III Symplectic Geometry

L1.1

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§ Symplectic linear algebra

Antisymmetric bilinear maps (skew-symmetric)

V m-dim-l real vector space,

and suppose $\omega : V \times V \to \mathbb{R}$ is a bilinear map

The map is skew-symmetric if

$$\Omega(u,v) = -\Omega(v,u), \quad \forall u,v \in V$$

Theorem (standard form of antisymmetric bilinear maps)

Ω antisymmetric; then there is a basis

$u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V s.t.

$$\Omega(u_i, v) = 0 \quad \forall i = 1, \dots, k, \ v \in V$$

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \quad \forall i,j = 1, \dots, n$$

$$\Omega(e_i, f_j) = \delta_{ij} \quad \forall i,j = 1, \dots, n$$

Remarks (1) Basis is not unique

(although it is often called "canonical")

(2) In matrix notation

$$\Omega(u,v) = [\,-\, u \,-\,] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Id \\ 0 & -Id & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Pf (induction proof)

Let $U = \{ u \in V \mid \Omega(u,v) = 0 \text{ for all } v \}$

and choose a basis $u_1, \dots, u_k$ for U.

Choose a complementary subspace W (s.t. $V = U \oplus W$)

Take any non-zero $e_1 \in W$

Then there is $f_1 \in W$ s.t $\Omega(e_1, f_1) \neq 0$

Wlog $\Omega(e_1, f_1) = 1$

Let now $W_1 = \text{span}\{e_1, f_1\}$.

$$W_1^{\Omega} = \{ w \in W \mid \Omega(w,v) = 0 \ \forall v \in W_1 \}$$

Claim $W_1 \cap W_1^{\Omega} = \{0\}$

Suppose $v = ae_1 + bf_1 \in W_1 \cap W_1^{\Omega}$

$$\left.\begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned}\right\} \text{ so } v = 0$$

Claim $W = W_1 \oplus W_1^{\Omega}$

Suppose $v \in W$ has $\Omega(v, e_1) = c$,

$\Omega(v, f_1) = d$.

Then $v = \underbrace{(de_1 - cf_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^{\Omega}}$

Continue as follows: let $e_2 \in W_1^{\Omega}$, $e_2 \neq 0$.

There is an $f_2 \in W_1^{\Omega}$ s.t. $\Omega(e_2, f_2) \neq 0$

Wlog $\Omega(e_2, f_2) = 1$ so let $W_2 = \operatorname{span}\{e_2, f_2\}$. etc

We eventually stop since $\dim V < \infty$,

and obtain a splitting $V = U \oplus W_1 \oplus \cdots \oplus W_n$

where all summands are orthogonal wrt Ω and where each W_i has a basis e_i, f_i s.t. $\Omega(e_i, f_i) = 1$. $\square$

Remarks (1) $\dim U$ does not depend on any choices, call it k

(2) $\dim V = m = 2n + k$

$2n$ is called the rank of Ω

Symplectic vector spaces

$\Omega : V \times V \to \mathbb{R}$ bilinear

Def $\tilde{\Omega} : V \to V^*$, $\tilde{\Omega}(v)(u) = \Omega(v, u)$

$\ker \tilde{\Omega} = U$ from above

Def A skew-symmetric bilinear form Ω is symplectic, or (non-degenerate) if $\tilde{\Omega}$ is bijective i.e. $U = \{0\}$

The form Ω is called a linear symplectic structure on V, and the pair (V, Ω) is called a symplectic vector space.

The following are immediate :

1. $k = \dim U = 0$, $\Rightarrow$ $\dim V = 2n$ <u>even</u>

2. by the theorem, a symplectic vector space has a basis $e_1, \dots, e_n, f_1, \dots, f_n$ s.t.

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}$$

$$\text{i.e. } \Omega(u, v) = [-\, u \,-] \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

<u>Subspaces</u>

<u>Def</u> A subspace is called <u>symplectic</u> if $\Omega|_W$ $(\Omega|_{W \times W})$ is non-degenerate.

(For instance $\operatorname{span}\{e_1, f_1\}$)

A subspace is called <u>isotropic</u> if $\Omega|_W = 0$

(For instance $\operatorname{span}\{e_1, e_2\}$)

The <u>prototype example</u> is $(\mathbb{R}^{2n}, \Omega_0)$

with basis $e_1 = (1, 0, \dots, 0)$ $\quad f_1 = (0, \dots, 0, \overset{(n+1)^{st}}{1}, \dots, 0)$

$\vdots$

$e_n = (0, \dots, \underset{n^{th}}{1}, 0, \dots, 0)$ $\quad f_n = (0, \dots, 0, 1)$

being canonical.

<u>Def</u> A linear symplectomorphism φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\sim} V'$ s.t. $\varphi^* \Omega' = \Omega$ $\quad \left((\varphi^* \Omega')(u, v) = \Omega'(\varphi(u), \varphi(v)) \right)$

i.e. $\Omega(u, v) = \Omega'(\varphi(u), \varphi(v))$

<u>Remark</u> If $\varphi : V \to V'$ linear satisfies $\varphi^* \Omega' = \Omega$
then φ is injective : if $\varphi(u) = 0$
then $\Omega(u, v) = \Omega'(\varphi(u), \varphi(v)) = 0$
for all v
(so since Ω is non-deg) $u \overset{\therefore}{=} 0$

(V, Ω) symplectic vector space, $\dim V = 2n$

$\Omega \times V \times V \longrightarrow \mathbb{R}$ s.t.

$\Omega(u,v) = -\Omega(v,u)$

From theorem in last lecture

$\Rightarrow (V, \Omega)$ symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$

Symplectic manifolds

M - manifold, a smooth 2-form ω on M

1) $p \in M$, $\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ bilinear, skew-symmetric

2) $p \mapsto \omega_p$ smooth

From a local point of view, coordinates $x_1, \dots, x_n$

$$\sum f_{ij}^{(x)} \underbrace{dx_i \wedge dx_j} = \omega \qquad dx_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

f_{ij} smooth functions

* ω_p symplectic $\forall p$ (non-degenerate) $\Rightarrow \dim M = 2n$

* $d\omega = 0$ (closed)

d: exterior derivative

$\underbrace{d\omega} \overset{?}{=}$ define it first on each $f_{ij}\, dx_i \wedge dx_j$

and extend by linearity

$$d(f_{ij}\, dx_i \wedge dx_j) = \sum_{k=1}^{n} \frac{\partial f_{ij}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j$$

$d^2 = 0$

<u>Def</u> The 2-form ω is <u>symplectic</u> if it is closed, and ω_p is symplectic for each p

<u>Def</u> A <u>symplectic manifold</u> is a pair (M, ω) where M is a mfd and ω is a symplectic form

<u>Example</u> $M = \mathbb{R}^{2n}$, with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$

The form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is symplectic

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$

is a symplectic basis for all p.

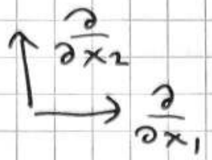
<u>Vector fields</u> M

$p \bullet \longrightarrow V(p) \subseteq T_p M$

$p \mapsto V(p)$ smooth

$x_1, \dots, x_n$

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$



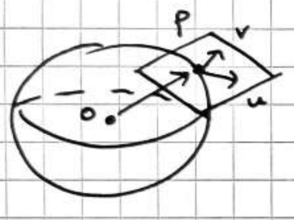
$M = \mathbb{C}^n$, coordinates $z_1, \dots, z_n$

$$\omega = \frac{i}{2} \sum_{i=1}^{n} dz_k \wedge d\bar{z}_k$$

$z_k = x_k + i y_k$, $\mathbb{C}^k \cong \mathbb{R}^{2k}$

$$\begin{aligned} dz_k \wedge d\bar{z}_k &= (dx_k + i\,dy_k) \wedge (dx_k - i\,dy_k) \\ &= -i\,dx_k \wedge dy_k + i\,dy_k\,dx_k \\ &= -2i\,dx_k \wedge dy_k \end{aligned}$$

<u>Example</u> $M = S^2$ = unit vectors in $\mathbb{R}^3$



$u, v \in T_p S^2$

$\omega(u,v) = \langle p, u \times v \rangle$ — cross product in $\mathbb{R}^3$

- top form always closed
- non-degenerate $\omega_p(u,v) = 0 \iff u \times v = 0$

if $u \neq 0$, take $v = u \times p$, then $u \times (u \times p) \neq 0$

<u>Symplectomorphisms</u> (M_1, ω_1) & (M_2, ω_2) symplectic mfd

A symplectomorphism is a diffeomorphism, $\varphi : M_1 \to M_2$

s.t. $\varphi^* \omega_2 = \omega_1$, i.e.

$$(\varphi^* \omega_2)_p (u,v) = (\omega_2)_{\varphi(p)} \big(D\varphi_p(u), D\varphi_p(v) \big)$$

$$\| \qquad (\omega_1)_p (u,v)$$

$u, v \in T_p M_1$

$D\varphi_p : T_p M_1 \xrightarrow{\sim} T_{\varphi(p)} M_2$

"Ideal goal": classify all symplectic mfds up to symplectomorphism

Locally, any symplectic mfd looks like $(\mathbb{R}^{2n}, \omega_0)$

<u>Theorem</u> (Darboux, to be proven) Given any point p in M $\exists$ a coordinate chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ around p s.t.

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Cotangent bundles

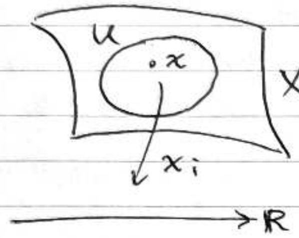
X n-dim manifold

$M = T^*X = \{(x,\xi) : x \in X, \xi \in T_x^* X\}$

$x \in X$, $T_x X$ $\hookrightarrow$ dual to $T_x X$

$\xi : T_x X \to \mathbb{R}$ linear

$(U, x_1, \dots, x_n)$, $x_i : U \to \mathbb{R}$ coordinate chart, $U \subset X$ open



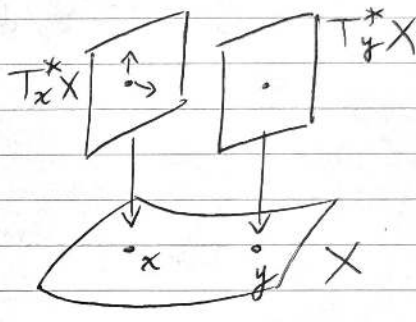
Fix $x \in U$, and consider $(dx_1)_x, \dots, (dx_n)_x$ as linear maps $T_x X \to \mathbb{R}$ i.e. elements of $T_x^* X$.

Moreover this gives a basis of $T_x^* X$.

So if $\xi \in T_x^* X$, then we can uniquely write

$$\xi = \sum_i \xi_i (dx_i)_x$$

for $\xi_i \in \mathbb{R}$.



Get $T^*U \to \mathbb{R}^{2n}$

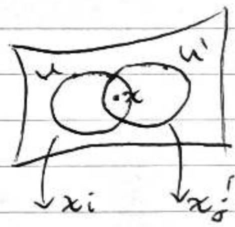
$(x,\xi) \mapsto (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$

This chart $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is a coordinate chart on T^*X,

$(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ are the <u>cotangent coordinates</u> associated with $(x_1, \dots, x_n)$ on U.

Classical mechanics $(q_1, \dots, q_n, p_1, \dots, p_n)$

Transition functions on overlaps are smooth



Given $(U, x_1, \dots, x_n)$, $(U', x_1', \dots, x_n')$, $x \in U \cap U'$, $\xi \in T_x^* X$,

then $\xi = \sum_i \xi_i (dx_i)_x$.

Write $x_i = x_i(x_1', \dots, x_n')$,

so $(dx_i)_x = \sum_j \frac{\partial x_i}{\partial x_j'} (dx_j')_x$

$$\therefore \xi = \sum_{ij} \xi_i \frac{\partial x_i}{\partial x_j'} (dx_j')_x = \sum_j \underbrace{\Big(\sum_i \xi_i \overset{\text{smooth}}{\frac{\partial x_i}{\partial x_j'}} \Big)}_{\xi_j'} (dx_j')_x$$

$\Rightarrow$ transition functions are smooth

Define a 2-form ω on T^*U by

$$\omega = \sum_i dx_i \wedge d\xi_i$$

(classical mechanics: $\sum_i dq_i \wedge dp_i$)

This is coordinate independent:

Consider $\alpha = \sum_i \xi_i \, dx_i$ on $T^*\mathcal{U}$

a 1-form.

Then $d\alpha = \sum_i d\xi_i \wedge dx_i = -\omega$

$\downarrow$ exterior derivative

(classical mechanics, $\alpha = \sum p_i dq_i$)

$$\boxed{\begin{array}{l} X = \text{"configuration space"} \\ T^*X = \text{"phase space"} \end{array}}$$

<u>Claim</u> α is intrinsically defined (hence so is ω)

<u>Proof</u> Let $(\mathcal{U}, x_1, \dots, x_n)$, $(\mathcal{U}', x_1', \dots, x_n')$ be coordinate charts on X, and consider the associated cotangent coordinates.

On $\mathcal{U} \cap \mathcal{U}'$, we have the relation

$$\xi_j' = \sum_i \xi_i \frac{\partial x_i}{\partial x_j'}$$

So
$$\alpha = \sum_i \xi_i \, dx_i = \sum_i \xi_i \Big(\sum_j \frac{\partial x_i}{\partial x_j'} dx_j' \Big)$$
$$= \sum_j \Big(\sum_i \xi_i \frac{\partial x_i}{\partial x_j'} \Big) dx_j' = \sum_j \xi_j' \, dx_j' . \qquad \square$$

<u>Coordinate-free definitions</u>

$$\begin{array}{ccc} M = T^*X & \ni & (x, \xi), \\ \downarrow \pi & & \downarrow \\ X & \ni & x \end{array}$$

α - 1-form on T^*X

given $(x,\xi) \in T^*X$, act on $\gamma \in T_{(x,\xi)} T^*X$

$$\alpha_{(x,\xi)}(\gamma) = \xi\big((d\pi)_{(x,\xi)}(\gamma) \big)$$

$$(d\pi)_{(x,\xi)} : T_{(x,\xi)} T^*X \to T_x X$$

α is called the canonical 1-form

or tautological 1-form

or Liouville 1-form

<u>Exercise</u>: Check that in local coordinates $\alpha = \sum \xi_i \, dx_i$

The canonical symplectic form is $\omega := -d\alpha$

(T^*X, ω) <u>exact</u>

X_1, X_2 manifolds of dim n

$M_i = T^*X_i$

Let $f : X_1 \to X_2$ be a diffeomorphism.

$$\begin{tikzcd} T^*X_1 \arrow[r, dashed, "f_\#"] \arrow[d, "\pi_1"] & T^*X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

Want to define $f_\# : T^*X_1 \to T^*X_2$.

$$(df)_{x_1} : T_{x_1} X_1 \xrightarrow{\sim} T_{f(x_1)} X_2$$

$$(df)^*_{x_1} : T^*_{x_2} X_2 \xrightarrow{\sim} T_{x_1} X_1 \quad \text{dual map}$$

$$f_{\#}(x_1, \xi_1) =: (x_2, \underbrace{[(df_{x_1})^*]^{-1}(\xi_1)}_{\xi_2})$$

<u>Note</u>: the diagram commutes, $f_{\#}$ is smooth, and $(f_{\#})^{-1} = (f^{-1})_{\#}$ is smooth

<u>Proposition</u>: $f_{\#}^* \alpha_2 = \alpha_1$ whence $f_{\#}^* \omega_2 = \omega_1$

Cotangent bundles

T^*X, α canonical 1-form

$$\begin{array}{c} T^*X \\ \downarrow \pi \\ X \end{array}, \quad \pi(x,\xi) = x$$

$$\alpha_{(x,\xi)}(v) = \xi\left(d\pi_{(x,\xi)}(v)\right)$$

$$v \in T_{(x,\xi)}T^*X$$

$\alpha = \sum \xi_i \, dx_i$

$\omega = -d\alpha$

$f: X_1 \longrightarrow X_2$ diffeo

$$\begin{tikzcd} T^*X_1 \arrow[r, "f_\#"] \arrow[d, "\pi_1"'] & T^*X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

commutes

$f_\#(x_1,\xi_1) = (x_2,\xi_2)$

w/ $x_2 = f(x_1)$

$$\xi_2 = \left[(df_{x_1})^*\right]^{-1}(\xi_1)$$

Proposition $f_\#^* \alpha_2 = \alpha_1$

Proof $$(f_\#^*\alpha_2)_{(x_1,\xi_1)}(v) \overset{\text{def}^n \text{ of } f_\#^*}{=} \alpha_{2\,\underbrace{f_\#(x_1,\xi_1)}_{(x_2,\xi_2)}}\left((df_\#)_{(x_1,\xi_1)}(v)\right)$$

$v \in T_{(x_1,\xi_1)}T^*X$

$$\overset{\text{def}^n \text{ of } \alpha_2}{=} \xi_2\left((d\pi_2)_{(x_2,\xi_2)}(df_\#)_{(x_1,\xi_1)}(v)\right)$$

$$= \xi_2\left(d(\pi_2 \circ f_\#)_{(x_1,\xi_1)}(v)\right)$$

$$= \xi_2\left(d(f \circ \pi_1)_{(x_1,\xi_1)}(v)\right)$$

$$= \underbrace{\xi_2\left((df)_{x_1}\right.}_{\xi_1 \text{ by def}^n \text{ of } \xi_2}\left.(d\pi_1)_{(x_1,\xi_1)}(v)\right)$$

$$\overset{\text{def}^n \text{ of } \alpha_1}{=} (\alpha_1)_{(x_1,\xi_1)}(v). \qquad \square$$

Corollary $f_\#^*\omega_2 = \omega_1$

Proof $$\underbrace{d f_\#^* \alpha_2}_{f_\#^* \underbrace{d\alpha_2}_{-\omega_2}} = \underbrace{d\alpha_1}_{-\omega_1} \qquad \square$$

$\Rightarrow$ Any diffeo $f: X_1 \to X_2$ "lifts" to a symplectomorphism of cotangent bundles.

$$\underset{\text{group}}{\mathrm{Diff}(X)} \to \underset{\text{group of all symplectomorphism}}{\mathrm{Symp}(T^*X)}$$
$$f \mapsto f_\#$$

is a group homomorphism (check!)
injective but not surjective (Ex Sheet #1)

Twisted cotangent bundles

a variation

$$\begin{array}{c} T^*X \\ \downarrow \pi \\ X, \sigma \end{array} \qquad \omega_\sigma := \omega + \pi^*\sigma$$

- ω_σ is symplectic

σ any closed 2-form on X, $d\sigma = 0$

- ω_σ is exact $\Leftrightarrow$ σ is exact
very cool!!

Lagrangian Submanifolds

Some terminology: M and X manifolds, $\dim X < \dim M$

Def A (smooth) map $i: X \to M$ is an immersion if
$$di_p : T_pX \to T_{i(p)}M$$
is injective for all $p \in X$.

An embedding is an immersion which is a homeomorphism onto its image.

A closed embedding is a proper injective immersion. (f^{-1}(cpct) is cpct)

Exercise: $i: X \to M$ is a closed embedding iff i is an embedding & $i(X)$ is closed.

Def A submanifold of M is a manifold X with a closed embedding $i: X \to M$.

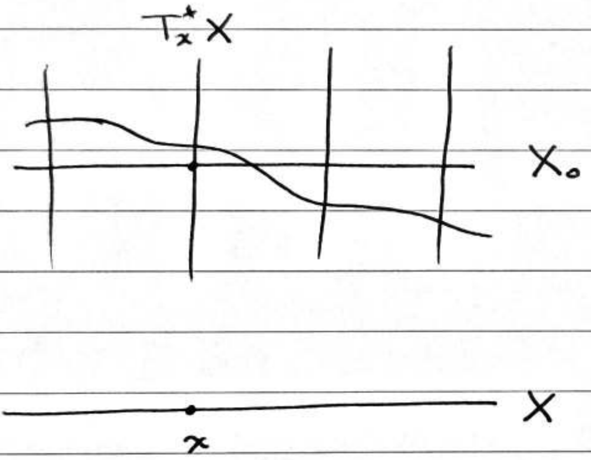
Notation Given a submanifold, we identify X with $i(X)$ and regard i as inclusion

<u>Def</u> (M, ω) symplectic manifold
A submanifold Y of M is a <u>Lagrangian submanifold</u>
if $\forall p \in Y$, $T_p Y$ is a Lagrangian subspace of $T_p M$,
i.e. $\omega_p|_{T_pY} = 0$ and $\dim Y = \frac{1}{2} \dim M$

Equivalently, $i^*\omega = 0$, & $\dim Y = \frac{1}{2} \dim M$.
<u>Example</u> $M = T^*X$, $\alpha = \sum \xi_i \, dx_i$, $\omega = -d\alpha$
zero section of T^*X

$$X_0 = \{(x, 0) \in T^*X : x \in X\}$$

n-dimensional submanifold of T^*X



$i^*\alpha = 0$
apply d
$\therefore$ $i^*\omega = 0$
$\Rightarrow$ X_0 is <u>Lagrangian</u>

$T_x^* X \hookrightarrow T^*X$ are also Lagrangian ✓ $d\pi$ kills T_x^*X
vertical fibres

Let μ be a 1-form, viewed as a section $s_\mu : X \longrightarrow T^*X$

$$X_\mu = \{(x, \mu_x) : x \in X\} \subset T^*X$$

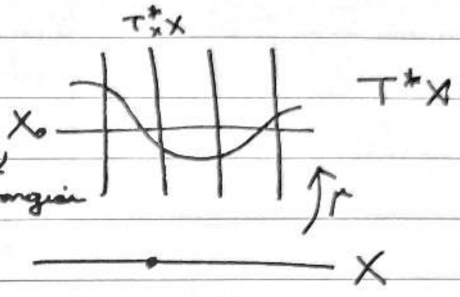
$s_\mu : X \to T^*X$, $s_\mu(x) = (x, \mu_x)$
<u>Proposition</u> $s_\mu^* \alpha = \mu$

Lagrangian Submanifolds

T^*X, μ 1-form, $\mu : X \to T^*X$

$$X_\mu = \{ (x, \mu_x) : x \in X \} \subset T^*X$$

$s_\mu : X \to T^*X$, $s_\mu(x) = (x, \mu_x)$



Proposition : $s_\mu^* \alpha = \mu$

Proof :

$$(s_\mu^* \alpha)_x (v) \overset{\text{def}^n \text{ of } s_\mu^*}{=} \alpha_{(x,\mu_x)}((ds_\mu)_x v)$$

$v \in T_x X$

$$= \mu_x((d\pi)_{(x,\mu_x)}(ds_\mu)_x v)$$

$$= \mu_x(d(\pi \circ s_\mu)_x v)$$

$\downarrow$ using $\pi \circ s = 1$

$$= \mu_x(v) \qquad \square$$

Q : when is X_μ Lagrangian ?

$$\begin{tikzcd} X \arrow[rr, "s_\mu"] \arrow[dr, "\tau"', "\cong"] & & T^*X \\ & X_\mu \arrow[ur, "\iota"'] & \end{tikzcd}$$

$$X_\mu \text{ Lagrangian} \iff \iota^* \underbrace{d\alpha}_{-\omega} = 0$$

$\downarrow$ since τ diffeo

$$\iff \tau^* \iota^* d\alpha = 0$$
$$\iff (\iota \circ \tau)^* d\alpha = 0$$
$$\iff s_\mu^* d\alpha = 0$$
$$\iff d s_\mu^* \alpha = 0$$
$$\iff d\mu = 0$$

X_μ is Lagrangian iff μ is a <u>closed 1-form</u>.

If μ is exact, $\mu = df$, f is called a <u>generating function</u> for X_{df}.

<u>Note</u>: if X is closed (compact & $\partial X = \emptyset$)

$$\#(X_{df} \cap X_0) \geq 2$$

$$\tilde{\omega} := (pr_1)^* \omega_1 - (pr_2)^* \omega_2$$

$$\begin{tikzcd} & M_1 \times M_2 \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

Note: it is easy to check that $\tilde{\omega}$ is a symplectic form (do it!)

$$\Gamma_\varphi = \text{graph}(\varphi) = \{ (p, \varphi(p)) : p \in M_1 \}$$

Proposition A diffeo $\varphi : M_1 \to M_2$ is a symplecto iff Γ_φ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.

Proof $\gamma : M_1 \longrightarrow M_1 \times M_2$

$p \longmapsto (p, \varphi(p))$ embedding

Γ_φ Lagrangian $\Leftrightarrow$ $\gamma^* \tilde{\omega} = 0$

$$\begin{aligned} \gamma^* \tilde{\omega} &= \gamma^* \big((pr_1)^* \omega_1 - (pr_2)^* \omega_2 \big) \\ &= (pr_1 \circ \gamma)^* \omega_1 - (pr_2 \circ \gamma)^* \omega_2 \\ &= \omega_1 - \varphi^* \omega_2 . \end{aligned}$$

$\square$

Generating functions

- a machine for constructing symplectomorphisms

X_1, X_2, $M_i = T^* X_i$, α_i, ω_i, $i = 1, 2$

$$\left. \begin{aligned} M_1 \times M_2 &= T^* X_1 \times T^* X_2 \\ &\cong T^* (X_1 \times X_2) \end{aligned} \right\} \text{check!}$$

and $\alpha = (pr_1)^* \alpha_1 + (pr_2)^* \alpha_2$

$$\therefore \omega = -d\alpha = (pr_1)^* \omega_1 + (pr_2)^* \omega_2$$

To describe $\tilde{\omega} = (pr_1)^* \omega_1 - (pr_2)^* \omega_2$, we define an involution $\sigma_2 : M_2 \to M_2$, $(x, \xi) \mapsto (x, -\xi)$.

Conormal bundles

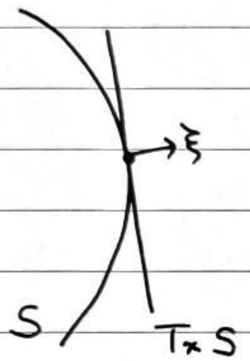
S any k-dimensional submanifold of X

Def The conormal space at $x \in S$ is

$$N^*_x S = \{ \xi \in T^*_x X : \xi(v) = 0 \ \forall v \in T_x S \}$$

The conormal bundle

$$N^*S = \{ (x, \xi) \in T^*X : x \in S, \ \xi \in N^*_x S \}$$



Exercise N^*S is an n-dimensional submanifold of T^*X (need IFT)

$S = \{x\} \to N^*S = T^*_x X$

$S = X \to N^*S = X_0$

Lagrangian

Proposition N^*S is a Lagrangian submanifold, in fact $\iota : N^*S \to T^*X$ has $\iota^*\alpha = 0$

Proof $\pi|_{N^*S} : N^*S \to S$

$$d\pi_{(x,\xi)}\big|_{T_{(x,\xi)}N^*S} : T_{(x,\xi)} N^*S \to T_x S$$

$$\underbrace{\alpha_{(x,\xi)}}_{\in N^*S} (\underbrace{\mu}_{\in T_{(x,\xi)}N^*S}) = \xi\big(\underbrace{d\pi_{(x,\xi)}(\mu)}_{\in T_x S} \big) \underset{\text{def}^n \text{ of } N^*S}{=} 0$$

$\square$

Weinstein : "everything is Lagrangian"

— o —

Another view of symplectomorphisms.

(M_1, ω_1), (M_2, ω_2) symplectic mfds

$\varphi : M_1 \xrightarrow{\sim} M_2$ diffeo, $\varphi^*\omega_2 = \omega_1$

$\sigma_2^* \alpha_2 = -\alpha_2$

Let $\sigma = id \times \sigma_2 : M_1 \times M_2 \longrightarrow M_1 \times M_2$

Then $\sigma^* \tilde{\omega} = (pr_1)^* \omega_1 + (pr_2)^* \omega_2 = \omega$.

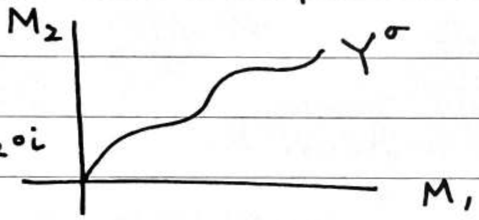
$Y \subset (M_1 \times M_2, \omega)$ Lagrangian

$\therefore$ $Y^\sigma = \sigma(Y)$ is Lagrangian in $(M_1 \times M_2, \tilde{\omega})$

We get the following recipe:

1) Start with a Lag^n submanifold of $(M_1 \times M_2, \omega)$
2) Twist to get Y^σ Lag^n in $(M_1 \times M_2, \tilde{\omega})$
3) Check if Y^σ is the graph of some diffeo $M_1 \to M_2$
4) Then φ s.t. $Y^\sigma = \Gamma_\varphi$ is a symplecto

$$\begin{tikzcd} & Y^\sigma \arrow[dl, "pr_1 \circ i"'] \arrow[d, "pr_2 \circ i"] \\ M_1 \arrow[r, "?"] & M_2 \end{tikzcd}$$



Step 3: need to check that $(pr_i) \circ \iota$ are diffeos

If so, done: $\varphi = ((pr_2) \circ \iota) \circ ((pr_1) \circ \iota)^{-1}$.

We got the following recipe:

$$M_1 = T^*X_1 \longrightarrow M_2 = T^*X_2$$

- Start with a Lagrangian submanifold Y of $M_1 \times M_2$ with ω the natural form on $M_1 \times M_2 \cong T^*(X_1 \times X_2)$
- Twist to get Y^σ a Lagrangian submfd of $M_1 \times M_2$, $\tilde{\omega}$ $\quad$ $\sigma_2(x,\xi) = (x,-\xi)$, $\sigma = id \times \sigma_2$
- Check if Y^σ is the graph of a diffeo φ
- φ is a symplectomorphism

Any $f \in C^\infty(X_1 \times X_2, \mathbb{R})$ gives rise to an exact Lagrangian graph $Y_f = Y_{df} = \{ (x,y, df_{(x,y)}) : (x,y) \in X_1 \times X_2 \}$

$$T^*_{(x,y)}(X_1 \times X_2) \cong T^*_x X_1 \times T^*_y X_2$$

$$df_{(x,y)} = (d_x f, d_y f)$$

with $d_x f \in T^*_x X_1$ and $d_y f \in T^*_y X_2$.

So we can write

$$Y_f = \{ (x,y, d_x f, d_y f) : (x,y) \in X_1 \times X_2 \}$$

$$Y_f^\sigma = \{ (x,y, d_x f, -d_y f) : (x,y) \in X_1 \times X_2 \}$$

Def If Y_f^σ is the graph of a diffeo $\varphi : M_1 \to M_2$ then we call φ the symplectomorphism generated by f, and f is a generating function for φ.

Note Y_f^σ is the graph of φ iff $\forall (x,\xi) \in M_1$, $\forall (y,\eta) \in M_2$, $\varphi(x,\xi) = (y,\eta) \iff \xi = d_x f,\ \eta = -d_y f$ (*)

Let's look at this locally:

$(U, x_1, \dots, x_n)$ on X,

$(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on T^*X,

$(V, y_1, \dots, y_n)$ on Y,

$(T^*V, y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ on T^*Y

(*) become
$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i}(x,y) & (*) \\ \eta_i = -\dfrac{\partial f}{\partial y_i}(x,y) & (**) \end{cases}$$

If there is a solution $y = \varphi_1(x,\xi)$ of (*)
we input this into (**) to get $\eta = \varphi_2(x,\xi)$

$$\varphi(x,\xi) = (\varphi_1(x,\xi), \varphi_2(x,\xi))$$

By the implicit function theorem, to solve (*) locally in terms of (x,ξ), we need

$$\det\left[\frac{\partial}{\partial y_j}\left(\frac{\partial f}{\partial x_i}\right)\right]_{i,j=1}^{n} \neq 0$$

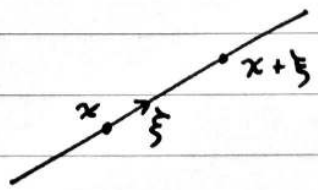
Locally this iff, but globally this could be challenging

<u>Example</u> $X_1 = U_1 = \mathbb{R}^2$, $X_2 = U_2 = \mathbb{R}^2$

$$f(x,y) = \frac{-|x-y|^2}{2} \in C^\infty(X_1 \times X_2, \mathbb{R})$$

$$\begin{aligned} \xi_i &= \frac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i &= -\frac{\partial f}{\partial y_i} = y_i - x_i \end{aligned} \iff \begin{aligned} y_i &= x_i + \xi_i \\ \eta_i &= \xi_i \end{aligned}$$

$$\therefore \varphi(x,\xi) = (x+\xi, \xi)$$



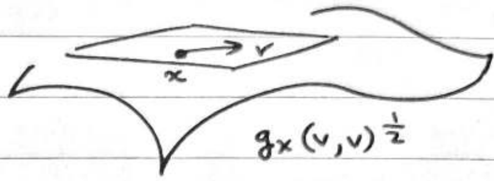
"geodesic flow" of $\mathbb{R}^n$
time 1-map

$$\phi_t(x,\xi) = (x + t\xi, \xi), \quad t \in \mathbb{R}$$

X, g : g - Riemannian metric

g_x is a positive definite symmetric inner product in $T_x X$

$x \mapsto g_x$ smooth



$\gamma : [a,b] \to X$

$$\text{length}(\gamma) = \int_a^b \left[g_x(\dot\gamma(t), \dot\gamma(t)) \right]^{\frac{1}{2}} dt$$

$$= \int_a^b |\dot\gamma(t)|_{\gamma(t)} \, dt$$

$\to$ geodesics : curves γ which locally minimise length

$\downarrow$

$$d(x,y) = \inf_\gamma \{ L(\gamma) : \gamma \text{ from } x \text{ to } y, \ C^\infty \text{ say} \}$$

distance function

$$f : X \times X \to \mathbb{R}$$
$$(x,y) \mapsto -\tfrac{1}{2}(d(x,y))^2$$

2 geometric assumptions on (X,g) :

(i) <u>complete</u> i.e. geodesics defined $\forall t \in \mathbb{R}$

or (X,d) a complete metric space

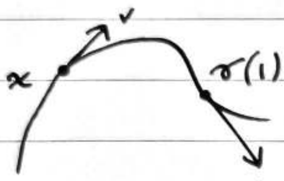
(ii) (X,g) is geodesically convex:

between any two points there is a unique geodesic (up to reparametrisation)

Under (i), (ii), f is C^∞ and generates a symplectomorphism equal to the time 1-map of the geodesic flow.

The metric gives an isomorphism

$$\tilde{g}_x : T_x X \xrightarrow{\sim} T_x^* X$$
$$v \longmapsto g_x(v, \cdot)$$



$\dot{\gamma}(1) \in T_{\gamma(1)} X$

$$\begin{array}{ccc} (x, v) & \longmapsto & (\gamma(1), \dot{\gamma}(1)) \\ {\scriptstyle \tilde{g}_x^{-1}} \uparrow & & \downarrow {\scriptstyle \tilde{g}_{\gamma(1)}} \\ (x, \xi) & \overset{\varphi}{\longmapsto} & (\gamma(1), p(1)) \end{array}$$

$$p(1) \in T^*_{\gamma(1)} X$$

$\varphi : M \to M$

Recurrence

__Periodic points__: $M = T^*X$, X – n dim^{l}

$f : X \times X \to \mathbb{R}$ generates $\varphi : M \to M$

$$(x, d_x f) \mapsto (\cancel{x}\; y, -d_y f)$$

What are the fixed points of φ?

__Prop^n__ Define $\psi : X \to \mathbb{R}$

$$x \mapsto f(x, x)$$

There is a 1-1 correspondence between the fixed points of φ and __critical__ points of ψ.
(points x s.t. $d\psi_x = 0$)

<u>Recall</u>: $M = T^*X$, ω, X n-dim

$f : X \times X \to \mathbb{R}$ generates $\varphi(x, d_x f) = (y, -d_y f)$

$\psi : X \to \mathbb{R}$, $x \mapsto f(x,x)$

<u>Prop</u>: There is a 1-1 correspondence between fixed points of φ and critical points of ψ.

$\ulcorner d\psi = 0 \lrcorner$

<u>Proof</u>: $d\psi_{x_0} = (d_x f + d_y f)_{(x_0, x_0)}$
$= 0$

iff $(d_x f)_{(x_0,x_0)} = -(d_y f)_{(x_0,x_0)}$

Letting $\xi = (d_x f)_{(x_0,x_0)}$ we see $\varphi(x_0, \xi) = (x_0, \xi)$.

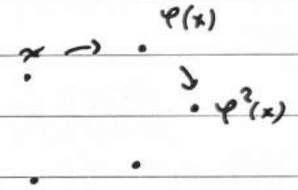
And we can reverse the argument. $\square$

<u>Example</u>: $f(x,y) = -\frac{1}{2}|x-y|^2$

$\varphi(x, \xi) = (x + \xi, \xi)$

$\psi(x) \equiv 0$ corresponding to $\varphi(x, 0) = (x, 0)$

Iterate: $\varphi^N = \underbrace{\varphi \circ \cdots \circ \varphi}_{N \text{ times}} : M \to M$



$\{\text{fixed points of } \varphi^N\} \xleftrightarrow[1-1]{} \{\text{critical points of } \psi^{(N)}(x) = f^{(N)}(x,x)\}$

if $f^{(N)}$ is a generating function for φ^N.

$\{\text{periodic points of } \varphi\}$

<u>Q</u>: What is the generating function for $\varphi \circ \varphi$?

Define, for fixed $x, y \in X$,

$$X \to \mathbb{R}$$
$$z \mapsto f(x,z) + f(z,y)$$

Suppose this map has a unique critical point $z_0(x,y)$ and it is <u>non-degenerate</u>.

Let $f^{(2)}(x,y) := f(x, z_0) + f(z_0, y)$.

<u>Proposition</u> The function $f^{(2)}$ is smooth and is a generating function for φ^2 if we assume that for each $\xi \in T_x^* X$ there is a unique $y \in X$ s.t. $(d_x f^{(2)})_{(x,y)} = \xi$.

<u>Proof:</u> z_0 critical point if

$$(d_y f)_{(x,z_0)} + (d_x f)_{(z_0,y)} = 0$$

This gives z_0 (implicitly)
Want to invoke IFT so $z_0(x,y)$ is C^∞.
Need $\det\left[\frac{\partial}{\partial z_i}\left(\frac{\partial f}{\partial y_j}\Big|_{(x,z_0)} + \frac{\partial f}{\partial x_j}\Big|_{(z_0,y)}\right)\right] \neq 0$

⌜which it does by assumption⌟

If this holds then $f^{(2)}$ is smooth.

$$\varphi^{(2)}(x, d_x f^{(2)}_{(x,y)}) = \varphi(\varphi(x, \underbrace{d_x f^{(2)}_{(x,y)}}_{d_x f_{(x,z_0)}}))$$

⌜use def^n via crit pt⌟

$$= \varphi(z_0, -d_y f_{(x,z_0)})$$
$$= \varphi(z_0, d_x f_{(z_0,y)})$$
$$= (y, -d_y f_{(z_0,y)})$$
$$= (y, -d_y f^{(2)}_{(\cancel{z_0}\, x,y)})$$

$\Rightarrow f^{(2)}$ is a generating function for φ^2. $\square$

φ^3 ? $\quad X \times X \longrightarrow \mathbb{R}$

$$(z,u) \longmapsto f(x,z) + f(z,u) + f(u,y)$$

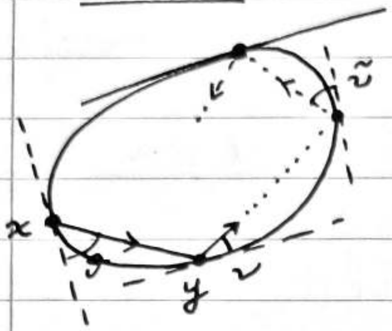
Suppose ~~it~~ this function has a unique critical point (z_0,u_0). Then

$$f^{(3)}(x,y) = f(x,z_0) + f(z_0,u_0) + f(u_0,y)$$

is a generating function for φ^3.

etc.

Billiards convex billiard table



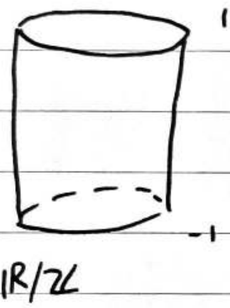
$\gamma: \mathbb{R} \to \mathbb{R}^2$ smooth

$\gamma(s+1) = \gamma(s) \quad \forall s \in \mathbb{R}$

$\left|\frac{d\gamma}{ds}\right| = 1$

convex: the tangent to γ at each point has $\operatorname{im}\gamma$ on one side

$$\varphi: \mathbb{R}/\mathbb{Z} \times (-1,1) \longrightarrow \mathbb{R}/\mathbb{Z} \times (-1,1) \qquad \left(\text{for } \begin{matrix} v = \cos\theta \\ w = \cos\nu \end{matrix}\right)$$
$$(x, v) \longmapsto (y, w)$$



annulus $A \circlearrowleft$

$$f: \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \|\gamma(x) - \gamma(y)\|$$

smooth outside the diagonal

f is a generating function for φ!

$$\checkmark \quad \begin{cases} \dfrac{\partial f}{\partial x} = v & \text{(check!)} \\ \dfrac{\partial f}{\partial y} = -w \end{cases}$$

$\Rightarrow \varphi$ preserves area $dx \wedge d\nu = \sin\theta \, dx \wedge d\theta$

Theorem (Poincaré recurrence) Suppose $\varphi: A \to A$ is an area-preserving diffeo of a finite-area manifold A.
Let $p \in A$, U be a nghbd of p.
Then $\exists\, q \in U$ and a positive integer N s.t. $\varphi^N(q) \in U$.
(we always return)

Proof $U_0 = U$, $U_N = \varphi^N(U_0)$
If all disjoint then since $\text{Area}(U_N) = \text{Area}(U) > 0$ for all N, $\divideontimes$ to finite area.

L7.4

So $\exists\, i,j \geq 0,\ i \neq j$ s.t. $\varphi^i(U) \cap \varphi^j(U) \neq \emptyset$

So $\varphi^{|i-j|}(U) \cap U \neq \emptyset$. $\square$

Preliminaries for local forms

1) Isotopies & vector fields

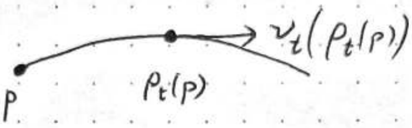
- M manifold, $\rho : M \times \mathbb{R} \to M$ smooth

<u>Def</u> ρ is an <u>isotopy</u> if for each $t \in \mathbb{R}$,

$$\rho_t : M \to M$$
$$p \mapsto \rho(p,t) \quad \text{is a diffeo, and } \rho_0 = id.$$

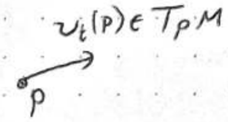
$$v_t(p) \overset{\text{def}}{=} \left.\frac{d}{ds}\right|_{s=t} \rho_s(q) \quad \text{where } q = \rho_t^{-1}(p)$$

i.e. $v_t \circ \rho_t = \dfrac{d\rho_t}{dt}$



Then v_t is a time-dependent vector field.

- Conversely, if v_t is given,



look for a solution to

$$\dot{\gamma}(t) = v_t(\gamma(t))$$

$$\begin{cases} \dot{x} = F(x(t),t) \\ x(0) \text{ given} \end{cases}$$
time-dep ODE

Big theorem $\Rightarrow \exists! \rho_t$ s.t.

$$\begin{cases} \dfrac{\partial \rho_t(x)}{dt} = v_t(\rho_t(x)) \\[2ex] \rho_0(x) = x \end{cases}$$

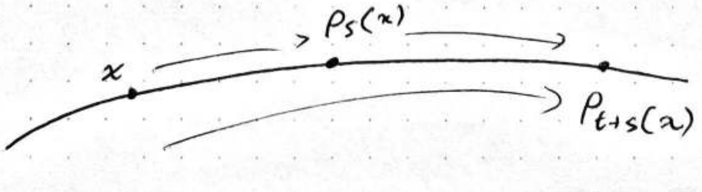
- If M is compact, or more generally v_t is compactly supported ($\exists$ cpct set $K \subset M$ s.t. $v_t(x) = 0 \;\forall x \notin K, \forall t$)
then $\exists$ smooth solutions for all t.

If M is compact, then get

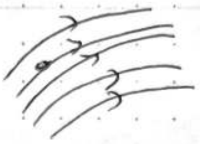
$$\left\{ \begin{matrix} \text{isotopies} \\ \rho_t \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{time dep} \\ \text{vector fields } v_t \end{matrix} \right\}$$

If v_t is independent of t, then we say v_t is autonomous, in which case the associated isotopy is called a <u>flow</u>; we have the group

- law $\rho_{t+s} = \rho_t \circ \rho_s$
(group action of $\mathbb{R}$ on M).



$\mathbb{R} \to \mathrm{Diff}(M)$ group hom
$t \mapsto \rho_t$

M partitions into orbits 

- **Lie derivative** $\mathcal{L}_{v_t} : \Omega^k(M) \to \Omega^k(M)$

$$\mathcal{L}_{v_t}\omega := \frac{d}{dh}\Big|_{h=0} \left(\rho_{t+h} \circ \rho_t^{-1}\right)^* \omega$$

$$= \lim_{h\to 0} \left[\frac{(\rho_{t+h}\circ\rho_t^{-1})^*\omega - \omega}{h} \right]$$

Then $\rho_t^*\left(\mathcal{L}_{v_t}\omega\right) = \frac{d}{dh}\Big|_{h=0} \rho_{t+h}^*\omega$

$\omega, \ t \mapsto \rho_t^*\omega \in \Omega^k(M)$

- Cartan's magic formula:

$$\mathcal{L}_{v_t}\omega = \iota_{v_t} d\omega + d\,\iota_{v_t}\omega$$

⌜ Aside: $(\underbrace{\iota_X \overset{\uparrow}{\omega}}_{(k-1)\text{ form}})_x (v_1, \dots, v_{k-1}) = \omega_x(X, v_1, \dots, v_{k-1})$ ⌟

k-form

This is usually proved for the autonomous case, but also holds in this time-dependent case. (see ES #2)

- **Proposition** For a smooth family of forms $\omega_t, t \in \mathbb{R}$, we have

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right)$$

Proof Set $f(x,y) = \left(\rho_x^*\omega_y\right)_p$, $x, y \in \mathbb{R}$

$$\frac{d}{dt} f(t,t) = \frac{d}{dx} f(x,t) + \frac{d}{dy} f(t,y)$$ ⌜chain rule⌟

$$= \frac{d}{dx}\left(\rho_x^*\omega_t\right)_p\Big|_{x=t} + \frac{d}{dy}\left(\rho_t^*\omega_y\right)_p\Big|_{y=t}$$

$$= \rho_t^*\left(\mathcal{L}_{v_t}\omega_t\right) + \rho_t^*\frac{d\omega_t}{dt}$$ □

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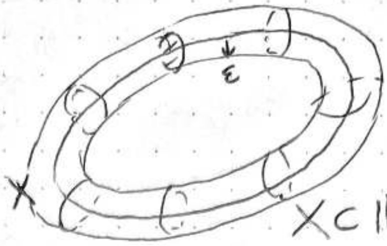
2) Tubular neighbourhood theorem



$X \subset \mathbb{R}^n$

$\dim X = k < n$

$i : X \hookrightarrow M$, $\dim X = k < \dim M = n$

$T_xX \hookrightarrow T_xM$, $x \in X$

$\therefore$ can define $N_xX = T_xM / T_xX$

an $(n-k)$-dim v space

Normal bundle $NX = \{(x, v) : x \in X, v \in N_xX\}$

is a smooth vector bundle, over X.

$\dim NX = n$

The zero section of NX, $\iota_0 : X \hookrightarrow NX$

$$x \longmapsto (x, 0)$$

A neighbourhood U_0 of the zero section X in NX is called <u>convex</u>

if $U_0 \cap N_xX$ is convex $\forall x \in X$.

<u>Theorem</u> (tubular ngbd thm)

$\exists$ a convex ngbd U_0 of X in NX

a ngbd U of X in M

and a diffeo
$$\begin{tikzcd} NX \supseteq U_0 \arrow[r, "\simeq\ \varphi"] & U \subseteq M \\ X \arrow[u, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$
s.t. diagram commutes.

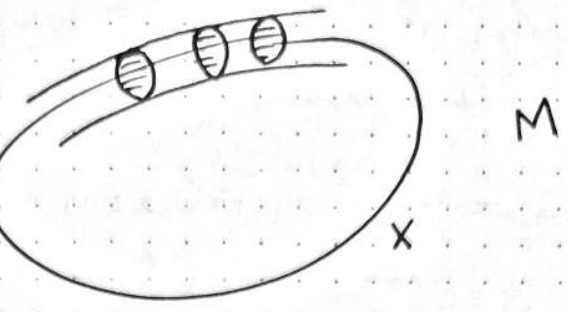
$\pi_0 : U_0 \to X$, $\pi_0^{-1}(x)$ are convex
$(x, v) \longmapsto x$

so $\pi = \pi_0 \circ \varphi^{-1} : U \to X$ is called the tubular ngbd fibration

$X \subset M$ convex

$\mathcal{U}_0$ ngbd $NX \to X$

$\mathcal{U}$ ngbd M



$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\tilde{}", "\varphi"'] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes

$N_x X = T_x M / T_x X$

$i_0(x) = (x, 0)$ TNT

3) Homotopy formula / invariance

$\rho_t : M \circlearrowleft$ isotopy, $\rho_0 = id$

ω closed form, degree k, $d\omega = 0$

$\rho_t^* \omega$ also closed, de Rahm class ?

$[\rho_t^* \omega]$ vs $[\omega]$ how are they related ?

$$\rho_1^*\omega - \omega = \int_0^1 \big(\underbrace{\tfrac{d}{dt} \rho_t^* \omega}_{\rho_t^* \mathcal{L}_{v_t} \omega} \big)\, dt$$

$\ulcorner\, v_t \circ \rho_t = \frac{d\rho_t}{dt} \,\lrcorner$

Cartan

$$\overset{\downarrow}{=} \int_0^1 \rho_t^* \left(d\, \iota_{v_t} \omega + \iota_{v_t} d\omega \right) dt$$

$\ulcorner\, v_t(p) = \frac{d\rho_t}{dt}(\rho_t^{-1}(p))$

$= \frac{d}{ds}\Big|_{s=t} \rho_s(\rho_t^{-1}(p)) \,\lrcorner$

$$= d \int_0^1 \rho_t^* \iota_{v_t} \omega \, dt + \int_0^1 \rho_t^* \iota_{v_t} d\omega \, dt$$

$$Q : \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\omega \mapsto \int_0^1 \rho_t^* (\iota_{v_t} \omega)\, dt$$

Then $\rho_1^* \omega - \omega = dQ\omega + Qd\omega$,

$$\boxed{\rho_1^* - id = dQ + Qd}$$ Homotopy Formula

ü

If ω is closed, then

$$\rho_1^*\omega - \omega = dQ\omega$$

$$\therefore [\rho_1^*\omega] = [\omega]$$

$F : M \times [0,1] \to M$ smooth

$f(x) = F(x,0)$

$g(x) = F(x,1)$

Then $g^* - f^* = dQ + Qd$.

<u>Moser's theorem</u> M <u>compact</u>, ω_1 & ω_0 symplectic forms s.t. $[\omega_1] = [\omega_0]$ and $\omega_t = (1-t)\omega_0 + t\omega_1$ <u>symplectic</u> $\forall\, t$ in

Then $\exists$ isotopy $\rho_t : M \circlearrowleft$, $t \in [0,1]$ s.t.

$$\rho_t^*\omega_t = \omega_0 \quad \forall\, t \in [0,1]$$

In particular $\varphi = \rho_1$ has $\varphi^*\omega_1 = \omega_0$.

<u>Proof</u> If we had such a ρ_t then

$$\frac{d}{dt}\rho_t^*\omega_t = 0$$

But

$$\text{LHS} = \rho_t^*\mathcal{L}_{v_t}\omega_t + \rho_t^*\frac{d\omega_t}{dt}$$

$$\Rightarrow \mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0$$

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\mu \quad \text{for } \mu \text{ some 1-form}$$

$$\mathcal{L}_{v_t}\omega_t = d\iota_{v_t}\omega_t + \iota_{v_t}\underbrace{d\omega_t}_{\text{zero}}$$

$$\Rightarrow d\iota_{v_t}\omega_t + d\mu = 0$$

$$d(\iota_{v_t}\omega_t + \mu) = 0$$

In particular if $\iota_{v_t}\omega_t + \mu = 0$ then we integrate v_t and the result follows.

Moser trick: find v_t s.t. $\iota_{v_t}\omega_t + \mu = 0$

Integrate v_t (using M compact) to get an isotopy ρ_t.

Trace back previous calculation to get $\frac{d}{dt}\rho_t^*\omega_t = 0$.

$\iota_X \omega = \mu$ Fix $x \in M$

$$\omega_x(X, \cdot) = \mu_x(\cdot) \qquad (*)$$

ω_x non-degenerate means

$\exists!\ X$ s.t. $(*)$ holds $\square$

$H: M \to \mathbb{R}$, C^∞ ω symplectic

$$\exists!\ X \text{ s.t. } \iota_X\omega = dH$$

X is Hamiltonian vector field of the Hamiltonian H

Let ~~$*$~~ ρ_t be the flow of X.

Then
$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_X\omega$$

$$\begin{aligned}\mathcal{L}_X\omega &= \iota_X d\omega + d\overbrace{\iota_X\omega}^{dH} \\ &= \cancel{ddH} + \cancel{\iota_X d\omega} \quad \text{if } d\omega = 0 \\ &= 0\end{aligned}$$

$\therefore \rho_t^*\omega = \omega$ so Hamiltonian flows preserve the symplectic form

$\ulcorner \dim M = 2n \lrcorner$

$\therefore \rho_t^*\omega^n = \underbrace{\omega^n}_{\text{volume}}$ (Liouville's theorem: Hamiltonian motions preserve volume)

If we examine the proof of Moser's theorem, need for ω_t a smooth curve of closed 2-forms connecting ω_0, ω_1,

- $\frac{d}{dt}[\omega_t] = 0$
- ω_t symplectic $\forall t$ $\Rightarrow$ same proof works

So, given two symplectic forms ω_0, ω_1 which are connected by a smooth 1-parameter family ω_t of symplectic forms s.t. $\frac{d}{dt}[\omega_t] = 0$,

then $\exists \rho_t : M \circlearrowleft$ isotopy s.t. $\rho_t^* \omega_t = \omega_0$.

Moser theorem : M compact

(Rephrase) $c \in H^2(M;\mathbb{R})$

$S_c = \{\omega \text{ symplectic s.t. } [\omega] = c\}$

Moser theorem $\Rightarrow$ all symplectic forms in a path-connected component of S_c are symplectomorphic

Theorem (Moser theorem - relative version)

M manifold, $i : X \hookrightarrow M$ a compact submanifold, ω_0, ω_1 symplectic forms on M s.t. $\omega_0|_p = \omega_1|_p$ for all $p \in X$.

Then $\exists$ ngbds U_0, U_1 of X in M and a diffeo

$\varphi : U_0 \to U_1$ s.t.

$$\begin{tikzcd} U_0 \arrow[rr, "\varphi"] & & U_1 \\ & X \arrow[ul, "i_0"] \arrow[ur, "i_1"'] & \end{tikzcd}$$

commutes

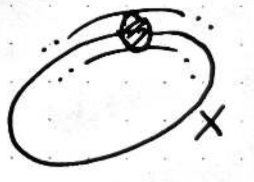
and $\varphi^* \omega_1 = \omega_0$.

Proof: Needs the following

Relative Poincaré Lemma U tubular ngbd of X in M $i : X \hookrightarrow M$, ω closed k-form on U s.t. $i^*\omega = 0$

Then ω is exact, and $\omega = d\mu$ for $\mu \in \Omega^{k-1}(U)$ s.t. $\mu_x = 0$ for all $x \in X$

$U \sim$ convex ngbd in NX



Proof: $\pi : U \to X$, $\exists\, Q$ (homotopy) s.t. $id - \pi^* i^* = dQ + Qd$

where $Q(\omega) = \int_0^1 \rho_t^*(\iota_{v_t}\omega)\, dt$, $\rho_t(x,v) = (x, tv)$

(Check this!)

So choose $\mu = Q\omega$ & from the formula, using $v_t(x,0) = 0$, get $\mu_x = 0$ $\forall x \in X$. $\square$

Pick a tubular ngbd U_0 of X in M.

The form $\omega_1 - \omega_0$ is closed in U_0, and $(\omega_1 - \omega_0)_x = 0$ for all $x \in X$.

RPL $\Rightarrow \exists \mu \in \Omega^1(U_0)$ s.t. $\omega_1 - \omega_0 = d\mu$, $\mu_p = 0$ $\forall p \in X$

Let now $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\, d\mu$

- By shrinking U_0 if necessary, we may assume that ω_t is symplectic $\forall t \in [0,1]$.

Solve Moser eq^n

$$\iota_{v_t}\omega_t = -\mu \qquad \forall t \in [0,1]$$

"it really will be!"

By shrinking U_0 again (if necessary), we get an isotopy

$\rho_t : U_0 \longrightarrow M, \quad t \in [0,1]$ Note $\rho_t|_X = id$.

Just set $U_1 = \rho_1(U_0)$.

And $\rho_1^* \omega_1 = \omega_0$. □

- **Theorem** (Darboux) (M, ω) symplectic manifold, $p \in M$

Then we can find a coordinate system $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p s.t. on U,

$$\omega = \Sigma_i\, dx_i \wedge dy_i$$

Proof: We want to apply the relative moser theorem to $X = \{p\}$.

Use any symplectic basis on T_pM to construct coordinates $(x_1', \ldots, x_n', y_1', \ldots, y_n')$ around p, say in U', s.t. at p,

$$\omega|_p = (\Sigma_i\, dx_i' \wedge dy_i')|_p$$

- Now on U', we have two symplectic forms

$$\omega_0 = \omega, \quad \omega_1 = \Sigma_i\, dx_i' \wedge dy_i'$$

By the relative Moser theorem, $\exists$ ngbds U_0, U_1 of p and a diffeo $\varphi : U_0 \rightarrow U_1$ s.t. $(\varphi(p) = p)$ &

$$\varphi^*(\Sigma_i\, dx_i' \wedge dy_i') = \omega$$

$$\| $$

$$\sum_i d(\underbrace{x_i' \circ \varphi}_{x_i}) \wedge d(\underbrace{y_i' \circ \varphi}_{y_i})$$

Set $x_i = x_i' \circ \varphi$, $y_i = y_i' \circ \varphi$ as coords on U_0. □

-

Def 1) (M, ω_0) & (M, ω_1) are symplectomorphic if $\exists$ diffeo $\varphi : M \circlearrowleft$ s.t. $\varphi^* \omega_1 = \omega_0$

2) (M, ω_0) & (M, ω_1) are strongly isotopic if $\exists$ isotopy $\rho_t : M \circlearrowleft$ s.t. $\rho_1^* \omega_1 = \omega_0$

3) (M, ω_0) & (M, ω_1) are deformation equivalent if $\exists$ smooth path $\{\omega_t\}$ of symplectic forms connecting ω_0 to ω_1

4) (M, ω_0) & (M, ω_1) are isotopic if they are deformation equivalent with $[\omega_t]$ indep of t

Note: $(2) \Rightarrow (1)$

$(2) \Rightarrow (4)$ (Homotopy invariance)

$(4) \Rightarrow (3)$

$(4) \Rightarrow (2)$ by Moser if M is compact

Weinstein Lagrangian neighbourhood theorem

We want a good model for a ngbd of a Lagrangian submanifold X inside M.

Linear Algebra Preliminaries

U, W n-dim^l vector spaces, $\Omega : U \times W \to \mathbb{R}$ bilinear map

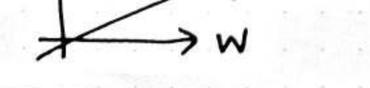
$\tilde{\Omega} : U \to W^*$, $\tilde{\Omega}(u)(w) = \Omega(u, w)$

Ω is non-degenerate $\Leftrightarrow$ $\tilde{\Omega}$ is a bijection

Proposition (V, Ω) $2n$-dim symp v. space and $U \subset V$ a Lagrangian subspace. Let W be any subspace complementary to U. Then from W we can canonically construct a Lagrangian complement $\tilde{W}$ to U.

Proof: $\Omega' : U \times W \to \mathbb{R}$ is non-degenerate pairing

Get $\tilde{\Omega}' : U \to W^*$. Look for linear maps $A : W \to U$ s.t.



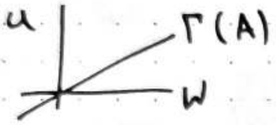
$W' = \{(w, Aw), w \in W\}$ is Lagrangian

<u>Proposition</u> (V, Ω) 2n-dim symplectic vector space
$U \subset V$ Lagrangian & W a complementary subspace
Then from W we can <u>canonically</u> construct a lagrangian complement to U.

<u>Proof</u> (ctd.) $U \times W \xrightarrow{\Omega} \mathbb{R}$ non-degenerate

$$\therefore \ \tilde{\Omega} : U \xrightarrow{\sim} W^*$$

Looking for $W' = \{ w + Aw : w \in W \}$ for $A : W \to U$ linear.



W' Lagrangian

$$\Updownarrow$$

$$\omega(w_1 + Aw_1, w_2 + Aw_2) = 0 \quad \forall\, w_1, w_2 \in W$$

$$\parallel$$

$$\omega(w_1, w_2) + \omega(Aw_1, w_2) + \omega(w_1, Aw_2)$$

Enough for
$$\Omega(w_1, w_2) = \Omega(Aw_2, w_1) - \Omega(Aw_1, w_2)$$
$$= \tilde{\Omega}'(Aw_2)(w_1) - \tilde{\Omega}'(Aw_1)(w_2)$$

Let $A' = \tilde{\Omega}' \circ A$. Look for A' s.t.

$$\Omega(w_1, w_2) = A'(w_2)(w_1) - A'(w_1)(w_2) \qquad (*)$$

Canonical choice: $A'(w) \underset{\text{def}}{=} -\frac{1}{2} \Omega(w, \cdot)$

Then set $A = (\tilde{\Omega}')^{-1} \circ A'$. $\square$

<u>Proposition</u> V a 2n-dimensional vector space, Ω_0, Ω_1 symplectic forms. Let $U \subset V$ be Lagrangian for both Ω_0, Ω_1 and let W be any complement to U in V. Then from W we can <u>canonically</u> construct a linear isomorphism $L : V \xrightarrow{\sim} V$ s.t. $L|_U = id_U$ & $L^* \Omega_1 = \Omega_0$.

<u>Proof</u> By the previous proposition, we can canonically construct W_0, W_1 complements to U s.t. W_i is Lagrangian for Ω_i, $i = 1, 2$.

$W_0 \times U \xrightarrow{\Omega_0} \mathbb{R}$ $\qquad$ give isomorphisms $\qquad$ $\tilde{\Omega}_0 : W_0 \to U^*$

$W_1 \times U \xrightarrow{\Omega_1} \mathbb{R}$ $\qquad\qquad\qquad\qquad\qquad$ $\tilde{\Omega}_1 : W_1 \to U^*$

$$\begin{tikzcd} W_0 \arrow[r, "\tilde{\Omega}_0"] \arrow[d, "B"'] & U^* \arrow[d, "id"] \\ W_1 \arrow[r, "\tilde{\Omega}_1"'] & U^* \end{tikzcd}$$

Define $B : W_0 \to W_1$ s.t.

$$\Omega_1 (B w_0 , u) = \Omega_0 (w_0 , u)$$

$\forall\ w_0 \in W_0 ,\ u \in U$.

Extend B to V via $L : U \oplus W_0 \longrightarrow U \oplus W_1$

$$(u, w_0) \longmapsto (u, Bw_0)$$

Check that L is symplecto:

$(L^* \Omega_1)(u + w_0 ,\ u' + w_0')$

$= \Omega_1 (u + Bw_0 ,\ u' + Bw_0')$

$= \Omega_1 (Bw_0 , u') + \Omega_1 (u , Bw_0')$ $\qquad$ using U, W_1 Lag for Ω_1

$= \Omega_0 (w_0 , u') + \Omega_0 (u , w_0')$

$= \Omega_0 (u + w_0 ,\ u' + w_0')$ $\qquad$ using U, W_0 Lag for Ω_0 $\qquad \square$

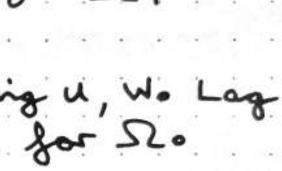
<u>Theorem</u> (Weinstein Lagrangian Neighbourhood Theorem)

M a $2n$-dimensional manifold, X a <u>compact</u> n-dimensional submanifold & $i : X \to M$ inclusion. Let ω_0, ω_1 be two symplectic forms on M s.t. $i^*\omega_0 = i^*\omega_1 = 0$ (X is lagrangian wrt ω_0, ω_1)

Then $\exists$ ngbds U_0, U_1 of X in M and a diffeo $\varphi : U_0 \xrightarrow{\sim} U_1$ s.t.

$$\begin{tikzcd} U_0 \arrow[rr, "\varphi"] & & U_1 \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes, and $\varphi^* \omega_1 = \omega_0$.



<u>Proof</u> Let g be a Riemannian metric on M.

Fix a point $p \in X$, and let $V = T_pM$, $U = T_pX$.

So $U \subset V$ is a Lagrangian subspace wrt $\omega_0|_p, \omega_1|_p$.

Take $W = U^\perp$ the orthogonal complement wrt the metric g.

By our last proposition, can canonically construct a linear iso $L_p : T_pM \to T_pM$ s.t. $L_p|_{T_pX} = id_{T_pX}$ and $L_p^*(\omega_1|_p) = \omega_0|_p$.

Since the constructions were canonical, $p \mapsto L_p$ is smooth, $p \in X$.

<u>Whitney Extension Theorem</u>

$\Downarrow$

$\exists$ ngbd N of X and an embedding $h : N \to M$ with $h|_X = id_X$ and $dh_p = L_p$ for $p \in X$.

$$(h^*\omega_1)_p = (dh_p)^*(\omega_1)_p = (L_p)^*(\omega_1)_p = (\omega_0)_p$$

equality as elements of T_p^*M, $\forall\, p \in X$.

By the relative Moser theorem, $\exists$ ngbd $U_0 \ni X$ and an embedding $f : U_0 \to N$ s.t. $f|_X = id_X$ and $f^*h^*\omega_1 = \omega_0$.

Set $\varphi = h \circ f$. $\square$

<u>Whitney extension theorem</u> M n-dim manifold, X a k-dim submanifold. Suppose for each $p \in X$ we are given a linear iso $L_p : T_pM \to T_pM$, $L_p|_{T_pX} = id_{T_pX}$ and with $p \mapsto L_p$ smooth.

Then $\exists$ an embedding of some ngbd N of X in M s.t. $h|_X = id_X$, $dh_p = L_p$ for $p \in X$.

<u>Observation</u> (V, Ω) symplectic vector space

$U \subset V$ Lagrangian subspace

$$\Omega' : (V/U) \times U \longrightarrow \mathbb{R}$$
$$[v], u \longmapsto \Omega(v,u)$$

· well-defined
· non-degenerate

Hence $\tilde{\Omega}' : V/U \longrightarrow U^*$ <u>natural iso</u>

$X \subset (M, \omega)$

Lagrangian   symplectic

So $T_x X \subset T_x M$ Lagrangian.

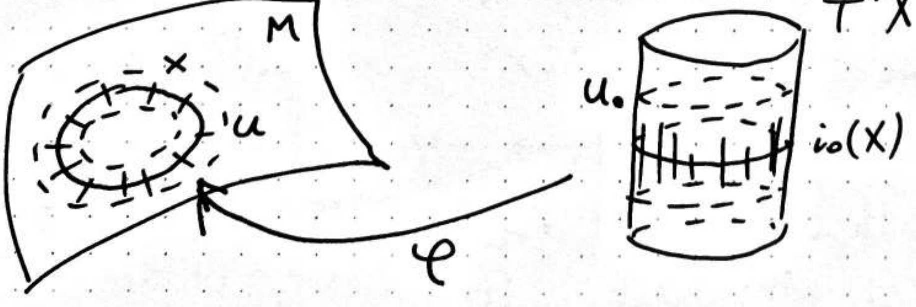
$$\therefore \; \underbrace{T_xM / T_xX}_{= N_x X} \cong T_x^* X \quad \text{naturally via } \omega$$

So $NX \cong T^*X$ in a canonical way.

<u>Theorem</u> (Weinstein tubular neighbourhood theorem, WTNT)
If $X \subset (M, \omega)$ is a compact Lagrangian submanifold, and we let ω_0 be the canonical symplectic form on T^*X, $i_0 : X \to T^*X$ the zero section, $i : X \to M$ inclusion, then there are ngbds U_0 of $i_0(X)$ in T^*X
U of X in M
and a diffeo $\varphi : U_0 \longrightarrow U$
s.t.

$$\begin{tikzcd} U_0 \arrow[r, "\varphi"] & U \\ X \arrow[u, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes, and $\varphi^* \omega = \omega_0$.



Pf Since $NX \cong T^*X$ we can find a ngbd N_0 of X in T^*X, and a ngbd N of X in M with a diffeo $\psi : N_0 \to N$ preserving X. (This is just TNT)

Let ω_0 be the canonical symp form on T^*X,

$$\omega_1 = \psi^*\omega;$$

two symp forms on N_0 which have $i_0(X)$ Lagrangian.

By the Weinstein Lagrangian Neighbourhood Theorem, there are ngbds U_0, U_1 of X in N_0 and a diffeo $U_0 \xrightarrow{\theta} U_1$ s.t. $\theta^*\omega_1 = \omega_0$, $\theta|_X = id$.

Take $\varphi = \psi \circ \theta$ & $U = \varphi(U_0)$.

Then $\varphi^*\omega = \theta^*\psi^*\omega = \omega_0$. $\square$

Application ① (M, ω), $\mathrm{Symp}(M,\omega)$

$$\mathrm{Symp}(M,\omega) = \{ f : M \circlearrowleft : f^*\omega = \omega \}$$

$T_{id}\,\mathrm{Symp}(M,\omega)$?

Def $f_i : X \to Y$ smooth maps between mfds

Say $f_i \to f$ [redacted] in the C^0 topology if $f_i \to f$ uniformly on compact subsets.

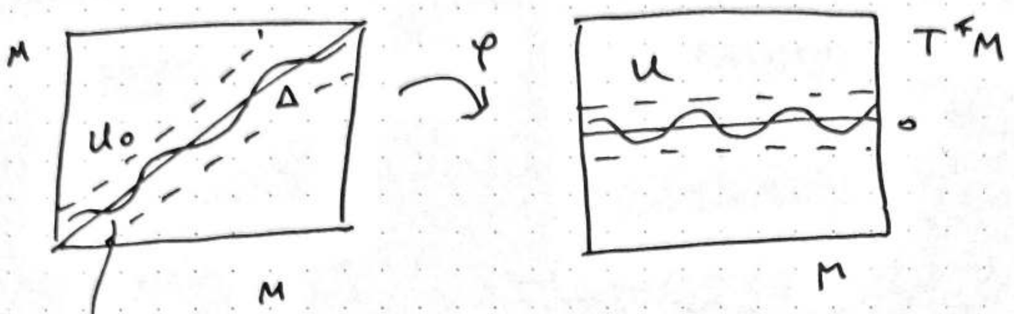
Say $f^i \to f$ [redacted] in the C^1 topology if $Df_i \to Df$ in the C^0 topology, as maps $TX \to TY$.

What can we say about $f \in \mathrm{Symp}(M,\omega)$ which is C^1-close to the identity?

(M, ω) compact, $f \in \mathrm{Symp}(M,\omega)$

$$\left.\begin{array}{l} \mathrm{Graph}\, f \\ \mathrm{Graph}\, id = \underset{\underset{M}{\wr\wr}}{\Delta} \end{array}\right\} \text{Lagrangian submanifolds of } (M \times M,\ pr_1^*\omega - pr_0^*\omega)$$

WTNT $\Rightarrow$ $\exists$ a ngbd of Δ in $(M\times M, \tilde{\omega})$, $\mathcal{U}_0$
which is symplectomorphic to a
neighbourhood $\mathcal{U}$ of M in (T^*M, ω_0)



$\varphi(p,p) = (p,0)$

$j: M \hookrightarrow \mathcal{U}_0$
graph of f

C^1-close $\Rightarrow$ $\varphi(\text{graph} f)$ is section of $T^*M \to M$

$j_0 = \varphi \circ j$

$\therefore$ $\exists$ 1-form $\mu: M \to T^*M$
whose graph gives $j_0(M)$

This form μ must be closed since $j_0(M)$ is Lagrangian
This can be reversed; go from μ to symplecto [for μ close to the zero section]

Conclusion: A small C^1-ngbd of id in $\text{Symp}(M,\omega)$
is homeo to a C^1-ngbd of the zero section
in $\{$ closed 1-forms $M \to T^*M \}$

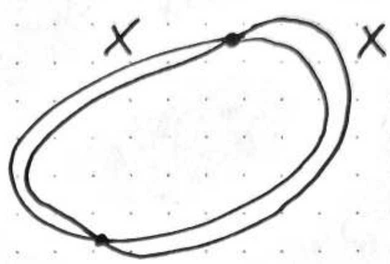
$\therefore$ $T_{id}\,\text{Symp}(M,\omega) \cong$ Closed 1-forms on M

Theorem (M,ω) compact with $H^1(M,\mathbb{R}) = \{0\}$
Then any symplectomorphism which is C^1-close to the identity has at least two fixed points.

Proof From the above, $\text{graph}(f) \longleftrightarrow$ closed 1 form on M, μ
If $H^1(M,\mathbb{R}) = 0$ then $\mu = dH$ for some $H \in C^\infty(M,\mathbb{R})$.
Compact $\Rightarrow$ H has ≥ 2 critical points. $\square$
$\text{Graph}(f) \cap \Delta \longleftrightarrow \{p : \mu_p = dH_p = 0\}$

Lagrangian Intersection Problem

Theorem (M, ω) symplectic. Suppose $X \subset M$ is a compact Lagrangian submanifold with $H^1(X, \mathbb{R}) = 0$.
Then any Lagrangian submanifold of M which is C^1 close to X intersects X in at least two points.



Arnol'd Conjecture

$f \in Symp(M, \omega)$, (M, ω) compact

Def A Hamiltonian diffeo ($h_t : M \to \mathbb{R}$, $h_{t+1} = h_t \; \forall t \in \mathbb{R}$) given by flow of v_t where $\omega(v_t, \cdot) = dh_t(\cdot)$

Let ρ_t be the associated isotopy of v_t, then $f := \rho_1$.

Note: fixed points of $f \leftrightarrow$ 1-periodic solutions of v_t

A fixed point is non-degenerate if $df_p : T_pM \to T_pM$ does not have 1 as an eigenvalue i.e. $\det(df_p - id) \neq 0$

$$\#\{p : f(p) = p \;\&\; \text{non-degenarete}\} \geq \sum_{i=0}^{2n} \dim H^i(M, \mathbb{R})$$

---

1 Topological "reasons"

Lefschetz fixed point theorem, $f : X \circlearrowleft$, $f \sim id$

$$\#\text{fixed points} \geq \chi(M) = \sum_{i=0}^{2n} (-1)^i \dim H^i(M, \mathbb{R})$$

2 h is autonomous, h indep of t

Morse theory gives the answer

Conjecture proved in this form: Conely-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian

Floer homology

Gromov

Contact manifolds

"Odd-dimensional analogue of symplectic manifolds"

Def^n (Contact element) $p \in M$, $H_p \subset T_pM$ where H_p is a hyperplane i.e. of codimension 1

H_p determines an element $\alpha_p \in T_p^*M \setminus \{0\}$ up to multiplication by a non-zero scalar, s.t.

$$H_p = \ker \alpha_p \quad , \ulcorner \alpha_p : T_pM \to \mathbb{R} \text{ linear}, \ \alpha_p \neq 0$$

$$\ker \alpha_p = \ker \alpha_p' \iff \alpha_p = \lambda \alpha_p', \text{ some } \lambda \neq 0 \lrcorner$$

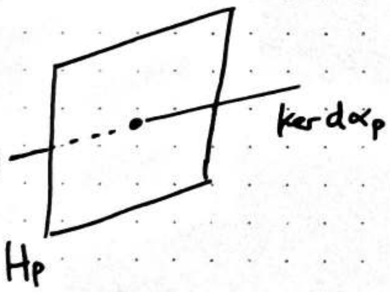
Suppose now H is a smooth field of hyperplanes, so locally $H_p = \ker \alpha_p$ for some 1-form α called a locally defining 1-form for H.

Def^n A <u>contact structure</u> on M is a smooth field of tangent hyperplanes $H \subset TM$ s.t. for any locally defining 1-form α, we have $d\alpha|_H$ is non-degenerate (i.e. symplectic).

(M, H) is called a <u>contact manifold</u>, and α is called a <u>local contact form</u>

$$T_pM = \underbrace{H_p}_{\substack{\ker\alpha, \\ \text{even max} \\ \text{dimension}}} \oplus \underbrace{\ker d\alpha_p}_{\text{1-dim}}$$

dim T_pM odd $= 2n+1$



↑ depends on α

$d\alpha_p|_H$ non-degenerate $\Rightarrow$ $(d\alpha_p)^n$ is a volume form on H_p

<u>Remarks</u> 1) To have a globally defined α we just need TM/H to be orientable.

If this happens, since H is orientable, then M must be orientable too

Then $\alpha \wedge (d\alpha)^n$ becomes a volume form on M

Take a basis $\underbrace{e_1, f_1, \dots, e_n, f_n}_{\text{basis of } H}$, r of T_pM

$$(\alpha \wedge (d\alpha)^n)(e_1, f_1, \dots, e_n, f_n, r)$$
$$= \underbrace{\alpha(r)}_{\substack{\text{non}\\ \text{-zero}}} \underbrace{(d\alpha)^n(e_1, f_1, \dots, e_n, f_n)}_{\text{non-zero}}$$

<u>Def^n</u> $(M, H_1), (N, H_2)$ 2 contact manifolds. A <u>contactomorphism</u> is a diffeo $f : M \to N$ s.t. $f_* H_1 = H_2$.

<u>Examples</u> 1) $\mathbb{R}^3$, $\alpha = x\,dy + dz$

$$H_{(x,y,z)} = \ker \alpha_{(x,y,z)} = \left\{ a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \;:\; bx + c = 0 \right\}$$

$$\alpha \wedge d\alpha = (x\,dy + dz) \wedge (dx \wedge dy) \qquad \ulcorner \alpha\left(\tfrac{\partial}{\partial z}\right) = 1 \lrcorner$$
$$= dx \wedge dy \wedge dz \qquad \text{volume form in } \mathbb{R}^3$$

2) $\mathbb{R}^{2n+1}$, $\alpha = \sum_{i=1}^{n} x_i\,dy_i + dz$ for coords $(x_1, y_1, \dots, x_n, y_n, z)$

contact form

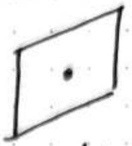
<u>Theorem</u> (Contact Darboux) Let (M, H) be a contact manifold and $p \in M$. Then there is a coordinate system

$$(\mathcal{U}, x_1, y_1, \dots, x_n, y_n, z)$$

centred at p, s.t.

$$\alpha = \sum x_i \, dy_i + dz$$

is a locally contact form for H.



$H = \ker \alpha$

$d\alpha|_H$ symp

$\therefore \dim M = 2n+1$

$\alpha \wedge (d\alpha)^n$ vol form

<u>Proof</u> ES 3. $\square$

<u>Theorem</u> (Gray stability) M compact, α_t a smooth 1-parameter family of global contact forms on M.
Let $H_t = \ker \alpha_t$.
Then $\exists$ isotopy $\rho : M \times \mathbb{R} \to M$ s.t $H_t = (\rho_t)_*(H_0)$ $\forall t$.

<u>Proof</u> ES 3. $\square$

<u>More examples</u>

1) Co-sphere bundle & geodesic flow

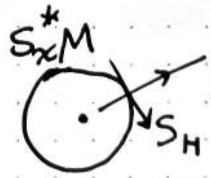
(X, g) where g a Riemannian metric

$$\begin{array}{c} S^*X = \{(x,\xi) \in T^*X : |\xi|_g = 1\} \\ \cap \\ T^*X \end{array} \quad \text{co-sphere bundle}$$

$$\begin{array}{ll} \pi : S^*X \to X & ; \quad \pi^{-1}(x) \cong S^{n-1} \\ (x,\xi) \mapsto x & \quad n = \dim X \end{array}$$

Hamiltonian $H : T^*X \longrightarrow \mathbb{R}$, $(x,\xi) \longmapsto \frac{1}{2}|\xi|_x^2$ "kinetic energy"

As in ES#1, define v via $\iota_v d\alpha = \alpha$ where α is the canonical 1-form on T^*X. $\lceil \omega = -d\alpha \rfloor$



Flow of v is $(x,\xi) \mapsto (x, e^t\xi)$.

X_H - Hamiltonian vector field of H

$$\iota_{X_H}(-d\alpha) = dH$$

<u>Note</u> X_H is clearly tangent to S^*X as

$$S^*X = H^{-1}(1/2).$$

$$-d\alpha(X_H, \nu) = dH(\nu) = \frac{d}{dt}\Big|_{t=0} H(x, e^t\xi)$$

$$= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} |e^t\xi|_x^2 = |\xi|_x^2$$

On S^*X, $d\alpha(X_H, \nu) = -1$.

So $\{X_H, \nu)$ is a symplectic 2-plane.

Then $\{X_H, \nu\}^\omega$ is symplectic of dimension $2n-2$.

<u>Claim</u> $\ker(\alpha|_{S^*X}) = \{X_H, \nu\}^\omega$

<u>Pf</u> Count dimensions and check that $\ker(\alpha|_{S^*X}) \supset \{X_H, \nu\}^\omega$ $\square$

$\therefore$ α on S^*X is a <u>contact form</u>

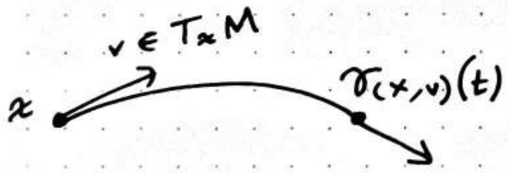
~~[illegible]~~ $\alpha(X_H) = d\alpha(\nu, X_H) = 1$

On S^*X: $\alpha(X_H) = 1$ and $\iota_{X_H} d\alpha = 0$

<u>Reeb vector field</u>.

X_H - geodesic vector field : flow of X_H, geodesic flow

If X is compact, its flow is defined $\forall t \in \mathbb{R}$.

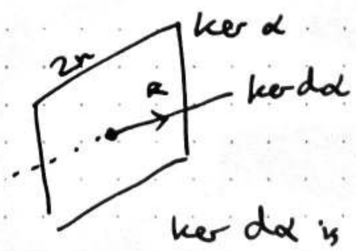


$\phi_t(x,v) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))$ geodesic flow on TM

<u>Reeb vector fields</u> M^{2n+1}, α global contact form: $\alpha \wedge (d\alpha)^n \neq 0$

The Reeb vector field associated with α is the unique vector field defined by

$$\begin{cases} \alpha(R) = 1 \\ \iota_R\, d\alpha = 0 \end{cases}$$

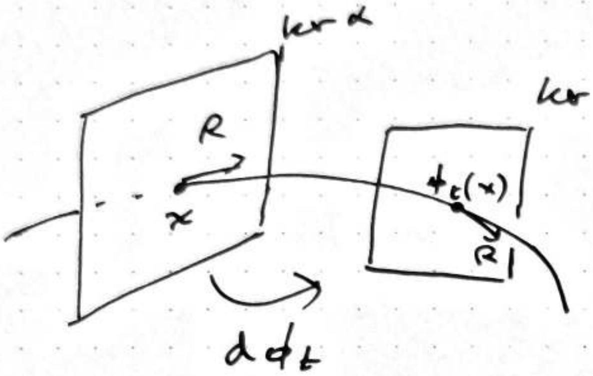


ker $d\alpha$ is 1-dim'l

Claim The flow of R preserves α

i.e. $\mathcal{L}_R \alpha = 0$

Pf $\mathcal{L}_R \alpha = d \underbrace{\underbrace{\iota_R \alpha}_{1}}_{\text{zero}} + \underbrace{\iota_R d\alpha}_{\text{zero}} = 0.$ $\square$



$(\mathbb{R}^{2n+1}, \alpha)$, $\alpha = \sum x_i dy_i + dz$,

$R = \frac{\partial}{\partial z}$ $\qquad d\alpha = \sum dx_i \wedge dy_i$

2) $S^{2n-1} \overset{\iota}{\subseteq} \mathbb{R}^{2n}$ $\qquad (x_1, y_1, \ldots, x_n, y_n)$ coords on $\mathbb{R}^{2n}$

$\quad\| $

$\{\sum x_i^2 + y_i^2 = 1\}$ $\qquad \sigma = \frac{1}{2}\sum(x_i dy_i - y_i dx_i)$ 1-form on $\mathbb{R}^{2n}$

Claim $\alpha = \iota^*\sigma$ is a contact form on S^{2n-1}

Pf $H = \frac{1}{2}\sum_i (x_i^2 + y_i^2)$, $\quad S^{2n-1} = H^{-1}(\frac{1}{2})$

$dH = \sum_i (x_i dx_i + y_i dy_i)$

Let v be defined by

$d\sigma = \sum_i dx_i \wedge dy_i$ (symp)

$\iota_v d\sigma = \sigma$

And $\iota_{X_H} d\sigma = dH$.

Compute v & X_H.

$S^{2n-1} \subseteq \mathbb{R}^{2n}$, coords $(x_1, y_1, \dots, x_n, y_n)$

$\sigma = \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$

- Claim $\alpha = \iota^* \sigma$ is a contact form on S^{2n-1}

Pf (cont.) $d\sigma = \sum_i dx_i \wedge dy_i$, $H = \frac{1}{2} \sum (x_i^2 + y_i^2)$,

$$\begin{cases} \imath_\nu d\sigma = \sigma \\ \imath_{X_H} d\sigma = dH \end{cases} \qquad dH = \sum (x_i dx_i + y_i dy_i)$$

Let us compute ν and X_H explicitly.

$$W = \sum a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$$

$$\imath_W (dx_i \wedge dy_i) = dx_i(W)\, dy_i - dy_i(W)\, dx_i$$
$$= a_i\, dy_i - b_i\, dx_i$$

- $W = \nu$: $\Rightarrow a_i = \frac{1}{2} x_i$, $b_i = +\frac{1}{2} y_i$

$W = X_H$: $\Rightarrow a_i = y_i$, $b_i = -x_i$

flow of ν?

$\dot{x}_i = \frac{1}{2} x_i$, $\dot{y}_i = \frac{1}{2} y_i$

$p \longmapsto p e^{\frac{1}{2}t}$

Hence $d\sigma(X_H, \nu) = dH(\nu)$

$$= \left.\frac{d}{dt}\right|_{t=0} \frac{e^t}{2} \sum_{i=1}^{n} x_i^2 + y_i^2$$

- $$= H.$$

So on S^{2n-1}, $d\sigma(X_H, \nu) = \frac{1}{2} \neq 0$

$\Rightarrow \{X_H, \nu\}$ is symplectic

$\Rightarrow \ker \alpha = \{X_H, \nu\}^{d\sigma}$ symplectic on S^{2n-1} $\Rightarrow \alpha$ contact

Reeb vector field: $\begin{cases} \alpha(R) = 1 \\ d\alpha(R, \cdot) = 0 \end{cases}$

$$\sigma(X_H) = \frac{1}{2} \sum_i (-x_i^2 - y_i^2) = -H$$

- So on S^{2n-1}, $\alpha(X_H) = -\frac{1}{2}$.

Hence $\boxed{R = 2 \sum_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)}$

flow of R ?

$$\dot{x}_i = -2y_i \, , \; \dot{y}_i = 2x_i$$

$z_i = x_i + \sqrt{-1}\, y_i$, so $\dot{z}_i = 2\sqrt{-1}\, z_i$

$$z_i = e^{2t\sqrt{-1}} z_i(0)$$

$$\phi_t(z_1, \dots, z_n) = e^{2t\sqrt{-1}}(z_1, \dots, z_n)$$

All orbits of R are <u>periodic</u>, Hopf fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n-1} \\ & & \downarrow \\ & & \mathbb{CP}^{n-1} \end{array}$$

<u>Symplectisation</u> (M, H) contact manifold with global contact form α

$\pi : M \times \mathbb{R} \to M$, and define $\omega = d(e^s \pi^* \alpha)$
$\quad \; s \qquad \alpha$

Then ω is <u>symplectic</u> (ES# 3)

If R is the Reeb vector field of α, then

$$\omega\left(R, \tfrac{\partial}{\partial s}\right) = -e^s \neq 0$$

<u>Conjectures of Weinstein & Seifert</u>

S^3, Seifert 1948: X any vector field on S^3 s.t. X is non-zero for all $x \in S^3$.

Does X have a closed orbit ?

P. Schweitzer 1974: C^1-counterexample [priest in Rio]

Kristine Kuperberg 1994: C^∞ - "

Volume-preserving? Greg Kuperberg 1997: C^1-counterexample [son]

<u>C^∞ - open</u>

Ω : volume form on S^3, demand $\mathcal{L}_X \Omega = 0$

$$\mathcal{L}_X\Omega = d\iota_X \Omega + \cancel{\iota_X d\Omega} \rightarrow \text{zero}$$

On S^3, since $H^2(S^3,\mathbb{R}) = 0$, must have

$$\iota_X \Omega = d\alpha \quad \text{for some 1-form } \alpha$$

Clearly you don't own an airfrier.

$$\iota_X d\alpha = 0$$

<u>If</u> $\alpha(X) > 0$ then $R := \dfrac{X}{\alpha(X)}$ satisfies $\begin{cases} \alpha(R) = 1 \\ \iota_R d\alpha = 0 \end{cases}$

And $\alpha \wedge d\alpha \neq 0$ $\quad \ulcorner \alpha \wedge d\alpha = \alpha(X)\cdot \Omega \lrcorner$

<u>Weinstein conjecture</u> ('78) : Any Reeb vector field admits a periodic orbit.

<u>Hofer '93</u> : S^3 ✓

<u>Taubes '06</u> : any 3-manifold ✓

<u>Hamiltonian Mechanics</u>

(M,ω) symplectic mfd, $H : M \longrightarrow \mathbb{R}$

<u>Def</u> X_H is defined by $\boxed{\iota_{X_H}\omega = dH}$

Assume flow of X_H is defined $\forall t \in \mathbb{R}$,

$\rho_t : M \circlearrowleft$ flow, $\rho_{t+s} = \rho_t \circ \rho_s$, $t,s \in \mathbb{R}$

$$\mathcal{L}_{X_H}\omega = d\underbrace{\iota_{X_H}\omega}_{dH} + \iota_{X_H} d\omega = 0$$

So $\boxed{\rho_t^*\omega = \omega}$.

<u>Energy is preserved</u>: $\dfrac{d}{dt} H(\rho_t x) \overset{!}{=} 0$

$$\begin{aligned} &\frac{d}{dt} H(\rho_t x) \\ &\quad \| \\ &dH(X_H(\rho_t x)) \\ &\quad \| \\ &\omega(X_H, X_H) = 0 \end{aligned}$$

<u>Def</u>: Let X be any v. field on (M,ω) s.t. $\mathcal{L}_X \omega = 0$

Then X is said to be <u>symplectic</u> $\Leftrightarrow$ $\iota_X\omega$ is closed

<u>Hamiltonian</u> $\Leftrightarrow$ $\iota_X \omega$ is exact

Example $M = \mathbb{T}^2$, $\omega = d\theta_1 \wedge d\theta_2$

$\qquad\quad\ \ \| $
$\quad\ S^1 \times S^1$

Then $X = \partial/\partial\theta_1, \partial/\partial\theta_2$ are symplectic but not Hamiltonian

(M,ω)

$H: M \to \mathbb{R}$

$\cdot\ \iota_{X_H} = dH$

symplectic vector field X

$$\mathcal{L}_X \omega = 0$$
$$\parallel$$
$$d\,\underbrace{\iota_X \omega}$$

Classical mechanics

$\mathbb{R}^{2n} = T^* \mathbb{R}^n$ $\quad \{\underbrace{q_1, \dots, q_n}_{\text{position variables}}, \underbrace{p_1, \dots, p_n}_{\text{momentum variables}}\}$

$$\omega_0 = \sum_{j=1}^{n} dq_j \wedge dp_j$$

$H: \mathbb{R}^{2n} \to \mathbb{R}$, Hamiltonian

$(q(t), p(t))$ is an integral curve of X_H if

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$ "Hamilton equations"

We need to check that

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

$\iota_{X_H} \omega_0 = ?$

$$\left\ulcorner\ \iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\iota_X \beta) \right\lrcorner$$

$$\iota_{X_H}(dq_j \wedge dp_j)$$
$$= \iota_{X_H}(dq_j) \cdot dp_j - \iota_{X_H}(dp_j) \cdot dq_j$$
$$= \frac{\partial H}{\partial p_j} \cdot dp_j + \frac{\partial H}{\partial q_j} \cdot dq_j$$

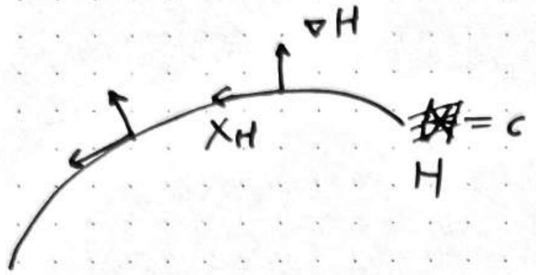
So $\iota_{X_H} \omega_0 = dH$ ✓

$$J : \partial/\partial q_i \mapsto -\partial/\partial p_i$$
$$\partial/\partial p_i \mapsto \partial/\partial q_i$$

$J^2 = -1$

"complex structure" on $\mathbb{R}^{2n}$

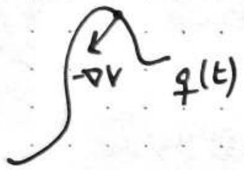
$$\boxed{X_H = J \nabla H}, \quad \nabla H = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial}{\partial q_j} + \frac{\partial H}{\partial p_j}\frac{\partial}{\partial p_j}$$



} symplectic gradient

$\underline{n=3}$ $\quad V : \mathbb{R}^3 \to \mathbb{R}$ potential

$$m\frac{d^2q}{dt} = -\nabla V(q) \qquad \text{"Newton's 2nd law"}$$



$$\mathbb{R}^6 = T^*\mathbb{R}^3,$$

$$H = \frac{1}{2m}|p|^2 + V(q) \qquad \text{"Hamiltonian"}$$

Hamilton eq's:

$$\begin{cases} \dot{q}_i = \frac{1}{m} p_i \\ \dot{p}_i = -\frac{\partial V}{\partial q_i} \end{cases}$$

⌐ $\frac{1}{2}m|\dot{q}|^2$?

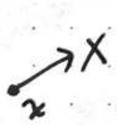
V, g IP space
$\downarrow$
$V \xrightarrow{\cong} V^*$
$v \mapsto \langle v, \cdot \rangle$ ⌐

$\sharp, \flat$

Poisson bracket

Lie bracket: $X : C^\infty(M) \to C^\infty(M)$

$$f \mapsto X(f) = df(X)$$



1^{st} order diff operator

$$[X, Y] = XY - YX$$

Exercise $\mathcal{L}_{[X,Y]}\alpha = [\mathcal{L}_X, \mathcal{L}_Y]\alpha$

$$= \mathcal{L}_X \mathcal{L}_Y \alpha - \mathcal{L}_Y \mathcal{L}_X \alpha \qquad \text{(cool!)}$$

Proposition (M, ω) symplectic manifold. Let X, Y be symplectic vector fields. Then $[X, Y]$ is Hamiltonian, with Hamiltonian function $\omega(Y, X)$.

<u>Proof</u> $\iota_{[X,Y]}\omega = \mathcal{L}_X \iota_Y \omega - \iota_Y \underbrace{\mathcal{L}_X \omega}_{\text{zero}}$

X, Y symp $\Downarrow$ $\mathcal{L}_X\omega = \mathcal{L}_Y\omega = 0$

$$= d\,\iota_X \iota_Y \omega + \iota_X \underbrace{d\,\iota_Y \omega}_{\text{zero}}$$

$$= d\big(\omega(Y, X)\big) \qquad \square$$

$\mathfrak{X}(M)$ – all C^∞ vector fields

$\mathfrak{X}^{sym}(M)$ – " symplectic v. fields

$\mathfrak{X}^{Ham}(M)$ – " Hamiltonian "

$$(\mathfrak{X}^{Ham}, [\cdot,\cdot]) \subseteq (\mathfrak{X}^{Sym}, [\cdot,\cdot]) \subseteq (\mathfrak{X}, [\cdot,\cdot])$$

inclusions of <u>Lie algebras</u>

<u>Def^n</u> The <u>Poisson bracket</u> of two functions
$f, g \in C^\infty(M)$ is

$$\{f, g\} = \omega(X_f, X_g)$$

Proposition $\Rightarrow$ $-X_{\{f,g\}} = [X_f, X_g]$

The Poisson bracket satisfies

1) Jacobi identity $\{f, \{g, h\}\} + \text{cyc} = 0$
2) Leibniz $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (exercise)

$$C^\infty(M) \longrightarrow \mathfrak{X}^{Ham}(M)$$
$$H \longmapsto X_H$$
$$\{\cdot,\cdot\} \longmapsto -[\cdot,\cdot]$$

is a Lie algebra anti-homomorphism

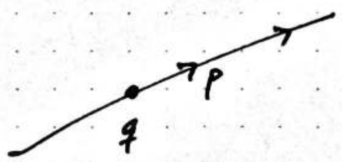
Integrals of motion f is constant along the orbits of X_H iff $\{f, H\} = 0$

Proof
$$\{f, H\} = \omega(X_f, X_H)$$
$$= -\omega(X_H, X_f)$$
$$= df(X_H)$$
$$= -dH(X_f) \qquad \square$$

Jacobi identity $\Rightarrow$ Poisson bracket of integrals of motion is again an integral of motion.

Example : $M^4 = T^*\mathbb{T}^2$, $H(q,p) = \frac{1}{2}|p|^2$

$$\{H, p_i\} = 0$$



Integrable system

(M, ω, H), $\dim M = 2n$

Suppose $(f_1 = H, f_2, \dots, f_n)$ (are first integrals) s.t. $\{f_i, f_j\} = 0$ and $\{df_1, \dots, df_n\}$ lin indep on an open, dense subset of M.

$F = (f_1, \dots, f_n) : M \to \mathbb{R}^n \ni c$

$F^{-1}(c)$ in nice situation, $F^{-1}(c)$ is a Lagrangian torus (if compact say)

Arnol'd-Liouville Theorem