$$

$\square$

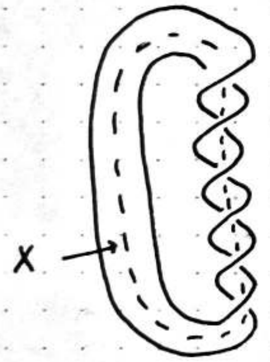
<u>Def</u> $A = A^+$ is the <u>Seifert matrix</u> of K determined by S and $\{x_1, \dots, x_{2g}\}$.

Then $\Delta_K(t) \sim \det(A^+ - tA^-) = \det(A - tA^T)$.

<u>Ex</u> How to compute Seifert matrix

Key example is k-twisted band

$S \simeq S^1 \times [-1,1]$



$k=3$ $\quad H_1(S) = \langle x \rangle$, $x = S^1 \times 0$

$\operatorname{lk}(\iota_+(x), x) = k$, $\quad k = \operatorname{lk}(\partial_1 S, \partial_2 S)$

$= \frac{1}{2} w(D)$ $\quad$ ↑ oriented parallel

<u>Proof</u> $\nu(X) \simeq S^1 \times D^2$ and $\partial_1(S), \partial_2(S)$

are 2 sections of $\nu_{S^3, X}|$ unit sphere bundle

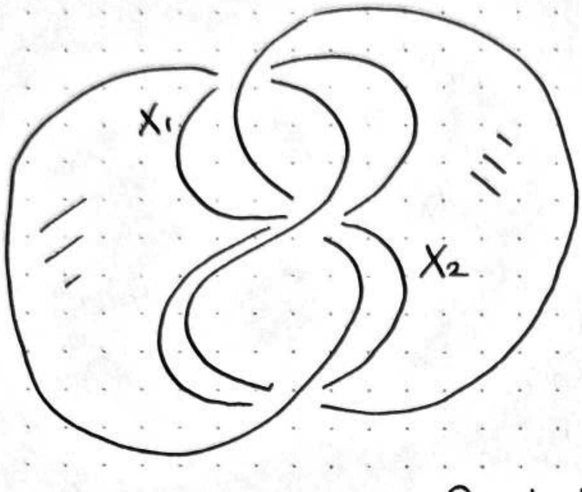
$S = s \subset V_{S^3/X}$ where s is the total space of a section of the normal bundle

$\iota_+, \iota_-(x)$ are sections of $V_{S/X}$ which are $\perp$ to s

$\Rightarrow \iota_\pm(x) \sim \partial_1 S, \partial_2 S$ so $\operatorname{lk}(\iota_\pm(x), x) = \operatorname{lk}(\partial_1(S), x)$

$= \operatorname{lk}(\partial_1(S), \partial_2(S)) = k$.

Ex 2 $K = +$ trefoil



wrt this basis, Seifert matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$lk(L_+(x_1), x_1) = 1$
since $\nu(x_1) \subset S$
is a 1 - twisted band

Similarly $lk(L_+(x_2), x_2) = 1$

$lk(L_\pm(x_1), x_2)$

Check $\det(A - tA^T) = \det\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)$

$$= \det \begin{pmatrix} 1-t & 1 \\ -t & 1-t \end{pmatrix}$$

$$= (1-t)^2 + t = t^2 - t + 1$$

Symmetry Write $t = q^2$, so

$$\det(A - q^2 A^T) \sim \det(q^{-1}A - qA^T) \leftarrow \hat{\Delta}(K)|_q$$

$$\sim \det(-qA + q^{-1}A^T) \leftarrow \hat{\Delta}(K)|_{-q^{-1}}$$

So $\hat{\Delta}_K(q) = \hat{\Delta}_K(-q^{-1})$

$\Rightarrow \Delta_K(t)$ can be normalized so it's symmetric under $t \mapsto t^{-1}$

E.g. $\Delta_{Tref} \sim t - 1 + t^{-1}$

Normalized Alexander Polynomial

- $L \hookrightarrow S^3$ an oriented link

  $S \hookrightarrow S^3$ is a Seifert surface for L

  $\langle x_1, \dots, x_k \rangle = H_1(S) \rightsquigarrow$ Seifert matrix A

Def^n: $\hat{\Delta}_L(q) = \det(q^{-1}A - qA^T)$

Last time: $\tilde{\Delta}_K(q) \sim \Delta_K(q^2)$

$\hat{\Delta}_L(-q^{-1}) = \hat{\Delta}_L(q)$ (symmetry)

Symmetry determines $\hat{\Delta}(q)$ up to sign.

- $\Delta_T(t) \sim t^2 - t + 1$

$\Rightarrow \hat{\Delta}_T(q) \sim \pm(q^2 - 1 + q^{-2})$

Ex:

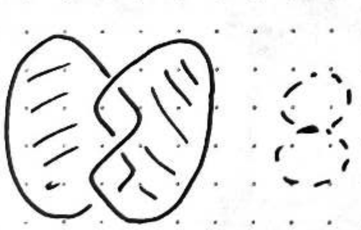
$\rightsquigarrow$ Seifert matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\therefore \hat{\Delta}(q) = \det\left(\begin{pmatrix} q^{-1} & q^{-1} \\ 0 & q^{-1} \end{pmatrix} - \begin{pmatrix} q & 0 \\ q & q \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -(q-q^{-1}) & q^{-1} \\ -q & -(q-q^{-1}) \end{pmatrix}$$

$$= (q-q^{-1})^2 + 1$$

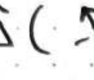
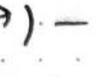
$$= q^2 - 1 + q^{-2}$$



$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

all entries of A are same as in previous Ex, except bottom right

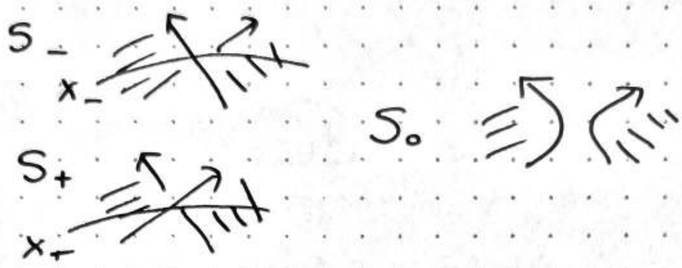
$\hat{\Delta}_K(q) = 1$

Prop $\hat{\Delta}($$) - \hat{\Delta}($$) = (q - q^{-1})\hat{\Delta}($$)$

(Conway skein relation)

Proof Apply Seifert's algorithm to get Seifert surfaces $S_\pm, S_0$ for $D_\pm, D_0$.

$S_\pm$ are obtained by adding a 1-handle w/ ± 1-twist to S_0.

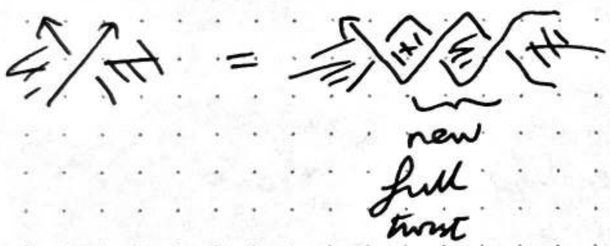


If $H_1(S_0) = \langle x_1, \dots, x_k \rangle$

then $H_1(S_\pm) = \langle x_1, \dots, x_k, x_\pm \rangle$ where $x_\pm$ goes over the new 1-handle.

Claim $lk(\iota_+(x_+), x_+) = lk(\iota_+(x_-), x_-) + 1$ [what should be a minus ?]

Pf If ν_{S_-/x_-} is a k-twisted band, then



ν_{S_+/x_+} is a band with $k+1$ twists

$\Rightarrow$ We have Seifert matrices

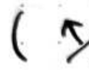
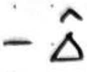
$$A_\pm = \begin{bmatrix} A_0 & x \\ y & k + \frac{1}{2} \pm \frac{1}{2} \end{bmatrix}$$

$$\hat{\Delta}_{D_\pm}(q) = \det(q^{-1}A_\pm - qA_\pm^T)$$

$$= \det\begin{pmatrix} q^{-1}A_0 - qA_0^T & z \\ w & (q^{-1} - q)(k + \frac{1}{2} \pm \frac{1}{2}) \end{pmatrix}$$

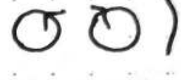
Expand det's along bottom row, all terms are the same except last one:

$$\hat{\Delta}_{D_-}(q) - \hat{\Delta}_{D_+}(q) = -(q^{-1} - q)\det(q^{-1}A_0 - qA_0^T)$$

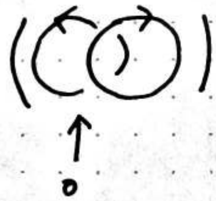
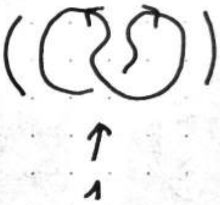
$\therefore$ $\hat{\Delta}($  $) - \hat{\Delta}($  $) = (q^{-1}-q)\,\hat{\Delta}($  $)$. $\square$

<u>Cor</u>: $\hat{\Delta}_K(1) = \hat{\Delta}_K(\text{U}) = 1$ (Exercise)

$\Rightarrow$ $\hat{\Delta}_K$ is fully determined by $\Delta_K(t)$ i.e. it does not depend on choice of S, x_i.

<u>Ex</u>: 1) $\hat{\Delta}($  $) - \hat{\Delta}($  $) = \hat{\Delta}($  $)(q - q^{-1})$

so $\hat{\Delta}($  $) = 0$

2) $\hat{\Delta}($  $) - \hat{\Delta}($  $) = (q-q^{-1})\,\hat{\Delta}($  $)$

$\therefore$ $\hat{\Delta}($  $) = q - q^{-1}$.

3) $\hat{\Delta}($  $) - \hat{\Delta}($  $) = (q-q^{-1})\,\hat{\Delta}($  $)$

$\therefore$ $\hat{\Delta}($  $) = 1 + (q-q^{-1})^2 = q^2 - 1 + q^{-2}$. $\square$

3.3 Framings & Surgery

Suppose $N = \partial M^n$ & $j : \underbrace{S^{k-1} \times D^{n-k}}_{= \partial_A H_n^k} \hookrightarrow N$ is embedding.

Then we have $N[j] : N \to N'$ obtained by surgery on N using j.

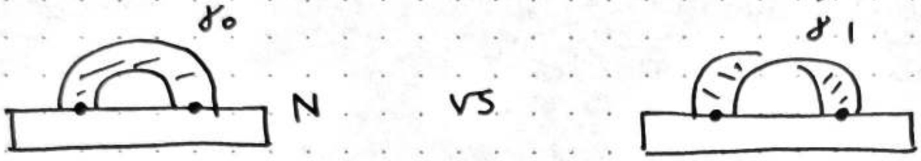
~~Observe~~ Recall that $j_0 \sim_i j_1 \Rightarrow N[j_0] \cong N[j_1]$

Observe that if

$j_0 \sim_i j_1$ then $j_0|_{S^{k-1}\times 0} \sim_i j_1|_{S^{k-1}\times 0}$ (restrict isotopy)

Converse is false.

$n=2, k=1$ $\quad S^0 \hookrightarrow N$



$j_0|_{S^0\times 0} = j_1|_{S^0\times 0}$

but $N[j_0] \not\cong N[j_1]$ so $j_0 \not\sim_i j_1$

<u>Def</u>

Defⁿ Suppose $C: S^{k-1} \hookrightarrow N^{n-1}$ is an embedding. A <u>framing</u> of C is a trivialisation of $\nu_{N/C}$

i.e. a bundle map $f: C \times \mathbb{R}^{n-k} \longrightarrow \nu_{N/C}$

$$\begin{tikzcd} C \times \mathbb{R}^{n-k} \arrow[r, "f"] \arrow[dr] & \nu_{N/C} \arrow[d] \\ & C \end{tikzcd}$$

If $j: S^{k-1} \times D^{n-k} \longrightarrow N$ is an embedding, then j determines a framing f_j of $C(j) = j|_{S^{k-1}\times 0}$ via $f_j = dj \circ \iota$ where

$$\iota: S^{k-1} \times \mathbb{R}^{n-k} \longrightarrow S^{k-1} \times T_0 D^{n-k} \subset T(S^{k-1} \times D^{n-k})|_{S^{k-1}\times 0}$$

j is embedding $\Rightarrow$ dj is injective

$\Rightarrow dj \circ \iota$ is bundle iso

<u>Tubular Ngbd Thm</u>: $\Rightarrow$ if $C(j_0) = C(j_1)$ and $f_{j_0} = f_{j_1}$ then $j_0 \sim_i j_1$

<u>Def</u> : Framings f_0, f_1 of $C : S^{k-1} \hookrightarrow N$ are homotopic if there's a family of bundle maps

$$\begin{array}{ccc} f_t : S^{k-1} \times \mathbb{R}^{n-k} \times I & \longrightarrow & \nu_{N/C} \\ \downarrow & & \downarrow \\ S^{k-1} & \longrightarrow & C \end{array}$$

each f_t a framing for all t.

<u>Model Case</u> : $N = S^{k-1} \times \mathbb{R}^{n-k}$

$$C : S^{k-1} \hookrightarrow N$$
$$x \longmapsto (x, 0)$$

There's a bijection

$$\{\text{continuous } A : S^{k-1} \to GL_{n-k}(\mathbb{R})\} \longleftrightarrow \{\text{framings of } C\}$$
$$A \longmapsto f_A$$

$$f_A : S^{k-1} \times \mathbb{R}^{n-k} \longrightarrow S^{k-1} \times \mathbb{R}^{n-k}$$
$$(x ; v) \longmapsto (x ; A(x) v)$$

<u>Recall</u>: $C : S^{k-1} \hookrightarrow N^{n-1}$

- A <u>framing</u> of C is a trivialisation $f : S^{k-1} \times \mathbb{R}^{n-k} \to \nu_{N/C}$

Framings f_0, f_1 are homotopic $f_0 \sim f_1$ if they are connected by a smooth family of framings f_t, $t \in [0,1]$

<u>Def</u>: $Fr(C) = \{\text{framings of } C\}/\sim$

$Fr(C) \neq \phi \iff \nu_{N/C}$ is trivial

<u>Model Case</u>: $C_0 : S^{k-1} \hookrightarrow S^{k-1} \times \mathbb{R}^{n-k}$

$$x \longmapsto (x, 0)$$

<u>Lemma</u>: There's a bijection

- $$\{\text{smooth } A : S^{k-1} \to GL_{n-k}(\mathbb{R})\} \leftrightarrow Fr(C_0) \quad \text{not up to homotopy!}$$

$$A \longmapsto f_A, \quad f_A(x,v) = (x, Av)$$

<u>Proof</u> Easy to check f_A is a framing.

Conversely, a framing f has $f|_{x \times \mathbb{R}^{n-k}}$ is a linear map, so given by a matrix $A_f(x)$. $\square$ ∴✓

Similarly have bijections

- homotopies A_t between $A_0, A_1 : S^{k-1} \to GL_{n-k}(\mathbb{R})$ $\longleftrightarrow$ homotopies of framings

$$\therefore \quad [S^{k-1}, GL_{n-k}(\mathbb{R})] \longleftrightarrow Fr(C_0)$$

Lie group so ∴ not connected so...

$$= \pi_{k-1}(GL_{n-k}(\mathbb{R}))$$

$$= \pi_{k-1}(O(n-k)) \quad \text{since } GL_{n-k}(\mathbb{R}) \text{ d. retracts to } O(n-k)$$

- Let $Emb_{C_0}(S^{k-1} \times D^{n-k}, N_0) = \left\{ \begin{array}{l} \text{embeddings} \\ j : S^{k-1} \times D^{n-k} \hookrightarrow N_0 \end{array} \middle| \; j|_{S^{k-1} \times 0} = C_0 \right\} / \sim$

$(*)$; $\sim$: isotopies preserving $(*)$

<u>Lemma</u> There's a well-defined surjective map

$$\Phi : Fr(C_0) \to Emb_{C_0}(S^{k-1} \times D^{n-k}, N_0)$$

given by $\Phi([f]) = [f|_{S^{k-1}\times D^{n-k} \subset S^{k-1}\times \mathbb{R}^{n-k}}]$ [TNT]

Proof To check Φ is well defined, must show that if $f_0 \sim f_1$, then $j_{f_0} \sim_i j_{f_1}$

But if $f_t = f A_t$ is a homotopy, then

$J_t(x,v) = (x, A_t(x)v)$ is an isotopy since $A_t : S^{k-1} \hookrightarrow GL_{n-k}(\mathbb{R})$

Φ is surjective: $j \in \mathrm{Emb}_{C_0}(S^{k-1}\times D^{n-k}, N_0)$, get f_j ~~with~~ "derivative"

Tubular Ngbd Thm: $\Phi([f_j]) \sim_i f_j$. $\square$

Cor: If $C : S^{k-1} \hookrightarrow N$ has trivial normal bundle, then

1) there's a bijection (of sets) $Fr(C) \longleftrightarrow \pi_{k-1}(O(n-k))$

2) there's a surjective map

$$Fr(C) \longrightarrow \mathrm{Emb}_C(S^{k-1}\times D^{n-k}, N)$$

Proof Choose tubular ngbd $\nu(C)$

$\mathrm{int}\,\nu(C) \cong C \times \mathbb{R}^{n-k}$, use lemma. $\square$

Summary: Given a $C : S^{k-1} \hookrightarrow N$ and a class of framings $[f]$ of C, get $j_{C,[f]} : S^{k-1}\times D^{n-k} \hookrightarrow N$

well-defined up to isotopy

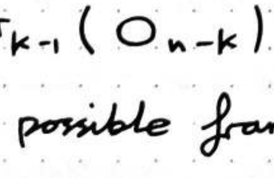
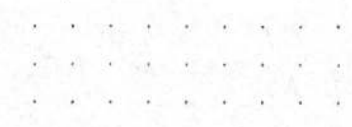
and hence a handle attachment

$N[C,[f]]$ well-defined up to diffeo

Ex: $n=2, k=1$

$\pi_{k-1}(O_{n-k}) = \pi_0(O(1)) = \mathbb{Z}/2$

2 possible framings

 vs 

If $\operatorname{im} C \simeq S^0$ is contained in 1 component of N, one of these is orientable, the other is knot. (comedy)

<u>Focus on</u> : ~~At~~ $n=4$, $k=2$

$$\pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$$

If $K : S^1 \hookrightarrow N^3$ w/ trivial normal bundle,

$Fr(K) \simeq \mathbb{Z}$.

Concrete description for $K : S^1 \hookrightarrow S^3$

$$Fr(K) \longleftrightarrow \left\{ \begin{array}{c}\text{non-vanishing sections} \\ s : K \to \nu_{S^3/K}\end{array} \right\} \Big/ \text{homotopy}$$

$$f \longmapsto f(e_1)$$

$$(s, s^\perp) \longleftarrow s$$

{here avoid whole $\pi_0 O(2)$ issue}

$$\longleftrightarrow \quad \lambda \in \partial\nu(K) \text{ with } \lambda \sim_i K \text{ in } \nu(K) \Big/ \sim$$

$$\underset{\text{Exercise on ES 2}}{\longleftrightarrow} \quad [\lambda] \in H_1(\partial\nu(K)) \text{ with } \lambda \cdot m = 1$$

Seifert longitude ℓ gives a preferred class $\ell \in H_1(\partial\nu(K))$ with $\ell \cdot m = 1$.

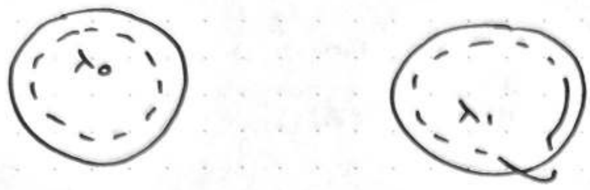
$S \cap \partial\nu(K)$ for S Seifert surface

i.e. $\lambda = \lambda_n = \ell + nm$, $n \in \mathbb{Z}$ since $\lambda = a\ell + nm$, $\lambda \cdot m = 1 \Rightarrow a = 1$

$$Fr(C) \longleftrightarrow \{\lambda_n = \ell + n\,m\}$$

<u>Remark</u> $lk(\lambda_n, K) = n$

<u>Proof</u> $lk(\lambda_n, K) = [\lambda_n]\cdot[S] = \underbrace{[\ell]\cdot[S]}_{0} + n\underbrace{[m]\cdot[S]}_{1}$ □

Ex: $K =$ 

$(\lambda_n, U) = T(2, 2n)$

But 

blue curve $\neq \lambda_0$, it's $\lambda_{lk(\text{blue, white})} = \lambda_{-3}$

Def^n A framed link $\hat{L} \hookrightarrow S^3$ is an unoriented ? link L together with an integer ~~n_i~~ α_i attached to each component L_i of L.

~~n_i~~ α_i determines a framing $\lambda_{\alpha_i} = \ell + \alpha_i m$ on L_i where $\ell = \partial S$, S a Seifert surface for L_i (ignore other cpts of L)

Def If $\hat{L}$ is a framed link, let $W(\hat{L})$ be the 4-manifold obtained by attaching 2-handles along the L_i's with framing λ_{α_i} (to B^4)

$S^3_{\hat{L}} = \partial W(\hat{L})$ is the manifold obtained by framed surgery on $\hat{L}$.

1^{st} part of lecture $\Rightarrow$ if $\hat{L}, \hat{L}'$ are isotopic framed links, then $W(\hat{L}) \simeq W(\hat{L}')$

Moral Lots of links, hence lots of 3 and 4-manifolds

Observe; W is result of attaching n 2-handles to B^4 (0-handle)

$\Rightarrow W(\hat{L}) \sim X$ cell cx with 1 0-cell, $|\hat{L}| = n$ 2-cells

$X = \vee^n S^2$ so $\sim$ type of W only sees # cpts of L

Consider a framed link $\hat{L} \subset S^3$ with components $L_1, \dots, L_n$ and framings $\lambda_{\alpha_i} = l_i + \alpha_i m_i$.

Form $W(\hat{L}) = B^4 \cup_J \coprod_{i=1}^{n} H(i)$, $H(i) \simeq H_4^2$

and $H(i)$ is attached along L_i with framing λ_{α_i}.

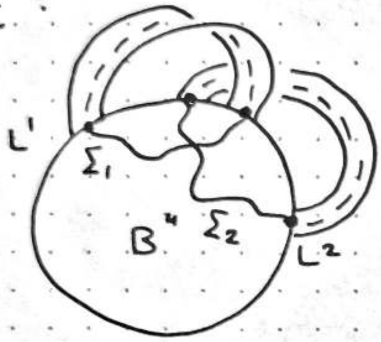
$\partial W(\hat{L}) = S^3_{\hat{L}} = E_L \cup_\varphi \coprod_{i=1}^{n} \partial_B H(i)$

where $\partial_B H(i) = S^1 \times D^2$, $\varphi : \partial_B H(i) \to \partial \nu(L_i) \subset \partial E_L$

$\varphi_*(S^1 \times \circ) \to \lambda_{\alpha_i} \in H_1(\partial \nu(L_i))$

maybe $1 \times \partial D^2$

Picture:



Suppose $\Sigma_i \hookrightarrow B^4$ is a smoothly embedded orientable surface with $\partial \Sigma_i = L_i$.

(E.g. Σ_i is a Seifert surface of L_i pushed into B^4)

Let $\hat{\Sigma}_i = \Sigma_i \cup_{L_i} D^2 \times 0 \subset B^4 \cup_{\nu(L_i)} H(i) \subset W(\hat{L})$

$\hat{\Sigma}_i$ is a closed oriented surface $\simeq \Sigma_i \cup_{L_i} D^2$

orientation is inherited from Σ_i

So $[\hat{\Sigma}_i] \in H_2(W(\hat{L}))$.

Recall: $W(\hat{L})$ dr's to $\bigvee_{i=1}^{n} S^2$

So $H_2(W(\hat{L})) \cong \mathbb{Z}^n$

Lemma 1 $\{[\hat{\Sigma}_1], \dots, [\hat{\Sigma}_n]\}$ are a basis for $H_2(W(\hat{L}))$

Proof The deformation retraction $p : W(\hat{L}) \to \bigvee_{i=1}^{n} S^2$ acts on $\hat{\Sigma}_i$ by

$$\hat{\Sigma}_i \to \hat{\Sigma}_i / \Sigma_i \simeq S^2 \xrightarrow{f_i} \bigvee_{i=1}^{n} S^2$$

where f_i is the inclusion of i^{th} S^2.

The map $H_2(\hat{\Sigma}_i) \to H_2(\hat{\Sigma}_i / \Sigma_i) \simeq H_2(S^2)$ is an iso^m

So $p_*([\hat{\Sigma}_i]) = f_{i*}([S^2])$, these generate $H_2(\bigvee_{i=1}^{n} S^2)$. $\square$

Lemma 2 $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_j] = \begin{cases} lk(L_i, L_j) & \text{if } i \neq j \\ \alpha_i & \text{if } i = j \end{cases}$

Proof $H(i) \cap H(j) = \phi$ if $i \neq j$

$$\Rightarrow [\hat{\Sigma}_i]\cdot[\hat{\Sigma}_j] = [\Sigma_i]\cdot[\Sigma_j]$$
$$= lk(L_i, L_j) \quad (ES1)$$

[⚠ To choose an orientation on $\hat{\Sigma}_i$, choose one on L_i, demand $\partial\Sigma_i = L_i$]

If $i = j$, consider

$$\hat{\Sigma}_i' = \Sigma_i' \cup_{\lambda_{\alpha_i}} D^2 \times p \quad (p \in D^2 \setminus 0)$$

where $\Sigma_i' \hookrightarrow B^4$ has $\partial \Sigma_i' = \lambda_i$

Then $p_*([\hat{\Sigma}_i']) = f_{i*}([S^2])$

$$\Rightarrow [\hat{\Sigma}_i'] = [\hat{\Sigma}_i]$$

so
$$[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_i] = [\hat{\Sigma}_i]\cdot[\hat{\Sigma}_i']$$
$$= [\Sigma_i]\cdot[\Sigma_i']$$
$$= lk(L_i, \lambda_{\alpha_i})$$
$$= \alpha_i$$

□

Def^n $B = (b_{ij})$ where $b_{ij} = \begin{cases} lk(L_i, L_j) & i \neq j \\ \alpha_i & i = j \end{cases}$

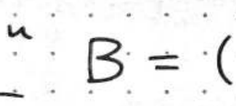
is the <u>linking matrix</u> of the oriented framed link $\hat{L}$ in S^3.

It's the symmetric matrix which gives the intersection form on $W(\hat{L})$ wrt the basis $[\hat{\Sigma}_i]$.

[⚠ the intersection form is an invariant of $(X, \partial X)$ via P.D.. use $H_2(X) \to H^2(X, \partial X)$.]

Ex :

1)  $B = (n)$

2)  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3)  3 5 $\quad B = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$

<u>Prop</u>: $H_1(S^3_{\hat{L}}) \simeq \operatorname{coker} B$

<u>Proof</u>: Let $W = W(\hat{L})$, consider LES of pair $(W, \partial W)$

$$\begin{array}{ccccccc} H_2(W) & \xrightarrow{f_*} & H_2(W, \partial W) & \longrightarrow & H_1(\partial W) & \longrightarrow & \underbrace{\cancel{H_1(W)}}_{=0} \\ & \searrow^{\beta} & \wr\!\!\mid \text{ P.D.} & & & & \\ & & H^2(W) & & & & \end{array}$$

We see that $H_1(\partial W) = \operatorname{coker} f_* \simeq \operatorname{coker} \beta$

Let $[\hat{\Sigma}_1]^*, \dots, [\hat{\Sigma}_n]^*$ be the basis of $H^2(W) \simeq \mathbb{Z}^n$ dual to $[\hat{\Sigma}_1], \dots, [\hat{\Sigma}_n]$ i.e. $\langle [\hat{\Sigma}_j]^*, [\hat{\Sigma}_i] \rangle = \delta_{ij}$.

If $\beta([\hat{\Sigma}_j]) = \sum \beta_{ij} [\hat{\Sigma}_i]^*$, then

$$\begin{aligned} \beta_{ij} &= \langle \beta([\hat{\Sigma}_j]), [\hat{\Sigma}_i] \rangle \\ &= \langle \text{P.D. } f_*[\hat{\Sigma}_j], [\hat{\Sigma}_i] \rangle \\ &= [\hat{\Sigma}_j] \cdot [\hat{\Sigma}_i] \\ &= b_{ij} \end{aligned}$$

, so β is given by the matrix B wrt bases $\{[\hat{\Sigma}_i]\}$, $\{[\hat{\Sigma}_i]^*\}$. $\square$

<u>Ex</u>: $\hat{L} = K$ w/ framing n

$B = (n)$ so $H_1(S^3_{K,n}) \simeq \mathbb{Z}/n$

$$\Rightarrow H_*(S^3_{K,n}) = \begin{cases} \mathbb{Z}, & * = 0,3 \\ \mathbb{Z}/n, & * = 1 \\ 0, & \text{o/w} \end{cases} \quad \text{if } n \neq 0$$

$H_2(S^3_{K,n}) \simeq H^1(S^3_{K,n}) = 0$ by PD and UCT

If $n=0$, then

$$H_*(S^3_{K,0}) = \begin{cases} \mathbb{Z}, & *=0,1,2,3 \\ 0, & \text{o/w} \end{cases}$$

Does not depend on K.

If $n=0$, consider

$$\pi_1(S^3_{K,0}) \xrightarrow{1.1} H_1(S^3_{K,0}) \simeq \mathbb{Z}$$

Let $p : \widetilde{S^3_{K,0}} \to S^3_{K,0}$ be the covering map corresponding to $\ker 1.1$. This has deck group $\mathbb{Z}$,

so $H_1(\widetilde{S^3_{K,0}})$ is a module over $\mathbb{Z}[G_{\text{Deck}}] = \mathbb{Z}[t^{\pm 1}]$

<u>Propn</u> $H_1(\widetilde{S^3_{K,0}}) \simeq H_1(\widetilde{E}_K)$

as modules over $R = \mathbb{Z}[t^{\pm 1}]$

<u>Proof</u> Let $Y_K = S^3_{K,0}$. Then

$$Y_K = E_K \cup_{\partial E_K} S^1 \times D^2 \qquad (1)$$

So $\widetilde{Y}_K = \widetilde{E}_K \cup_{\widetilde{\partial E_K}} \widetilde{S^1 \times D^2}$ where $\widetilde{X}$ is a covering space of X. (2)

M-V for (1) says

$$H_1(\partial E_K) \xrightarrow{i_{1*}+i_{2*}} H_1(E_K) \oplus H_1(S^1 \times D^2) \to H_1(Y_K) \to 0$$

$$H_2(\cancel{E_K}) \oplus H_2(\cancel{S^1 \times D^2}) \to H_2(Y_K)$$

$0 \qquad 0 \qquad H_2(Y_K) = \mathbb{Z}$

So $i_{2*}(l) = 0$ [so l generates kernel of $i_{1*}+i_{2*}$]

$i_{1*} + i_{2*}(m) = 1 \oplus 1$

$$H_1(\partial E_K) \to H_1(Y_K) = \mathbb{Z}$$
$$m \mapsto 1$$

This implies that $\widetilde{\partial E_K} = S^1 \times \mathbb{R}$, $\langle S^1 \rangle = l$

$\widetilde{E}_K$ <u>is</u> the infinite cyclic cover of E_K,

$\widetilde{S_1 \times D^2}$ is $\mathbb{R} \times D^2$

M-V for (2) is

$$\begin{array}{ccccc} H_1(S^1 \times \mathbb{R}) & \xrightarrow{\tilde{f}_*} & H_1(\widetilde{E_K}) \oplus H_1(\mathbb{R} \times \overset{\circ}{D^2}) & \to H_1(\widetilde{Y_K}) & \to 0 \\ \| & & & & \\ \mathbb{Z}\cdot[\ell] & \longmapsto & 0 \quad \oplus \quad 0 & & \end{array}$$

$\tilde{f}_* = 0 \implies H_1(\widetilde{E_K}) \simeq H_1(\widetilde{Y_K})$. $\square$

<u>Cor</u> If $\Delta(K) \neq \Delta(K')$ then

$S^3_{K,0} \not\cong S^3_{K',0}$ even though $H_*(S^3_{K,0}) \cong H_*(S^3_{K',0})$

3.5) Application + Examples

Dehn Twists:

Let $A = S^1 \times [-1,1]$, with product orientation (iii)

w/ blackboard orientation

Consider $\tau : A \to A$

$$\tau(z,t) = (e^{i\pi(t+1)} z, t)$$

$$\tau|_{\partial A} = id_{\partial A}$$

τ is the model Dehn twist.

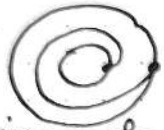


image of $\{1\} \times [-1,1]$

If Σ is a oriented surface and $\alpha : S^1 \hookrightarrow \Sigma$ (has trivial normal bundle), choose an orientation preserving $\varphi : \nu(\alpha) \xrightarrow{\sim} A$

Define $\tau_\alpha : \Sigma \to \Sigma$ by $\tau_\alpha(x) = \begin{cases} \varphi^{-1}\tau\varphi(x) & \text{if } x \in \nu(\alpha) \\ x & \text{if } x \notin \nu(\alpha) \end{cases}$

τ_α is the (!) Dehn twist along α

Exercise τ_α acts on $H_1(\Sigma)$ by

$$(\tau_\alpha)_* x = x + (\alpha \cdot x)\alpha \qquad \text{(iii)}$$

right vs left Dehn twist

Fact: Isotopy class of τ_α does not depend on choice of φ, $\nu(\alpha)$, isotopies of α, orientation of α

Does depend on orientation of Σ.

Knots on Surfaces

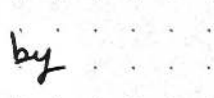
Σ is an oriented surface, $\alpha : S^1 \hookrightarrow \Sigma$ an embedded loop

$M = \Sigma \times [-1,1]$

$K = \alpha \times 0 \subset M$

$\nu(\alpha) \simeq A$

$\nu(K) \simeq A \times [-1,1]$

$\nu(K)$ () $\times S^1$

framing curve
$l = 0 \times S^1$

meridian $m = \partial \square$

A framing of K is determined by a non-vanishing section of $\nu M/\alpha$. The surface Σ gives a preferred section

$$\longleftrightarrow \quad \Sigma \cap \partial\nu(K)$$

All other framings are of form $\lambda_n = l + n.m$

Study $M_{K,1}$ = surgery on K w/ framing $\lambda_1 = l + m$

<u>Lemma</u> Suppose

$$\varphi : \partial(S^1 \times D^2) \xrightarrow{\sim} \partial(S^1 \times D^2)$$

Then φ extends to

$$\Phi : S^1 \times D^2 \xrightarrow{\sim} S^1 \times D^2$$

if and only if

$$\varphi_*([1 \times \partial D^2]) = \pm [1 \times \partial D^2]$$

<u>Proof</u> If φ extends, $\iota \circ \varphi = \Phi \circ \iota$

where $\iota : \partial(S^1 \times D^2) \to S^1 \times D^2$ is inclusion.

$\iota_*([1 \times \partial D^2]) = 0$ (& generate kernel)

So must have $\iota_*(\varphi_*([1 \times \partial D^2])) = 0$

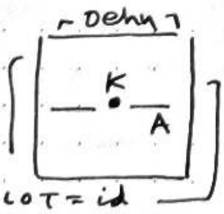
i.e. $\varphi_*([1 \times \partial D^2]) = \pm [1 \times \partial D^2]$

Other direction: ES 2, Q 2. $\square$

<u>Lemma</u> $M = A \times I$, $K = S^1 \times 0 \times 0$

Then there is a diffeo

$\Phi : M_{K,1} \to M$ s.t.



$$\begin{tikzcd} M_{K,1} \arrow[rr, "\Phi"] & & M \\ & \partial M \arrow[ul, "\iota"] \arrow[ur, "\iota \circ \tau"'] & \end{tikzcd}$$

where $\iota : \partial M \to M$ is inclusion, and τ is Dehn twist with $A = A \times I \subset \partial M$

Proof $E_K \simeq T^2 \times I$,

so $M_K = E_K \cup_{T^2 \times (-1)} S^1 \times D^2 \simeq S^1 \times D^2 \overset{!}{=} M$

So to check existence of Φ, enough to check that

$$\Phi|_{\partial M_{K,1}} (\underbrace{[1 \times \partial D^2]}_{\ell + m}) \overset{?}{=} \underbrace{[1 \times \partial D^2]}_{= m} \in H_1(M)$$

since we took $M_{K,1}$

So this follows from

$$\tau(\ell + m) = (\ell + m) + \ell(\ell \cdot (\ell + m))$$
$$= \ell + m - \ell = m$$

□

Cor $M = \Sigma \times [-1,1]$

$K = \alpha \times 0$

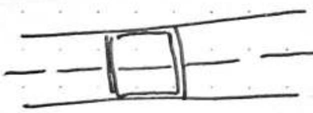
Then $\exists$ ~~Φ~~ $\Psi : M_{K,1} \to M$ s.t.

$$\begin{tikzcd} M_{K,1} \arrow[r, "\Psi"] & M \\ & \Sigma \arrow[ul, "\iota_+"] \arrow[u, "\iota_+ \circ \tau_a"'] \end{tikzcd} \qquad \begin{tikzcd} M_{K,1} \arrow[r, "\Psi"] & M \\ & \Sigma \arrow[ul, "\iota_-"] \arrow[u, "\iota_-"'] \end{tikzcd}$$

where $\iota_\pm : \Sigma \to \Sigma \times \{\pm 1\} \subset M$ are inclusions

Proof ~~Define~~ Choose a tubular ngbd of K as in lemma

Define $\Psi(x) = \begin{cases} \Phi(x), & x \in \nu(K) \\ x, & x \notin \nu(K) \end{cases}$



$\Sigma \times 0$

$\Phi|_{\sqcup \text{ boundary}} = id$, so extends by identity to rest of $\Sigma \times I$.

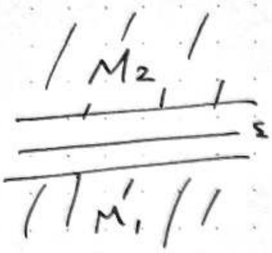
□

So now suppose $\Sigma \subset Y^3$, so

$$Y = M_1 \cup_{\varphi_1} \Sigma \times [-1,1] \cup_{\varphi_2} M_2$$

where $\rho_1 : \partial M_1 \simeq \Sigma \times \{-1\}$

$\rho_2 : \partial M_2 \simeq \Sigma \times \{+1\}$



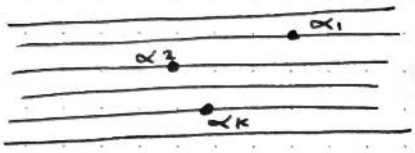
$K = \alpha \times 0 \subset \Sigma \times [-1,1]$

Then $Y_{K,1} = M_1 \cup_{\rho_1} M_{K,1} \cup_{\rho_2} M_2$

$$= M_1 \cup_{\rho_1} \Sigma \times [-1,1] \cup_{\tau_\alpha \circ \rho_2} M_2$$

More generally, if $\alpha_1, \dots, \alpha_k \hookrightarrow \Sigma$

$\hat{L} =$



$\Sigma \times [-1,1]$

where all α's have framing 1.

Then $M_{\hat{L}} = M_1 \cup_{\rho_1} \Sigma \times [-1,1] \cup_{\tau_{\alpha_k} \circ \tau_{\alpha_{k-1}} \circ \dots \circ \tau_{\alpha_1} \circ \rho_2} M_2$

Theorem (Dehn, Lickorish)

Any o.preserving diffeo $\sigma : \Sigma \to \Sigma$ is up to isotopy a composition of Dehn twists

Thm (Lickorish-Wallace)

If Y is an orientable closed 3-manifold, then $Y = S^3_{\hat{L}}$ for some framed link $\hat{L} \subset S^3$

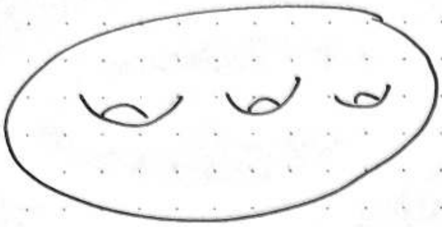
Proof Y admits a handle decomposition

$\Rightarrow Y$ has a Heegaard splitting $Y = H_g \cup_\varphi H_g$ (ES3)

where H_g is an orientable 3-dim'l handlebody with 1 0-handle, g 1-handles

$\partial H_g = \Sigma_g$, $\varphi : \Sigma_g \to \Sigma_g$ diffeo

S^3 has a Heegaard splitting of genus g

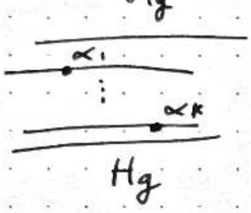


"the outside is also an H_g" ◡̈

$$\therefore S^3 = H_g \cup_{\varphi_0} H_g$$

Write $\varphi = \tau_{\alpha_k}^{\pm} \circ \tau_{\alpha_{k-1}}^{\pm} \circ \cdots \circ \tau_{\alpha_1}^{\pm 1} \circ \varphi_0$ (by Dehn)

Take $\hat{L} =$



$\subset S^3$

where α_i has framing ± 1 according to exponent of τ_{α_i}

Then $S^3_{\hat{L}} = H_g \cup_{\tau_{\alpha_k}^{\pm} \cdots \tau_{\alpha_1}^{\pm} \varphi_0} H_g = H_g \cup_\varphi H_g = Y$. $\square$