Today: 3 topics

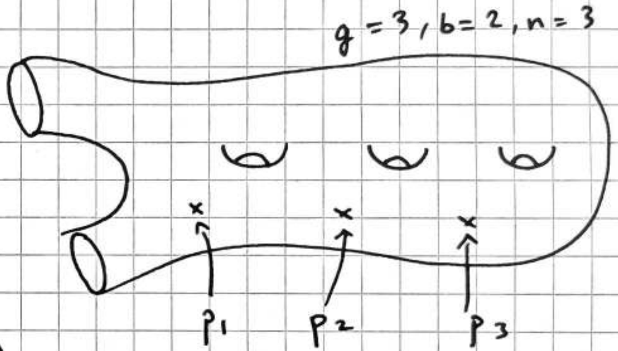
- Surfaces of finite type
  - ↳ Basic objects of study
- Mapping Class Groups
  - ↳ The definition!
- Context & Motivation
  - ↳ In case you don't care yet!

1.1 Surfaces of finite type

Let S be a surface, i.e. a (smooth) 2-dimensional real manifold.

We'll always assume S is also of finite type.

Finite type: $$S = \underset{\text{compact, possibly } \partial\overline{S} \neq \emptyset}{\overline{S}} \setminus \underset{\text{"punctures"}}{\{p_1, \dots, p_n\}}$$



Theorem 1.1 (Classification of surfaces of finite type)

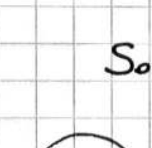
Every connected orientable surface of finite type is diffeo^{c} to some $S_{g,n,b}$, the surface obtained by connect-summing $g \geq 0$ copies of the torus T^2 with the 2-sphere S^2, and removing b open discs* and n points.

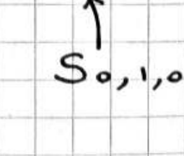
S is:
- closed if $S \cong S_{g,0,0}$ (i.e. no punctures, $\partial S = \emptyset$)
- compact if $S \cong S_{g,0,b}$ (i.e. no punctures)

The Euler characteristic $\chi(S) = 2 - 2g - n - b$ is often useful.

Example 1.2 If $\chi(S) > 0$ then $g = 0$ and $n + b \leq 1$.

It follows that S is one of $\underset{S_{0,0,0}}{S^2}$, $\underset{S_{0,1,0}}{\mathbb{C}}$ or the compact 2-disc $\underset{S_{0,0,1}}{D^2}$.





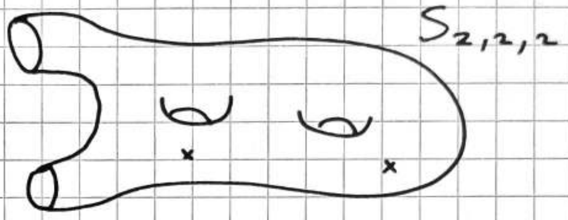
$\chi(S) = 0$:

Example 1.3 If $\chi(S) = 0$ then either $g = 1$ and $n+b = 0$ or $g = 0$ and $n+b = 2$. Therefore, S is one of:

- the 2-torus T^2  $\cong S_{1,0,0}$
- the punctured disc D^2_*  $\cong S_{0,1,1}$
- the twice-punctured sphere (aka $\mathbb{C}_*$) $\cong S_{0,2,0}$
- the annulus $A = S^1 \times [-1,1]$  $\cong S_{0,0,2}$

For all other surfaces ~~of finite~~ S of finite type, we have $\chi(S) < 0$.

So this is "generic".



1.2 Mapping class groups

S a surface

The group $\text{Homeo}(S) := \{ \phi : S \underset{\text{homeo}}{\xrightarrow{\cong}} S \}$ is obviously interesting.

But it's also huge. (uncountable) uncountably so (!)

Homeos $\phi, \psi : S \to S$ are isotopic if there is a continuous map

$$S \times [0,1] \longrightarrow S$$
$$(p,t) \longmapsto \phi_t(p)$$

s.t.
- $\phi_0 = \phi$, $\phi_1 = \psi$, and
- each $\phi_t : S \to S$ is a homeomorphism.

Furthermore, if $\partial S \neq \emptyset$, we require
- $\phi_t|_{\partial S} = \text{id}_{\partial S}$ for all $t \in [0,1]$

Remark $\text{Homeo}_0(S) := \{ \phi : S \xrightarrow{\cong} S \mid \phi \text{ isotopic to } \text{id}_S \}$ is a normal subgroup of $\text{Homeo}(S)$.

(Indeed, it's the component of the identity in a natural topology)

<u>Def 1.4</u> Let $\mathrm{Homeo}^+(S, \partial S) := \{ \phi : S \xrightarrow[\text{homeo}]{\sim} S \mid$
(S connected, orientable, and of finite type)

ϕ is orientation-preserving
$\phi|_{\partial S} = \mathrm{id}_{\partial S} \}$

Consider $\mathrm{Homeo}_0(S, \partial S) \triangleleft \mathrm{Homeo}^+(S, \partial S)$.

Then the mapping class group of S is

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S) / \mathrm{Homeo}_0(S, \partial S)$$

Related definitions:

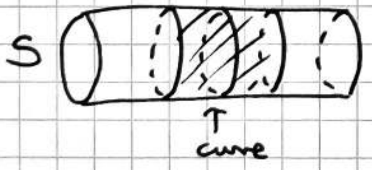
We could have used diffeomorphisms & smooth isotopies, or just homotopies $\uparrow$ instead of isotopies.
(rel. ∂S)

$\rightsquigarrow$ same group !

If we forget the "orientation-preserving" hypothesis, we get the extended MCG $\mathrm{Mod}^{\pm}(S)$.

Note: $|\mathrm{Mod}^{\pm}(S) : \mathrm{Mod}(S)| = 2$

In this course, we will cheat:
we will use continuous maps & homeos, but assume some nice properties that are easier to prove for smooth maps, e.g. regular ngbds



~ Jordan-Schonflies

1.3 Context & Motivation

Hopefully you already think MCGs are obviously fascinating
Just in case, I'll give 3 motivating contexts

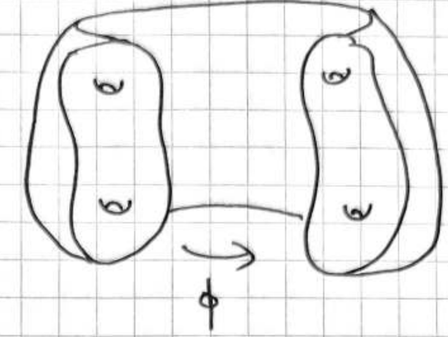
Topology :

Any homeo $\phi : S \to S$ defines a surface bundle

$$M_\phi := S \times [0,1]$$

where $(p,1) \sim (\phi(p), 0)$.

This is an important source of interesting 3-manifolds.



Moduli Space

Many mathematicians care a lot about

$\mathcal{M}_g$ = "space of all conformal structures / complex structures on the surface S_g" .

In a sense,

$$\mathrm{Mod}(S_g) = \pi_1(\mathcal{M}_g) .$$

Analogy with matrices

Since $\mathbb{Z}^2 \cong \pi_1(T^2)$, surfaces (or their fundamental groups) can be thought of as "non-commutative analogues" of $\mathbb{Z}^n$.

This leads to analogies:

$$GL_n(\mathbb{Z}) \leftrightsquigarrow \mathrm{Mod}^{\pm}(S_g)$$

$$SL_n(\mathbb{Z}) \leftrightsquigarrow \mathrm{Mod}(S_g)$$

2. Surfaces & Hyperbolic Geometry

- The Hyperbolic Plane
- Hyperbolic Structures

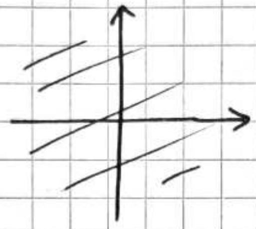
2.1 | 3 important examples of complete, simply connected surfaces of constant curvature K :

$K = +1$

$S^2 =$

$K = 0$

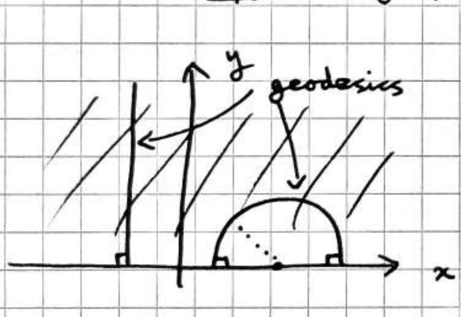
$\mathbb{C} =$



$K = -1$

$\mathbb{H}^2$ hyperbolic plane

Recall The upper half-plane model of $\mathbb{H}^2$, $\mathbb{U}$ \mathbb{U}



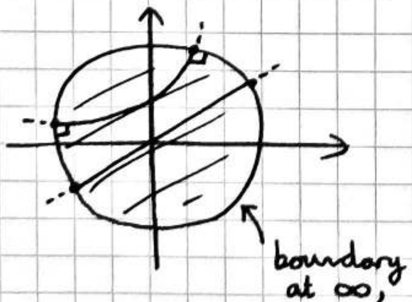
$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{metric}$$

l_+ is $\{x = 0\}$

$\circlearrowright$

$$\mathrm{Isom}^+(\mathbb{H}^2) = PSL_2(\mathbb{R}) \qquad z \mapsto \frac{az+b}{cz+d}$$

The disc model $\mathbb{D}$ of $\mathbb{H}^2$ is obtained by conjugating the previous picture by $z \mapsto \frac{z-i}{z+i}$



$$ds^2 = 4\,\frac{dx^2 + dy^2}{(1-r^2)^2}$$

$\Downarrow$

radial symmetry !

$\partial_\infty \mathbb{H}^2$ or $\partial \mathbb{H}^2$

Write $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2 \cong D^2$.

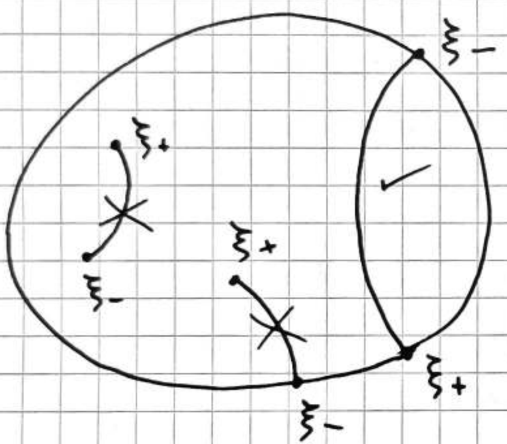
Every $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ extends to a continuous
$$\bar{f} : \overline{\mathbb{H}}^2 \longrightarrow \overline{\mathbb{H}}^2 .$$

Next, let's classify $f \in \mathrm{Isom}^+(\mathbb{H}^2) \setminus \{id\}$ by thinking about $\# \mathrm{Fix}(\bar{f})$.

Brouwer FPT : $\# \mathrm{Fix}(\bar{f}) \geqslant 1$

Möbius transf : $\# \mathrm{Fix}(\bar{f}) < 3$

Case 1 $\# \mathrm{Fix}(\bar{f}) = 2$, say $\mathrm{Fix}(\bar{f}) = \{\xi_+, \xi_-\}$



If $\xi_\pm \in \mathbb{H}^2$ then f fixes an arc, whence $\# \mathrm{Fix}(f) = \infty > 2$ ⨳

In this case, f is called hyperbolic / loxodromic. f preserves the unique geodesic $\mathrm{Axis}(f) := (\xi_+, \xi_-)$ and translates by some $\tau(f)$.

$\{\xi_\pm\} \to \{0, \infty\}$

Remark Can conjugate so that $\mathrm{Axis}(f) = \ell^+ \subset \mathbb{U}$ and $f(z) = e^{\tau(f)} z$. See also ExSheet 1, q4.

Case 2 $\# \mathrm{Fix}(\bar{f}) = 1$, say $\mathrm{Fix}(\bar{f}) = \{\xi\}$

(a) $\xi \in \mathbb{H}^2$

Then f is called elliptic. Conjugate ξ to $0 \in \mathbb{D}$.

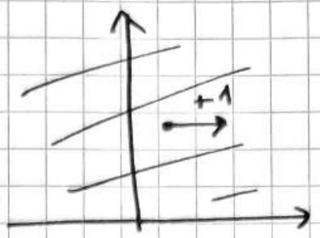
$\Rightarrow f(z) = e^{i\theta} z$, some $\theta \in \mathbb{R}$, a rotation

(b) $\xi \in \partial \mathbb{H}^2$

f is called parabolic

Conjugate $\xi \mapsto \infty \in \overline{\mathbb{U}}$.

So $f(z) = z \pm 1$ (after conjugating).



2.2| For now assume S is closed (i.e. $S_{g,0,0} = S_g$) and that $g \geqslant 2$.

Defⁿ A geometric structure on S is a Riemannian metric of constant curvature $\kappa \in \{\pm 1, 0\}$.

Gauss-Bonnet:

$$2\pi \chi(S) = \int_S \kappa \, dA$$

$$\Rightarrow \operatorname{sign}(\kappa) = \operatorname{sign}(\chi(S))$$

If $\chi(S) \geqslant 0$ (i.e. $g \leqslant 2$) we can construct geometric structures explicitly.

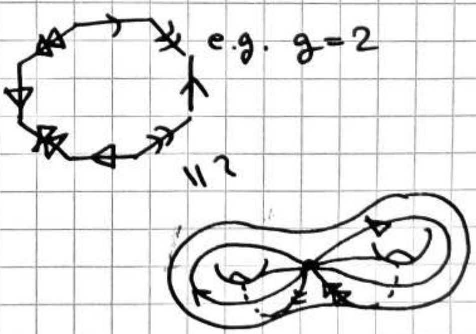
$\chi(S) < 0$

Thm 2.1 Let S be a closed surface of genus $g \geqslant 2$. There is an action (free even) of $\pi_1 S$ by isometries on $\mathbb{H}^2$ s.t. $\pi_1 S \backslash \mathbb{H}^2 \underset{\text{diffeo}}{\cong} S$

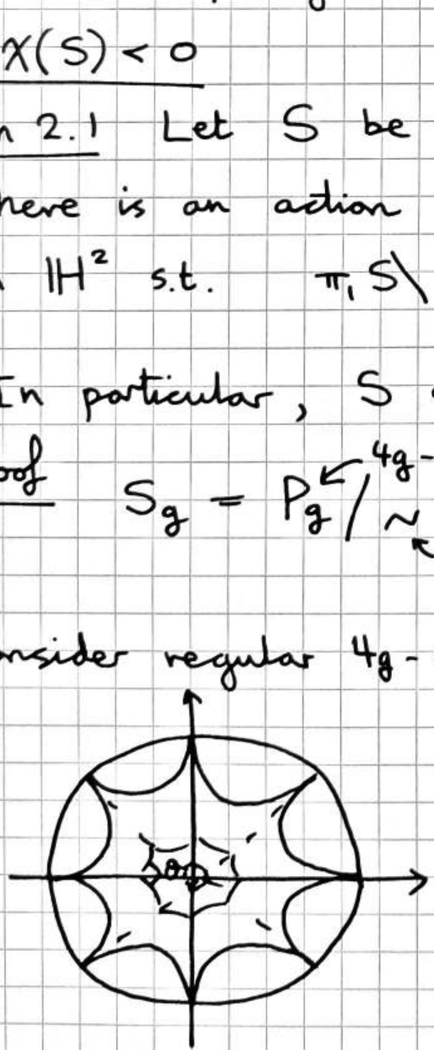
In particular, S admits a metric of curvature -1.

Proof $S_g = \overset{\text{4g-gon}}{P_g} / \underset{\text{identifies pairs of points in } \partial P_g}{\sim}$

e.g. $g=2$



Consider regular $4g$-gons in $\mathbb{H}^2$:



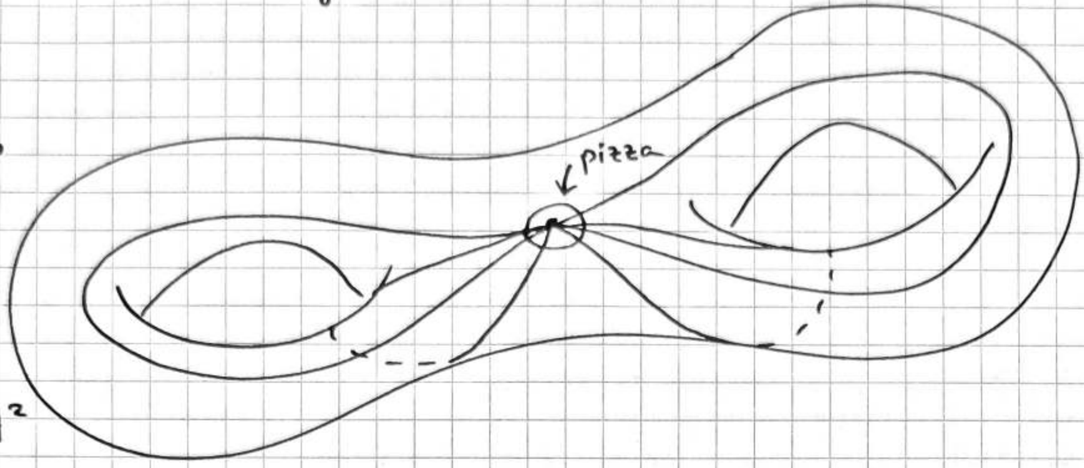
As we vary radius,

$\theta \to 0$

$\theta \to \left(1 - \frac{2}{4g}\right)\pi$

By IVT (using $g \geqslant 2$), $\exists$ polygon s.t. $\theta = \frac{2\pi}{4g}$

Put this metric on P_g



By Diff Geo
+ Alg Top,
get $\tilde{S} \cong \mathbb{H}^2$
and
$S \cong \pi_1 S \backslash \mathbb{H}^2$

↑

also use that $\mathbb{H}^2$ is
the only complete simply connected
hyperbolic surface

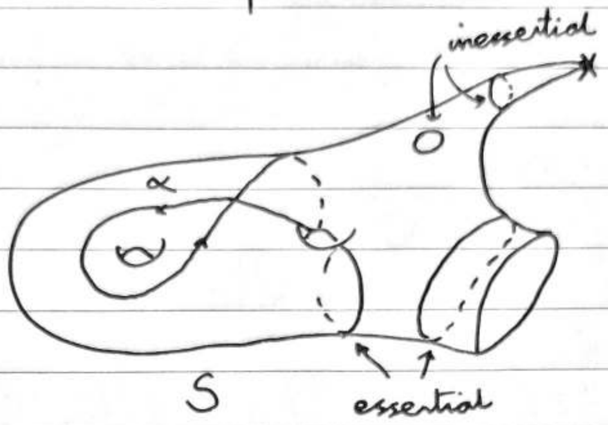
□

3. Curves & Isometries

A closed curve on S is a cts map

$$\alpha : S^1 \longrightarrow S$$

A curve α is inessential if $\alpha \simeq$ pt or a puncture. O/w α is essential.



⌜$\alpha \simeq$ punct includes wrapping several times⌟

For S hyperbolic, covering space theory gives

curve on S $\longmapsto$ isom. of $\mathbb{H}^2$

↳ $\underset{\text{convex}}{\tilde{S} \subseteq \mathbb{H}^2}$, $\pi_1 S \curvearrowright \tilde{S}$ by isometry

$S \cong \pi_1 S \backslash \tilde{S}$ $\boxed{\text{finite area complete}}$

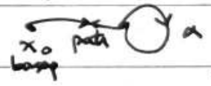
⌜$\left\{ \begin{array}{l} \text{non-trivial} \\ \text{conj} \\ \text{classes} \\ \text{in } \pi_1 S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{non-trivial} \\ \text{free hom} \\ \text{classes of} \\ \text{oriented} \\ \text{closed} \\ \text{curves in S} \end{array} \right\}$⌟

In detail:

$\alpha : S^1 \longrightarrow S$

$\rightsquigarrow$ conjugacy class $[\alpha] \subseteq \pi_1 S$

⌜fix basepoint once and for all, then $[\alpha]$ given by



⌟

Theorem 2.1 $\pi_1 S \hookrightarrow \mathrm{Isom}^+ \mathbb{H}^2$

∴ α can be thought of as an element of $\mathrm{Isom}^+(\mathbb{H}^2)$ up to conjugacy

Lemma 3.1 S hyp surface, α closed curve on S

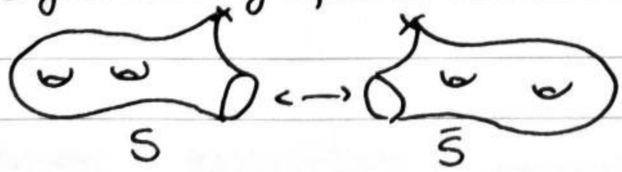
(i) α elliptic ⟹ α is homotopic to a point

(ii) α parabolic ⟹ α " puncture

(iii) α hyperbolic ⟹ α essential

In particular, $\pi_1 S$ is torsion-free

Proof By doubling, we can assume $\partial S = \emptyset$.

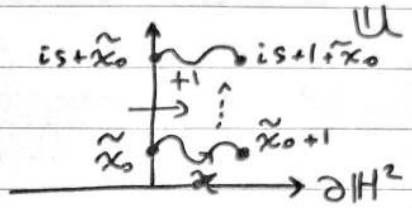


(i) α elliptic

$\Rightarrow \alpha$ fixes a point in $\mathbb{H}^2 \cong \widetilde{S}$

$\Rightarrow \alpha = 1_{\pi_1 S}$ because $\pi_1 S \curvearrowright \widetilde{S}$ is free //

(ii) α parabolic



WLOG $\mathbb{H}^2 = \mathbb{U}$, $\alpha(z) = z+1$

Pick a basepoint x_0 for α and a lift $\tilde{x}_0 \in \widetilde{S} \equiv \mathbb{H}^2$.

Lift the curve α at $\tilde{x}_0$.

For any $s \in [0,\infty)$, define a path

$$\tilde{\alpha}_s(t) = \tilde{\alpha}(t) + is$$

This descends to a loop α_s on S. ~~that~~

The family of curves α_s defines a homotopie that takes α to a puncture.

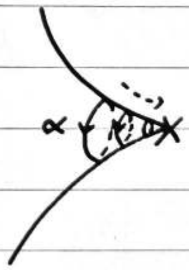
(iii) Suppose α is hyperbolic.

Then WTS $\alpha \underset{①}{\not\simeq} pt$, $\alpha \underset{②}{\not\simeq}$ puncture

① $\alpha \simeq pt \Rightarrow \alpha = 1_{\pi_1 S} = 1_{Isom^+(\mathbb{H}^2)}$

$\Rightarrow \alpha$ elliptic$^{(!)}$, not hyperbolic

② Suppose $\alpha \simeq$ puncture.

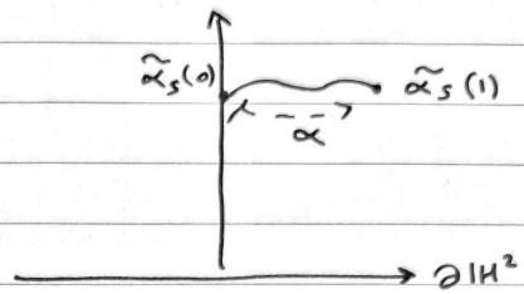


Then $\exists$ family of curves α_s homotoping α to a puncture s.t. length$(\alpha_s) \longrightarrow 0$ as $s \longrightarrow \infty$,

by completeness + finite area.

Now lift the paths α_s to paths $\tilde{\alpha}_s$ in $\mathbb{H}^2$

As an isometry of $\mathbb{H}^2$,
α sends $\tilde{\alpha}_s(0) \mapsto \tilde{\alpha}_s(1)$.



Therefore,

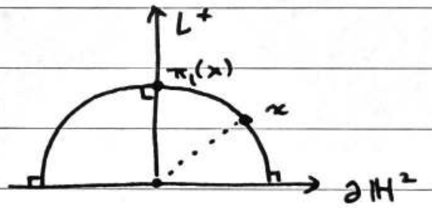
$$\begin{aligned} d(\tilde{\alpha}_s(0), \tilde{\alpha}_s(1)) &\leq \text{length}(\tilde{\alpha}_s) \\ &= \text{length}(\alpha_s) \\ &\longrightarrow 0 \quad \text{as } s \to \infty. \end{aligned}$$

$\therefore$ α is elliptic or parabolic $\square$

Lemma 3.2 Let S be a hyperbolic surface, and let α be an essential curve on S. There is a unique geodesic representative (up to reparametrisation) in the homotopy class of α.

Idea: for any geodesic $L \subset \mathbb{H}^2$ $\exists$ orthogonal projection $\pi_L : \mathbb{H}^2 \to L$

WLOG $L = L^+$



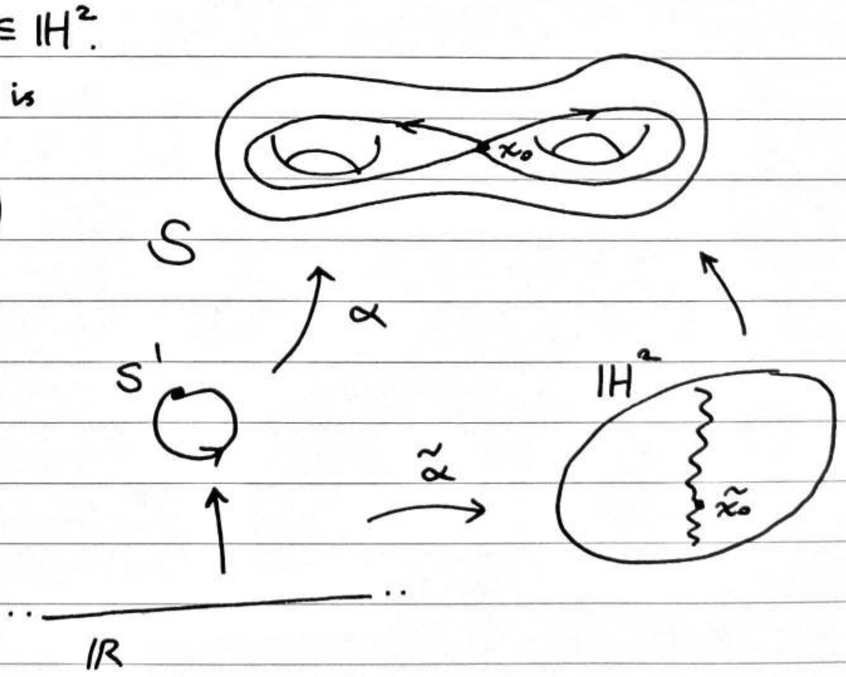
Proof Fix a basepoint $x_0 \in S$ for α, and a lift $\tilde{x}_0 \in \mathbb{H}^2$.
Now LIFT & UNWRAP α to a continuous line in $\tilde{S} \subseteq \mathbb{H}^2$.
The map $\tilde{\alpha} : \mathbb{R} \to \mathbb{H}^2$ is equivariant, i.e.

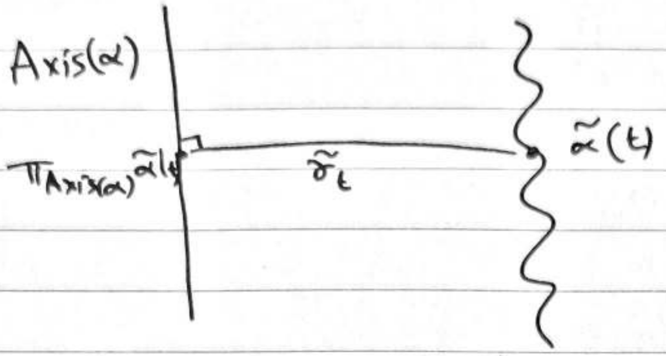
$$\tilde{\alpha}(t+1) = \underset{\substack{\uparrow \\ \text{isometry}}}{\alpha}(\tilde{\alpha}(t))$$

For each $t \in \mathbb{R}$, let

$$\tilde{\gamma}_t : [0,1] \to \mathbb{H}^2$$

be the unique constant speed geodesic from $\tilde{\alpha}(t)$ to $\pi_{\text{Axis}(\alpha)}(\tilde{\alpha}(t))$





Because of equivariance, $\tilde{\gamma}_t$ descends to a homotopy

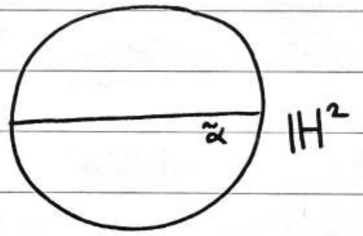
$$\alpha \underset{\gamma}{\simeq} \beta$$

where β is in the image of Axis(α) in S.
Reparametrising $\beta \simeq$ geodesic.
This proves existence.

To prove uniqueness, suppose α, β are both geodesics, $\alpha \simeq \beta$.
Lift to geodesics $\tilde{\alpha}, \tilde{\beta}$ in $\mathbb{H}^2$:

$$d(\tilde{\alpha}(t), \tilde{\beta}(t)) < C$$

$\therefore \tilde{\beta}$ has the same endpoints in $\partial \mathbb{H}^2$

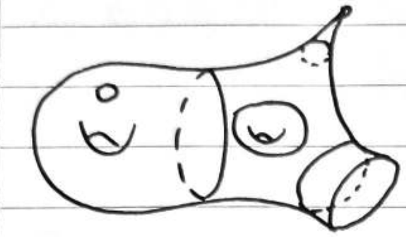


$$\Rightarrow \tilde{\alpha} = \tilde{\beta} \Rightarrow \alpha = \beta. \quad \square$$

4. Intersections of simple closed curves

4.1 Simple closed curves & intersection number

Def^n A curve on S is simple if $\alpha : S^1 \to S$ is injective.

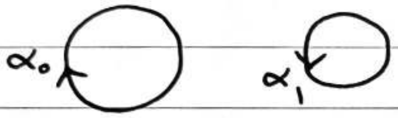


Def^n An isotopy between two sccs α_0, α_1 is a homotopy $\alpha_\cdot$ s.t. each α_t is a simple closed curve

An ambient isotopy from α_0 to α_1 is an isotopy $\phi_t : S \xrightarrow{\sim} S$ s.t.

$$\phi_0 = id_S \ \& \ \alpha_1 = \phi_1 \circ \alpha_0 .$$

e.g.



homotopic but not isotopic

Lemma 4.2 Two essential simple closed curves on an oriented surface S are homotopic iff they are ambient isotopic.

Pf Next time ... □

Consider T^2, $\pi_1 T^2 \cong \mathbb{Z}^2$.

Let us call $g \in G$ imprimitive if $g = h^n$ for some $h \in G$, $n > 1$. Otherwise g is primitive.

Lemma 4.3 Homotopy classes of ⌜essential⌟ simple closed curves on T^2 correspond to primitive elements of $\pi_1 T^2 = \mathbb{Z}^2$.

Pf See example sheet 1, q 8 (?) □

Lemma 4.4 Let S be an orientable hyperbolic surface and α an essential simple closed curve on S. And let γ be the unique geodesic rep $\gamma \simeq \alpha$. Then γ is also simple.

Pf Use a criterion for injectivity:

$\alpha : S^1 \to S$ is injective iff for any lift $\tilde{\alpha} : \mathbb{R} \to \tilde{S}$ of α to the universal cover, something torsion

(i) $\tilde{\alpha} : \mathbb{R} \hookrightarrow \tilde{S}$

(ii) $\forall g \in \pi_1 S, \quad g\tilde{\alpha} \cap \tilde{\alpha} \neq \emptyset \Rightarrow g \in \langle \alpha \rangle \leq \pi_1 S$

So if γ is not injective, then

$\exists g \in \pi_1 S \setminus \langle \alpha \rangle$

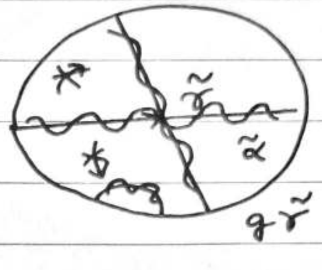
s.t. $g\tilde{\alpha} \cap \tilde{\alpha} = \emptyset$,

but $g\tilde{\gamma} \cap \tilde{\gamma} \neq \emptyset$.

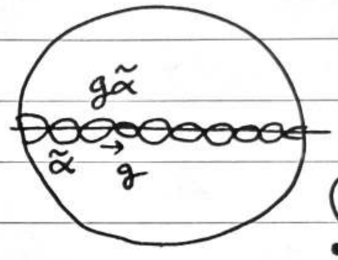
Case analysis (see picture)

$\Rightarrow g\tilde{\gamma} = \tilde{\gamma}$

i.e. $g\,\text{Axis}(\gamma) = \text{Axis}(\gamma)$

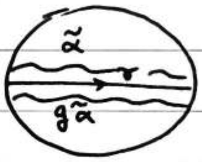


? weirdly presented: g ~~translates~~ in axis(γ)



so $g\tilde{\alpha}$ has to cross $\tilde{\alpha}$ (using g orientation-preserving)

NOT:



☐

Lemma 4.5 If α is an essential scc on a hyperbolic S then α is primitive. Moreover,

$$\langle \alpha \rangle = \underset{\substack{\uparrow \\ \text{centraliser}}}{C(\alpha)} \leq \pi_1 S$$

Proof Lemma 3.2 + Lemma 4.4 $\Rightarrow$ wlog α is geodesic

Now $\alpha \curvearrowright \text{Axis}(\alpha) \subseteq \mathbb{H}^2$ as translation by $\tau(\alpha)$.

Recall that

$$\text{Axis}(\alpha) = \{ x \in \mathbb{H}^2 \mid d(\alpha x, x) = \tau(\alpha) \}.$$

Consider $g \in C(\alpha)$. For any $x \in \text{Axis}(\alpha)$,

$$\begin{aligned} d(gx, \alpha g x) &= d(gx, g\alpha x) \\ &= d(x, \alpha x) \\ &= \tau(\alpha) \end{aligned}$$

$\therefore gx \in \text{Axis}(\alpha)$

In conclusion $C(\alpha) \curvearrowright \text{Axis}(\alpha)$.

Therefore, $\alpha : S^1 \hookrightarrow S^\#$ can be factored:

$$\begin{tikzcd} S^1 \arrow[r, hook, "\alpha"] \arrow[d, phantom, "\parallel"] & S^\# \\ \langle\alpha\rangle\backslash\text{Axis}(\alpha) \arrow[dr, "j"'] & \\ & C(\alpha)\backslash\text{Axis}(\alpha) \arrow[uu] \end{tikzcd}$$

Therefore the covering map j is injective

$\therefore |C(\alpha):\langle\alpha\rangle| = 1$

$\therefore \langle\alpha\rangle = C(\alpha)$. $\square$

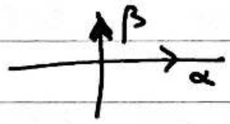
<u>Def^n 4.6</u> Let α, β be sccs on S.

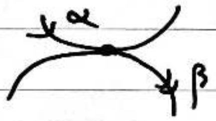
Their <u>(geometric) intersection number</u> is

$$i(\alpha,\beta) := \min_{\substack{\alpha' \simeq \alpha \\ \beta' \simeq \beta}} \#(\alpha' \cap \beta')$$

We may always assume that

α, β intersect <u>transversely</u>:

- $\#(\alpha \cap \beta) < \infty$ AND
- 

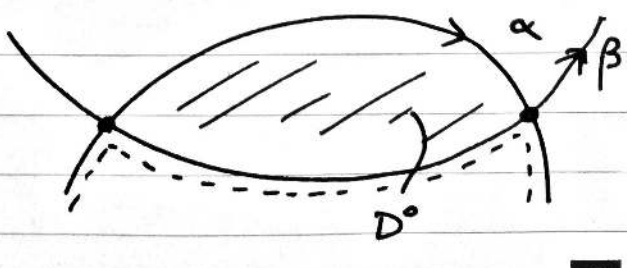
NOT 

<u>Def^n</u> If $i(\alpha,\beta) = \#(\alpha \cap \beta)$ then we say they are in <u>minimal position</u>

<u>4.2 Bigons</u>

<u>Def^n 4.7</u> Let α, β be transverse sccs, on S. A <u>bigon</u> is an embedded disc $D \hookrightarrow S$ s.t.

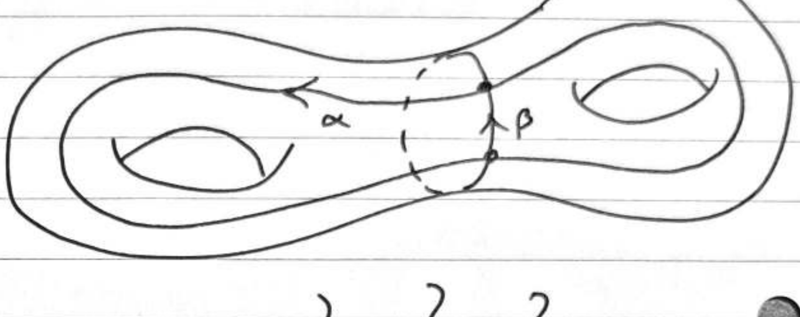
- $(\alpha \cup \beta) \cap D^\circ = \phi$
- $\partial D = a \cup b$ where $a \cap b = \{2 \text{ pts}\}$ and $a \subseteq \alpha$, $b \subseteq \beta$

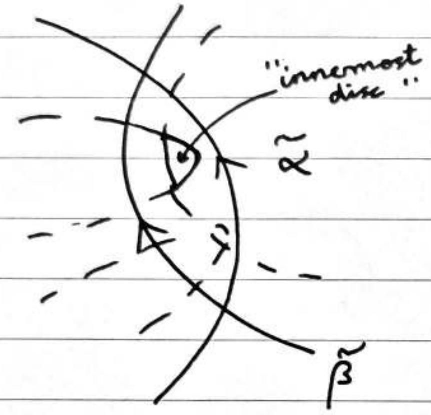


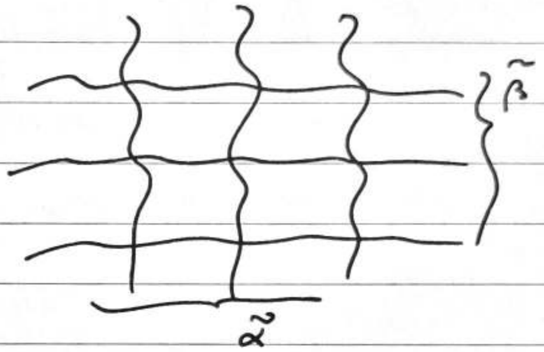
Lemma 4.8 If α, β are transverse essential sccs on an orientable surface S without bigons, then any pair of lifts $\tilde{\alpha}, \tilde{\beta}$ to $\tilde{S}$ intersect at most once.

Pf Let $\tilde{\alpha}, \tilde{\beta}$ be a pair of lifts to $\tilde{S}$, planar surface.

Suppose $\tilde{\alpha}, \tilde{\beta}$ intersect at least twice.







Passing to an "innermost disc", find a bigon $D \hookrightarrow \tilde{S}$.

Claim: $D \hookrightarrow \tilde{S} \longrightarrow S$ is injective

i.e. $\forall\, g \in \pi_1 S,\ g(D) \cap D \neq \emptyset \Rightarrow g = 1 \in \pi_1 S$

$\therefore$ suppose $g(D) \cap D \neq \emptyset$

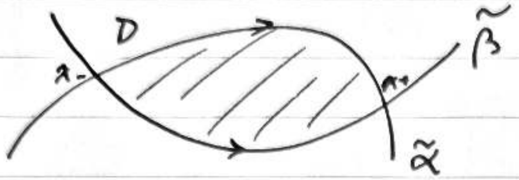
Think about $g(\partial D)$ which divides $\tilde{S}$ into two pieces.

So there are 2 cases:

① $D \subset g(D)$

② $D \cap g(\partial D) \neq \emptyset$

Proof of Lemma 4.8 (cont.)



$\hookrightarrow \tilde S \longrightarrow S$

Claim $D \longrightarrow S$ is an embedding
i.e. $\forall\, g \in \pi_1 S$, $gD \cap D \neq \emptyset$ then $g = 1_{\pi_1 S}$

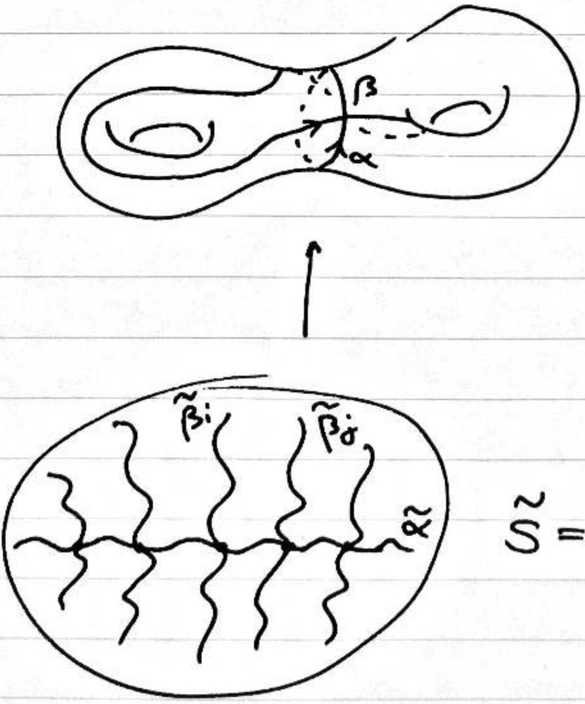
Pf Since g preserves "colours",
$g(x_-) = x_-$ or x_+
If $g(x_-) = x_+$ then g is not orientation-preserving $\#$
$\therefore g(x_-) = x_- \Rightarrow g = 1_{\pi_1 S}$ since $\pi_1 S \curvearrowright \tilde S$ freely □

The bigon criterion & topological type

5.1 The bigon criterion

Propⁿ 5.1 (The bigon criterion)
Transverse, essential SCCS α, β in S are in minimal position $\Longleftrightarrow$ there are no bigons

Proof "$\Rightarrow$" obvious
"$\Leftarrow$" give the proof in the case S is closed and hyperbolic.



Look at a single fixed lift $\tilde\alpha$ of α and all lifts $\{\tilde\beta_i\}$ of β that intersect $\tilde\alpha$.

$\tilde S = \mathbb{H}^2$

Note that

$\mathbb{Z} \cong \langle \alpha \rangle$ acts on the picture

and $\#(\alpha \cap \beta) = \#\ \mathbb{Z}$-orbits of $\{\tilde{\beta}_i\}$

It remains to prove:

modifying α, β by homotopies doesn't change whether or not a given pair of lifts $\tilde{\alpha}, \tilde{\beta}$ intersect.

Since S is closed, α, β essential,

$\alpha, \beta \curvearrowright \mathbb{H}^2$ as hyperbolic isometries

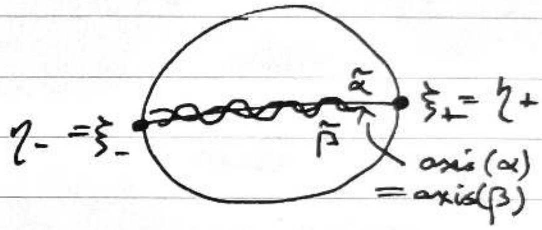
Note that $\lim_{t \to \pm\infty} \tilde{\alpha}(t) = \xi_\pm \in \partial \mathbb{H}^2$

where $\text{axis}(\tilde{\alpha}) = (\xi_-, \xi_+)$.

Similarly, $\text{axis}(\beta) = (\eta_-, \eta_+)$

conj? → where $\eta_\pm = \lim_{t \to \infty} \tilde{\beta}(t)$

<u>Claim</u> $\{\xi_\pm\} \cap \{\eta_\pm\} = \phi$



<u>Pf</u> If $\{\xi_\pm\} = \{\eta_\pm\}$

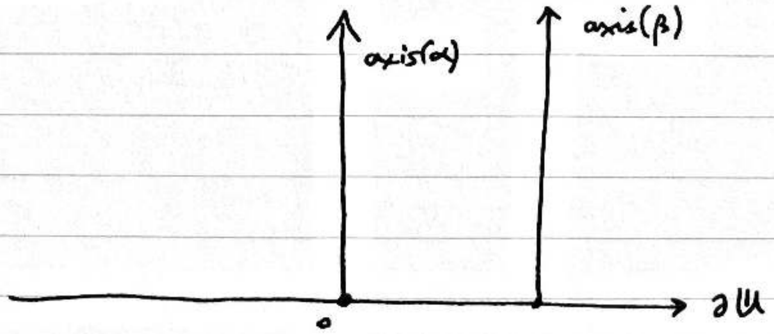
then $\text{axis}(\alpha) = \text{axis}(\beta)$

$\therefore\ \#\ \tilde{\alpha} \cap \tilde{\beta} = \infty$ because it has free $\mathbb{Z}$-action

If say $\xi_+ = \eta_+,\ \xi_- \neq \eta_-$

then wlog $\xi_+ = \eta_+ = \infty \in \overline{\mathbb{U}}$

& picture is:



⌜conjugate $z \mapsto z + c$ by α to get $z \mapsto z + \text{small } c$ ⨳ to discrete $\pi_1 S$ action⌟

Explicit computation:

$\alpha \beta \alpha^{-1} \beta^{-1}(z)$

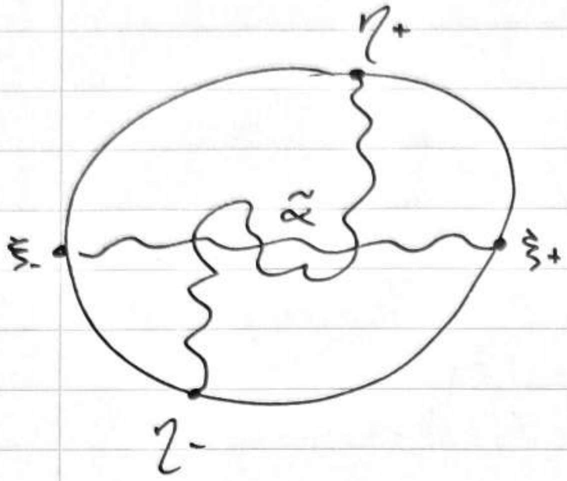
$= z + c$

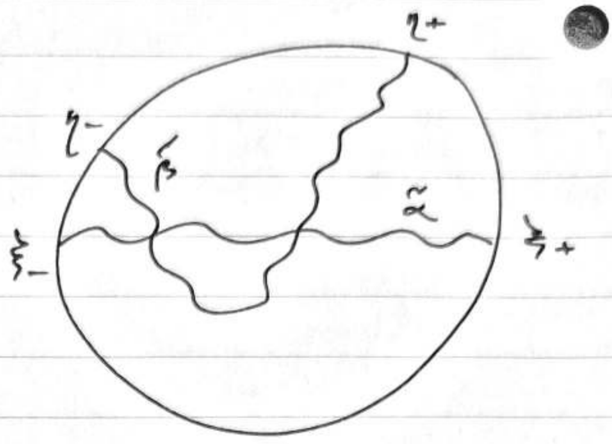
is parabolic

$\Rightarrow S$ has punctures

⨳

Next, note that the intersection pattern of $\{\xi_\pm\}, \{\eta_\pm\}$ in $\partial \mathbb{H}^2$ determines the parity of $\#(\tilde{\alpha} \cap \tilde{\beta})$.





Since there are no bigons, by Lemma 4.8,

$\{\xi_\pm\}, \{\eta_\pm\}$

determine whether or not $\tilde{\alpha}$ & $\tilde{\beta}$ intersect.

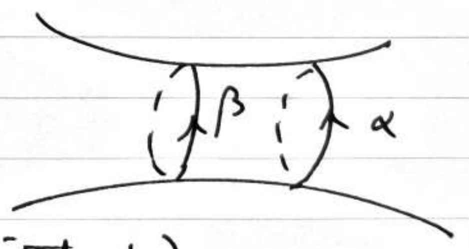
Finally, modifying α, β via homotopies moves $\tilde{\alpha}, \tilde{\beta}$ bounded distances, so keeps $\eta_\pm, \xi_\pm$ fixed. □

Corollary 5.2 If α, β are distinct simple closed geodesics on hyperbolic S then α, β are in minimal position.

5.2 The annulus criterion

Prop^n 5.3 (Annulus criterion)

Let α, β be essential disjoint simple closed curves on S. If $\alpha \simeq \beta$ then α & β together bound an embedded annulus in S.



Proof See notes. □

Pf of Lemma 4.2

$(\underset{\text{essential}}{\alpha_0} \simeq \alpha_1 \iff \alpha_0, \alpha_1$ ambient isotopic$)$

Make α_0, α_1 transverse.

By induction, on $\#(\alpha_0 \cap \alpha_1)$,

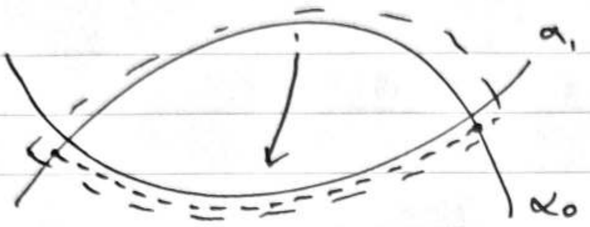
base case $\#(\alpha_0 \cap \alpha_1) = 0$: use annulus criterion

Inductive step

If $\#(\alpha_0 \cap \alpha_1) > 0$ but $\alpha_0 \simeq \alpha_1 \Rightarrow i(\alpha_0, \alpha_1) = 0$.

So not in minimal position.

So $\exists$ bigon.



reparametrise in disc

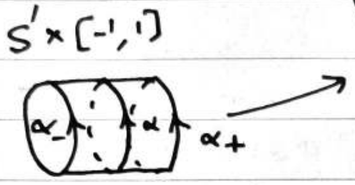
Therefore, can reduce $\#(\alpha_0 \cap \alpha_1)$ by an ambient isotopy. $\square$

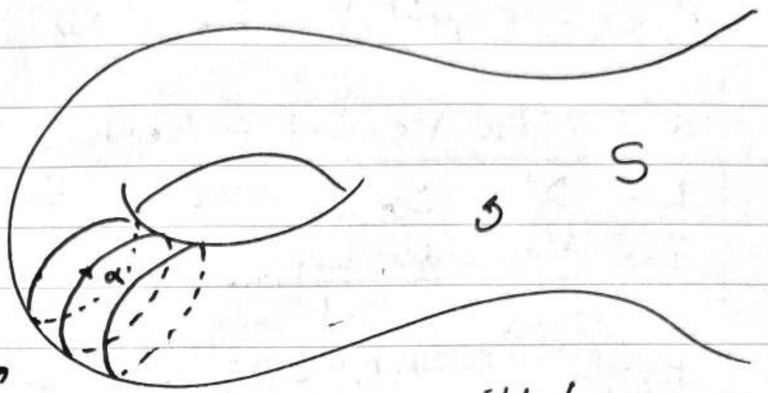
5.3 Topological Type

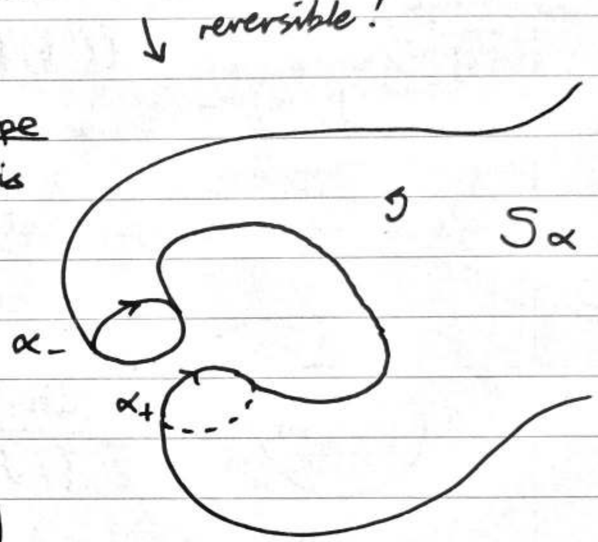
Def^n 5.4 Any essential SCC α on S has a (closed) annulus neighbourhood $\overline{N(\alpha)} \cong S^1\times[-1,1] \hookrightarrow S$

The cut surface

$$S_\alpha = S \setminus \underset{\overset{\uparrow}{\text{int } \overline{N(\alpha)}}}{N(\alpha)}$$



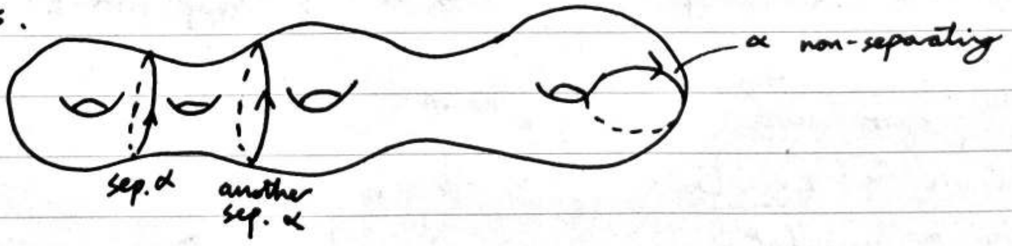




Def^n 5.5 The topological type of a SCC on a surface S is the homeo class of the cut surface S_α.
If S_α is connected, say α is non-separating.

Note: $$\chi(S_\alpha) = \chi(S) - \chi(\overline{N(\alpha)}) + 2\underbrace{\chi(S^1)}_{\text{zero}}$$

Example 5.6 $S = S_g = S_{g,0,0}$
There is 1 topological type of (essential) non-separating curve α and $\lfloor \frac{g}{2} \rfloor$ top type of essential separating curves.

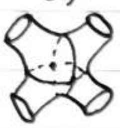


6. Change of coordinates & the Alexander Lemma

6.1 Change of coordinates

[do we need to assume non-separating]

c.f.



Prop 6.1 (Change of coordinates) Essential sccs α, β on S have the same topological type iff there is an orientation preserving homeo $\phi: S \to S$ (fixing ∂S) s.t. $\phi \circ \alpha = \beta$.

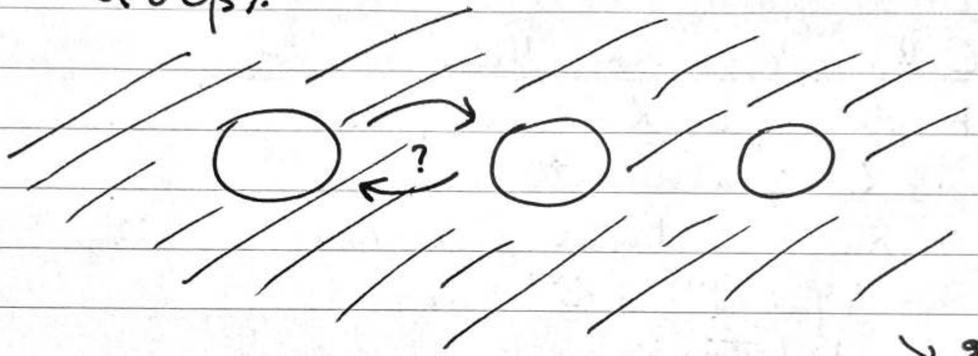
Proof "$\Leftarrow$" Easy

"$\Rightarrow$" Suppose $\phi: S_\alpha \xrightarrow{\cong} S_\beta$ exists.

Every S_β admits an orientation reversing homeo ι.

So wlog ϕ is orientation-preserving.

Next note that Homeo(S_β) acts as the full symmetry group in $\pi_0(\partial S_\beta)$.



↘ genus, punctures etc.

Therefore wlog ϕ preserves ∂S and sends $\alpha_\pm \to \beta_\pm$.

Now extend $\phi: S_\alpha \to S_\beta$

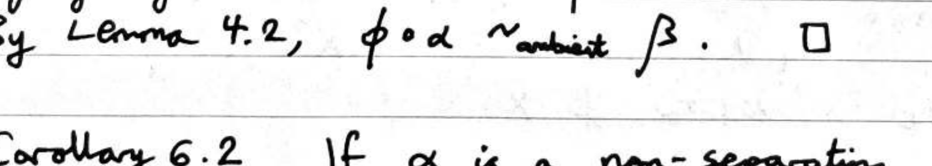
to $\phi: S \to S$

by gluing in $\overline{N(\alpha)} \to \overline{N(\beta)}$.

By Lemma 4.2, $\phi \circ \alpha \sim_{\text{ambient}} \beta$. $\square$

Corollary 6.2 If α is a non-separating simple closed curve on a closed surface S then $\exists \beta$ on S s.t. $i(\alpha, \beta) = 1$.

Pf: By change of coordinates, $\exists \phi: S \to S$ s.t. $\phi \circ \alpha$ is this curve here:



Now $i(\alpha \beta) = i(\phi \cdot \alpha, \phi \cdot \beta)$.

Let $\beta = \phi^{-1} \circ \beta'$. $\square$

Also applies to pairs of curves:

⌜cut twice⌟

Prop 6.3 Suppose that α_1, β_1 and α_2, β_2 are pairs of sccs on S with $i(\alpha_1, \beta_1) = i(\alpha_2, \beta_2) = 1$

Then $\exists\, \phi \in \mathrm{Homeo}^+(S, \partial S)$ s.t. $\phi \circ \alpha_1 = \alpha_2,\ \phi \circ \beta_1 = \beta_2$

Proof See printed notes. □

6.2 The Alexander Lemma

Let $D^2 = S_{0,0,1}$, the closed disc.

The Alexander Lemma:

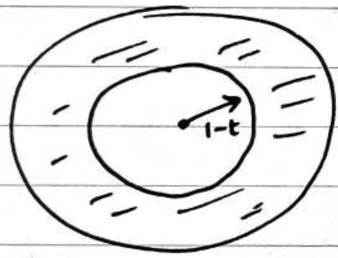
$\mathrm{Mod}(D^2) \cong 1$

Proof Let $f : D^2 \to D^2$ be a homeo s.t. $f|_{\partial D^2} = id$

Then

$$f_t(x) = \begin{cases} (1-t) f\left(\frac{x}{1-t}\right), & 0 \leq |x| \leq 1-t \\ x, & |x| \geq 1-t \end{cases}$$

defines an isotopy between $f = f_0$, $id = f_1$ □



The same proof gives

Lemma 6.5 $\mathrm{Mod}(D^2_* = S_{0,1,1}) \cong 1$

(c.f. Sheet 1 q 1)

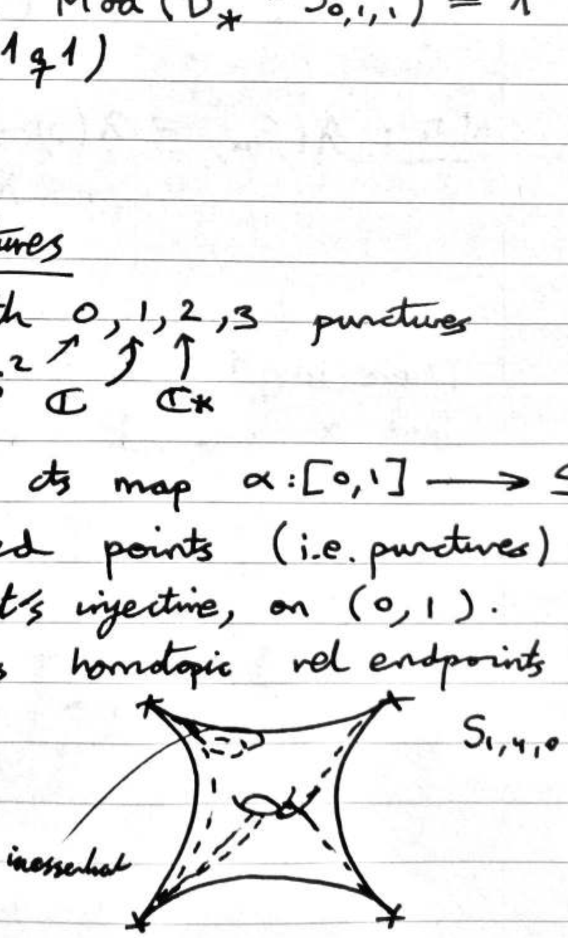
6.3 Spheres with few punctures

Compute MCG of spheres with $\underset{S^2}{0}, \underset{\mathbb{C}}{1}, \underset{\mathbb{C}^*}{2}, 3$ punctures

Def^n 6.6 A (proper) arc is a cts map $\alpha : [0,1] \longrightarrow S$ s.t. $\alpha(0)$, $\alpha(1)$ are marked points (i.e. punctures) on S.

A proper arc is simple if it's injective, on $(0,1)$.

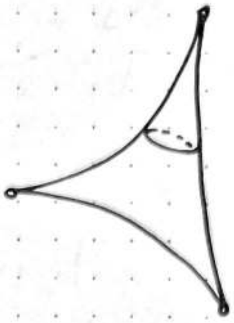
An arc is essential unless it is homotopic rel endpoints into a puncture.



Lemma 6.7

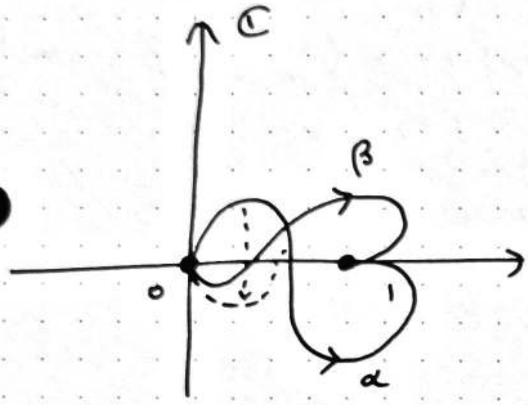
If α, β are (essential) simple proper arcs, on $S_{0,3,0}$, each with distinct endpoints s.t. the endpoints of α = the endpoints of β. Then $\alpha \underset{\text{isot}}{\simeq} \beta$.

$S = S_{0,3,0}$



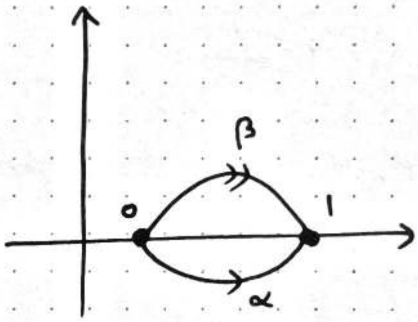
Pf $S_{0,3,0} \cong \mathbb{C} \setminus \{0,1\}$

Assume that α, β are transverse.



By an innermost disk argument, α, β bound a bigon. If $\#(\alpha, \beta) > 2$, then pushing over this bigon reduces $\#(\alpha, \beta)$.

If $\alpha, \beta = 2$ then the $\alpha \cup \beta$ bound a disc in $\mathbb{C}$ and $\alpha \underset{\text{iso}}{\simeq} \beta$.



□

Note: For any $S_{g,n,b}$, the action on the punctures defines a homomorphism $\mathrm{Mod}(S_{g,n,b}) \to \mathrm{Sym}(n)$.

This map is always surjective.

Prop 6.8 The natural hom

$$\mathrm{Mod}(S_{0,3,0}) \to \mathrm{Sym}(3)$$

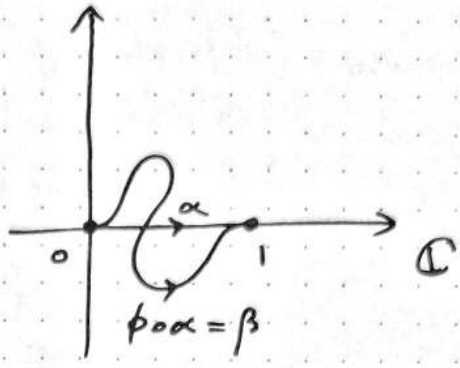
is an isomorphism.

Pf ETS that the map is injective.

That is, given $\phi \in \mathrm{Homeo}^+(S_{0,3,0})$ fixing all the punctures, we need to prove that $\phi \underset{\text{iso}}{\simeq} \mathrm{id}_{S_{0,3,0}}$.

Let $\alpha : [0,1] \longrightarrow \overset{\mathbb{C}\setminus\{0,1\}}{\overset{\shortparallel}{S_{0,3,0}}}$ be the "standard" arc.

$\alpha(t) = t \in \mathbb{C}$,



Lemma 6.7 $\Rightarrow$ $\phi\circ\alpha \underset{\text{isd}}{\simeq} \alpha$

$\therefore$ after modifying ϕ by an ambient isotopy, wlog $\phi\circ\alpha = \alpha$.

$\therefore$ ϕ induces a homeo

$\bar{\phi}$ of the cut surface $S_\alpha \cong D^2_*$

$\underset{\text{Alexander Lemma}}{\Longrightarrow}$ $\bar{\phi} \underset{\text{iso}}{\simeq} \mathrm{id}_{D^*_2}$

Regluing along α we get that $\phi \underset{\text{iso}}{\simeq} \mathrm{id}_{S_{0,3,0}}$ $\square$

<u>Cor 6.9</u> If $S \cong S_{0,n,0}$ for $n \le 3$ then $\mathrm{Mod}(S_{0,n,0}) \longrightarrow \mathrm{Sym}(n)$ is an iso.

In particular $\mathrm{Mod}(S^2) \cong 1$

$\mathrm{Mod}(\mathbb{C}) \cong 1$

$\mathrm{Mod}(\mathbb{C}_*) \cong \mathbb{Z}/2$.

<u>Pf</u> See notes. $\square$

7. Infinite Mapping Class Groups

7.1 The annulus

$$A \cong [0,1]\times S^1 \cong S_{0,0,2}$$

<u>Prop 7.1</u> $\mathrm{Mod}(A) \cong \mathbb{Z}$

<u>Coordinates on A</u>

$S^1 \underset{\text{standard}}{\hookrightarrow} \mathbb{C}$ ← standard orientation

Identify $A \cong [0,1]\times S^1$.

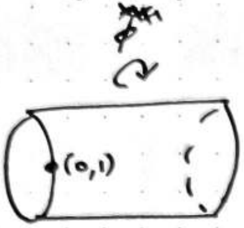
In these coordinates; the universal cover is

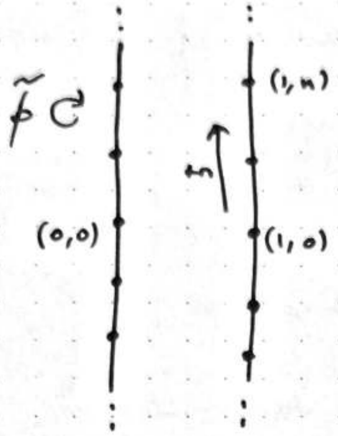
$$\tilde{A} \cong [0,1]\times\mathbb{R} \to [0,1]\times S^1 \cong A$$
$$(x\,,\,y) \mapsto (x, e^{2\pi i\cdot y})$$

Proof of prop

Consider $\phi : A \longrightarrow A$ s.t. $\phi|_{\partial A} = \mathrm{Id}_{\partial A}$

Let $\tilde\phi : \tilde A \longrightarrow \tilde A$ be the unique lift of ϕ that fixes $(0,0) \in \tilde A$





Let $\tilde\phi_1$ be the restriction of $\tilde\phi$ to the line $\{1\} \times \mathbb{R}$

Note that $\tilde\phi_1$ lifts $\mathrm{id}_{\{1\}\times S^1}$

Hence $\tilde\phi_1$ must send

$(1, y) \longmapsto (1,+y n)$, some fixed $n \in \mathbb{Z}$

Since n varies continuously with ϕ, if $\phi \simeq \psi$ then ψ is also associated to n.

So we have defined a map

$\mathrm{Mod}(A) \to \mathbb{Z}$.

To show the map is a homomorphism, uniqueness of lifts

$$\Downarrow$$

$$\tilde\phi \circ \tilde\psi = \widetilde{\phi \circ \psi}$$

Therefore $\tilde\phi_1 \circ \tilde\psi_1 = \widetilde{(\phi\circ\psi)}_1$

$\Rightarrow$ get a homomorphism.

Surjective Let $n \in \mathbb{Z}$ and consider

$$\tilde\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Note $\tilde\phi(\tilde A) = \tilde A$ is a homeo $\tilde A \to \tilde A$

Since $\tilde\phi((x,y) + \{0\}\times\mathbb{Z}) = \tilde\phi(x,y) + \{0\}\times\mathbb{Z}$

$\Rightarrow \tilde\phi$ descends to a homeo ϕ of A.

Note that $\tilde\phi(1,0) = (1,n)$

$\Rightarrow [\phi] \mapsto n$ under $\mathrm{Mod}(A) \to \mathbb{Z}$

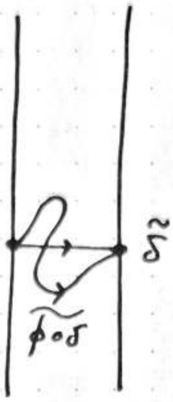
Injectivity Let $\delta(t) = (t,1)$

$[0,1] \to A$

Suppose
$$\begin{tikzcd} {[\phi]} \arrow[drr, bend right] & \in \mathrm{Mod}(A) \arrow[r] & \mathbb{Z} \\ & & 0 \end{tikzcd}$$

Consider $\tilde{\phi}$, $\tilde{\delta}$, $\widetilde{\phi \circ \delta}$

arcs in $\tilde{A}$, both start at $(0,0)$, end at $(1,0)$



$\therefore$ as before, they bound a bigon and we may iteratively modify $\underline{\underline{\phi}}$ via isotopy until $\phi \circ \delta = \delta$.

Cut along δ.

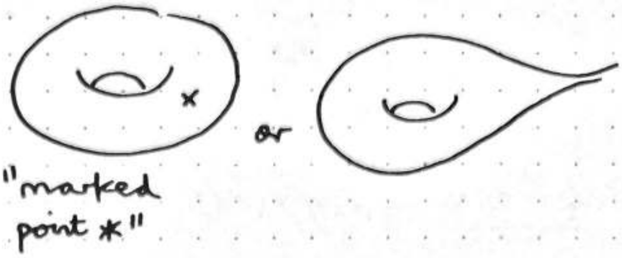
Induced $\bar{\phi} : A_\delta \to A_\delta$ isotopic to id_{A_δ} by Alexander Lemma.

Reglue, $\phi \simeq \mathrm{id}$

□

7.2 The punctured torus

Compute $\mathrm{Mod}(\underset{\uparrow S_{1,1,0}}{T^2_*})$



$\phi \in \mathrm{Homeo}^+(T^2_*)$ can be interpreted as a homeo of T^2 fixing $*$

$\therefore$ ϕ defies $\phi_* \in \mathrm{Aut}(\pi_1(T^2, *))$

$$= \mathrm{Aut}(\mathbb{Z}^2) \cong GL_2(\mathbb{Z})$$

We obtain a homomorphism

$$\mathrm{Mod}(T^2_*) \longrightarrow GL_2(\mathbb{Z})$$

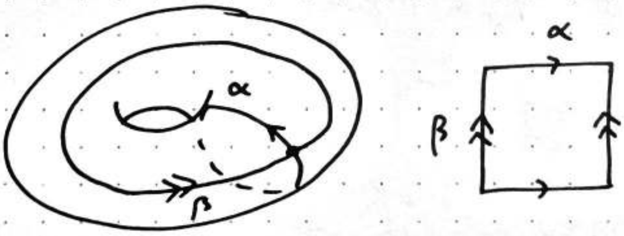
Thm 7.2 The above hom induces an isomorphism

$$\mathrm{Mod}(T^2_*) \xrightarrow{\sim} SL_2(\mathbb{Z})$$

Pf Need to show that the above hom is injective & surjects onto $SL_2\mathbb{Z}$.

(Injectivity)

Consider ϕ an element of the kernel of $\mathrm{Mod}(T^2_*) \to SL_2\mathbb{Z}$.

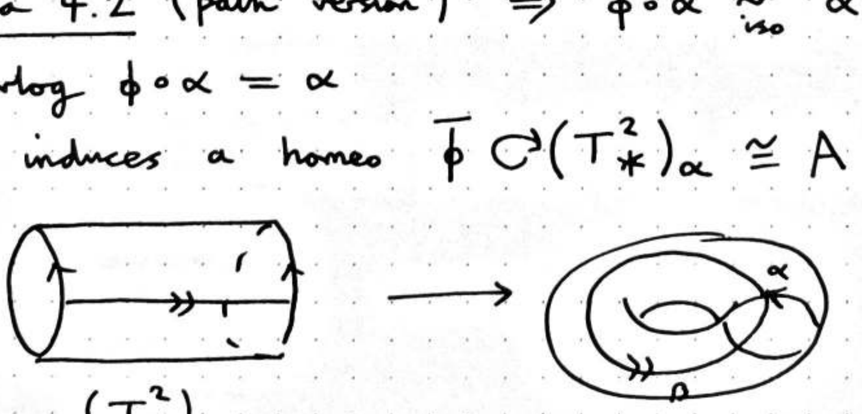


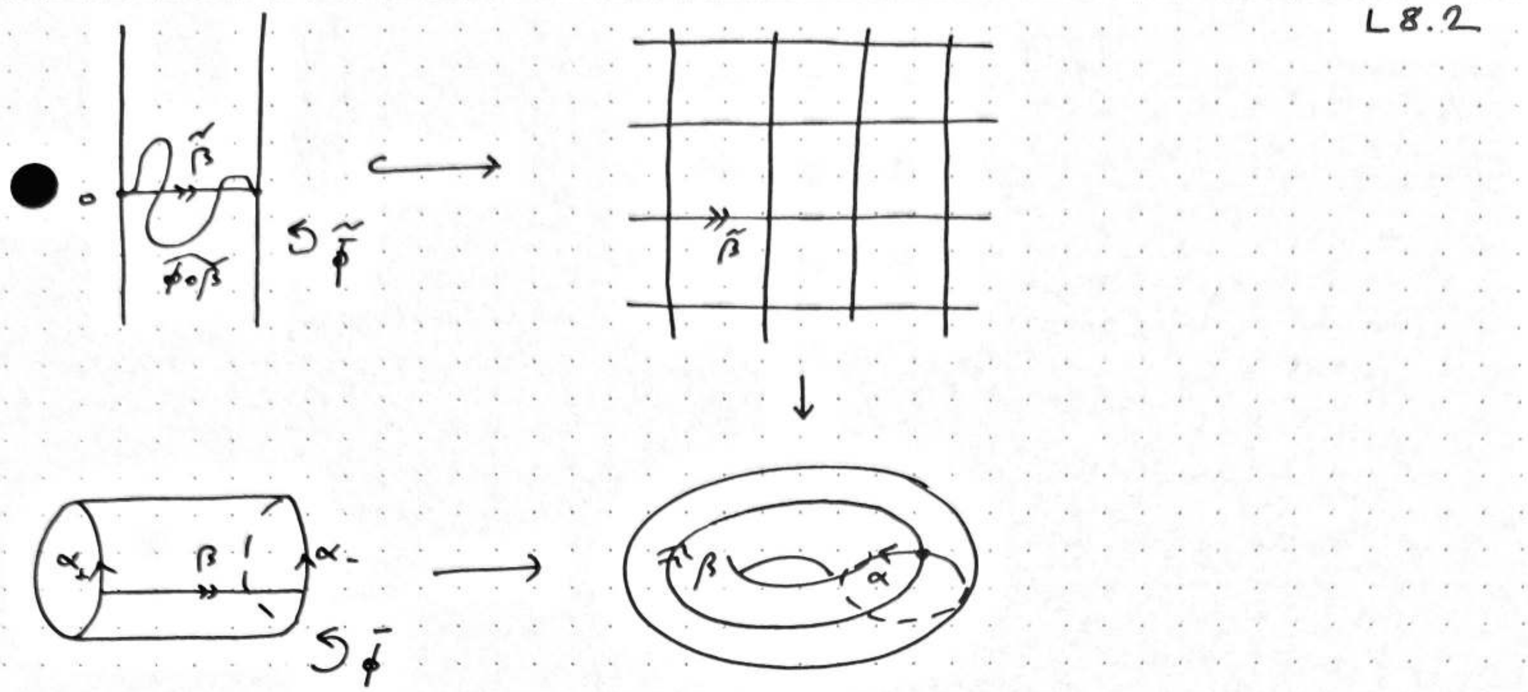
So $\phi \circ \alpha \simeq \alpha$, $\phi \circ \beta \simeq \beta$ (as paths / in $\pi_1(T^2, *)$)

Lemma 4.2 (path version) $\Rightarrow$ $\phi \circ \alpha \underset{\text{iso}}{\sim} \alpha$

So wlog $\phi \circ \alpha = \alpha$

$\therefore$ ϕ induces a homeo $\bar{\phi} \circlearrowright (T^2_*)_\alpha \cong A$





Since $\phi\circ\beta \simeq \beta$, HLL ⇒ $\bar\phi = 0$ in ~~Mod(T^2_*~~ $\mathrm{Mod}((T^2_*)_\alpha)$

So $\tilde\phi \simeq \mathrm{id}$ on $(T^2_*)_\alpha$

Reglue to get $\phi \simeq \mathrm{id}$ on T^2_*. //

(Image of $\mathrm{Mod}(T^2_*) \hookrightarrow GL_2\mathbb{Z}$)

If $A \in SL_2\mathbb{Z}$ then can define $\phi_A \in \mathrm{Homeo}^+(T^2_*)$ by

$\tilde\phi_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

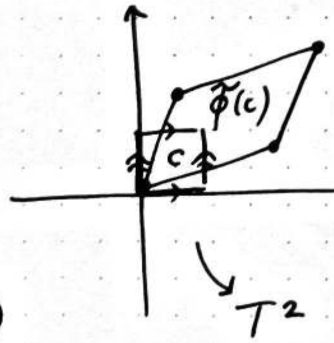
$\begin{pmatrix}x\\y\end{pmatrix} \mapsto A\begin{pmatrix}x\\y\end{pmatrix}$, descending to T^2_*. ⌜$* = 0 + \mathbb{Z}^2$⌟

And $(\phi_A)_* = A$.

Finally, suppose $\phi \mapsto A$ in $GL_2\mathbb{Z}$.

Consider the action of ϕ_* on $H_2(T^2) \cong \mathbb{Z}$

Computation: ϕ_* is mult^n by $(\det A)$



count area etc

□

8.1 The torus

T^2 has a special feature :

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 \quad \text{is a group !}$$

Corollary 8.1 $\text{Mod}(T^2) \cong SL_2\mathbb{Z}$

Proof There is a natural map / hom

$$\text{Mod}(T^2_*) \longrightarrow \text{Mod}(T^2)$$

Given by FORGETTING A PUNCTURE

Let's take $* = 0$

Let's prove this is a group iso

(Surjective) Suppose $\phi \in \text{Homeo}^+(T^2)$

Let $\alpha : [0,1] \to T^2$ be any its path from 0 to $\phi(0)$.

Now $\phi_t(x) = \phi(x) - \alpha(t)$

is an isotopy from ϕ to a homeo fixing 0.

(Injective) Let $\phi_0, \phi_1 \in \text{Homeo}^+(T^2)$

By surjectivity, wlog $\phi_0(0) = \phi_1(0) = 0$.

Suppose ϕ_t is an isotopy in T^2 from ϕ_0 to ϕ_1.

Then $\beta(t) = \phi_t(0)$ is a path from $\phi_0(0) = 0$ to $\phi_1(0) = 0$.

(a loop in T^2 based at 0)

Set $\phi_t'(x) = \phi_t(x) - \beta(t)$.

Then ϕ_t' is an isotopy in T^2_* from ϕ_0 to ϕ_1. ☐

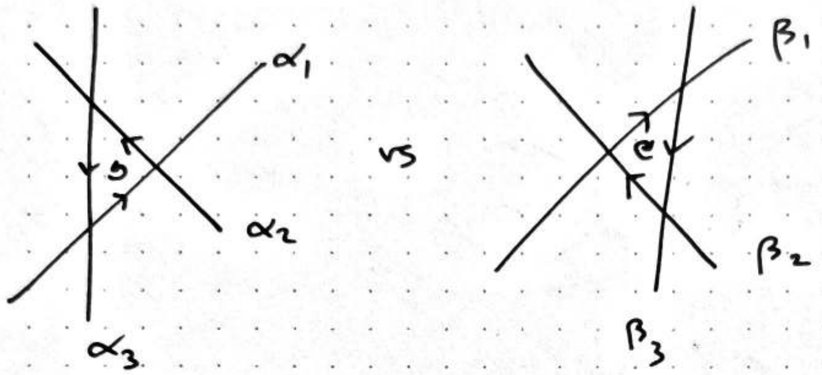
EX class 1 stuff on Moodle

8.2 Pairwise isotopy

If α is an essential scc and $\phi \circ \alpha \underset{iso}{\not\simeq} \alpha$ then $\phi \neq e \in \mathrm{Mod}(S)$.

Goal: Analyze actions of ϕ on a large set of curves / arcs in order to certify when $\phi \underset{iso}{\sim} id_S$.

But, there's a problem:



Fortunately, we shall see that this is the only problem.

Also, let's write $\alpha \approx \beta$ if $\exists f \in \mathrm{Homeo}(I)$ or $\mathrm{Homeo}(S^1)$ (as appropriate) s.t. $\alpha = \beta \circ f$.

Lemma 8.3 Let S be a surface, and $\{\alpha_1, \dots, \alpha_n\}$, $\{\beta_1, \dots, \beta_n\}$ two collections of essential sccs or proper arcs.

Suppose that:

(i) $\{\alpha_i\}$ are pairwise in minimal position (NO BIGONS)
(& the same for the $\{\beta_i\}$)

(ii) $\{\alpha_i\}$ are pairwise non-isotopic (NO ANNULI)
(& same for the $\{\beta_i\}$)

(iii) for distinct indicies i, j, k at least one of
$\alpha_i \cap \alpha_j$, $\alpha_j \cap \alpha_k$, $\alpha_k \cap \alpha_i$ is empty (NO TRIANGLES)
(& same for the $\{\beta_i\}$)

Then if $\alpha_i \underset{iso}{\simeq} \beta_i$ for all i, then $\exists$ ambient isotopy ϕ of S s.t. $\alpha_i \approx \phi \circ \beta_i$ for all i.

Proof Induction on n.

Base case: $n=1$ Lemma 4.2

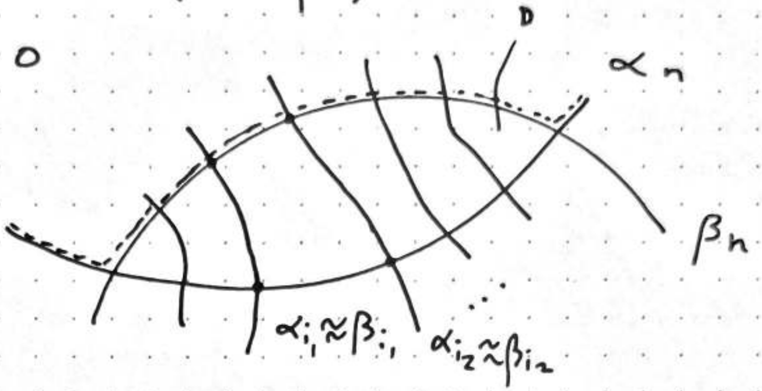
Inductive step: wlog $\alpha_i \approx \beta_i$ for all $i<n$

Task: make $\alpha_n \approx \beta_n$

Proceeds by induction on $\#(\alpha_n \cap \beta_n)$.

Suppose $\#(\alpha_n \cap \beta_n) > 0$

$\Rightarrow \exists$ bigon $D \hookrightarrow S$



No bigons

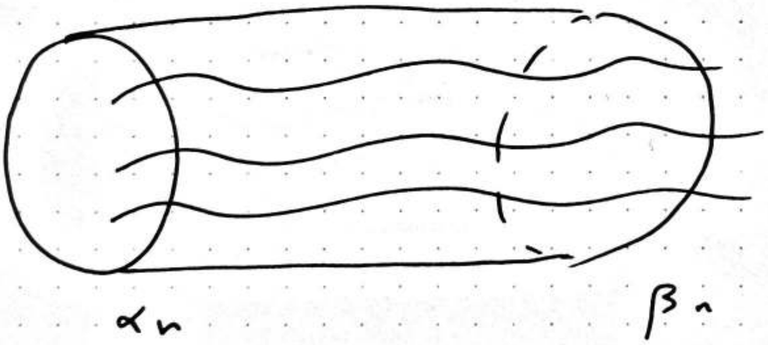
+ No triangles

$\Downarrow$

the picture of D looks like so: ⇢

$\therefore$ can push α_n over D, reparametrising the α_i, $i<n$ reducing $\#(\alpha_n \cap \beta_n)$

If $\alpha_n \cap \beta_n = \phi$, then get a similar picture with an annulus A instead of a bigon D:



("no annuli" needed)

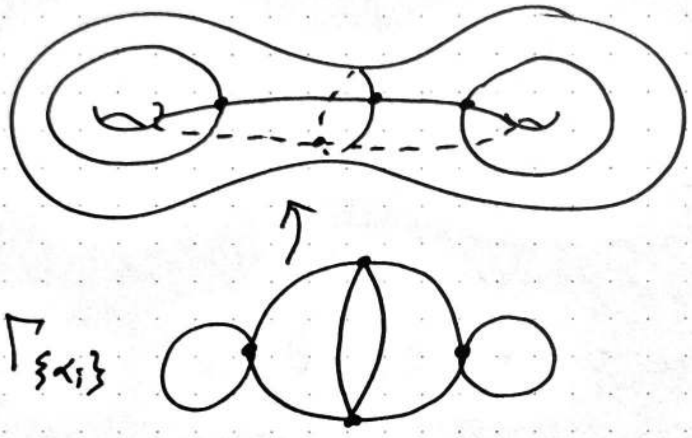
Pushing off A, reparametrising α_i, $i<n$ we get $\alpha_n \approx \beta_n$ as req^d □

8.3 The Alexander Method

Def^n 8.4 A transverse collection of simple curves & arcs $\{\alpha_i\}$ is said to *fill* a surface S if each component of the cut surface $S_{\{\alpha_i\}}$ is homeo to D^2 or D^2_*.

For such a collection, the *structure graph* $\Gamma_{\{\alpha_i\}}$ is the union of $\bigcup_i \alpha_i \cup \partial S$ with vertices placed at each intersection, and each puncture.

Example:



Prop^n 8.3 (The Alex. Method) Let $\{\alpha_i\}$ be a collection of n essential simple closed curves & proper arcs on S that fills, and with no bigons, no annuli, no Δ's.

(i) If $\phi \in \mathrm{Homeo}^+(S, \partial S)$ & $\sigma \in \mathrm{Sym}(n)$ are s.t. $\phi \circ \alpha_i \underset{iso}{\simeq} \alpha_{\sigma(i)}$ then ϕ induces an automorphism $\phi_\Gamma \in \mathrm{Aut}(\Gamma_{\{\alpha_i\}})$

(ii) If $\phi_\Gamma = id_\Gamma$ then $\phi \underset{iso}{\simeq} id_S$

In particular, under the hypotheses of (i), $o(\phi) < \infty$ in $\mathrm{Mod}(S)$.

Proof: (i) By Lemma 8.3, after an ambient isotopy, ϕ preserves $\bigcup_i \alpha_i \cup \partial S$ setwise

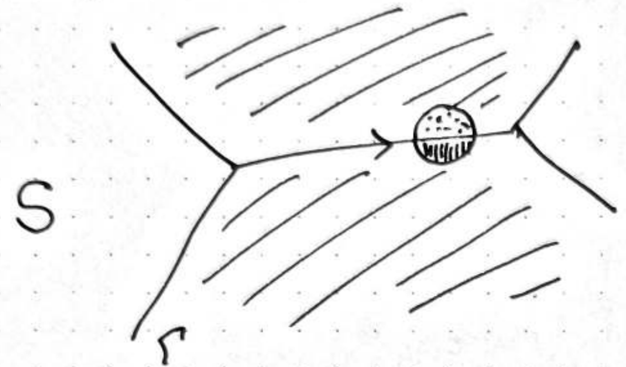
$\therefore$ ϕ induces ϕ_Γ

(⇐) Suppose $\phi_\Gamma = \mathrm{id}_{\Gamma_{\{\alpha_i\}}}$

Then after an ambient isotopy, $\phi|_\Gamma = \mathrm{id}_{\Gamma\{\alpha_i\}}$

Since ϕ is orientation preserving,

ϕ also preserves the components of $S_{\{\alpha_i\}}$.



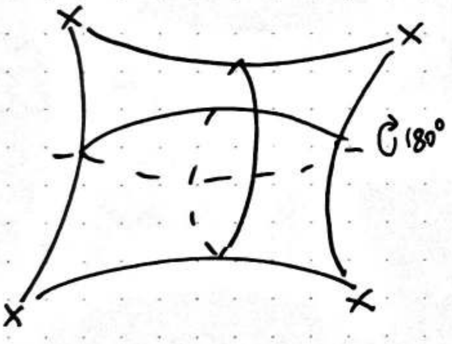
∴ Alexander Lemma

⇒ ϕ can be modified by an isotopy on each component, after which $\phi = \mathrm{id}_S$ □

Exercise 8.6 (= Q1? on sheet 2)

Such collections exist.

Minimal tricky example



∃ϕ which preserves the curves pairwise up to reparametrisation but is not iso to id

⌜b/c ϕ_Γ not the identity automorphism⌟

9. Dehn twists

9.1 Definition & their action on curves

The proof of Prop 7.1 told us:

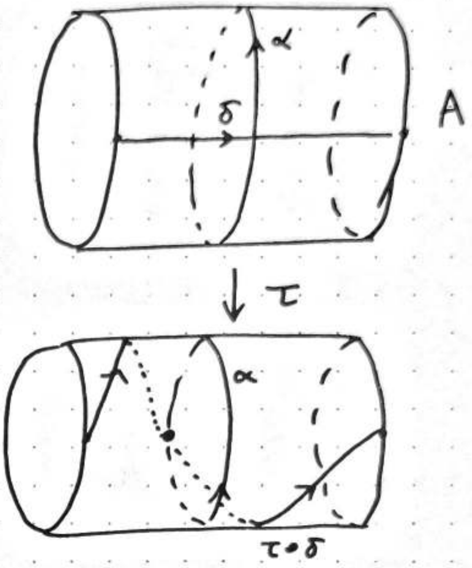
$$\mathrm{Mod}(A) \cong \mathbb{Z} \quad \text{generated by } [\tau]$$

where $\tau : A \to A \equiv [0,1] \times S^1$

$$(x, e^{2\pi i y}) \longmapsto (x, e^{2\pi i (x+y)})$$

<u>Def^n</u> The mapping class $[\tau] \in \mathrm{Mod}(A)$ is called a <u>left</u> Dehn twist.

<u>Remark</u> Switching the orientation on A replaces τ by the right Dehn twist τ^{-1}

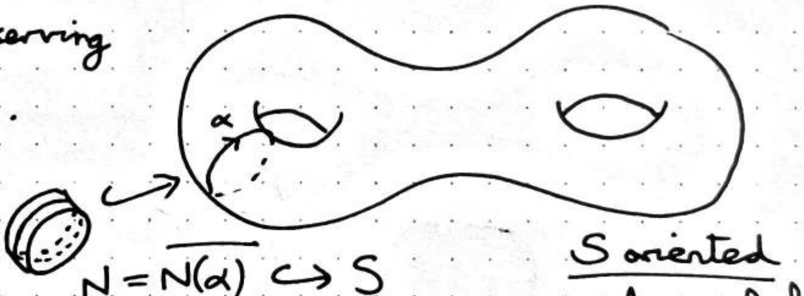


<u>Def^n</u> Let α be an essential scc on S and N an annular neighbourhood of α.

Choose an orientation-preserving homeomorphism $i : A \xrightarrow{\sim} N$.

Now the homeo

$$\tau_\alpha(x) = \begin{cases} i \circ \tau \circ i^{-1} ; & x \in N \\ x ; & x \notin N \end{cases}$$



$N = N(\alpha) \hookrightarrow S$

<u>S oriented</u> surface of finite type

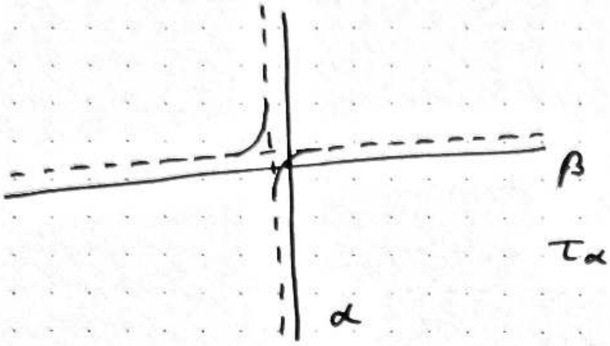
defines the (left) Dehn twist $\underset{T_\alpha}{\uparrow}$ in α, as the induced element of $\mathrm{Mod}(S)$.

<u>Remark</u> T_α is independent of orientation on α: $T_\alpha = T_{\alpha^{-1}}$

<u>Lemma 9.2</u> T_α is independent of the choices made in its definition, & also only depends on α up to isotopy.

<u>Pf</u> Routine, see notes. $\square$

To work with Dehn twists, we need to compute their actions on curves, which we do via surgery diagrams.



9.2 Order & intersection number

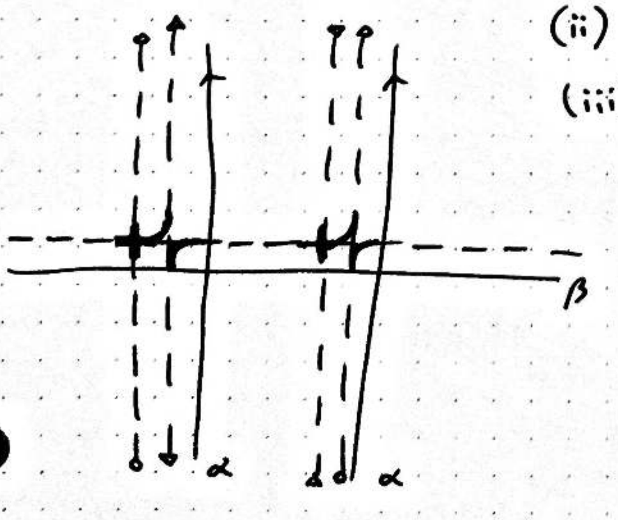
Prop 7.1 told us $T_\alpha \in \mathrm{Mod}(A)$ has ∞ order.
What about $T_\alpha \in \mathrm{Mod}(S)$ for arbitrary S?
Note that if $T_\alpha^n = 1$ for some n, then $T_\alpha^n \beta \underset{iso}{\sim} \beta$ for any essential scc β.

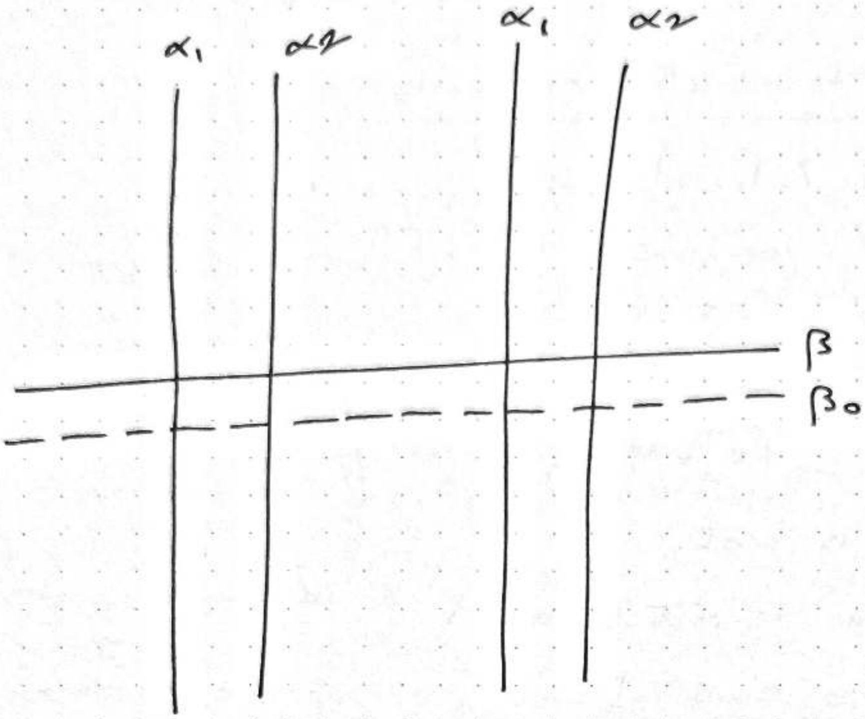
Lemma 9.3 Let α be an essential scc and β any essential scc or simple proper arc. Then

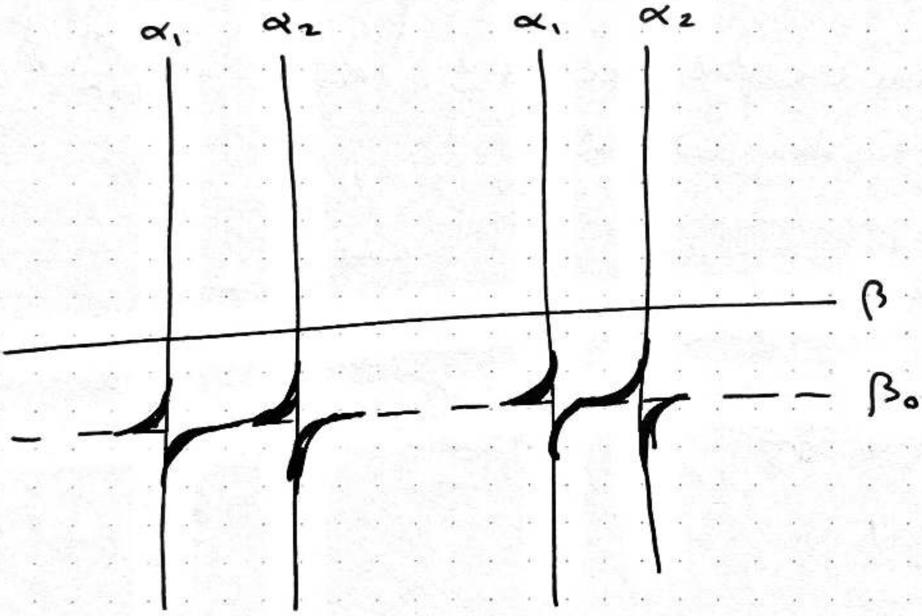
$$i(T_\alpha^k(\beta), \beta) = |k| \, i(\alpha,\beta)^2 \quad \text{for any } k \in \mathbb{Z}$$

Proof Wlog $\#(\alpha \cap \beta) = i(\alpha,\beta)$.
Let $\beta' = \tau_\alpha^k \circ \beta$, made transverse.
β' can be constructed explicitly as follows.

(i) Take $\beta_0 = \beta$ moved slightly to the left $\overset{\uparrow}{\text{(or right)}}$
(ii) Take $|k| \cdot i(\alpha,\beta)^2$ copies of α
(iii) Perform surgery



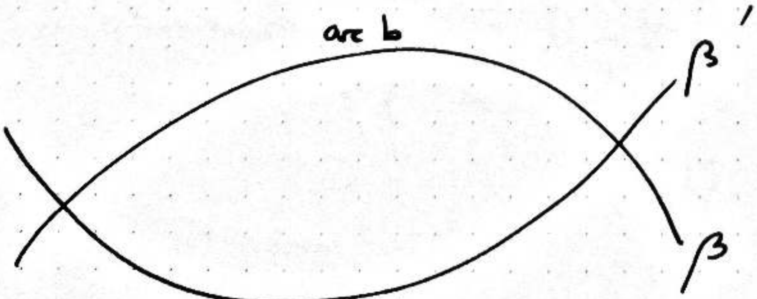




Clearly $\#(\beta' \cap \beta) = |k| \; i(\alpha, \beta)^2$.

It remains to prove β, β' are in minimal position.

If not, β, β' bound a bigon.



Orientation $\Rightarrow b, b'$ meet on the same side of the picture

If it's the LHS, $b' \subset \alpha \Rightarrow \beta$ & α bound a bigon
$\Rightarrow$ ⨳ to minimal position

If b' is in the RHS, move β_0 to the other side of β and repeat the argument. $\square$

Last Time:

Lemma 9.3

$$i(T_\alpha^k(\beta),\beta) = |k|\, i(\alpha,\beta)^2$$

Prop^n 9.4: If α is an essential scc on S, then $o(T_\alpha)=\infty$.

Pf Need

Claim $\exists$ essential scc or proper arc β on S s.t. $i(\alpha,\beta)>0$

Pf An exercise in change of coordinates.
(c.f. lemma 9.3 when α is non-separating) $\square$

Now for any $k \neq 0$,

$$i(T_\alpha^k(\beta),\beta) = |k|\, i(\alpha,\beta)^2 \neq 0$$

$$\Rightarrow T_\alpha^k(\beta) \underset{iso}{\not\simeq} \beta \Rightarrow T_\alpha^k \neq e \text{ in } \mathrm{Mod}(S) \qquad \square$$

10.1 Basic properties of Dehn twists

Lemma 10.1 α,β essential sccs on S

$T_\alpha = T_\beta \iff \alpha \underset{iso}{\simeq} \beta$ (up to reversing orientation of α)

Proof ($\Leftarrow$) Lemma 9.2

($\Rightarrow$) **Claim** $\exists \gamma$ s.t. $i(\alpha,\gamma)>0$ but $i(\beta,\gamma)=0$

The lemma follows from the claim:

$$i(T_\alpha(\gamma),\gamma) = i(\alpha,\gamma)^2 > 0$$

but

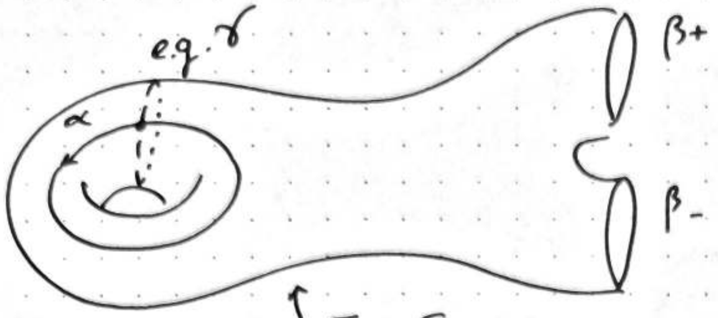
$$i(T_\beta(\gamma),\gamma) = i(\beta,\gamma)^2 = 0$$

$$\therefore T_\alpha \neq T_\beta$$

Pf of claim Suppose $i(\alpha,\beta)>0$. Then take $\gamma=\beta$. ☺

o/w $i(\alpha,\beta)=0$ so wlog $\alpha\cap\beta=\emptyset$.

i.e. $\alpha \subseteq S_\beta$



It follows that

$$\Sigma \ncong D^2, D^2_*, A$$

Now argue by cases in the classification of surfaces. □ □

Remark 10.2 Note that, by def^n, if $\phi \in \mathrm{Homeo}^+(S, \partial S)$, then $[\phi] T_\alpha [\phi^{-1}] = T_{\phi \circ \alpha}$

In particular, T_α, T_β are conjugate iff α, β are in the same $\mathrm{Mod}(S)$-orbit.

I.e. of the same topological type ⌜cf. boundary⌟

Lemma 10.3 Let $[\phi] \in \mathrm{Mod}(S)$ and α, β essential sccs on S. Then

(i) ϕ centralises T_α $\iff$ $\phi \circ \alpha \underset{iso}{\simeq} \alpha^{\pm 1}$

(ii) T_α & T_β commute $\iff$ $i(\alpha, \beta) = 0$

Pf Follows from the above. □

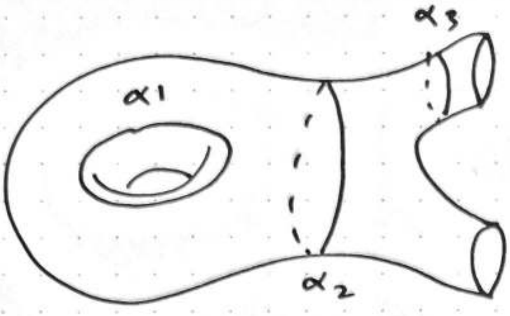
10.2 Multi-twist subgroups

Def^n 10.4 A multicurve on S is a finite set of essential $\overset{\text{pairwise disjoint}}{\downarrow}$ pairwise non-isotopic sccs on S. $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$

A mapping class

$$T_{\alpha_1}^{k_1} \cdots T_{\alpha_n}^{k_n}$$

is called a multi-twist.



$\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3$

<u>Prop^n 10.5</u> If α is a multi-curve, there is a natural hom

$$\mathbb{Z}^n \longrightarrow \mathrm{Mod}(S)$$

$$(k_1, \dots, k_n) \mapsto T_{\alpha_1}^{k_1} \cdots T_{\alpha_n}^{k_n},$$

which is injective.

<u>Pf</u> Idea is to find β an essential scc or proper arc s.t. $i(\alpha_1, \beta) > 0$ but $i(\alpha_i, \beta) = 0$ for $i > 1$.
Then proceed by induction on n.
Use change of coordinates to find β. □

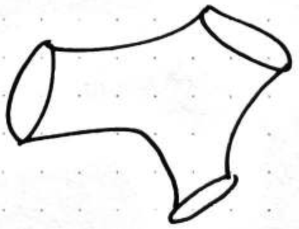
In particular, consider what happens when $\partial S \neq \emptyset$, in which case it defines a multicurve (as long as $S \neq A$).

<u>Corollary 10.6</u> If $S = S_{g,n,b}$, $S \neq A$, then the multi-twist subgroup defined by ∂S is a <u>central</u> subgroup of $\mathrm{Mod}(S)$ isomorphic to $\mathbb{Z}^b$.

<u>Pf</u> By Prop 10.5, just need to prove it's central.
... Central by Lemma 10.3 (i). □

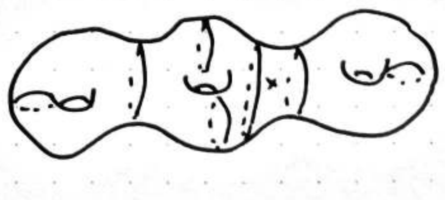
10.3 Pair of pants

The <u>pair of pants</u> is $S_{0,0,3}$



Compare & contrast with $\mathrm{Mod}(S_{0,3,0}) \cong \mathrm{Sym}(3)$.

Twice-punctured disk & once-punctured annulus are also important.



Thm 10.7 If $S = S_{0,n,b}$ with $n+b=3$, then the kerel of the natural map

$$\mathrm{Mod}(S) \to \mathrm{Sym}(n)$$

is equal to the multi-twist subgroup in the boundary.

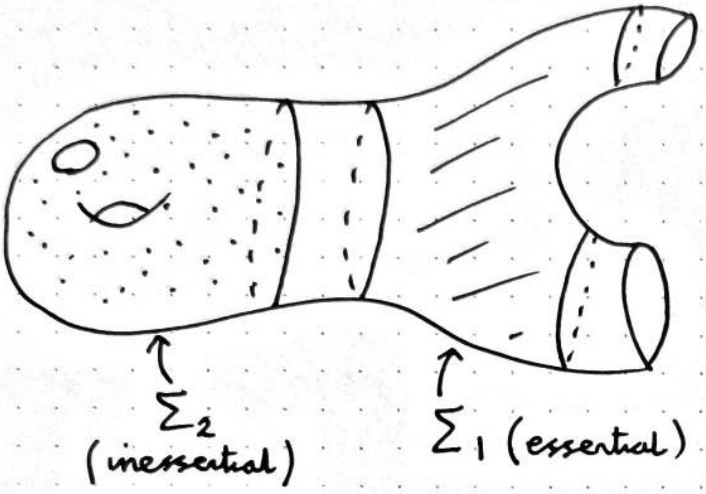
In particular:
- $\mathrm{Mod}(S_{0,0,3}) \cong \mathbb{Z}^3$
- $\mathrm{Mod}(S_{0,1,2}) \cong \mathbb{Z}^2$
- $\mathrm{Mod}(S_{0,2,1}) \cong \mathbb{Z} \times \underset{\uparrow_{\mathrm{Sym}(2)}}{\mathbb{Z}/2}$

Thm 10.7

$$\mathrm{Mod}(S_{0,0,3}) \cong \mathbb{Z}^3$$

Pf Omitted. $\square$

11.1 The inclusion homomorphism

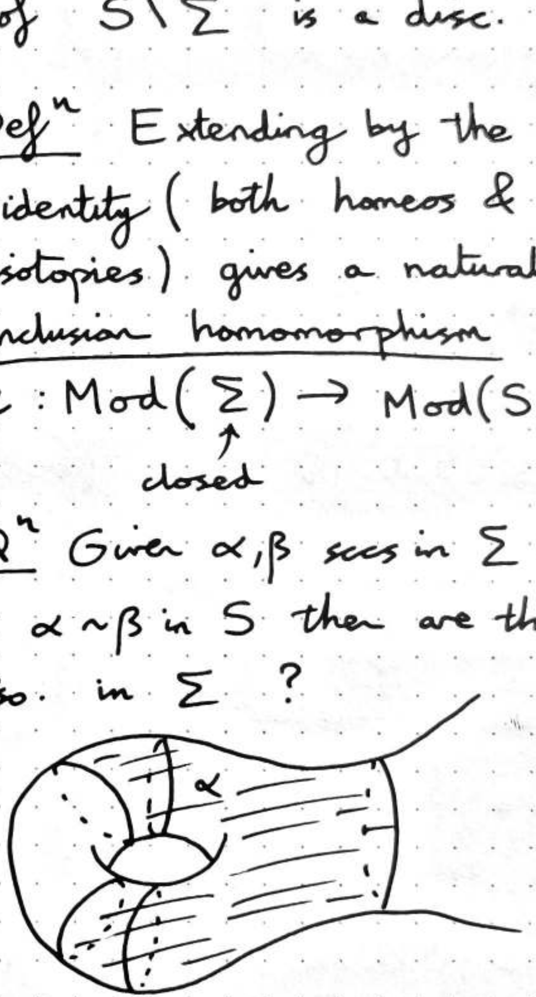


Def^n A closed, connected (important) subsurface $\Sigma \subseteq S$ is essential if no component of $S \setminus \Sigma$ is a disc.

Def^n Extending by the identity (both homeos & isotopies) gives a natural inclusion homomorphism

$$\iota : \mathrm{Mod}(\underset{\text{closed}}{\underset{\uparrow}{\Sigma}}) \to \mathrm{Mod}(S)$$

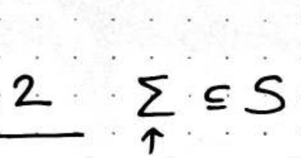
Q^n Given α, β sccs in $\Sigma \subseteq S$, if $\alpha \sim \beta$ in S then are they iso. in Σ ?



Lemma 11.2 $\underset{\text{closed connected essential}}{\underset{\uparrow}{\Sigma}} \subseteq S$

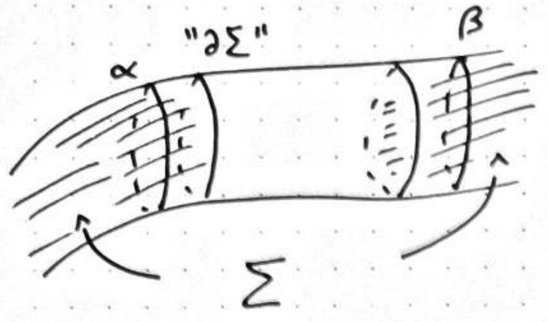
If α, β sccs in Σ, not iso into $\partial\Sigma$, and $\alpha \underset{\text{iso}}{\simeq} \beta$ in S, then $\alpha \underset{\text{iso}}{\simeq} \beta$ in Σ.

Pf: If $\alpha \underset{\text{iso}}{\simeq} \beta$ in S, then we can make them disjoint by inductively pushing α over bigons D in S.



Since Σ essential, $D \subseteq \Sigma$
$\Rightarrow$ this can be done in Σ

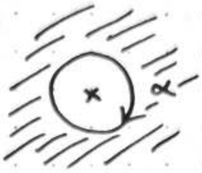
Now suppose α, β are disjoint. By the annulus criterion, α, β together bound $A \subseteq S$.



If $A \not\subseteq \Sigma$, then α, β are both isotopic into $\partial\Sigma$. □

$$T_\alpha T_\beta^{-1} \xmapsto[\iota]{} 1$$

When might ι fail to be injective ?



$$\begin{tikzcd} T_\alpha \in \mathrm{Mod}(\Sigma) \arrow[d, mapsto] \\ 1 \in \mathrm{Mod}(S) \end{tikzcd}$$

Theorem 11.3 Connected, essential closed $\Sigma \subseteq S$. Then

$$\mathrm{Ker}\ \iota = \langle T_{\alpha_1}, \ldots, T_{\alpha_m}, T_{\beta_1^+} T_{\beta_1^-}^{-1}, \ldots, T_{\beta_n^+} T_{\beta_n^-}^{-1} \rangle$$

where $\alpha_1, \ldots, \alpha_m$ are the components of $\partial\Sigma$ bounding D^2_* in S, & $\beta_1^\pm, \ldots, \beta_n^\pm$ are pairs of components of $\partial\Sigma$ that together bound an annulus in S.

Pf STP that $\ker \iota$ is contained in the boundary multi-twist subgroup of $\mathrm{Mod}(\Sigma)$.

(Then apply Prop^n 10.5)

~~If $g=0$~~ Consider $\Sigma = S_{g,n,b}$.

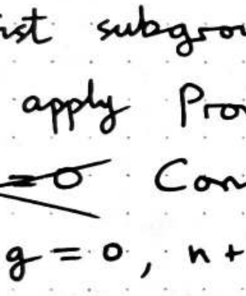
If $g=0$, $n+b \leq 3$ then the result follows from earlier results, especially Thm 10.7.

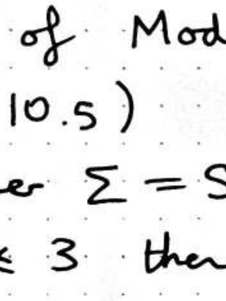
Otherwise, $g > 0$ or $n+b \geqslant 4$.

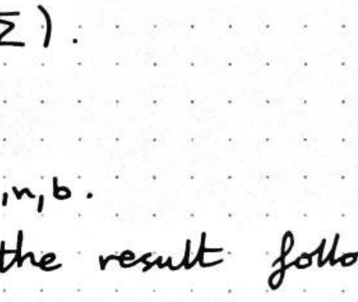
Claim $\exists$ collection of essential sccs $\{\gamma_i\}$ on Σ satisfying

(i) $\nexists$ bigons, annuli or triangles

(ii) every complementary component of $\Sigma \setminus \bigcup_i \gamma_i$ looks like :



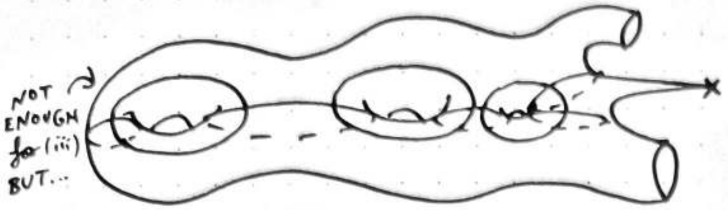




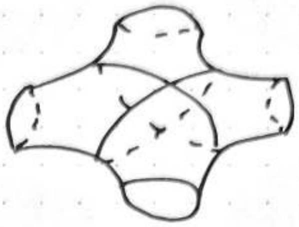
($i\frac{1}{2}$) no γ_i iso into $\partial\Sigma$

(iii) each γ_i intersects the boundary of each complementary region in a connected arc [can get away with less ?]

Pf of claim: $g \geq 1$



$g=0,\ n+b \geq 4$



///

Now consider $\varphi \in \ker L \subseteq \mathrm{Mod}(\Sigma)$

For each γ_i,

$\varphi \circ \gamma_i \sim \gamma_i$ in S

Lemma 11.2 $\Rightarrow \varphi \circ \gamma_i \sim \gamma_i$ in Σ

∴ by the Alex. method, φ induces a graph automorphism

$\varphi_\Gamma \in \mathrm{Aut}(\Gamma_{\{\gamma_i\}})$

RTP that $\varphi_\Gamma = 1$

By construction, φ_Γ preserves each γ_i

Note that $\partial\Sigma \neq \emptyset$ (or there's nothing to prove)

∴ $\exists \geq 1$ annular complementary component

Item (iv) $\Rightarrow \varphi_\Gamma$ fixes each edge of $\Gamma_{\{\gamma_i\}}$ adjacent to an annular component.

Since ϕ is orientation-preserving, if φ_Γ fixes an edge then it fixes every edge meeting it.

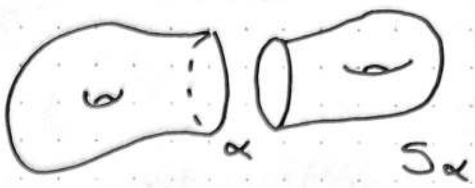
Now Γ connected $\Rightarrow \varphi_\Gamma = \mathrm{id}_{\Gamma_{\{\gamma_i\}}}$ as required. ☒

In conclusion, after an isotopy, φ is supported on the complementary regions of the structure graph. In particular, φ can be written as a product of Dehn twists supported on the annular regions, as required.

11.2 Cut surfaces & stabilisers

Cut surfaces need not be connected.
However, the MCG still makes sense

$$\mathrm{Mod}(S_\alpha) = \prod_{\Sigma\in\pi_0(S_\alpha)} \mathrm{Mod}(\Sigma)$$



For α a multicurve on S, we also have a natural subgroup of $\mathrm{Mod}(S)$, namely the (oriented) stabiliser $\mathrm{Mod}_\alpha(S)$ where $[\phi]\in\mathrm{Mod}_\alpha(S)$ if $\phi\circ\alpha \underset{\text{iso}}{\simeq} \alpha$. ⌜not $\phi\circ\alpha\sim\alpha$'⌟

<u>Prop^n 11.4</u> S connected surface of finite type
α a multicurve on S with n components.
There is a central extension

$$1\to\mathbb{Z}^n\to\mathrm{Mod}(S_\alpha)\xrightarrow{\text{inclusion hom}}\mathrm{Mod}_\alpha(S)\to 1$$

where $\mathbb{Z}^n$ is generated by the multi-twists

$$T_{\alpha_i^+}\circ T_{\alpha_i^-}^{-1} \quad \text{where } \alpha=\alpha_1\sqcup\alpha_2\sqcup\cdots\sqcup\alpha_n.$$

<u>Proof</u> Routine from what we have already seen. □ ⌜yeah fair⌟

12. Forgetting boundary cpts & punctures

12.1 Capping i.e. turning ∂ cpts into punctures

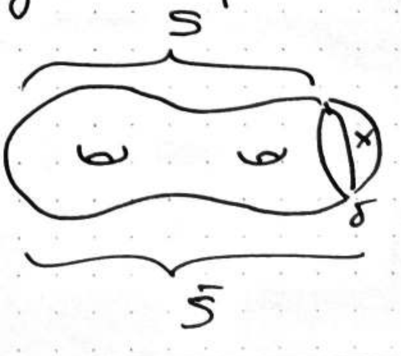
Recall that $\exists$ surjective hom

$$\mathrm{Mod}(S_{g,n,b}) \twoheadrightarrow \mathrm{Sym}(n).$$

<u>Def^n 12.1</u> The <u>pure mapping class group</u> of $S=S_{g,n,b}$ is the kernel of the above hom, denoted $\mathrm{PMod}(S)$.

<u>Capping</u> is the operation of ~~including~~ taking a component $\delta\subseteq\partial S$ and writing down the inclusion

$$S\hookrightarrow S\cup_{\delta=\partial D^2_*} D^2_* = \bar{S}$$



Corollary 12.2 With the above notation, there is a central extension

$$1 \hookrightarrow \langle T_\gamma \rangle \longrightarrow \mathrm{PMod}(S) \underset{\uparrow}{\longrightarrow} \mathrm{PMod}(\bar{S}) \longrightarrow 1 .$$

inclusion hom

Proof Routine given what we have already done. □

[only surj needs major work]

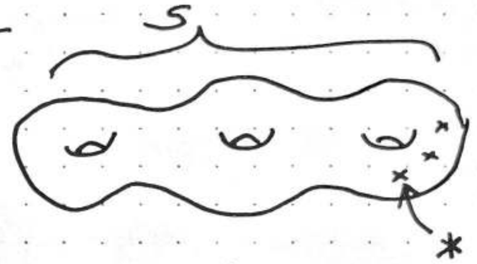
12.2 The Birman exact sequence

This answers the question: "How do we remove a puncture?"

Theorem 12.3 (Birman exact sequence)

Let S be a hyperbolic surface. There is a SES

$$1 \to \pi_1(S,*) \to \mathrm{PMod}(S_*) \to \mathrm{PMod}(S) \to 1$$



$S_* = S \setminus *$

forgetful map

$\mathrm{PMod}(S_*) \longrightarrow \mathrm{PMod}(S)$

We'll give a low-tech proof by analysing $\mathrm{Aut}(\pi_1(S,*))$.

Defn 12.4 Let G be any group, $\gamma \in G$. A conjugating automorphism

$$i_\gamma(g) = \gamma g \gamma^{-1}$$

is called an inner automorphism of G.

These form a subgroup $G/Z(G) \cong \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.

In fact, $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$, and the quotient gets called the outer automorphism group of G:

$$\mathrm{Out}(G) := \mathrm{Aut}(G) / \mathrm{Inn}(G).$$

In summary, there is an algebraic SES

$$1 \to G/Z(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 .$$

In our setting:

$$1 \to \pi_1(S,*) \to \mathrm{Aut}(\pi_1(S,*)) \to \mathrm{Out}(\pi_1(S,*)) \to 1 .$$

Think of $* \in S_*$ as a basepoint.

Since $[\phi] \in PMod(S_*)$ is represented by a homeo ϕ fixing $*$, we get an induced isomorphism $\phi_* \in Aut(\pi_1(S,*))$

This defines a homomorphism

$$\begin{aligned} PMod(S_*) &\longrightarrow Aut(\pi_1(S,*)) \\ [\phi] &\longmapsto f_* \end{aligned}$$

Recall that ~~forgetting~~ changing $* \in S$ corresponds to acting on $\pi_1 S$ by an inner automorphism.

Therefore we also get

$$\begin{aligned} PMod(S) &\longrightarrow Out(\pi_1(S,*)) \\ [\phi] &\longmapsto \underset{\uparrow\, \text{coset of } Inn(\pi_1(S,*))}{[\phi_*]} \end{aligned}$$

Birman Exact Sequence

S hyperbolic, ∃ s.e.s.

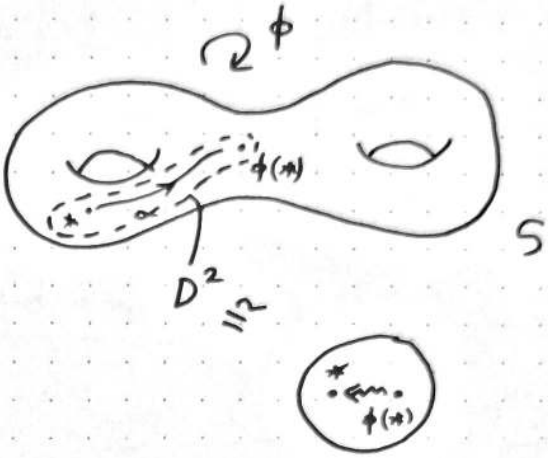
$$1 \to \pi_1(S,*) \to PMod(S_*) \to PMod(S) \to 1$$

———— . ————

Lemma 12.5 For a connected S,

$$PMod(S_*) \longrightarrow PMod(S)$$

is surjective.



Pf RTP: any $[\phi] \in PMod(S)$ can be represented by ϕ' s.t. $\phi'(*) = *$.

Let α be a path $* \rightsquigarrow \phi(*)$ & $D = N(\alpha)$ a reg nbhd of α.

Then ∃ ambient isotopy of D^2 (fixing ∂D^2) and taking $*$ to $\phi(*)$.

Now extend ψ to an isotopy of S by id on $S \setminus D^2$.

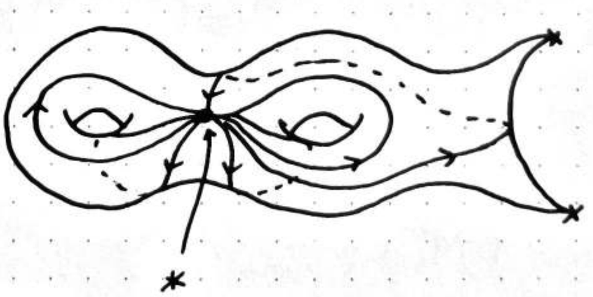
$\Rightarrow \phi = \psi_0^{-1} \circ \phi \sim \psi_1^{-1} \circ \phi$

$\Rightarrow$ result. □

Lemma 12.6

The natural map $PMod(S_*) \longrightarrow Aut(\pi_1 S)$ is injective, if $\partial S = \phi$.

Proof Consider a standard generating set for $\pi_1(S,*)$, $\{\alpha_i\}$.



⌜no Δ's in S_*⌟

Note: α_i fill, have no bigons, no annuli, no Δ's

$\Rightarrow$ Alex method applies.

Also $\Gamma_{\{\alpha_i\}}$ is a bouquet of circles $\Rightarrow \phi_\Gamma = id$ if ϕ preserves the α_i and their orientations.

⌜no $\phi\alpha_i \sim \alpha_i^{\pm}$
yes $\phi\alpha_i \sim \alpha_i$⌟

$[\phi] \in \ker(PMod(S_*) \to Aut(\pi_1 S))$

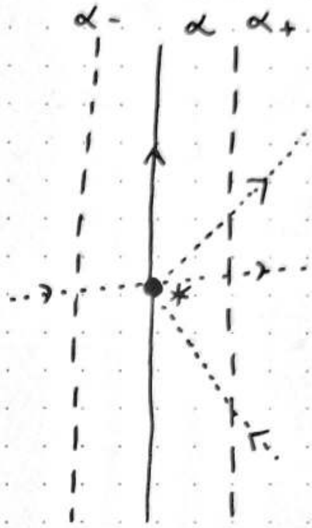
$\Rightarrow \phi$ preserves each α_i up to isotopy $\Rightarrow \phi \sim id$ by Alex. □

Lemma 12.7 Let α be an oriented, simple, non-sep, closed curve on S based at $*$.

- The map $\pi_1(S) \longrightarrow \mathrm{Aut}(\pi_1(S))$ sends α to the automorphism induced by

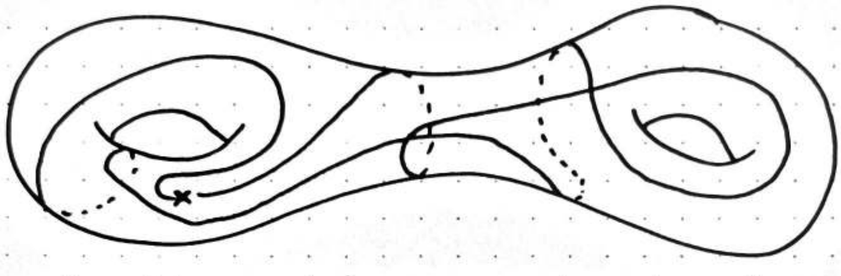
$$T_{\alpha_+} \circ T_{\alpha_-}^{-1}.$$

α pushed to the right — α pushed to the left



In particular, $\pi_1(S) \leq \mathrm{PMod}(S_*)$, both viewed as subgroups of $\mathrm{Aut}(\pi_1 S)$.

-



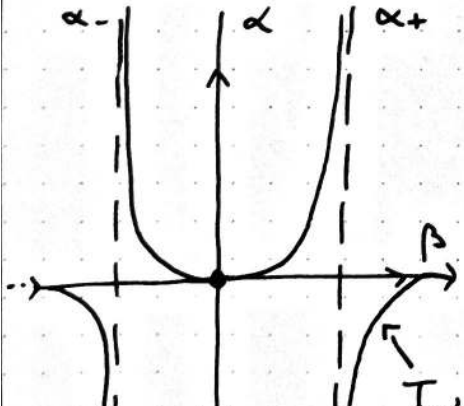
"point-pushing mapping class"

Proof It suffices to check the action on β_i, where we extend α to a generating set $\{\alpha, \beta_i\}$ for $\pi_1 S$.

It suffices to check

$$\tau_{\alpha_+} \circ \tau_{\alpha_-}^{-1} \circ \beta \simeq \alpha \cdot \beta \cdot \cancel{f}\,\overline{\alpha} \quad \text{for all } \beta = \beta_i.$$

- There are 2 cases:

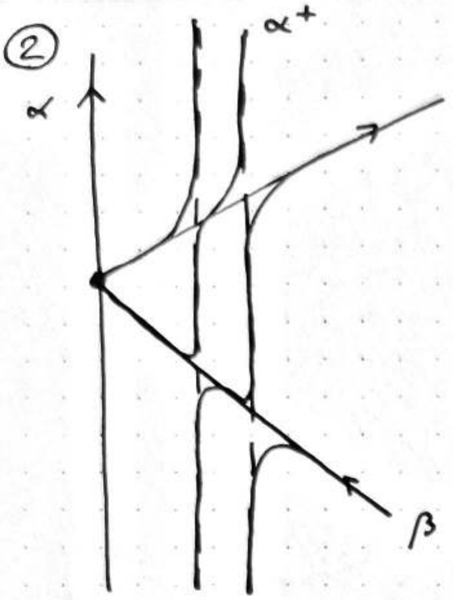


①

$\tau_{\alpha_+} \circ \tau_{\alpha_-}^{-1} \circ \beta \simeq \alpha \cdot \beta \cdot \overline{\alpha}$

-

②



Now we get

$$\begin{tikzcd} & & \text{PMod}(S_*) \arrow[r] \arrow[d, hook] & \text{PMod}(S) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1(S,*) \arrow[r] \arrow[ur] & \text{Aut}(\pi_1 S) \arrow[r] & \text{Out}(\pi_1 S) \arrow[r] & 1 \end{tikzcd}$$

☐

and check exactness at the top.

(✓)

13. Generation by Dehn twists

13.1 Genus 0 case

Corollary 13.1 (Dehn 1938)

Let $S = S_{0,n,b}$. ∃ finite set of sccs on S s.t. the Dehn twists in that finite set generate $\text{PMod}(S)$.

In particular $\text{Mod}(S)$ is f.g.

Proof First suppose $b = 0$, $S = S_{0,n,0}$.

Proceed by induction on n.

Base case: $n \leq 3$: $\text{PMod}(S) = 1$ (prop^n 6.8)

Inductive step: if $S = S_{0,n,0}$ then $S_* = S_{0,n+1,0}$

BES: $1 \longrightarrow \pi_1(S,*) \longrightarrow \text{PMod}(S_{0,n+1,0}) \rightarrow \text{PMod}(S_{0,n,0}) \rightarrow 1$

Dehn twists in $\text{PMod}(S_{0,n,0})$ lift to
" $\text{PMod}(S_{0,n+1,0})$.

By Lemma 12.7, $\pi_1(S,*)$ is generated by Dehn twists in $\text{PMod}(S_{0,\underset{\overset{\uparrow}{n+1\,?}}{n},0})$.

Corollary 13.1

$PMod(S_{0,n,b})$ are generated by finite sets of Dehn twists, & $Mod(-)$ is f.g.

Pf Induction on n using BES gives $PMod(S_{0,n,0})$. Use the capping exact sequence to handle $S_{0,n,b}$ for arbitrary n, b.

For the last part, since

$$|Mod(-) : PMod(-)| = n! < \infty$$

$\Rightarrow$ $Mod(-)$ is f.g. $\square$

Corollary 13.2 Let $S = S_{g,n,b}$.

There is a finite set of essential sccs on S s.t. Dehn twists in this set generate PMod(S) iff the same is true for $\underset{S_g}{\overset{}{PMod(S_{g,0,0})}}$

$S_{g,0,0} \cong S_g$

Goal: Build & study "the complex of curves" on which $Mod(S_g)$ acts.

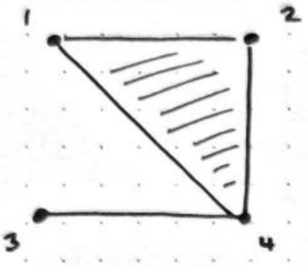
13.2 Definition & connectivity

Def^n 13.3 Let S be a surface of finite type (orientable). The complex of curves associated to S is the following simplicial cx $\mathcal{C}(S)$:

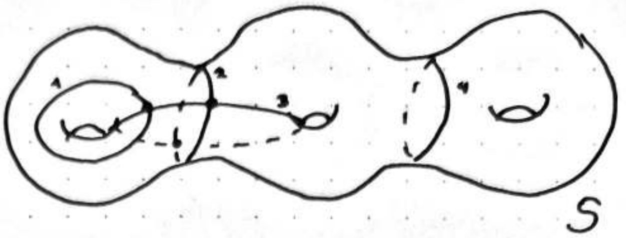
(i) vertices of $\mathcal{C}(S)$ are unoriented isotopy classes of essential sccs not homotopic into ∂S,

(ii) a collection $\{[\alpha_0], \ldots, [\alpha_n]\}$ spans a simplex iff $[\alpha_i]$ have mutually disjoint representatives, i.e. $i(\alpha_i, \alpha_j) = 0 \ \forall i, j$

Example



part of $\mathcal{C}(S)$



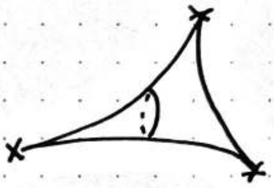
⌈we had to fill in [1, 2, 4]⌋

Remark : There is a natural action of Mod(S) on $\mathcal{C}(S)$ defined by $[\phi]\cdot[\alpha] = [\phi\circ\alpha]$.

Henceforth, $\partial S = \phi$.

Example 13.4 If $g=0$, $n \leq 3$,

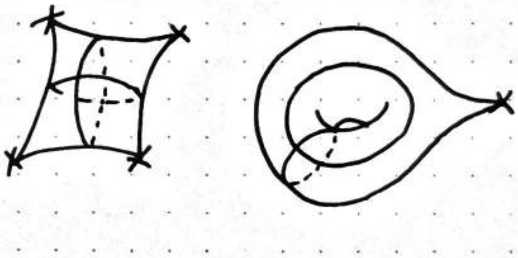
$\mathcal{C}(S) = \phi$.



Example 13.5 If $g=0$, $n=4$

or

$g=1$, $n\leq 1$



In these cases, $\mathcal{C}(S)$ is ∞ but has no edges $\Rightarrow$ disconnected.

Theorem 13.6 If $3g+n \geqslant 5$, then $\mathcal{C}(S) \neq \emptyset$ and is connected.

Proof Let α, β be essential sccs on S.

We need to find a sequence of essential sccs

$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$

s.t. $i(\alpha_j, \alpha_{j+1}) = 0$ for $j=0,\ldots,n-1$.

The proof is by induction on $i(\alpha,\beta)$.

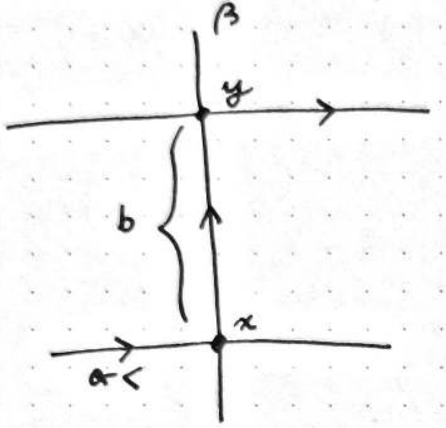
Base case : · $i(\alpha,\beta)=0$. Trivial.

· $i(\alpha,\beta)=1$. By change of coordinates, looks like «one-holed torus», take γ.

Inductive step:

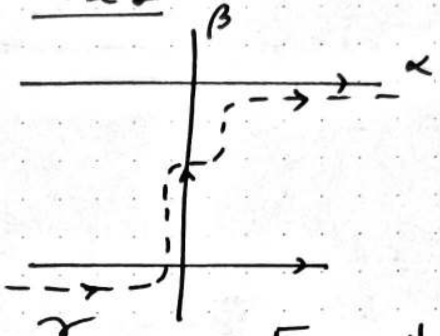
Suppose $i(\alpha,\beta) \geqslant 2$, wlog in minimal position etc.

$\exists\, x \neq y \in \alpha \cap \beta$ consecutive intersection points on β



There are two cases depending on the orientation of the crossing at x.

Case 1:

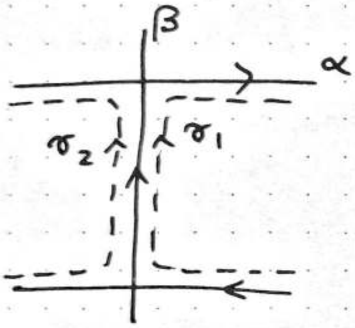


From the picture
$\#(\alpha,\gamma) = 1$
$\#(\beta,\gamma) < \underset{\overset{\|}{i(\alpha,\beta)}}{\#(\alpha,\beta)}$

So $i(\alpha,\gamma) \leq 1$,
$i(\beta,\gamma) < i(\alpha,\beta)$.

Induction $\Rightarrow$ done.

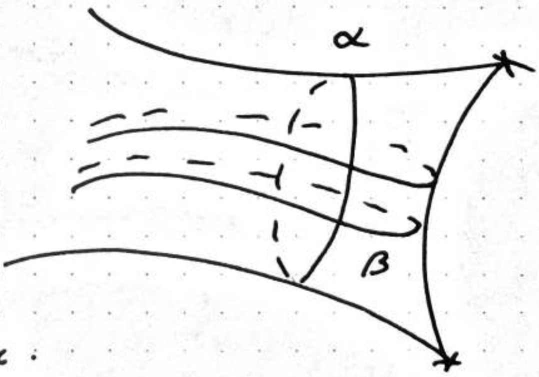
Case 2:



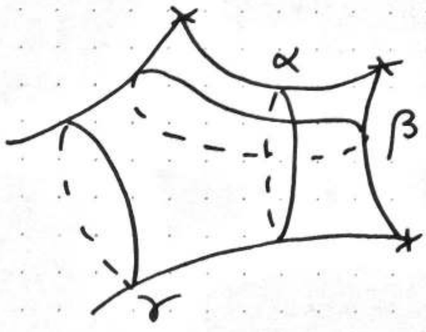
By construction
$i(\alpha,\gamma_i) = 0$
& $i(\beta,\gamma_i) < i(\alpha,\beta)$.

RTP one of γ_1, γ_2 is essential.

The bad case is where both bound punctured discs:



Now reverse roles of α, β.
We win unless β also bounds a twice punctured disc.



Now γ is essential unless $S = S_{0,4,0}$.

□

14.1 Non-separating curves

Corollary 14.1 Let $S = S_g$, $g \geqslant 2$.

Every pair of isotopy classes of non-separating sccs α, β on S is joined in $\mathcal{C}(S)$ by a sequence of non-sep curves

$$\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$$

(i.e. $i(\alpha_i, \alpha_{i-1}) = 0 \ \forall i$)

Pf Thm 13.6 gives us

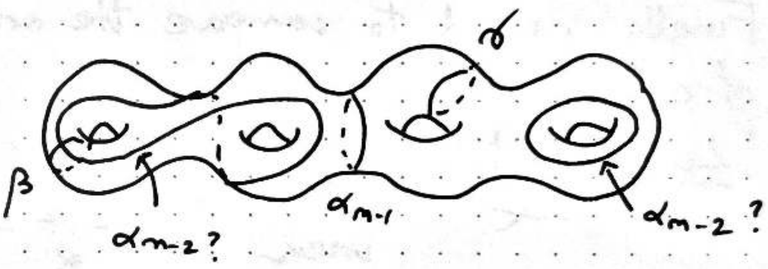
$$\alpha = \alpha_0, \alpha_1, \dots, \alpha_{m-1}, \alpha_m = \beta$$

Proceed by induction on m.

If α_{m-1} is non-separating, done by induction.

So assume α_{m-1} is separating.

If α_{m-2}, β are in different components of $S_{\alpha_{m-1}}$, then remove α_{m-1} from our list.

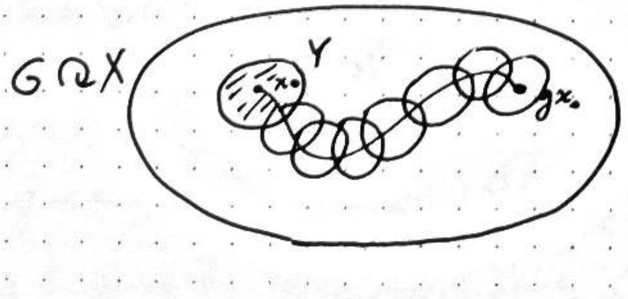


If α_{m-2}, β are in the same component of $S_{\alpha_{m-1}}$, let γ be a non-separating scc in the other component.

Replace α_{m-1} by γ; done by induction. □

14.2 Generation by Dehn twists

Lemma 14.2: Let X be a path-connected topological space, $G \curvearrowright X$ by homeos. Suppose $Y \subset X$ is open and $G \cdot Y = X$. Then $\{ g \in G \mid gY \cap Y \neq \emptyset \}$ generates G.



Pf Omitted, but see Q5 of Ex Sheet 3 □

Let α be some non-separating scc on S.

Consider $\{\beta \mid$ ~~$\beta \subseteq S$~~ $\beta \cap \alpha = \phi$, β non-sep in $S\}$. [also $\beta \not\simeq_{\text{iso}} \alpha$]

Rmk By change of coordinates, there are only finitely many $\underset{\underset{\text{stabilise}}{\uparrow}}{\mathrm{Mod}_\alpha}(S)$ - orbits of β's
$\nearrow$?

Let $\{\beta_1, \dots, \beta_k\}$ be a finite set of orbit reps.

Let $\phi_1, \dots, \phi_k \in \mathrm{Mod}(S)$ s.t. $\phi_i(\alpha) = \beta_i$.

Lemma 14.3 If $S = S_g$, $g \geqslant 2$, then

$$\mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \cup \{\phi_1, \dots, \phi_k\}$$

generates $\mathrm{Mod}(S)$.

Proof Let $g \in \mathrm{Mod}(S)$.

Corollary 14.1 gives a sequence of non-separating sccs.

$$\alpha = \alpha_0, \dots, \alpha_m = g\alpha$$

Since the α_i all have the same top type, $\exists g_i \in \mathrm{Mod}(S)$ s.t.

$\alpha_i = g_i \alpha$, and seq becomes

$$\alpha, g_1\alpha, \dots, g_{m-1}\alpha, g\alpha$$

By induction on m, $g_{m-1} \in \langle \mathrm{Stab}_{\mathrm{Mod}(S)}\alpha, \phi_1, \dots, \phi_k \rangle$

Consider $\gamma = g_{m-1}^{-1} g_m \alpha$.

Note $i(\alpha, \gamma) = i(g_{m-1}\alpha, g_{m-1}\gamma)$

$= i(g_{m-1}\alpha, g_m\alpha)$

$= 0$

So after an isotopy, $\alpha \cap \gamma = \phi$

$\therefore \exists h \in \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha)$ s.t. $h\gamma = \underset{\phi_j \alpha}{\underset{\shortparallel}{\beta_j}}$, some j.

$\therefore g\alpha = g_{m-1}\gamma = g_{m-1} h^{-1} \phi_i \alpha$

$\Rightarrow g \in g_{m-1} h^{-1} \phi_i \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \subseteq \langle$ desired gen set $\rangle$. $\square$

Lemma 14.4: For any pair of disjoint non-sep scc's α, β on S, $\exists$ seq of Dehn twists taking α to β.

Pf: Claim: $\exists$ scc γ on S s.t.

$$i(\alpha,\gamma) = i(\beta,\gamma) = 1$$

Pf: Exercise □

Now Q8 of Sheet 2 says

$$T_\alpha T_\gamma(\alpha) = \gamma$$
$$T_\gamma T_\beta(\gamma) = \beta.$$

$$\therefore \quad T_\gamma T_\beta T_\alpha T_\gamma(\alpha) = \beta. \qquad \square$$

Finally, need to compare the oriented & unoriented stabilisers of α.

Lemma 14.5 If α, β are essential sccs, $i(\alpha,\beta) = 1$, then $T_\beta T_\alpha^2 T_\beta(\alpha) = \alpha^{-1}$.

Idea of proof: Happens on a punctured torus, so use surgery diagrams □

Putting the pieces together:

Theorem 14.6 (Dehn ~1930, Lickorish ~1960)
Let S be any connected oriented surface of finite type. There is a finite set of sccs $\alpha_1, \dots, \alpha_n$ on S s.t. $T_{\alpha_1}, \dots, T_{\alpha_n}$ generate $PMod(S)$.
In particular, $Mod(S)$ is f.g.