Topics in Symplectic Topology (IV)

~ Ailsa Keating

Def M^{2n} is a symplectic manifold if it's equipped with a closed, non-degenerate 2-form ω.

$(\omega \in \Omega^2(M))$ $(d\omega = 0)$

$\Omega^2(M)$: space of all 2-forms

⌈ in local coords $x_1, \dots, x_{2n}$
$\omega = \sum f_{ij} \, dx_i \, dx_j$ ⌋

⌈ $\dfrac{\partial f_{ij}}{\partial x_k} + \dfrac{\partial f_{jk}}{\partial x_i} + \dfrac{\partial f_{ki}}{\partial x_j} = 0$ ⌋

Rmk ① TM tangent bundle

$\omega \in \Omega^2(M) \leftrightarrow$ alternating pairing on T_pM $(\simeq \mathbb{R}^{2n})$ for each $p \in M$

- non-deg: pairing is non-deg in the linear algebra sense

Note: Linear algebra: given $\mathbb{R}^{2n}$, there's a unique non-deg alternating form up to change of basis

Why study symplectic manifolds?

① classical mechanics

X: space that a particle lives in

T^*X: space of (position, momentum) for that particle. This is naturally symplectic.

Aside: cotangent bundles

X manifold, local coords $x_1, \dots, x_k$

local coords on T^*X: $\overset{q_1}{\widehat{x_1}}, \dots, \overset{q_k}{\widehat{x_k}}, \overset{p_1}{\widehat{dx_1}}, \dots, \overset{p_k}{\widehat{dx_k}}$ ...

Claim T^*X has a natural symplectic form

Pf Set $\theta = \sum p_i \, dq_i \in \Omega^1(T^*X)$, $\omega = d\theta$

Check: these patch together to give a closed non-deg 2-form on T^*X

(Pf: Jacobian on both sides cancel out)

Ex $X = \mathbb{R}^k$, $q_1, \dots, q_k$

$T^*\mathbb{R}^k \cong \mathbb{R}^{2k}$ $(q_1, \dots, q_k, p_1, \dots, p_k)$

$\theta = \sum_i p_i \, dq_i$, $\omega = d\theta = \sum_i dp_i \, dq_i$ $\left(\leftrightarrow \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \text{ matrix for pairing} \right)$

② Classification of Lie groups (Cartan–Lie) has the following upshot: the following additional structures on TM "make sense":

- metric
- volume form
- symplectic
- contact (odd-dim-l versions of symplectic)
- conformal versions of these

(–.)

③ Lots of smooth algebraic varieties are symplectic (e.g. Kähler)

Exs · smooth subvarieties of $\mathbb{C}^n$ or $\mathbb{CP}^n$

symplectic form: $\mathbb{C}^n$, $\sum dx_j\, dy_j$; this restricts to a symplectic form on subvarieties (!) ⊛ woah

$z_j = x_j + iy_j$

similarly for $\mathbb{CP}^n$: have a form on each coord chart, which patch together (Fubini–Study)

<u>Slogan</u>: symplectic geometry is a good framework for studying families of varieties (e.g. deforming the coeffs of a polynomial eq.) focus of this course

<u>Aside</u>: $\mathbb{C}^n$, $T_p\mathbb{C}^n \simeq \mathbb{C}^n$

(*) $\cup$ $\qquad \cup$

X cx subvariety $\qquad \mathbb{C}^k$ cx vector subspace

restriction is non-deg

<u>Hamiltonian vector fields</u> (M^{2n}, ω) symplectic

$f \in C^\infty(M)$

$df \in \Omega^1(M)$ i.e. a one-form

There is a unique vector field X s.t.

$$\underbrace{\iota_X \omega}_{\omega(\cdot, X)} = df$$

<u>Rmk</u> You just need $\iota_X\omega$ to be closed

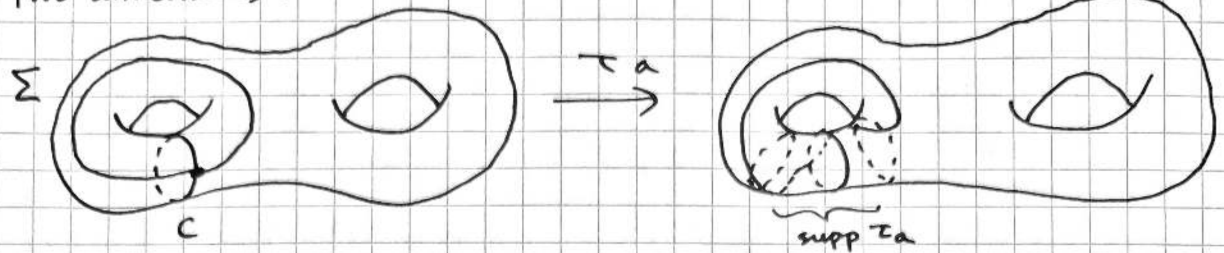
<u>Claim</u> The flow of X preserves ω

<u>Pf</u> Cartan's formula

$$\underbrace{\mathcal{L}_X \omega}_{\frac{d}{dt}(\phi_t^* \omega)} = d\underbrace{\iota_X \omega}_{df} + \iota_X \overbrace{d\omega}^{\text{zero}} = 0. \quad \square$$

Dehn Twists (take 1)

- Two dimensions:



τ_a: orientation preserving diffeo of Σ, defined up to smooth isotopy

Fact: In the smooth category, Dehn twists along simple closed curves generate the mapping class group. $(\pi_0 \, \mathrm{Diff}^+ \Sigma)$

π_0: connected components [equiv classes under isotopy]
Diff: diffeos
$+$: orientation preserving

- Two dimensions, ctd.: upgrade to a symplectomorphism

Def (M, ω) symplectic mfd, $f \in \mathrm{Diff}(M)$ is called a symplectomorphism if $f^*\omega = \omega$

Ex (L1) If $f = \varphi_1$ is the time 1 flow of a Hamiltonian vector field, then f is a symplecto.

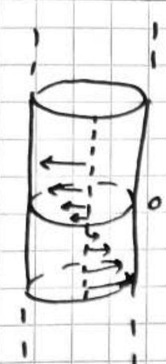
Σ^2 is symplectic iff orientable, and $\omega = d\mathrm{vol}$ (some volume form)

Note:

Check: • all 2 forms on Σ^2 are closed

- non-degeneracy $\Leftrightarrow$ ω is a volume form ($d\mathrm{vol}_p > 0$, all $p \in \Sigma$) ($\Rightarrow \Sigma$ is orientable.)
- conversely: given orientable 2-mfd, have volume form, so any one is symplectic

Fact (follows from Moser's theorem)
If $f \in \mathrm{Diff}(\Sigma)$ preserves orientation, then it can be smoothly isotoped to preserve the symplectic form.

- 

T^*S^1 coords q on base, p dual on fiber

0 section

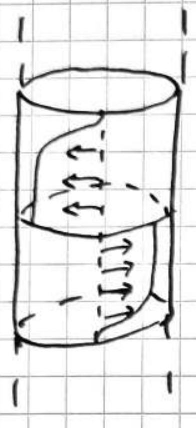
$$H = \tfrac{1}{2}|p|^2 \quad \text{"Hamiltonian function"}$$

$H \in C^\infty(T^*S^1)$

what's the associated vector field?

$(X_H \text{ s.t. } \omega(-, X_H) = dH)$

<u>Rk</u> the flow of X_H is geodesic flow for standard metric ("round")



$$\tilde{H} = |p| \cdot \psi \quad \leftarrow \text{cutoff}$$

$X_{\tilde{H}}$ not defined on zero section

<u>Note</u>: at $t = \pi$, the flow of $X_{\tilde{H}}$ extends over the zero section! (even after cutoff)

This is a Dehn twist in the zero section.

⌜ Why does this preserve ω?

- Hamiltonian flows preserve ω
- zero section? locally, $\tau_{\text{zero-sec}}$ is $\iota^*: T^*S^1 \to T^*S^1$ is induced by $\iota: S^1 \to S^1$ ⌟

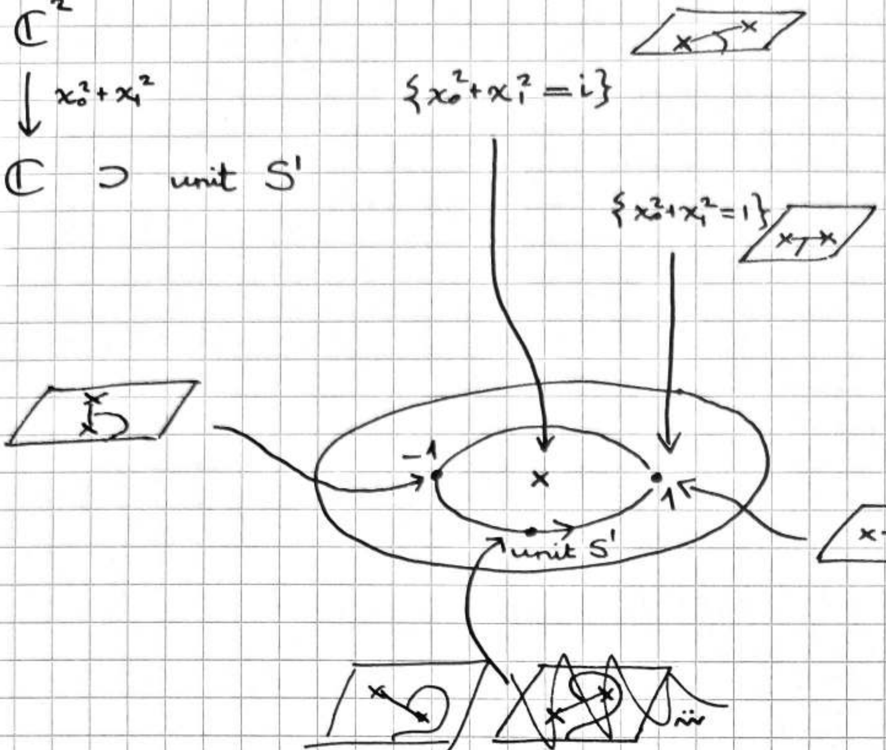
- Dehn twists from one-parameter families in alg geometry

<u>Lemma</u>: $\{x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{C}^{n+1}$ $\overset{\text{sympl}}{\cong}$ T^*S^n w/ standard symplectic forms on either side

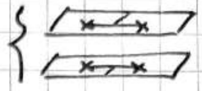
real locus $\updownarrow$ zero section

<u>Pf</u> Exercise if unseen.

$$\begin{array}{c} \mathbb{C}^2 \\ \big\downarrow {\scriptstyle x_0^2+x_1^2} \\ \mathbb{C} \supset \text{unit } S^1 \end{array}$$

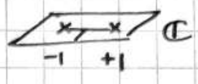


$\{x_0^2+x_1^2=1\}$



$\downarrow x_0$

$\mathbb{C}$



⌈gluing along a branch cut:

T^*S^1 ⌋

$\downarrow$

Dehn twist in zero secⁿ!

<u>Slogan</u> D.T. is monodromy of the S^1 family

⌈Q: something about compact support, Dehn ≃ id on cylinder T^*S^1 ⌋

Dehn twists in higher dimensions

Recall: T^*S^n $\left(\cong \{x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{C}^{n+1}\right)$

symplectic, $\omega = \sum dp_i \wedge dq_i$

$q_1, \dots, q_n$ local coords on S^n

$p_1, \dots, p_n$ dual coords on T^*S^n fibers

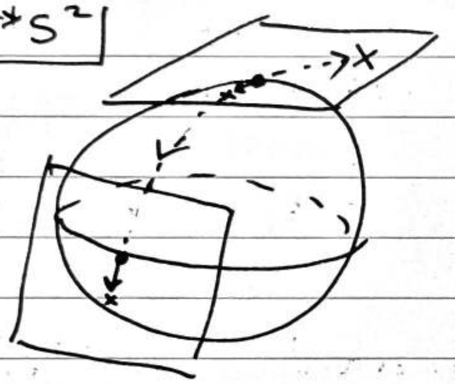
T^*S^n, Hamiltonian function $\tilde{H} = \|p\| = (\sum p_i^2)^{1/2}$ ← from metric

(Check this is well-defined as a f^n on T^*S^n) ⚠ really need metric!

corresponding Ham. vector field $X_{\tilde{H}}$

take $\{q_i\}$ s.t. $\{\frac{\partial}{\partial q_i}\}$ o.n. then $\|p\|$ is metric induced on T^*S^n

$\boxed{T^*S^2}$



- $X_{\tilde{H}}$ not defined over zero section
- "unit" vector field in direction given by $\xi \in T^*_pS^n \cong T_pS^n$
- flow of $X_{\tilde{H}}$? travelling at constant speed in dir^n determined by cotangent vector

- At $t = \pi$, the flow of $X_{\tilde{H}}$ extends over the zero section; send $x \mapsto -x$

This is because the round metric on S^n has periodic geodesic flow

⌜extension is symplecto⌟

$\tau_{S^n} \in \pi_0 \, \mathrm{Symp}_c \, T^*S^n$

(π_0: up to isotopy (within); Symp_c: cptly supported symplectos)

- time π flow of $X_{\tilde{H}\cdot\psi}$ where ψ is a bump function, 1 in a nghd of zero section, 0 at far away from zero section
- extend over the zero section with antipodal map

- $H_n(D^*S^n, \partial)$ homology relative to boundary

A compactly supported homeomorphism gives a well-defined ~~[illegible]~~ hom of relative homology

<u>Exercise:</u> $\tau_{S^n}^k([D_p^* S^n]) = \begin{cases} [F] + (-1)^k [0\text{-sec}^n] & \text{if } n \text{ even} \\ [F] + k[0\text{-sec}^n] & \text{if } n \text{ odd} \end{cases}$

$= [F]$

class of a cotangent fiber

Topological world (Kronheimer)

$n = 2,6$: $\tau_{S^n}^2 \simeq id$

isotopic to id, rel ∂

other n $\tau_{S^n}^n \simeq id$

(but $\tau_{S^n}^2 \not\simeq id$)

(Krylov, Kaufman)

Symplectic world

<u>Seidel</u>: $\tau_{S^n} \in \pi_0 \mathrm{Symp}^c T^*S^n$ has infinite order !

4D Dehn twists in smooth setting

Thm (Kronheimer)

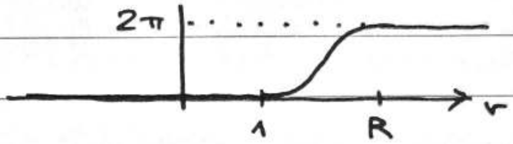
$\tau_{S^2}^2 \simeq \mathrm{id} \in \pi_0 \mathrm{Diff}^c(T^*S^2)$
smoothly

Pf · Notation: $R^{\theta}_{(u,v)}$ rotation of $\mathbb{R}^3$ through θ

where: · axis of rotation orthogonal to u, v $(u \wedge v \neq 0)$

· θ angle of the rotation; convention (u,v) ordered positively

Pick $\theta : \mathbb{R} \to \mathbb{R}$ s.t. $\quad$ $R > 1$ fixed
$r \in$



A smooth representative for τ^2 is given as follows.

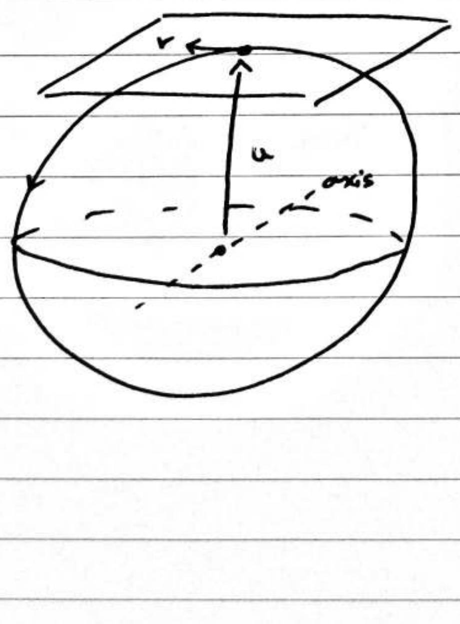
T^*S^2 $\quad$ (identify $TS^2 \cong T^*S^2$ with, say, round metric)

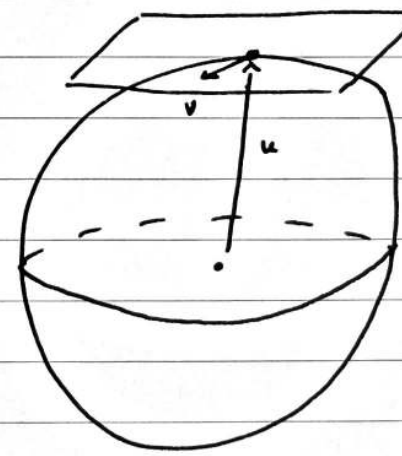
$(u,v) \in$ $\quad$ $u, v \in \mathbb{R}^3$, $\|u\| = 1$, $v \perp u$

$$(u,v) \longmapsto \begin{cases} R^{\theta(\|v\|)}_{(u,v)}(u,v), & v \neq 0, \\ (u,v), & v = 0. \end{cases}$$

⚠ We're working in smooth category

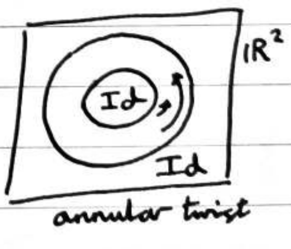
τ





$\tau^2 \simeq S_1$

· preserves cotangent fibers



annular twist

Observe: for $v \neq 0$, $u, v, u \wedge v$ are linearly independent

Slogan: isotope axes of rotation from $u \wedge v$ to u

$$S_t(u,v) = \begin{cases} R^{\theta(\|v\|)}_{((1-t)u + t\, u\wedge v,\, v)}(u,v), & v \neq 0 \\ (u,v), & v = 0. \end{cases}$$

$t=0 : \tau^2$

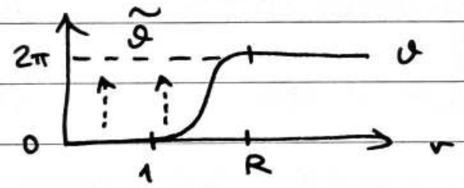
smooth isotopy

all S_t have compact support by assumption on θ

S_1 uses $R^{\theta(\|v\|)}_{(u\wedge v,\, v)}$ rotation in axis u with angle θ

Now we can isotope (smoothly) from S_1 to id, via compactly supported maps, by changing θ to $\tilde{\theta}$ the constant 2π

(read: linearly interpolate)



$\smile$ $\square$

Remarks 1) Can check similarly (slogan "octonions") that the square of $\tau^2_{S^6} \simeq \mathrm{id} \in \pi_0 \operatorname{Diff}^c(T^*S^6)$

2) All other (even) dimensions, smooth order is 4 or 8.

(4 when Kervaire sphere is trivial in dim $2n+1$, 8 otherwise)

As a compactly supported homeomorphism: order 4 (unless $n = 2, 6$)

Aside on Kervaire sphere

Thm (Milnor) There exist manifolds homeomorphic but not diffeomorphic to S^n (for $n = \cancel{8}\, 7$ in original)

$\mathbb{C}^{L+1}$, $x_0, \dots, x_L$

$\{ x_0^2 + \dots + x_{L-1}^2 + x_L^3 = 1 \} \cap S_N^{2L+1}$ = Kervaire sphere

real dimension $2L$

smooth symplectic manifold, called "the A_2 Milnor fibre"

this is called the link of this fibre

Hamiltonian flow for $H \in C^\infty(T^*S^n, \mathbb{R})$
$(H = |p|)$, called φ_t^H

Assume $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$

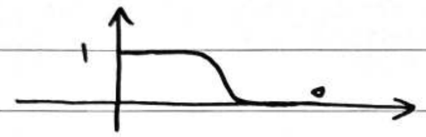
Hamiltonian flow of $\psi \circ H$, called $\varphi_t^{\psi \circ H}$

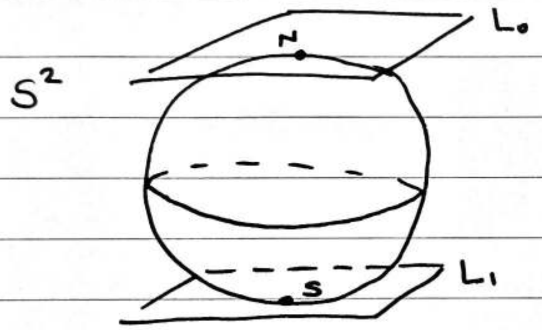
<u>Lemma</u> $\varphi_t^{\psi \circ H}(x) = \varphi^H_{t\psi'(H(x))}(x)$

what?? no :(

<u>Pf</u> Def^{n}s of flows + chain rule [Check] $\square$

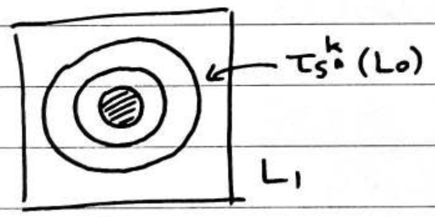
$\tau_{S^n}^k \curvearrowright T^*S^n$

ψ: 



- Look at $\tau_{S^n}^k(L_0) \cap L_1$

<u>Check</u>: for suitable choice of ψ, this is a disc plus k concentric circles.



"k odd"
(k=5 in pic)

<u>Fact</u>: there is generalised form of homology in a symplectic setting, where

<u>input</u>: 2 Lagrangians

<u>output</u>: abelian group [ring $\mathbb{Z}$] invariant under compactly supported isotopies

In our setting, each copy of S^1 contributes 2 generators

$\Rightarrow$ no power of τ_{S^n} can be isotopic to the identity

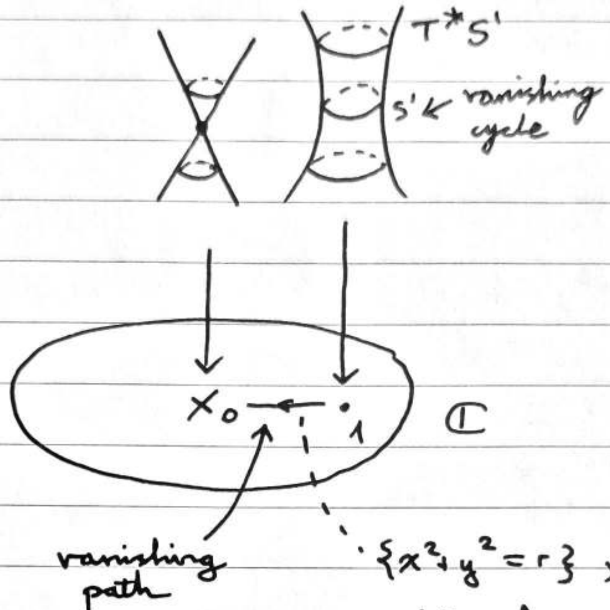
$\ulcorner$ $HF(L_1, \tau^k L_0) \cong \mathbb{Z}^{2k}$

Floer rank grows linearly $\lrcorner$

T^*S^n revisited (towards Lefschetz fibrations)

Warm-up

$$\begin{array}{c} \mathbb{C}^2 \\ \pi \downarrow x^2+y^2 \\ \mathbb{C} \end{array}$$



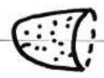
$\{x^2+y^2=0\}$
$=$
$\{x+iy=0\}$
$\cup \{x-iy=0\}$
"two copies of $\mathbb{C}$ intersecting transversely"

$\{x^2+y^2=r\}$, $r\in[0,1]$
radius of zero section $\sqrt{r}$ goes to zero

Consider union of vanishing cycles over $[0,1]$.

This is $D = \{(x,y)\in\mathbb{C}^2 \mid x,y\in\mathbb{R},\ x^2+y^2\leq 1\}$

a disc! Call it Lefschetz ~~thimbk~~ thimble.



D is Lagrangian in $\mathbb{C}^2$

i.e. $\omega|_D = 0$ and half-dimensional

<u>Aside</u>: symplectic parallel transport fibres F_a to F_b along a path γ missing 0

<u>Need</u>: given $v\in T_tB$, need a lift to $\cancel{T_pF_t}$ $T_p\mathbb{C}^2$ for all $p\in\pi^{-1}(t)=F_t$

$\left\{\begin{array}{l}\text{lift means}\\ D\pi(\tilde{v})=v\end{array}\right\}$

1) Natural way to do this symplectically.

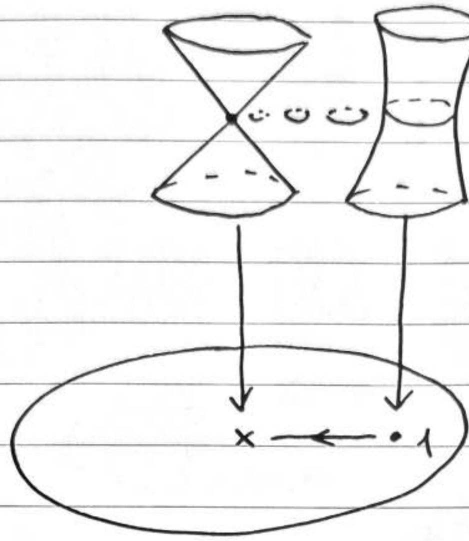
$$T_zF_p \underset{\substack{\text{symp}\\\text{vector}\\\text{subspace}}}{\leqslant} T_z\mathbb{C}^2$$

Let $(T_zF_p)^\perp$ be its orthogonal complement. Done.

2) <u>Exercise</u>: the map $F_a\to F_b$ induced is symplecto. ← cool! appropriate use of Cartan

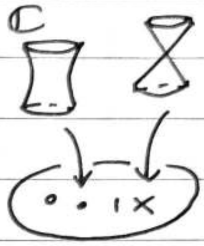
Recall:

$$\begin{array}{cc} \mathbb{C}^2 & x,y \\ \downarrow & \downarrow \\ \mathbb{C} & x^2+y^2 \end{array}$$



Lefschetz thimble (D^2)

equivalent to
$\{x^2+y^2+w=1\} \subseteq \mathbb{C}^3$
$\downarrow$ $proj_3$
$\mathbb{C}$

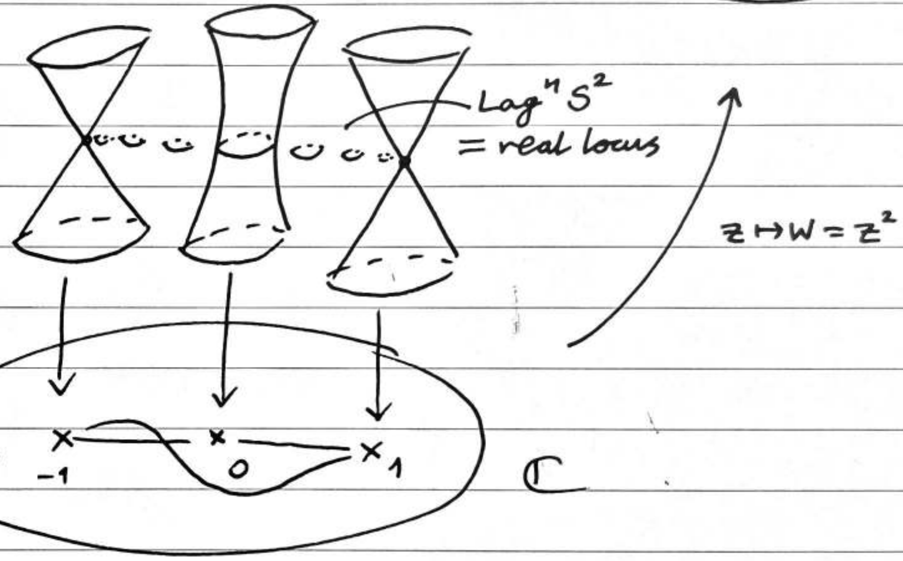


$\{x^2+y^2+z^2=1\} \subseteq \mathbb{C}^3$

$\cong T^*S^2$

$\downarrow proj_3$

$\mathbb{C}$



Relation b/w the two: branch double cover on base, & pull it back to fibers.

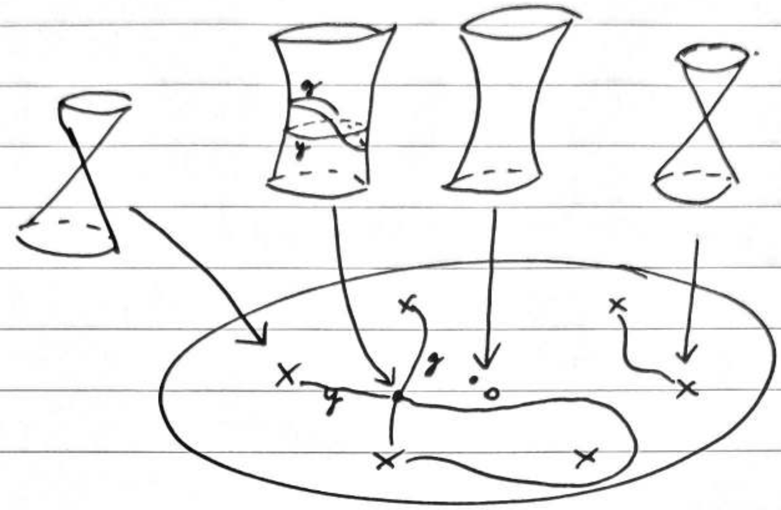
Recall: deforming the vanishing path gives a different vanishing cycle which is Hamiltonian isotopic to the original one
Up to smooth isotopy rel endpoints $\exists!$ path joining -1 & 1 in $\mathbb{C}$. Any choice of path gives a Lagrangian sphere.
Deforming the path back to the straight line path; the spheres are all Hamiltonian isotopic

⌜In fact: for paths close enough to the straight line, $\Gamma(\underset{\alpha}{\cancel{A}}) \subset T^*S^2$ is your sphere⌟

$\{x^2 + y^2 + z^d = 1\} \subseteq \mathbb{C}^d$

$\downarrow \text{proj}_3$

$\mathbb{C}$



Any embedded segment in the base with endpoints at d^{th} roots of unity gives a Lag^n sphere in the total space

· Path in the same : matching path

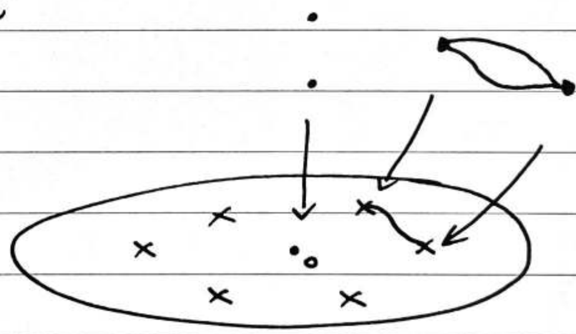
· Lagrangian sphere : matching cycle

Dehn twists in matching cycles: toy model (T^*S^0)

$\{x^2 + z^d = 1\} \subseteq \mathbb{C}^2$

$\downarrow \text{proj}_2$

$\mathbb{C}$

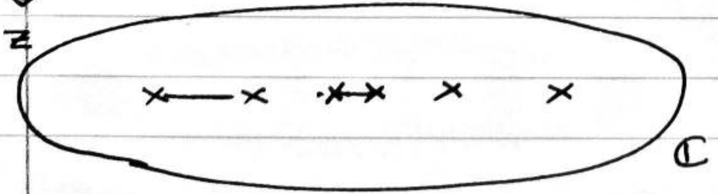


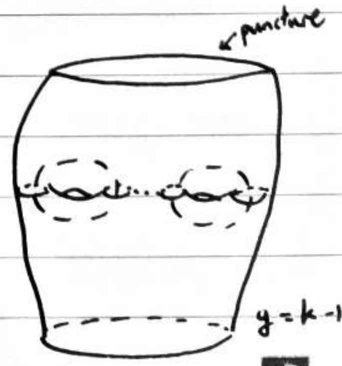
Pictorial simplification: replace z^d by a poly of degree d with distinct real roots, $p(z)-1$ say.

$\{x^2 + p(z) = 1\}$

$\downarrow$

$\mathbb{C} \ni z$

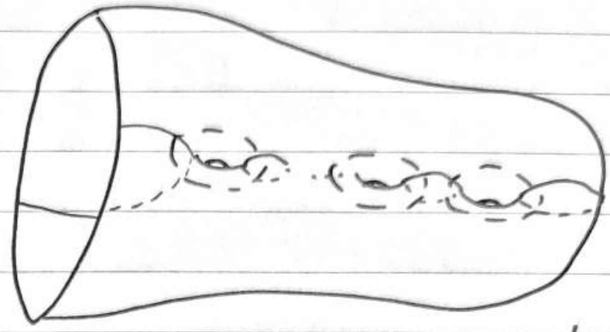




L6.3

d odd

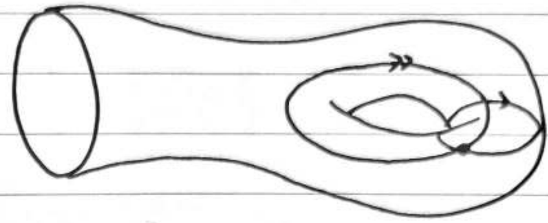
$d = 2k+1$

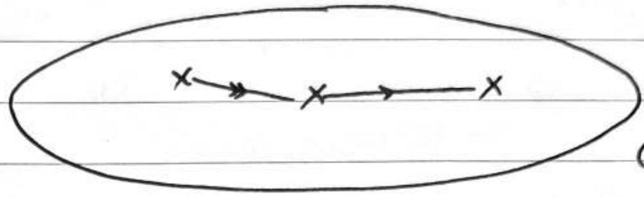


$k+1$ S^1's

Braid groups acting by Dehn twists

2D toy case from last time





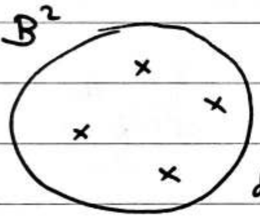
$\mathbb{C}$

$$\underset{\substack{\uparrow \\ \text{braid group} \\ \text{on } d \text{ strands}}}{Br_d} = \pi_1 \underbrace{Conf(d, B^2)}$$

braid group on d strands

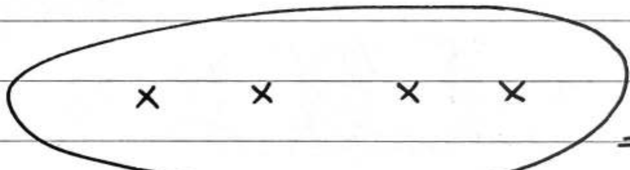
configuration space of d marked points (distinct, unordered) in a 2-disc

B^2

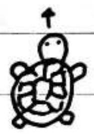


$d=4$

wlog: basepoint (for describing π_1) is the following conf^n:

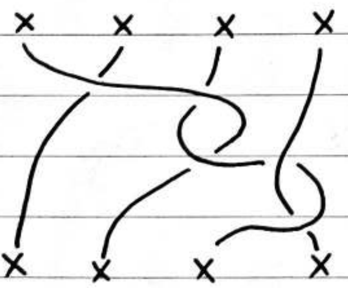


$= *$



tortuga

⌜ d strings, fixed at top and bottom ends



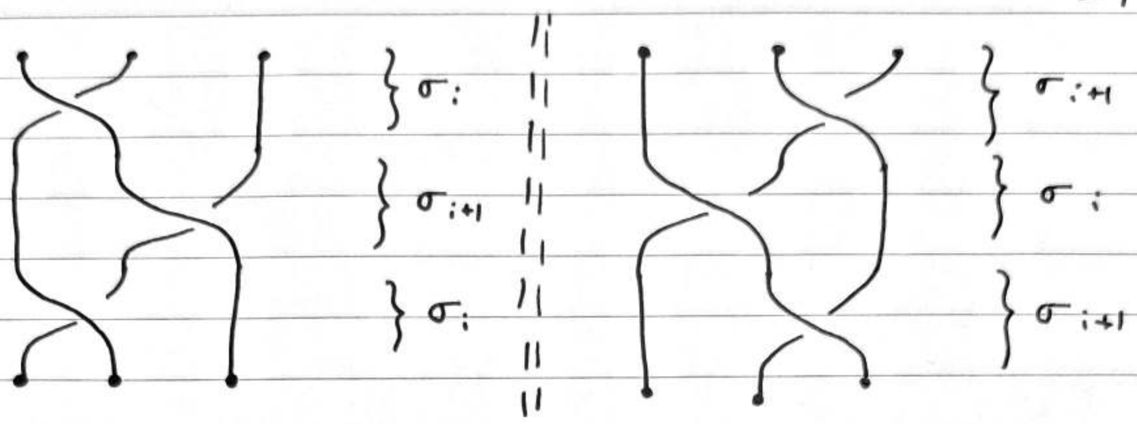
$\downarrow t$ ⌟

- Generators: $\frac{1}{2}$ twists in adjacent strings

σ_i :

$\{\sigma_1, \ldots, \sigma_{d-1}\}$

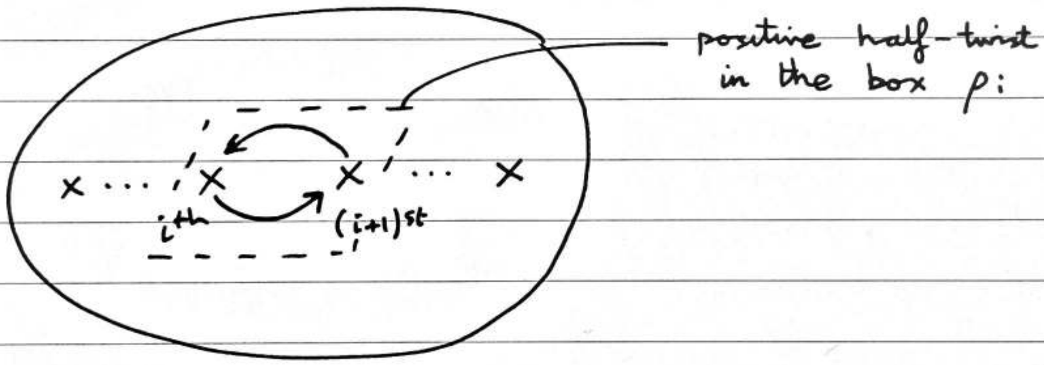
- Relations:
  - → $|i-j|>1$ implies $\sigma_i\sigma_j = \sigma_j\sigma_i$
  - → $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ ("braid relation")



these are the same braid

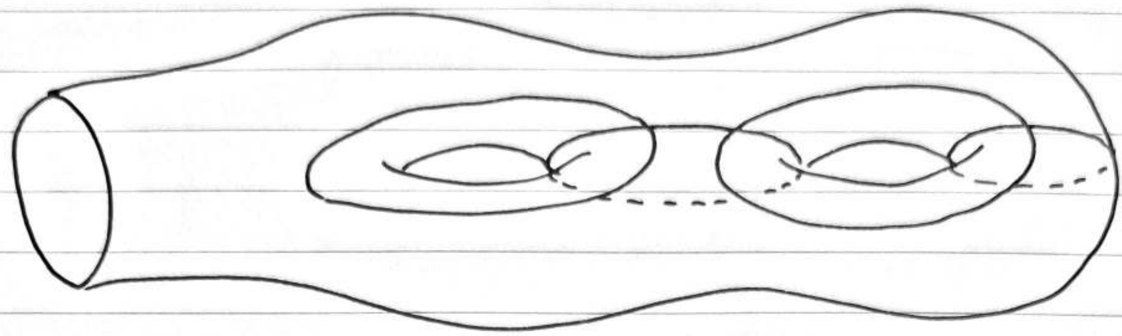
Theorem (E. Artin): The above generators & relations give a presentation for Br_d.

σ_i on $(B^2,$ marked points$)$



Positive half-twist can be pulled back to act on R. surface $\{x^2+z^d=1\} \subseteq \mathbb{C}^2$

From the discussion on monodromy of $\mathbb{C}^2 \to \mathbb{C}$, $(x,y) \mapsto x^2+y^2$, we get that ρ_i lifts to the Dehn twist τ_i in the i^{th} matching S^1.



Br_5 acts via Dehn twists in these 4 curves!
(in particular, get braid relation $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$.

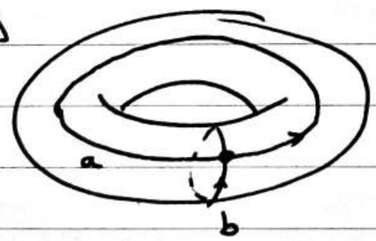
- Br_d acts faithfully on $(B^2, d \text{ pts})$ (†)
- Question:

$$Br_d \circlearrowright \{x^2+z^d=1\}$$

by Dehn twists

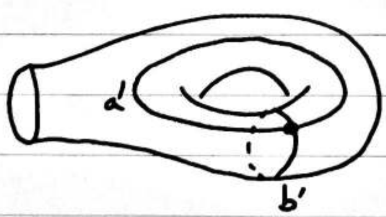
Q: Is this faithful?
A: Yes! Birman-Hilden: if a product of Dehn twists is isotopic to the identity, then it is $\mathbb{Z}/2$-equivariantly isotopic to the identity.
$\Rightarrow$ (†) implies the action on $\{x^2+z^d=1\}$ is faithful

⚠



Check: $(\tau_a \tau_b)^6 \simeq Id$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\begin{pmatrix}1&0\\-1&1\end{pmatrix}$

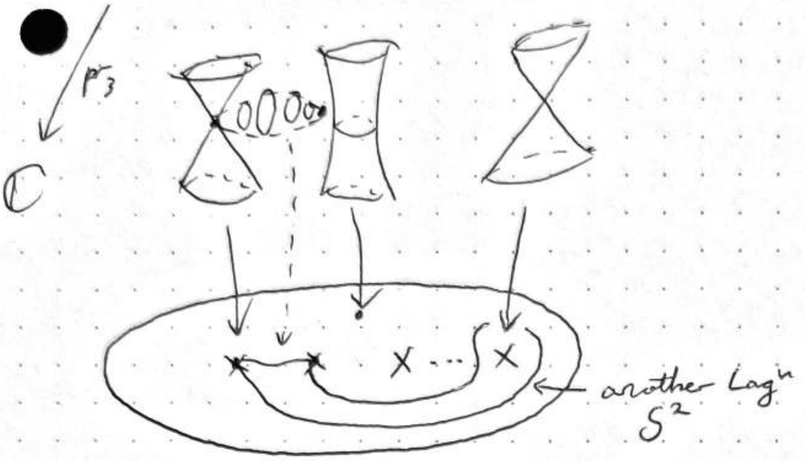
vs



No product of +ve twists can be isotopic to id.

Braid group actions by Dehn twists in higher dimension (esp. dim 4)

$\{x^2+y^2+z^d=1\} \in \mathbb{C}^3$



$Br_d = \pi_1 \mathrm{Conf}(d, B)$

(ρ_i)

- $\frac{1}{2}$ twist action on the base of π lifts (by pullback) to a diffeo of the total space M_{d-1}

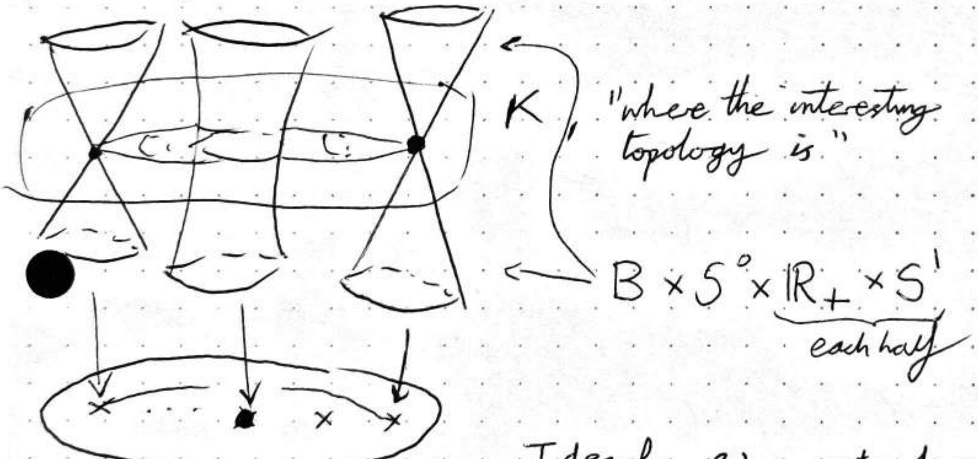
Prop / using Moser-type manipulations, can arrange for

1. cut-off the pullback of ρ_i to get a compactly supported map say $\tilde{\rho}_i$
2. $\tilde{\rho}_i \omega = \omega$
3. $\tilde{\rho}_i \simeq \tau_{S_i}$, where S_i is the matching sphere b/w the i^{th}, $(i+1)^{st}$ marked points

« a generalised fibration – a Lefschetz fibration »

Rk If S, S' are Lagrangian isotopic S^2's then $\tau_S = \tau_{S'} \in \pi_0 \mathrm{Symp}_c M$

Pf (sketch) Idea: trivialise the fibration outside some big compact set



defining $\tilde{\rho}_i$

- on K, use pullback ρ_i
- interpolate to the identity outside some larger K', using triviality of the fibration
- identity outside K'

Idea for 2): want 1 parameter family ω_t of exact symplectic forms s.t. $\omega_0 = \tilde{\rho}_i^* \omega$, $\omega_1 = \omega$

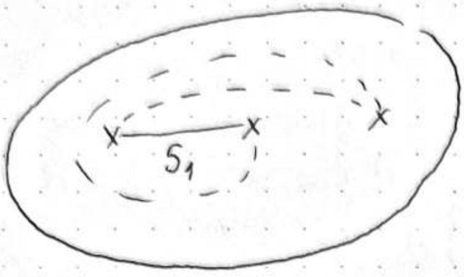
Loosely, we look at forms "$\omega_{fibre} + \omega_{base} + \xi$" where ξ does not affect being symplectic

$\cancel{\begin{pmatrix} \omega_F & * \\ 0 & \omega_{base} \end{pmatrix}}$

(e.g. $(\omega')^2 = \omega_{fib} \wedge \omega_{base}$)

⌈ correct framework: work with block-diagonal almost cx structures ⌋ $\square$

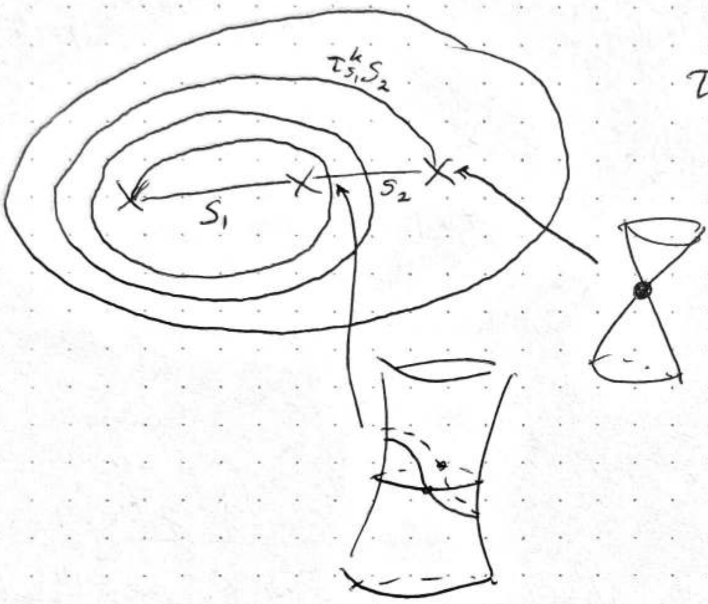
(S_1, S_2 Lag" spheres over x—x—x)



$\tau_{S_1} S_2$, $\tau_{S_1}^2 S_2$

$S_2 \pitchfork S_1 = \{pt\}$

[can locally arrange for it to look like a cotangent fibre]



$\tau_{S_1}^k S_2$

We recover the picture from the proof that τ_{S_1} had infinite order

Thm (Khovanov - Seidel)

Suppose S, S' are Lag matching spheres corr to matching paths γ, γ'.

Assume γ, γ' are in "minimal position"

Then

$$\mathrm{rk}\, HF(S, S') = \begin{array}{l} 2\#\text{ interior intersection} \\ + \#\text{ end intersection} \end{array}$$

(Idea: the intersection numbers being preserved $\Rightarrow$ iso to id on the base c.f. MCG)

Cor $Br_d \hookrightarrow \pi_0 \mathrm{Symp}_c(M_{d-1})$

Thm (Evans) No

Thm: $\pi_0 \mathrm{Symp}_c(M_{d-1}) \longleftarrow \mathrm{Br}_d$ is an isomorphism (Evans, Wu)

Two further viewpoints on M_{d-1}

A) Another example of a Lefschetz fibration:

$$\{x^2+y^2+\underbrace{z^d}_{p(z)} = 1\} = M_{d-1}$$

$p(z)$ a Morse function of degree d

$$\mathrm{proj}_x = \pi \downarrow$$

$\mathbb{C}$

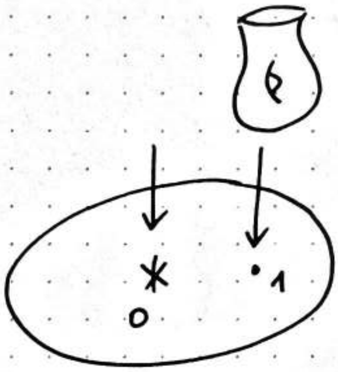
What does this look like?

$$\begin{array}{cc} \mathbb{C}^2 & (y,z) \\ \downarrow & \downarrow \\ \mathbb{C} & (y^2+p(z) = f(y,z)) \end{array}$$

fibre : F_{d-1}

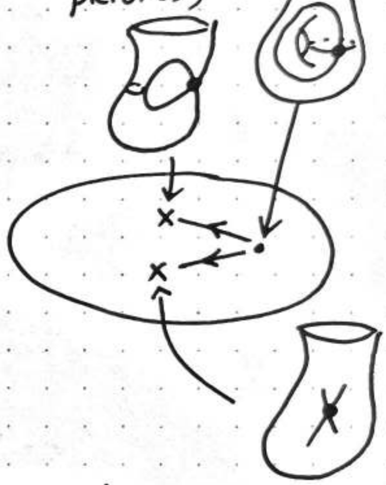
Critical pts / values

$$\begin{array}{cc} \mathbb{C}^2 & (x,z) \\ \downarrow & \downarrow \\ \mathbb{C} & x^2+z^d \end{array}$$

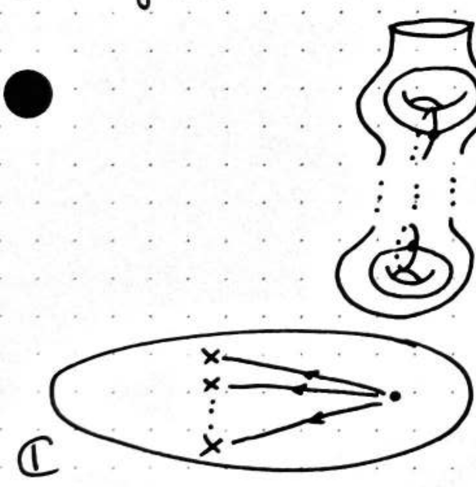


Morsify z^d

(d=3 for pictures)

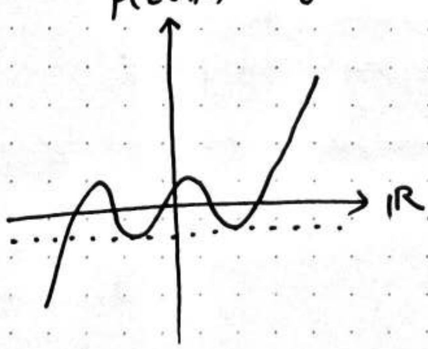


More general d:

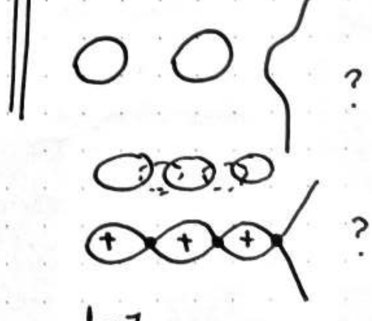


Why should I (!) expect this chain?

One approach: "work with real no.s for as long as possible"



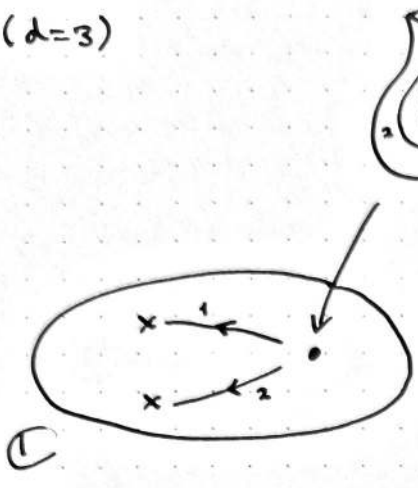
$x^2+p(z)=0$ real locus in $\mathbb{R}^2$

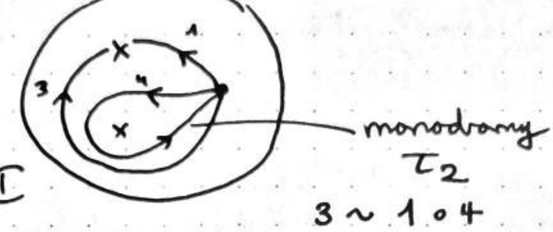


$d=7$

What if I (!) change the vanishing paths?

($d=3$)



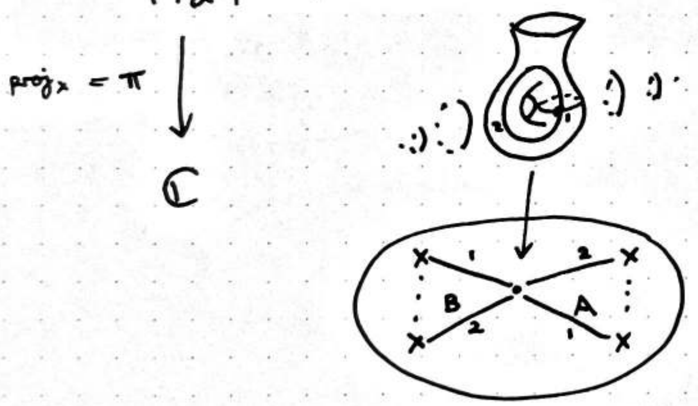


$3 \sim 1 \circ 4$

$\therefore$ vanishing cycle for 3 is $\tau_2^{\pm 1} a$

In general: changing vanishing path induces change of v. cycles by DT in each other

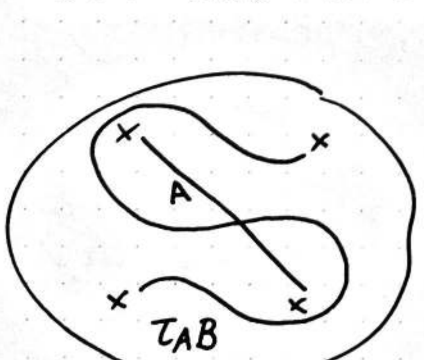
$M_{d-1} = \{x^2+y^2+z^d=1\}$



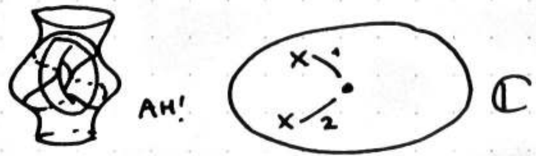
matching cycles
$\Downarrow$
Lagrangian spheres

⚠ Embedded segments with endpoints on critical values will almost never be matching paths

· Dehn twists: same local model, $\frac{1}{2}$ twist in matching cycle



Lefschetz fibration: symplectic 4-fold determined by
- central fibre (punct Riem surface)
- ordered collection of v. cycles



<u>Def</u> A <u>Weinstein domain</u> is a triple (W, λ, φ) where:

- W is a compact manifold-with-boundary
- $d\lambda = \omega$ is a symplectic form on W
- $\varphi : W \to \mathbb{R}$ is a Morse function s.t. ∂W is a regular level set
- X_λ (v. field s.t. $\omega(-, X_\lambda) = -\lambda$) is gradient-like for φ

⌈$X_\lambda = 0$ at crit⌋ → ⌈i.e. $X_\lambda \cdot \varphi > 0$ everywhere⌋

<u>Ex</u> smooth variety $Y \subset \mathbb{C}^n$, standard λ

$\varphi = \sum |z_i|^2$ possibly perturbed a tiny bit

$Y \cap B_R(0) := W$

<u>Def</u> An <u>abstract Lefschetz fibration</u> is $(\underset{F^{2n-2}}{\cancel{W}} ; L_1, \dots, L_k) = W_{abs}$

- $\underset{F}{\cancel{W}}$ a Weinstein domain
- $L_i \subset \underset{F}{\cancel{W}}$ exact Lagrangian spheres in $\underset{F}{\cancel{W}}$, parametrised

<u>Fact</u>: Given this data, can construct a geometric realisation $|W_{abs}| = W$ a Weinstein manifold, & Lefschetz fibration w/ smooth fibre and ordered collection of v. cycles $L_1, \dots, L_k$

<u>Thm</u> (Giroux-Pardon) Every Weinstein domain is Weinstein deformation equivalent to the total space of an abstract Lefschetz fibration

(<u>Def</u> $(N^{2n}, \omega = d\lambda)$ symplectic. $L^n \subset N^{2n}$ Lag submanifold

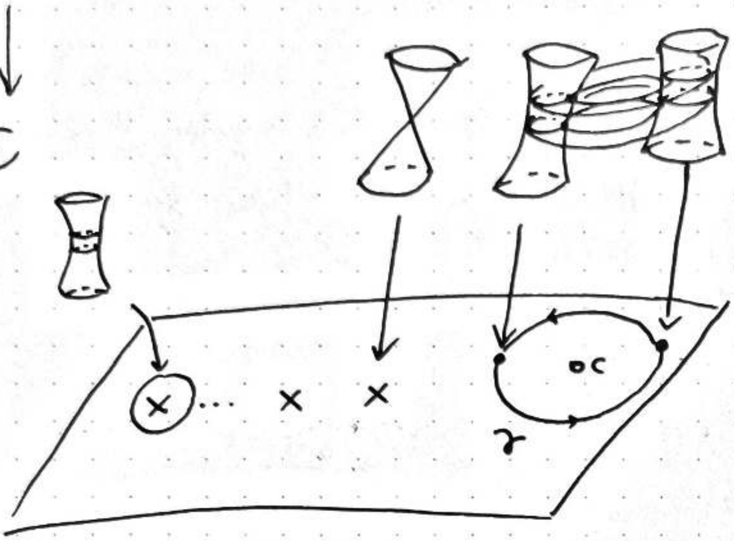
$\omega|_L \equiv 0$ so $\lambda|_L \in \Omega^1(L)$ is closed, so defines

$$[\lambda|_L] \in H^1_{dR}(L)$$

say L is <u>exact</u> if $[\lambda|_L] = 0$)

Ⓑ Lagrangian torus fibrations

$$\begin{array}{c} M_{d-1} \\ \pi \downarrow \\ \mathbb{C} \end{array}$$



tori around loop in base

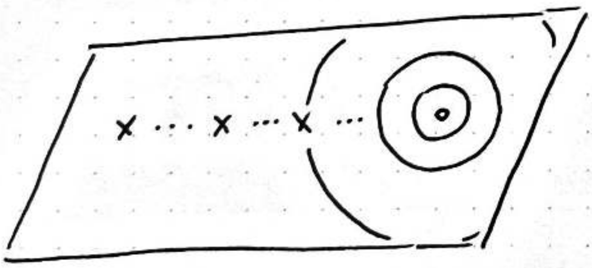
- Base: S^1 curve γ not enclosing any critical values
- $M_{d-1}|_{\pi^{-1}(\gamma)} \simeq$ [drawing] $\times \gamma$
- $\mathbb{R}^1$'s worth of Lagrangian tori given by $S^1_r \times \gamma$, $r =$ symp area between S^1_r & waist curve

Let c be a smooth value for π. Consider a system of concentric circles in the base of π centred at c.

Define $\rho : M_{d-1} \setminus \pi^{-1}(c) \longrightarrow \mathbb{R}^{k}_{>0} \times \mathbb{R}$

$$w \;\cancel{x}\; \longmapsto \big(d(\pi(\underset{w}{\cancel{x}}), c),\ \underbrace{\text{height } \overset{\in \mathbb{R}}{r}, \ x \in S^1_r}\big)$$

how to make this consistent ?

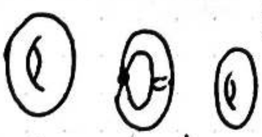


Note:

$x^2 + y^2 = k$ (constant) $\quad \Big\| \quad XY = k$

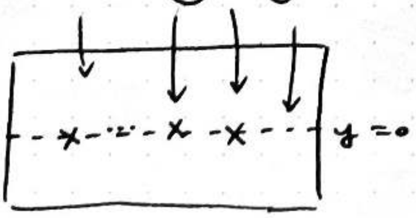
$e^{i\theta}.(X, Y) = (e^{i\theta}X, e^{-i\theta}Y)$

$$\begin{array}{c} M_{d-1} = \{x^2 + y^2 + p(z) = 1\} \\ \text{proj}_z \downarrow \\ \mathbb{C} \end{array}$$



Given $(x, y) \in \pi^{-1}(pt)$, this gives a preferred $S^1 \subset \pi^{-1}(pt)$ containing it.

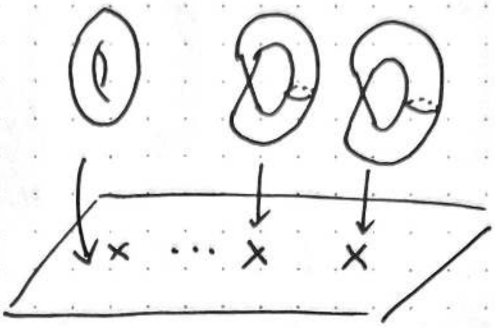
$$\begin{array}{c} M_{d-1} \setminus \pi^{-1}(c) \\ \downarrow \rho \\ \mathbb{R}^2 \end{array}$$



Last time:

$$\begin{array}{l} M_{d-1} \setminus \pi^{-1}(c) \\ \downarrow \rho \\ \mathbb{R}^2 \end{array}$$

Lagrangian torus fibration w/ d crit fibers, pinched tori



Remarks

1. $H_i = \pi_i \circ \rho : N_{d-1} \to \mathbb{R}$

Check: H_1 & H_2 Poisson commute:

X_{H_i} vector field on N_{d-1} s.t. $\iota_{X_{H_i}} \omega = dH_i$

$\omega(X_{H_1}, X_{H_2}) = 0$ Slogan: the flows of the 2 v. fields commute

$\quad \parallel$

$\{H_1, H_2\}$ "Poisson bracket"

2. (W^{2n}, ω) symplectic, $H_1, \dots, H_n$ non-degenerate Hamiltonian functions which pairwise Poisson commute: "integrable Hamiltonian system"

Arnol'd-Liouville thm

Suppose (W^{2n}, ω) symplectic, $\sigma : W^{2n} \to B \underset{\text{open}}{\subset} \mathbb{R}^n$ regular Lagrangian fibration (fibres compact Lag)

Then: - any fiber is a torus $T^n = F$

- locally: $F \times U \to U$ with $U \overset{\text{open}}{\subset} B$

with standard symp form from T^*U mod Λ:

$$\begin{tikzcd} (T^*U, \omega_{std}) \arrow[r, "\tilde{\sigma}"] \arrow[d] & U \\ \underbrace{T^*U/\Lambda^*}_{\text{inherits } \omega_{std}} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

$\Lambda^*_p = \mathbb{Z}^n \subset T^*_p U$

$T^*U \cong U \times \mathbb{R}^n$

Check: $\Lambda^* \circlearrowright T^*U$ by translation

These are symplecto's

Duistermaat: generalises A-L to regular Lag fibrations (all fibres smooth cpct) over an arbitrary base

The fibration must be of the form

$$T^*B/\Lambda^* \longrightarrow B \underset{\text{open}}{\subset} \mathbb{R}^n$$

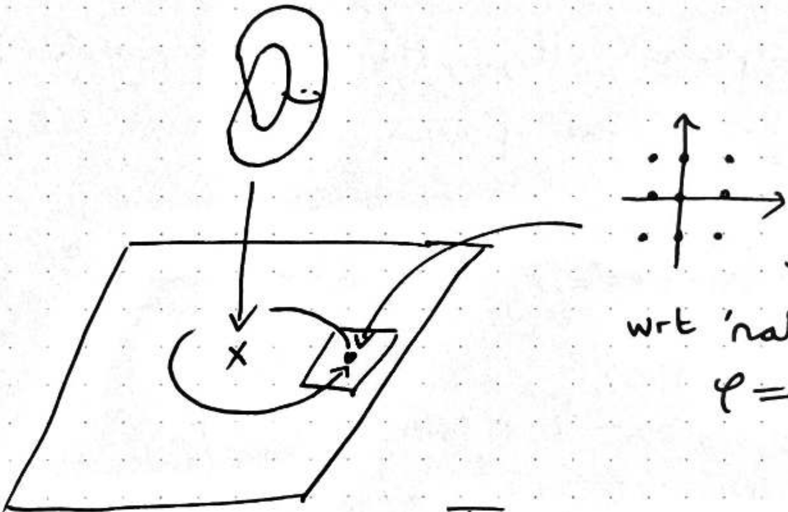
where $\Lambda^*_p \subset T^*_p B$ is a full lattice $(\simeq \mathbb{Z}^n)$ (smoothly varying)

s.t. the local sections of T^*B given by Λ^* are Lagrangian

[$\xi : V \to T^*V$ section
translation by ξ is symplecto iff $d\xi = 0$ iff ξ Lagrangian]

(Book: Evans, Lagrangian torus fibrations)

Toy example w/ one critical fibre



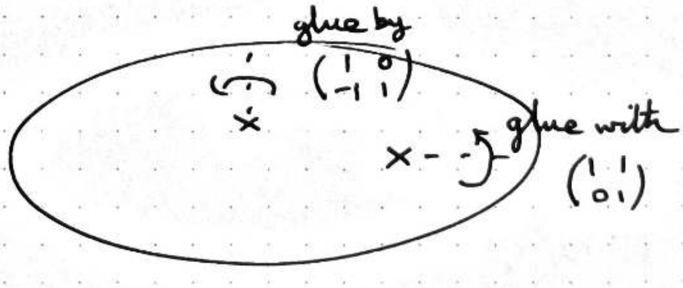
$\mathbb{Z}^2 \subset \mathbb{R}^2$

$\circlearrowleft SL(2,\mathbb{Z}) \ni \varphi$

wrt 'natural' basis

$\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

To get more examples; can glue together conjugating φ by elt of $GL(2,\mathbb{Z})$

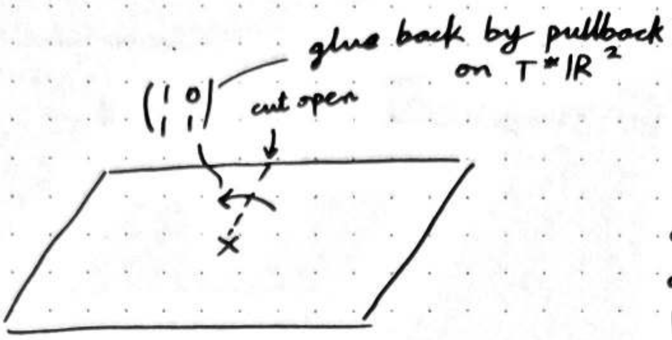


Building Lagrangian Torus Fibrations

Toy case: $T^*\mathbb{R}^2/\mathbb{Z}^2$

$$\downarrow$$

$$\mathbb{R}^2$$



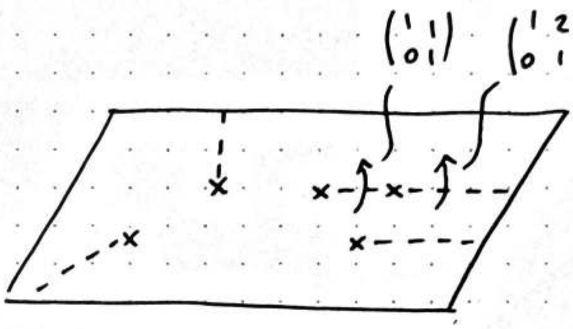
At the marked point: can fill in w/ a pinched Lag^n torus, local model from last time

Base is an integral affine manifold

transition functions of the form

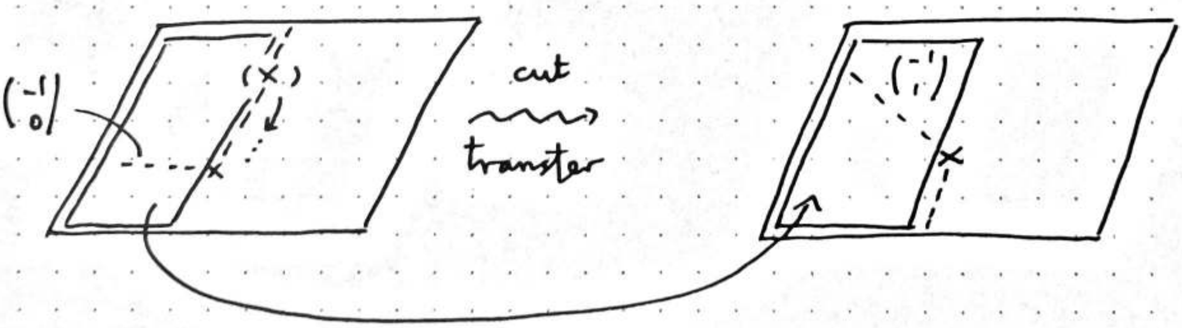
$$x \mapsto Ax + b, \quad A \in GL_2\mathbb{Z}, \quad b \in \mathbb{R}^2$$

- Can add:
  - more cuts w/ same invariant dir^n
  - cuts w/ other invariant dir^n



Modifications:

1) Nodal slide: "cut a bit more or a bit less along the invariant dir^n"



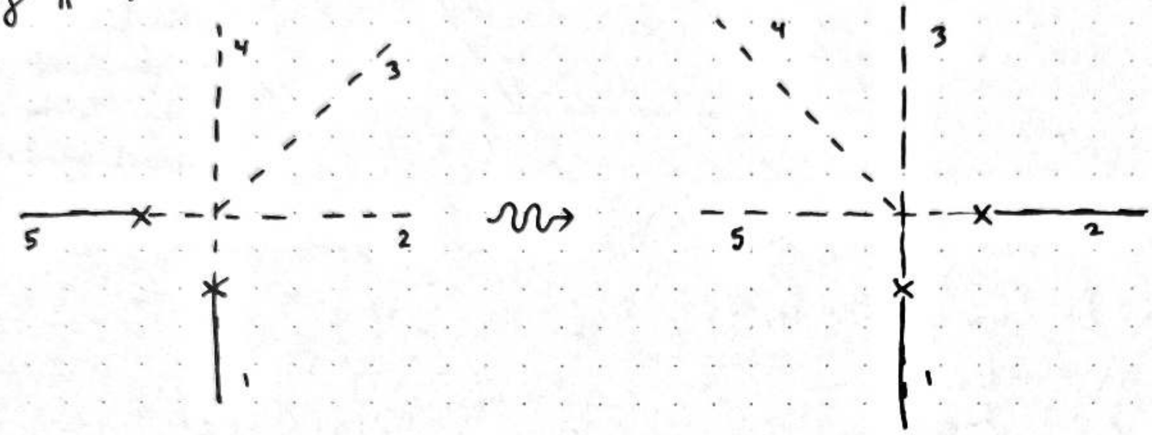
Assume all of our invariant directions / cuts are colinear

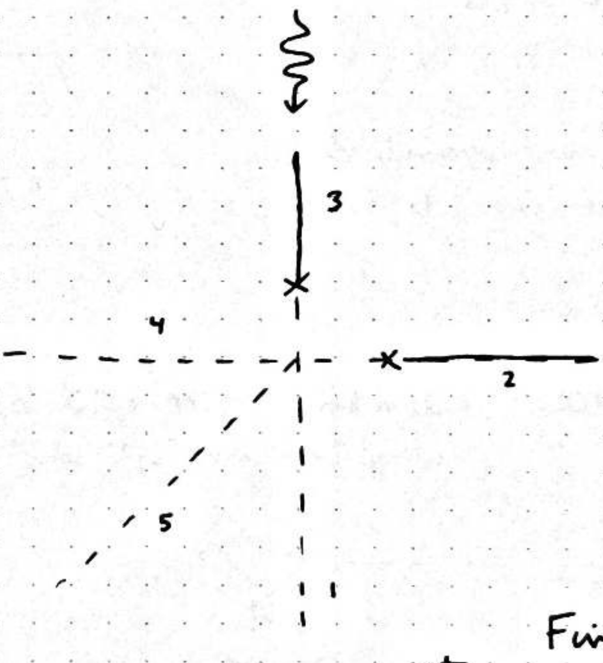
$PL(2,\mathbb{Z})$ piecewise integral transf^n of $\mathbb{Z}^2$

Fact: This is generated by the one above (used for cut transfer) and its $GL(2,\mathbb{Z})$ conjugates, AND elements of $GL(2,\mathbb{Z})$

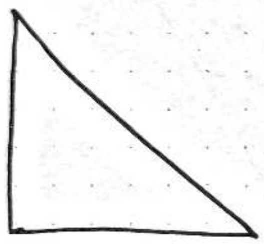
Observe: Suppose we have a sequence of cut transfers & we get back to the configuration we started with. By Moser, this defines a symplectomorphism of the total space.

Where to get these loops? Factorisations of birational transformations of $\mathbb{P}^2$.



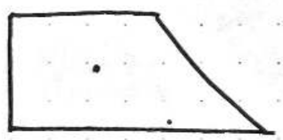


etc. Five cut transfers: get back to original labels



$\mathbb{P}^2$

$\downarrow$ blowup



$\mathbb{P}^2 \# \overline{\mathbb{P}}^2 / \Delta$

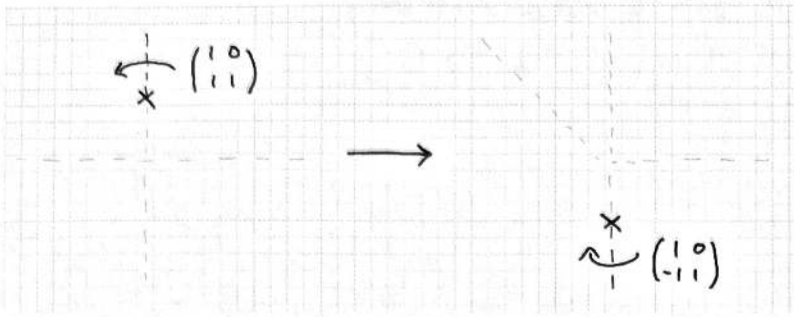
↑ toric divisor

$\mathbb{P}^2 \# \overline{\mathbb{P}}^2 \setminus E$

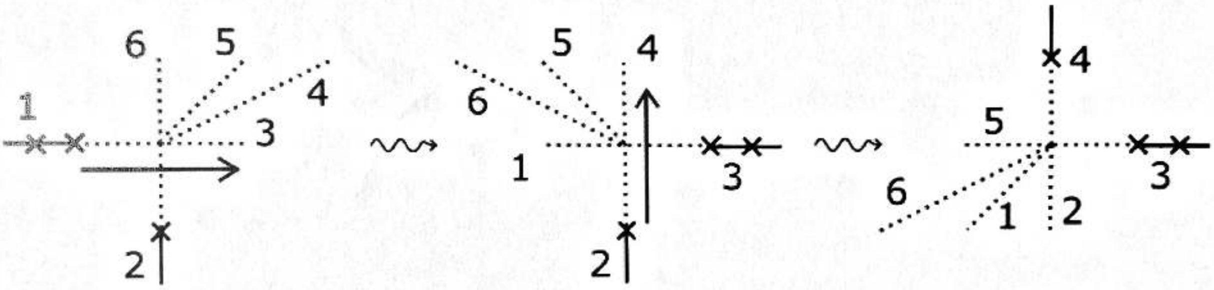
↑ smooth anticanonical divisor, given by smoothing corners of Δ

Fact: This gives a symplecto which is isotopic to Id, but not through fiber-preserv maps

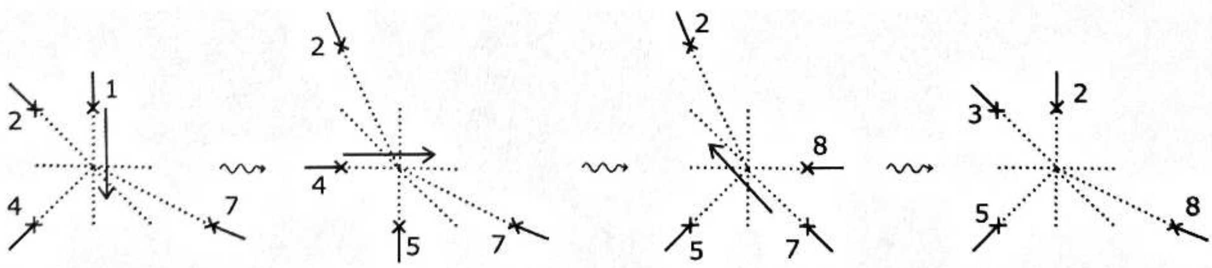
Model nodal slide + cut transfer:



~~First~~ Third **example:** Repeat these ~~five~~ three times to get a compactly supported map. It's actually isotopic to the identity!



Second example: Repeat these eight times to get a compactly supported map.



Check: Can calculate

- final symplecto ρ has non-trivial action on $H_2(M, \partial M)$
- $\nexists$ Lag^n spheres in M for homology reasons

$\rightsquigarrow$ ρ can't be a product of Dehn twists